# Om Mistry ENGR123 Test 1:

$P \wedge Q$ (P and Q)
True only when both P and Q are true

| P | Q | $P \wedge Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Tautology: Proposition-that's always true
Contradiction: Proposition-that's always false
Contingent: Neither Tautology or Contradiction

| P | Q | $P \leftrightarrow Q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Conjunction And $P \wedge Q$ (P and Q) try when P and Q are true

| P | Q | $P \vee Q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

Equivalence: P if and only if Q

Laws of Logic:

Double negation: $P \equiv \neg\neg P$

De Morgan's laws:
$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

$P \to Q \equiv \neg P \vee Q$

Commutative laws:
$P \wedge Q \equiv Q \wedge P$
$P \vee Q \equiv Q \vee P$

Idempotent laws:
$P \wedge P \equiv P$
$P \vee P \equiv P$

Distributive laws:
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

$P \vee Q$ (P or Q) true if one of P or Q is true

| P | Q | P xor Q |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

P xor Q true when only one is true In English the word "or" has two different meanings. If someone said that they want "fish or steak for dinner", you would not expect them to eat both. This is the exclusive use of "or"

| P | ¬P |
|---|---|
| 0 | 1 |
| 1 | 0 |

¬P (not P or it is not the case that P) True when P is false

| P | Q | $P \to Q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

P -> Q (P implies Q)
If P then Q

| P | Q | $Q \to P$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

Q -> P, Q if P, if P then Q
Note that P → Q and Q → P have very different meanings. "All engineering students are undergrads" is not the same as "all undergrads are engineering students"

$\neg Q \to \neg P$

| P | Q | $\neg Q \to \neg P$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Double negation: $P \equiv \overline{\overline{P}}$

De Morgan's laws:
$\overline{P \cdot Q} \equiv \overline{P} + \overline{Q}$
$\overline{P + Q} \equiv \overline{P} \cdot \overline{Q}$

$P \to Q \equiv \overline{P} + Q$

Commutative laws:
$P \cdot Q \equiv Q \cdot P$
$P + Q \equiv Q + P$

Idempotent laws:
$P + P \equiv P$
$P \cdot P \equiv P$

Distributive laws:
$P + (Q \cdot R) \equiv (P + Q) \cdot (P + R)$
$P \cdot (Q + R) \equiv (P \cdot Q) + (P \cdot R)$

Associati
$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

Contrapositive: $(P \to Q) \equiv (\neg Q \to \neg P)$

Tautology: if $\mathbb{T}$ is a tautology, then
$P \vee \mathbb{T} \equiv \mathbb{T}$
$P \wedge \mathbb{T} \equiv P$

Contradiction: if $\mathbb{F}$ is a contradiction, ther
$P \vee \mathbb{F} \equiv P$
$P \wedge \mathbb{F} \equiv \mathbb{F}$

Inverter AND gate OR gate

- The truth-table for implication has some counter-intuitive properties, called false premises
- People are often surprised that $P \to Q$ is true when $P$ is false and $Q$ is true. For example,
  - If I swim regularly, then I will get fit.
- It is quite possible that you do not swim regularly but get fit for a different reason (e.g. running regularly).
- True or False: "If Wellington is the largest city in China then Toronto is the capital of France"?
- The above example shows an even weirder property of implication: $P \to Q$ is true when both $P$ and $Q$ are false. In computer science, this is known as "*Garbage in, garbage out*"

Let $\preceq$ be a partial order on $A$. If $a \preceq b$ or $b \preceq a$ then we say $a$ and $b$ are comparable. Otherwise, they are incomparable.
A total order is an order where all elements are comparable.

**Example**
Total orders:
- $\leq$ on $\mathbb{Z}$ or $\mathbb{R}$
- humans, with relation $a \preceq b$ if $b$ is taller than $a$ (measured to picometers)

**Not** total orders:
- The subset relation on a powerset
- the "ancestor+self" relation on humans

If Alex got an $A$ then she passed ENGR123.
Alex passed ENGR123.
Alex got an $A$.

More formally:
$$\frac{\begin{array}{l} P \to Q \\ Q \end{array}}{P}$$
Invalid!

All human beings are mortal
Socrates is a human being
Socrates is mortal

More formally:
$$\frac{\begin{array}{l} \forall x,\ P(x) \to Q(x) \\ P(\text{Socrates}) \end{array}}{Q(\text{Socrates})}$$
Valid!

## Some rules of inference

- *Modus ponens*
$$\frac{\begin{array}{l} P \\ P \to Q \end{array}}{Q}$$
- *Modus tollens*
$$\frac{\begin{array}{l} P \to Q \\ \neg Q \end{array}}{\neg P}$$
- *Or-elimination*
$$\frac{\begin{array}{l} P \vee Q \\ \neg P \end{array}}{Q}$$
- *And-elimination*
$$\frac{P \wedge Q}{P}$$

## Some rules of inferenc

- *Transitivity*
$$\frac{\begin{array}{l} P \to Q \\ Q \to R \end{array}}{P \to R}$$
- *Or-introduction*
$$\frac{P}{P \vee Q}$$
- *Contrapositive*
$$\frac{P \to Q}{\neg Q \to \neg P}$$
- *Implies-introduction*
$$\frac{Q}{P \to Q}$$

### Direct Proof

*Goal:* To prove $P \to Q$
*Strategy:* Assume $P$
⋮
$Q$

**Theorem**
*If $x$ is even and $y$ is even then $x + y$ is also even.*

**Theorem**
*If $n$ is an odd integer, then so is $n^2$.*

### Proof by contrapositive

*Goal:* To prove $P \to Q$
*Strategy:* Assume $\neg Q$
⋮
$\neg P$

**Theorem**
*ose that $n \in \mathbb{N}_+$ and $d$ divides $n$.*
*s odd then $d$ is odd.*

**rem**
*$+ 2$ is odd, then so is $n$.*

### f by contradiction

*l:* To prove $P \to Q$
*Strategy:* Assume $P$ and $\neg Q$
⋮
$R$
⋮
$\neg R$
nclusion: $(P \wedge \neg Q) \to (R \wedge \neg R)$ which is a contradiction.
erefore, $P$ cannot is true while $Q$ is false.
$P \to Q$.

**orem**
*$^3 + 5$ is odd then $n$ is even.*

**orem**
*is irrational.*

| R | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | × | × | × | |
| 2 | × | × | | × |
| 3 | × | | | |
| 4 | | | × | × |

- The **identity relation** on $A$ is
$$id_A = \{(a,a) : a \in A\} = \{(x,y) \in A \times A : x = y\}$$
In words, the identity relation is "each element is related to itself, and nothing else".
- The **empty relation** from $A$ to $B$ is
$$\emptyset \subset A \times B$$
e.g. let $A$ be the set of humans and $B$ be the set of dinosaurs. The relation between people and dinosaurs, $xRy$ if "they were born at the same nanosecond" is probably empty. In other words, not one person is related to a dinosaur in this sense.
- The **universal relation** $R = A \times B$.
"everything in $A$ is related to everything in $B$"

**Example**
Let $H$ be the set of all humans and
- $R_P = \{(x,y) \mid x \text{ is a parent of } y\} \subset H \times H$.
- $R_A = \{(x,y) \mid x \text{ is an ancestor of } y\} \subset H \times H$.
- $R_S = \{(x,y) \mid x \text{ is a sibling of } y\} \subset H \times H$.
- $R_B = \{(x,y) \mid x \text{ is a brother of } y\} \subset H \times H$.

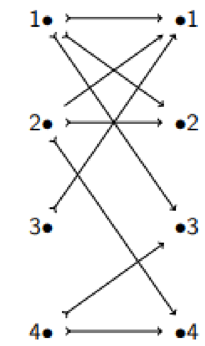
We sometimes write $aRb$ as a shorthand for $(a,b) \in$

- The **inverse relation** of $R \subset A \times B$ is
$$R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}.$$
Check: The inverse relation of ancestor is descendant; the inverse relationship of parent is child; the inverse relationship of sibling is sibling.
Check: $(R^{-1})^{-1} = R$.

## Visualizing relations

The relation
$$R = \{(1,1),(1,2),(2,1),(1,3),(2,2),(2,4),(3,1),(4,3),(4$$
can be visualized using a table:

Or a directed graph:



### Composing relations

If $R \subset A \times B$ and $S \subset B \times C$ are two relations, then the **composite** relation $RS \subset A \times C$ is given by
$$RS = \{(a,c) : \exists b \in B \text{ such that } aRb \wedge bSc\}.$$
We often write $a(RS)c$ instead of $(a,c) \in RS$.
The definition looks complicated, but is actually just common sense.

**Example**
If $R_B$ is the relation "is a brother of" and $R_P$ is the relation "is a p
then $R_B R_P$ is the relation "is an uncle of". In this case, the defini
that Bill is an uncle of Jane if and only if there exists some perso
that Bill is a brother of $x$ and $x$ is a parent of Jane.
- What is $R_P R_B$?
- The relation $R_P R_P$ ="grandparent".
  What is $R_A R_A$?
  What is $R_S R_S$?
- Check: $(RS)^{-1} = S^{-1}R^{-1}$.

### Properties of relations

**Example**
- *symmetric*: $R_S$
  "if I'm your sibling then you're my sibling"
- *not* symmetric: $R_B$ (brother)
  "just because I'm your brother does not mean you're my brother – you could be my sister"
- *not* symmetric: $R_A$ and $R_P$ (ancestor and parent)
  "just because I'm your ancestor does not mean you're my ancestor"

### Properties of relations

Relation $R$ on a set $A$ is
- **reflexive** if: $(a,a) \in R$ for all $a \in A$.
- **symmetric** if: $(a,b) \in R$ whenever $(b,a) \in R$.
- **antisymmetric** if: $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$.
- **transitive** if: $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$.

### Equivalence Relations

**Definition:** *Equivalence Relations*
An equivalence relation is a relation $R \subset A \times A$ that is
- reflexive: $a \sim a$ for all $a \in A$
- symmetric: if $a \sim b$ then $b \sim a$
- transitive: if $a \sim b$ and $b \sim c$, then $a \sim c$.

We often use $\sim$, instead of $R$, to denote equivalence relations.

- An equivalence relation is something "like" equality. Note that equality is reflexive (because $a = a$ for any $a$) symmetric (because if $a = b$ then $b = a$) and transitive (because if $a = b$ and $b = c$ then $a = c$).
- Equivalence relations give us a mathematically precise way to talk about relations between objects that are "approximately equal" in some sense or other. Examples are given below.

### Properties of relations

**Example**
- *reflexive*: $R_{Height} = \{(x,y) \in H \times H : x$ and $y$ are same heigh
  "I am the same height as myself"
- *not* reflexive: $R_P$ (parent)
  "I am not my own parent"
- *transitive*: $R_A$ (ancestor) and $R_S$ (sibling)
  "my ancestor's ancestor is my ancestor"
- *not* transitive: $R_P$ (parent)
  "my parent's parent is not my parent"

## Equivalence classes

**Definition:** *Equivalence class*
Let $\sim$ be an equivalence relation on $A$. Then the set
$$[a]_\sim = \{x \in A : x \sim a\}$$
is called the $\sim$ equivalence class of $a$.

### What is this all about?

- Equivalence relations capture the notion of abstraction. Abstraction is just a fancy word for "ignoring some details and paying attention to others".
- For example, if all I care about is peoples' ages – not their name, gender, intelligence, etc – then two people of the same age are *identical* as far as I'm concerned. And thus I *group* people by age.
- Similarly, if all I care about is the absolute value of numbers (and not their sign) then the numbers 2 and −2 are *identical* to me and I will group them together.
- Writing down an equivalence relation is a mathematical way of say "I care about *this aspect of things* and *nothing else*". It follows automatically that you're grouping things together.

e the equivalence classes for:
: identity relation $id_A$.
: universal relation $A \times A$
t $H$="humans" and $R$ ="have the same age".
$a, b) \in \mathbb{R} \times \mathbb{R} : a^2 = b^2\}$
$a, b) \in \mathbb{Z} \times \mathbb{Z} : a + b$ is even$\}$
t $\mathcal{P}$="all propositions".
$= \{(P,Q) \in \mathcal{P} \times \mathcal{P} : P \equiv Q\}$
e "pure-sibling+self" relation

**Example**
- $A = \{1,2,3,4,5,6\}$ and $\mathcal{C} = \{\{1,2,3\},\{4\},\{5,6\}\}$.
- $A = \mathbb{Z}$ and $\mathcal{C} = \{$even numbers, odd numbers$\}$.
- $A = \mathbb{Z}$ and $\mathcal{C} = \{\{0\},\{-1,1\},\{-2,2\},\{-3,3\},\ldots\}$.
- $A$ ="humans" and $\mathcal{C} = \{$people <1 year, people 1-2 years, ...$\}$
- $A$ ="humans" and $\mathcal{C} = \{$people from Texas, everyone else$\}$

## Partitions ↔ equivalence relations

We can convert back and forth between partitions and equivalence relations.
- Partition → equivalence relations
  Check:
  If $\sim$ is an equivalence relation on $A$ then
$$\mathcal{C} = \{[a]_\sim : a \in A\}$$
  is a partition of $A$.
- Partition ← equivalence relation
  Check:
  If $\mathcal{C}$ is a partition of $A$, then the relation
$$a \sim b \text{ iff } \left(\exists C \in \mathcal{C} \text{ satisfying } a \in C \text{ and } b \in C\right)$$
  is an equivalence relation on $A$.

1. Propositions and predicates

(a) Complete the following truth table, showing all working [4 marks]

| $P$ | $Q$ | $R$ | $(P \wedge \neg Q) \rightarrow (R \rightarrow P)$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 1 1 1 |
| 0 | 0 | 1 | 0 1 1 0 |
| 0 | 1 | 0 | 0 0 1 1 |
| 0 | 1 | 1 | 0 0 1 0 |
| 1 | 0 | 0 | 1 1 1 1 |
| 1 | 0 | 1 | 1 1 1 1 |
| 1 | 1 | 0 | 0 0 1 1 |
| 1 | 1 | 1 | 0 0 1 1 |

(b) Decide whether $(P \wedge \neg Q) \rightarrow (R \rightarrow P)$ is a tautology, a contradiction or contingent (with reasoning) [2 marks]

Tautology, the result is always true, no matter what variables are.

(c) Explain why $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N})(m \leq n)$ is true. [2 marks]

There is a smallest natural number $m$. This is true when $m=0$

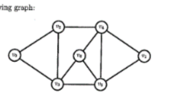
(d) Explain why $(\exists m \in \mathbb{Z})(\forall n \in \mathbb{Z})(m \leq n)$ is false. [2 marks]

This is false, there is no smallest integer

- For any $m$, set $n=m-1$
- Consider $(\forall m \in \mathbb{Z})(\exists n \in \mathbb{Z})(m > n)$, the negation of the original statement. This is true, every integer has another that is smaller.

6. Graphs.

Let $G$ be the following graph:



(a) List the degrees of the vertices of $G$.

| vertex | degree |
|---|---|
| $V_0$ | 2 |
| $V_1$ | 2 |
| $V_2$ | 3 |
| $V_3$ | 4 |
| $V_4$ | 4 |
| $V_5$ | 4 |
| $V_6$ | 3 |

(b) Does $G$ have an Eulerian walk? Explain.

Yes, it has exactly two vertices of odd degree, $V_2$ and $V_6$ and is connected

(c) Does $G$ have an Eulerian walk that starts at $v_0$? Explain

No, if a connected graph has two vertices of odd degree, any Eulerian walk must start and finish at those vertices.

2. Proofs. [4 marks]

Give a direct proof that if a number is odd, then its square is also odd i.e.

If $m$ is odd, then $m^2$ is odd

Let $m = 2k+1$, where $k \in \mathbb{Z}$

Then $m^2 = (2m+1)^2$
$= 4m^2 + 4m + 1$
$= 2(2m^2+2m)+1$

(as $\mathbb{Z}$ is closed under multiplication and addition, $2m^2+2m \in \mathbb{Z}$) Not required

$m^2 = 2t+1$ for $t = 2m^2+2m \in \mathbb{Z}$

Hence $m^2$ is odd

3. Relations

(a) Let $A = \{0,1,2,3,4\}$ and

$$R = \{(a,b) \in A \times A : a+b \text{ is odd}\}.$$

Check whether $R$ is reflexive, symmetric, antisymmetric and/or transitive. In each case where the relation doesn't have the property, give a counter-example. [4 marks]

Not reflexive, $(1,1) \notin R$ as $1+1=2$ which is not odd

It is symmetric, if $a+b$ is odd then $b+a$ is odd as $a+b=b+a$

It is not antisymmetric, $(1,2) \in R$ and $(2,1) \in R$, but $1 \neq 2$

It is not transitive, $(1,2) \in R$ and $(2,3) \in R$, $1+2$ is odd, $2+3$ is odd but $(1,3) \notin R$, $1+3$ is even

2. Proofs

Give a proof by contrapositive that, if the product of two numbers is even, then one of the numbers is even i.e.

If $mn$ is even, then either $m$ is even or $n$ is even

[4 marks]

The contrapositive statement is, if neither $m$ nor $n$ is even, then $mn$ is not even.

Suppose both $m$ and $n$ are odd.

Then $m = 2s+1$ and $n = 2t+1$

Hence $mn = (2s+1)(2t+1)$
$= 4st + 2s + 2t + 1$
$= 2(2st+s+t)+1$

As $\mathbb{Z}$ is closed under $\times$ & $+$, $2st+s+t$ is an integer

So $mn$ is odd.

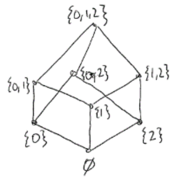
(b) Let $A = \{0,1,2,\ldots,8\}$, and let

$$E = \{(a,b) \in A \times A : 3 \text{ divides } b-a\}.$$

List the $E$-equivalence classes. [2 marks]

$\{0,3,6\}$, $\{1,4,7\}$ and $\{2,5,8\}$

(c) Let $A = \mathcal{P}(\{0,1,2\})$ (the collection of all subsets of $\{0,1,2\}$). Draw the Hasse diagram of the partial ordering $R = \{(X,Y) \in A \times A : X \subseteq Y\}$. [4 marks]



(d) Suppose that we attempt to carry out Fleury's algorithm on $G$, starting at $v_0$, and always taking the next permissible vertex with smallest possible index.

What is the path created by the algorithm until it gets stuck? [2 marks]

$V_0 \rightarrow V_2 \rightarrow V_3 \rightarrow V_5 \rightarrow V_1$ bridge but ok

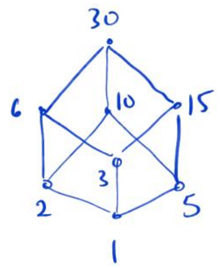
$V_3 \leftarrow V_6 \leftarrow V_5 \leftarrow V_4$

bridge but only other option is also a bridge — bridge but ok — avoided bridge to $V_2$

ii. A. What is required for a relation $S$ (on a set $A$) to be a partial order? [2 marks]

Reflexive, Antisymmetric and Transitive

B. Suppose $S$ is the divisibility relation on $\{1,2,3,5,6,10,15,30\}$ (the factors of 30). Draw the Hasse diagram of $S$. [4 marks]



4. Functions

(a) Give an example of a function $f: \mathbb{N} \rightarrow \mathbb{N}$ which is 1-1 but not onto. [3 marks]

$f(x) = 2x$

is 1-1 as $f(x_1) = f(x_2) \rightarrow 2x_1 = 2x_2 \rightarrow x_1 = x_2$

but not onto as there is no $x$ which satisfies $f(x)=1$

(b) Let $A = \{0,1,2\}$. Is there a function $g: A \rightarrow A$ which is 1-1 but not onto? [1 marks]

No.

Assume $g$ is 1-1,

then $|im(g)| \geq |dom(g)|$ (as $g$ is 1-1)

but $|im(g)| \leq |A|$ (as the codomain of $g$ is $A$)

and $|dom(g)| = |A|$ (as the domain of $g$ is $A$)

So $|A| = |dom(g)| \leq |im(g)| \leq |A|$

Hence $|im(g)| = |A|$ and $g$ is onto.

5. Induction.

Use induction to show that

$$2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1 \text{ for } n \geq 0$$

[5 marks]

Base case $n=0$

$LHS = 2^0 = 1$ and $RHS = 2^{0+1} - 1 = 2^1 - 1 = 1$

So $LHS = RHS$

Induction Hypothesis $n=k$

Assume $2^0 + 2^1 + \cdots + 2^k = 2^{k+1} - 1$

Induction Step $n=k+1$

Try to show that $2^0 + 2^1 + \cdots + 2^{k+1} = 2^{k+2} - 1$

$LHS = 2^0 + 2^1 + \cdots + 2^{k+1}$
$= (2^0 + 2^1 + \cdots + 2^k) + 2^{k+1}$
$= 2^{k+1} - 1 + 2^{k+1}$ by IH
$= 2 \cdot 2^{k+1} - 1$
$= 2^{k+2} - 1$
$= RHS$

(b) The relation $h = \{(1,2),(2,1),(3,2)\}$ is a function from $\{1,2,3\}$ to $\{1,2\}$.

Determine whether $h$ is one-to-one or onto. Make sure to state your decision clearly, and give some reasoning. [2 marks]

$h$ is not 1-1 as $(1,2) \in h$ and $(3,2) \in h$

$h$ is onto as the range $\{2,1,2\}$ equals the codomain $\{1,2\}$

(c) Suppose $f$ is a 1-1 function from $A$ to $B$, and $g$ is a 1-1 function from $B$ to $C$. Prove that $g \circ f$ is 1-1 (from $A$ to $C$). [2 marks]

We'll check $g \circ f(x_1) = g \circ f(x_2)$ implies $x_1 = x_2$

Suppose $g \circ f(x_1) = g \circ f(x_2)$

$\leftrightarrow g(f(x_1)) = g(f(x_2))$

$\rightarrow f(x_1) = f(x_2)$ as $g$ is 1-1

$\rightarrow x_1 = x_2$ as $f$ is 1-1

1. Propositions and predicates

(a) Complete the following truth table [6 marks]

| $P$ | $Q$ | $R$ | $(Q \rightarrow R) \vee (P \rightarrow Q)$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 1 1 |
| 0 | 0 | 1 | 1 1 1 |
| 0 | 1 | 0 | 0 1 1 |
| 0 | 1 | 1 | 1 1 1 |
| 1 | 0 | 0 | 1 1 0 |
| 1 | 0 | 1 | 1 1 0 |
| 1 | 1 | 0 | 0 0 1 |
| 1 | 1 | 1 | 1 1 1 |

(1) (3) (2)

(b) Decide whether $(Q \rightarrow R) \vee (P \rightarrow Q)$ is a tautology, a contradiction or contingent (with reasoning) [2 marks]

It is contingent, it has a mixture of true and false values.

(c) Explain why $(\forall m \in \mathbb{Z})(\exists n \in \mathbb{Z})(m \text{ divides } n)$ is true. [2 marks]

For each integer $m$, we want an integer $n$ such that $\exists t \in \mathbb{Z}(n = tm)$

For each $m$, set $n$ equal to $m$, then $m$ divides $n$.

3. Relations

(a) i. Suppose $R$ is a relation on $A = \{0,1,2,3,4\}$ defined by

$aRb$ iff $a \times b$ is even

Check whether $R$ is reflexive, symmetric, antisymmetric and/or transitive. [4 marks]

In each case where the relation doesn't have the property, give a short reason why or an example.

$R$ is only symmetric.

$R$ is not reflexive as $1 \times 1$ is not even, so $1R1$ is not true.

$R$ is not antisymmetric e.g. $1R2$ and $2R1$, but $1 \neq 2$.

$R$ is not transitive e.g. $1R2$ and $2R3$ but $1 \times 3$ is not even, so $1R3$ is not true

2. Consider the following (short) jumbled argument about birds, using the predicate P for parrot, G for gigantic and K for Kea:

(1) All parrots are gigantic;
(2) Nothing gigantic is fearful;
(3) All kea are parrots.

(a) Rewrite each statement using predicates. [3 marks]

(b) State the contrapositive version of (2). [2 marks]

(c) Derive a conclusion, and order all the statements so that the conclusion follows logically from the premises. [2 marks]

a)

(1) $\forall x(P(x) \rightarrow G(x))$

(2) $\forall y(G(y) \rightarrow \neg F(y))$ OR $\forall y(F(y) \rightarrow \neg G(y))$

(3) $\forall z(K(z) \rightarrow P(z))$

b) alternative in (2)

c) (3) (2) (1) Should be 3, 2, 1

$\forall a(K(a) \rightarrow \neg F(a))$

No kea are fearful.

| $P$ | $Q$ | $R$ | $P \implies \neg(Q \vee \neg R)$ | $(P \wedge Q) \iff (R \implies Q)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 0 1 1 | 0 0 1 |
| 0 | 0 | 1 | 1 1 0 0 | 0 1 0 |
| 0 | 1 | 0 | 1 0 1 1 | 0 0 1 |
| 0 | 1 | 1 | 1 0 1 0 | 0 0 1 |
| 1 | 0 | 0 | 0 0 1 1 | 0 0 1 |
| 1 | 0 | 1 | 1 1 0 0 | 0 1 0 |
| 1 | 1 | 0 | 0 0 1 1 | 1 1 1 |
| 1 | 1 | 1 | 0 0 1 0 | 1 1 1 |
| | | | 4 3 2 1 | 1 3 2 |

4. Induction.

Use induction to show that

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right) = \frac{1}{n}, \text{ for } n \geq 2$$

[6 marks]

Base Case: $n=2$

$LHS = 1 - \frac{1}{2} = \frac{1}{2}$, $RHS = \frac{1}{2}$ — they match!

Induction Hypothesis: $n=k$

$(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{k}) = \frac{1}{k}$

Induction Step: check $n=k+1$

$LHS = (1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{k+1})$
$= (1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{k})(1-\frac{1}{k+1})$
$= \frac{1}{k}(1-\frac{1}{k+1})$ by IH
$= \frac{1}{k}(\frac{k+1}{k+1}-\frac{1}{k+1})$
$= \frac{1}{k}(\frac{k}{k+1}) = \frac{1}{k+1}$

5. Eulerian walks

(a) Check that the graph $G_1$ (in figure 1 on page 9) is Eulerian (it suffice). [1 mark]

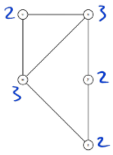


Figure 1: $G_1$

All are even except two odd degree vertices, So $G_1$ is Eulerian.

(b) Use Fleury's algorithm to find an eulerian walk, under the following restrictions.

- Start at vertex $w$.
- Where-ever possible, choose the vertex that comes first alphabetically e.g. choose y over z.
- Make sure to indicate bridges, taken or not!

[4 marks]

$w \rightarrow v \xrightarrow{B} x \rightarrow y \xrightarrow{B} z \xrightarrow{B} w \xrightarrow{B} x$

B indicates a bridge.

3. Determine the truth values of the following statements, provide a brief explanation in each case. [4 marks]

(a) $(\exists a \in \mathbb{N})(\forall b \in \mathbb{N})(a^2 > b)$

(b) $(\forall b \in \mathbb{N})(\exists a \in \mathbb{N})(a^2 > b)$

(a) False. Suppose $b = a^2+1$. Clearly $a^2 > a^2+1$ is false. So no $a$ exists that works for all $b$.

(b) True. In general, let $a = b+1$. $(b+1)^2 > b$ is true when $b^2+b+1 > 0$ which is true for all $b \in \mathbb{N}$

4. What is the negation of

$$(\exists a \in \mathbb{N})(\forall b \in \mathbb{N})(a^2 > b)$$

[2 marks]

$\neg(\exists a \in \mathbb{N})(\forall b \in \mathbb{N})(a^2 > b)$

$\equiv (\forall a \in \mathbb{N})\neg(\forall b \in \mathbb{N})(a^2 > b)$

$\equiv (\forall a \in \mathbb{N})(\exists b \in \mathbb{N})\neg(a^2 > b)$

$\equiv (\forall a \in \mathbb{N})(\exists b \in \mathbb{N})(a^2 \leq b)$

5. (a) What properties must a relation satisfy to be a partial order? [3 marks]

(b) Let $R$ be the relation on the set of movie actors, where $(a,b) \in R$ iff they have acted in a movie together.

Show that $R$ is reflexive and symmetric, but not transitive. [3 marks]

(c) Explain why $\{1,2\},\{2,3,4\},\{4,5,6\}$ is not a partition of $\{1,2,3,4,5,6\}$ [2 marks]

(a) Reflexive, Antisymmetric & Transitive.

(b) Reflexive: movie actors have all acted in a movie with themself.

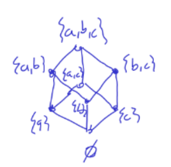
Symmetric: If $a$&$b$ have acted together, so have $b$&$a$!

Not transitive: Could be different movies

(c) The element 2 appears in two cells $\{1,2\}$ and $\{2,3,4\}$ when it should appear in precisely one cell.

6. Let $X = \mathcal{P}(\{a,b,c\})$ be the powerset of $\{a,b,c\}$.

Draw a Hasse diagram of the partial order $R$ on $X$, where $(A,B) \in R$ iff $A \subseteq B$. [3 marks]



| $P$ | $Q$ | $R$ | $\neg P \leftrightarrow (Q \wedge R)$ | $P \rightarrow (Q \rightarrow (\neg R \wedge P))$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 0 0 | 1 1 1 0 |
| 0 | 0 | 1 | 1 0 0 | 1 1 0 0 |
| 0 | 1 | 0 | 1 0 0 | 1 0 1 0 |
| 0 | 1 | 1 | 1 1 1 | 1 0 0 0 |
| 1 | 0 | 0 | 0 1 0 | 1 1 1 1 |
| 1 | 0 | 1 | 0 1 0 | 1 1 0 0 |
| 1 | 1 | 0 | 0 1 0 | 1 1 1 1 |
| 1 | 1 | 1 | 0 0 1 | 0 0 0 0 |
| | | | 1 Fin 2 | Fin 3 1 2 |

5. (a) What properties must a relation satisfy to be an equivalence relation? [3 marks]

(b) Let $R \subseteq \mathbb{Z} \times \mathbb{Z}$ be the relation on integers given by

$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x+y \text{ is even}\}$

i. Is $R$ a partial order? Explain why or why not. [2 marks]

ii. Is $R$ a function? Explain why or why not. [2 marks]

(a) Reflexive, Symmetric, Transitive

(b) $R$ is reflexive, $a+a=2a$ is always even.

$R$ is symmetric, if $a+b=2k$ then $b+a=2k$ as well!

$R$ is transitive, if $a+b=2k$ and $b+c=2l$, then

$(a+b)+(b+c) = 2k+2l$
$a+2b+c = 2k+2l$
$a+c = 2k+2l-2b$
$= 2(k+l-b)$

So $a+c$ is even!

6. Prove by contraposition that if the product of two natural numbers is greater than 100, then at least one of them is greater than or equal to 10. [5 marks]

Suppose both natural numbers are not greater than or equal to 10.

Then $x < 10$ and $y < 10$.

So $xy < 10 \times 10 = 100$

Hence the product is less than 100.

So the product is not greater than or equal to 100

7. Prove by contradiction that the sum of two positive numbers is positive. [5 marks]

Suppose $x > 0$, $y > 0$ and $x+y \not> 0$

Then $x+y > 0+0 = 0$

but $x+y \not> 0$

Contradiction!