an example

$2 R 1$ and $1 R 4$ but
$2 \not R 4$

R is not transitive.

3) S on all people

$x R y$ iff x and y are related by blood

mum S child and child S dad
but mum $\not S$ dad

not transitive. ax^2+bx+c

Equivalence Relations

Def: R on A is eq. relation if

1) R is reflexive: aRa for any $a \in A$
2) R is symmetric if aRb then bRa
3) R is transitive if aRb and bRc then aRc

Notation R, $\sim$, $\equiv$, $\approx$, $\cong$

Example some sort of colour blindness on all colours

xRy if this person can not distinguish them

1) xRx as this person sees the same colour identically
2) xRy means that this person sees x, y the same. Hence yRx
3) xRy and $yRz \Rightarrow x, y, z$ are seen as the same xRz

Examples: R is on all VuW students

xRy iff x's and y's surnames start with the same letter

1) Refl. xRx, as x surname starts ~~tha~~ with the same letter as x

2) Sym. xRy then x, y surnames start with the same letter

Thus yRx

3) Transitivity xRy and yRz means that x, y, z surnames start

with the same letter. So $x R t$

2) R on $\mathbb{Z}$

xRy iff $x+y$ is even

$1R3$, $2R4$

1) Reflexivity.

Let $x \in \mathbb{Z}$

$x+x$

Case 1: x is even

$x+x$ = even + even = even

Case 2: x is odd

$x+x$ = odd + odd = even

Alternate

$x+x = 2x$

$\uparrow$ even

So $x+x$ is even and xRx

2) Symetry: if xRy It means

$x+y$ is even $\Rightarrow$ $y+x$ is also even Hence yRx

3) Transitivity: Let xRy and yRz.

It means that

$$\begin{array}{r} x+y = \text{even} \\ + \quad y+z = \text{even} \\ \hline x+y+y+z = \text{even} \end{array} \Rightarrow$$

$\Rightarrow -2y + x + 2y + z$ = even $- 2y$ $\Rightarrow$ $x+z$ = even $- \underbrace{2y}_{\text{even}}$ =

= even $\Rightarrow$ $x+z$ = even Therefore xRz

R is eq. rel.

3) S on $\mathbb{R}$

xSy iff $x \leq y$

It is not an. eq. relation.

S is not sym. because

$1S5$ but $5 \not S 1$

$1 \leq 5$ $5 \not\leq 1$

5) Let $a \equiv b \pmod{n}$ iff

$a-b$ is divisible by n

$1 \equiv 5 \pmod{4} \Rightarrow 1-5=-4$ is div by 4

$7 \equiv 14 \pmod{7}$ $7-14=-7$ is div by 7

remainders $7/4$

$\Rightarrow 7 =$ somethin div by 4 + remaide

$= \underbrace{4}_{} + 3$ ← remaider

whole part $7/4 = \underline{1}$

$7-4\cdot 1 = 3$

whole $(13/3) = 4$

remainder =

$13-4\cdot 3 = 1$

$13 = 4\cdot 3 + 1$

Example: $a \equiv b \pmod{5}$

$a-b$ is div. by 5

1 and 21

$\overset{\shortparallel}{5\cdot 0+1}$ $\overset{\shortparallel}{5\cdot 4+1}$

0   1   2   3   4   5

$0\cdot 3+\underline{0}$ $0\cdot 3+\underline{1}$ $0\cdot 3+\underline{2}$ $3\cdot 1+\underline{0}$ $3\cdot 1+\underline{1}$ $3\cdot 1+\underline{2}$

0   1   2   3   4   5   6   7   8

0   1   2   0   1   2   0   1   2

rem

R on $\mathbb{Z}$

xRy iff $x \equiv y \pmod{13}$

1) $x \in \mathbb{Z}$ $x - x = 0$ is div by 13, so $x \equiv x \pmod{13} \Rightarrow xRx$

2) xRy. It means $x \equiv y \pmod{13}$ or $x - y$ is div by 13

Thus $y - x = -(x-y)$ is also divisible by 13 $\Rightarrow y \equiv x \pmod{13}$

yRx

3) xRy and yRz. It means

$x \equiv y \pmod{13}$ and $y \equiv z \pmod{13}$

$x - y$ is div by 13 $y - z$ is div by 13

$x - z = x - z + 0 = x - z \underbrace{- y + y}_{0} = \underbrace{(x-y)}_{\text{div. by } 13} + \underbrace{(y-z)}_{\text{div by } 13} \Rightarrow$ div by 13

$x - z$ is div by 13

xRz

R is eq. rel.

Equivalence classes

R is an eq. rel on A

Def: $[a]_\sim = [a]_R =$

$= \{x : x \in A \text{ and } aRx\}$

equivalence class of a

Examples

1) R on $\{0,1,2,3,4\}$, R is eq. rel.

xRy iff $x+y$ is even

$[0]_R = \{2,4,0\}$

$[a]_R = \{a, \ldots\ldots\}$

$[2]_R = \{0,2,4\} = [0]_R$

$[4]_R = \{0,2,4\}$ $\quad [0]_R = [2]_R = [4]_R$

$[1]_R = \{1,3\} = [3]_R$

R is on $\mathbb{Z}$

xRy iff $x+y$ is even

$[0]_R = \{x \in \mathbb{Z}, x \text{ is even}\} = [2]_R = [-2]_R$
$= [4]_R$

$[1]_R = \{x \in \mathbb{Z} : x \text{ is odd}\}$

2) R is a relation on $\mathbb{Z}$

xRy iff $x \equiv y \pmod 5$

$x-y$ is div by 5

$[0]_R = \{x \in \mathbb{Z} : x \text{ is div by } 5\} =$

$= \{\ldots, -10, -5, 0, 5, 10, 15, \ldots\}$

$[1]_R = \{x \in \mathbb{Z}$: s.t. x is a multiple of $5+1$ (or the remainder is $1\}$ $= \{1, 6, 11, 16, 21, 26, \ldots -4, -9, -14, \ldots\}$