(a) Show that $k = 1/21$.

The probability density function (pdf) of a continuous random variable is given by the derivative of its cumulative distribution function (CDF). So, to find k, we first need to find the CDF.

The CDF F(x) is given as:

$F(x) = \int[1, x]\ kx^2\ dx$

Now, let's calculate this integral from 1 to x:

$F(x) = (k/3) * [x^3 - 1^3]$ $F(x) = (k/3) * (x^3 - 1)$

Now, we can calculate F(4):

$F(4) = (k/3) * (4^3 - 1) = (k/3) * (64 - 1) = (k/3) * 63$

Since F(4) should equal 1 (as it's the CDF of the entire probability space), we have:

$(k/3) * 63 = 1$

Now, solve for k:

$k/3 = 1/63$

Multiply both sides by 3:

$k = 3/63$

Simplify the fraction:

$k = 1/21$

So, we have shown that $k = 1/21$.

(b) Find $P(X \leq 3)$.

To find $P(X \leq 3)$, we need to calculate the CDF at $x = 3$:

$F(3) = (1/21) * (3^3 - 1) = (1/21) * (27 - 1) = (1/21) * 26 = 26/21$

(c) Find $P(X = 2)$.

Since X is a continuous random variable, the probability of X taking a specific value (like 2) is zero. In other words, $P(X = 2) = 0$ for a continuous random variable.

(d) Find E(X), $E(X^2)$, and Var(X).

To find the expected value (E(X)), we integrate X times the pdf from 1 to 4:

$E(X) = \int[1, 4]\ x * (1/21) * x^2\ dx$

$E(X) = (1/21) * \int[1, 4]\ x^3\ dx$

$E(X) = (1/21) * [(1/4) * x^4] \mid [1, 4]$

$E(X) = (1/21) * [(1/4) * 4^4 - (1/4) * 1^4]$

$E(X) = (1/21) * (256/4 - 1/4)$

$E(X) = (1/21) * (64 - 1)$

$E(X) = (1/21) * 63$

$E(X) = 3$

Now, let's find $E(X^2)$:

$E(X^2) = \int[1, 4]\ x^2 * (1/21) * x^2\ dx$

$E(X^2) = (1/21) * \int[1, 4]\ x^4\ dx$

$E(X^2) = (1/21) * [(1/5) * x^5] \mid [1, 4]$

$E(X^2) = (1/21) * [(1/5) * 4^5 - (1/5) * 1^5]$

$E(X^2) = (1/21) * [(1024/5) - (1/5)]$

$E(X^2) = (1/21) * (1023/5)$

$E(X^2) = 73$

Now, calculate Var(X):

$Var(X) = E(X^2) - (E(X))^2$ $Var(X) = 73 - 3^2$ $Var(X) = 73 - 9$ $Var(X) = 64$

So, $Var(X) = 64$.

2. (a) Show that $A = 3$.

To find the value of A, we need to ensure that the probability density function (pdf) integrates to 1 over its entire range. In this case, the range is $x \geq 1$.

$\int[1,\infty)\ f(x)\ dx = 1$

$\int[1,\infty)\ A/x^4\ dx = 1$

Let's calculate this integral:

$\int[1,\infty)\ A/x^4\ dx = A * [(x^{(-3)})/(-3)] \mid [1,\infty)$

Using the fundamental theorem of calculus:

$A * [(0 - (1^{(-3)})/(-3))] = 1$

$A * [(0 + 1/3)] = 1$

$A * (1/3) = 1$

$A = 3$

So, we have shown that $A = 3$.

(b) Find $P(3 \leq X \leq 4)$.

To find $P(3 \leq X \leq 4)$, we need to integrate the pdf from 3 to 4:

$P(3 \leq X \leq 4) = \int[3, 4]\ (3/x^4)\ dx$

$P(3 \leq X \leq 4) = 3 * \int[3, 4]\ (1/x^4)\ dx$

Now, calculate the integral:

$P(3 \leq X \leq 4) = 3 * [-x^{(-3)}/(-3)]\ |\ [3, 4]$

$P(3 \leq X \leq 4) = 3 * [(-1/3^3) - (-1/4^3)]$

$P(3 \leq X \leq 4) = 3 * [(1/27) - (1/64)]$

$P(3 \leq X \leq 4) = 3 * [(64 - 27)/(27 * 64)]$

$P(3 \leq X \leq 4) = (3 * 37)/(27 * 64)$

(c) Find $E(X)$.

To find the expected value ($E(X)$), we integrate X times the pdf from 1 to ∞:

$E(X) = \int[1, \infty)\ x * (3/x^4)\ dx$

$E(X) = 3 * \int[1, \infty)\ x^{(-3)}\ dx$

Now, calculate the integral:

$E(X) = 3 * [-x^{(-2)}/(-2)]\ |\ [1, \infty)$

$E(X) = 3 * [0 - (-1/2)]$

$E(X) = 3 * (1/2)$

$E(X) = 3/2$

So, $E(X) = 3/2$.

(d) Find $E(X^3)$.

To find $E(X^3)$, we integrate X^3 times the pdf from 1 to ∞:

$E(X^3) = \int_{[1, \infty)} x^3 * (3/x^4)\, dx$

$E(X^3) = 3 * \int_{[1, \infty)} x^{-1}\, dx$

Now, calculate the integral:

$E(X^3) = 3 * [\ln|x|] \mid [1, \infty)$

$E(X^3) = 3 * [\ln(\infty) - \ln(1)]$

Since $\ln(\infty)$ is undefined (approaches infinity), we have:

$E(X^3) = 3 * (\infty - 0)$

$E(X^3) = \infty$

So, $E(X^3)$ is infinite in this case.

3. (a) To find the cumulative distribution function (CDF) $F(x)$, we need to integrate the probability density function (pdf) $f(x)$ from 0 to x:

$F(x) = \int_{[0, x]} 5e^{-5t}\, dt$

Now, let's calculate this integral:

$F(x) = [-e^{-5t}] \mid [0, x]$ $F(x) = -(e^{-5x} - e^{0})$ $F(x) = 1 - e^{-5x}$

So, the full expression for the CDF is:

$F(x) = \{1 - e^{-5x}, x \geq 0\ 0, x < 0$

(b) To find the probability that the inter-arrival time is at least 10, we need to calculate $P(X \geq 10)$. Using the CDF we found in part (a):

$P(X \geq 10) = 1 - F(10)$ $P(X \geq 10) = 1 - (1 - e^{-5 * 10})$ $P(X \geq 10) = 1 - (1 - e^{-50})$ $P(X \geq 10) = 1 - 1 + e^{-50}$ $P(X \geq 10) = e^{-50}$

(c) To find $E(X)$ using integration by parts, we can use the formula for the expected value of a continuous random variable:

$E(X) = \int_{[0, \infty)} x * f(x)\, dx$

In this case, $f(x) = 5e^{-5x}$. Let's use integration by parts:

$E(X) = \int_{[0, \infty)} x * 5e^{-5x}\, dx$

Let $u = x$ and $dv = 5e^{-5x}\, dx$. Then, we have:

$du = dx$ and $v = -e^{-5x}$

Now, apply integration by parts:

$E(X) = (uv - \int v\, du) \mid [0, \infty)$

$E(X) = (x * (-e^{-5x})) - \int(-e^{-5x} * dx)) \mid [0, \infty)$

Now, calculate the limits of integration:

As x approaches ∞, both terms in the bracket approach 0.

As x approaches 0, the first term is 0, and the second term is 1 (integral of $-e^{-5x}$).

$E(X) = 0 - (-1) = 1$

So, $E(X) = 1$.

4. (a) The cumulative distribution function (CDF) F(y) for an exponential distribution with rate parameter λ is given by:

$F(y) = 1 - e^{-\lambda y}$

In this case, $\lambda = 1/2$, so the full expression for F(y) is:

$F(y) = 1 - e^{-y/2}$

(b) To evaluate $P(Y \leq 3)$, we simply use the CDF from part (a):

$P(Y \leq 3) = F(3) = 1 - e^{-3/2}$

(c) To evaluate $P(Y > 4)$, we can use the complement rule, which states that $P(Y > 4) = 1 - P(Y \leq 4)$. We already found $P(Y \leq 4)$ in part (b):

$P(Y > 4) = 1 - (1 - e^{-2}) = e^{-2}$

(d) To evaluate $P(2 < Y < 6)$, we can use the CDF:

$P(2 < Y < 6) = F(6) - F(2) = (1 - e^{-3}) - (1 - e^{-1})$

(e) The mean length of time that the insect spends on a plant, denoted as E(Y), for an exponential distribution with rate parameter λ is given by:

$E(Y) = 1/\lambda$

In this case, $\lambda = 1/2$, so:

$E(Y) = 1 / (1/2) = 2$

So, the mean length of time that the insect spends on a plant is 2 minutes.

5. To find these probabilities and the value of x, we can use the standard normal distribution (Z) and standardize the values of X using the formula:

$Z = (X - \mu) / \sigma$

Where:

- Z is the standardized value of X,
- X is the random variable we want to find the probability for,
- μ is the mean of X,
- σ is the standard deviation of X.

In this case, $\mu = 20$ and $\sigma^2 = 25$, so $\sigma = 5$.

(a) $P(X \leq 16)$:

Standardize 16: $Z = (16 - 20) / 5 = -0.8$

Now, find $P(Z \leq -0.8)$ using a standard normal distribution table or calculator.

(b) $P(X \leq 24)$:

Standardize 24: $Z = (24 - 20) / 5 = 0.8$

Now, find $P(Z \leq 0.8)$ using a standard normal distribution table or calculator.

(c) $P(17 \leq X \leq 23)$:

Standardize 17 and 23: $Z1 = (17 - 20) / 5 = -0.6$ $Z2 = (23 - 20) / 5 = 0.6$

Now, find $P(-0.6 \leq Z \leq 0.6)$ using a standard normal distribution table or calculator. This is the probability between Z1 and Z2.

(d) Find x when $P(X \geq x) = 0.4$:

First, find the corresponding Z value for the 0.4 probability. In a standard normal distribution, $P(Z \geq z) = 0.4$ corresponds to $z \approx 0.2533$.

Now, use the standardization formula to find x:

$0.2533 = (x - 20) / 5$

Solve for x:

$x - 20 = 0.2533 * 5$ $x - 20 = 1.2665$

$x \approx 20 + 1.2665$ $x \approx 21.2665$

So, $x \approx 21.27$ when $P(X \geq x) = 0.4$.