Give a *direct* proof that if a number is odd, then its square is also odd i.e.

if m is odd, then m^2 is odd

Let $m = 2k+1$, where $k \in \mathbb{Z}$

Then $m^2 = (2m+1)^2$
$= 4m^2 + 4m + 1$
$= 2(2m^2 + 2m) + 1$

(as $\mathbb{Z}$ is closed under multiplication and addition, $2m^2 + 2m \in \mathbb{Z}$) Not required

$m^2 = 2t + 1$ for $t = 2m^2 + 2m \in \mathbb{Z}$

Hence m^2 is odd

(a) Let $A = \{0, 1, 2, 3, 4\}$ and

$$R = \{(a,b) \in A \times A : a + b \text{ is odd}\}.$$

Check whether R is reflexive, symmetric, antisymmetric and/or transitive. In each case where the relation doesn't have the property, give a counter-example. [4 marks]

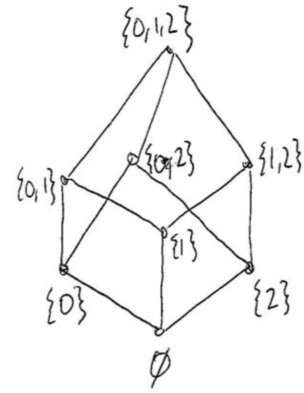
Not reflexive, $(1,1) \notin R$ as $1+1=2$ which is not odd

It is symmetric, if $a+b$ is odd then $b+a$ is odd, as $a+b = b+a$

It is not antisymmetric, $(1,2) \in R$ and $(2,1) \in R$, but $1 \neq 2$

It is not transitive, $(1,2) \in R$ and $(2,3) \in R$
$1+2$ is odd " $2+3$ is odd
but $(1,3) \notin R$
$1+3$ is even

Let $A = \mathcal{P}(\{0,1,2\})$ (the collection of all subsets of $\{0,1,2\}$). Draw the Hasse diagram of the partial ordering $R = \{(X,Y) \in A \times A : X \subseteq Y\}$. [4 marks]



Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is 1-1 but not onto.

[3 ı

$f(x) = 2x$

is 1-1 as $f(x_1) = f(x_2) \rightarrow 2x_1 = 2x_2$
$\rightarrow x_1 = x_2$

but not onto as there is no x which satisfies $f(x) = 1$

(b) Let $A = \{0, 1, 2, \ldots, 8\}$, and let

$$E = \{(a,b) \in A \times A : 3 \text{ divides } b - a\}.$$

List the E-equivalence classes.

$\{0,3,6\}$, $\{1,4,7\}$ and $\{2,5,8\}$

5. Induction.

Use induction to show that

$$2^0 + 2^1 + \cdots 2^n = 2^{n+1} - 1 \text{ for } n \geq 0$$

[5 marks]

Base case $n=0$
$LHS = 2^0 = 1$ and $RHS = 2^{0+1} - 1 = 2^1 - 1 = 1$
So $LHS = RHS$

Induction Hypothesis $n = k$
Assume $2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$

Induction Step $n = k+1$
Try to show that $2^0 + 2^1 + \cdots + 2^{k+1} = 2^{k+2} - 1$
$LHS = 2^0 + 2^1 + \cdots + 2^{k+1}$
$= (2^0 + 2^1 + \cdots + 2^k) + 2^{k+1}$
$= 2^{k+1} - 1 + 2^{k+1}$ by IH
$= 2 \cdot 2^{k+1} - 1$
$= 2^{k+2} - 1$
$= RHS$

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Let $A = \{0, 1, 2\}$. Is there a function $g : A \to A$ which is 1-1 but not onto? [1 marks]

No.

Assume g is 1-1,
then $|im(g)| \geq |dom(g)|$ (as g is 1-1)
but $|im(g)| \leq |A|$ (as the codomain of g is A)
and $|dom(g)| = |A|$ (as the domain of g is A)

So $|A| = |dom(g)| \leq |im(g)| \leq |A|$
Hence $|im(g)| = |A|$
and g is onto.

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Explain when a partial order R on set A is an equivalence relation.

A relation is a partial order and an equivalence relation if it is reflexive, transitive, symmetric and antisymmetric. Which relation can be symmetric and antisymmetric at the same time.

Suppose xRy. Then by symmetry yRx. Since R is antisymmetric, yRx, and xRy you should have $x = y$. It means that the only pairs you have in the relation are (x,x), where $x \in A$. Finally,

$$R = \{(x,x) : x \in A\}.$$

As was shown above R is symmetric and antisymmetric. R is clearly reflexive. What about transitivity? To check transitivity consider xRy and yRz, but in the relation we have only pairs where element related to itself. Hence $x = y$ and $y = z$. Therefore $x = y = z$ and xRz. R is transitive.

Let S be a relation on $\mathbb{R}$.

$$xSy \text{ iff } x^3 \leq y^3.$$

Is it a partial order? If yes is it a total order? Explain your answer.

Reflexivity: Let $x \in \mathbb{R}$. Then $x^3 \leq x^3$ and, hence, xSx. So S is reflexive.

Antisymmetry: Suppose xSy and ySx. It means that $x^3 \leq y^3$ and $y^3 \leq x^3$. It is possible only if $x = y$. Thus S is antisymmetric.

Transitivity: Let xSy and ySz. It means that $x^3 \leq y^3$ and $y^3 \leq z^3$. Thus, $x^3 \leq y^3 \leq z^3$ and $x^3 \leq z^3$. Therefore xSz.

S is a partial order.

S is a total order because it is a partial order and for any two elements $x, y \in \mathbb{R}$ you have $x^3 \leq y^3$ or $x^3 \geq y^3$, which is the same as xSy or ySx.

Suppose R is an equivalence relation on $\mathbb{Z}$.

$$aRb \text{ iff } a \equiv b (\text{mod } 2).$$

Describe sets $[0]_R, [1]_R, [2]_R, [3]_R, [4]_R, [5]_R, [-1]_R, [-2]_R, [-3]_R$.

$[0]_R$ consists of all $x \in \mathbb{Z}$ such that $0Rx$, that is, $0 \equiv x(\text{mod } 2)$. It means that $0 - x$ should be divisible by 2 or simply x is an even number. Thus,

$$[0]_R = \{x : \ x \in \mathbb{Z} \text{ and } x \text{ is even}\}$$

Any integer from $[0]_R$ will have an equivalence class equal to $[0]_R$. Therefore,

$$[0]_R = [2]_R = [4]_R = [-2]_R.$$

$[1]_R$ consists of all $x \in \mathbb{Z}$ such that $1Rx$, that is, $1 \equiv x(\text{mod } 2)$. It means that $1 - x$ should be divisible by 2 or simply x is an odd number. Thus,

$$[1]_R = \{x : \ x \in \mathbb{Z} \text{ and } x \text{ is odd}\}$$

Any integer from $[1]_R$ will have an equivalence class equal to $[1]_R$. Therefore,

$$[1]_R = [3]_R = [5]_R = [-1]_R = [-3]_R.$$