I. V. Savelyev

PHYSICS

A General Course

MECHANICS

MOLECULAR PHYSICS

I

Mir Publishers
Moscow

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A GENERAL COURSE

(In three volumes)

VOLUME I

MECHANICS
MOLECULAR PHYSICS

MIR PUBLISHERS
MOSCOW

Translated from Russian by G. Leib

First published 1980
Revised from the 1977 Russian edition
Second printing 1985
Third printing 1989

Printed in the Union of Soviet Socialist Republics

ISBN 5-03-000901-9, 1977
ISBN 5-03-000900-0, 1980

AUTHOR'S PREFACE TO THE ENGLISH EDITION

The present book is the first volume of the three-volume general course in physics. The course is a result of twenty five year's work in the Department of General Physics of the Moscow Institute of Engineering Physics. I was in constant personal contact with my students not only at lectures, but also, perhaps even more importantly, at exercises, consultations, and examinations. These fruitful contacts helped me refine and improve the exposition of the various topics in the course.

The advice and friendly criticism of my colleagues in the department has also been a great help. I would like to make a special mention of the part played by N. B. Narozhny, who, in particular, is the author of the original and comparatively simple statistical derivation of the equation $\mathrm{d}S = \mathrm{d}'Q/T$ [Eq. (**??**)].

In writing the book, I have done everything in my power to acquaint students with the basic ideas and methods in physics and to teach them how to think physically. This is why the book is not encyclopedic in its nature. It is mainly devoted to explaining the meaning of physical laws and showing how to apply them consciously. What I have tried to achieve is a deep knowledge of the fundamental principles of physics rather that a shallower acquaintance with the a wide range of questions.

While using the book, try not to memorise the material formalistically and mechanically, but logically, *i.e.*, memorise the material by thoroughly understanding it. I have tried to present physics not as a science for "cramming", not as a certain volume of information to be memorised, but as a clever, logical, and attractive science. It is left to the reader to judge the extent to which I have succeeded in doing this.

Acknowledging the fact that a thick volume by its very appearance makes a student despondent, I have done my utmost to limit the size of the course. This was achieved by carefully choosing the material which in my opinion should be included in a general course of physics. I also tried to be concise, but not at the expense of clarity.

Notwithstanding my desire to reduce the size, I considered it essential to include a number of mathematical sections in the course: on vectors, linear differential equations, the basic concepts of the theory of probability, etc. This was done to impart a "physical" tinge to the relevant concepts and relations. In addition, the mathematical "inclusions" make it possible to go on with the physics even if, as is often the case, the relevant material has not yet been covered in a mathematics course.

The present course is intended above all for higher technical schools with an extended syllabus in physics. The material has been arranged, however, so that the book can be used as a teaching aid for higher technical schools with ordinary syllabus simply omitting some sections.

Igor Savelyev

Moscow, July, 1979

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INTRODUCTION

Physics is a science dealing with the most general properties and forms of motion of matter.

A classical definition of matter was given by V. Lenin in his book *Materialism and Empirio-Criticism*: "Matter is a philosophical category denoting the objective reality which is given to man by his sensations, and which is copied, photographed and reflected by our sensations, while existing independently of the"[1]. Two propositions are significant in this definition, namely, (1) matter is what exists objectively, *i.e.*, independently of anyone's consciousness or sensations, and (2) matter is copied and reflected by our sensations and, consequently, is cognizable.

It follows from the definition of physics that it concentrates knowledge accumulated on the most general properties and phenomena of the world surrounding us. As academician S. Vavilov noted in one of his articles, "the extremely common character of a considerable part of the contents of physics, its facts and laws drew physics and philosophy together from time immemorial.… Sometimes physical statements have such a nature that they are difficult to distinguish and separate from philosophical statements, and a physicist must be a philosopher".

Two kinds of matter are known at present: substance and field. The first kind of matter—substance—includes, for example, atoms, molecules, and all bodies built of them. Electromagnetic, gravitational, and other fields form the second kind of matter. The different kinds of matter can change into each other. For instance, an electron and a positron (representatives of substance) may transform into photons (*i.e.*, into an electromagnetic field). The reverse process is also possible.

Matter is in continuous motion, which is understood to mean any change in general in dialectical materialism[2]. Motion is an inalienable property of matter,

[1]V. I. Lenin. *Collected Works*, Vol. 14, p. 130. Moscow, Foreign Languages Publishing House (1962).

[2]Dialectical materialism is the name given to the Marxist-Leninist philosophy. The fundamental issue of any philosophy as to what is primary—matter or consciousness—is solved by dialectic materialism in favour of matter when it states that matter is primary and consciousness is secondary. The method of this philosophy is dialectics. It considers matter in constant motion and develop-

which, like matter itself, cannot be created or destroyed. Matter exists and moves in space and in time, which are forms of existence of matter.

The laws of physics are established by generalizing experimental facts. They express the objective regularities existing in nature. These laws are customarily expressed in the form of quantitative relationships between various physical quantities.

The fundamental method of investigation in physics is the running of an experiment, *i.e.*, the observation of the phenomenon being studied in accurately controlled conditions. The latter must permit one to watch the course of the phenomenon and reproduce it each time when these conditions are repeated. Phenomena can be produced experimentally that are not observed in nature. For example, more than ten of the chemical elements known at present have meanwhile not been discovered in nature and were obtained artificially by means of nuclear reactions.

Hypotheses are enlisted to explain experimental data. A hypothesis is a scientific assumption advanced to explain a definite fact or phenomenon and requiring verification and proving to become a scientific theory or law. The correctness of a hypothesis is verified by running the corresponding experiments and by determining whether the corollaries following from the hypothesis agree with the results of experiments and observations. A hypothesis that has successfully passed such verification and has been proved becomes a scientific law or theory.

A physical theory is a system of basic ideas summarizing experimental data and reflecting the objective regularities of nature. A physical theory explains a whole field of natural phenomena from a single viewpoint.

Physics is subdivided into the so-called classical physics and quantum physics. The term classical is applied to the physics whose creation was completed at the beginning of the 20th century. Classical physics was founded by Isaac Newton (1642-1727), who formulated the fundamental laws of classical mechanics. Newtonian mechanics proved to be exceedingly fruitful and mighty, and physicists acquired the conviction that any physical phenomenon can be explained with the aid of Newton's laws.

The edifice of classical physics built up by the end of the last century was very harmonious. Most physicists were convinced that they already knew everything about nature that could be known. The most perspicacious physicists, however, understood that the edifice of classical physics had weak spots. For example, the British physicist William Thomson (Lord Kelvin, 1824-1907) said that there are two

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ment whose source is contained in the internal contradictions inherent in objects and phenomena themselves.

dark clouds on the horizon of the cloudless sky of classical physics—the unsuccessful attempts to set up a theory of blackbody radiation, and the contradictory behaviour of ether—the hypothetical medium in which light waves were supposed to propagate. The persistent attempts to surmount these difficulties led to unexpected results. To solve these problems, which were beyond the possibilities of classical physics, it became necessary to revise quite radically the established, habitual notions and introduce concepts that were alien to the spirit of classical physics. Max Planck (1858-1947) succeeded in solving the problem of blackbody radiation in 1900 by introducing the concept of light emission in separate portions—quanta. Thus, at the threshold of the 20th century, the concept of the quantum appeared. It plays an exceedingly important part in modern physics and has resulted in the creation of quantum mechanics.

The contradictory nature of the experimental facts relating to ether induced Albert Einstein (1879-1955) to revise the notions of space and time that were considered to be obvious from Newton's times. The result was the appearance of the theory of relativity. The latter gives equations of motion appreciably differing from those of Newtonian mechanics for bodies travelling with speeds that are noticeable in comparison with the speed of light.

The year 1897 saw the discovery of the electron. The atoms of all the chemical elements were found to contain these particles. Thus, atoms, previously considered indivisible, appeared to have a complicated structure.

The beginning of the 20th century was thus marked in physics by the radical breaking down of numerous habitual concepts and notions. New physical discoveries and theories destroyed the notions of the structure of matter formed by many physicists. Some of them interpreted this as the vanishing of matter. Many physicists lapsed into idealism, and a crisis began in physics.

V. Lenin in his book *Materialism and Empirio-Criticism* written in 1908 gave annihilating criticism of "physical" idealism. He showed that the new discoveries indicate not the vanishing of matter, but the vanishing of the limit up to which matter was known before that time. "Matter disappears", wrote Lenin, "means that the limit within which we have hitherto known matter disappears and that our knowledge is penetrating deeper; properties of matter are likewise disappearing which formerly seemed absolute, immutable, and primary (impenetrability, inertia, mass, etc.) and which are now revealed to be relative and characteristic only of certain states of matter. For the sole 'property' of matter with whose recognition philosophical materialism is bound up is the property of being an objective reality, of existing outside the mind."[3].

[3]V. I. Lenin. *Collected Works*, Vol. 14, p. 260. Moscow, Foreign Languages Publishing House (1962).

The process of recognizing the world is infinite. Our knowledge at any given stage of development of science is due to the historically achieved level of cognition and cannot be considered as final or complete. It is of necessity relative knowledge, *i.e.*, requires further development, further verification, and more precise definition. At the same time, any truly scientific theory, notwithstanding its relativity and incompleteness, contains elements of absolute, *i.e.*, complete, knowledge, and thus signifies a step in the cognition of the objective world. For instance, mechanics based on Newton's laws is not correct, strictly speaking. But for a certain range of phenomena, this mechanics is quite satisfactory. Thus, the development of science did not cross out Newtonian mechanics. It only established the limits within which it is correct. Newtonian mechanics formed a constituent part of the general edifice of the physical science.

The beginning of the 20th century is characterized by persistent attempts to penetrate into the internal structure of atoms. The key to determining their structure was found to be the studying of atomic spectra. The theory of the atom developed by Niels Bohr (1885-1962) in 1913 was the first striking success in explaining the observed spectra. This theory, however, has obvious features of inconsistency: in addition to the motion of an electron in an atom obeying the laws of classical mechanics, the theory imposes special quantum restrictions on this motion. The theory soon had to pay for this lack of consistency. After the first successes in explaining the spectra of the simplest atom—that of hydrogen—it was found that Bohr's theory is unable to explain the behaviour of atoms with two or more electrons.

The need to develop a new comprehensive theory of atoms became pressing. A bold hypothesis of Louis de Broglie put forward in 1924 placed the cornerstone in such a theory. It was known by that time that light, while being a wave process, also exhibits a corpuscular nature in a number of cases, *i.e.*, behaves like a stream of particles. De Broglie put forth the idea that the particles of a substance, in turn, should display wave properties too in definite conditions. De Broglie's hypothesis soon received a brilliant experimental confirmation—it was proved that a wave process is associated with the particles of a substance, and it must be taken into account when considering the mechanics of an atom. A result of this discovery was the development by Erwin Schrödinger (1887-1961) and Werner Heisenberg (1901-1976) of a new physical theory—wave or quantum mechanics. The latter achieved striking successes in explaining atomic processes and the structure of a substance. Results were obtained that showed excellent agreement with experimental data when 'it was found possible to surmount the mathematical difficulties.

The latest decades were noted by remarkable achievements in the field of studying the atomic nucleus. Scientists and engineers have mastered nuclear processes to such an extent that the practical use of nuclear energy has become possible. One of the leading places in this field belongs to Soviet physics. Particularly, the first atomic power plant in the world was erected in the USSR.

Finally, in recent years, the walls of laboratories created by the hands of man were moved apart beyond the limits of our globe. On October 4, 1957, an artificial satellite of the Earth was launched in the Soviet Union the first time in history. It was a small laboratory outfitted with apparatus for scientific research. April 12, 1961, saw the first flight of a man into outer space. The first Soviet cosmonaut, Yuri Gagarin, flew around the Earth and landed safely. The first space rockets were built in the Soviet Union. They left the field of the Earth's attraction and transmitted to the Earth by means of radio signals valuable results of studying outer space and, particularly, photographs of the reverse side of the Moon. In 1969, U.S. astronauts landed on the Moon. In 1975, two Soviet automatic spaceships made a soft landing on Venus and transmitted valuable information on the physical conditions on this planet, and also photographs of its surface.

There is no doubt that the nearest future will be marked with new fundamental discoveries in the science of physics.

PART A

THE PHYSICAL FUNDAMENTALS OF MECHANICS

Chapter 1

KINEMATICS

1.1. Mechanical motion

Mechanical motion is the simplest form of motion of matter. It consists in the movement of bodies or their parts relative to one another. We can see movements of bodies everywhere in our ordinary life. This is why mechanical notions are so clear. This also explains the fact that mechanics was the first of all the natural sciences to be developed very broadly.

A combination of bodies separated for consideration is called a **mechanical system**. The bodies to be included in a system depend on the nature of the problem being solved. In a particular case, a system may consist of a single body.

It was indicated above that motion in mechanics is defined as the change in the mutual arrangement of bodies. If we imagine a separate isolated body in a space where no other bodies are present, then we cannot speak of the motion of the body because there is nothing with respect to which the body could change its position. It thus follows that if we intend to study the motion of a body, then we must indicate with respect to what other bodies the given motion occurs.

Motion occurs both in space and in time (space and time are inalienable forms of existence of matter). Consequently, to describe motion, we must also determine time. We use a timepiece (watch or clock) for this purpose.

A combination of bodies that are stationary relative to one another with respect to which motion is being considered and a timepiece indicating the time forms a **reference frame**.

The motion of the same body relative to different reference frames may have a different nature. For example, let us imagine a train gaining speed. Suppose that a passenger is walking with a constant velocity along the corridor of one of the cars of the train. The motion of the passenger relative to the ear will be uniform, and relative to the Earth's surface it will be accelerated.

To describe the motion of a body means to indicate for every moment of time the position of the body in space and its velocity. To set the state of a mechanical system, we must indicate the positions and the velocities of all the bodies forming the system. A typical problem of mechanics consists in determining the states of a system at all the following moments of time t when we know the state of the system at a certain initial moment t_0 and also the laws governing the motion.

It must be noted that no physical problem can be solved absolutely exactly. An approximate solution is always obtained. The degree of approximation is determined by the nature of the problem and the object to be achieved. In solving a problem approximately, we disregard the factors that are not significant in the given case. For example, we may often disregard the dimensions of the body whose motion is being studied. For instance, it is quite possible to disregard the Earth's dimensions when treating its motion about the Sun. This allows us to considerably simplify our description of the motion because the Earth's position in space can be determined by a single point.

A body whose dimensions may be disregarded in the conditions of a given problem is called a **point particle** (or simply a **particle**). Whether or not we may consider a given body as a particle depends not on the dimensions of the body, but on the conditions of the problem. The same body in some cases may be treated as a particle, but in others it must be considered as an extended body.

When speaking about a body as a particle, we disengage ourselves from its dimensions. Another abstraction which we have to do with in mechanics is a perfectly rigid body. Absolutely undeformable bodies do not exist in nature. Any body deforms to a greater or smaller extent, *i.e.*, changes its shape and dimensions, under the action of forces applied to it. The deformations of bodies when considering their movements may often be disregarded, however. If this is done, then the body is called perfectly rigid. Thus, a body whose deformations may be disregarded in the conditions of a given problem is called a **perfectly rigid**, or simply a **rigid body**.

Any motion of a rigid body can be resolved into two basic kinds of motion—**translational motion** and **circular motion**.

Translational motion (translation) is defined as motion in which any straight line associated with the moving body remains parallel to itself (Fig. 1.1).

In circular motion (rotation), all the points of a body move in circles whose centers are on a single straight line called the axis of rotation (Fig. 1.2). The axis of rotation can be outside a body (see Fig. 1.2b).

Since when treating a body as a particle we ignore its length, the concept of

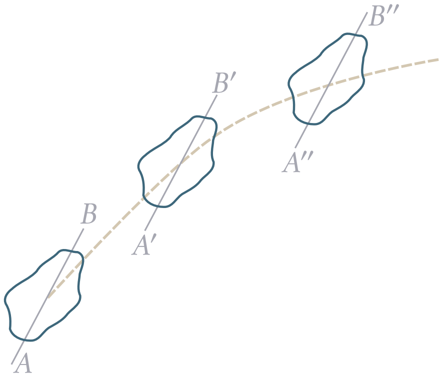

Fig. 1.1

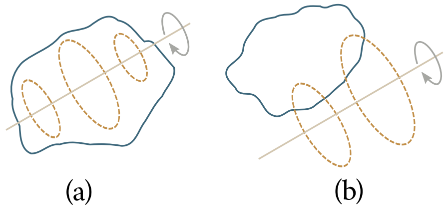

Fig. 1.2

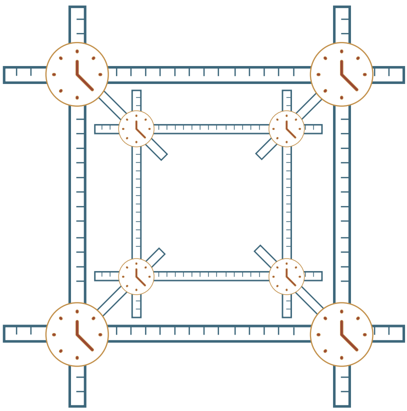

Fig. 1.3

circular motion about an axis passing through such a body cannot be applied to it.

To acquire the possibility of describing motion quantitatively, we have to associate a **coordinate system** (for example a Cartesian one) with the bodies forming a reference frame. Hence, the position of a particle can be determined by setting the three numbers x, y, and z—the Cartesian coordinates of the particle. A coordinate system can be made by forming a rectangular lattice from identical rods or rules graduated to a definite scale: (Fig. 1.3). Identical clocks synchronized with one another must be placed at the lattice points. The position of a particle and the moment of time corresponding to this position are recorded on the graduated rods and the clock closest to the particle.

It is simpler to treat a point particle than an extended body. We shall therefore first study the mechanics of a particle, and then go over to the mechanics of a rigid body. We shall start with kinematics, and then delve into dynamics. We remind our reader that **kinematics** studies the motion of bodies without regard to what causes this motion. **Dynamics** studies the motion of bodies with a view to what causes this motion to have the nature it does, *i.e.*, with a view to the interactions between bodies.

1.2. Vectors

Definition of a Vector. Vectors are defined as quantities characterized by a numerical value and a direction and also as ones that are added according to the triangle or parallelogram method[1]. The last requirement is a very significant one. We can indicate quantities characterized by a numerical value and a sense of direction but that are added in a different way than vectors. We shall take as an example the rotation of a body about an axis through the finite angle φ. Such rotation can be depicted in the form of a segment of length φ directed along the axis about which rotation is occurring and pointing in a direction associated with that of rotation according to the right-hand screw rule. The top portion of Fig. 1.4 shows two consecutive turns of the sphere through the angles $\pi/2$ depicted by the segments φ_1 and φ_2. The first turn about axis 1—1 transfers point A of the sphere to position A', and the second turn about axis 2—2 transfers it to position A''. The same result, *i.e.*, transfer of point A to position A'', can be achieved by turning the sphere about axis 3—3 (see the bottom portion of Fig. 1.4) through the angle π. Hence, such a turn should be considered as the sum of the turns <p1 and <p 2 • It cannot be obtained from the segments φ_1 and φ_2, however, by adding them according to the parallelogram method. Such addition gives a segment of length $\pi/\sqrt{2}$ instead of the required length π. Rotation through the angle $\pi/\sqrt{2}$ transfers point A to point A'''. It thus follows that the turns through finite angles depicted by the directed segments do not have the properties of vectors.

The numerical value of a vector is called its magnitude. Figuratively speaking, the magnitude of a vector indicates its length. The magnitude of a vector is a scalar, and always a positive one.

Vectors are represented graphically by arrows. The length of an arrow determines to the established scale the magnitude of the relevant vector, and the arrow points in the direction of the vector.

Vectors are customarily distinguished by setting their symbols in boldface type, for example, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{v}$ and $\boldsymbol{F}$. The same symbols set in italics signify the magnitude of

[1]According to a stricter definition, a vector is a combination of three quantities that transform when the coordinate axes rotate according to a definite law.

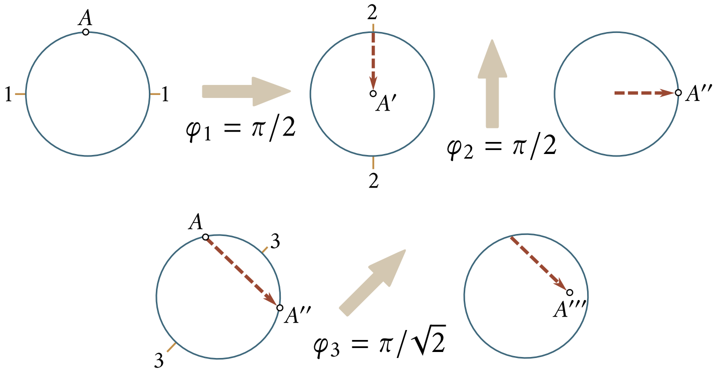

Fig. 1.4

the relevant vectors, for example, a is the magnitude of the vector $\boldsymbol{a}$[2]. It is sometimes necessary to express the magnitude by placing a vertical bar (an absolute value sign) on each side of the symbol for the vector. Thus, $|a|$ is the magnitude of the vector $\boldsymbol{a}$. This representation is used, for example, to show the magnitude of the sum of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$:

$$|\boldsymbol{a}_1 + \boldsymbol{a}_2| = \text{magnitude of the vector } (\boldsymbol{a}_1 + \boldsymbol{a}_2). \tag{1.1}$$

In this case, the notation $a_1 + a_2$ signifies the sum of the magnitudes of the vectors being added, which in general does not equal the magnitude of the sum of the vectors (the two sums will be equal only when the vectors being added have the same direction).

Vectors directed along parallel straight lines (in the same or in opposite directions) are called **collinear**. Vectors in parallel planes are called **coplanar**. Collinear vectors can be arranged along the same straight line and coplanar vectors can be brought into one plane by parallel translation.

Collinear vectors equal in magnitude and having the same direction are considered to equal each other[3].

Vector Addition and Subtraction. It is more convenient to add vectors in practice without constructing a parallelogram. Examination of Fig. 1.5 shows that we can achieve the same result if we bring the tail of the second vector in contact

[2]In handwriting, vectors are denoted by arrows over their symbols (for example, $\vec{a}$. In this case, the same letter without the arrow stands for the magnitude of the vector.

[3]What is meant are the so-called **free vectors**, *i.e.*, vectors that can be drawn from any point in space. Also distinguished are **slip vectors** whose tail can be placed at any point on the straight line along which the vector is directed, and localized vectors, which are applied to a definite point. The last two kinds of vectors can be expressed through free vectors. This is why vector calculus is based on the concept of the free vector, usually called simply a vector.

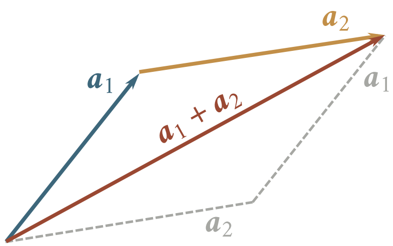

Fig. 1.5

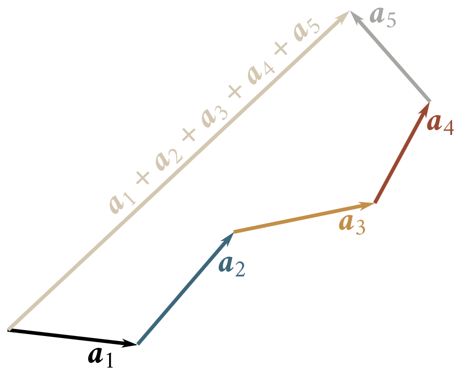

Fig. 1.6

with the tip of the first one, and then draw the resultant vector from the tail of the first vector to the tip of the second one. It is very good to use this procedure when we have to add more than two vectors (Fig. 1.6).

The difference of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as such a vector $\boldsymbol{c}$ which when added to the vector $\boldsymbol{b}$ gives the vector $\boldsymbol{a}$ (Fig. 1.7—the vector $-\boldsymbol{b}$ depicted by a dash line will be treated below The magnitude of the difference of two vectors, like the magnitude of a sum [see Eq. (1.1)], may be written only with the aid of vertical bars:

$$|\boldsymbol{a}_1 - \boldsymbol{a}_2| = \text{magnitude of the vector } (\boldsymbol{a}_1 - \boldsymbol{a}_2), \tag{1.2}$$

because the notation $a_1 - a_2$ signifies the difference of the magnitudes of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, which, generally speaking, does not equal the magnitude of the vector difference.

Multiplication of a Vector by a Scalar. Multiplication of the vector $\boldsymbol{a}$ by the scalar α yields a new vector $\boldsymbol{b} = \alpha\, \boldsymbol{a}$ whose magnitude is $|\alpha|$ times that of the vector a (*i.e.*, $b = |\alpha| a$). The direction of the vector $\boldsymbol{b}$ either coincides with that of the vector $\boldsymbol{a}$ (if $\alpha > 0$), or is opposite to it (if $\alpha < 0$). It follows from the above that multiplication by -1 reverses the direction of a vector. Consequently, the vectors $\boldsymbol{a}$ and $-\boldsymbol{a}$ have the same magnitudes, but are opposite in direction. It is simple to see with the aid of Fig. 1.7 that subtraction of the vector $\boldsymbol{b}$ from the vector $\boldsymbol{a}$ is equivalent to addition of the vector $-\boldsymbol{b}$ to the vector $\boldsymbol{a}$.

It follows from our definition of multiplication of a vector by a scalar that any vector $\boldsymbol{a}$ can be represented in the form

$$\boldsymbol{a} = a\, \hat{\boldsymbol{e}}_a, \tag{1.3}$$

where a is the magnitude of the vector $\boldsymbol{a}$ and $\hat{\boldsymbol{e}}_a$, is vector with a magnitude of unity and of the same direction as $\boldsymbol{a}$ (Fig. 1.8).

The vector $\hat{\boldsymbol{e}}_a$ is called the unit vector of the vector $\boldsymbol{a}$. The unit vector can be

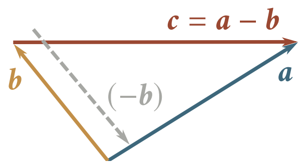

Fig. 1.7

Fig. 1.8

represented in the form

$$\hat{\boldsymbol{e}}_a = \frac{\boldsymbol{a}}{a}, \tag{1.4}$$

whence it follows that it is a dimensionless quantity.

Unit vectors can be compared not only with vectors, but also with any direction in space. For example, $\hat{\boldsymbol{e}}_x$ is the unit vector of the coordinate axis x, $\hat{\boldsymbol{e}}_n$ is the unit vector of a normal to a curve or surface, and $\hat{\boldsymbol{e}}_\tau$ is the unit vector of a tangent to a curve.

Linear Relation Between Vectors. Let us consider three non-collinear vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ that are in one plane. A glance at Fig. 1.9 shows that any of them (for instance, $\boldsymbol{c}$) can be expressed through the other two with the aid of the relation

$$\boldsymbol{c} = \alpha \boldsymbol{a} + \beta \boldsymbol{b}, \tag{1.5}$$

where α and β are scalars (for the case shown in the figure, $\alpha > 1$ and $-1 < \beta < 0$). Hence, we conclude that any vector $\boldsymbol{c}$ that is in the same plane as the non-collinear vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be expressed through the latter with the aid of linear relation (1.5). When the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are fixed, any third vector is unambiguously determined by the two quantities α and β.

Assume that we have three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$, each of which is not coplanar with the other two.[4] By analogy with Eq. (1.5), we can see quite easily that any vector $\boldsymbol{d}$ can be represented as a linear combination of the given vectors:

$$\boldsymbol{d} = \alpha \boldsymbol{a} + \beta \boldsymbol{b} + \gamma \boldsymbol{c}, \tag{1.6}$$

When the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are fixed, any vector $\boldsymbol{d}$ is unambiguously determined by the three quantities α, β and γ, each of which may be either positive or negative.

Projection of a Vector. Let us consider a direction in space that we shall set by the axis l (Fig. 1.10). Let the vector $\boldsymbol{a}$ make the angle φ with the axis l[5]. The quantity

$$a_l = a \cos \varphi \tag{1.7}$$

[4]Two vectors are always coplanar. This follows from the fact that their tails can he made to coincide by translation, and they will thus be in one plane.

[5]If the straight line along which the vector $\boldsymbol{a}$ is directed and the axis l do not intersect, the angle φ should be found by drawing a straight line parallel to the vector $\boldsymbol{a}$ and intersecting the axis l. The angle between this line and the axis l will be the angle φ we are interested in.

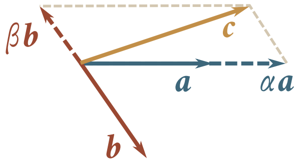

Fig. 1.9

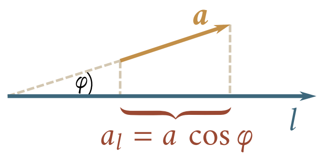

Fig. 1.10

(where a is the magnitude of the vector) is called the projection of the vector $\boldsymbol{a}$ onto the axis$l$. A projection is designated by the same symbol as its vector, with the addition of a subscript showing the direction onto which the vector has been projected.

A projection of a vector is an algebraic quantity. If the vector makes an acute angle with the given direction, then $\cos \varphi > 0$, and the projection is positive. If the angle φ is obtuse, then $\cos \varphi < 0$, and, consequently, the projection is negative. When a vector is at right angles to a given axis, its projection equals zero.

The projection of a vector has a simple geometrical meaning. It equals the distance between the projections of the tail and the tip of the segment depicting the given vector onto the given axis. When $\varphi < \pi/2$, this distance is assumed to be positive, and when $\varphi > \pi/2$, it is negative.

Let $\boldsymbol{a} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 + \boldsymbol{a}_4$ (Fig. 1.11). It is easy to see from the figure that the projection of the resultant vector a onto a direction l equals the sum of the projections of the separate vectors being added:

$$a_l = a_{1l} + a_{2l} + a_{3l} + a_{4l}. \tag{1.8}$$

We must remind our reader that when adding the projections of the vectors shown in Fig. 1.11, the distances 0—1, 1—2, and 2—3 have to be taken with the plus sign, and the distance 3—4 with the minus sign. Equation (1.8) holds for any number of addends.

Expressing a Vector Through Its Projections onto the Coordinate Axes. Let us take Cartesian coordinate axes and consider the vector $\boldsymbol{a}$ in a plane at right angles to the z-axis (Fig. 1.12). We shall introduce the unit vectors of the coordinate axes, *i.e.*, the unit vectors $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$ and $\hat{\boldsymbol{e}}_z$ ($\hat{\boldsymbol{e}}_z$ is not shown in the drawing, it is perpendicular to the plane of the drawing and directed toward us). It must be noted that these three unit vectors completely determine a system of coordinates and are therefore called the **basis of the coordinate system**.

Inspection of Fig. 1.12 shows that the vector $\boldsymbol{a}$ can be represented in the form of a linear combination of the unit vectors $\hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{e}}_y$ [see Eq. (1.5)]:

$$\boldsymbol{a} = a_x \hat{\boldsymbol{e}}_x + a_y \hat{\boldsymbol{e}}_y.$$

The projections of the vector onto the coordinate axes play the part of the coefficients α and β. In the example being considered, the projection a_x is negative,

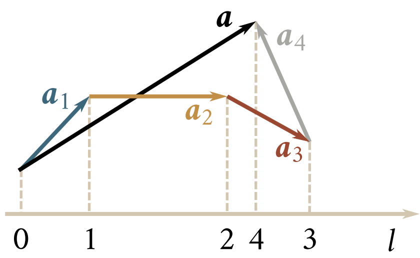

Fig. 1.11

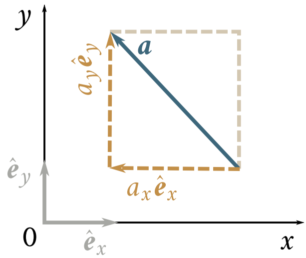

Fig. 1.12

therefore the vector $a_x\hat{\boldsymbol{e}}_x$ has a direction opposite to that of the unit vector $\hat{\boldsymbol{e}}_x$.

We took the vector $\boldsymbol{a}$ perpendicular to the z-axis owing to which $a_z = 0$. In the general case when all three projections of a vector differ from zero, we have

$$\boldsymbol{a} = a_x\hat{\boldsymbol{e}}_x + a_y\hat{\boldsymbol{e}}_y + a_z\hat{\boldsymbol{e}}_z, \tag{1.9}$$

Thus, any vector can be expressed through its projections onto the coordinate axes and the unit vectors of these axes. Therefore, the projections of a vector onto the coordinate axes are called its **components**.

The components a_x, a_y, a_z equal (with an accuracy to the sign) the sides of a right parallelepiped in which the vector $\boldsymbol{a}$ is the major diagonal (Fig. 1.13). We therefore have

$$a^2 = a_x^2 + a_y^2 + a_z^2. \tag{1.10}$$

Assume that $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$. Representing each of these vectors in accordance with Eq. (1.9), we get

$$c_x\hat{\boldsymbol{e}}_x + c_y\hat{\boldsymbol{e}}_y + a_z\hat{\boldsymbol{e}}_z = (a_x + b_x)\hat{\boldsymbol{e}}_x + (a_y + b_y)\hat{\boldsymbol{e}}_y + (a_z + b_z)\hat{\boldsymbol{e}}_z$$

(we have factored out $\hat{\boldsymbol{e}}_x$, $\hat{\boldsymbol{e}}_y$, and $\hat{\boldsymbol{e}}_z$)· Equal vectors have identical projections onto the coordinate axes. On these grounds, we can write that

$$c_x = a_x + b_x, \quad c_y = a_y + b_y, \quad a_z = a_z + b_z \tag{1.11}$$

[compare with Eq. (1.8)]. Equations (1.11) express analytically the rule of vector addition. They hold for any number of addends.

Position Vector. The position vector (or radius vector) $\boldsymbol{r}$ of a point is defined as the vector drawn from the origin of coordinates to the given point (Fig. 1.14). Its projections onto the coordinate axes equal the Cartesian coordinates of the given point:

$$r_x = x, \quad r_y = y, \quad r_z = z. \tag{1.12}$$

Consequently, in accordance with Eq. (1.9), the position vector can be represented in the form

$$\boldsymbol{r} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z. \tag{1.13}$$

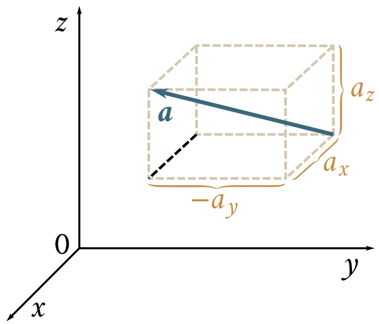

Fig. 1.13

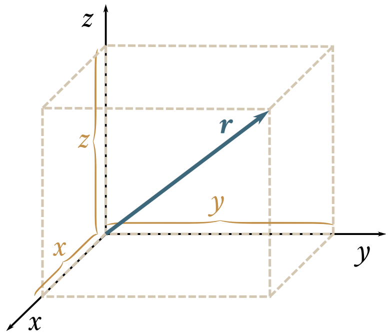

Fig. 1.14

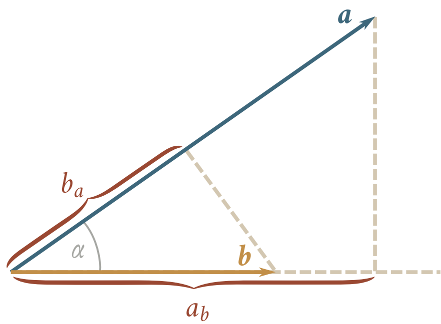

Fig. 1.15

By Eq. (1.10), we have

$$r^2 = x^2 + y^2 + z^2. \tag{1.14}$$

The Scalar Product of Vectors. Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be multiplied by each other in two ways. One of them results in a scalar quantity, and the other in a certain new vector. Accordingly, two products of vectors are distinguished—the scalar product and the vector product. It must be noted that *the operation of dividing a vector by a vector does not exist.*

The scalar product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the scalar quantity equal to the product of the magnitudes of these vectors and the cosine of the angle α between them:

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab \sin \alpha \tag{1.15}$$

(Fig. 1.15). When writing a scalar product, the symbols of the vectors being multiplied are usually written next to each other with dot between them (this is why a scalar product is also called a dot product; sometimes nothing is used between the symbols)[6]. Equation (1.15) expresses an algebraic quantity: when α is acute, we have

[6]The dot symbol between vectors is preferred in the LaTeXversion to adopt a more modern ap-

$\boldsymbol{a}\cdot\boldsymbol{b} > 0$, and when it is obtuse, we have $\boldsymbol{a}\cdot\boldsymbol{b} < 0$. The scalar product of mutually perpendicular vectors ($\alpha = \pi/2$) equals zero.

It must be noted that by the square of a vector is always meant the scalar product of this vector by itself:

$$\boldsymbol{a}^2 = \boldsymbol{a}\cdot\boldsymbol{a} = aa\cos\alpha = a^2. \tag{1.16}$$

Thus, the square of a vector equals the square of its magnitude. In particular, the square of any unit vector equals unity:

$$\hat{\boldsymbol{e}}_x^2 = \hat{\boldsymbol{e}}_y^2 = \hat{\boldsymbol{e}}_z^2 = 1. \tag{1.17}$$

We shall note in passing that owing to the unit vectors being mutually perpendicular, scalar products such as $\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_k$, equal zero if $i \neq k$.

The Kronecker symbol or delta δ_{ik} is very convenient. It is determined as follows:

$$\delta_{ik} = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases} \tag{1.18}$$

When this symbol is used, the properties of the scalar products of the coordinate axis unit vectors established above can be expressed by a single formula:

$$\hat{\boldsymbol{e}}_i\cdot\hat{\boldsymbol{e}}_k = \delta_{ik} \quad (i, k = x, y, z) \tag{1.19}$$

where the subscripts i and k can assume any of the values x, y and z independently of each other.

It follows from the definition (1.15) that a scalar product is commutative, *i.e.*, it does not depend on the sequence of the multipliers:

$$\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}. \tag{1.20}$$

Equation (1.15) can be written in several ways:

$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\alpha = (a\cos\alpha)\,b = a\,(b\cos\alpha).$$

Examination of Fig. 1.15 shows that $a\cos\alpha$ equals a_b—the projection of the vector $\boldsymbol{a}$ onto the direction of the vector $\boldsymbol{b}$. Similarly, $b\cos\alpha = b_a$—the projection of the vector $\boldsymbol{b}$ onto the direction of the vector $\boldsymbol{a}$. We can therefore say that the scalar product of two vectors is defined as the scalar quantity equal to the product of the magnitude of one of the vectors being multiplied and the projection of the second vector onto the direction of the first one:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_b b = ab_a. \tag{1.21}$$

Taking into account that the projection of the sum of vectors equals the sum of the projections of the vectors being added, we can write that

$$\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}+\ldots) = a(\boldsymbol{b}+\boldsymbol{c}+\ldots)_a = a(b_a+c_a+\ldots) = ab_a+ac_a+\ldots = ab+ac+\ldots. \tag{1.22}$$

proach.

Hence, it follows that the scalar product of vectors is distributive—the product of the vector $\boldsymbol{a}$ and the sum of several vectors equals the sum of the products of the vector $\boldsymbol{a}$ and each of the added vectors taken separately.

Let us represent the vectors being multiplied in the form of Eq. (1.9) and take advantage of the distributive nature of a scalar product. We get

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} &= (a_x\hat{\boldsymbol{e}}_x + a_y\hat{\boldsymbol{e}}_y + a_z\hat{\boldsymbol{e}}_z)(b_x\hat{\boldsymbol{e}}_x + b_y\hat{\boldsymbol{e}}_y + b_z\hat{\boldsymbol{e}}_z)\\ &= a_xb_x\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_x + a_xb_y\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_y + a_xb_z\hat{\boldsymbol{e}}_x\cdot\hat{\boldsymbol{e}}_z + a_yb_x\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_x + a_yb_y\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_y\\ &+ a_yb_z\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_z + a_zb_x\hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_x + a_zb_y\hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_y + a_zb_z\hat{\boldsymbol{e}}_z\cdot\hat{\boldsymbol{e}}_z.\end{aligned}$$

Now let us take Eq. (1.19) into consideration. As a result, we get an expression for a scalar product through the projections of the vectors being multiplied:

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_xb_x + a_yb_x + a_zb_z. \tag{1.23}$$

It must be noted that when the coordinate axes are rotated, the projections of vectors onto these axes change. The quantity $ab\cos\alpha$ does not depend on the choice of the axes, however. We thus conclude that the changes in the projections of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, when the axes are rotated, are of a nature such that their combination of the form of Eq. (1.23) remains invariant (unchanged):

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_xb_x + a_yb_x + a_zb_z = \text{inv}. \tag{1.24}$$

It is a simple matter to see that the projection of the vector $\boldsymbol{a}$ onto the direction l [see Eq. (1.7)] can be represented in the form

$$a_l = \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_l, \tag{1.25}$$

where $\hat{\boldsymbol{e}}_l$ is the unit vector of the direction l. Similarly,

$$a_x = \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_x, \quad a_y = \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_y, \quad a_z = \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_z. \tag{1.26}$$

The Vector Product. The vector product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the vector $\boldsymbol{c}$ determined by the equation

$$\boldsymbol{c} = ab\sin(\alpha)\hat{\boldsymbol{n}}, \tag{1.27}$$

where a y b magnitudes of the vectors being multiplied, α, is the angle between the vectors, $\hat{\boldsymbol{n}}$, is the unit vector of a normal[7] to the plane containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Fig. 1.16).

The direction of $\hat{\boldsymbol{n}}$ is chosen so that the sequence of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\hat{\boldsymbol{n}}$ forms a right-handed system. This signifies that if we look along the vector $\hat{\boldsymbol{n}}$, then the shortest path in rotation from the first multiplier to the second one will be clockwise. In Fig. 1.16, the vector $\hat{\boldsymbol{n}}$ is directed beyond the drawing, and it is therefore depicted by a circle with a cross[8]. The direction of the vector $\boldsymbol{c}$ coincides with that

[7]The symbol $\hat{\boldsymbol{n}}$ is simpler and more illustrative than *vecunin*.

[8]We shall depict vectors perpendicular to the plane of a drawing by a circle with a cross in it if

of $\hat{\boldsymbol{n}}$.

A vector product is usually designated in one of two ways:

$$[\boldsymbol{a}, \boldsymbol{b}] \quad \text{or} \quad \boldsymbol{a} \times \boldsymbol{b}$$

the latter notation resulting in the term cross product sometimes being used to signify a vector product. We shall use the latter notation[9]. Thus, according to Eq. (1.27), we have

$$\boldsymbol{a} \times \boldsymbol{b} = (ab \sin \alpha)\hat{\boldsymbol{n}}. \tag{1.28}$$

A glance at Fig. 1.16 shows that the magnitude of a vector product has a simple geometrical meaning—the expression $ab \sin \alpha$ numerically equals the area of a parallelogram constructed on the vectors being multiplied.

We determined the direction of the vector $\boldsymbol{a} \times \boldsymbol{b}$ by relating it to the direction of rotation from the first multiplier to the second one. When considering vectors such as the position vector $\boldsymbol{r}$, the velocity $\boldsymbol{v}$, and the force $\boldsymbol{F}$, the choice of their direction is quite obvious—it follows from the nature of these quantities themselves. Such vectors are called **polar** or **true**. Vectors of the type $\boldsymbol{a} \times \boldsymbol{b}$ whose direction is related to that of rotation are called axial or **pseudovectors**. When conditions change, for example, upon going over from a right-hand system of coordinates to a left-hand one, the directions of pseudovectors are reversed, while those of true vectors remain unchanged.

It must be borne in mind that a vector product will be a pseudovector only when both of the vectors being multiplied are true (or both are pseudovectors). The vector product of a true vector and a pseudovector will be true. Reversing of a condition determining the direction of a pseudovector will lead in this case to a change in the sign in front of the vector product and also to a change in the sign of one of the multipliers. As a result, the quantity expressed by the vector product remains unchanged.

Since the direction of a vector product is determined by the direction of rotation from the first multiplier to the second one, the result of vector multiplication depends on the order of the multipliers. Transposition of the multipliers leads to reversing of the direction of the resultant vector. Thus, a vector product does not have the property of commutativity:

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}. \tag{1.29}$$

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the vector is directed away from us, and by a circle with a point at its centre if the vector is directed toward us. For clarity, we can imagine a vector in the form of an arrow with a tapered tip and cross-shaped feathers on its tail. Thus, when the vector is directed toward us (the arrow is flying toward us), we see a circle with a point; when the vector is directed away from us (the arrow is flying away from us), we see a circle with a cross.

[9]To avoid confusion, in the LaTeXversion, we shall use the cross product symbol.

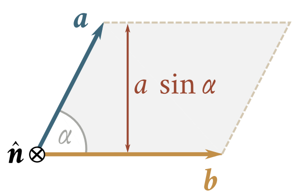

Fig. 1.16

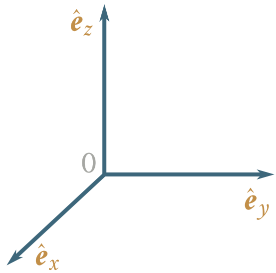

Fig. 1.17

A vector product can be proved to be distributive, *i.e.*, it can be shown that

$$\boldsymbol{a} \times (\boldsymbol{b}_1 + \boldsymbol{b}_2 + \ldots)] = \boldsymbol{a} \times \boldsymbol{b}_1 + \boldsymbol{a} \times \boldsymbol{b}_2 + \ldots. \tag{1.30}$$

Let us consider the vector products of the unit vectors of the coordinate axes (Fig. 1.17). In accordance with the definition (1.28), we have

$$\begin{aligned}
&\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_x = \hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_z = 0,\\
&\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y = -\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_x = \hat{\boldsymbol{e}}_z,\\
&\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_z = -\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_y = \hat{\boldsymbol{e}}_x,\\
&\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_x = -\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_y.
\end{aligned} \tag{1.31}$$

Representing the vectors being multiplied in the form of Eq. (1.9) and taking advantage of the distributivity of a vector product, we get:

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= (a_x\hat{\boldsymbol{e}}_x + a_y\hat{\boldsymbol{e}}_y + a_z\hat{\boldsymbol{e}}_z) \times (b_x\hat{\boldsymbol{e}}_x + b_y\hat{\boldsymbol{e}}_y + b_z\hat{\boldsymbol{e}}_z)\\
&= a_xb_x\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_x + a_xb_y\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y + a_xb_z\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_z\\
&+ a_yb_x\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_x + a_yb_y\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_y + a_yb_z\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_z\\
&+ a_zb_x\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_x + a_zb_y\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_y + a_zb_z\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_z
\end{aligned}$$

Taking into account relation (1.31), we arrive at the following expression:

$$\boldsymbol{a} \times \boldsymbol{b} = \hat{\boldsymbol{e}}_x(a_yb_z - a_zb_y) + \hat{\boldsymbol{e}}_y(a_zb_x - a_xb_z) + \hat{\boldsymbol{e}}_z(a_xb_y - a_yb_x). \tag{1.32}$$

The above expression can be represented in the form of a determinant

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{1.33}$$

Scalar Triple Product. A scalar triple product of three vectors is defined as the expression $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$, *i.e.*, the scalar product of the vector $\boldsymbol{a}$ and the vector product of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. According to the definitions (1.15) and (1.28), we have

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = a\{bc \sin(\boldsymbol{b}, \boldsymbol{c})\} \cos(\boldsymbol{a}, \hat{\boldsymbol{n}}).$$

Here $(\boldsymbol{b}, \boldsymbol{c})$ is the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$, and $(\boldsymbol{a}, \hat{\boldsymbol{n}})$ is the angle between the vector

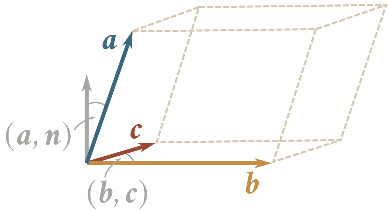

Fig. 1.18

$\boldsymbol{a}$ and the unit vector $\hat{\boldsymbol{n}}$ determining the direction of the vector $\boldsymbol{b} \times \boldsymbol{c}$. Inspection of Fig. 1.18 shows that the expression $bc \sin(\boldsymbol{b}, \boldsymbol{c})$ numerically equals the area of the base of a parallelepiped constructed on the vector being multiplied, while the expression $a \cos(\boldsymbol{a}, \hat{\boldsymbol{n}})$ numerically equals the altitude of this parallelepiped taken with the plus sign if the angle $(\boldsymbol{a}, \hat{\boldsymbol{n}})$ is acute, and with the minus sign if it is obtuse. Consequently, the expression $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ has a simple geometrical meaning—it numerically equals the volume of a parallelepiped constructed on the vectors being multiplied [taken with the plus or minus sign depending on the value of the angle $(\boldsymbol{a}, \hat{\boldsymbol{n}})$]. In calculating the volume of a parallelepiped, the result cannot depend on which of its faces is taken as the base. Hence, it follows that

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}). \tag{1.34}$$

Thus, a scalar triple product permits cyclic transposition of the multipliers, *i.e.*, substitution for each of the multipliers of the one following it in the cycle:

$$\begin{tikzcd} \boldsymbol{a} \arrow[dr] & \\ \boldsymbol{c} \arrow[u] & \boldsymbol{b} \arrow[l] \end{tikzcd}$$

Vector Triple Product. Let us consider a vector triple product of the three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$

$$\boldsymbol{d} = \boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}.$$

Any vector product is perpendicular to both multipliers. Therefore, the vector $\boldsymbol{d}$ is perpendicular to the unit vector $\hat{\boldsymbol{n}}$ determining the direction of the vector $\boldsymbol{b} \times \boldsymbol{c}$. Hence, it follows that the vector $\boldsymbol{d}$ is in the plane formed by the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ and, consequently, can be represented as a linear combination of these vectors:

$$\boldsymbol{d} = \alpha \boldsymbol{b} + \beta \boldsymbol{c}$$

[see Eq. (1.5)]. We find from the relevant calculations that $\alpha = \boldsymbol{a} \cdot \boldsymbol{c}$ and $\beta = -\boldsymbol{a} \cdot \boldsymbol{b}$. Thus,

$$\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}). \tag{1.35}$$

Derivative of a Vector. Let us consider a vector that changes in time according to a known law $\boldsymbol{a}(t)$. The projections of this vector onto the coordinate axes are preset functions of time. Hence,

$$\boldsymbol{a}(t) = \hat{\boldsymbol{e}}_x a_x(t) + \hat{\boldsymbol{e}}_y a_y(t) + \hat{\boldsymbol{e}}_z a_z(t) \tag{1.36}$$

(we assume that the coordinate axes do not rotate in space so that their unit vectors do not change with time).

Let the vector projections receive the increments Δa_x, Δa_y, Δa_z during the time Δt. The vector therefore receives the increment $\Delta\boldsymbol{a} = \hat{\boldsymbol{e}}_x \Delta a_x + \hat{\boldsymbol{e}}_y \Delta a_y + \hat{\boldsymbol{e}}_z \Delta a_z$. The rate of change of the vector $\boldsymbol{a}$ with time can be characterized by the ratio of $\Delta\boldsymbol{a}$ to Δt:

$$\frac{\Delta\boldsymbol{a}}{\Delta t} = \hat{\boldsymbol{e}}_x \frac{\Delta a_x}{\Delta t} + \hat{\boldsymbol{e}}_y \frac{\Delta a_y}{\Delta t} + \hat{\boldsymbol{e}}_z \frac{\Delta a_z}{\Delta t}. \tag{1.37}$$

This expression gives the mean rate of change of $\boldsymbol{a}$ during the time interval Δt. Let us assume that $\boldsymbol{a}$ changes continuously with time, without any jumps. Consequently, the smaller the interval Δt, the more accurately does the value of Eq. (1.37) characterize the rate of change in $\boldsymbol{a}$ at the moment t preceding the interval Δt. Therefore, the rate of change in the vector $\boldsymbol{a}$ at the moment t equals the limit of Eq. (1.37) obtained when Δt tends to zero:

$$\begin{aligned}\text{the rate of change in } \boldsymbol{a} &= \lim_{\Delta t\to 0} \frac{\Delta\boldsymbol{a}}{\Delta t} \\ &= \hat{\boldsymbol{e}}_x \lim_{\Delta t\to 0} \frac{\Delta a_x}{\Delta t} + \lim_{\Delta t\to 0} \hat{\boldsymbol{e}}_y \frac{\Delta a_y}{\Delta t} + \lim_{\Delta t\to 0} \hat{\boldsymbol{e}}_z \frac{\Delta a_z}{\Delta t}.\end{aligned} \tag{1.38}$$

If there is a function $f(t)$ of the argument t, then the limit of the ratio of the increment of the function Δf to the increment of the argument Δt obtained when Δt tends to zero is called the derivative of the function f with respect to t and is designated by the symbol $\mathrm{d}f/\mathrm{d}t$. Expression (1.38) can therefore be written as follows:

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \hat{\boldsymbol{e}}_x \frac{\mathrm{d}a_x}{\mathrm{d}t} + \hat{\boldsymbol{e}}_y \frac{\mathrm{d}a_y}{\mathrm{d}t} + \hat{\boldsymbol{e}}_z \frac{\mathrm{d}a_z}{\mathrm{d}t}. \tag{1.39}$$

The result obtained signifies that the projections of the vector $\mathrm{d}\boldsymbol{a}/\mathrm{d}t$ onto the coordinate axes equal the time derivatives of the projections of the vector $\boldsymbol{a}$:

$$\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{\text{pr. }x} = \frac{\mathrm{d}a_x}{\mathrm{d}t}, \quad \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{\text{pr. }y} = \frac{\mathrm{d}a_y}{\mathrm{d}t}, \quad \left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\right)_{\text{pr. }z} = \frac{\mathrm{d}a_z}{\mathrm{d}t}, \quad . \tag{1.40}$$

It is customary practice in physics to denote time derivatives by the symbol of the corresponding quantity with a dot over it, for example,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \dot{\varphi}, \quad \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \ddot{\varphi}, \quad \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \dot{\boldsymbol{a}}, \quad \frac{\mathrm{d}^2\boldsymbol{a}}{\mathrm{d}t^2} = \ddot{\boldsymbol{a}}. \tag{1.41}$$

Using this notation, we can write equation (1.39) as follows:

$$\dot{\boldsymbol{a}} = \hat{\boldsymbol{e}}_x \dot{a}_x + \hat{\boldsymbol{e}}_y \dot{a}_y + \hat{\boldsymbol{e}}_z \dot{a}_z. \tag{1.42}$$

If we take the position vector $\boldsymbol{r}(t)$ of a moving point as $\boldsymbol{a}(t)$, then by Eq. (1.42) we have

$$\dot{\boldsymbol{r}} = \hat{\boldsymbol{e}}_x \dot{r}_x + \hat{\boldsymbol{e}}_y \dot{r}_y + \hat{\boldsymbol{e}}_z \dot{r}_z, \tag{1.43}$$

where x, y, z are functions of t, namely, $x = x(t)$, $y = t(t)$, $z = z(t)$.

The differential ("increment") of the function $f(t)$ is defined as the expression

$$\mathrm{d}f = f' \,\mathrm{d}t, \tag{1.44}$$

where f' is the derivative of f with respect to t. According to Eq. (1.39), the differential of the vector $\boldsymbol{a}$ is determined by the equation

$$\mathrm{d}\boldsymbol{a} = \hat{\boldsymbol{e}}_x \mathrm{d}a_x + \hat{\boldsymbol{e}}_y \mathrm{d}a_y + \hat{\boldsymbol{e}}_z \mathrm{d}a_z. \tag{1.45}$$

In particular,

$$\mathrm{d}\boldsymbol{r} = \hat{\boldsymbol{e}}_x \mathrm{d}x + \hat{\boldsymbol{e}}_y \mathrm{d}y + \hat{\boldsymbol{e}}_z \mathrm{d}z. \tag{1.46}$$

It must be noted that the increment of a function during a very short, but finite interval Δt approximately equals

$$\Delta f \approx f' \Delta t = \frac{\mathrm{d}f}{\mathrm{d}t} \Delta t. \tag{1.47}$$

In the limit, when $\Delta t \to 0$, the approximate equation (1.47) transforms into the accurate equation (1.44).

A similar equation to (1.47) can also be written for the vector function

$$\Delta \boldsymbol{a} \approx \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} \Delta t. \tag{1.48}$$

Derivative of the Product of Functions. We shall consider the function $\boldsymbol{b}(t)$ that equals the product of the scalar function $\varphi(t)$ and the vector function $\boldsymbol{a}(t)$, *i.e.*, $\boldsymbol{b}(t) = \varphi(t)\boldsymbol{a}(t)$ or, more briefly, $\boldsymbol{b} = \varphi \boldsymbol{a}$. Let us find the increment of the function $\boldsymbol{b}$:

$$\Delta \boldsymbol{b} = \Delta(\varphi \boldsymbol{a}) = (\varphi + \Delta\varphi)(\boldsymbol{a} + \Delta\boldsymbol{a}) - \varphi\boldsymbol{a} = \varphi\Delta\boldsymbol{a} + \boldsymbol{a}\Delta\varphi + \Delta\varphi\Delta\boldsymbol{a}.$$

Representing the increments of the functions in the form of expressions (1.47) and (1.48), we get:

$$\Delta \boldsymbol{b} \approx \varphi \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} \Delta t + \boldsymbol{a} \frac{\mathrm{d}\varphi}{\mathrm{d}t} \Delta t + \frac{\mathrm{d}\varphi}{\mathrm{d}t} \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} (\Delta t)^2$$

whence

$$\frac{\Delta \boldsymbol{b}}{\Delta t} \approx \varphi \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} + \boldsymbol{a} \frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{\mathrm{d}\varphi}{\mathrm{d}t} \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} \Delta t.$$

In the limit when $\mathrm{d}t$ tends to zero, this approximate equation transforms into an accurate one. Thus,

$$\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} = \lim_{\Delta t\to 0}\left(\varphi\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} + \boldsymbol{a}\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\Delta t\right).$$

The first two addends do not depend on Δt and therefore do not change when going over to the limit. The limit of the third addend equals zero. Hence, substituting $\varphi\boldsymbol{a}$ for $\boldsymbol{b}$, we obtain

$$\frac{\mathrm{d}(\varphi\boldsymbol{a})}{\mathrm{d}t} = \varphi\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} + \boldsymbol{a}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \varphi\dot{\boldsymbol{a}} + \dot{\varphi}\boldsymbol{a}. \tag{1.49}$$

Now let us consider the scalar product of two vector functions $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$. The increment of this product is

$$\begin{aligned}\Delta(\boldsymbol{ab}) &= (\boldsymbol{a}+\Delta\boldsymbol{a})(\boldsymbol{b}+\Delta\boldsymbol{b}) - \boldsymbol{ab}\\ &= \boldsymbol{a}\Delta\boldsymbol{b} + \boldsymbol{b}\Delta\boldsymbol{a} + \Delta\boldsymbol{a}\Delta\boldsymbol{b}\\ &\approx \boldsymbol{a}\dot{\boldsymbol{b}}\Delta t + \boldsymbol{b}\dot{\boldsymbol{a}}\Delta t + \dot{\boldsymbol{a}}\dot{\boldsymbol{b}}(\Delta t)^2\end{aligned}$$

Hence

$$\frac{\mathrm{d}(\boldsymbol{ab})}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{\Delta(\boldsymbol{ab})}{\Delta t} = \lim_{\Delta t\to 0}(\boldsymbol{a}\dot{\boldsymbol{b}} + \boldsymbol{b}\dot{\boldsymbol{a}} + \dot{\boldsymbol{a}}\dot{\boldsymbol{b}}\Delta t)$$

or finally

$$\frac{\mathrm{d}(\boldsymbol{ab})}{\mathrm{d}t} = \boldsymbol{a}\dot{\boldsymbol{b}} + \boldsymbol{b}\dot{\boldsymbol{a}}. \tag{1.50}$$

Multiplying Eq. (1.50) by dt, we get a differential:

$$\frac{\mathrm{d}(\boldsymbol{ab})}{\mathrm{d}t} = \boldsymbol{a}\dot{\boldsymbol{b}} + \boldsymbol{b}\dot{\boldsymbol{a}}. \tag{1.51}$$

Let us calculate the derivative and the differential of the square of a vector function. According to Eqs. (1.50) and (1.51), we have

$$\frac{\mathrm{d}\boldsymbol{a}^2}{\mathrm{d}t} = 2\boldsymbol{a}\dot{\boldsymbol{a}}, \tag{1.52}$$

$$\mathrm{d}(\boldsymbol{a}^2) = 2\boldsymbol{a}\,\mathrm{d}\boldsymbol{a}, \tag{1.53}$$

Taking into account that $\boldsymbol{a}^2 = a^2$ [see Eq. (1.16)], we can write:

$$2\boldsymbol{a}\,\mathrm{d}\boldsymbol{a} = \mathrm{d}(a^2) \quad \text{or} \quad \boldsymbol{a}\,\mathrm{d}\boldsymbol{a} = \mathrm{d}\left(\frac{a^2}{2}\right). \tag{1.54}$$

Finally, let us consider the derivative of the vector product of the functions $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$. The increment of the function being considered is

$$\begin{aligned}\Delta\boldsymbol{a}\times\boldsymbol{b} &= [(\boldsymbol{a}+\Delta\boldsymbol{a}), (\boldsymbol{b}+\Delta\boldsymbol{b}) - \boldsymbol{a}\times\boldsymbol{b}]\\ &= [\boldsymbol{a}, \Delta\boldsymbol{b}] + [\Delta\boldsymbol{a}, \boldsymbol{b}] + [\Delta\boldsymbol{a}, \Delta\boldsymbol{b}]\\ &\approx [\boldsymbol{a}, \dot{\boldsymbol{b}}\Delta t] + [\dot{\boldsymbol{a}}\Delta t, \boldsymbol{b}] + [\dot{\boldsymbol{a}}\Delta t, \dot{\boldsymbol{b}}\Delta t].\end{aligned}$$

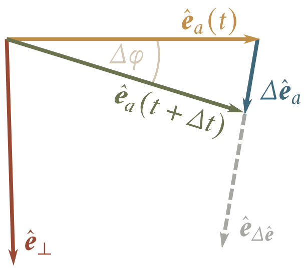

Fig. 1.19

Correspondingly,

$$\frac{\mathrm{d}\boldsymbol{a}\times\boldsymbol{b}}{\mathrm{d}t} = \lim_{\Delta t\to 0}\{[\boldsymbol{a},\dot{\boldsymbol{b}}] + [\dot{\boldsymbol{a}},\boldsymbol{b}] + [\dot{\boldsymbol{a}},\dot{\boldsymbol{b}}]\Delta t\}.$$

After a limit transition, we arrive at the equation

$$\frac{\mathrm{d}\boldsymbol{a}\times\boldsymbol{b}}{\mathrm{d}t} = [\boldsymbol{a},\dot{\boldsymbol{b}}] + [\dot{\boldsymbol{a}},\boldsymbol{b}]. \tag{1.55}$$

Derivative of a Unit Vector. Let us consider the unit vector $\hat{\boldsymbol{e}}_a$ of the vector $\boldsymbol{a}$. It is obvious that the vector $\hat{\boldsymbol{e}}_a$ can change only in direction. Assume that during the very short interval Δt the vector $\boldsymbol{a}$ and together with it the unit vector $\hat{\boldsymbol{e}}_a$ rotate through the angle $\Delta\varphi$ (Fig. 1.19). At a low value of $\Delta\varphi$, the magnitude of the vector $\Delta\hat{\boldsymbol{e}}_a$ approximately equals the angle $\Delta\varphi$, namely, $|\Delta\hat{\boldsymbol{e}}_a| \approx \Delta\varphi$ (the segment depicting $\Delta\hat{\boldsymbol{e}}_a$ is the base of an isosceles triangle with sides equal to unity). We must note that the smaller is $\Delta\varphi$, the more accurate is our approximate equation. The vector $\Delta\varphi$ itself can be represented in the form

$$\Delta\hat{\boldsymbol{e}}_a = |\Delta\hat{\boldsymbol{e}}_a|\cdot\hat{\boldsymbol{e}}_{\Delta\boldsymbol{e}} \approx \Delta\varphi\cdot\hat{\boldsymbol{e}}_{\Delta\boldsymbol{e}}$$

where $\hat{\boldsymbol{e}}_{\Delta\boldsymbol{e}}$ is the unit vector of the vector $\Delta\hat{\boldsymbol{e}}_a$. When $\Delta\varphi$ tends to zero, the unit vector $\hat{\boldsymbol{e}}_{\Delta\boldsymbol{e}}$ will rotate and in the limit coincide with the unit vector $\hat{\boldsymbol{e}}_\perp$ perpendicular to $\hat{\boldsymbol{e}}_a$ (see Fig. 1.19).

The derivative of $\hat{\boldsymbol{e}}_a$ with respect to t, by definition, is

$$\frac{\mathrm{d}\hat{\boldsymbol{e}}_a}{\mathrm{d}t} = \lim_{\Delta t\to 0}\frac{\Delta\hat{\boldsymbol{e}}_a}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta\varphi}{\Delta t}\hat{\boldsymbol{e}}_{\Delta\boldsymbol{e}} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{\boldsymbol{e}}_\perp.$$

Thus,

$$\dot{\hat{\boldsymbol{e}}}_a = \dot{\varphi}\hat{\boldsymbol{e}}_\perp. \tag{1.56}$$

The quantity $\dot{\varphi} = \mathrm{d}\varphi/\mathrm{d}t$ is the angular velocity of rotation of the vector $\boldsymbol{a}$ (see Sec. 1.5). The unit vector $\hat{\boldsymbol{e}}_\perp$ is in the plane in which the vector $\boldsymbol{a}$ is rotating at the given moment, and its sense is in the direction of rotation.

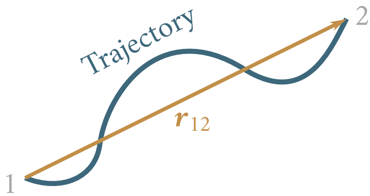

Fig. 1.20

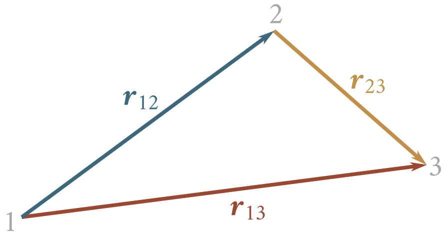

Fig. 1.21

1.3. Velocity and Speed

A point particle in motion travels along a certain line. The latter is called its path or trajectory[10]. Depending on the shape of a trajectory, we distinguish rectilinear or straight motion, circular motion, curvilinear motion, etc.

Assume that a point particle (in the following we shall call it simply a particle for brevity's sake) travelled along a certain trajectory from point 1 to point 2 (Fig. 1.20). The path between points 1 and 2 measured along the trajectory is called the distance travelled by the particle. We shall denote it by the symbol s.

The straight line between points 1 and 2, *i.e.*, the shortest distance between these points, is called the displacement of the particle. We shall denote it by the symbol $\boldsymbol{r}_{12}$. Let us assume that a particle completes two successive displacements $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{23}$ (Fig. 1.21). It is natural to call such a displacement $\boldsymbol{r}_{13}$ the sum of the first two that leads to the same result as they do together. Thus, displacements are characterized by magnitude and direction and, besides, are added by using the

[10]It must be noted that the concept of a trajectory can be applied only to a "classical" particle to which accurate values of its coordinate and momentum (*i.e.*, velocity) can be ascribed at each moment of time. According to quantum mechanics, real particles can be characterized with the aid of a coordinate and momentum only with a certain accuracy. The limit of this accuracy is determined by the equation of Heisenberg's uncertainty principle: $\Delta x \Delta p \gtrsim \hbar$. Here Δx is the uncertainty in the coordinate of a particle, Δp is the uncertainty in its momentum, and $\hbar$ is Planck's constant h divided by 2π, *i.e.*, $\hbar = h/2\pi = 1.05 \times 10^{-34}$ J s. The sign $\gtrsim$ signifies "greater than a value of the order of". Replacing the momentum with the product of the mass and the velocity, we can write $\Delta x \Delta v \gtrsim \hbar/m$. It can be seen from this relation that the smaller the mass of a particle, the more uncertain do its coordinate and velocity become, and, consequently, the less applicable is the concept of trajectory. For macroscopic bodies (*i.e.*, bodies formed by a very great number of molecules), the uncertainties in the coordinate and velocity do not exceed the practically attainable accuracy of measuring these quantities. Hence, the concept of trajectory may be applied to such bodies without any reservations. For microparticles (electrons, protons, neutrons, separate atoms and molecules), the concept of trajectory either cannot be applied at all, or can be applied with a limited accuracy, depending on the conditions in which motion occurs. For example, the motion o electrons in a cathode-ray tube can approximately be considered as occurring along certain trajectories.

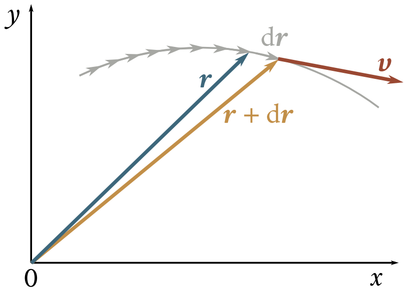

Fig. 1.22

parallelogram method. Hence, it follows that displacement is a vector.

In everyday life, we use the terms **speed** and **velocity** interchangeably, but in physics there is an important distinction between them. Speed depends on the distance travelled, and velocity on the displacement. Speed is the distance travelled by a particle in unit time. If a particle travels identical distances during equal time intervals that may be as small as desired, its motion is called uniform. In this case, the speed of the particle at each moment can be calculated by dividing the distance s by the time t.

Velocity is a vector quantity characterizing not only how fast a particle travels along its trajectory, but also the direction in which the particle moves at each moment. Let us divide a trajectory into infinitely small portions of length $\mathrm{d}s$. An infinitely small displacement $\mathrm{d}\boldsymbol{r}$ corresponds to each of these portions (Fig. 1.22). Dividing this displacement by the corresponding time interval $\mathrm{d}t$, we get the instantaneous velocity at the given point of the trajectory:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \dot{\boldsymbol{r}}. \tag{1.57}$$

Thus, the velocity is the derivative of the position vector of the particle with respect to time. The displacement $\mathrm{d}\boldsymbol{r}$ coincides with an infinitely small element of the trajectory. Consequently, the vector $\boldsymbol{v}$ is directed along a tangent to the trajectory (see Fig. 1.22).

Reasoning more strictly, to derive equation (1.57) we must proceed as follows.· Having fixed a certain moment of time t, let us consider the increment of the position vector $\Delta\boldsymbol{r}$ during the small time interval Δt[11] following t (Fig. 1.23). The ratio $\Delta\boldsymbol{r}/\Delta t$ gives the average value of the velocity during the time Δt. If we take smaller

[11]The symbol Δ (delta) is used in two cases: (a) for designating the increment of a quantity. In the case being considered, $\Delta\boldsymbol{r}$ is the increment of the position vector $\boldsymbol{r}$ during the time Δt; (b) for designating a fraction of a quantity. For example, Δt is a fraction of the total time t during which motion occurs, and Δs is a fraction of the entire distance s travelled by the particle.

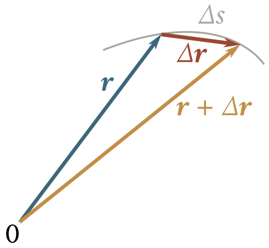

Fig. 1.23

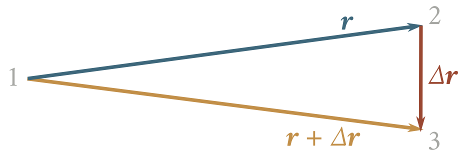

Fig. 1.24

and smaller intervals Δt, the ratio $\Delta \boldsymbol{r}/\Delta t$ in the limit will give us the value of the velocity $\boldsymbol{v}$ at the moment t:

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{1.58}$$

We have arrived at equation (1.57).

Let us find the magnitude of the expression (1.58), *i.e.*, the magnitude of the velocity $\boldsymbol{v}$:

$$v = |\boldsymbol{v}| = \left| \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} \right| = \lim_{\Delta t \to 0} \frac{|\Delta \boldsymbol{r}|}{\Delta t}. \tag{1.59}$$

We cannot write Δr instead of $|\Delta \boldsymbol{r}|$ in this formula. The vector $\Delta \boldsymbol{r}$ is in essence the difference between two vectors ($\boldsymbol{r}$ at the moment $t + \Delta t$ minus $\boldsymbol{r}$ at the moment t). Therefore, its magnitude may be written only with the aid of vertical bars [see Eq. (1.2)]. The symbol $|\Delta \boldsymbol{r}|$ signifies the magnitude of the increment of the vector $\boldsymbol{r}$, whereas Δr is the increment of the magnitude of the vector $\boldsymbol{r}$, *i.e.*, $\Delta|\boldsymbol{r}|$. These two quantities, generally speaking, do not equal each other:

$$|\Delta \boldsymbol{r}| \neq \Delta|\boldsymbol{r}| = \Delta r.$$

The following example will illustrate this. Assume that the vector $\boldsymbol{r}$ receives such an increment $\Delta \boldsymbol{r}$ that its magnitude does not change, *i.e.*, $|\boldsymbol{r} + \Delta \boldsymbol{r}| = |\boldsymbol{r}|$ (Fig. 1.24). Consequently, the increment of the magnitude of the vector equals zero ($\Delta|\boldsymbol{r}| = \Delta r = 0$). At the same time, the magnitude of the increment of the vector $\boldsymbol{r}$, *i.e.*, $\Delta|\boldsymbol{r}|$, differs from zero (it equals the length of $2 - 3$). What has been said above holds for any vector $\boldsymbol{a}$: in the general case $\Delta|\boldsymbol{a}| \neq \Delta|a|$.

Inspection of Fig. 1.23 shows that the distance Δs, generally speaking, differs in value from the magnitude of the displacement $|\Delta \boldsymbol{r}|$. If we take increments of the distance Δs and the displacement $\Delta \boldsymbol{r}$ corresponding to smaller and smaller time intervals Δt, then the difference between Δs and $|\Delta \boldsymbol{r}|$ will diminish, and their ratio in the limit will become equal to unity:

$$\lim_{\Delta t \to 0} \frac{\Delta s}{|\Delta \boldsymbol{r}|} = 1.$$

On these grounds, we can substitute Δs for $|\Delta \boldsymbol{r}|$ in equation (1.59), which gives us the expression

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{1.60}$$

Thus, the magnitude of the velocity equals the derivative of the distance with respect to time.

It is evident that the quantity which in everyday life we call the speed is actually the magnitude of the velocity $\boldsymbol{v}$. In uniform motion, the magnitude of the velocity remains constant ($v = $ constant), whereas the direction of the vector $\boldsymbol{v}$ changes arbitrarily (in particular it may be constant).

In accordance with Eq. (1.57), the elementary displacement of a particle is

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t. \tag{1.61}$$

Sometimes for clarity's sake, we shall denote an elementary displacement by the symbol $\mathrm{d}\boldsymbol{s}$, *i.e.*, write Eq. (1.61) in the form

$$\mathrm{d}\boldsymbol{s} = \boldsymbol{v}\,\mathrm{d}t. \tag{1.62}$$

The velocity vector, like any other vector, can be represented in the form

$$\boldsymbol{v} = v_x \hat{\boldsymbol{e}}_x + v_y \hat{\boldsymbol{e}}_y + v_z \hat{\boldsymbol{e}}_z \tag{1.63}$$

where v_x, v_y, v_z are the projections of the vector $\boldsymbol{v}$ onto the coordinate axes. At the same time, the vector $\dot{\boldsymbol{r}}$ equal to $\boldsymbol{v}$, according to Eq. (1.43), can be written as follows:

$$\dot{\boldsymbol{r}} = \dot{x}\hat{\boldsymbol{e}}_x + \dot{y}\hat{\boldsymbol{e}}_y + \dot{z}\hat{\boldsymbol{e}}_z \tag{1.64}$$

It follows from a comparison of Eqs. (1.63) and (1.64) that

$$v_x = \dot{x}, \quad v_y = \dot{y}, \quad v_z = \dot{z}. \tag{1.65}$$

Consequently, the projection of the velocity vector onto a coordinate axis equals the time derivative of the relevant coordinate of the moving particle. Taking Eq. (1.10) into account, we get:

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}. \tag{1.66}$$

The velocity vector can be written in the form $\boldsymbol{v} = v\hat{\boldsymbol{e}}_v$, where v is the magnitude of the velocity, and $\hat{\boldsymbol{e}}_v$ is the unit vector of $\boldsymbol{v}$. Let us introduce the unit vector $\hat{\boldsymbol{\tau}}$ of the tangent to a trajectory with its sense the same as that of $\boldsymbol{v}$. Hence, obviously, the unit vectors $\hat{\boldsymbol{e}}_v$ and $\hat{\boldsymbol{\tau}}$ will coincide, and we can write the following expression:

$$\boldsymbol{v} = v\hat{\boldsymbol{e}}_v = v\hat{\boldsymbol{\tau}}. \tag{1.67}$$

Let us obtain still another expression for $\boldsymbol{v}$. For this purpose, we shall introduce the position vector in the form of $\boldsymbol{r} = r\hat{\boldsymbol{e}}_r$ into Eq. (1.57). According to Eq. (1.49), we have

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\hat{\boldsymbol{e}}}_r. \tag{1.68}$$

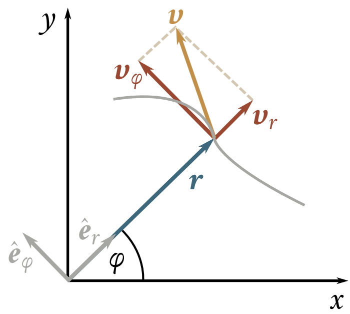

Fig. 1.25

We shall limit ourselves, for simplicity, to the case when the trajectory is a plane curve, *i.e.*, a curve such that all its points are in a single plane. Let this plane be the plane x, y. In Eq. (1.68), the vector $\boldsymbol{v}$ written in the form of two components (Fig. 1.25). The first of them, which we shall designate $\boldsymbol{v}_r$, is

$$\boldsymbol{v}_r = \dot{r}\hat{\boldsymbol{e}}_r. \tag{1.69}$$

It is directed along the position vector $\boldsymbol{r}$ and characterizes the rate of change of the magnitude of $\boldsymbol{r}$. The second component, which we shall designate $\boldsymbol{v}_\varphi$, is

$$\boldsymbol{v}_\varphi = r\dot{\hat{\boldsymbol{e}}}_r. \tag{1.70}$$

It characterizes the rate of change of the direction of the position vector.

Using Eq. (1.56), we can write that

$$\dot{\hat{\boldsymbol{e}}}_r = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{\boldsymbol{e}}_\varphi = \dot{\varphi}\hat{\boldsymbol{e}}_\varphi$$

where φ is the angle between the position vector and the x-axis, and $\hat{\boldsymbol{e}}_\varphi$ is a unit vector perpendicular to the position vector with its sense in the direction of growth of the angle φ [in Eq. (1.56) the symbol $\hat{\boldsymbol{e}}_\perp$ was used for this unit vector]. Using this value in Eq. (1.70), we get:

$$\boldsymbol{v}_\varphi = r\dot{\varphi}\hat{\boldsymbol{e}}_\varphi. \tag{1.71}$$

We have introduced the symbols $\boldsymbol{v}_\varphi$ and $\hat{\boldsymbol{e}}_\varphi$ to underline the fact that the component $\boldsymbol{v}_\varphi$ and the corresponding unit vector are related to a change in the angle φ.

The vectors $\boldsymbol{v}_r$ and $\boldsymbol{v}_\varphi$ are obviously mutually perpendicular. Hence,

$$v = \sqrt{v_r^2 + v_\varphi^2} = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}. \tag{1.72}$$

Now let us consider how to calculate the distance travelled by a particle from the moment of time t_1 to t_2 if we know the speed at each moment. Let us divide the interval $t_2 - t_1$ into N small, but not necessarily equal intervals: Δt_1, Δt_2, …, Δt_N. The total distance s travelled by a particle can be represented as the sum of the distances $\Delta s_1, \Delta s_2, \ldots, \Delta s_N$ travelled during the relevant time intervals Δt:

$$s = \Delta s_1 + \Delta s_2 + \ldots + \Delta s_N = \sum_{i=1}^{N} \Delta s_i.$$

In accordance with Eq. (1.60), each of the addends can approximately be represented in the form

$$\Delta s_i \approx v_i \Delta t_i$$

where Δt_i is the time interval during which the distance Δs_i was travelled, and v_i is one of the values of the speed during the time Δt_i. Hence,

$$s \approx \sum_{i=1}^{N} v_i \Delta t_i. \tag{1.73}$$

This expression will be obeyed more accurately with diminishing time intervals Δt_i. In the limit when all the Δt_i's tend to zero (the number of intervals Δt_i will correspondingly grow unlimitedly), the approximate equation will become accurate:

$$s = \lim_{\Delta t_i \to 0} \sum_{i=1}^{N} v_i \Delta t_i.$$

This expression is a definite integral of the function $v(t)$ taken within the limits from t_1 to t_2. Thus, the distance travelled by a particle during the interval from t_1 to t_2 is

$$s = \int_{t_1}^{t_2} v(t)\,\mathrm{d}t. \tag{1.74}$$

It must be underlined that here we are speaking of the speed. If we take an integral of the velocity $\boldsymbol{v}(t)$, we get the vector of the displacement of the particle from the point where it was at the moment t_1 to the point where it was at the moment t_2:

$$\int_{t_1}^{t_2} \boldsymbol{v}(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d}\boldsymbol{r} = \boldsymbol{r}_{12} \tag{1.75}$$

[see Eq. (1.61)].

If we plot the dependence of v on t (Fig. 1.26), then the distance travelled can be represented as the area of the figure confined between the curve $v(t)$, the straight lines $t = t_1$ and $t = t_2$, and the t-axis. Indeed, the product $v_i \Delta t_i$ numerically equals the area of the i-th strip. The sum Eq. (1.73) equals the area of the figure confined on top by the broken line formed by the top edges of all such strips. When all the Δt_i's tend to zero, the width of a strip diminishes (their number grows simultaneously), and the broken line will coincide with the curve $v = v(t)$ in the limit. Thus, the distance travelled during the time from the moment t_1 to the moment t_2 numeri-

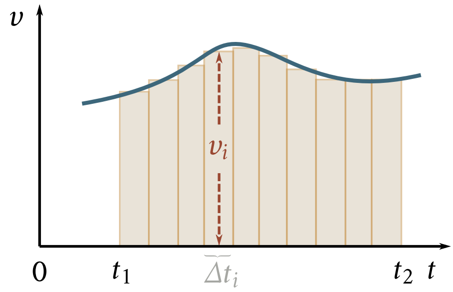

Fig. 1.26

cally equals the area confined between the curve of the function $v = v(t)$, the time axis, and the straight lines $t = t_1$ and $t = t_2$.

It should be noted that the average value of the speed during the time from $t = t_1$ to $t = t_2$, by definition, is

$$\langle v \rangle = \frac{s}{t_2 - t_1}.$$

(The symbol $\langle\rangle$ embracing the v indicates an average.) Introducing into this equation the expression (1.74) for s, we get

$$\langle v \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\,\mathrm{d}t. \tag{1.76}$$

The average values of any scalar or vector functions are calculated in a similar way. For example, the average value of the velocity is

$$\langle \boldsymbol{v} \rangle = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \boldsymbol{v}(t)\,\mathrm{d}t = \frac{\boldsymbol{r}_{12}}{t_2 - t_1}. \tag{1.77}$$

[see Eq. (1.75)]. The average value of the function $y(x)$ within the interval from x_1 to x_2 is determined by the expression

$$\langle y \rangle = \frac{1}{t_2 - t_1} \int_{x_1}^{x_2} y(t)\,\mathrm{d}x. \tag{1.78}$$

1.4. Acceleration

The velocity $\boldsymbol{v}$ of a particle can change with time both in magnitude and in direction. The rate of change of the vector $\boldsymbol{v}$, like the rate of change of any function of time, is determined by the derivative of the vector $\boldsymbol{v}$ with respect to t. Denoting this derivative by the symbol $\boldsymbol{a}$, we get:

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \dot{\boldsymbol{v}}. \tag{1.79}$$

The quantity determined by equation (1.79) is called the **acceleration** of the particle.

It must be noted that the acceleration $\boldsymbol{a}$ plays the same part with respect to $\boldsymbol{v}$ as the vector $\boldsymbol{v}$ does with respect to the position vector $\boldsymbol{r}$.

Equal vectors have identical projections onto the coordinate axes. Consequently, for example,

$$a_x = \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_{\mathrm{pr.}\,\boldsymbol{x}} = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \dot{v}_x$$

[see Eqs. (1.40)]. At the same time according to Eqs. (1.65), we have $v_x = \dot{x} = \mathrm{d}x/\mathrm{d}t$. Therefore,

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}.$$

What we have obtained is that the projection of the acceleration vector onto the x-axis equals the second derivative of the coordinate x with respect to time: $a_x = \ddot{x}$. Similar expressions are obtained for the projections of the acceleration onto the y- and z-axes. Thus,

$$a_x = \ddot{x}, \quad a_y = \ddot{y}, \quad a_z = \ddot{z}. \tag{1.80}$$

Using Eq. (1.67) for $\boldsymbol{v}$ in (1.79), we get:

$$\boldsymbol{a} = \frac{\mathrm{d}(v\hat{\boldsymbol{\tau}})}{\mathrm{d}t}. \tag{1.81}$$

We remind our reader that $\hat{\boldsymbol{\tau}}$ is the unit vector of a tangent to a trajectory having the same direction as $\boldsymbol{v}$. According to Eq. (1.49),

$$\boldsymbol{a} = \dot{v}\hat{\boldsymbol{\tau}} + v\dot{\hat{\boldsymbol{\tau}}}. \tag{1.82}$$

Hence, the vector $\boldsymbol{a}$ can be represented in the form of the sum of two components. One of them has the direction $\hat{\boldsymbol{\tau}}$, *i.e.*, is tangent to the trajectory. It is therefore designated $\boldsymbol{a}_{\hat{\tau}}$ and is called the **tangential acceleration**. It equals

$$\boldsymbol{a}_{\hat{\tau}} = \dot{v}\hat{\boldsymbol{\tau}}. \tag{1.83}$$

The second component equal to $v\dot{\hat{\boldsymbol{\tau}}}$ is directed, as we shall show below, along a normal to the trajectory. It is therefore designated $\boldsymbol{a}_{\hat{n}}$ and is called the **normal acceleration**. Thus,

$$\boldsymbol{a}_{\hat{n}} = v\dot{\hat{\boldsymbol{\tau}}}. \tag{1.84}$$

In studying the properties of the two components, we shall restrict ourselves for the sake of simplicity to the case when the trajectory is a plane curve.

The magnitude of the tangential acceleration (1.83) is

$$a_{\hat{\tau}} = |\dot{v}|. \tag{1.85}$$

If $\dot{v} > 0$ (the velocity grows in magnitude), then the vector $\boldsymbol{a}_{\hat{\tau}}$ has the same direction as $\hat{\boldsymbol{\tau}}$ (*i.e.*, the same direction as $\boldsymbol{v}$). If $\dot{v} < 0$ (the velocity decreases with time), then

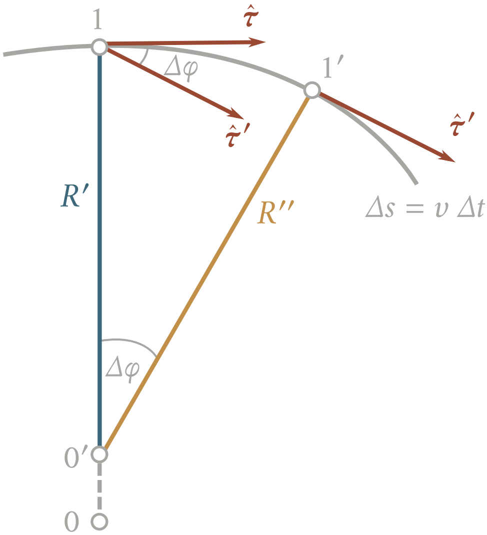

Fig. 1.27

the vectors $\boldsymbol{v}$ and $\boldsymbol{a}_{\hat{\tau}}$ have opposite directions. In uniform motion, $\dot{v} = 0$, and, therefore, tangential acceleration is absent.

To determine the properties of the normal acceleration [Eq. (1.84)], we must find out what $\dot{\hat{\boldsymbol{\tau}}}$, *i.e.*, the rate of change with time of the direction of a tangent to the trajectory, is determined by. It is easy to understand that this rate will grow with an increasing curvature of the trajectory and a higher velocity of a particle along it.

The degree of bending of a plane curve is characterized by its curvature C determined by the expression

$$C = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta s} = \frac{\mathrm{d}\varphi}{\mathrm{d}s} \tag{1.86}$$

where $\Delta\varphi$ is the angle between tangents to the curve at points spaced Δs apart (Fig. 1.27). Thus, the curvature determines the rate of turning of a tangent in motion along a curve.

The reciprocal of the curvature C is called the **radius of curvature** at the given point of the curve and is designated R: The degree of bending of a plane curve is characterized by its curvature C determined by the expression

$$R = \frac{1}{C} = \lim_{\Delta\varphi \to 0} \frac{\Delta s}{\Delta\varphi} = \frac{\mathrm{d}s}{\mathrm{d}\varphi}. \tag{1.87}$$

The radius of curvature is the radius of a circle that coincides at the given spot with the curve on an infinitely small portion of it. The centre of this circle is defined as the centre of curvature for the given point of the curve.

The radius and centre of curvature at point 1 (see Fig. 1.27) can be determined as follows. Take point $1'$ near point 1. Draw the tangents $\hat{\boldsymbol{\tau}}$ and $\hat{\boldsymbol{\tau}}'$ at these points. The perpendiculars to the tangents will intersect at a certain point $0'$. We must note that for a curve which is not a circle the distances R' and R'' will differ somewhat from each other. If point $1'$ is brought closer to point 1, the point of intersection $0'$ of the perpendiculars will move along the straight line R' and in the limit will be at point 0. It is exactly the latter that will be the centre of curvature for point 1. The distances R' and R" will tend to a common limit R equal to the radius of curvature. Indeed, if points 1 and $1'$ are close to each other, we can write that $\Delta\varphi \approx \Delta s/R'$ or $R' \approx \Delta s/\Delta\varphi$. In the limit when $\Delta\varphi \to 0$, this approximate equation will transform into the strict equation $R = \mathrm{d}s/\mathrm{d}\varphi$ coinciding with the definition of the radius of curvature [see Eq. (1.87)].

Let us now turn to the calculation of an [see Eq. (1.84)]. According to Eq. (1.56),

$$\dot{\hat{\boldsymbol{\tau}}} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\hat{\boldsymbol{n}} \tag{1.88}$$

where $\hat{\boldsymbol{n}}$ is the unit vector of the normal to the trajectory with its sense in the direction of rotation of the vector $\hat{\boldsymbol{\tau}}$ when a particle travels along the trajectory [in Eq. (1.56) a similar unit vector was designated $\hat{\boldsymbol{e}}_\perp$]. The quantity $\mathrm{d}\varphi/\mathrm{d}t$ can be related to the radius of curvature of the trajectory and the speed of the particle $\boldsymbol{v}$. It follows from Fig. 1.27 that

$$\Delta\varphi \approx \frac{\Delta s}{R'} = \frac{v'\,\Delta t}{R'}$$

where $\Delta\varphi$ is the angle of rotation of the vector $\hat{\boldsymbol{\tau}}$ during the time Δt (coinciding with the angle between the perpendiculars R' and R''), and v' is the average speed over the distance Δs. Hence,

$$\frac{\Delta\varphi}{\Delta s} \approx \frac{v'}{R'}.$$

In the limit when Δt tends to zero, the approximate equation will become a strict one, the average speed v' will transform into the instantaneous speed v at point 1, and R' will become the radius of curvature R. As a result, we get the equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{v}{R} = vC \tag{1.89}$$

(C is the curvature). Hence, the rate of rotation of the velocity vector, as we assumed, is proportional to the curvature of the trajectory and the speed of a particle along its trajectory.

Using Eq. (1.89) in (1.88), we find that $\dot{\hat{\boldsymbol{\tau}}} = (v/R)\hat{\boldsymbol{n}}$. And at last, introducing this expression into Eq. (1.84), we arrive at the final equation for the normal acceleration:

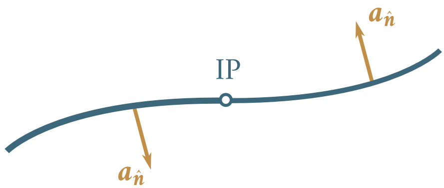

Fig. 1.28

$$\boldsymbol{a}_{\hat{n}} = \frac{\mathrm{d}v^2}{\mathrm{d}R}\hat{\boldsymbol{n}}. \tag{1.90}$$

Thus, the acceleration vector when a particle travels along a plane curve is determined by the following expression:

$$\boldsymbol{a} = \boldsymbol{a}_{\hat{\tau}} + \boldsymbol{a}_{\hat{n}} = \dot{v}\hat{\tau} + \frac{v^2}{R}\hat{\boldsymbol{n}}. \tag{1.91}$$

The magnitude of the vector $\boldsymbol{a}$ is

$$a = \sqrt{a_{\hat{\tau}}^2 + a_{\hat{n}}^2} = \sqrt{\dot{v}^2 + \left(\frac{v^2}{R}\right)^2}. \tag{1.92}$$

In rectilinear motion, the normal acceleration is absent. It must be noted that an vanishes at the inflection point of a curvilinear trajectory (at point IP in Fig. 1.28). At both sides of this point, the vectors $\boldsymbol{a}_{\hat{n}}$ have different directions. The vector $\boldsymbol{a}_{\hat{n}}$ cannot change in a jump. Its direction reverses smoothly, and it becomes equal to zero at the inflection point. Assume that a particle is travelling uniformly with an acceleration constant in magnitude. Since in uniform motion the magnitude of the velocity does not change, we have $\boldsymbol{a}_{\hat{\tau}} = 0$, so that $\boldsymbol{a} = \boldsymbol{a}_{\hat{n}}$. The constant magnitude of an signifies that v^2/R = constant. Hence, we conclude that R = constant (v = constant because the motion is uniform). This means that the particle is travelling along a curve of constant curvature, *i.e.*, a circle. Thus, when the acceleration of a particle is constant in magnitude and is directed at each moment of time at right angles to the velocity vector, the trajectory of the particle will be a circle.

1.5. Circular Motion

The rotation of a body through a certain angle φ can be given in the form of a straight line whose length is φ and whose direction coincides with the axis about which the body is rotating. To indicate the direction of rotation about a given axis, it is related to the line depicting rotation by the **right-hand screw rule**: the line should be directed so that when looking along it (Fig. 1.29) we see clockwise rotation (when rotating the head of a right-hand screw clockwise, we cause it to

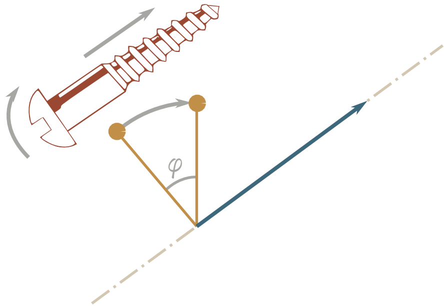

Fig. 1.29

move away from us). We showed in Sec. 1.2 (see Fig. 1.4) that rotations through finite angles are not added by the parallelogram method and are therefore not vectors. Matters are different for rotations through very small angles $\Delta\boldsymbol{\varphi}$. The distance travelled by any point of a body when rotated through a very small angle can be considered as a straight line (Fig. 1.30). Consequently, two small circular motions $\Delta\boldsymbol{\varphi}_1$ and $\Delta\boldsymbol{\varphi}_2$ performed sequentially, as can be seen from the figure, result in the same displacement $\Delta\boldsymbol{r}_3 = \Delta\boldsymbol{r}_1 + \Delta\boldsymbol{r}_2$ of any point of the body as the circular motion $\Delta\boldsymbol{\varphi}_3$ obtained from $\Delta\boldsymbol{\varphi}_1$ and $\Delta\boldsymbol{\varphi}_2$ by addition using the parallelogram method. Hence it follows that very small circular motions can be considered as vectors (we shall denote these vectors by $\Delta\boldsymbol{\varphi}$ or $\mathrm{d}\boldsymbol{\varphi}$). The direction of the rotation vector is associated with the direction of rotation of a body. Consequently, $\mathrm{d}\boldsymbol{\varphi}$ is not a true vector, but a pseudovector.

The vector quantity

$$\boldsymbol{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\varphi}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t} \tag{1.93}$$

(where Δt is the time during which the circular motion $\Delta\boldsymbol{\varphi}$ is performed) is called the angular velocity of a body[12]. The angular velocity $\boldsymbol{\omega}$ is directed along the axis about which the body is rotating in a direction determined by the right-hand screw rule (Fig. 1.31) and is a pseudovector. The magnitude of the angular velocity, *i.e.*, the angular speed, equals $\mathrm{d}\varphi/\mathrm{d}t$. Circular motion at a constant angular velocity is called uniform. For uniform circular motion, we have $\omega = \varphi t$, where φ is the finite angle of rotation during the time t (compare with $v = s/t$). Thus, in uniform circular motion, ω shows the angle through which a body rotates in unit time.

Uniform circular motion can be characterized by the **period of revolution** T. It is defined as the time during which a body completes one revolution, i.e. rotates through the angle 2π rad, or 360°. Since the time interval $\Delta t = T$ corresponds to

[12]The velocity $\boldsymbol{v}$ considered in Sec. 1.3 is sometimes called linear.

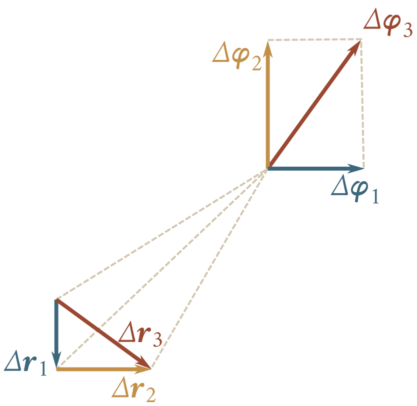

Fig. 1.30

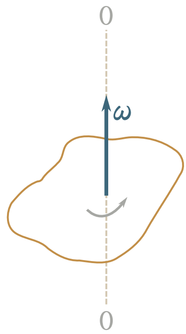

Fig. 1.31

the angle of rotation $\Delta\varphi = 2\pi$, we have

$$\omega = \frac{2\pi}{T} \tag{1.94}$$

whence

$$T = \frac{2\pi}{\omega}. \tag{1.95}$$

The number of revolutions in unit time ν is evidently equal to

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{1.96}$$

It follows from Eq. (1.96) that the angular velocity equals 2π multiplied by the number of revolutions per unit time:

$$\omega = 2\pi\nu. \tag{1.97}$$

The concepts of the period of revolution and the number of revolutions per unit time can also be retained for non-uniform circular motion. Here, we must understand the instantaneous value of T to signify the time during which a body would perform one revolution if it rotated uniformly with the given instantaneous value of the angular velocity, and ν to signify the number of revolutions which a body would complete in unit time in similar conditions.

The vector $\boldsymbol{\omega}$ may vary either as a result of a change in the speed of rotation of a body about its axis (in this case it changes in magnitude), or as a result of turning of the axis of rotation in space (in this case $\boldsymbol{\omega}$ changes in direction). Assume that during the time Δt the vector $\boldsymbol{\omega}$ receives the increment $\Delta\boldsymbol{\omega}$. The change in the angular velocity vector with time is characterized by the quantity

$$\boldsymbol{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{\omega}}{t} = \frac{d\boldsymbol{\omega}}{dt} \tag{1.98}$$

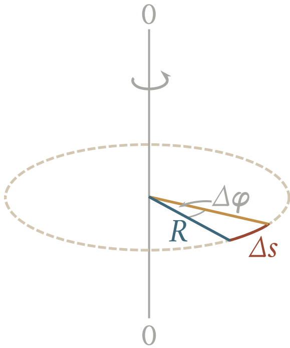

Fig. 1.32

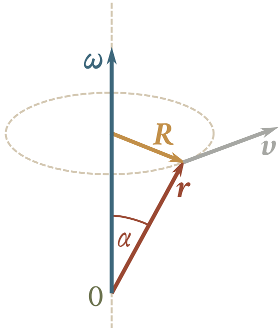

Fig. 1.33

called the **angular acceleration**. The latter, like the angular velocity, is a pseudovector.

Different points of a body in circular motion have different linear velocities $\boldsymbol{v}$. The velocity of each point continuously changes its direction. The speed v is determined by the speed of rotation of the body ω and the distance R to the point being considered from the axis of rotation. Assume that during a small interval of time the body turned through the angle $\Delta\varphi$ (Fig. 1.32). The point at the distance R from the axis travels the path $\Delta s = R\Delta\varphi$. The linear speed of the point is

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} R\frac{\Delta\varphi}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = R\frac{\mathrm{d}\varphi}{\mathrm{d}t} = R\omega.$$

Thus,

$$v = \omega R. \tag{1.99}$$

Equation (1.99) relates the linear and the angular speeds. Let us find an expression relating the relevant velocities $\boldsymbol{v}$ and $\boldsymbol{\omega}$. We shall determine the position of the point of the body being considered by the position vector $\boldsymbol{r}$ drawn from the origin of coordinates on the axis of rotation (Fig. 1.33). Examination of the figure shows that the vector product $\boldsymbol{\omega} \times \boldsymbol{r}$ coincides in direction with the vector $\boldsymbol{v}$ and its magnitude is $\omega r \sin\alpha = \omega R$. Hence,

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}. \tag{1.100}$$

The normal acceleration of the points of a rotating body is $a_{\hat{\boldsymbol{n}}} = v^2/R$. Introducing into this expression the value of v from Eq. (1.99), we get

$$a_{\hat{\boldsymbol{n}}} = \omega^2 R. \tag{1.101}$$

If we introduce the vector $\boldsymbol{R}$ drawn to the given point of the body from the axis of rotation at right angles to the latter (see Fig. 1.33), then Eq. (1.101) can be given a vector form:

$$\boldsymbol{a}_{\hat{\boldsymbol{n}}} = -\omega^2 \boldsymbol{R}. \tag{1.102}$$

There is a minus sign in this formula because the vectors $\boldsymbol{a}_{\hat{\boldsymbol{n}}}$ and $\boldsymbol{R}$ have opposite directions.

Let us assume that the axis of rotation of a body does not turn in space. According to Eq. (1.85), the magnitude of the tangential acceleration is $|\mathrm{d}v/\mathrm{d}t|$. Using equation (1.99) and taking into account that the distance to the point being considered from the axis of rotation R = constant, we can write

$$a_{\hat{\tau}} = \left|\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}\right| = \left|\lim_{\Delta t \to 0} \frac{\Delta(\omega R)}{\Delta t}\right| = R\left|\lim_{\Delta t \to 0} \frac{\Delta(\omega)}{\Delta t}\right| = R\alpha$$

where α, is the magnitude of the angular acceleration. Consequently, the magnitude of the tangential acceleration is related to the magnitude of the angular acceleration as follows:

$$a_{\hat{\tau}} = \alpha R. \tag{1.103}$$

Thus, the normal and tangential accelerations grow linearly with an increasing distance to a point from the axis of rotation.

Chapter 2

DYNAMICS OF A POINT PARTICLE

2.1. Classical Mechanics. Range of Its Applicability

Kinematics describes the motion of bodies without being concerned with why a body moves exactly in a given way (for example, uniformly along a circle, or with uniform acceleration along a straight line), and not in a different one.

Dynamics studies the motion of bodies in connection with its causes (the interactions between bodies) resulting in the occurrence of a specific kind of motion.

The so-called classical or Newtonian mechanics is based on three laws of dynamics that were formulated by Isaac Newton in 1687.

Newton's laws (like all other laws of physics) were the result of generalizing a great amount of experimental facts. Their correctness (although it covers a very extensive range of phenomena, the latter are nevertheless limited) is confirmed by the agreement of the corollaries following from them with experimental results.

Newtonian mechanics achieved such great successes during two centuries that many physicists of the 19th century were convinced in its omnipotence. It was considered that the explanation of any physical phenomenon required its reduction to a mechanical process obeying Newton's laws. With the development of science, however, new facts were uncovered for which no place could be found within the confines of classical mechanics. These facts were explained in new theories—the special theory of relativity and quantum mechanics.

The special theory of relativity advanced by Albert Einstein in 1905 radically revised Newton's notions of space and time. This revision resulted in the creation of "high-speed mechanics" or, as it is called, relativistic mechanics. The new mechanics did not result, however, in complete negation of the old Newtonian mechanics. The equations of relativistic mechanics in their limit (for speeds small in comparison with the speed of light) transform into the equations of classical mechanics.

Thus, classical mechanics has entered relativistic mechanics as a particular case of it and has retained its previous significance for describing motions occurring at speeds much smaller than that of light.

Matters are similar with the relation between classical and quantum mechanics. The latter took root in the twenties of the present century as a result of the development of physics of the atom. The equations of quantum mechanics also result in those of classical mechanics in their limit (for masses that are great in comparison with the masses of atoms). Consequently, classical mechanics is also a part of quantum mechanics and is a limiting case of it.

Thus, the development of science has not eliminated classical mechanics, but has only shown its limited applicability. Classical mechanics based on Newton's laws is the mechanics for bodies of large (compared with the mass of atoms) masses travelling at low (compared with the speed of light) speeds.

2.2. Newton's First Law. Inertial Reference Frames

Newton's first law is formulated as follows: *every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by external forces to change that state*. Both states named are distinguished by the acceleration of the body equalling zero. Therefore, the first law can also be formulated as follows: the velocity of every body remains constant (in particular, it equals zero) until the action of other bodies on this body causes it to change.

Newton's first law is obeyed not in any reference frame. We have already noted that the nature of motion depends on the choice of the reference frame. Let us consider two frames of reference moving with respect to each other with a certain acceleration. If a body is at rest relative to one of them, then it will obviously travel with acceleration relative to the other one. Consequently, Newton's first law cannot be obeyed simultaneously in both frames.

A reference frame in which Newton's first law is obeyed is called an **inertial** one. The law itself is quite often called the **law of inertia**. A reference frame in which Newton's first law is not obeyed is called a non-inertial reference frame. There is an infinite multitude of inertial frames. Any reference frame moving uniformly in a straight line (*i.e.*, with a constant velocity) relative to an inertial frame will also be an inertial one. This will be discussed in greater detail in Sec. 2.7.

It has been established experimentally that the reference frame whose centre coincides with the Sun and whose axes are directed toward appropriately selected stars is an inertial one. This system is defined as a **heliocentric reference frame** (*helios* means Sun in Greek). Any reference frame moving uniformly in a straight line relative to the heliocentric frame will be an inertial one.

The Earth moves relative to the Sun and stars along a curvilinear trajectory having the shape of an ellipse. Curvilinear motion always occurs with a certain acceleration. The Earth also rotates about its axis. For these reasons, a reference frame associated with the Earth's surface travels with acceleration relative to the heliocentric reference frame, and is not inertial. The acceleration of such a frame, however, is so small that it may be considered practically inertial in a great number of cases. But sometimes the non-inertial nature of a reference frame associated with the Earth significantly affect the nature of mechanical phenomena being considered relative to it. We shall treat some of these cases on a later page.

2.3. Mass and Momentum of a Body

The action of other bodies on a given one causes its velocity to change, *i.e.*, imparts an acceleration to it. Experiments show that the same action imparts accelerations differing in magnitude to different bodies. Every body resists attempts to change its state of motion. This property of bodies is called **inertia**. It is characterized quantitatively by a physical quantity called the **mass** of a body.

To find the mass of a body, we must compare it with that of the body taken as the standard of mass. We can also compare the mass of the given body with that of a body having a known mass (found by comparing it with the standard). The masses m_1 and m_2 of two point particles can be compared as follows. We place the particles in conditions allowing us to ignore their interaction with other bodies. A system of bodies interacting only with one another and not interacting with other bodies is called **isolated**. We are therefore considering an isolated system of two particles. If we make these particles interact (for example, by colliding with each other), their velocities receive the increments $\Delta\boldsymbol{v}_1$ and $\Delta\boldsymbol{v}_2$. Experiments show that these increments are always directed oppositely, *i.e.*, differ in their sign. The ratio of the magnitudes of the velocity increments, however, does not depend on the method and intensity of interaction of the given two bodies or particles[1]. This ratio is inversely proportional to the ratio of the masses of the bodies being considered:

$$\frac{|\Delta\boldsymbol{v}_1|}{\Delta\boldsymbol{v}_2} = \frac{m_2}{m_1} \tag{2.1}$$

(the velocity of the body with the greater inertia, *i.e.*, with the larger mass, changes less). Taking into account the relative direction of the vectors $\Delta\boldsymbol{v}_1$ and $\Delta\boldsymbol{v}_2$, Eq. (2.1)

[1]This holds for the case when the initial and final velocities of the particles are small in comparison with the speed of light c.

can be written in the form

$$m_1 \Delta \boldsymbol{v}_1 = -m_2 \Delta \boldsymbol{v}_2. \tag{2.2}$$

In Newtonian mechanics (*i.e.*, in mechanics based on Newton's laws), the mass of a body is assumed to be a constant quantity not depending on its velocity. At velocities smaller than the speed of light c (when $v \ll c$), this assumption is obeyed in practice. Taking advantage of the constancy of mass, we can write Eq. (2.2) as follows:

$$\Delta(m_1 \boldsymbol{v}_1) = -\Delta(m_2 \boldsymbol{v}_2). \tag{2.3}$$

The product of the mass and the velocity of a body is called its **momentum**. Using the symbol $\boldsymbol{p}$ for it, we get

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{2.4}$$

Definition (2.4) holds for point particles and extended bodies in translational motion. When we have to do with an extended body whose motion is not translational, we must imagine the body as a combination of particles of masses Δm_i, determine the momenta $\Delta m_i \boldsymbol{v}_i$ of these particles, and then add these momenta vectorially. The result will be the total momentum of the body:

$$\boldsymbol{p} = \sum_i m_i \boldsymbol{v}_i. \tag{2.5}$$

In translation of a body, all the $\boldsymbol{v}_i$'s are the same, and Eq. (2.5) transforms into (2.4).

Substituting the momenta $\boldsymbol{p}$ for the products $m\boldsymbol{v}$ in Eq. (2.3), we get $\Delta \boldsymbol{p}_1 = \Delta \boldsymbol{p}_2$, whence $\Delta(\boldsymbol{p}_1 + \boldsymbol{p}_2) = 0$. When the increment of a quantity equals zero, this signifies that the quantity itself remains unchanged. We have thus arrived at the conclusion that *the total momentum of an isolated system of two interacting particles remains constant*:

$$\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = \text{constant}. \tag{2.6}$$

The above statement forms the **law of conservation of momentum**. We shall consider this law in greater detail in Sec. 3.10.

We must note here that in relativistic mechanics (see Chap. 8) the expression for the momentum is more complicated than Eq. (2.4):

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}. \tag{2.7}$$

Here m is the so-called **rest mass** of a body (its mass at $v = 0$), and c is the speed of light in a vacuum. Equation (2.7) can be interpreted to state that the mass of a body does not remain constant (as is assumed in Newtonian mechanics), but changes with the speed according to the law

$$m(v) = \frac{m}{\sqrt{1 - v^2/c^2}} \tag{2.8}$$

Hence, Eq. (2.7) can be written as follows:

$$\boldsymbol{p} = m(v)\,\boldsymbol{v} \tag{2.9}$$

i.e., in a form similar to Eq. (2.4).

The mass $m(v)$ determined by Eq. (2.8) is called the **relativistic mass**. We shall designate it by the symbol m_{r} in the following.

2.4. Newton's Second Law

Newton's second law states that *the rate of change of the momentum of a body equals the force* $\boldsymbol{F}$ *acting on the body*:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}. \tag{2.10}$$

Equation (2.10) is called the **equation of motion of a body**.

Substituting $m\boldsymbol{v}$ for $\boldsymbol{p}$ according to Eq. (2.4) and taking into account that in Newtonian mechanics the mass is assumed to be constant, we can write Eq. (2.10) in the form

$$m\boldsymbol{a} = \boldsymbol{F} \tag{2.11}$$

where $\boldsymbol{a} = \dot{\boldsymbol{v}}$. We have thus arrived at a different formulation of Newton's second law: *the product of the mass of a body and its acceleration equals the force acting on the body*.

Equation (2.11) has called forth and is continuing to call forth many controversies among physicists. To date, there is no generally adopted interpretation of this relation. The complication consists in that there are no independent ways of determining the quantities m and $\boldsymbol{F}$ in Eq. (2.11). To determine one of them (m or $\boldsymbol{F}$), we have to use Eq. (2.11) relating it to the other one and to the acceleration $\boldsymbol{a}$. For example, according to S. Khaikin[2], "Since to establish a way of measuring the mass of a body we use the same second law of Newton (the magnitude of the mass of a body is determined by simultaneously measuring the force and the acceleration), then Newton's second law contains, on the one hand, a statement that the acceleration is proportional to the force, and on the other, a definition of the mass of a body as the ratio of the force acting on it to the acceleration imparted to this body".

R. Feynman states the following about the meaning of Newton's second law:

[2] S. E. Khaikin. Fizicheskie osnovy mekhaniki (The Physical Fundamentals of Mechanics). Moscow, Fizmatgiz (1963), p. 104.

"Let us ask, 'What is the meaning of the physical laws of Newton, which we write as $F = ma$? What is the meaning of force, mass, and acceleration?' Well, we can intuitively sense the meaning of mass, and we can define acceleration if we know the meaning of position and time. We shall not discuss these meanings, but shall concentrate on the new concept of force. The answer is equally simple: 'If a body is accelerating, then there is a force on it'. That is what Newton's laws say, so the most precise and beautiful definition of force imaginable might simply be to say that force is the mass of an object times the acceleration…". However, "if we have discovered a fundamental law, which asserts that the force is equal to the mass times the acceleration, and then *define* the force to be the mass times the acceleration, we have found out nothing…, such things certainly cannot be the content of physics, because they are definitions going in a circle…no prediction whatsoever can be made from a definition.… The real content of Newton's laws is this: that the force is supposed to have some independent properties, in addition to the law $\boldsymbol{F} = m\boldsymbol{a}$; but the specific independent properties that the force has were not completely described by Newton or by anybody else…"[3].

We must underline the fact that Newton's second law (like his other two laws) is an experimental one. It took shape as a result of generalization of the data of experiments and observations.

In a particular case when $\boldsymbol{F} = 0$ (*i.e.*, in the absence of action on a body by other bodies), the acceleration, as follows from Eq. (2.11), also equals zero. This conclusion coincides with Newton's first law. Therefore, the first law is contained in the second one as a particular case of it. Notwithstanding this circumstance, the first law is formulated independently of the second one because it contains in essence the postulate (statement) of the existence of inertial reference frames.

In conclusion, we shall note that upon an independent choice of the units of mass, force, and acceleration, the second law must be written in the form

$$m\boldsymbol{a} = k\boldsymbol{F} \tag{2.12}$$

where k is a constant of proportionality.

2.5. Units and Dimensions of Physical Quantities

The laws of physics, as we have already noted, establish quantitative relations between physical quantities. To establish such relations, it is necessary to be able to measure various physical quantities.

To measure a physical quantity (for example, speed) means to compare it with

[3]R. P. Feynman, R. B. Leighton, M. Sands. The Feynman Lectures on Physics. Reading, Mass., Addison-Wesley (1965), p. 12-1.

a quantity of the same kind (in our example with speed) taken as a unit.

Generally speaking, we could establish a unit for every physical quantity arbitrarily, regardless of other quantities. We can limit ourselves, however, to an arbitrary choice of the units for only a few (at least three) quantities taken as the basic ones. Any quantities can be taken as the basic ones in principle. The units of all other quantities can be established with the aid of these basic units using for this purpose the physical laws relating the relevant quantity to the basic ones or to quantities for which the units have already been established in this way.

Let us consider the following example to explain what has been said above. Assume that we have already established the units for mass and acceleration. Equation (2.12) expresses the law relating these quantities to a third physical quantity —force. We choose the unit of force so that the proportionality constant in this equation will equal unity. Equation (2.12) thus acquires a simple form:

$$m\boldsymbol{a} = \boldsymbol{F}. \tag{2.13}$$

It follows from Eq. (2.13) that the established unit of force is a force such that a body of unit mass receives an acceleration of unity under its action [substitution of $F = 1$ and $m = 1$ in Eq. (2.13) gives $a = 1$].

When units are selected in this way, physical relations acquire a simpler form. The combination of units themselves forms a definite system.

There are several systems differing in the selection of the basic units. Systems based on the units of length, mass, and time are called **absolute**.

USSR State Standard GOST 9867-61 in force from January 1, 1963, provides for the use of the International System of Units, designated SI, in the USSR. This system of units has been introduced as preferable in all fields of science, engineering, and the national economy, and also in education. The basic units of the SI system include the unit of length—the metre (its symbol is m), the unit of mass—the kilogramme (kg), and the unit of time—the second (s). The SI system is thus an absolute one. In addition to the three units indicated above, the other basic units of this system are the unit of current—the ampere (A), the unit of thermodynamic temperature—the kelvin (K), the unit of luminous intensity—the candela (cd), and the unit of the amount of substance—the mole (mol). These units will be treated in the corresponding parts of our course.

The metre is defined as the length equal to $1,650,763.73$ wavelengths in vacuum of the radiation corresponding to the transition between the levels $2p_{10}$ and $5d_5$ of the krypton-86 atom[4] (the orange line of krypton-86). The metre approximately equals $1/40,000,000$th of the length of an Earth's meridian. Multiple and submultiple units are also used, namely, the kilometre ($1\text{ km} = 103\text{ m}$), the cen-

[4]The meaning of these symbols will be explained in Vol. III in the part "Atomic Physics".

timetre (1 cm = 10^{-2} m), the millimetre (1 mm = 10^{-3} m), the micrometre (1 μm = 10^{-6} m), etc.

The kilogramme is the mass of a platinum and iridium[5] body kept in the International Chamber of Weights and Measures at Sèvre (near Paris). This body is called the international prototype of the kilogramme. The mass of this prototype is close to that of 1000 cm of pure water at 4 °C. A gramme equals one-thousandth of a kilogramme.

The second is the duration of 9, 192, 631, 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom. The second approximately equals 1/86, 400 of the mean solar day.

Physics also employs an absolute system of units called the cgs system. The basic units in this system are the centimetre, gramme, and second.

The units of the quantities that we introduced in kinematics (velocity and acceleration) are derived from the basic units. Thus, the unit of velocity (or speed) is the velocity of a body in uniform motion that travels a distance of unit length (metre or centimetre) in unit time (second). This unit is designated m s^{-1} in the SI system and cm s^{-1} in the cgs system. The unit of acceleration is the acceleration of uniformly varying motion when the velocity of a body changes by one unit (one m s^{-1} or cm s^{-1}) in unit time (s). This unit is designated m s^{-2} in the SI system and cm s^{-2} in the cgs system.

The unit of force in the SI system is called the newton (N). According to Eq. (2.13), one newton equals the force that imparts an acceleration of 1 m s^{-2} to a body with a mass of 1 kg. The unit of force in the cgs system is called the dyne (dyn). One dyne equals the force that imparts an acceleration of 1 cm s^{-2} to a body with a mass of 1 g. The newton and dyne are related as follows:

$$1\,\mathrm{N} = 1\,\mathrm{kg} \times 1\,\mathrm{m\,s^2} = 10^3\,\mathrm{g} \times 10^2\,\mathrm{cm\,s^2}$$

The mk(force)s system (usually called the technical system) is widely used in engineering. The basic units of this system are the metre, the unit of force—the kilogramme-force (kgf), and the second. The kilogramme-force is defined as the force that imparts an acceleration of 9.80655 m s^{-1} (the normal acceleration of free fall) to a mass of 1 kg. It follows from this definition that 1 kgf = 9.80655 N (approximately 9.81 N).

The unit of mass in the mk(force)s system, according to Eq. (2.13), should be taken equal to the mass of a body that receives an acceleration of 1 m s^{-2} when acted upon by a force of 1 kgf. Although many names have been proposed for this

[5]An alloy of platinum and iridium has a high hardness and resistance to corrosion (*i.e.*, is virtually not subjected to the chemical action of the surroundings).

unit[6], none of them has been legalized, and it is designated kgf s^2 m^{-1}. It is obvious that 1 kgf s^2 m^{-1} = 9.80655 kg (approximately 9.81 kg).

It follows from the way a system of units is constructed that a change in the basic units leads to a change in the derived ones. If, for example, the minute is taken as the unit of time instead of the second, *i.e.*, the unit of time is increased 60 times, then the unit of velocity will diminish 60 times, and the unit of acceleration will diminish 3600 times.

The relation showing how a unit of a quantity changes when the basic units are changed is called the dimension of this quantity. The dimension of an arbitrary physical quantity is designated by its symbol placed in brackets. For example, $[v]$ stands for the dimension of velocity. Special symbols are used for the dimensions of the basic quantities: L for the length, M for the mass, and T for the time. Thus, designating the length by l, the mass by m, and the time by t, we can write

$$[l] = \mathrm{L}, \quad [m] = \mathrm{M}, \quad [t] = \mathrm{T}.$$

When these symbols are used, the dimension of an arbitrary physical quantity has the form $\mathrm{L}^{\alpha}\mathrm{M}^{\beta}\mathrm{T}^{\gamma}$ (α, β and γ may be either positive or negative, and in particular may equal zero). This notation signifies that when the unit of length is increased n_1 times, the unit of a given quantity grows n_1^{α} times (accordingly, the number that expresses the value of the quantity in these units diminishes n_1^{α} times). When the unit of mass is increased n_2 times, the unit of the given quantity grows n_2^{β} times. Finally, when the unit of time is increased n_3 times, the unit of the given quantity grows n_3^{γ} times.

Since physical laws cannot depend on the selection of the units for the quantities figuring in them, the dimensions of both sides of the equations expressing these laws must be the same. This condition can be used, first, for verifying the correctness of physical relations obtained, and, second, for establishing the dimensions of physical quantities. For example, speed is determined by the formula $v = \Delta s/\Delta t$. The dimension of Δs is L, and that of Δt is T. The dimension of the right-hand side of the above formula is $[\Delta s]/[\Delta t] = \mathrm{L}/\mathrm{T} = \mathrm{LT}^{-1}$. The dimension of the left-hand side must be the same. Hence,

$$[v] = \mathrm{LT}^{-1}. \tag{2.14}$$

This relation is called a dimension formula, and its right-hand side the dimension of the relevant quantity (in our case, of speed).

[6]See L. A. Sena. Units of Physical Quantities and Their Dimensions. 2nd ed., Moscow, Mir Publishers (1975), pp. 9, 54

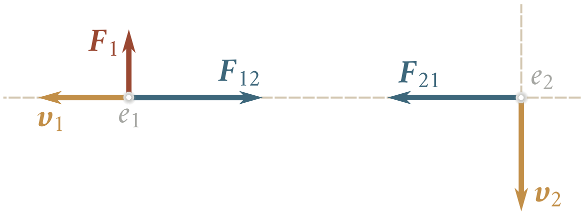

Fig. 2.1

The relation $a = \Delta v/\Delta t$ permits us to establish the dimension of acceleration:

$$[a] = \frac{[\Delta v]}{[\Delta t]} = \frac{\mathrm{LT}^{-1}}{\mathrm{T}} = \mathrm{LT}^{-2}. \tag{2.15}$$

The dimension of force is

$$[F] = [m][a] = \mathrm{MLT}^{-2}. \tag{2.16}$$

The dimensions of all other quantities are established in a similar way.

2.6. Newton's Third Law

Any action of bodies on one another has the nature of mutual interaction: if body 1 acts on body 2 with the force $\boldsymbol{F}_{21}$ then body 2, in turn, acts on body 1 with the force $\boldsymbol{F}_{12}$.

Newton's third law states that *the forces exerted by interacting bodies on each other are equal in magnitude and opposite in direction*. Using the above symbols for such forces, the third law can be expressed in the form of the equation

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}. \tag{2.17}$$

It follows from Newton's third law that forces appear in pairs: for any force applied to a body there is another force equal in magnitude and opposite in direction applied to the second body interacting with the first one.

Newton's third law is not always correct. It is observed quite strictly in contact interactions (*i.e.*, interactions observed upon the direct contact of bodies), and also when bodies at rest that are a certain distance apart interact with each other.

As an example of the violation of Newton's third law, we can take a system of two charged particles e_1 and e_2 moving at the given moment as shown in Fig. 2.1. It is proved in electrodynamics that apart from the force of electrostatic interaction $\boldsymbol{F}_{12}$ obeying the third law, the magnetic force $\boldsymbol{F}_1$ will also be exerted on the first particle. Only the force $\boldsymbol{F}_{21}$ equal to $-\boldsymbol{F}_{12}$ acts on the second particle. The magnitude of the magnetic force acting on the second particle for the case shown in the figure equals zero. It must be noted that for speeds of particles that are much

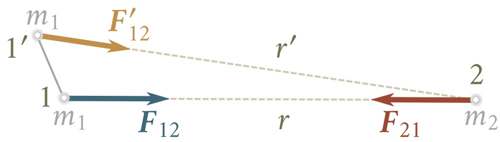

Fig. 2.2

smaller than the speed of light in a vacuum (when $v_1 \ll c$ and $v_2 \ll c$) the force $\boldsymbol{F}_1$ is negligibly small in comparison with the force $\boldsymbol{F}_{12}$, so that Newton's third law is virtually correct in this case too.

Now let us consider a system of two electrically neutral particles m_1 and m_2 at the distance r from each other. Owing to universal gravitation, these particles attract each other with the force

$$F = G\frac{m_1 m_2}{r^2}. \tag{2.18}$$

In this case, the particles interact via a gravitational field. Say, the first particle sets up in the space surrounding it a field which manifests itself in that the particle m_2 placed at a point of this field experiences a force of attraction to the first particle. Similarly, the second particle sets up a field which manifests itself in its action on the first particle. Experiments show that the changes in the field due, for instance, to a change in the position of the particle producing it propagate in space not instantaneously, but at a finite, though very high, speed equal to the speed of light in a vacuum c.

Let us assume that the particles m_1 and m_2 are initially at rest at positions 1 and 2 (Fig. 2.2). The forces of interaction $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{21}$ are equal in magnitude and opposite in direction. Now assume that the particle m_1 moves very rapidly (with a speed almost equal to c) to position $1'$. At this point, the particle m_1 will experience the force $\boldsymbol{F}'_{12}$ smaller in magnitude ($r' > r$) and directed differently than $\boldsymbol{F}_{12}$ (we remind our reader that the field of the particle m_2 remains constant). The force $\boldsymbol{F}_{21}$ will continue to act on the second particle until the disturbance of the field due to the displacement of m_1 reaches point 2. Consequently, Newton's third law was violated while the particle m_1 was in motion some time after it stopped at point $1'$.

If the particle m_1 moved from point 1 to point $1'$ with the speed v much smaller than c ($v \ll c$), or the speed of propagation of field disturbances were infinitely great, then the instantaneous values of the field at point 2 would correspond to the positions of the particle m_1 at the same moment of time, and, consequently, no violations of the third law would be observed.

Newtonian mechanics in general holds only for speeds that are much smaller than the speed of light (when $v \ll c$). Therefore, within the confines of this mechanics, the speed of propagation of field disturbances is considered to be infinite,

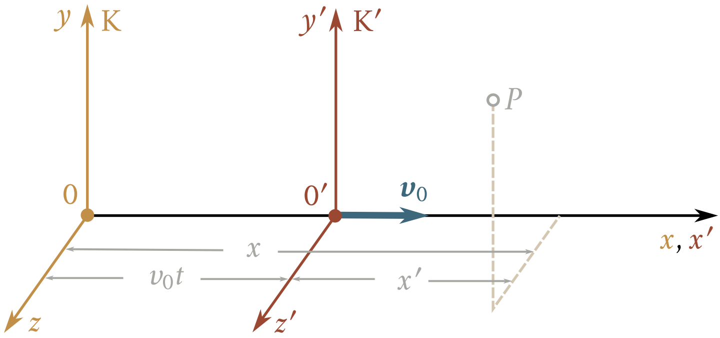

Fig. 2.3

and Newton's third law is always obeyed.

2.7. Galileo's Relativity Principle

Let us consider two reference frames moving relative to each other with the constant velocity $\boldsymbol{v}_0$. One of these frames, designated in Fig. 2.3 by the letter K, will conditionally be considered fixed. Therefore the second frame K′ will move uniformly in a straight line. Let us choose the coordinate axes x, y, z of frame K and the axes x', y', z' of frame K′ so that the axes x and x' coincide, while the axes y and y', and also z and z' are parallel to each other.

Let us find the relation between the coordinates x, y, z of a point P in frame K and the coordinates x', y', z' of the same point in frame K′. If we begin to count the time from the moment when the origins of the coordinates of the two frames coincided, then as follows from Fig. 2.3, we have $x = x' + v_0 t$. In addition, it is quite obvious that $y = y'$ and $z = z'$. Adding to these relations the assumption adopted in classical mechanics that the time flows the same in both frames, *i.e.*, that $t = t'$, we get a group of four equations:

$$x = x' + v_0 t, \quad y = y', \quad z = z', \quad t = t'. \tag{2.19}$$

They are called **Galilean transformations**.

The first and last of Eqs. (2.19) are correct only at values of v_0 that are small in comparison with the speed of light in a vacuum c (*i.e.*, $v_0 \ll c$). At values of v_0 comparable with c, the Galilean transformations must be replaced with the more general Lorentz transformations (see Sec. 8.2). Equations (2.19) are assumed to be accurate within the confines of Newtonian mechanics.

Differentiating Eqs. (2.19) with respect to time, we find the relation between the velocities of point P relative to the reference frames K and K′:

$$\begin{aligned} \dot{x} &= \dot{x}' + v_0, \quad &\text{or} \quad v_x &= v'_x + v_0 \\ \dot{y} &= \dot{y}', \quad &\text{or} \quad v_y &= v'_y \\ \dot{z} &= \dot{z}', \quad &\text{or} \quad v_z &= v'_z. \end{aligned} \tag{2.20}$$

The three scalar relations Eq. (2.20) are equivalent to the following relation between the velocity vector $\boldsymbol{v}$ relative to frame K and the velocity vector $\boldsymbol{v}'$ relative to frame K′:

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}_0. \tag{2.21}$$

To convince ourselves in the truth of this equation, it is sufficient to project vector equation (2.21) onto the axes x, y, z. As a result, we get equations (2.20).

Equations (2.20) and (2.21) give the rule of velocity addition in classical mechanics. It must be borne in mind that Eq. (2.21), like any other vector equation, remains correct upon an arbitrary selection of the mutual directions of the coordinate axes of the frames K and K′. Equations (2.20), however, are obeyed only when the axes are chosen as shown in Fig. 2.3.

We noted in Sec. 2.2 that any reference frame moving relative to an inertial frame with a constant velocity will also be inertial. Now we are in a position to prove this statement. To do this, let us differentiate Eq. (2.21) with respect to time. Taking into account that $\boldsymbol{v}_0$ is constant, we get

$$\dot{\boldsymbol{v}} = \dot{\boldsymbol{v}}', \quad \text{or} \quad \boldsymbol{a} = \boldsymbol{a}'. \tag{2.22}$$

Hence it follows that the acceleration of a body in all reference frames moving uniformly in a straight line relative to one another is the same. Therefore, if one of these frames is inertial (this signifies that in the absence of forces $\boldsymbol{a} = 0$), then the others will also be inertial ($\boldsymbol{a}'$ also equals zero).

The fundamental equation of mechanics (2.21) is characterized by containing only the acceleration of the kinematic quantities. It does not contain the velocity. As we have established above, however, the acceleration of a body in two arbitrarily selected inertial reference frames K and K′ is the same. Hence it follows according to Newton's second law that the forces acting on a body in frames K and K′ will also be the same. Consequently, *the equations of dynamics do not change upon transition from one inertial reference frame to another one, i.e.*, they are said to be invariant with respect to the transformation of the coordinates corresponding to the transition from one inertial reference frame to another. From the viewpoint of mechanics, all inertial reference frames are absolutely equivalent, and none of them can be preferred to others. In practice, this manifests itself in that we cannot establish by any mechanical experiments conducted within the limits of a given reference frame whether it is in a state of rest or in uniform straight-line motion. For example, if we are in a car of a train running uniformly in a straight line without jolts we cannot determine whether the car is moving or at rest without looking out of a window. The free fall of bodies, the motion of objects that we throw, and all other mechanical processes in this case will occur in the same way as if the car were standing.

These circumstances were already established by Galileo Galilei (1564-1642). The statement that all mechanical phenomena in different inertial reference frames proceed identically, owing to which no mechanical experiments allow us to determine whether the given reference frame is at rest or is moving uniformly in a straight line, is called **Galileo's relativity principle**.

2.8. Forces

Four kinds of interactions are distinguished in modern physics: (1) gravitational (or interaction due to universal gravitation), (2) electromagnetic (achieved via electric and magnetic fields), (3) strong or nuclear (ensuring the binding of particles in an atomic nucleus), and (4) weak interaction (responsible for many processes of elementary particle decay).

Within the confines of classical mechanics, we have to do with gravitational and electromagnetic forces, and also with elastic and friction forces. The latter two kinds of forces are determined by the nature of the interaction between the molecules of a substance. The forces of interaction between molecules have an electromagnetic origin. Consequently, elastic and friction forces are electromagnetic in their nature.

Gravitational and electromagnetic forces are fundamental—they cannot be reduced to other simpler forces. Elastic and friction forces, on the other hand, are not fundamental.

The laws of the fundamental forces are exceedingly simple. The magnitude of the gravitational force is determined by Eq. (2.18). The magnitude of the force with which two point charges q_1 and q_2 at rest interact is determined by Coulomb's law:

$$F = k\frac{q_1 q_2}{r^2} \tag{2.23}$$

(k is a constant of proportionality depending on the units chosen for the quantities in the formula).

If the charges are moving, then magnetic forces act on them in addition to the force defined by Eq. (2.23). The magnetic force acting on a point charge q moving with the velocity $\boldsymbol{v}$ in a magnetic field of induction $\boldsymbol{B}$ is determined by the formula

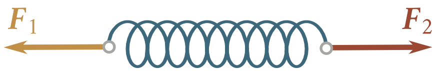

Fig. 2.4

$$F = k'q\,(\boldsymbol{v} \times \boldsymbol{B}) \tag{2.24}$$

(k' is a constant of proportionality).

Equations (2.18), (2.23), and (2.24) are accurate ones. For elastic and friction forces we can obtain only approximate empirical formulas that are considered in the following sections.

2.9. Elastic Forces

Any real body becomes deformed, *i.e.*, changes its dimensions and shape, under the action of forces applied to it. If the body regains its initial dimensions and shape after the action of the forces stops, the deformation or strain is called **elastic**. Elastic deformations are observed when the force producing the deformation does not exceed a definite limit, called the **elastic limit**, for each concrete body.

Let us take a spring of length l_0 in its undeformed state and apply the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ to its ends that are equal in value and opposite in direction (Fig. 2.4). Under the action of these forces, the spring will stretch over a certain distance Δl, after which equilibrium will set in. In the state of equilibrium, the external forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ will be balanced by the elastic forces set up in the spring as a result of its deformation. Experiments show that with small deformations, the stretching of the spring Δl is proportional to the stretching force: $\Delta l \propto F$ (here $F = F_1 = F_2$). Accordingly, the elastic force is proportional to the elongation of the spring:

$$F = k\,\Delta l. \tag{2.25}$$

The constant of proportionality k is called the **spring constant**.

The statement that the elastic force and deformation are proportional to each other is called **Hooke's law**.

Elastic strains are set up throughout the entire spring. Any part of the spring acts on another part with a force determined by Eq. (2.25). Therefore, if we cut the spring in half, an identical elastic force will appear in each half with the elongation being half the original value. Hence, we conclude that with a given material of the spring and a given coil size the magnitude of the elastic force is determined not by the absolute elongation of the spring Δl, but by its relative elongation $\Delta l / l_0$.

Elastic strains, but of the opposite sign, are also set up when a spring is compressed. Let us generalize Eq. (2.25) as follows. We shall rigidly fix one end of a

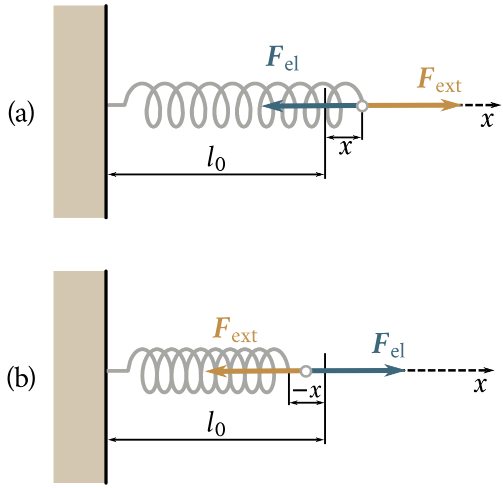

Fig. 2.5

spring (Fig. 2.5) and shall consider the elongation of the spring as the coordinate x of its other end measured from its position corresponding to the undeformed spring[7]. In addition, by F we shall understand the projection of the elastic force $\boldsymbol{F}_{el}$ onto the x-axis. We can thus write that

$$F = -kx \tag{2.26}$$

(inspection of Fig. 2.5 shows that the projection of the elastic force onto the x-axis and the coordinate x always have opposite signs).

Homogeneous bars behave in tension or uniaxial compression like a spring. If we apply the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ ($F_1 = F_2 = F$) to the ends of a bar, these forces being directed along its axis and acting uniformly over the entire cross section, then the length of the bar l_0 will receive either a positive (in stretching) or a negative (in compression) increment[8] Δl (Fig. 2.6). It is quite natural to take the relative change in the length of the bar as the quantity characterizing its deformation:

$$\varepsilon = \frac{\Delta l}{l_0}. \tag{2.27}$$

Experiments show that for bars of a given material the relative elongation or strain in elastic deformation is proportional to the force per unit cross-sectional

[7]In Fig. 2.5b, the distance over which the end of the spring moved is designated $-x$. The reason is that the distance is a positive quantity, while the coordinate x in this case, however, is a negative one.

[8]A change in the length of the bar is attended by a corresponding change in its cross-sectional dimensions.

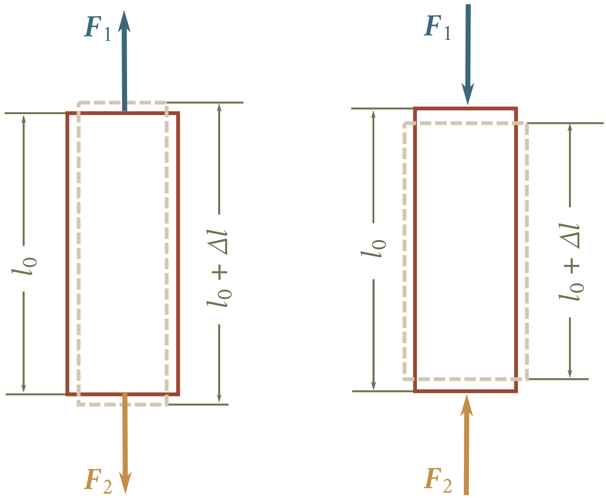

Fig. 2.6

area of the bar:

$$\varepsilon = \alpha \frac{\Delta l}{S}. \tag{2.28}$$

The constant of proportionality a is called the **compliance coefficient**.

The quantity equal to the ratio between the force and the area of the surface it is acting on is called the stress. Owing to the interaction of the parts of a body with one another, the stress is transmitted to all points of the body—the entire volume of the body, for example a bar, will be in a stressed state. If the force is directed along a normal to the surface, the stress is called **normal**. If it is directed along a tangent to the surface it is acting on, the stress is called **tangential** (**shear**). The normal stress is designated by the symbol σ, the tangential or shear stress by the symbol τ.

The ratio F/S in Eq. (2.28) is the normal stress σ. Hence, this equation can be written in the form

$$\varepsilon = \alpha\sigma. \tag{2.29}$$

In addition to the compliance coefficient α, the elastic properties of a material are also characterized by its reciprocal $E = 1/\alpha$, which is called the **modulus of elasticity** or **Young's modulus**. It is measured in pascals (the pascal is the unit of pressure in the SI system—1 Pa = 1 N m^{-2}).

Using $1/E$ instead of α in Eq. (2.9), we get

$$\varepsilon = \frac{\alpha}{E} \tag{2.30}$$

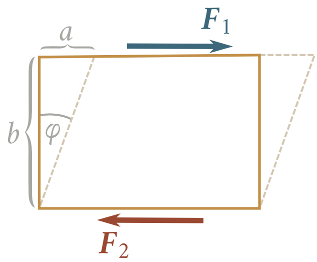

Fig. 2.7

from which it follows that Young's modulus equals such a normal stress at which the relative elongation or strain will equal unity (*i.e.*, the increment of the length Δl will be equal to the original length l_0) if so great elastic deformations were possible (actually a bar will fail at considerably smaller stresses, and the elastic limit is reached still earlier).

Solving Eq. (2.28) with respect to F and using $\Delta l/l_0$ instead of ε and $1/E$ instead of α, we get

$$F = \frac{E\,S}{l_0}\Delta l = k\,\Delta l \tag{2.31}$$

where k is a constant quantity for a given bar. Equation (2.31) expresses Hooke's law for a bar [compare with Eq. (2.26)]. Do not forget that this law is obeyed only until the elastic limit is reached.

In conclusion, let us briefly consider shear strain. Let us take a homogeneous body having the shape of a rectangular parallelepiped and apply to its opposite faces the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ ($F_1 = F_2 = F$) directed parallel to these faces (Fig. 2.7). If the action of the forces is uniformly distributed over the entire surface of the corresponding face, then in any cross section parallel to these faces the tangential (shear) stress

$$\tau = \frac{F}{S} \tag{2.32}$$

will appear (S is the area of a face). The action of the stresses will cause the body to deform so that one face will move relative to another over the distance a. If we mentally divide the body into elementary layers parallel to the faces we are considering, then each layer will be shifted with respect to its neighbours. For this reason, such deformation is called **shear**.

In shear, any straight line originally perpendicular to the layers will turn through the angle φ. Shear is characterized by the quantity

$$\gamma = \frac{a}{b} = \tan\varphi \tag{2.33}$$

called the **relative shear** (what a and b are is clear from Fig. 2.7). Upon elastic deformations, the angle φ is very small. We can therefore assume that $\tan\varphi \approx \varphi$. Consequently, the relative shear γ equals the angle of shear φ.

Experiments show that the relative shear is proportional to the tangential stress:

$$\gamma = \frac{1}{G}\tau. \tag{2.34}$$

The coefficient G depends only on the properties of a material and is called the **shear modulus**. It equals such a tangential (shear) stress at which the angle of shear will be 45 degrees ($\tan\varphi = 1$) if the elastic limit were not exceeded at such great deformations. The shear modulus G, like Young's modulus E, is measured in pascals (Pa).

2.10. Friction Forces

Forces of friction appear when contacting bodies or their parts move relative to one another. The friction occurring in the relative movement of two contacting bodies is called **external**; the friction between parts of the same continuous body (for example, a fluid) is called **internal**.

The force of friction appearing when a solid body moves relative to a fluid (liquid or gas) medium must be related to the category of internal friction forces because in this case the layers of the fluid in direct contact with the body are carried along with it at the body's velocity. The motion of the body is influenced by the friction between these layers of the fluid and other of its layers that are external relative to them.

Friction between the surfaces of two solids in the absence of any intermediate layer, for instance, a lubricant between them, is called **dry**. Friction between a solid and a fluid, and also between the layers of a fluid, is called **viscous** (or **liquid**).

Two kinds of dry friction are distinguished: **sliding** and **rolling**.

Forces of friction are directed along a tangent to the surfaces (or layers) in contact so that they resist the relative displacement of these surfaces (layers). If, for example, two layers of a liquid slide over each other with different velocities, then the force applied to the faster layer is directed oppositely to the direction of motion, while the force acting on the slower layer is directed along its motion.

Dry Friction. In dry friction, a force of friction appears not only when one surface slides over another one, but also when attempts are made to set up such sliding motion. In the latter case, we have to do with the **force of static friction**. Let us consider two contacting bodies 1 and 2 of which the latter is fixed in place (Fig. 2.8). Body 1 is pressed against body 2 by the force $\boldsymbol{F}_{\hat{n}}$ directed along a normal

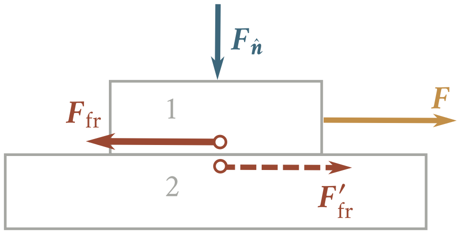

Fig. 2.8

to the surface of contact of the bodies. It is called the **normal force** and may be due to the pressure of the body's weight, or to other reasons. Let us try to move body 1 by acting on it with an external force $\boldsymbol{F}$. We shall find that for every concrete pair of bodies and every value of the normal force there is a definite minimum value F_0 of the force $\boldsymbol{F}$ at which body 1 first begins to move. At values of the external force F within the limits from 0 to F_0, the body remains at rest. According to Newton's second law, this is possible if the force $\boldsymbol{F}$ is balanced by a force equalling it in magnitude and opposite in direction, which is exactly the force of static friction $\boldsymbol{F}_{\text{fr}}$ (see Fig. 2.8). It automatically[9] acquires a value equal to that of the external force F (provided that the latter does not exceed F_0). The quantity F_0 is the maximum possible value of the force of static friction.

It must be noted that in accordance with Newton's third law body 2 must also experience the force of static friction $\boldsymbol{F}'_{\text{fr}}$ (it is shown by a dashed arrow in Fig. 2.8) equal in magnitude to the force $\boldsymbol{F}_{\text{fr}}$ but directed oppositely.

If the external force $\boldsymbol{F}$ exceeds F_0 in magnitude, the body begins to slide. Its acceleration is determined by the resultant of two forces: the external one $\boldsymbol{F}$ and the force of sliding friction $\boldsymbol{F}_{\text{fr}}$ whose magnitude depends to a certain extent on the sliding speed. The nature of this relation is determined by the nature and state of the contacting surfaces. The kind of the speed dependence of the force of friction shown in Fig. 2.9 is encountered most frequently. The graph shows both static and sliding friction. The force of static friction, as we have already noted, may range from 0 to F_0, which is shown by the vertical line in the graph. In accordance with Fig. 2.9, the force of sliding friction first diminishes somewhat with increasing speed, and then begins to grow. With special processing of contacting surfaces, the force of sliding friction may be virtually independent of the speed. In this case, the curved portion of the graph in Fig. 2.9 will transform into a horizontal line beginning at the point F_0.

[9]This occurs in the same way as a spring acted upon by a stretching force automatically acquires an elongation such that the elastic force balances the external one.

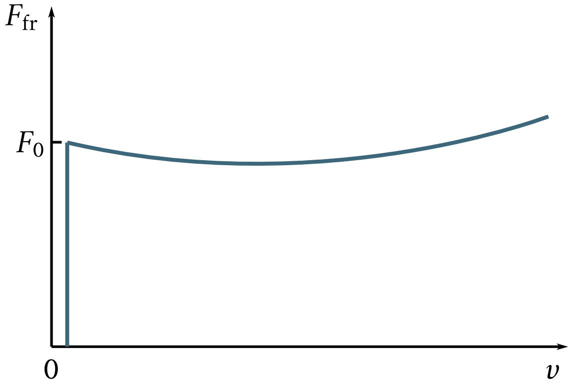

Fig. 2.9

The laws of dry friction consist in the following: the maximum force of static friction, and also the force of sliding friction do not depend on the area of contact between bodies and are approximately proportional to the magnitude of the normal force pressing the contacting surfaces together:

$$F_{fr} = f\,F_{\hat{\boldsymbol{n}}}. \tag{2.35}$$

The dimensionless proportionality constant f is called the **coefficient of friction** (static or sliding friction, as the case may be). It depends on the nature and state of the contacting surfaces, particularly on their roughness. The coefficient of sliding friction is a function of speed.

Friction forces play a very great part in nature. Friction is often a great help to us in our everyday life. Let us remember the tremendous difficulties encountered by pedestrians and vehicles on ice-covered pavements and roads, when the friction between the pavement surface and the pedestrians' soles or the wheels of the vehicles considerably diminishes. If there were no friction forces, our furniture would have to be fastened to the floor like on a ship on a rolling sea because upon the most minute deviation of the floor from a horizontal position it would slide in the direction of the slope. Our reader can give numerous similar examples of how helpful friction is.

The part played by friction is often extremely negative, and measures have to be taken to reduce it as much as possible. This relates, for example, to the friction in bearings or between the hub of a wheel and an axle.

The most radical way of reducing forces of friction is to replace sliding friction with rolling friction. The latter appears, for example, between a cylindrical or spherical body rolling over a flat or curved surface. Rolling friction formally obeys the same laws as sliding friction, but the coefficient of friction in this case is much

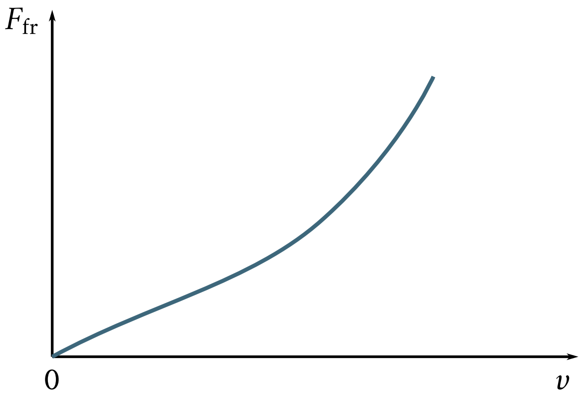

Fig. 2.10

lower.

Viscous Friction and Resistance of the Medium. Unlike dry friction, viscous (internal) friction is characterized by the force of viscous friction vanishing together with the velocity. Therefore, no matter how small an external force is, it can impart a relative velocity to the layers of a viscous medium. The laws which the forces of friction between the layers of a medium obey will be considered in the chapter devoted to fluid mechanics (Chap. **??**).

In this section, we shall limit ourselves to a treatment of the friction forces between a solid and a viscous (fluid) medium. It must be borne in mind that apart from the forces of friction proper, the motion of bodies in a fluid is attended by the so-called forces of **resistance of the medium** that can be much greater than the forces of friction. We have no possibility of considering the causes of these forces in detail. We shall only treat the laws obeyed jointly by forces of friction and resistance of the medium. We shall conditionally call the total force the force of friction. The speed dependence of this force is shown in Fig. 2.10.

At low velocities, the force grows linearly with the velocity:

$$\boldsymbol{F}_{\text{fr}} = -k_1 \boldsymbol{v} \tag{2.36}$$

(the minus sign signifies that the force is directed oppositely to the velocity). The value of the coefficient k_1 depends on the shape and dimensions of a body, the state of its surface, and on the property of the fluid called its viscosity. For example, this coefficient is much higher for glycerine than for water.

At high velocities, the linear law transforms into a quadratic one, *i.e.*, the force begins to grow in proportion to the square of the velocity:

$$\boldsymbol{F}_{\text{fr}} = -k_2\, v^2\, \hat{\boldsymbol{e}}_v \tag{2.37}$$

($\hat{\boldsymbol{e}}_v$ is the unit vector of the velocity). The value of the coefficient k_2 depends on the shape and dimensions of a body.

The magnitude of the velocity at which the law (2.36) transforms into (2.37) depends on the shape and dimensions of a body, and also on the viscosity and density of the fluid.

2.11. Force of Gravity and Weight

The force of attraction to the Earth causes all bodies to fall with the same acceleration relative to the Earth's surface, which is designated by the symbol g. This signifies that in a reference frame associated with the Earth, any body of mass m is acted upon by the force

$$\boldsymbol{P} = m\boldsymbol{g} \tag{2.38}$$

called the force of gravity[10]. When a body is at rest relative to the Earth's surface, the force $\boldsymbol{P}$ is balanced by the reaction[11] $\boldsymbol{F}_\mathrm{r}$ of the suspension or support preventing falling of the body ($\boldsymbol{F}_\mathrm{r} = -\boldsymbol{P}$). According to Newton's third law, the body in this case acts on the suspension or support with the force $\boldsymbol{W}$ equal to $-\boldsymbol{F}_\mathrm{r}$, *i.e.*, with the force

$$\boldsymbol{W} = \boldsymbol{P} = m\boldsymbol{g}$$

The force $\boldsymbol{W}$ with which a body acts on its suspension or support is called the **weight of the body**. This force equals $m\boldsymbol{g}$ only when the body and its support (or suspension) are stationary relative to the Earth. If they are moving with a certain acceleration $\boldsymbol{a}$, their weight $\boldsymbol{W}$ will not equal $m\boldsymbol{g}$. This can be explained by the following example. Let a suspension in the form of a spring fastened to a frame move together with a body with the acceleration $\boldsymbol{a}$ (Fig. 2.11). The equation of motion of the body will therefore have the form

$$\boldsymbol{P} + \boldsymbol{F}_\mathrm{r} = m\boldsymbol{a} \tag{2.39}$$

where $\boldsymbol{F}_\mathrm{r}$ is the reaction of the suspension, *i.e.*, the force with which the spring acts on the body. According to Newton's third law, the body acts on the spring with a force equal to $-\boldsymbol{F}_\mathrm{r}$, which by definition is the weight of the body $\boldsymbol{W}$ in these conditions. Substituting the force $-\boldsymbol{W}$ for the reaction $\boldsymbol{F}_\mathrm{r}$ and the product $m\boldsymbol{g}$ for the force of gravity $\boldsymbol{P}$ in Eq. (2.39), we get

$$\boldsymbol{W} = m(\boldsymbol{g} - \boldsymbol{a}). \tag{2.40}$$

[10] Owing to the non-inertial nature of a reference frame associated with the Earth, the force of gravity will differ somewhat from the force with which a body is attracted to the Earth. This will be treated in greater detail in Sec. 4.2.

[11] Reactions are forces with which a given body is acted upon by bodies restricting its motion.

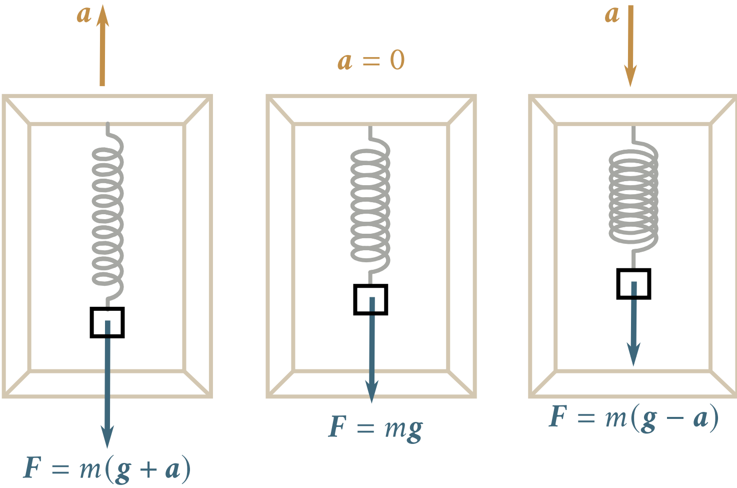

Fig. 2.11

Equation (2.40) determines the weight of a body in the general case. It holds for a suspension or a support of any kind.

Let us assume that our body and the suspension are moving in a vertical direction (Fig. 2.11 is based on this assumption).

We project Eq. (2.40) onto the direction of a plumb line:

$$W = m(g \pm a). \tag{2.41}$$

In this expression, W, g, and a are the magnitudes of the corresponding vectors. The plus sign corresponds to a directed upward, and the minus sign to a directed downward.

It follows from Eq. (2.41) that the magnitude of the weight $\boldsymbol{W}$ may be either greater or smaller than the force of gravity $\boldsymbol{P}$. In free fall of the frame with the suspension, $\boldsymbol{a} = \boldsymbol{g}$, and the force $\boldsymbol{W}$ with which the body acts on the suspension vanishes. A state of weightlessness sets in. A spaceship orbiting around the Earth with its engines switched off travels, like the freely falling frame, with the acceleration $\boldsymbol{g}$. As a result, the bodies inside it are in a state of weightlessness—they exert no pressure on the bodies in contact with them.

It must be noted that the force of gravity $\boldsymbol{P}$ is often confused with the weight of a body $\boldsymbol{W}$. This is due to the fact that with a stationary support, the forces $\boldsymbol{P}$ and $\boldsymbol{P}$ coincide both in magnitude and in direction (they both equal $m\boldsymbol{g}$). It must be remembered, however, that these forces are applied to different bodies: $\boldsymbol{P}$ is applied to a body itself, whereas $\boldsymbol{W}$ is applied to the suspension or support restricting the free motion of the body in the field of the Earth's gravitational forces. In addition,

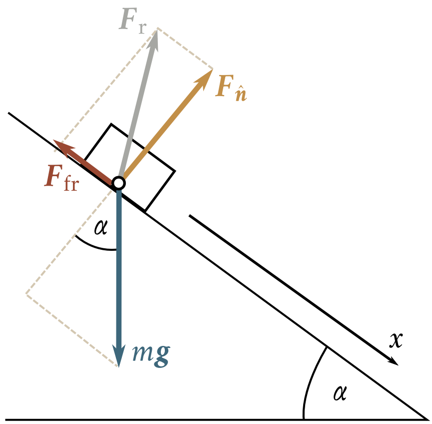

Fig. 2.12

the force $\boldsymbol{P}$ always equals $m\boldsymbol{g}$ regardless of whether the body is moving or at rest, whereas the force of weight $\boldsymbol{W}$ depends on the acceleration with which the support and the body are moving. It may be either greater or smaller than $m\boldsymbol{g}$, and, in particular, in a state of weightlessness it vanishes.

The relation (2.40) between the mass and the weight of a body provides a way of comparing the masses of bodies by weighing them: the ratio of the weights of bodies determined in identical conditions (usually at $\boldsymbol{a} = 0$) at the same point on the Earth's surface equals the ratio of the masses of these bodies:

$$W_1 : W_2 : W_3 : \ldots = m_1 : m_2 : m_3 \ldots .$$

It will be shown in Sec. 4.2 that the acceleration of free fall g and the force of gravity P depend on the latitude of a locality. They also depend on the altitude, and diminish with an increasing distance from the centre of the Earth.

2.12. Practical Application of Newton's Laws

To compile an equation of motion, we must first of all establish what forces act on the body being considered. It is necessary to determine the action of other bodies on the given one that must be taken into account. For example, for a body sliding down an inclined plane (Fig. 2.12), the action exercised by the Earth is important (it is characterized by the force $m\boldsymbol{g}$), and also the action exercised by the plane (it is characterized by the force of the reaction $\boldsymbol{F}_r$).

Never take into account "moving", "rolling down", "centripetal", "centrifugal"[12] and similar forces. To prevent an error, characterize a force according to the "source"

[12]This does not relate to the term "centrifugal force of inertia" (see Sec. 4.2.

causing it to appear, and not according to the action it produces. This means that behind every force we must see the body whose action sets up the force. This will eliminate the typical error consisting in that the same force is taken into account twice under different names.

In the example we are considering (see Fig. 2.12), it is good to resolve the force of the reaction $\boldsymbol{F}_{\mathrm{r}}$ into two components—the normal force $\boldsymbol{F}_{\hat{\boldsymbol{n}}}$ and the friction force $\boldsymbol{F}_{\mathrm{fr}}$. This, in particular, is useful in connection with the fact that the force of friction is proportional to the magnitude of the force $\boldsymbol{F}_{\hat{\boldsymbol{n}}}$ [see Eq. (2.35)].

Having determined the forces acting on a body, we write an equation of Newton's second law. In our example, it has the form

$$m\boldsymbol{a} = m\boldsymbol{g} + \boldsymbol{F}_{\mathrm{r}} = m\boldsymbol{g} + \boldsymbol{F}_{\hat{\boldsymbol{n}}} + \boldsymbol{F}_{\mathrm{fr}}. \tag{2.42}$$

To perform calculations, we must pass over from vectors to their projections onto the correspondingly chosen directions, using the following properties of projections:

(1) equal vectors have identical projections;
(2) the projection of a vector obtained by multiplying another vector by a scalar equals the product of the projection of this second vector and the scalar;
(3) the projection of a sum of vectors equals the sum of the projections of the vectors being added [see Eq. (1.8)].

Let us project the vectors in Eq. (2.42) onto the direction x shown in Fig. 2.12. The projections of the vectors are $a_x = a$ (a is the magnitude of the vector $\boldsymbol{a}$), $g_x = g \sin\alpha$, $F_{\hat{\boldsymbol{n}}x} = 0$, $F_{\mathrm{r}x} = -fF_{\hat{\boldsymbol{n}}x} = -fmg\cos\alpha$. Consequently, we arrive at the equation

$$ma = mg\sin\alpha - fmg\cos\alpha$$

from which it is a simple matter to find a.

In more complicated cases, we have to project the vectors onto several directions and solve the resulting system of algebraic or differential equations.

Chapter 3

LAWS OF CONSERVATION

3.1. Quantities Obeying the Laws of Conservation

Bodies forming a mechanical system may interact with one another and with bodies not belonging to the given system. Accordingly, the forces acting on the bodies of a system can be divided into **internal** and **external** ones. We shall define internal forces as the forces with which a given body is acted upon by the other bodies of the system, and external forces as those produced by the action of bodies not belonging to the system. If external forces are absent, the relevant system is called **closed**.

There are functions of the coordinates and velocities of the particles[1] forming a system for closed systems that retain constant values upon motion. These functions are called motion integrals.

The number of motion integrals that can be formed for a system of N particles between which there are no rigid constraints is $6N - 1$. Only those of them are of interest to us that have the property of additivity. This property consists in that the value of a motion integral for a system comprising parts whose interaction may be disregarded equals the sum of the values for each part. There are three additive motion integrals. The first is called **energy**, the second—**momentum**, and the third—**angular momentum**.

Thus, three physical quantities do not change in closed systems, namely, energy, momentum, and angular momentum. Accordingly, there are three **laws of conservation**—that of energy conservation, that of momentum conservation, and that of angular momentum conservation. These laws are closely associated with the fundamental properties of space and time.

The conservation of energy is based on the **uniformity of time**, *i.e.*, the equiv-

[1]We remind our reader that by a particle here we mean a point particle.

alence of all moments of time. The equivalence should be understood in the sense that the substitution of the moment of time t_2 for the moment t_1 without a change in the values of the coordinates and velocities of the particles does not change the mechanical properties of a system. This signifies that after such a substitution, the coordinates and velocities of the particles have the same values at any moment of time $t_2 + t$ as they would have had before the substitution at the moment $t_1 + t$.

The conservation of momentum is based on the uniformity of space, *i.e.*, the identical properties of space at all points. This should be understood in the sense that a translation of a closed system from one place in space to another without changing the mutual arrangement and velocities of the particles does not change the mechanical properties of the system (it is assumed that the closed nature of the system is not violated at the new place).

Finally, the conservation of angular momentum is based on the **isotropy of space**, *i.e.*, the identical properties of space in all directions. This should be understood in the sense that rotation of a closed system as a whole does not affect its mechanical properties.

The laws of conservation are a powerful means of research. It is often extremely difficult to accurately solve equations of motion. In these cases, the laws of conservation permit us to obtain numerous important data on how mechanical phenomena proceed without having to solve equations of motion. The laws of conservation do not depend on the nature of the acting forces. This is why they can help us obtain much important information on the behaviour of mechanical systems even when the forces are unknown.

In the following sections, we shall obtain the laws of conservation on the basis of Newton's equations. It must be borne in mind, however, that the laws of conservation have a much more general nature than Newton's laws. The laws of conservation remain strictly correct even when Newton's laws (particularly the third one) are violated. We stress the fact that the laws of energy, momentum, and angular momentum conservation are accurate laws that are also strictly obeyed in the relativistic realm.

3.2. Kinetic Energy

Let us now pass over to finding the additive integrals of motion. We shall first consider the simplest system consisting of a single point particle.

The equation of motion of the particle is

$$m\dot{\boldsymbol{v}} = \boldsymbol{F}. \tag{3.1}$$

Here $\boldsymbol{F}$ is the resultant of the forces acting on the particle. Multiplying Eq. (3.1) by the displacement of the particle $\mathrm{d}\boldsymbol{s} = \boldsymbol{v}\,\mathrm{d}t$, we get

$$m\boldsymbol{v}\dot{\boldsymbol{v}}\,\mathrm{d}t = \boldsymbol{F}\,\mathrm{d}\boldsymbol{s}. \tag{3.2}$$

The product $\dot{\boldsymbol{v}}\,\mathrm{d}t$ is the increment of the velocity of the particle $\mathrm{d}\boldsymbol{v}$ during the time $\mathrm{d}t$. Accordingly

$$m\boldsymbol{v}\dot{\boldsymbol{v}}\,\mathrm{d}t = m\boldsymbol{v}\,\mathrm{d}\boldsymbol{v} = m\mathrm{d}\left(\frac{v^2}{2}\right) = \mathrm{d}\left(\frac{mv^2}{2}\right) \tag{3.3}$$

Performing such a substitution in Eq. (3.2), we arrive at the expression

$$\mathrm{d}\left(\frac{mv^2}{2}\right) = \boldsymbol{F}\,\mathrm{d}\boldsymbol{s}. \tag{3.4}$$

If the system is closed, *i.e.*, $\boldsymbol{F} = 0$, then $\mathrm{d}(mv^2/2) = 0$, while the quantity

$$E_\mathrm{k} = \frac{mv^2}{2} \tag{3.5}$$

itself remains constant. This quantity is called the **kinetic energy** of the particle. For an isolated particle, the kinetic energy is an integral of motion[2].

Multiplying the numerator and denominator of Eq. (3.5) by m and taking into consideration that the product mv equals the momentum p of a body, the expression for the kinetic energy can be given the form

$$E_\mathrm{k} = \frac{p^2}{2m}. \tag{3.6}$$

If the force $\boldsymbol{F}$ acts on a particle, its kinetic energy does not remain constant. In this case in accordance with Eq. (3.4), the increment of the particle's kinetic energy during the time $\mathrm{d}t$ equals the scalar product $\boldsymbol{F}\,\mathrm{d}\boldsymbol{s}$ ($\mathrm{d}\boldsymbol{s}$ is the displacement of the particle during the time $\mathrm{d}t$). The quantity

$$\mathrm{d}A = \boldsymbol{F}\,\mathrm{d}\boldsymbol{s} \tag{3.7}$$

is called the **work** done by the force $\boldsymbol{F}$ over the path $\mathrm{d}s$ ($\mathrm{d}s$ is the magnitude of the displacement $\mathrm{d}\boldsymbol{s}$). The scalar product (3.7) can be represented as the product of the projection of the force onto the direction of the displacement F_s and the elementary distance $\mathrm{d}s$. Consequently, we can write that

$$\mathrm{d}A = F_\mathrm{s}\,\mathrm{d}s. \tag{3.8}$$

It is clear from the above that work characterizes the change in energy due to the action of a force on a moving particle.

[2]For a single isolated particle, any power of the velocity remains constant. But for a system of several interacting particles, it is exactly quantities of the form of Eq. (3.5) that are addends in the additive integral of motion.

Let us integrate Eq. (3.4) along a certain trajectory from point 1 to point 2:

$$\int_1^2 \mathrm{d}\left(\frac{mv^2}{2}\right) = \int_1^2 \boldsymbol{F}\,\mathrm{d}\boldsymbol{s}.$$

The left-hand side is the difference between the values of the kinetic energy at points 2 and 1, *i.e.*, the increment[3] of the kinetic energy along path 1-2. Taking this into account, we get:

$$E_{\mathrm{k},2} - E_{\mathrm{k},1} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \int_1^2 \boldsymbol{F}\,\mathrm{d}\boldsymbol{s}. \tag{3.9}$$

The quantity

$$A = \int_1^2 \boldsymbol{F}\,\mathrm{d}\boldsymbol{s} = \int_1^2 F_s\,\mathrm{d}s \tag{3.10}$$

is the work of the force $\boldsymbol{F}$ over path 1-2. We shall sometimes denote this work by the symbol A_{12} instead of A.

Thus, *the work of the resultant of all the forces acting on a particle produces an increment of the particle's kinetic energy*:

$$A_{12} = E_{\mathrm{k},2} - E_{\mathrm{k},1}. \tag{3.11}$$

It follows from Eq. (3.11) that energy has the same dimension as work. Accordingly, energy is measured in the same units as work (see the following section).

3.3. Work

Let us consider the quantity that we called work in greater detail. Equation (3.7) can be written in the form

$$\mathrm{d}A = \boldsymbol{F}\,\mathrm{d}\boldsymbol{s} = F\cos\alpha\,\mathrm{d}s \tag{3.12}$$

where α is the angle between the direction of the force and that of the displacement of the point of application of the force.

If the force and the direction of the displacement make an acute angle ($\cos\alpha > 0$), the work is positive. If the angle α is obtuse ($\cos\alpha < 0$), the work is negative. When $\alpha = \pi/2$, the work equals zero. This especially clearly shows that the concept of work in mechanics appreciably differs from our ordinary notion of it. In the ordinary meaning, any effort, particularly muscular strain, is always attended by

[3]The change in a quantity a can be characterized either by its increment or its decrement. The increment of the quantity a, which we shall designate by Δa is defined as the difference between the final (a_2) and initial (a_1) values of this quantity: increment $= \Delta a = a_2 - a_1$. The decrement of the quantity a is the difference between its initial (a_1) and final (a_2) values: increment $= a_1 - a_2 = -\Delta a$. The decrement of a quantity equals its ·increment with the opposite sign. The increment and decrement are algebraic quantities.

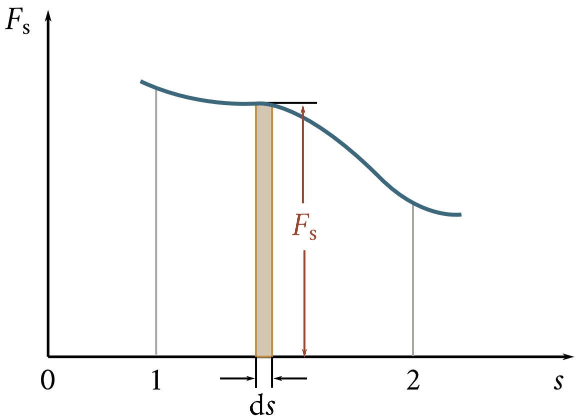

Fig. 3.1

work being done. For example, in order to hold a heavy load while standing still, and, moreover, to carry this load along a horizontal path, a porter spends much effort, *i.e.*, "does work". The work as a mechanical quantity in these cases, however, equals zero.

Figure 3.1 is a plot of the projection of the force onto the direction of displacement F_s as a function of the position of the particle on its trajectory (the axis of abscissas has been taken as the s-axis, the length of the part of this axis between points 1 and 2 equals the total length of the path). Examination of the figure shows that the elementary work $\mathrm{d}A = F_s\,\mathrm{d}s$ equals numerically the area of the shaded strip, while the work A over path 1-2 equals numerically the area of the figure confined by the curve F_s, the vertical lines from points 1 and 2 and the s-axis (compare with Fig. 1.26).

Let us use this result to find the work done in the deformation of a spring obeying Hooke's law [see Fig. 2.5 and Eq. (2.26)]. We shall begin with stretching of the spring. We shall do this very slowly so that the force $\boldsymbol{F}_{\mathrm{ext}}$ which we act on the spring with may be considered equal in magnitude to the elastic force $\boldsymbol{F}_{\mathrm{el}}$ all the time. Hence, $\boldsymbol{F}_{x,\mathrm{ext}} = -\boldsymbol{F}_{x,\mathrm{el}} = kx$, where x is the elongation of the spring (Fig. 3.2). A glance at the figure shows that the work required to cause the elongation x of the spring is

$$A = \frac{kx^2}{2}. \tag{3.13}$$

When the spring is compressed by the amount x, work of the same magnitude and sign is done as in stretching by x. The projection of the force $\boldsymbol{F}_{\mathrm{ext}}$ in this case is negative ($\boldsymbol{F}_{\mathrm{ext}}$ is directed to the left, x grows to the right, see Fig. 3.2), and all the dx's are also negative. As a result, the product $\boldsymbol{F}_{x,\mathrm{ext}}\,\mathrm{d}x$ is positive.

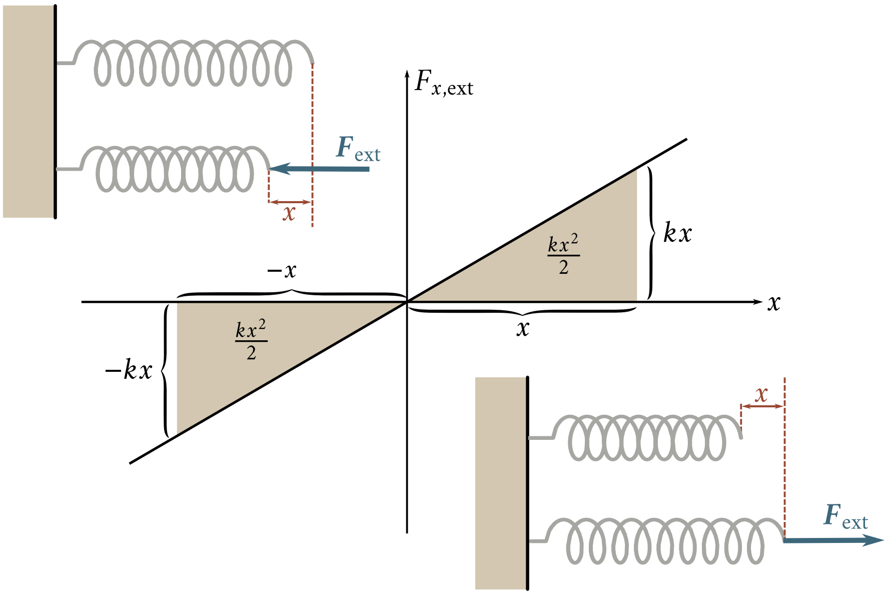

Fig. 3.2

In a similar way, we can find an expression for the work done upon the elastic stretching or compression of a bar. According to Eq. (2.31), this work is

$$A = \frac{1}{2}\frac{ES}{l_0}(\Delta l)^2 = \frac{1}{2}ESl_0\left(\frac{\Delta l}{l_0}\right)^2 = \frac{1}{2}EV\varepsilon^2 \tag{3.14}$$

where $V = Sl_0$ is the volume of the bar, and $\varepsilon = \Delta l/l_0$ is the relative elongation [see Eq. (2.27)].

Assume that several forces whose resultant is $\boldsymbol{F} = \sum_i \boldsymbol{F}_i$ act simultaneously on a body. It follows from the distributivity of a scalar product of vectors [see Eq. (1.20)] that the work dA done by the resultant force over the path ds can be represented in the form

$$\mathrm{d}A = \left(\sum_i \boldsymbol{F}_i\right)\mathrm{d}s = \sum_i \boldsymbol{F}_i\,\mathrm{d}\boldsymbol{s} = \sum_i \mathrm{d}A_i. \tag{3.15}$$

This signifies that the work of the resultant of several forces equals the algebraic sum of the work done by each force separately.

The elementary displacement d$\boldsymbol{s}$ can be represented as $\boldsymbol{v}\mathrm{d}t$. We can therefore write the expression for the elementary work in the form

$$\mathrm{d}A = \boldsymbol{F}\boldsymbol{v}\,\mathrm{d}t. \tag{3.16}$$

The work done during the interval from t_1 to t_2 can thus be calculated by the for-

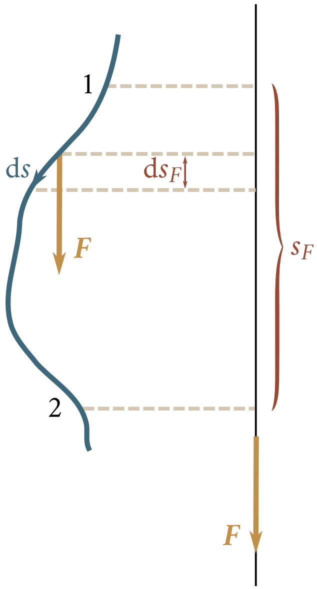

Fig. 3.3

mula

$$A = \int_{t_1}^{t_2} \boldsymbol{F}\, \mathrm{d}t. \tag{3.17}$$

In accordance with Eq. (1.21), we have $\boldsymbol{F}\, \mathrm{d}\boldsymbol{s} = F\, \mathrm{d}s_F$, where $\mathrm{d}s_F$, is the projection of the elementary displacement d$\boldsymbol{s}$$s$ onto the direction of the force $\boldsymbol{F}$. The formula for work can therefore be written as follows:

$$\mathrm{d}A = F\, \mathrm{d}s_F. \tag{3.18}$$

If the force has a constant magnitude and direction (Fig. 3.3), then the vector $\boldsymbol{F}$ in the expression for work may be put outside the integral. The result is

$$A = \boldsymbol{F} \int_1^2 \mathrm{d}\boldsymbol{v} = \boldsymbol{F} \cdot \boldsymbol{s} = F s_F \tag{3.19}$$

where $\boldsymbol{s}$ is the vector of the displacement from point 1 to point 2, and s_F is its projection onto the direction of the force.

The work done in unit time is called power. If the work $\mathrm{d}A$ is done in the time $\mathrm{d}t$, then the power is

$$P = \frac{\mathrm{d}A}{\mathrm{d}t}. \tag{3.20}$$

Taking $\mathrm{d}A$ as given by Eq. (3.16), we get the following expression for the power:

$$P = \boldsymbol{F} \cdot \boldsymbol{v} \tag{3.21}$$

according to which the power equals the scalar product of the force vector and the vector of the velocity with which the point of application of the force is moving.

Units of Work and Power. The unit of work is the work done by a force equal to unity and acting in the direction of the displacement over a unit distance. Consequently,

(1) in the SI system, the unit of work is the joule (J)—the work done by a force of 1 N over a distance of 1 m;
(2) in the cgs system, the relevant unit is the erg—the work done by a force of 1 dyn over a distance of 1 cm;
(3) in the mkg(force)s system, the unit is the kilogramme(force)m (kgf m)—the work done by a force of 1 kgf over a distance of 1 m.

The units of work are related as follows:

$$1\,\mathrm{J} = 1\,\mathrm{N} \times 1\,\mathrm{m} = 10^5\,\mathrm{dyn} \times 10^2\,\mathrm{cm} = 10^7\,\mathrm{erg}$$

$$1\,\mathrm{kgf\,m} = 1\,\mathrm{kgf} \times 1\,\mathrm{m} = 9.81\,\mathrm{N} \times 1\,\mathrm{m} = 9.81\,\mathrm{J}.$$

The unit of power is the power at which 1 unit of work is done in unit time. The unit of power in the SI system is the watt (W) equal to one joule per second ($\mathrm{J\,s^{-1}}$). The unit of power in the cgs system ($\mathrm{erg\,s^{-1}}$) has no special name. The relation between the watt and the $\mathrm{erg\,s^{-1}}$ is $1\,\mathrm{W} = 10^7\,\mathrm{erg\,s^{-1}}$.

The unit of power in the mkgforce)s system is the (metric) horsepower (hp), equal to $75\,\mathrm{kgf\,m\,s^{-1}}$, $1\,\mathrm{hp} = 736\,\mathrm{W}$ (do not confuse this unit with the British or U.S. horsepower equal to $550\,\mathrm{ft\text{-}lb\,s^{-1}}$ or 746 W).

A system of prefixes is used, especially in the SI system, to denote multiples and submultiples of units. The names and symbols of these prefixes and the relevant factor by which the basic unit is multiplied are indicated in Table 3.1.

For example, the unit of work called the megajoule is equivalent to 10^6 joules ($1\,\mathrm{MJ} = 10^6\,\mathrm{J}$), and the unit of power called the microwatt is equivalent to 10^{-6} watt ($1\,\mu\mathrm{W} = 10^{-6}\,\mathrm{W}$. Similarly, 1 micrometer (formerly called the micron) is equivalent to 10^{-6} m ($1\,\mu\mathrm{m} = 10^{-6}\,\mathrm{m}$), and $1\,\mathrm{pN} = 10^{-12}\,\mathrm{N}$.

3.4. Conservative Forces

If a particle is subjected to the action of other bodies at every point of space, the particle is said to be in a field of forces. For example, a particle near the Earth's surface is in the field of gravity forces—at every point of space the force $\boldsymbol{P} = m\boldsymbol{g}$ acts on it.

Let us consider as a second example the charged particle e in the electric field set up by the fixed point charge q (Fig. 3.4). A feature of this field is that the direction of the force acting on the particle at any point of space passes through a fixed centre

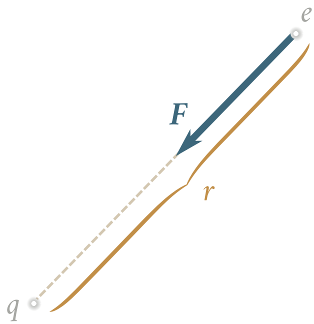

Fig. 3.4

(the charge q), while the magnitude of the force depends only on the distance to this centre, *i.e.*, $F = F(r)$ [see Eq. (2.23)]. A field of forces with such properties is called a **central** one.

If at every point of a field the force acting on a particle is identical in magnitude and direction ($\boldsymbol{F}$ = constant), the field is called **homogeneous**.

A field that changes with time is called **non-stationary**. A field that remains constant with time is called **stationary**.

For a stationary field, the work done on a particle by the forces of the field may depend only on the initial and final positions of the particle and not depend on the path along which the particle moved. Forces having such a property are called **conservative**.

It follows from the work of conservative forces being independent of the path that the work of such forces along a closed path equals zero. To prove this, let us

Table 3.1: Prefixes for Multiples and Submultiples of Units

Name	Symbol	Factor by which unit is multiplied	Name	Symbol	Factor by which unit is multiplied
Tera	T	10^{12}	Centi	c	10^{-2}
Giga	G	10^{9}	Milli	m	10^{-3}
Mega	M	10^{6}	Micro	μ	10^{-6}
Kilo	k	10^{3}	Nano	n	10^{-9}
Hecto	h	10^{2}	Pico	p	10^{-12}
Deca	da	10^{1}	Femto	f	10^{-15}
Deci	d	10^{-1}	Atto	a	10^{-18}

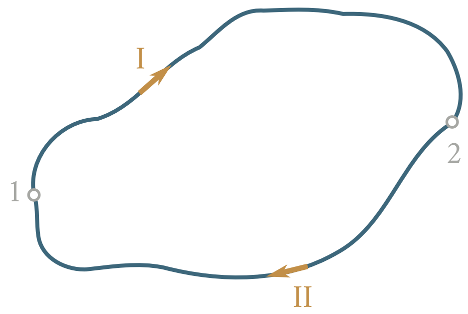

Fig. 3.5

divide an arbitrary closed path into two parts: path I along which a particle passes from point 1 to point 2, and path II along which the particle passes from point 2 to point 1 (Fig. 3.5). We have chosen points 1 and 2 arbitrarily. The work along the entire closed path equals the sum of the work done on each of the parts:

$$A = (A_{12})_{\mathrm{I}} + (A_{21})_{\mathrm{II}}. \tag{3.22}$$

It is easy to see that the work $(A_{21})_{\mathrm{II}}$ differs from $(A_{12})_{\mathrm{I}}$ only in its sign. Indeed, reversing of the direction of motion results in $\mathrm{d}\boldsymbol{s}$ being replaced with $-\mathrm{d}\boldsymbol{s}$, and as a consequence the value of the integral $\int \boldsymbol{F}\,\mathrm{d}\boldsymbol{s}$ reverses its sign. Thus, Eq. (3.22) can be written in the form

$$A = (A_{12})_{\mathrm{I}} - (A_{21})_{\mathrm{II}}.$$

and since the work does not depend on the path, *i.e.*, $(A_{12})_{\mathrm{I}} = (A_{21})_{\mathrm{II}}$, we arrive at the conclusion that $A = 0$.

From the equality to zero of the work over a closed path, it is easy to obtain that the work A_{12} is independent of the path. This can be done by reversing the above reasoning.

Thus, conservative forces can be defined in two ways: (1) as forces whose work does not depend on the path along which a particle passes from one point to another, and (2) as forces whose work along any closed path equals zero.

We shall prove that the force of gravity is conservative. This force at any point has the same magnitude and direction—vertically downward (Fig. 3.6). Therefore, regardless of the path along which the particle moves (for example I or II in the figure), the work A_{12} according to Eq. (3.19) is determined by the expression

$$A_{12} = m\,(\boldsymbol{g}\cdot\boldsymbol{s}_{12}) = mg(s_{12})_{\mathrm{pr.}\,\boldsymbol{g}}.$$

Inspection of Fig. 3.6 shows that the projection of the vector $\boldsymbol{s}_{12}$ onto the direction $\boldsymbol{g}$ equals the difference between the heights $h_1 - h_2$. Hence, the expression for the

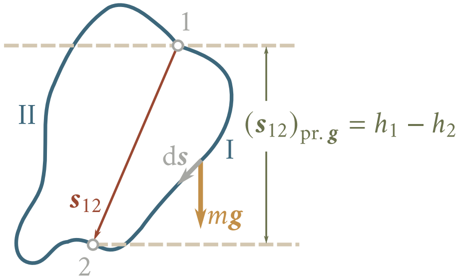

Fig. 3.6

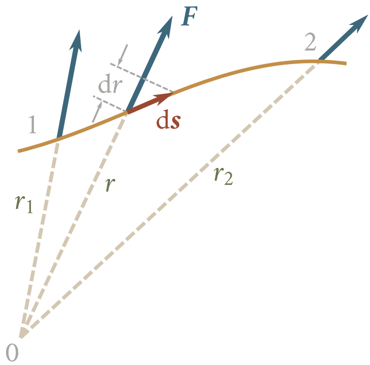

Fig. 3.7

work can be written in the form

$$A_{12} = mg(h_1 - h_2). \tag{3.23}$$

This expression obviously does not depend on the path. Hence it follows that the force of gravity is conservative. Q.E.D.[4]

It is a simple matter to see that the same result is obtained for any stationary homogeneous field.

The forces acting on a particle in a central field are also conservative. By Eq. (3.18), the elementary work over the path ds (Fig. 3.7) is

$$\mathrm{d}A = F(r)\,\mathrm{d}s_F.$$

But the projection of ds onto the direction of the force at a given point, *i.e.*, onto the direction of the position vector $\boldsymbol{r}$, is dr—the increment of the distance from the particle to the force centre O, namely, $\mathrm{d}s_F = \mathrm{d}r$. Hence, $\mathrm{d}A = F(r)\,\mathrm{d}r$, and the work along the entire path is

$$A_{12} = \int_{r_1}^{r_2} F(r)\,\mathrm{d}r. \tag{3.24}$$

Equation (3.24) depends only on the form of the function $F(r)$ and on the values of r_1 and r_2. It does not depend in any way on the form of the trajectory, whence it follows that the forces are conservative.

For our reader not to form the erroneous idea that any force depending only on the coordinates of a point is conservative, let us consider the following example. Assume that the components of a force are determined by the equations

$$F_x = ay, \quad F_y = -ax, \quad F_z = 0. \tag{3.25}$$

[4]Q.E.D. is an abbreviation of the Latin phrase "quod erat demonstrandum", literally meaning "what was to be shown".

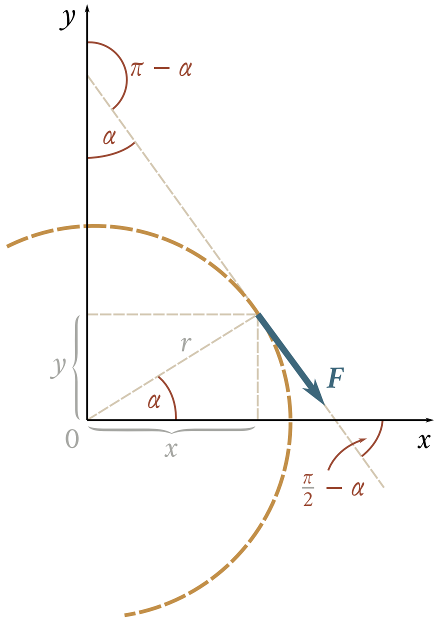

Fig. 3.8

This force has a magnitude equal to $F = ar$, and is directed along a tangent to a circle of radius r (Fig. 3.8). Indeed, as follows from the figure, for a force of such a magnitude and direction, we have

$$F_x = ar\cos\left(\frac{\pi}{2} - \alpha\right) = ar\sin\alpha = ar\,\frac{y}{r} = ay,$$
$$F_y = ar\cos(\pi - \alpha) = ar\cos\alpha = -ar\,\frac{x}{r} = -ax,$$

which coincides with the values given by Eqs. (3.25). Let us take a closed path in the form of a circle of radius r with its centre at the origin of coordinates. The work of the force along this path evidently equals $F \times 2\pi r = ar \times 2\pi r = 2\pi ar^2$, *i.e.*, does not equal zero. Consequently, the force is not conservative.

Forces of friction are typical non-conservative ones. Since the force of friction $\boldsymbol{F}$ and the velocity of a particle $\boldsymbol{v}$ are directed oppositely[5], then the work of the force of friction on each part of the path is negative:

$$\mathrm{d}A = \boldsymbol{F}\,\mathrm{d}\boldsymbol{s} = (\boldsymbol{F}\cdot\boldsymbol{v})\,\mathrm{d}t = -Fv\,\mathrm{d}t = -F\,\mathrm{d}s < 0.$$

[5]Here we have in view friction between a moving body and a stationary (relative to the reference frame) one. The forces of friction may sometimes be positive. This occurs, for instance, when the force of friction is due to the interaction of a given body with another one moving in the same direction, but with a higher velocity.

Therefore, the work along any closed path will also be negative (*i.e.*, other than zero). Hence it follows that the forces of friction are not conservative.

It must be noted that a field of conservative forces is a particular case of a potential force field. A field of forces is called **potential** if it can be described with the aid of the function $V(x, y, z, t)$, whose gradient [see the following section, Eq. (3.31)] determines the force at each point of the field: $\boldsymbol{F} = \nabla V$ [compare with Eq. (3.32)]. The function V is called the **potential function** or the **potential**.

When a potential does not depend explicitly on the time, *i.e.*, $V = V(x, y, z)$, the potential field is stationary, and its forces are conservative. In this case

$$V(x, y, z) = -E_{\mathrm{p}}(x, y, z)$$

where $E_{\mathrm{p}}(x, y, z)$ is the potential energy of a particle (see the following section).

For a non-stationary force field described by the potential $V(x, y, z, t)$, the potential and conservative forces cannot be considered identical.

3.5. Potential Energy in an External Force Field

Let us consider the case when the work of field forces does not depend on the path, but depends only on the initial and final positions of a particle in the field. A value of a certain function $E_{\mathrm{p}}(x, y, z)$ can be assigned to each point of the field such that the difference between the values of this function at points 1 and 2 will determine the work of the forces when the particle passes from the first point to the second one:

$$A_{12} = E_{\mathrm{p},1} - E_{\mathrm{p},2}. \tag{3.26}$$

We can assign this function as follows. We take an arbitrary value of the function equal to $E_{\mathrm{p},0}$, for an initial point 0. We assign the value

$$E_{\mathrm{p}}(P) = E_{\mathrm{p},0} + A_{\mathrm{p},0} \tag{3.27}$$

to any other point P. Here $A_{\mathrm{p},0}$ is the work done on a particle by the conservative forces when it is moved from point P to point 0. Since the work is independent of the path, the value of $E_{\mathrm{p}}(P)$ determined in this way will be unambiguous. It must be noted that the function $E_{\mathrm{p}}(P)$ has the dimension of work (or energy).

In accordance with Eq. (3.27), the values of the function at points 1 and 2 are

$$E_{\mathrm{p},1} = E_{\mathrm{p},0} + A_{10}, \quad E_{\mathrm{p},2} = E_{\mathrm{p},0} + A_{20}.$$

Let us form the difference between these values and take into account that $A_{20} = -A_{02}$ (see the preceding section). As a result, we get

$$E_{\mathrm{p},1} - E_{\mathrm{p},2} = A_{10} - A_{20} = A_{10} + A_{02}.$$

The sum $A_{10} + A_{02}$ gives the work done by the forces of the field when the particle moves from point 1 to point 2 along a trajectory passing through point 0. However, the work done to move the particle from point 1 to point 2 along any other trajectory (including one not passing through point 0) will be the same. Hence, the sum $A_{10} + A_{02}$ can be written simply in the form A_{12}. As a result, we get Eq. (3.26).

We can thus use the function E_p to determine the work done on a particle by conservative forces along any path beginning at arbitrary point 1 and terminating at arbitrary point 2.

Assume that only conservative forces act on the particle. Consequently, the work done on the particle along path 1-2 can be represented in the form of Eq. (3.26). According to Eq. (3.11), this work produces an increment of the kinetic energy of the particle. We thus arrive at the equation

$$E_{k,2} - E_{k,1} = E_{p,1} - E_{p,2}$$

whence it follows that

$$E_{k,2} + E_{p,2} = E_{k,1} + E_{p,1}$$

The result obtained signifies that the quantity

$$E = E_k + E_p \tag{3.28}$$

for a particle in the field of conservative forces remains constant, *i.e.*, is an integral of motion.

It follows from Eq. (3.28) that E_p is an addend in the motion integral having the dimension of energy. In this connection, the function $E_p(x, y, z)$ is called the **potential energy** of a particle in an external force field. The quantity E equal to the sum of the kinetic and potential energies is called the **total mechanical energy** of the particle.

According to Eq. (3.26), the work done on a particle by conservative forces equals the decrement of the potential energy of the particle.

We can say in a different way that work is done at the expense of the store of potential energy. We can see from Eq. (3.27) that the potential energy is determined with an accuracy to a certain unknown additive constant $E_{p,0}$. This circumstance is of no significance, however, because all physical relations contain either the difference between the values of E_p for two positions of a body, or the derivative of the function E_p with respect to the coordinates. In practice, the potential energy of a body at a certain position is considered to equal zero, and the energy at other positions is taken with respect to this energy.

Knowing the form of the function $E_p(x, y, z)$, we can find the force acting on a particle at every point of a field. Let us consider the displacement of a particle parallel to the x-axis by the amount dx. This displacement is attended by work being done on the particle that is $dA = \boldsymbol{F}\, d\boldsymbol{s} = F_x\, dx$ (the displacement components

$\mathrm{d}y$ and $\mathrm{d}z$ equal zero). According to Eq. (3.26), the same work can be represented as the decrement of the potential energy: $\mathrm{d}A = -\mathrm{d}E_\mathrm{p}$. Equating the two expressions for the work, we obtain

$$F_x\,\mathrm{d}x = -\mathrm{d}E_\mathrm{p}$$

whence

$$F_x = -\frac{\mathrm{d}E_\mathrm{p}}{\mathrm{d}x}\ (y = \text{constant},\ z = \text{constant}).$$

The expression in the right-hand side is the derivative of the function $E_\mathrm{p}(x, y, z)$ calculated on the assumption that the variables y and z remain constant, and only the variable x changes. Such derivatives are called partial ones and are denoted, unlike derivative functions of one variable, by the symbol $\partial E_\mathrm{p}/\partial x$. Consequently, the component of the force along the x-axis equals the partial derivative of the potential energy with respect to the variable x taken with the opposite sign: $F_x = -\partial E_\mathrm{p}/\partial x$. Similar expressions are obtained for the components of the force along the y- and z-axes. Thus,

$$F_x = -\frac{\partial E_\mathrm{p}}{\partial x}, \quad F_y = -\frac{\partial E_\mathrm{p}}{\partial y}, \quad F_z = -\frac{\partial E_\mathrm{p}}{\partial z}. \tag{3.29}$$

Knowing its components, we can find the force vector:

$$\boldsymbol{F} = F_x\hat{\boldsymbol{e}}_x + F_y\hat{\boldsymbol{e}}_y + F_z\hat{\boldsymbol{e}}_z = -\frac{\partial E_\mathrm{p}}{\partial x}\hat{\boldsymbol{e}}_x - \frac{\partial E_\mathrm{p}}{\partial y}\hat{\boldsymbol{e}}_y - \frac{\partial E_\mathrm{p}}{\partial z}\hat{\boldsymbol{e}}_z. \tag{3.30}$$

A vector having the components $\partial\varphi/\partial x$, $\partial\varphi/\partial y$, $\partial\varphi/\partial z$, where φ is a scalar function of the coordinates x, y, z, is called the gradient of the function φ and is designated by the symbol grad φ or $\nabla\varphi$ (∇ stands for the **nabla operator**). It follows from the definition of the gradient that

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{\boldsymbol{e}}_x + \frac{\partial\varphi}{\partial y}\hat{\boldsymbol{e}}_y + \frac{\partial\varphi}{\partial z}\hat{\boldsymbol{e}}_z. \tag{3.31}$$

A comparison of Eqs. (3.30) and (3.31) shows that the conservative force equals the gradient of the potential energy taken with the opposite sign:

$$\boldsymbol{F} = -\nabla E_\mathrm{p}. \tag{3.32}$$

Assume that a particle which the force (3.32) acts on moves over the distance $\mathrm{d}\boldsymbol{s}$ having the components $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$. The force does the work

$$\mathrm{d}A = \boldsymbol{F}\,\mathrm{d}\boldsymbol{s} = -\nabla E_\mathrm{p}\,\mathrm{d}\boldsymbol{s} = -\left(\frac{\partial E_\mathrm{p}}{\partial x}\,\mathrm{d}x + \frac{\partial E_\mathrm{p}}{\partial y}\,\mathrm{d}y + \frac{\partial E_\mathrm{p}}{\partial z}\,\mathrm{d}z\right).$$

Taking into account that $\mathrm{d}A = -\mathrm{d}E_\mathrm{p}$, we get the following expression for the increment of the function E_p:

$$\mathrm{d}E_\mathrm{p} = \frac{\partial E_\mathrm{p}}{\partial x}\,\mathrm{d}x + \frac{\partial E_\mathrm{p}}{\partial y}\,\mathrm{d}y + \frac{\partial E_\mathrm{p}}{\partial z}\,\mathrm{d}z. \tag{3.33}$$

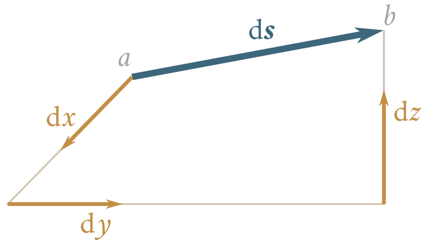

Fig. 3.9

An expression such as Eq. (3.33) is called the total differential of the relevant function.

The concept of the total differential plays a great part in physics. For this reason, we shall devote a few lines to it. The **total differential** of the single-valued function $f(x, y, z)$ is defined as the increment which this function receives in transition from a point with the coordinates x, y, z to a neighbouring point with the coordinates $x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z$. By definition, this increment equals

$$\mathrm{d}f(x, y, z) = f(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z) - f(x, y, z)$$

and, consequently, is determined only by the values of the function at the initial and final points. Hence, it cannot depend on the path along which the transition occurs. Let us take the broken line consisting of the segments $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ as such a path (Fig. 3.9). On the segment $\mathrm{d}xv$, the function $f(x, y, z)$ behaves like a function of one variable x, and receives the increment $(\partial f/\partial x)\,\mathrm{d}x$. Similarly, on the segments $\mathrm{d}y$ and $\mathrm{d}z$, the function receives the increments $(\partial f/\partial y)\,\mathrm{d}y$ and $(\partial f/\partial z)\,\mathrm{d}z$. The total increment of the function when passing from the initial point to the final one thus equals

$$\mathrm{d}f(x, y, z) = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z. \tag{3.34}$$

We have arrived at the expression for the total differential [compare with Eq. (3.33)].

Not any expression of the kind

$$P(x, y, z)\mathrm{d}x + Q(x, y, z)\mathrm{d}y + R(x, y, z)\mathrm{d}z$$

is a total differential of a certain function $f(x, y, z)$. Particularly, the expression for the work done by the force whose projections are given by Eqs. (3.25)

$$\mathrm{d}A = ay\,\mathrm{d}x - ax\,\mathrm{d}y \tag{3.35}$$

is not a total differential because there is no such function E_p for which $-\partial E_\mathrm{p}/\partial x = ay$, and $-\partial E_\mathrm{p}/\partial y = -ax$ [see Eqs. (3.25)]. Correspondingly, there is no function E_p whose decrement would determine the work (3.35).

It follows from the above that only forces complying with the condition (3.32)

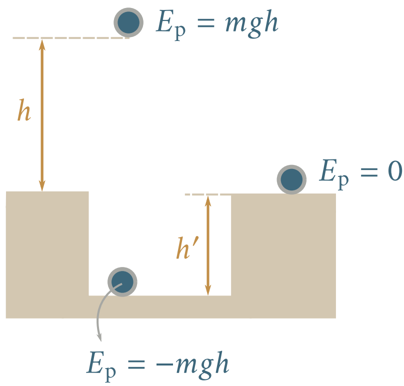

Fig. 3.10

can be conservative, *i.e.*, such forces whose components along the coordinate axes equal the derivatives of a certain function $E_p(x, y, z)$ with respect to the relevant coordinates taken with the opposite sign. This function is the potential energy of a particle.

The concrete form of the function $E_p(x, y, z)$ depends on the nature of the force field. Let us find as an example the potential energy of a particle in a field of forces of gravity. According to Eq. (3.23), the work done on a particle by the forces of this field is

$$A_{12} = mg(h_1 - h_2).$$

On the other hand, according to Eq. (3.26),

$$A_{12} = E_{p,1} - E_{p,2}.$$

Comparing these two expressions for the work, we arrive at the conclusion that the potential energy of a particle in a field of gravity forces is determined by the expression

$$E_p = mgh \tag{3.36}$$

where h is measured from an arbitrary level.

The zero of potential energy may be chosen arbitrarily. Therefore, E_p may have negative values. If we take the potential energy of a particle on the Earth's surface as zero, for example, then the potential energy of a particle lying on the bottom of a pit with a depth of h' will be $E_p = -mgh'$ (Fig. 3.10). It must be noted that the kinetic energy cannot be negative.

Assume that the non-conservative force $\boldsymbol{F}^*$ acts on a particle in addition to conservative forces. Hence, when the particle is moved from point 1 to point 2, the work done on it will be

$$A_{12} = \int_1^2 \boldsymbol{F}\, d\boldsymbol{s} + \int_1^2 \boldsymbol{F}^*\, d\boldsymbol{s} = A_{\text{cons}} + A_{12}^*$$

where A_{12}^* is the work of the non-conservative force. The work of the conservative forces A_{cons} can be represented as $E_{\text{p},1} - E_{\text{p},2}$. As a result, we find that

$$A_{12} = E_{\text{p},1} - E_{\text{p},2} + A_{12}^*$$

The total work of all the forces applied to the particle produces an increment of its kinetic energy [see Eq. (3.11)]. Consequently,

$$E_{\text{k},2} - E_{\text{p},1} = E_{\text{p},1} - E_{\text{p},2} + A_{12}^*$$

whence, taking into consideration that $E_\text{k} + E_\text{p} = E$, we get

$$E_2 - E_1 = A_{12}^*. \tag{3.37}$$

The result obtained signifies that the work of non-conservative forces is spent on an increment of the total mechanical energy of a particle.

If the kinetic energy of a particle is the same in its final and initial positions (in particular, it equals zero), then the work of the non-conservative forces produces an increment of the potential energy of the particle:

$$A_{12}^* = E_{\text{p},2} - E_{\text{p},1} \tag{3.38}$$

$(E_{\text{k},2} = E_{\text{k},1})$. This relation is useful when finding the difference between the values of the potential energy.

Let us consider a system consisting of N particles in the field of conservative forces when the particles do not interact with one another. Each of the particles has the kinetic energy $E_{\text{k},i} = m_i v_i^2/2$ (i is the number of the particle) and the potential energy $E_{\text{p},i} = E_{\text{p},i}(x_i, y_i, z_i)$. Considering the i-th particle independently of the other particles, we can find that

$$E_i = E_{\text{k},i} + E_{\text{p},i} = \text{constant}_i$$

Summing these equations for all the particles, we arrive at the relation

$$E = \sum_{i=1}^{N} E_i = \sum_{i=1}^{N} E_{\text{k},i} + \sum_{i=1}^{N} E_{\text{p},i} = \text{constant}. \tag{3.39}$$

This relation points to the additivity of the total mechanical energy for the system being considered.

According to Eq. (3.39), *the total mechanical energy of a system of non-interacting particles on which only conservative forces act remains constant*. This statement expresses the law of energy conservation for the above mechanical system.

If non-conservative forces $\boldsymbol{F}^*$ act on particles in addition to conservative forces, the total energy of the system does not remain constant, and

$$E_2 - E_1 = \sum_{i=1}^{N} (A^*_{12})_i \tag{3.40}$$

where $(A^*_{12})_i$ is the work done by the non-conservative force applied to the i-th particle when it moves from its initial position to its final one.

We established at the end of the preceding section that the work of friction forces is always negative. Therefore, when such forces are present in a system, the total mechanical energy of the system diminishes (dissipates), transforming into non-mechanical forms of energy (for example, into the internal energy of bodies, or, as is customarily said, into heat). This process is called the **dissipation** of energy. Forces leading to the dissipation of energy are called **dissipative**. Thus, friction forces are dissipative. In general, forces that always act oppositely to the velocities of particles and therefore cause their retardation are called dissipative.

We shall note that non-conservative forces are not necessarily dissipative ones.

3.6. Potential Energy of Interaction

Up to now, we treated systems of non-interacting particles. Now we shall pass over to the consideration of a system of two particles interacting with each other. Let $\boldsymbol{F}_{12}$ be the force with which the second particle acts on the first one, and $\boldsymbol{F}_{21}$ be the force with which the first particle acts on the second one. In accordance with Newton's third law, $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$.

Let us introduce the vector $\boldsymbol{R}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the position vectors of the particles (Fig. 3.11). The distance between the particles equals the magnitude of this vector. Assume that the magnitudes of the forces $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{21}$ depend only on the distance $\boldsymbol{R}12$ between the particles, and that the forces are directed along the straight line connecting the particles. We know that this holds for forces of gravitational and Coulomb interactions [see Eqs. (2.18) and (2.23)].

With these assumptions, the forces $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{21}$ can be represented in the form

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} = f(R_{12})\hat{\boldsymbol{e}}_{12} \tag{3.41}$$

where $\hat{\boldsymbol{e}}_{12}$ is the unit vector of $\boldsymbol{R}_{12}$ (Fig. 3.12), and $f(R_{12})$ is a certain function of R_{12} that is positive when the particles attract each other and negative when they repel each other.

Considering our system to be closed (there are no external forces), let us write the equations of motion for our two particles:

$$m_1\dot{\boldsymbol{v}}_1 = \boldsymbol{F}_{12}, \quad m_2\dot{\boldsymbol{v}}_2 = \boldsymbol{F}_{21}$$

Let us multiply the first equation by $\mathrm{d}\boldsymbol{r}_1 = \boldsymbol{v}_1\,\mathrm{d}t$, the second by $\mathrm{d}\boldsymbol{r}_2 = \boldsymbol{v}_2\,\mathrm{d}t$, and

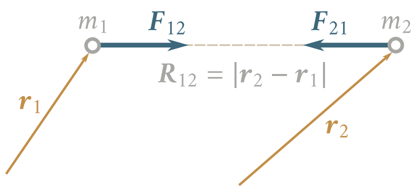

Fig. 3.11

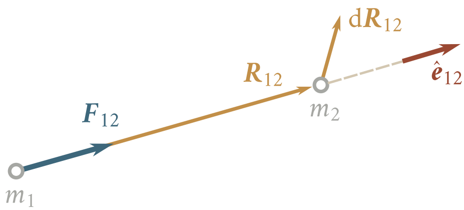

Fig. 3.12

add the resulting equations[6]. We get

$$m_1\boldsymbol{v}_1\dot{\boldsymbol{v}}_1\,\mathrm{d}t + m_2\boldsymbol{v}_2\dot{\boldsymbol{v}}_2\,\mathrm{d}t = \boldsymbol{F}_{12}\,\mathrm{d}\boldsymbol{r}_1 + \boldsymbol{F}_{21}\,\mathrm{d}\boldsymbol{r}_2. \tag{3.42}$$

The left-hand side of this equation is the increment of the kinetic energy of the system during the time $\mathrm{d}t$ [see Eq. (3.3)], and the right-hand side is the work of the internal forces during the same time. Taking into account that $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$, we can write the right-hand side as follows:

$$\mathrm{d}A_{\text{int}} = \boldsymbol{F}_{12}\,\mathrm{d}\boldsymbol{r}_1 + \boldsymbol{F}_{21}\,\mathrm{d}\boldsymbol{r}_2 = -\boldsymbol{F}_{12}\,\mathrm{d}(\boldsymbol{r}_2 - \boldsymbol{r}_1) = -\boldsymbol{F}_{12}\,\mathrm{d}\boldsymbol{R}_{12}. \tag{3.43}$$

Introducing Eq. (3.41) for $\boldsymbol{F}_{12}$ into the above equation, we get

$$\mathrm{d}A_{\text{int}} = -f(R_{12})\hat{\boldsymbol{e}}_{12}\,\mathrm{d}\boldsymbol{R}_{12}.$$

Examination of Fig. 3.12 shows that the scalar product $\hat{\boldsymbol{e}}_{12}\,\mathrm{d}\boldsymbol{R}_{12}$ equals $\mathrm{d}R_{12}$—the increment of the distance between the particles. Thus,

$$\mathrm{d}A_{\text{int}} = -f(R_{12})\,\mathrm{d}R_{12}. \tag{3.44}$$

The expression $f(R_{12})\,\mathrm{d}R_{12}$ can be considered as the increment of a certain function of R_{12}. Designating this function $E_\mathrm{p}(R_{12})$, we arrive at the equation

$$f(R_{12})\,\mathrm{d}R_{12} = \mathrm{d}E_\mathrm{p}(R_{12}). \tag{3.45}$$

Consequently,

$$\mathrm{d}A_{\text{int}} = \mathrm{d}E_\mathrm{p}. \tag{3.46}$$

With a view to everything said above, Eq. (3.42) can be written in the form $\mathrm{d}E_\mathrm{k} = -\mathrm{d}E_\mathrm{p}$, or

$$\mathrm{d}E = \mathrm{d}(E_\mathrm{k} + E_\mathrm{p}) = 0 \tag{3.47}$$

whence it follows that the quantity $E = E_\mathrm{k} + E_\mathrm{p}$ for the closed system being considered remains unchanged. The function $E_\mathrm{p}(R_{12})$ is the potential energy of interaction. It depends on the distance between the particles.

Let the particles move from their positions spaced $R_{12}^{(a)}$ apart to new positions spaced $R_{12}^{(b)}$ apart. In accordance with Eq. (3.46), the internal forces do the following

[6]Here it is expedient to use the symbol $\mathrm{d}\boldsymbol{r}$ for the displacement instead of $\mathrm{d}\boldsymbol{s}$.

work on the particles:

$$A_{\mathrm{ab,\,int}} = -\int_a^b \mathrm{d}E_\mathrm{p} = E_\mathrm{p}[R_{12}^{(a)}] - E_\mathrm{p}[R_{12}^{(b)}]. \tag{3.48}$$

It follows from Eq. (3.48) that the work of the forces (3.41) does not depend on the paths of the particles and is determined only by the initial and final distances between them (the initial and final configurations of the system). Forces of interaction of the form given by Eq. (3.41) are thus conservative.

If both particles move, the total energy of the system is

$$E = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + E_{\mathrm{p,ia}}(R_{12}) \tag{3.49}$$

where $E_{\mathrm{p,ia}}(R_{12})$ is the potential energy of interaction.

Assume that particle 1 is fixed at a certain point which we shall take as the origin of coordinates ($\boldsymbol{r}_1 = 0$). As a result, this particle will lose its ability to move, so that the kinetic energy will consist only of the single addend $m_2 v_2^2/2$. The potential energy will be a function only of $\boldsymbol{r}_2$. Therefore, Eq. (3.49) becomes

$$E = \frac{m_2 v_2^2}{2} + E_{\mathrm{p,ia}}(r_2). \tag{3.50}$$

If we consider the system consisting of only the single particle 2, then the function $E_{\mathrm{p,ia}}$ will play the part of the potential energy of particle 2 in the field of the forces set up by particle 1. In essence, however, this function is the potential energy of interaction of particles 1 and 2. In general, the potential energy in an external field of forces is essentially the energy of interaction between the bodies of the system and those producing a force field that is external relative to the system.

Let us again turn to a system of two interacting free ("unfixed") particles. If the external force $\boldsymbol{F}_1^*$ acts on the first particle in addition to the internal force, and the force $\boldsymbol{F}_2^*$ on the second particle, then the addends $\boldsymbol{F}_1^*\,\mathrm{d}\boldsymbol{r}_1^*$ and $\boldsymbol{F}_2^*\,\mathrm{d}\boldsymbol{r}_2^*$ will appear in the right-hand side of Eq. (3.42), and their sum will give the work of the external forces $\mathrm{d}A_\mathrm{ext}$. Equation (3.47) will correspondingly become

$$\mathrm{d}(E_\mathrm{k} + E_{\mathrm{p,ia}}) = \mathrm{d}A_\mathrm{ext}. \tag{3.51}$$

When the total kinetic energy of the particles remains constant (for example, equals zero), Eq. (3.51) becomes

$$\mathrm{d}E_{\mathrm{p,ia}} = \mathrm{d}A_\mathrm{ext} \tag{3.52}$$

(here $\mathrm{d}E_\mathrm{k} = 0$). Integration of this equation from configuration a to configuration b yields

$$E_{\mathrm{p,ia}}[R_{12}^{(b)}] - E_{\mathrm{p,ia}}[R_{12}^{(a)}] = \mathrm{d}A_{\mathrm{ab,ext}} \tag{3.53}$$

($E_{\mathrm{k},b} = E_{\mathrm{k},a}$) [compare with Eq. (3.38)].

Let us extend the results obtained to a system of three interacting particles. In this case, the work of the internal forces is

$$\mathrm{d}A_{\mathrm{int}} = (\boldsymbol{F}_{12} + \boldsymbol{F}_{13})\,\mathrm{d}\boldsymbol{r}_1 + (\boldsymbol{F}_{21} + \boldsymbol{F}_{23})\,\mathrm{d}\boldsymbol{r}_2 + (\boldsymbol{F}_{31} + \boldsymbol{F}_{32})\,\mathrm{d}\boldsymbol{r}_3. \tag{3.54}$$

Taking into account that $\boldsymbol{F}_{ik} = -\boldsymbol{F}_{ki}$ we can write Eq. (3.54) in the form

$$\begin{aligned}\mathrm{d}A_{\mathrm{int}} &= -\boldsymbol{F}_{12}\,\mathrm{d}(\boldsymbol{r}_2 - \boldsymbol{r}_1) - \boldsymbol{F}_{13}\,\mathrm{d}(\boldsymbol{r}_3 - \boldsymbol{r}_1) - \boldsymbol{F}_{23}\,\mathrm{d}(\boldsymbol{r}_3 - \boldsymbol{r}_2)\\ &= -\boldsymbol{F}_{12}\,\mathrm{d}\boldsymbol{R}_{12} - \boldsymbol{F}_{13}\,\mathrm{d}\boldsymbol{R}_{13} - \boldsymbol{F}_{23}\,\mathrm{d}\boldsymbol{R}_{23}\end{aligned} \tag{3.55}$$

where $\boldsymbol{R}_{ik} = \boldsymbol{r}_k - \boldsymbol{r}_i$.

Let us assume that the internal forces can be represented in the form $\boldsymbol{F}_{ik} = f_{ik}(R_{ik})\hat{\boldsymbol{e}}_{ik}$ [compare with Eq. (3.41)]. Hence,

$$\mathrm{d}A_{\mathrm{int}} = -f_{12}(R_{12})\hat{\boldsymbol{e}}_{12}\,\mathrm{d}\boldsymbol{R}_{12} - f_{13}(R_{13})\hat{\boldsymbol{e}}_{13}\,\mathrm{d}\boldsymbol{R}_{13} - f_{23}(R_{23})\hat{\boldsymbol{e}}_{23}\,\mathrm{d}\boldsymbol{R}_{23}.$$

Each of the products $\hat{\boldsymbol{e}}_{ik}\,\mathrm{d}\boldsymbol{R}_{ik}$ equals the increment of the distance between the corresponding particles $\mathrm{d}R_{ik}$. Consequently,

$$\begin{aligned}\mathrm{d}A_{\mathrm{int}} &= -f_{12}(R_{12})\,\mathrm{d}R_{12} - f_{13}(R_{13})\,\mathrm{d}R_{13} - f_{23}(R_{23})\,\mathrm{d}R_{23}\\ &= -\mathrm{d}[E_{\mathrm{p},12}(R_{12}) + E_{\mathrm{p},13}(R_{13}) + E_{\mathrm{p},23}(R_{23})] = -\mathrm{d}E_{\mathrm{p,ia}}.\end{aligned} \tag{3.56}$$

Here

$$E_{\mathrm{p,ia}} = E_{\mathrm{p},12}(R_{12}) + E_{\mathrm{p},13}(R_{13}) + E_{\mathrm{p},23}(R_{23}) \tag{3.57}$$

is the **potential energy of interaction of the system**. It consists of the energies of interaction of the particles taken in pairs.

Equating $\mathrm{d}E_{\mathrm{k}}$ to the sum of the work $\mathrm{d}A_{\mathrm{int}} = -\mathrm{d}E_{\mathrm{p,ia}}$ and $\mathrm{d}A_{\mathrm{ext}}$ we arrive at Eq. (3.51) in which by $E_{\mathrm{p,ia}}$ we must understand Eq. (3.57).

The result obtained is easily generalized for a system with any number of particles. For a system of N interacting particles, the potential energy of interaction consists of the energies of interaction of the particles taken in pairs:

$$\begin{aligned}E_{\mathrm{p,ia}} = {}& E_{\mathrm{p},12}(R_{12}) + E_{\mathrm{p},13}(R_{13}) + \ldots + E_{\mathrm{p},1N}(R_{1N})\\ &+ E_{\mathrm{p},23}(R_{23}) + E_{\mathrm{p},2N}(R_{2N}) + \ldots + E_{\mathrm{p},N-1,N}(R_{N-1,N}).\end{aligned} \tag{3.58}$$

This sum can be written as follows:

$$E_{\mathrm{p,ia}} = \sum_{i<k} E_{\mathrm{p},ik}(R_{ik}) \tag{3.59}$$

[note that in Eq. (3.58) the first subscript of each addend has a value smaller than the second one]. In connection with the fact that $E_{\mathrm{p},ik}(R_{ik}) = E_{\mathrm{p},ki}(R_{ki})$, the energy of interaction can also be represented in the form

$$E_{\mathrm{p,ia}} = \frac{1}{2}\sum_{i\neq k} E_{\mathrm{p},ik}(R_{ik}). \tag{3.60}$$

In the sums (3.59) and (3.60), the subscripts i and k take on values from 1 to N with observance of the condition that $i < k$ or $i \neq k$.

Assume that a system consists of four particles, and that the first particle interacts only with the second one and the third particle only with the fourth one. The total energy of this system will be

$$\begin{aligned} E &= E_{\mathrm{k},1} + E_{\mathrm{k},2} + E_{\mathrm{k},3} + E_{\mathrm{k},4} + E_{\mathrm{k},12} + E_{\mathrm{k},34} \\ &= (E_{\mathrm{k},1} + E_{\mathrm{k},2} + E_{\mathrm{k},12}) + (E_{\mathrm{k},3} + E_{\mathrm{k},4} + E_{\mathrm{k},34}) = E' + E''. \end{aligned} \tag{3.61}$$

Here E' is the total energy of the subsystem formed by particles 1 and 2, and E'' is the total energy of the subsystem formed by particles 3 and 4. In accordance with our assumption, there is no interaction between the subsystems. Equation (3.61) proves the additivity of energy (see the third paragraph of Sec. 3.1).

In conclusion, let us find the form of the function $E_{\mathrm{p,ia}}$ for the case when the force of interaction is inversely proportional to the square of the distance between the particles:

$$f(R_{12}) = \frac{\alpha}{R_{12}^2} \tag{3.62}$$

(α is a constant). We remind our reader that for attraction between the particles $\alpha > 0$, and for repulsion between them $\alpha < 0$ [see the text following Eq. (3.41)].

In accordance with Eq. (3.45)

$$\mathrm{d}E_{\mathrm{p,ia}} = f(R_{12})\,\mathrm{d}R_{12} = \frac{\alpha}{R_{12}^2}\,\mathrm{d}R_{12}.$$

Integration yields

$$E_{\mathrm{p,ia}} = -\frac{\alpha}{R_{12}} + \text{constant}. \tag{3.63}$$

Like the potential energy in an external field of forces, the potential energy of interaction is determined with an accuracy up to an arbitrary additive constant. It is usually assumed that when $R_{12} = \infty$, the potential energy becomes equal to zero [at such a distance the force Eq. (3.62) becomes equal to zero—the interaction between the particles vanishes]. Hence, the additive constant in Eq. (3.63) vanishes, and the expression for the potential energy of interaction acquires the form

$$E_{\mathrm{p,ia}} = -\frac{\alpha}{R_{12}}. \tag{3.64}$$

In accordance with Eq. (3.53), the following work must be done to move the particles away from each other from the distance R_{12} to infinity without changing their velocities:

$$A_{\mathrm{ext}} = E_{\mathrm{p,ia}}(\infty) - E_{\mathrm{p,ia}}(R_{12}).$$

Introduction of the corresponding values of the function Eq. (3.64) leads to the expression

$$A_{\mathrm{ext}} = 0 - \left(-\frac{\alpha}{R_{12}}\right) = \frac{\alpha}{R_{12}}. \tag{3.65}$$

When the particles are attracted to each other, we have $\alpha > 0$; accordingly, positive work must be done to move the particles away from each other.

Upon repulsion of the particles from each other, $\alpha < 0$, and the work (3.65) is negative. This work has to be done to prevent the particles that are repelling each other from increasing their velocity.

3.7. Law of Conservation of Energy

Let us combine the results obtained in the preceding sections. We shall consider a system consisting of N particles of masses $m_1, m_2, \ldots, m_N$. Assume that the particles interact with one another with the forces $\boldsymbol{F}_{ik}$, whose magnitudes depend only on the distance R_{ik} between the particles. We established in the preceding section that such forces are conservative. This signifies that the work done by these forces on the particles is determined by the initial and final configurations of the system. Assume that the external conservative force $\boldsymbol{F}_i$ and the external non-conservative force $\boldsymbol{F}_i^*$ act on the i-th particle in addition to the internal forces. The equation of motion of the i-th particle will therefore acquire the form

$$m_i \dot{\boldsymbol{v}}_i = \sum_{\substack{k=1 \\ (k \neq i)}}^{N} \boldsymbol{F}_{ik} + \boldsymbol{F}_i + \boldsymbol{F}_i^* \tag{3.66}$$

where $i = 1, 2, \ldots, N$.

Multiplying the i-th equation by $\mathrm{d}\boldsymbol{s}_i = \mathrm{d}\boldsymbol{r}_i = \boldsymbol{v}_i\,\mathrm{d}t$ and adding together all the N equations, we get

$$\sum_{i=1}^{N} m_i \boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_i = \sum_{i=1}^{N} \left[\sum_{\substack{k=1 \\ (k \neq i)}}^{N} \boldsymbol{F}_{ik}\right] \mathrm{d}\boldsymbol{r}_i + \sum_{i=1}^{N} \boldsymbol{F}_i\,\mathrm{d}\boldsymbol{s}_i + \sum_{i=1}^{N} \boldsymbol{F}_i^*\,\mathrm{d}\boldsymbol{s}_i. \tag{3.67}$$

The left-hand side is the increment of the kinetic energy of the system:

$$\sum_{i=1}^{N} m_i \boldsymbol{v}_i\,\mathrm{d}\boldsymbol{v}_i = \mathrm{d}\left[\sum_{i=1}^{N} \frac{m_i v_i^2}{2}\right] = \mathrm{d}E_{\mathrm{k}} \tag{3.68}$$

[see Eq. (3.3)]. It follows from Eqs. (3.54)-(3.59) that the first term of the right-hand side equals the decrement of the potential energy of interaction of the particles:

$$\sum_{i=1}^{N}\left[\sum_{\substack{k=1\\(k\neq i)}}^{N}\boldsymbol{F}_{ik}\right]\mathrm{d}\boldsymbol{r}_i=-\sum_{i<k}\boldsymbol{F}_{ik}\,\mathrm{d}\boldsymbol{R}_{ik}=-\mathrm{d}\left[\sum_{i<k}E_{\mathrm{p},ik}(R_{ik})\right]=-\mathrm{d}E_{\mathrm{p,ia}}. \tag{3.69}$$

According to Eq. (3.26), the second term in Eq. (3.67) equals the decrement of the potential energy of the system in the external field of the conservative forces:

$$\sum_{i=1}^{N}\boldsymbol{F}_i\,\mathrm{d}\boldsymbol{s}_i=-\mathrm{d}\left[\sum_{i=1}^{N}E_{\mathrm{p},i}(\boldsymbol{r}_i)\right]=-\mathrm{d}E_{\mathrm{p,ext}}. \tag{3.70}$$

Finally, the last term in Eq. (3.67) is the work of the non-conservative external forces:

$$\sum_{i=1}^{N}\boldsymbol{F}_i^*\,\mathrm{d}\boldsymbol{s}_i=\sum_{i=1}^{N}\mathrm{d}A_i^*=\mathrm{d}A_{\mathrm{ext}}^*. \tag{3.71}$$

Taking into account equations (3.68)-(3.71), we can write Eq. (3.67) as follows:

$$\mathrm{d}(E_{\mathrm{k}}+E_{\mathrm{p,ia}}+E_{\mathrm{p,ext}})=\mathrm{d}A_{\mathrm{ext}}^*. \tag{3.72}$$

The quantity

$$E=E_{\mathrm{k}}+E_{\mathrm{p,ia}}+E_{\mathrm{p,ext}} \tag{3.73}$$

is the total mechanical energy of the system. If external non-conservative forces are absent, the right-hand side of Eq. (3.72) will vanish, and, consequently, the total energy of the system remains constant:

$$E=E_{\mathrm{k}}+E_{\mathrm{p,ia}}+E_{\mathrm{p,ext}}=\text{constant}. \tag{3.74}$$

We have thus arrived at the conclusion that *the total mechanical energy of a system of bodies on which only conservative forces act remains constant*. This statement is the essence of one of the fundamental laws of mechanics—the **law of conservation of mechanical energy**.

For a closed system, *i.e.*, a system whose bodies experience no external forces, Eq. (3.74) has the form

$$E=E_{\mathrm{k}}+E_{\mathrm{p,ia}}=\text{constant}. \tag{3.75}$$

In this case, the law of conservation of energy is formulated as follows: *the total mechanical energy of a closed system of bodies between which only conservative forces act remains constant*.

If non-conservative forces, for example, forces of friction, act in a closed system in addition to conservative ones, then the total mechanical energy of the system is not conserved. Considering the non-conservative forces as external ones,

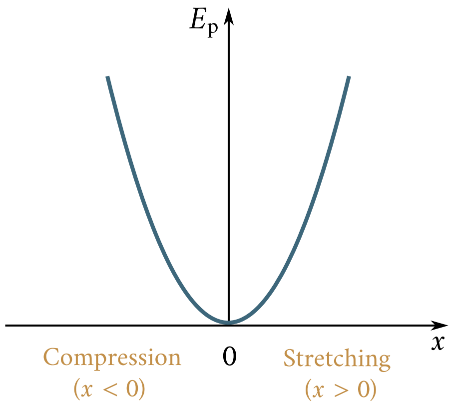

Fig. 3.13

we can write in accordance with Eq. (3.72) that

$$dE = d(E_k + E_{p,ia}) = dA_{non\text{-}cons}. \quad (3.76)$$

Integration of this equation yields

$$E_2 - E_1 = A_{12,non\text{-}cons}. \quad (3.77)$$

The law of energy conservation for a system of non-interacting particles was formulated in Sec. 3.5 [see the text following Eq. (3.39)].

3.8. Energy of Elastic Deformation

Not only a system of interacting bodies, but also a separately taken elastically deformed body (for example, a compressed spring, a stretched rod, etc.) can have potential energy. In this case, the latter depends on the mutual arrangement of separate parts of the body (for example, on the distance between adjacent coils of a spring).

According to Eq. (3.13), the work $A = kx^2/2$ must be done to stretch or compress a spring by the amount x. This work goes to increase the potential energy of the spring. Consequently, the potential energy of a spring depends on the elongation x as follows:

$$E_p = \frac{kx^2}{2} \quad (3.78)$$

where k is the spring constant (see Sec. 2.9). Equation Eq. (3.78) is based on the assumption that the potential energy of an undeformed spring equals zero. Figure 3.13 shows a plot of E_p against x.

The work determined by Eq. (3.14) is done in the elastic longitudinal deformation of a bar or rod. Accordingly, the potential energy of an elastically deformed rod is

$$E_{\mathrm{p}} = \frac{E\varepsilon^2}{2} V \tag{3.79}$$

where E is the Young's modulus, ε is the relative elongation and V is the volume of the rod.

Let us introduce the concept of the density of the energy of elastic deformation w_{e}, which we shall define as the ratio of the energy $\mathrm{d}E_{\mathrm{p}}$ to the volume $\mathrm{d}V$ in which it is confined:

$$w_{\mathrm{e}} = \frac{\mathrm{d}E_{\mathrm{p}}}{\mathrm{d}V}. \tag{3.80}$$

Since the rod is assumed to be homogeneous and its deformation is uniform, *i.e.*, identical at different points of the rod, the energy Eq. (3.79) is also distributed uniformly in the rod. We can therefore consider that

$$w_{\mathrm{e}} = \frac{E_{\mathrm{p}}}{V} = \frac{E\varepsilon^2}{2}. \tag{3.81}$$

This expression also gives the density of the energy of elastic deformation in stretching (or compression) when the deformation is not uniform. In the latter case to find the energy density at a certain point of a rod, the value of ε at this point must be introduced into Eq. (3.81).

On the basis of Eqs. (2.32)-(2.34), it is not difficult to obtain the following equation for the density of the energy of elastic deformation in shear:

$$w_{\mathrm{e}} = \frac{G\gamma^2}{2} \tag{3.82}$$

where G is the shear modulus and γ is the relative shear.

3.9. Equilibrium Conditions of a Mechanical System

Let us consider a point particle whose motion is restricted so that it has only one degree of freedom[7]. This signifies that its position can be determined with the aid of a single quantity, for example the coordinate x. We can take as an example a ball sliding without friction along a stationary wire bent in a vertical plane (Fig. 3.14a).

Another example is a ball attached to the end of a spring and sliding without friction along a horizontal guide wire (Fig. 3.15a). The ball is acted upon in each case by a conservative force: the force of gravity and the elastic force of the de-

[7]By the number of degrees of freedom of a mechanical system is meant the number of independent quantities with whose aid the position of the system can be set. This will be treated in greater detail in Sec. ??.

formed spring, respectively. Plots of the potential energy E_p against x are shown in Figs. 3.14b and 3.15b.

Since the balls move along the relevant wires without friction, the force with which the wire acts on the ball in each case is at right angles to the velocity of the ball and, consequently, does no work on the ball. Therefore, the energy is conserved

$$E = E_k + E_p = \text{constant}. \tag{3.83}$$

It follows from Eq. (3.83) that the kinetic energy can grow only at the expense of a reduction in the potential energy. Hence, if a ball is in a state such that its velocity equals zero and its potential energy is minimum, it will be unable to start moving without external action on it, *i.e.*, it will be in equilibrium.

Values of x equal to x_0 correspond to minima of E_p in the graphs (in Fig. 3.15, x_0 is the length of the undeformed spring). The condition of a minimum of the potential energy has the form

$$\frac{dE_p}{dx} = 0. \tag{3.84}$$

In accordance with Eq. (3.32), the condition (3.84) is equivalent to the fact that

$$F_x = 0 \tag{3.85}$$

(when E_p is a function of only one variable, we have $\partial E_p/\partial x = dE_p/dx$). Thus, the position corresponding to a minimum of the potential energy has the property that the force acting on the body equals zero.

In the case shown in Fig. 3.14, the conditions (3.84) and (3.85) are also observed for x equal to x_0 (*i.e.*, for a maximum E_p). The position of the ball determined by this value of x will also be an equilibrium one. This equilibrium, however, unlike that at $x = x_0$, will be unstable: it is sufficient to slightly move the ball out of this position, and a force will appear that will move it away from the position x_0. The forces appearing when the ball is displaced from its position of stable equilibrium (for which $x = x_0$) are directed so that they tend to return the ball to its equilibrium position.

Knowing the form of the function expressing the potential energy, we can arrive at a number of conclusions on the nature of motion of a particle. We shall explain this using the graph shown in Fig. 3.14b to describe the motion of our particle. If the total energy has the value shown in the figure, then the particle can move either within the limits from x_1 to x_2, or within the limits from x_3 to infinity. The particle cannot penetrate into the regions with $x < x_1$ and $x_2 < x < x_3$ because its potential energy cannot become greater than its total energy (if this occurred, then the kinetic energy would be negative). Thus, the region $x_2 < x < x_3$ is

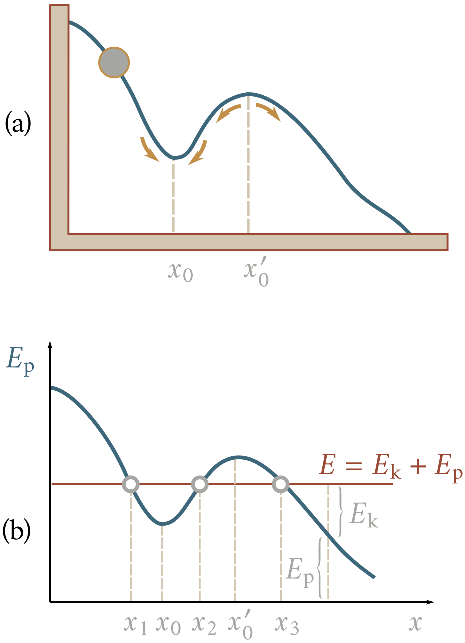

Fig. 3.14

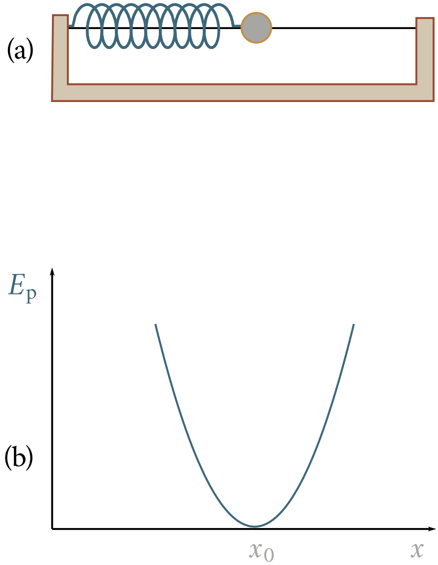

Fig. 3.15

a **potential barrier through** which the particle cannot penetrate having its given stock of total energy. The region $x_1 < x < x_2$ is called a **potential well**.

If a particle in its motion cannot move away to infinity, the motion is called **finite**. If the particle can travel any distance away, the motion is called **infinite**. A particle in a potential well performs finite motion. The motion of a particle with a negative total energy in the central field of forces of attraction will also be finite (it is assumed that the potential energy vanishes at infinity).

3.10. Law of Momentum Conservation

In the preceding sections, we considered the additive integral of motion called energy. Let us find another additive quantity that is conserved for a closed system. For this purpose, we shall consider a system of N interacting particles. Assume that external forces whose resultant is $\boldsymbol{F}_i$ act on the i-th particle in addition to the internal forces $\boldsymbol{F}_{ik}$. Let us write Eq. (2.10) for all the N particles:

$$\dot{\boldsymbol{p}}_1 = \boldsymbol{F}_{12} + \boldsymbol{F}_{13} + \ldots + \boldsymbol{F}_{1k} + \ldots + \boldsymbol{F}_{1N} + \boldsymbol{F}_1 = \sum_{k=2}^{N} \boldsymbol{F}_{1k} + \boldsymbol{F}_1$$

$$\dot{\boldsymbol{p}}_2 = \boldsymbol{F}_{21} + \boldsymbol{F}_{23} + \ldots + \boldsymbol{F}_{2k} + \ldots + \boldsymbol{F}_{2N} + \boldsymbol{F}_2 = \sum_{\substack{k=1 \\ (k\neq 2)}}^{N} \boldsymbol{F}_{2k} + \boldsymbol{F}_2$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_{i1} + \boldsymbol{F}_{i2} + \ldots + \boldsymbol{F}_{ik} + \ldots + \boldsymbol{F}_{iN} + \boldsymbol{F}_i = \sum_{\substack{k=1 \\ (k\neq i)}}^{N} \boldsymbol{F}_{ik} + \boldsymbol{F}_i$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$\dot{\boldsymbol{p}}_N = \boldsymbol{F}_{N1} + \boldsymbol{F}_{N2} + \ldots + \boldsymbol{F}_{Nk} + \ldots + \boldsymbol{F}_{N,N-1} + \boldsymbol{F}_N = \sum_{k=1}^{N-1} \boldsymbol{F}_{Nk} + \boldsymbol{F}_N$$

Let us find the sum of these N equations. Since $\boldsymbol{F}_{12} + \boldsymbol{F}_{21} = 0$, etc., only the external forces will remain in the right-hand side. We thus arrive at the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \ldots + \boldsymbol{p}_N) = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_N = \sum_{i=1}^{N} \boldsymbol{F}_i \tag{3.86}$$

The sum of the momenta of the particles forming a mechanical system is called the **momentum of the system**. Denoting this momentum by $\boldsymbol{p}$, we find that

$$\boldsymbol{p} = \sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i. \tag{3.87}$$

It follows from Eq. (3.87) that the momentum is an additive quantity.

Let us write Eq. (3.86) in the form

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \sum_{i=1}^{N} \boldsymbol{F}_i. \tag{3.88}$$

Hence it follows that in the absence of external forces, $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = 0$. Consequently, for a closed system, $\boldsymbol{p}$ is constant. This statement forms the content of the **law of momentum conservation**, which is formulated as follows: *the momentum of a closed system of point particles remains constant.*

It should be noted that the momentum also remains constant for an unclosed system provided that the sum of the external forces is zero [see Eq. (3.88)]. When the sum of the external forces does not equal zero, but the projection of this sum on a certain direction does equal zero, the component of the momentum in this direction is conserved. Indeed, upon projecting all the quantities of Eq. (3.88) onto a certain direction x, we find that

$$\frac{dp_x}{dt} = \sum_{i=1}^{N} F_{xi} \tag{3.89}$$

whence our statement follows. [We remind our reader that $(d\boldsymbol{p}/dt)_{\text{pr. }\boldsymbol{x}} = dp_x/dt$, see Eqs. (1.40)].

The momentum of a system of particles can be represented as the product of the total mass of the particles and the velocity of the centre of mass of the system:

$$\boldsymbol{p} = m\boldsymbol{v}_C \tag{3.90}$$

The **centre of mass** (or the **centre of inertia**) of a system is defined as the point C whose position is set by the position vector $\boldsymbol{r}_C$ determined as follows:

$$\boldsymbol{r}_C = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + \ldots + m_N\boldsymbol{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{\sum_{i=1}^{N} m_i\boldsymbol{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{m}\sum_{i=1}^{N} m_i\boldsymbol{r}_i \tag{3.91}$$

where m_i is the mass of the i-th particle, $\boldsymbol{r}_i$ the position vector determining the position of this particle, and m the mass of the system.

The Cartesian coordinates of the centre of mass equal the projections of $\boldsymbol{r}_C$ onto the coordinate axes:

$$x_C = \frac{1}{m}\sum_{i=1}^{N} m_i x_i, \quad y_C = \frac{1}{m}\sum_{i=1}^{N} m_i y_i, \quad z_C = \frac{1}{m}\sum_{i=1}^{N} m_i z_i. \tag{3.92}$$

It must be noted that in a homogeneous field of gravity forces the centre of mass coincides with the centre of gravity of a system.

We find the velocity of the centre of mass by time differentiation of the position vector (3.91):

$$\boldsymbol{v}_C = \dot{\boldsymbol{r}}_C = \frac{1}{m}\sum_{i=1}^{N} m_i\dot{\boldsymbol{r}}_i = \frac{1}{m}\sum_{i=1}^{N} m_i\boldsymbol{v}_i = \frac{\boldsymbol{p}}{m}$$

(see Eq. (3.87)]. Hence follows Eq. (3.90).

For a closed system, $\boldsymbol{p} = m\boldsymbol{v}_C =$ constant. Therefore, the centre of mass of a closed system either moves uniformly in a straight line, or remains stationary.

A reference frame in which the centre of mass is at rest is called a **centre-of-mass** frame or a **c.m.-frame**. This frame is obviously an inertial one.

A reference frame associated with measuring instruments is called a **laboratory** or an **l-frame**.

3.11. Collision of Two Bodies

When bodies collide with one another, they become deformed. The kinetic energy which the bodies had before the collision partially or completely transforms into the potential energy of elastic deformation and into the so-called internal energy of the bodies. An increase in the internal energy of bodies is attended by elevation of their temperature.

Two extreme kinds of collisions are distinguished: perfectly elastic and completely inelastic ones. A **perfectly elastic collision** is one in which the mechanical energy of the bodies does not transform into other non-mechanical kinds of energy. In such a collision, the kinetic energy transforms completely or partly into the potential energy of elastic deformation. Next the bodies return to their original shape, repelling each other. As a result, the potential energy of elastic deformation again transforms into kinetic energy, and the bodies fly apart with velocities whose magnitude and direction are determined by two conditions—conservation of the total energy and conservation of the total momentum of the system of bodies.

A **completely inelastic collision** is characterized by the fact that no potential energy of deformation is produced. The kinetic energy of the bodies completely or partly transforms into internal energy. After colliding, the bodies either move with the same velocity or are at rest. In a completely inelastic collision, only the law of conservation of momentum is observed. The law of conservation of mechanical energy is not observed—instead of it the law of conservation of the total energy of different kinds-mechanical and internal—is observed.

Let us first consider a completely inelastic collision of two particles (point particles) forming a closed system. Let the masses of the particles be m_1 and m_2, and their velocities before colliding $\boldsymbol{v}_{10}$ and $\boldsymbol{v}_{20}$. In view of the law of momentum conservation, the total momentum of the particles after the collision must be the same as before it:

$$m_1\boldsymbol{v}_{10} + m_2\boldsymbol{v}_{20} = m_1\boldsymbol{v} + m_2\boldsymbol{v} = (m_1 + m_2)\boldsymbol{v} \tag{3.93}$$

($\boldsymbol{v}$ is the identical velocity of both particles after colliding). It follows from Eq. (3.93) that

$$\boldsymbol{v} = \frac{m_1\boldsymbol{v}_{10} + m_2\boldsymbol{v}_{20}}{m_1 + m_2}. \tag{3.94}$$

For practical calculations, Eq. (3.94) must be projected onto the correspondingly selected directions.

Let us now consider a perfectly elastic collision. We shall limit ourselves to the case of a central collision of two homogeneous spheres. A collision is called central if the spheres before colliding travelled along the straight line passing through

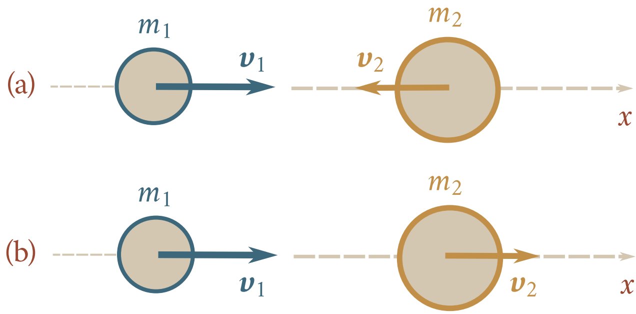

Fig. 3.16

their centres. A central collision of two spheres can take place (1) if the spheres are moving toward each other (Fig. 3.16a), or (2) if one of the spheres is overtaking the other one (Fig. 3.16b).

We shall assume that the spheres form a closed system or that the external forces applied to them balance each other. We shall also assume that the spheres do not rotate.

Let the masses of the spheres be m_1 and m_2, the velocities of the spheres before the collision be $\boldsymbol{v}_{10}$ and $\boldsymbol{v}_{20}$, and, finally, the velocities after the collision be $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The equations of conservation of energy and momentum are:

$$\frac{m_1\boldsymbol{v}_{10}^2}{2} + \frac{m_2\boldsymbol{v}_{20}^2}{2} = \frac{m_1\boldsymbol{v}_1^2}{2} + \frac{m_2\boldsymbol{v}_2^2}{2} \tag{3.95}$$

$$m_1\boldsymbol{v}_{10} + m_2\boldsymbol{v}_{20} = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2. \tag{3.96}$$

Taking into account that $(\boldsymbol{a}_2 - \boldsymbol{b}_2) = (\boldsymbol{a} - \boldsymbol{b})(\boldsymbol{a} + \boldsymbol{b})$, we can write Eq. (3.95) in the form

$$m_1(\boldsymbol{v}_{10} - \boldsymbol{v}_1)(\boldsymbol{v}_{10} + \boldsymbol{v}_1) = m_2(\boldsymbol{v}_2 - \boldsymbol{v}_{20})(\boldsymbol{v}_2 + \boldsymbol{v}_{20}). \tag{3.97}$$

Relation Eq. (3.96) can be transformed as follows:

$$m_1(\boldsymbol{v}_{10} - \boldsymbol{v}_1) = m_2(\boldsymbol{v}_2 - \boldsymbol{v}_{20}). \tag{3.98}$$

We can state, from considerations of symmetry, that the velocities of the spheres after the collision will be directed along the same straight line that was the path of the centres of the spheres before colliding. Consequently, all the vectors in Eqs. (3.97) and (3.98) are collinear. For the collinear vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, it follows from the equation $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{c}$ that $\boldsymbol{b} = \boldsymbol{c}$. Therefore, comparing Eqs. (3.97) and (3.98), we can conclude that

$$\boldsymbol{v}_{10} + \boldsymbol{v}_1 = \boldsymbol{v}_2 + \boldsymbol{v}_{20}. \tag{3.99}$$

Multiplying Eq. (3.99) by m_2 and subtracting the result from Eq. (3.98), then multiplying Eq. (3.99) by m_1 and adding the result to Eq. (3.98), we get the velocities of the spheres after the collision:

$$\boldsymbol{v}_1 = \frac{2m_2\boldsymbol{v}_{20} + (m_1 - m_2)\boldsymbol{v}_{10}}{m_1 + m_2}, \quad \boldsymbol{v}_2 = \frac{2m_1\boldsymbol{v}_{10} + (m_2 - m_1)\boldsymbol{v}_{20}}{m_1 + m_2}. \tag{3.100}$$

For numerical calculations, the relations (3.100) must be projected onto the x-axis along which the spheres are moving (see Fig. 3.16).

We must note that the velocities of the spheres after a perfectly elastic collision cannot be the same. Indeed, equating expressions (3.100) for $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and performing the relevant transformations, we get

$$\boldsymbol{v}_{10} = \boldsymbol{v}_{20}.$$

Consequently, for the velocities of the spheres to be the same after the collision, they must also be the same before it, but in this case no collision can take place. Hence, it follows that the condition of equality of the velocities of the spheres after the collision is incompatible with the law of conservation of energy.

Let us consider the case when the masses of the colliding spheres are equal: $m_1 = m_2$. It follows from Eqs. (3.100) that in this condition

$$\boldsymbol{v}_1 = \boldsymbol{v}_{20} \quad \boldsymbol{v}_2 = \boldsymbol{v}_{10}$$

i.e., the spheres exchange velocities when they collide. Particularly, if one of the spheres of the same mass, for instance, the second one, is stationary before the collision, then after it it will travel with the velocity which the first sphere originally had, while the first sphere after the collision will be stationary.

We can use Eqs. (3.100) to find the velocity of a sphere after an elastic collision with a stationary or a moving wall (which we can consider as a sphere of infinitely great mass m_2 and infinitely great radius). Dividing the numerator and denominator of Eqs. (3.100) by m_2 and disregarding the terms containing the factor m_1/m_2, we get

$$\boldsymbol{v}_1 = 2\boldsymbol{v}_{20} - \boldsymbol{v}_{10} \quad \boldsymbol{v}_2 = \boldsymbol{v}_{20}.$$

The result obtained shows that the velocity of the wall remains unchanged. The velocity of the sphere, however, if the wall is stationary ($\boldsymbol{v}_{20} = 0$), reverses. If the wall is moving, the magnitude of the velocity of the sphere also changes (it grows by $2v_{20}$ if the wall moves toward the sphere and diminishes by $2v_{20}$ if the wall moves away from the sphere catching up with it).

3.12. Law of Angular Momentum Conservation

We already know two additive quantities obeying laws of conservation: energy and momentum. Now we shall find a third quantity of this kind. For this purpose, we shall consider a system consisting of two interacting particles on which external

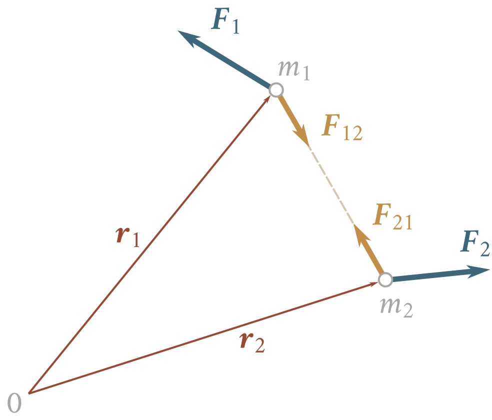

Fig. 3.17

forces also act (Fig. 3.17). The equations of motion of the particles have the form

$$m_1\dot{\boldsymbol{v}}_1 = \boldsymbol{F}_{12} + \boldsymbol{F}_1, \qquad m_2\dot{\boldsymbol{v}}_2 = \boldsymbol{F}_{21} + \boldsymbol{F}_2.$$

Let us find the vector product of the first equation and the position vector of the first particle $\boldsymbol{r}_1$ and of the second equation and the position vector of the second particle $\boldsymbol{r}_2$, placing the position vectors at the left:

$$\begin{cases} m_1(\boldsymbol{r}_1 \times \dot{\boldsymbol{v}}_1) = \boldsymbol{r}_1 \times \boldsymbol{F}_{12} + \boldsymbol{r}_1 \times \boldsymbol{F}_1 \\ m_2(\boldsymbol{r}_2 \times \dot{\boldsymbol{v}}_2) = \boldsymbol{r}_2 \times \boldsymbol{F}_{21} + \boldsymbol{r}_2 \times \boldsymbol{F}_2. \end{cases} \tag{3.101}$$

A vector product of the kind $\boldsymbol{r} \times \dot{\boldsymbol{v}}$ is equivalent to the expression $\mathrm{d}(\boldsymbol{r} \times \dot{\boldsymbol{v}})/\mathrm{d}t$. Indeed, according to Eq. (1.55)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \dot{\boldsymbol{v}}) = \boldsymbol{r} \times \dot{\boldsymbol{v}} + \dot{\boldsymbol{r}} \times \boldsymbol{v} = \boldsymbol{r} \times \dot{\boldsymbol{v}} \tag{3.102}$$

because $\dot{\boldsymbol{r}} \times \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{v} = 0$. Making such a substitution in Eqs. (3.101) and taking into account that $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$, we get the equations

$$\begin{cases} m_1 \dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_1 \times \boldsymbol{v}_1) = \boldsymbol{r}_1 \times \boldsymbol{F}_{12} + \boldsymbol{r}_1 \times \boldsymbol{F}_1 \\ m_2 \dfrac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_2 \times \boldsymbol{v}_2) = -\boldsymbol{r}_2 \times \boldsymbol{F}_{12} + \boldsymbol{r}_2 \times \boldsymbol{F}_2. \end{cases} \tag{3.103}$$

Mass is a constant scalar quantity. It can therefore be put inside the time derivative and into the vector product:

$$m\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \boldsymbol{v}) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times m\boldsymbol{v}) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \boldsymbol{p}).$$

With this in view, we shall find the sum of Eqs. (3.103). We get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2) = (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{F}_{12} + \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2.$$

The vectors $\boldsymbol{r}_1-\boldsymbol{r}_2$ and $\boldsymbol{F}_{12}$ are collinear. Consequently, their vector product equals zero. We thus obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2) = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2. \tag{3.104}$$

If the system is closed, the right-hand side of this relation vanishes, and, therefore,

$$\boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 = \text{constant}.$$

We have arrived at an additive quantity obeying a law of conservation that is called the angular momentum (or the moment of momentum) relative to point 0 (see Fig. 3.17).

For a separate particle, the angular momentum relative to point 0 is defined as the pseudovector

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times m\boldsymbol{v}. \tag{3.105}$$

The angular momentum of a system relative to point 0 is defined as the vector sum of the angular momenta of the particles in the system:

$$\boldsymbol{L} = \sum_i \boldsymbol{L}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i. \tag{3.106}$$

The projection of the vector (3.105) onto the z-axis is called the **angular momentum of a particle relative to this axis**:

$$L_z = (\boldsymbol{r} \times \boldsymbol{p})_{\mathrm{pr.},z}. \tag{3.107}$$

Similarly, the **angular momentum of a system relative to the z-axis** is defined as the scalar quantity

$$L_z = \sum_i (\boldsymbol{r}_i \times \boldsymbol{p}_i)_{\mathrm{pr.},z}. \tag{3.108}$$

A glance at Fig. 3.18 shows that the magnitude of the angular momentum vector of a particle is

$$L = rp\sin\alpha = lp \tag{3.109}$$

where $l = r\sin\alpha$ is the length of a perpendicular dropped from point 0 onto the straight line along which the momentum of the particle is directed. This length is called the arm of the momentum relative to point 0. It is assumed in Fig. 3.18 that point 0 relative to which the angular momentum is taken and the vector $\boldsymbol{p}$ are in the plane of the drawing. The vector $\boldsymbol{L}$ is at right angles to the plane of the drawing and is directed away from us.

Let us consider two typical examples.

1. Assume that a particle is moving along the straight line depicted in Fig. 3.18 by the dash line. In this case, the angular momentum of the particle can

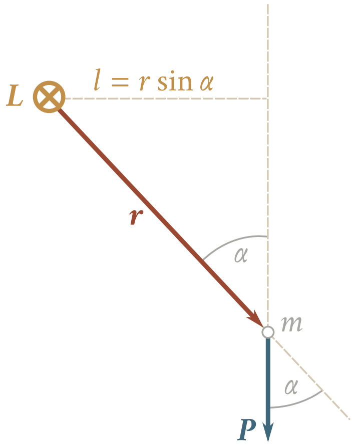

Fig. 3.18

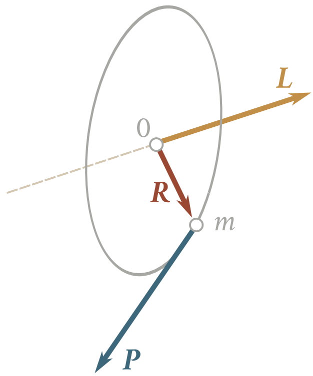

Fig. 3.19

change only in magnitude. The magnitude of the angular momentum is

$$L = mvl \tag{3.110}$$

the arm l remaining constant here.

2. A particle of mass m moves along a circle of radius R (Fig. 3.19). The magnitude of the angular momentum of the particle relative to the centre of the circle 0 is

$$L = mvR \tag{3.111}$$

The vector $\boldsymbol{L}$ is perpendicular to the plane of the circle. The direction of motion of the particle and the vector $\boldsymbol{L}$ form a right-handed system. Since the arm, which equals R, remains constant, the angular momentum can change only as a result of a change in the magnitude of the velocity. Upon uniform motion of the particle along the circle, the angular momentum remains constant both in magnitude and in direction.

The pseudovector

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{3.112}$$

is called the moment of the force $\boldsymbol{F}$ relative to point 0 (or the torque relative to this point) from which the position vector $\boldsymbol{r}$ is drawn to the point of application of the force (Fig. 3.20). Inspection of the figure shows that the magnitude of the moment of the force can be written in the form

$$M = rF \sin \alpha = lF \tag{3.113}$$

where $l = r \sin \alpha$ is the arm of the force (the moment or lever arm) relative to point 0 (*i.e.*, the length of a perpendicular dropped from point 0 onto the straight line

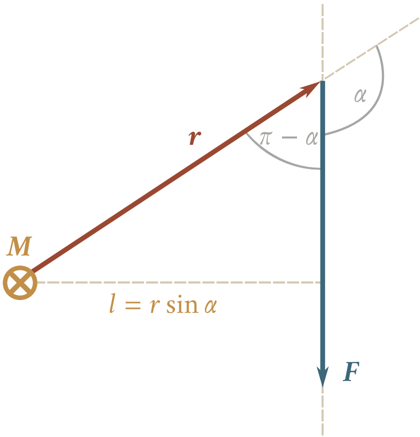

Fig. 3.20

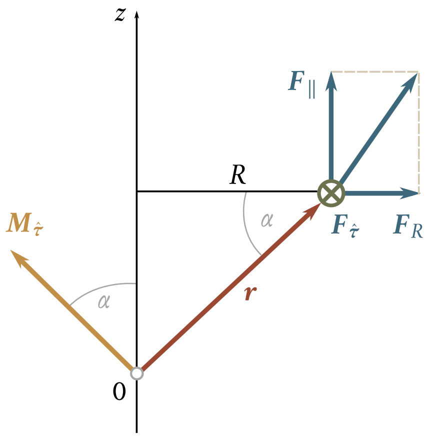

Fig. 3.21

along which the force acts).

The projection of the vector $\boldsymbol{M}$ onto an axis z passing through point 0 relative to which $\boldsymbol{M}$ has been determined is called the moment of the force (the torque) relative to this axis:

$$M_z = \boldsymbol{r} \times \boldsymbol{F}_{\mathrm{pr.},z}. \tag{3.114}$$

Let us resolve the force vector $\boldsymbol{F}$ (Fig. 3.21) into three mutually perpendicular components: $\boldsymbol{F}_{\|}$ parallel to the z-axis, $\boldsymbol{F}_R$ perpendicular to the z-axis and acting along a line passing through the axis, and, finally, $\boldsymbol{F}_{\hat{\tau}}$ perpendicular to the plane passing through the axis and the point of application of the force (this component is designated in the figure by a circle with a cross in it). If we imagine a circle of radius R with its centre on the z-axis, then the component $\boldsymbol{F}_{\hat{\tau}}$ will be directed along a tangent to this circle. The moment of the force $\boldsymbol{F}$ relative to point 0 equals the sum of the moments of the components: $\boldsymbol{M} = \boldsymbol{M}_{\|} + \boldsymbol{M}_R + \boldsymbol{M}_{\hat{\tau}}$. The vectors $\boldsymbol{M}_{\|}$, and $\boldsymbol{M}_R$ are perpendicular to the z-axis, therefore their projections onto this axis equal zero. The moment $\boldsymbol{M}_{\hat{\tau}}$ has a magnitude equal to $rF_{\hat{\tau}}$ and makes the angle α with the z-axis. The cosine of α is R/r. Hence, the moment of the component $\boldsymbol{F}_{\hat{\tau}}$ relative to the z-axis has the magnitude $\boldsymbol{M}_{\hat{\tau}} \cos\alpha = RF_{\hat{\tau}}$. The moment of the force $\boldsymbol{F}$ relative to the z-axis thus equals

$$M_z = RF_{\hat{\tau}}. \tag{3.115}$$

Up to now, we understood $F_{\hat{\tau}}$ to stand for the magnitude of the component $\boldsymbol{F}_{\hat{\tau}}$. But $F_{\hat{\tau}}$ can also be considered as the projection of the vector $\boldsymbol{F}$ onto the unit vector $\hat{\boldsymbol{\tau}}$ that is tangent to a circle of radius R and is directed so that motion along the circle in the direction of $\hat{\boldsymbol{\tau}}$ forms a right-handed system with the direction of the z-axis.

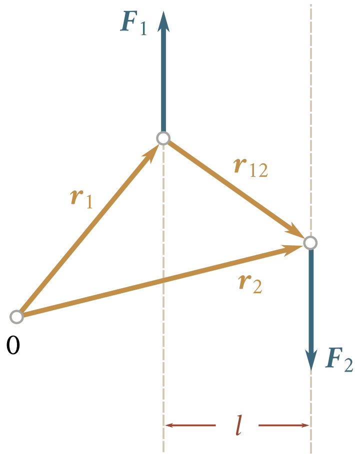

Fig. 3.22

With such an interpretation of $F_{\hat{\tau}}$, Eq. (3.115) will also determine the sign of M_z.

The moment $\boldsymbol{M}$ of a force characterizes the ability of the force to rotate a body about the point relative to which it is taken. We must note that when a body can rotate arbitrarily relative to point 0, the force will cause it to rotate about an axis that is perpendicular to the plane containing the force and point 0, *i.e.*, about an axis coinciding with the direction of the moment of the force relative to the given point.

The moment of a force relative to the z-axis characterizes the ability of the force to rotate a body about this axis. The components $\boldsymbol{F}_{\parallel}$ and $\boldsymbol{F}_R$ cannot cause rotation about the z-axis. Such rotation can be produced only by the component $\boldsymbol{F}_{\hat{\tau}}$, and the success of the rotation will grow with an increasing moment arm R.

Two equal, parallel and oppositely directed forces are called **a force couple** (Fig. 3.22). The distance l between the lines along which the forces act is called the arm of the couple. The total moment of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ forming the couple is

$$\boldsymbol{M} = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2.$$

Since $\boldsymbol{F}_1 = -\boldsymbol{F}_2$, we can write

$$\boldsymbol{M} = -\boldsymbol{r}_1 \times \boldsymbol{F}_2 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \boldsymbol{F}_2 = \boldsymbol{r}_{12} \times \boldsymbol{F}_2 \tag{3.116}$$

where $\boldsymbol{r}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the vector drawn from the point of application of the force $\boldsymbol{F}_1$ to the point of application of $\boldsymbol{F}_2$. Equation (3.116) does not depend on the choice of point 0. Consequently, the moment of a force couple relative to any point will be the same. The vector of the moment of a force couple is perpendicular to the plane containing the forces (see Fig. 3.22) and numerically equals the product of the magnitude of any of the forces and the arm.

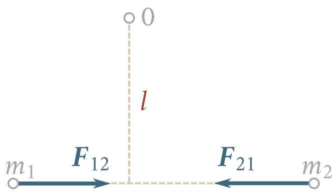

Fig. 3.23

Forces of interaction between particles act in opposite directions along the same straight line (Fig. 3.23). Their moments relative to an arbitrary point 0 are equal in magnitude and opposite in direction. Therefore, the moments of the internal forces balance one another in pairs, and the sum of the moments of all the internal forces for any system of particles, particularly for a solid body, always equals zero:

$$\sum \boldsymbol{M}_{\text{int}} = 0. \tag{3.117}$$

In accordance with definitions (3.106) and (3.112), we can write Eq. (3.104) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \sum \boldsymbol{M}_{\text{ext}}. \tag{3.118}$$

This equation is similar to Eq. (3.88). A comparison of these equations shows that just like the time derivative of the momentum of a system equals the sum of the external forces, so does the time derivative of the angular momentum equal the sum of the moments of the external forces.

It follows from Eq. (3.118) that in the absence of external forces $\mathrm{d}\boldsymbol{L}/\mathrm{d}t$. Hence, $\boldsymbol{L}$ is constant for a closed system. This statement is the content of the **law of angular momentum conservation**, which is formulated as follows: *the angular momentum of a closed system of point particles remains constant.*

We have proved Eq. (3.118) for a system of two particles. It can be generalized quite simply, however, for any number of particles. Let us write the equations of motion of the particles:

$$m_1\dot{\boldsymbol{v}}_1 = \sum_k \boldsymbol{F}_{1k} + \boldsymbol{F}_1$$

$$\ldots \quad \ldots \quad \ldots$$

$$m_i\dot{\boldsymbol{v}}_i = \sum_k \boldsymbol{F}_{ik} + \boldsymbol{F}_i$$

$$\ldots \quad \ldots \quad \ldots$$

$$m_N\dot{\boldsymbol{v}}_N = \sum_k \boldsymbol{F}_{Nk} + \boldsymbol{F}_N$$

Multiplying each of the equations by the corresponding position vector, we get [see Eq. (3.102)]:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_1 \times \boldsymbol{p}_1) &= \sum_k \boldsymbol{r}_1 \times \boldsymbol{F}_{1k} + \boldsymbol{r}_1 \times \boldsymbol{F}_1 \\
\ldots \quad \ldots \quad & \ldots \quad \ldots \quad \ldots \\
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_i \times \boldsymbol{p}_i) &= \sum_k \boldsymbol{r}_i \times \boldsymbol{F}_{ik} + \boldsymbol{r}_i \times \boldsymbol{F}_i \\
\ldots \quad \ldots \quad & \ldots \quad \ldots \quad \ldots \\
\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}_N \times \boldsymbol{p}_N) &= \sum_k \boldsymbol{r}_N \times \boldsymbol{F}_{Nk} + \boldsymbol{r}_N \times \boldsymbol{F}_N.
\end{aligned}$$

Let us add up all the N equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \boldsymbol{L}_i = \sum_{\substack{i=k \\ (i \neq k)}} \boldsymbol{r}_i \times \boldsymbol{F}_{ik} + \boldsymbol{r}_i \times \boldsymbol{F}_i.$$

The first sum in the right-hand side is the sum of the moments of all the internal forces, which, as we have shown, equals zero [see Eq. (3.117)]. The second sum in the right-hand side is the sum of the moments of the external forces. Consequently, we have arrived at Eq. (3.118).

We must note that the angular momentum also remains constant for an unclosed system provided that the total moment of the external forces equals zero [see Eq. (3.118)].

Projection of all the quantities in Eq. (3.118) onto a certain direction z yields

$$\frac{\mathrm{d}}{\mathrm{d}t}L_z = \sum M_{z,\mathrm{ext}} \tag{3.119}$$

according to which the time derivative of the angular momentum of the system relative to the z-axis equals the sum of the moments of the external forces relative to this axis.

It follows from Eq. (3.119) that when the sum of the moments of the external forces relative to an axis equals zero, the angular momentum of the system relative to this axis remains constant.

3.13. Motion in a Central Force Field

Let us consider a particle in a central force field. We remind our reader that the direction of the force acting on a particle at any point of such a field passes through point 0—the centre of the field—while the magnitude of the force depends only on the distance from this centre. It is easy to see that the dependence of the force

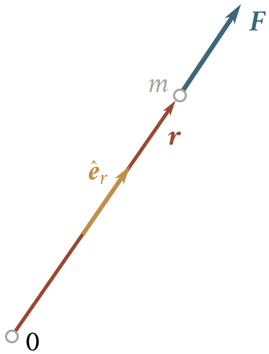

Fig. 3.24

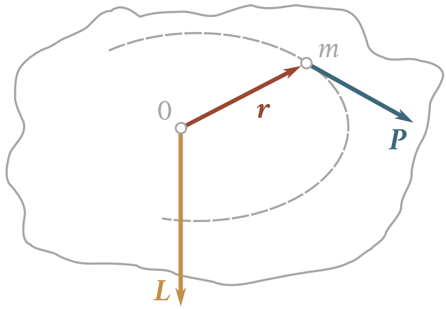

Fig. 3.25

$\boldsymbol{F}$ on $\boldsymbol{r}$ has the form

$$\boldsymbol{F} = f(r)\hat{\boldsymbol{e}}_r \tag{3.120}$$

where $\hat{\boldsymbol{e}}_r$ is the unit vector of the position vector (Fig. 3.24), and $f(r)$ is the projection of the force vector onto the direction of the position vector, *i.e.*, F_r. The function $f(r)$ is positive for a repulsive force, and negative for an attractive one. Figure 3.24 has been drawn for the case of repulsion of a particle from the force centre. Equation (3.120) naturally holds only if the origin of coordinates (*i.e.*, the point from which the position vectors are drawn) is at the centre of the field.

The moment of the force (3.120) relative to point 0 obviously equals zero. This follows from the fact that the moment arm equals zero. Hence, in accordance with Eq. (3.118), we see that the angular momentum of a particle moving in a central force field remains constant. The vector $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ at each moment of time is perpendicular to the plane formed by the vectors $\boldsymbol{r}$ and $\boldsymbol{p}$ (Fig. 3.25). If $\boldsymbol{L} =$ constant, this plane will be fixed. Thus, when a particle moves in a central force field, its position vector always remains in one plane. The vector $\boldsymbol{p}$ is also permanently in the same plane. Consequently, the trajectory of the particle is a plane curve. The plane containing the trajectory passes through the centre of the field (see Fig. 3.25).

Figure 3.26 shows a portion of the trajectory of the particle (the vector $\boldsymbol{L}$ is directed beyond the drawing). During the time dt, the position vector of the particle describes the shaded area dS. This area equals half the area of the parallelogram constructed on the vectors $\boldsymbol{r}$ and $\boldsymbol{v}\,\mathrm{d}t$. The area of the parallelogram, in turn, equals the magnitude of the vector product $\boldsymbol{r}\times\boldsymbol{v}\,\mathrm{d}t$ [see the text following Eq. (1.28)]. Thus, the area of the shaded triangle is

$$\mathrm{d}S = \frac{1}{2}|\boldsymbol{r} \times \boldsymbol{v}|\,\mathrm{d}t = \frac{1}{2m}|\boldsymbol{r} \times \boldsymbol{p}|\,\mathrm{d}t = \frac{1}{2m}L\,\mathrm{d}t$$

(we have put the scalar multiplier dt outside the symbol of the vector product).

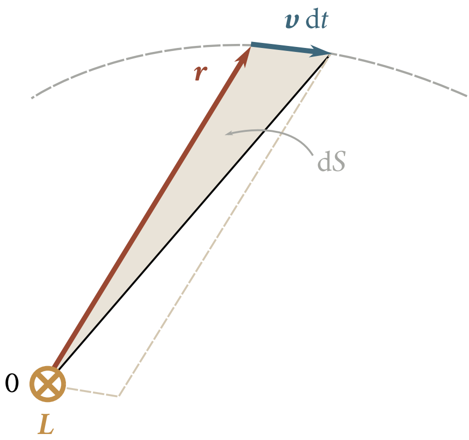

Fig. 3.26

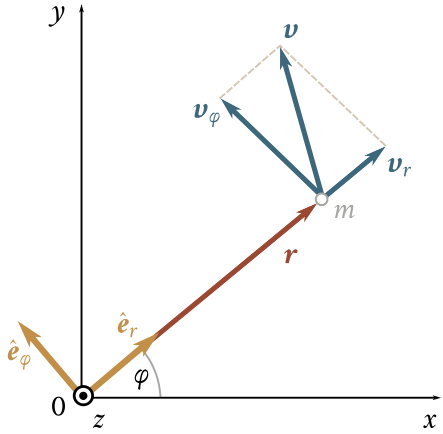

Fig. 3.27

Dividing both sides of the equation obtained by dt, we find that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}2m}. \tag{3.121}$$

The quantity $\mathrm{d}S/\mathrm{d}t$, *i.e.*, the area described by the position vector of a particle in unit time, is called the **sector velocity**. In a central force field, L = constant, hence the sector velocity of a particle also remains constant.

Let us find an expression for the angular momentum of a particle in the polar coordinates r and φ (Fig. 3.27). According to Eqs. (1.68)-(1.71), the vector velocity of the particle can he represented in the form

$$\boldsymbol{v} = \boldsymbol{v}_r + \boldsymbol{\varphi} = \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\varphi}\hat{\boldsymbol{e}}_\varphi. \tag{3.122}$$

Using this expression in the equation for $\boldsymbol{L}$, we get

$$\boldsymbol{L} = m(\boldsymbol{r} \times \boldsymbol{v}) = m(\boldsymbol{r} \times \boldsymbol{v}_r) + m(\boldsymbol{r} \times \boldsymbol{v}_\varphi).$$

The vectors $\boldsymbol{r}$ and $\boldsymbol{v}_r$ are collinear, therefore the first addend equals zero. Consequently

$$\boldsymbol{L} = m(\boldsymbol{r} \times \boldsymbol{v}_\varphi) = m(\boldsymbol{r} \times r\dot{\varphi}\hat{\boldsymbol{e}}_\varphi) = mr\dot{\varphi}(\boldsymbol{r} \times \hat{\boldsymbol{e}}_\varphi).$$

The vector product $\boldsymbol{r} \times \hat{\boldsymbol{e}}_\varphi$ equals $r\hat{\boldsymbol{e}}_z$, where $\hat{\boldsymbol{e}}_z$, is the unit vector of the z-axis (in Fig. 3.27 this unit vector is directed toward us). Thus,

$$\boldsymbol{L} = mr^2\dot{\varphi}\hat{\boldsymbol{e}}_z. \tag{3.123}$$

Hence we conclude that

$$L_z = mr^2\dot{\varphi} \tag{3.124}$$

where L_z is the projection of the angular momentum onto the z-axis. The magnitude of the angular momentum equals the magnitude of Eq. (3.124).

Let us now turn to the energy of a particle. Central forces are conservative (see Sec. 3.4). According to Eq. (3.30), the work of a conservative force equals the decrement of the potential energy of the particle E_p. Hence, for the force (3.120), the relation $\mathrm{d}A = -\mathrm{d}E_\mathrm{p}$ holds, *i.e.*,

$$\mathrm{d}E_\mathrm{p} = -\mathrm{d}A = f(r)\hat{\boldsymbol{e}}_r\,\mathrm{d}\boldsymbol{r} = -f(r)\,\mathrm{d}r.$$

Integration of this expression yields

$$E_\mathrm{p} = -\int f(r)\,\mathrm{d}r. \tag{3.125}$$

It follows from Eq. (3.125) that the potential energy of a particle in a field of central forces depends only on the distance r from the centre: $E_\mathrm{p} = E_\mathrm{p}(r)$.

Of special interest are forces inversely proportional to the square of the distance from the force centre. The function $f(r)$ in Eq. (3.120) has the following form for them:

$$f(r) = \frac{\alpha}{r^2} \tag{3.126}$$

where α is a constant quantity ($\alpha > 0$ corresponds to repulsion from the centre, and $\alpha < 0$ to attraction to the centre). Among such forces are gravitational and Coulomb ones.

Introduction of the function (3.126) into Eq. (3.125) yields

$$E_\mathrm{p} = -\alpha\int\frac{\mathrm{d}r}{r^2} = \frac{\alpha}{r} + C$$

where C is an integration constant. The potential energy is conventionally considered to vanish at infinity (*i.e.*, at $r = \infty$). In this condition, $C = 0$, and

$$E_\mathrm{p} = \frac{\alpha}{r}. \tag{3.127}$$

Thus, the total mechanical energy of a particle moving in a central field of forces that are inversely proportional to the square of the distance from the centre is determined by the expression

$$E = \frac{mv^2}{2} + \frac{\alpha}{r}. \tag{3.128}$$

Substituting the sum of the squares of the velocities $\boldsymbol{v}_r$ and $\boldsymbol{v}_\varphi$, for the square of the velocity $\boldsymbol{v}$ in accordance with Eq. (3.122), *i.e.*, substituting the expression $r^2 + r^2\dot{\varphi}^2$ for v^2, we obtain

$$E = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\varphi}^2}{2} + \frac{\alpha}{r}. \tag{3.129}$$

The energy and the angular momentum of a particle are conserved in a central field. Consequently, the left-hand sides of Eqs. (3.124) and (3.129) are constants. We

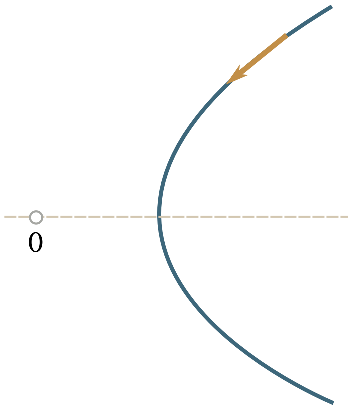

Fig. 3.28

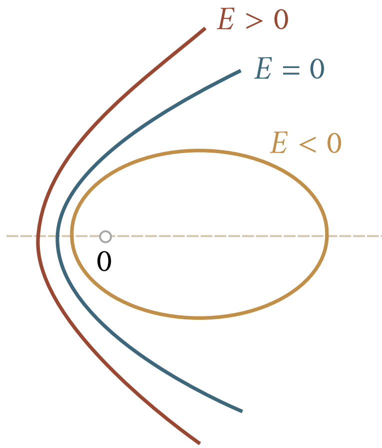

Fig. 3.29

have thus arrived at a system of two differential equations:

$$\begin{cases} mr^2\dot{\varphi}^2 = L_z = \text{constant} \\ m\dot{r}^2 + mr^2\dot{\varphi}^2 + \dfrac{2\alpha}{r} = 2E = \text{constant.} \end{cases} \tag{3.130}$$

After integrating these equations, we can find r and $\varphi 0$ as functions of t, *i.e.*, the trajectory and the nature of motion of the particle. It must be noted that Eqs. (3.130) contain the first time derivatives of r and φ. They are, therefore, much easier to solve than equations following from Newton's laws, which contain the second derivatives of the coordinates.

Solution of the system of equations (3.130) is beyond the scope of this book. We shall limit ourselves to giving the final result. The trajectory of the particle is a conical section, *i.e.*, an ellipse, or a parabola, or a hyperbola. Which of these curves is observed in a given concrete case depends on the sign of the constant α and the magnitude of the total energy of the particle.

For repulsion (*i.e.*, when $\alpha > 0$), the trajectory of the particle can only be a hyperbola (Fig. 3.28). If $L_z = 0$, the hyperbola degenerates into a straight line whose continuation passes through the force centre. We must note that when $\alpha > 0$, the total energy (3.128) cannot be negative.

For attraction (*i.e.*, when $alpha < 0$), the total energy may be either positive or negative; in particular, it may equal zero. When $E > 0$, the trajectory is a hyperbola (Fig. 3.29). When $E = 0$, the trajectory will be a parabola. This case takes place if a particle begins its motion from a state of rest at infinity (see Eq. (3.128)]. Finally, when $E < 0$, the trajectory will be an ellipse. At values of the energy and the angular momentum complying with the condition that $E = -m\alpha^2/(2L^2)$, the ellipse degenerates into a circle.

Motion along an ellipse is finite, and that along a parabola or hyperbola—infinite (see Sec. 3.9).

3.14. Two-Body Problem

A two-body problem is the name given to a problem on the motion of two interacting particles. The system formed by the particles is assumed to be closed. We learned in Sec. 3.10 that the centre of mass of a closed system is either at rest or moves uniformly in a straight line. We shall solve the problem in a centre-of-mass frame (a c.m. frame), placing the origin of coordinates at point C. In this case, $r_C = (m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2)/(m_1 + m_2) = 0$, *i.e.*,

$$m_1 \boldsymbol{r}_1 = -m_2 \boldsymbol{r}_2 \tag{3.131}$$

(Fig. 3.30a). Let us introduce the vector

$$\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \tag{3.132}$$

determining the whereabouts of the second particle relative to the first one (Fig. 3.30b). By simultaneously solving Eqs. (3.131) and (3.132), it is easy to find that

$$\boldsymbol{r}_1 = -\frac{m_2}{m_1 + m_2}\boldsymbol{r}, \quad \boldsymbol{r}_2 = \frac{m_1}{m_1 + m_2}\boldsymbol{r}. \tag{3.133}$$

Similarly to Eq. (3.59), we can write that $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} = f(r)\hat{\boldsymbol{e}}_r$, where $f(r)$ is a function of the distance between the particles. It is positive for forces of attraction (Fig. 3.30c) and negative for forces of repulsion. Let us write the equations of motion of our particles:

$$m_1 \ddot{\boldsymbol{r}}_1 = f(r)\hat{\boldsymbol{e}}_r, \quad m_2 \ddot{\boldsymbol{r}}_2 = -f(r)\hat{\boldsymbol{e}}_r$$

Division of the first equation by m_1, of the second one by m_2, and subtraction of the first equation from the second yield

$$\ddot{\boldsymbol{r}}_1 - \ddot{\boldsymbol{r}}_2 = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{\boldsymbol{e}}_r.$$

According to Eq. (3.132), the left-hand side is $\boldsymbol{r}$. Hence,

$$\ddot{\boldsymbol{r}} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{\boldsymbol{e}}_r. \tag{3.134}$$

Equation (3.134) can formally be considered as the equation of motion of an imaginary particle in a central force field. The position of the particle relative to the force centre is determined by the position vector $\boldsymbol{r}$. According to Eq. (3.134), the mass μ determined by the condition that

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{3.135}$$

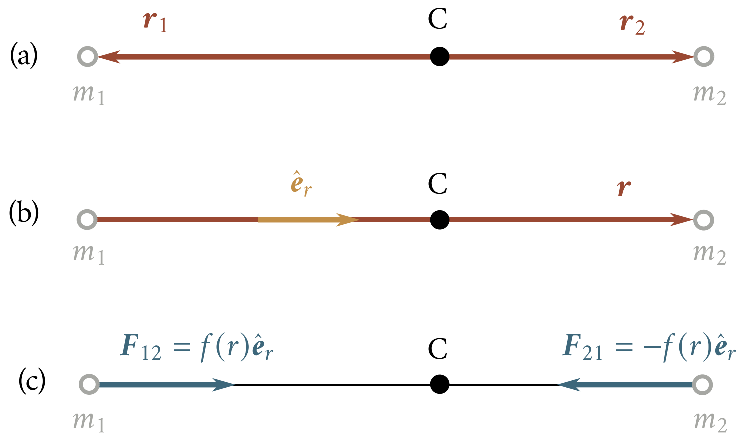

Fig. 3.30

must be ascribed to our imaginary particle. Hence,

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{3.136}$$

The quantity (3.136) is called the **reduced mass** of the particles.

A two-body problem thus consists in a problem on the motion of a single particle in a central force field. Finding $\boldsymbol{r}$ as a function of t from Eq. (3.134), we can use Eqs. (3.133) to determine $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$. The vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are laid off from the centre of mass C of the system. Therefore, to be able to use Eqs. (3.133), we must also lay off the position vector $\boldsymbol{r}$ of the imaginary particle from point C [for real particles the vector (3.132) is drawn from the first particle to the second one].

It can be seen from Eqs. (3.133) and Fig. 3.30 that both particles move relative to the centre of mass along geometrically similar trajectories[8]. The straight line joining the particles constantly passes through the centre of mass.

[8]When the force of interaction is inversely proportional to the square of the distance between the particles, these trajectories are ellipses, or parabolas, or hyperbolas (see Sec. 3.13).

Chapter 4

NON-INERTIAL REFERENCE FRAMES

4.1. Forces of Inertia

Newton's laws are obeyed only in inertial reference frames. A given body travels with the same acceleration a relative to all inertial frames. Any non-inertial reference frame travels with a certain acceleration relative to inertial frames, therefore the acceleration of a body in a non-inertial reference frame $\boldsymbol{a}'$ will differ from $\boldsymbol{a}$. Let us use the symbol $\boldsymbol{a}_0$ to denote the difference between the accelerations of a body in an inertial and a non-inertial reference frame:

$$\boldsymbol{a} - \boldsymbol{a}' = \boldsymbol{a}_0. \tag{4.1}$$

For a non-inertial frame in translational motion, $\boldsymbol{a}_0$ is the same for all points of space ($\boldsymbol{a}_0$ = constant) and is the acceleration of the non-inertial reference frame. For a rotating non-inertial frame, $\boldsymbol{a}_0$ will be different at different points of space [$\boldsymbol{a}_0 = \boldsymbol{a}_0(\boldsymbol{r}')$, where $\boldsymbol{r}'$ is the position vector determining the position of a point relative to the non-inertial reference frame].

Let the resultant of all the forces produced by the action of other bodies on the given body be $\boldsymbol{F}$. Hence, according to Newton's second law, the acceleration of the body relative to any inertial frame is

$$\boldsymbol{a} = \frac{1}{m}\boldsymbol{F}.$$

The acceleration of the body relative to a non-inertial frame, in accordance with Eq. (4.1), can be represented in the form

$$\boldsymbol{a}' = \boldsymbol{a} - \boldsymbol{a}_0 = \frac{1}{m}\boldsymbol{F} - \boldsymbol{a}_0.$$

Hence, it follows that even when $\boldsymbol{F} = 0$, the body will travel relative to the non-inertial reference frame with the acceleration—$\boldsymbol{a}_0$, *i.e.*, as if a force equal to —$m\boldsymbol{a}_0$ acted on it.

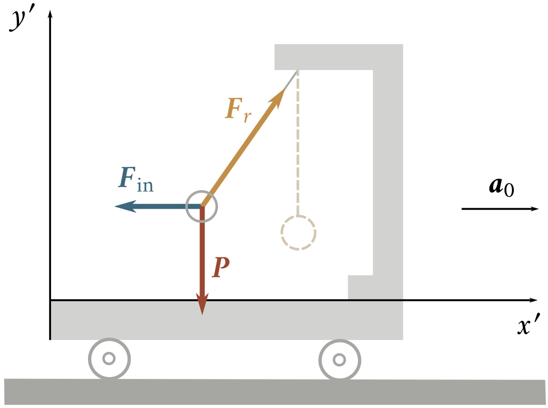

Fig. 4.1

What has been said above signifies that we can use Newton's equations in describing motion in non-inertial reference frames, if in addition to the forces due to the action of bodies on one another, we take into account the so-called **forces of inertia** $\boldsymbol{F}_{\text{in}}$. The latter should be assumed equal to the product of the mass of a body and the difference between its accelerations relative to the inertial and non-inertial reference frames taken with the opposite sign:

$$\boldsymbol{F}_{\text{in}} = -m(\boldsymbol{a} - \boldsymbol{a}') = m\boldsymbol{a}_0. \tag{4.2}$$

The equation of Newton's second law for a non-inertial reference frame will accordingly be

$$m\boldsymbol{a}' = \boldsymbol{F} + \boldsymbol{F}_{\text{in}}. \tag{4.3}$$

We shall explain our statement by the following example. Let us consider a cart with a bracket secured on it from which a ball is suspended on a string (Fig. 4.1). As long as the cart is at rest or is moving without acceleration, the string is vertical, and the force of gravity $\boldsymbol{P}$ is balanced by the reaction of the string $\boldsymbol{F}_{\text{r}}$. Now let us bring the cart into translational motion with the acceleration $\boldsymbol{a}_0$. The string will deviate from a vertical line through an angle such that the resultant of the forces $\boldsymbol{P}$ and $\boldsymbol{F}_{\text{r}}$ imparts an acceleration of $\boldsymbol{a}_0$ to the ball. The ball will be at rest relative to a reference frame associated with the cart, although the resultant of the forces $\boldsymbol{P}$ and $\boldsymbol{F}_{\text{r}}$ differs from zero. The absence of acceleration of the ball relative to this reference frame can be explained formally by the fact that in addition to the forces $\boldsymbol{P}$ and $\boldsymbol{F}_{\text{r}}$ whose sum equals $m\boldsymbol{a}_0$, the force of inertia $\boldsymbol{F}_{\text{in}} = -m\boldsymbol{a}_0$ also acts on the ball.

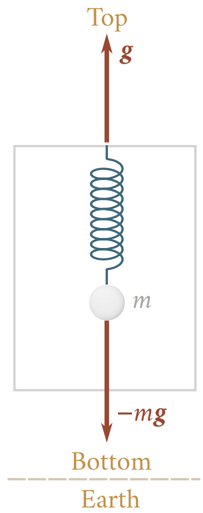

Fig. 4.2

The introduction of inertial forces permits us to describe the motion of bodies in any (both inertial and non-inertial) reference frames using the same equations of motion.

One must understand distinctly that the forces of inertia may never be treated on a par with such forces as elastic, gravitational, and friction ones, *i.e.*, with forces produced by the action on a body of other bodies. Forces of inertia are due to the properties of the reference frame in which mechanical phenomena are being considered. In this sense, they can be called fictitious forces.

The consideration of forces of inertia is not a necessity. Any motion, in principle, can always be considered relative to an inertial reference frame. In practice, however, it is exactly the motion of bodies relative to non-inertial reference frames, for instance, relative to the Earth's surface, that is often of interest to us. The use of inertial forces makes it possible to solve the relevant problem directly relative to such a reference frame, and this is frequently much simpler than consideration of the motion in an inertial frame.

A feature of inertial forces is that they are proportional to the mass of a body. Owing to this property, inertial forces are similar to gravitational ones. Imagine that we are in a closed cab removed from all external bodies and moving with the acceleration $\boldsymbol{g}$ in the direction which we shall call the "top" (Fig. 4.2). All the bodies in the cab will behave as if they experienced the force of inertia $-m\boldsymbol{g}$. In particular, a spring to whose end a body of mass m is fastened will stretch so that the

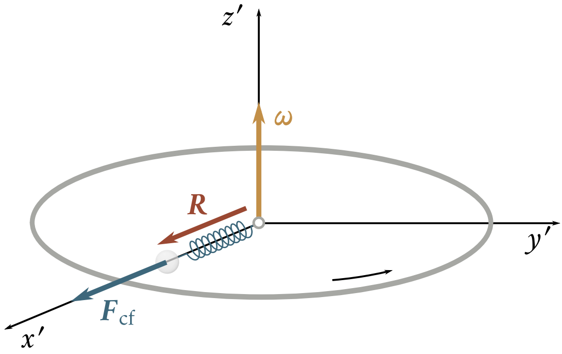

Fig. 4.3

elastic force balances the force of inertia $m\boldsymbol{g}$. The same phenomena will be observed, however, when the cab is stationary and is near the Earth's surface. Having no possibility of "looking out" of the cab, we would not be able to establish by any experiments conducted in the cab whether the force $-m\boldsymbol{g}$ is due to its accelerated motion or to the action of the Earth's gravitational field. On these grounds, we speak of the equivalence of forces of inertia and gravitation. This equivalence underlies Albert Einstein's general theory of relativity.

4.2. Centrifugal Force of Inertia

Let us consider a disk rotating about a vertical axis z' perpendicular to it with the angular velocity ω (Fig. 4.3). A ball fitted onto a spoke and connected to the centre of the disk by a spring rotates together with the disk. The ball occupies a position on the spoke such that the force $\boldsymbol{F}_{\rm spr}$ stretching the spring is equal to the product of the mass of the ball m and its acceleration $\boldsymbol{a}_n = -\omega^2 \boldsymbol{R}$ [see Eq. (1.102); $\boldsymbol{R}$ is a position vector drawn to the ball from the centre of the disk. Its magnitude R gives the distance from the centre of the disk to the ball]:

$$\boldsymbol{F}_{\rm spr} = -m\omega^2 \boldsymbol{R}. \tag{4.4}$$

The ball is at rest relative to the reference frame associated with the disk. This can be formally explained by the circumstance that apart from the force (4.4), the ball experiences the force of inertia

$$\boldsymbol{F}_{\rm cf} = m\omega^2 \boldsymbol{R}. \tag{4.5}$$

directed along a radius from the centre of the disk.

The force of inertia (4.5) set up in a rotating (relative to inertial frames) refer-

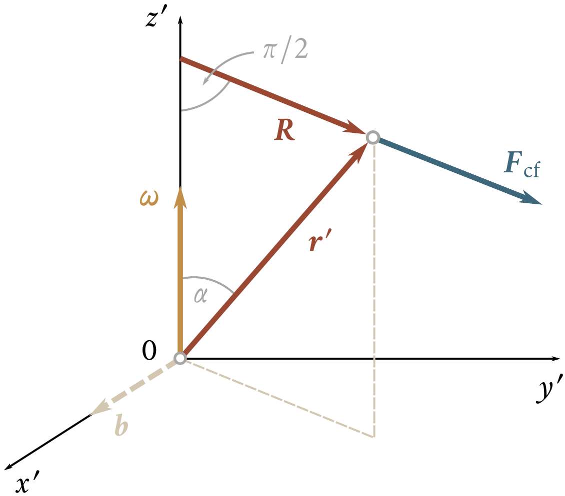

Fig. 4.4

ence frame is called the centrifugal force of inertia. This force acts on a body in a rotating reference frame regardless of whether the body is at rest in this frame (as we have assumed up to now) or is moving relative to it with the velocity $\boldsymbol{v}'$.

If the position of a body in a rotating reference frame is characterized by the position vector $\boldsymbol{r}'$, then the centrifugal force of inertia can be represented in the form of a vector triple product

$$\boldsymbol{F}_{\text{cf}} = m[\boldsymbol{\omega} \times (\boldsymbol{r}' \times \boldsymbol{\omega})]. \tag{4.6}$$

Indeed, the vector $\boldsymbol{b} = \boldsymbol{r}' \times \boldsymbol{\omega}$ is directed at right angles to the vectors $\boldsymbol{\omega}$ and $\boldsymbol{F}_{\text{cf}}$ "toward us" (Fig. 4.4), and its magnitude is $\omega r' \sin\alpha = \omega R$. The vector product of the mutually perpendicular vectors $m\boldsymbol{\omega}$ and $\boldsymbol{b}$ coincides in direction with $\boldsymbol{F}_{\text{cf}}$, and its magnitude is $m\omega b = m\omega^2 R = \boldsymbol{F}_{\text{cf}}$.

In the accurate solution of problems on the motion of bodies relative to the Earth's surface, account must be taken of the centrifugal force of inertia equal to $m\omega^2 R$, where m is the mass of a body, ω is the angular velocity of the Earth in its rotation about its axis, and R is the distance to the body from the Earth's axis (Fig. 4.5). When the height of bodies above the Earth's surface (their altitude) is not great, we may assume that $R = R_{\text{E}} \cos\varphi$ (R_{E} is the Earth's radius, and φ is the latitude of the locality). The expression for the centrifugal force of inertia thus becomes

$$F_{\text{cf}} = m\omega^2 R_{\text{E}} \cos\varphi. \tag{4.7}$$

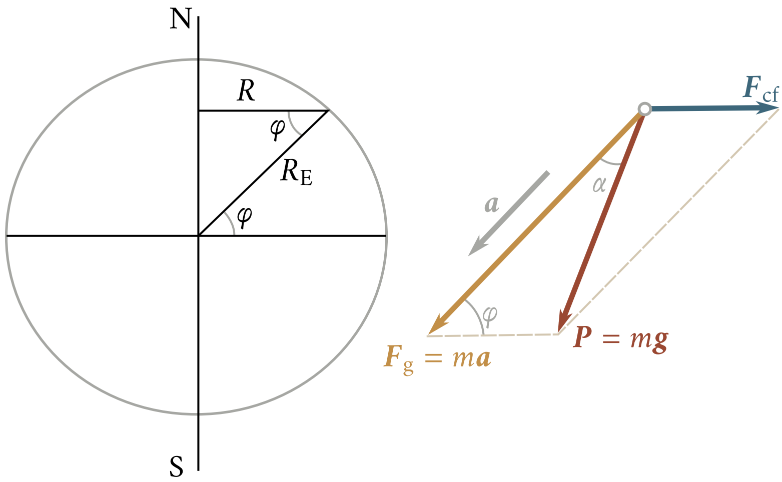

Fig. 4.5

The acceleration of free fall of bodies $\boldsymbol{g}$ observed relative to the Earth is due to the action of the force $\boldsymbol{F}_\mathrm{g}$ with which a body is attracted by the Earth, and of the force $\boldsymbol{F}_\mathrm{cf}$. The resultant of these forces

$$\boldsymbol{P} = \boldsymbol{F}_\mathrm{g} + \boldsymbol{F}_\mathrm{cf} \tag{4.8}$$

is the force of gravity equal to $m\boldsymbol{g}$ [see Eq. (2.38)].

The difference between the force of gravity $\boldsymbol{P}$ and the force of attraction to the Earth $\boldsymbol{F}_\mathrm{g}$ is not great because the centrifugal force of inertia is much smaller than $\boldsymbol{F}_\mathrm{g}$. Thus, for a mass of 1 kg, the maximum value of F_cf observed at the equator is

$$m\omega^2 R_\mathrm{E} = 1 \times \left(\frac{2\pi}{86400}\right)^2 \times 6.4 \times 10^6 = 0.035\ \mathrm{N}$$

whereas $\boldsymbol{F}_\mathrm{g}$ approximately equals 9.8 N, *i.e.*, is almost 300 times greater.

The angle α between the directions of $\boldsymbol{F}_\mathrm{g}$ and $\boldsymbol{P}$ (see Fig. 4.5) can be found by using the theorem of sines:

$$\frac{\sin\alpha}{\sin\varphi} = \frac{F_\mathrm{g}}{P} = \frac{m\omega^2 R_\mathrm{E}\cos\varphi}{mg} \approx \frac{0.035\ \mathrm{N}}{9.8\ \mathrm{N}}\cos\varphi \approx 0.0035\cos\varphi$$

whence

$$\sin\alpha \approx 0.0035\sin\varphi\cos\varphi \approx 0.0018\sin 2\varphi.$$

The sine of a small angle may be approximately replaced by the value of the angle itself. Such approximation yields

$$\alpha \approx 0.0018\sin 2\varphi. \tag{4.9}$$

Thus, the angle α varies within the limits from zero (at the equator, where $\varphi = 0$, and at the poles, where $\varphi = 90°$) to 0.0018 rad or 6′ (at a latitude of 45°).

The direction of the force $\boldsymbol{P}$ coincides with that of a string tensioned by a weight, which is called the direction of a plumb or the vertical direction. The force $\boldsymbol{F}_{\mathrm{g}}$ is directed toward the centre of the Earth. Therefore, a vertical line is directed toward the centre of the Earth only at the poles and the equator, and deviates at intermediate latitudes by the angle α determined by expression (4.9).

The difference $F_{\mathrm{g}} - P$ vanishes at the poles and reaches a maximum equalling 0.3% of the force F_{g} at the equator. Owing to the oblateness of the Earth, the force F_{g} varies somewhat with the latitude, being about 0.2% less at the equator than at the poles. As a result, the acceleration of free fall varies with the latitude within the limits from 9.780 m s^{-2} at the equator to 9.832 m s^{-2} at the poles. The value of $g = 9.80665$ m s^{-2} is taken as the standard one.

We must note that a freely falling body moves relative to an inertial, for example, a heliocentric, reference frame with the acceleration $\boldsymbol{a} = \boldsymbol{F}_{\mathrm{g}}/m$ (and not $\boldsymbol{g}$). A glance at Fig. 4.5 shows that from the equality of the acceleration g for different bodies we get the equality of the accelerations a. Indeed, the triangles constructed on the vectors $\boldsymbol{F}_{\mathrm{g}}$ and $\boldsymbol{P}$ for different bodies are similar (the angles α and φ for all bodies at the given point on the Earth's surface are identical). Consequently, the ratio F_{g}/P, which coincides with the ratio a/g is the same for all the bodies. Hence, it follows that we get identical values of a for the same g's.

4.3. Coriolis Force

When a body moves relative to a rotating reference frame, another force called the **Coriolis force** appears in addition to the centrifugal force of inertia.

The appearance of a Coriolis force can be detected in the following experiment. Let us take a horizontally arranged disk that can rotate about a vertical axis. We draw radial line OA on the disk (Fig. 4.6a). Let us launch a ball with the velocity $\boldsymbol{v}'$ in the direction from 0 to A. If the disk does not rotate, the ball will roll along the radius we have drawn. If the disk is rotated in the direction shown by the arrow, however, then the ball will roll along dash curve 0B, and its velocity relative to the disk $\boldsymbol{v}'$ will change its direction. Consequently, the ball behaves relative to the rotating reference frame as if it experiences the force $\boldsymbol{F}_{\mathrm{Cor}}$ perpendicular to the velocity $\boldsymbol{v}'$.

To make the ball roll on the rotating disk along the radius, we must install a guide, for instance, in the form of rib 0A (Fig. 4.6b). When the ball is rolling, the guide rib exerts the force $\boldsymbol{F}_{\mathrm{rib}}$ on it. The ball travels with a velocity constant in

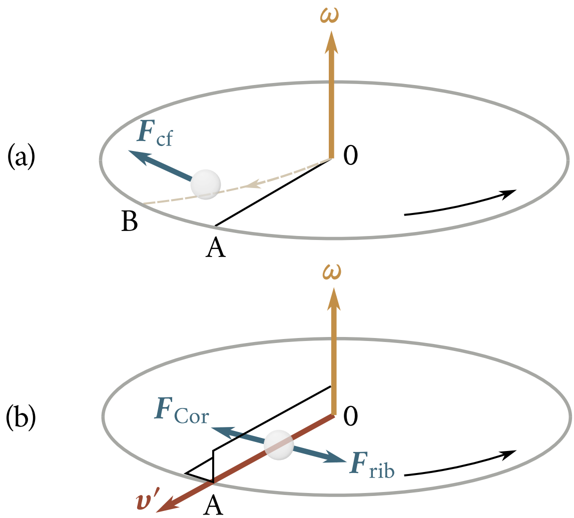

Fig. 4.6

direction relative to the rotating frame (disk). This can formally be explained by the fact that the force $\boldsymbol{F}_{\text{rib}}$ is balanced by the force of inertia $\boldsymbol{F}_{\text{Cor}}$ applied to the ball at right angles to the velocity $\boldsymbol{v}'$. It is exactly the force $\boldsymbol{F}_{\text{Cor}}$ that is the Coriolis force.

Let us first find an expression for the Coriolis force in the particular case when a particle m moves relative to a rotating reference frame uniformly along a circle in a plane perpendicular to the axis of rotation with its centre on this axis (Fig. 4.7). Let $\boldsymbol{v}'$ stand for the velocity of the particle relative to the rotating frame. The velocity $\boldsymbol{v}$ of the particle relative to a fixed (inertial) reference frame has the magnitude $v' + \omega R$ in case (a) and $|v - \omega R|$ in case (b), where ω is the angular velocity of the rotating frame, and R is the radius of the circle (see Eq. (1.99)].

For the particle to move relative to the fixed frame along a circle with the velocity $v = v' + \omega R$, it must experience the force $\boldsymbol{F}$ directed toward the centre of the circle, for example, the force of tension of the string by means of which the particle is tied to the centre of the circle (see Fig. 4.7a). The magnitude of this force is

$$F = ma_{\hat{\boldsymbol{n}}} = \frac{mv^2}{R} = \frac{m(v' + \omega R)^2}{R} = \frac{mv'^2}{R} + 2mv'\omega + m\omega^2 R. \tag{4.10}$$

The particle in this case moves relative to the rotating frame with the acceleration

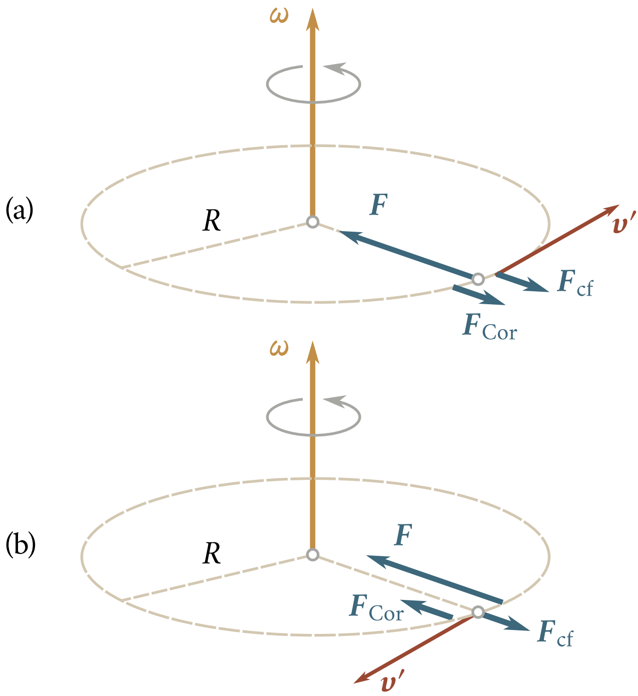

Fig. 4.7

$a'_{\hat{n}} = v'^2/R$, *i.e.*, as if it experienced the force

$$ma'_{\hat{n}} = \frac{mv^2}{R} = F - 2mv'\omega - m\omega^2 R \tag{4.11}$$

[see Eq. (4.10)]. Thus, the particle behaves in the rotating frame as if two other forces directed away from the centre acted on it in addition to the force $\boldsymbol{F}$ directed toward the centre. These two forces are $\boldsymbol{F}_{\text{cf}} = m\omega^2 R$ and $\boldsymbol{F}_{\text{Cor}}$ whose magnitude equals $2mv'\omega$ (Fig. 4.7a). It is easy to see that the force $\boldsymbol{F}_{\text{Cor}}$ can be represented in the form

$$\boldsymbol{F}_{\text{Cor}} = 2m(\boldsymbol{v}' \times \boldsymbol{\omega}). \tag{4.12}$$

The force (4.12) is exactly the Coriolis force. This force vanishes when $\boldsymbol{v}' = 0$. The force $\boldsymbol{F}_{\text{cf}}$ does not depend on $\boldsymbol{v}'$—as we have already noted, it acts both on bodies at rest and on moving ones.

For the case shown in Fig. 4.7b, we have

$$F = \frac{mv^2}{R} = \frac{m(v' - \omega R)^2}{R} = \frac{mv'^2}{R} - 2mv'\omega + m\omega^2 R.$$

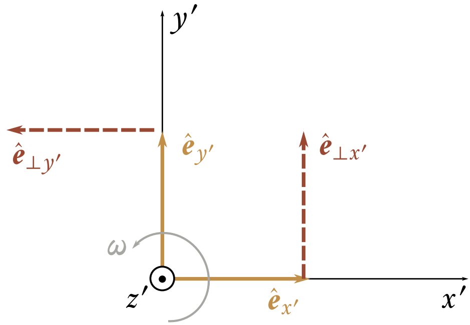

Fig. 4.8

Accordingly,

$$\frac{mv'^2}{R} = F + 2mv'\omega - m\omega^2 R.$$

Consequently, in a rotating frame, the particle behaves as if it experienced two forces $\boldsymbol{F}$ and $\boldsymbol{F}_{\text{Cor}}$ directed toward the centre of the circle, and also the force $\boldsymbol{F}_{\text{cf}} = m\omega^2 R$ directed away from the centre (see Fig. 4.7b). The force $\boldsymbol{F}_{\text{Cor}}$ in this case can be represented in the form of Eq. (4.12).

Now let us pass over to finding an expression for the Coriolis force when a particle moves arbitrarily relative to a rotating reference frame. Let us associate the coordinate axes x', y', z' with the rotating frame, and make the axis z' coincide with the axis of rotation (Fig. 4.8). The position vector of the particle can therefore be represented in the form

$$\boldsymbol{r}' = x'\hat{\boldsymbol{e}}'_x + y'\hat{\boldsymbol{e}}'_y + z'\hat{\boldsymbol{e}}'_z \tag{4.13}$$

where $\hat{\boldsymbol{e}}'_x$, $\hat{\boldsymbol{e}}'_y$ and $\hat{\boldsymbol{e}}'_z$ are the unit vectors of the coordinate axes. The unit vectors $\hat{\boldsymbol{e}}'_x$ and $\hat{\boldsymbol{e}}'_y$ rotate together with the reference frame with the angular velocity ω, whereas the unit vector $\hat{\boldsymbol{e}}'_z$ remains stationary.

The position of the particle relative to the fixed frame should be determined with the aid of the position vector $\boldsymbol{r}$. The symbols $\boldsymbol{r}'$ and $\boldsymbol{r}$, however, signify the same vector drawn from the origin of coordinates to the particle. An observer "living" in the rotating reference frame denoted this vector by $\boldsymbol{r}'$. According to his observations, the unit vectors $\hat{\boldsymbol{e}}'_x$, $\hat{\boldsymbol{e}}'_y$ and $\hat{\boldsymbol{e}}'_z$ are stationary, therefore when differentiating Eq. (4.13), he treats these unit vectors as if they are constants. A stationary observer uses the symbol $\boldsymbol{r}$. For him, the unit vectors $\hat{\boldsymbol{e}}'_x$ and $\hat{\boldsymbol{e}}'_y$ rotate with the velocity ω (the unit vector $\hat{\boldsymbol{e}}'_z$ is stationary). Therefore, when differentiating the expression (4.13) equal to $\boldsymbol{r}$, he must treat $\hat{\boldsymbol{e}}'_x$ and $\hat{\boldsymbol{e}}'_y$ as functions of t whose derivatives are

$$\dot{\hat{\boldsymbol{e}}}'_x = \omega \hat{\boldsymbol{e}}'_y, \quad \dot{\hat{\boldsymbol{e}}}'_y = -\omega \hat{\boldsymbol{e}}'_x \tag{4.14}$$

[see Fig. 4.8 and Eq. (1.56); the unit vector $\hat{\boldsymbol{e}}_{\perp x'}$ perpendicular to $\hat{\boldsymbol{e}}'_x$ equals $\hat{\boldsymbol{e}}'_y$ and the unit vector $\hat{\boldsymbol{e}}_{\perp y'}$ perpendicular to $\hat{\boldsymbol{e}}_{y'}$ equals $-\hat{\boldsymbol{e}}_{x'}$]. For the second time derivatives of the unit vectors, we get

$$\ddot{\hat{\boldsymbol{e}}}'_x = \omega \dot{\hat{\boldsymbol{e}}}'_y = -\omega^2 \hat{\boldsymbol{e}}'_x, \quad \ddot{\hat{\boldsymbol{e}}}'_y = \omega \dot{\hat{\boldsymbol{e}}}'_x = -\omega^2 \hat{\boldsymbol{e}}'_y. \tag{4.15}$$

Let us find the velocity of the particle relative to the rotating reference frame. To do this, we differentiate the position vector (4.13) with respect to time, considering the unit vectors as constants:

$$\boldsymbol{v}' = \dot{\boldsymbol{r}}' = \dot{x}' \hat{\boldsymbol{e}}'_x + \dot{y}' \hat{\boldsymbol{e}}'_y + \dot{z}' \hat{\boldsymbol{e}}'_z \tag{4.16}$$

If we now differentiate this expression, we get the acceleration of the particle relative to the rotating reference frame:

$$\boldsymbol{a}' = \dot{\boldsymbol{v}}' = \ddot{\boldsymbol{r}}' = \ddot{x}' \hat{\boldsymbol{e}}'_x + \ddot{y}' \hat{\boldsymbol{e}}'_y + \ddot{z}' \hat{\boldsymbol{e}}'_z. \tag{4.17}$$

Now we shall find the velocity of the particle relative to the fixed reference frame. For this purpose, we shall differentiate the position vector (4.13) "from the positions" of the stationary observer. Using the symbol $\boldsymbol{r}$ instead of $\boldsymbol{r}'$ (recall that $\boldsymbol{r} = \boldsymbol{r}'$), we get

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{x}' \hat{\boldsymbol{e}}'_x + x' \dot{\hat{\boldsymbol{e}}}'_x + \dot{y}' \hat{\boldsymbol{e}}'_y + y' \dot{\hat{\boldsymbol{e}}}'_y + \dot{z}' \hat{\boldsymbol{e}}'_z + z' \dot{\hat{\boldsymbol{e}}}'_z. \tag{4.18}$$

Differentiating this expression with respect to t, we find the acceleration of the particle relative to the fixed frame:

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{x}' \hat{\boldsymbol{e}}'_x + 2\dot{x}' \dot{\hat{\boldsymbol{e}}}'_x + x' \ddot{\hat{\boldsymbol{e}}}'_y + \ddot{y}' \hat{\boldsymbol{e}}'_y + 2\dot{y}' \dot{\hat{\boldsymbol{e}}}'_y + y' \ddot{\hat{\boldsymbol{e}}}'_y + \ddot{z}' \hat{\boldsymbol{e}}'_z + 2\dot{z}' \dot{\hat{\boldsymbol{e}}}'_z + z' \ddot{\hat{\boldsymbol{e}}}'_z.$$

Taking into account Eqs. (4.14), (4.15), and (4.17), we can transform the above expression into the form:

$$\boldsymbol{a} = \boldsymbol{a}' + 2\omega(\dot{x}' \hat{\boldsymbol{e}}'_y - \dot{y} \hat{\boldsymbol{e}}'_x) - \omega^2 (x' \hat{\boldsymbol{e}}'_x + \dot{y} \hat{\boldsymbol{e}}'_y). \tag{4.19}$$

Let us consider the vector product $\boldsymbol{\omega} \times \boldsymbol{v}'$. We shall represent it in the form of a determinant [see Eq. (1.33)]:

$$\boldsymbol{\omega} \times \boldsymbol{v}' = \begin{vmatrix} \hat{\boldsymbol{e}}'_x & \hat{\boldsymbol{e}}'_y & \hat{\boldsymbol{e}}'_z \\ \omega_x & \omega_y & \omega_z \\ v'_x & v'_y & v'_z \end{vmatrix}. \tag{4.20}$$

According to Eq. (4.16), $v_x = \dot{x}'$, $v_y = \dot{y}'$, $v_z = \dot{z}'$. In addition, for the direction of the coordinate axes that we have selected, we have $\omega_x = \omega_y = 0$, $\omega_z = \omega$. Introduction of these values into Eq. (4.20) yields

$$\boldsymbol{\omega} \times \boldsymbol{v}' = \begin{vmatrix} \hat{\boldsymbol{e}}'_x & \hat{\boldsymbol{e}}'_y & \hat{\boldsymbol{e}}'_z \\ 0 & 0 & \omega \\ \dot{x}' & \dot{y}' & \dot{z}' \end{vmatrix} = -\hat{\boldsymbol{e}}'_x \omega \dot{y}' + \hat{\boldsymbol{e}}'_y \omega \dot{x}'. \tag{4.21}$$

The result obtained shows that the second term of Eq. (4.19) can be written in the form $2\boldsymbol{\omega} \times \boldsymbol{v}'$. The expression in parentheses in the last term of Eq. (4.19) equals the component of the position vector $\boldsymbol{r}'$ perpendicular to the axis of rotation (to the axis z') [see Eq. (4.13)]. Let us denote this component by the symbol $\boldsymbol{R}$ (compare with Fig. 1.33). In view of everything said above, Eq. (4.19) can be written as follows:

$$\boldsymbol{a} = \boldsymbol{a}' + 2\boldsymbol{\omega} \times \boldsymbol{v}' - \omega^2 \boldsymbol{R}. \tag{4.22}$$

It follows from Eq. (4.22) that the acceleration of the particle relative to the fixed reference frame can be represented in the form of the sum of three accelerations: that relative to the rotating frame $\boldsymbol{a}'$, the acceleration equal to $-\omega^2 \boldsymbol{R}$[1], and the acceleration

$$\boldsymbol{a}_{\mathrm{Cor}} = 2\boldsymbol{\omega} \times \boldsymbol{v}' \tag{4.23}$$

called the **Coriolis acceleration**.

For a particle to move with the acceleration (4.22), bodies must act on it with the resultant force $\boldsymbol{F} = m\boldsymbol{a}$. According to Eq. (4.22)

$$m\boldsymbol{a}' = m\boldsymbol{a} - 2m\boldsymbol{\omega} \times \boldsymbol{v}' + m\omega^2 \boldsymbol{R} = \boldsymbol{F} + 2m\boldsymbol{v}' \times \boldsymbol{\omega} + m\omega^2 \boldsymbol{R} \tag{4.24}$$

(transposition of the multipliers changes the sign of the vector product). The result obtained signifies that when compiling an equation of Newton's second law for a rotating reference frame, in addition to the forces of interaction account must be taken of the centrifugal force of inertia determined by Eq. (4.25), and also of the Coriolis force which even in the most general case is determined by Eq. (4.12). We must note that the Coriolis force is always in a plane perpendicular to the axis of rotation.

It follows from a comparison of Eqs. (4.14), (4.16), and (4.18) that

$$\boldsymbol{v} = \boldsymbol{v}' + x'\dot{\hat{\boldsymbol{e}}}'_x + y'\dot{\hat{\boldsymbol{e}}}'_y = \boldsymbol{v}' + \omega(x'\hat{\boldsymbol{e}}'_y - y'\hat{\boldsymbol{e}}'_x).$$

Calculations similar to those which led us to Eq. (4.22) can help us see that the last term of the above expression equals $\boldsymbol{\omega} \times \boldsymbol{v}'$. Hence,

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{v}'. \tag{4.25}$$

When $\boldsymbol{v}' = 0$, this equation transforms into Eq. (1.100).

Examples of Motions in Which the Coriolis Force Manifests Itself. In interpreting phenomena associated with the motion of bodies relative to the Earth's surface, it is sometimes necessary to take account of the influence of Coriolis forces. For example, in the free fall of bodies, a Coriolis force acts on them that causes them to deviate to the East from a vertical line (Fig. 4.9). This force is the greatest at the

[1]The acceleration $\boldsymbol{a}_{\mathrm{tr}} = -\omega^2 \boldsymbol{R}$ is called transferable. It is the acceleration which a particle would have being at rest in a moving (in our case in a rotating) reference frame.

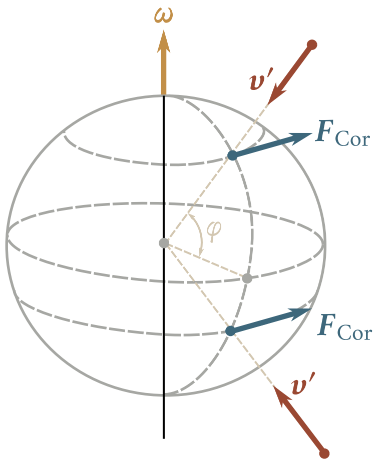

Fig. 4.9

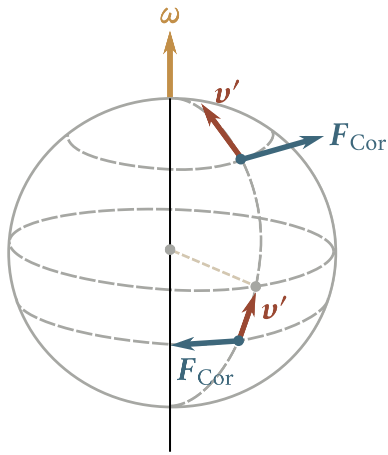

Fig. 4.10

equator and vanishes at the poles.

A flying projectile also experiences deviations due to Coriolis forces (Fig. 4.10). When a projectile is fired from a gun facing North, it will deviate to the East in the northern hemisphere and to the West in the southern one. If a projectile is fired along a meridian to the South, the deviations will be the reverse. If a projectile is fired along the equator, Coriolis forces will press it toward the Earth if the shot was directed to the West, and lift it if the shot was directed to the East. We invite our reader to convince himself that the Coriolis force acting on a body moving along a meridian in any direction (to the North or South) has a rightward direction relative to that of motion in the northern hemisphere and a leftward one in the southern hemisphere. This is why rivers always wash out their right banks in the northern hemisphere and their left banks in the southern one. This is also why the rails of a double-track railway wear differently.

The Coriolis forces also manifest themselves in the oscillations of a pendulum. Figure 4.11 shows the trajectory of a pendulum bob (it is assumed for simplicity's sake that the pendulum is at a pole). At the north pole, the Coriolis force will constantly be directed to the right in the direction of the pendulum's motion, and at the south pole to the left. As a result, the trajectory has the shape of a rosette.

As can be seen from the figure, the plane of oscillations of the pendulum turns clockwise relative to the Earth, and it completes one revolution a day. Relative to a heliocentric reference frame, the plane of oscillations remains unchanged, while the Earth rotates completing one revolution a day. It can be shown that at the latitude φ the plane of oscillations of a pendulum turns through the angle of $2\pi \sin\varphi$

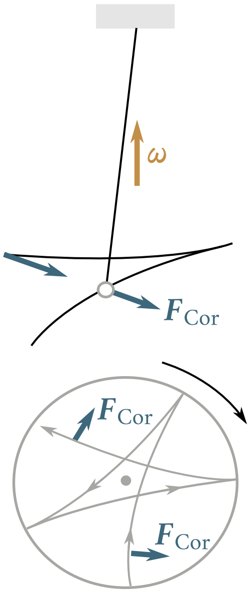

Fig. 4.11

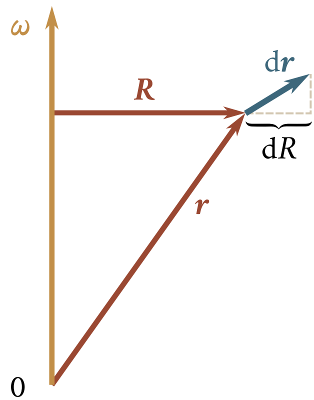

Fig. 4.12

in a day.

Thus, observations of the rotation of the plane in which a pendulum oscillates (pendulums intended for this purpose are called Foucault pendulums) provide a direct proof of the Earth's rotation about its axis.

4.4. Laws of Conservation in Non-Inertial Reference Frames

The equations of motion in a non-inertial frame do not differ in any way from those of motion in an inertial reference frame when the forces of inertia are taken into account. Therefore, all the corollaries following from the equations of motion, particularly Eqs. (3.77), (3.88), and (3.118), also hold in non-inertial reference frames.

Equation (3.77) acquires the following form for a non-inertial frame:

$$E_2 - E_1 = A_{12,\text{non-cons}} + A_{12,\text{in}} \tag{4.26}$$

where $A_{12,\text{in}}$ is the work done by the forces of inertia.

Equations (3.88) and (3.118) can be written as follows for a non-inertial frame:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \sum \boldsymbol{F}_{\text{ext}} + \sum \boldsymbol{F}_{\text{in}} \tag{4.27}$$

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum \boldsymbol{M}_{\text{ext}} + \sum \boldsymbol{M}_{\text{in}}. \tag{4.28}$$

Here $\boldsymbol{F}_{\text{ext}}$ is the force due to interaction, $\boldsymbol{F}_{\text{in}}$ is the force of inertia, $\boldsymbol{M}_{\text{ext}}$ and $\boldsymbol{M}_{\text{in}}$ are the moments of the above forces.

The centrifugal force of inertia $\boldsymbol{F}_{\text{cf}} = m\omega^2 \boldsymbol{R}$ is conservative. Indeed, the work of this force is

$$A_{12,\text{cf}} = \int_1^2 \boldsymbol{F}_{\text{cf}}\, \mathrm{d}\boldsymbol{r} = m\omega^2 \int_1^2 \boldsymbol{R}\, \mathrm{d}\boldsymbol{r}.$$

Inspection of Fig. 4.12 shows that the projection of the vector $\mathrm{d}\boldsymbol{r}$ on the direction of the vector $\boldsymbol{R}$ equals $\mathrm{d}R$—the increment of the magnitude of $\boldsymbol{R}$. Consequently, $\boldsymbol{R}\mathrm{d}\boldsymbol{r} = R\,\mathrm{d}R = \mathrm{d}R^2/2$. Thus,

$$A_{12,\text{cf}} = m\omega^2 \int_1^2 \mathrm{d}R^2/2 = m\omega^2 \frac{R_2^2}{2} - m\omega^2 \frac{R_1^2}{2}. \tag{4.29}$$

The expression obtained does not obviously depend on the path along which the displacement from point 1 to point 2 occurred.

The conservative nature of the force $\boldsymbol{F}_{\text{cf}}$ makes it possible to introduce the potential energy of a particle $E_{\text{p,cf}}$ (the centrifugal energy) whose decrement determines the work of the centrifugal force of inertia:

$$A_{12,\text{cf}} = E_{\text{p,cf,1}} - E_{\text{p,cf,2}} \tag{4.30}$$

[see Eq. (3.30)]. A comparison of Eqs. (4.29) and (4.30) shows that $E_{\text{p,cf}} = -m\omega^2 R^2/2+$ constant. We may assume that the constant equals zero. We thus get the following expression for the centrifugal energy:

$$E_{\text{p,cf}} = -\frac{1}{2} m\omega^2 R^2. \tag{4.31}$$

If we add Eq. (4.31) to the potential energy of a particle, then the work of the centrifugal force of inertia must not be included in the quantity $A_{12,\text{in}}$ in Eq. (4.26).

Chapter 5

MECHANICS OF A RIGID BODY

5.1. Motion of a Body

In Sec. 1.1, we acquainted ourselves with the two fundamental kinds of motion of a rigid body—translation and rotation.

In translation, all the points of a body receive displacements equal in magnitude and direction during the same time interval. Consequently, the velocities and accelerations of all the points are identical at every moment of time. It is therefore sufficient to determine the motion of one of the points of a body (for example, of its centre of mass) to completely characterize the motion of the entire body.

In rotation, all the points of a rigid body move along circles whose centres are on a single straight line called the axis of rotation. To describe rotation, we must set the position of the axis of rotation in space and the angular velocity of the body at each moment of time.

Any motion of a rigid body can be represented as the superposition of the two fundamental kinds of motion indicated above. We shall show this for plane motion, *i.e.*, motion when all the points of a body move in parallel planes. An example of plane motion is the rolling of a cylinder along a plane (Fig. 5.1).

The arbitrary displacement of a rigid body from position 1 to position 2 (Fig. 5.2) can be represented as the sum of two displacements—translation from position 1 to position $1'$ or $1''$, and rotation about the axis $0'$ or the axis $0''$. It is quite obvious that such a division of a displacement into translation and rotation can be performed in an infinite multitude of ways, but in any case rotation occurs through the same angle φ.

In accordance with the above, the elementary displacement of a point of a body ds can be resolved into two displacements—the "translational" one $\mathrm{d}\boldsymbol{s}_{\mathrm{tr}}$ and the

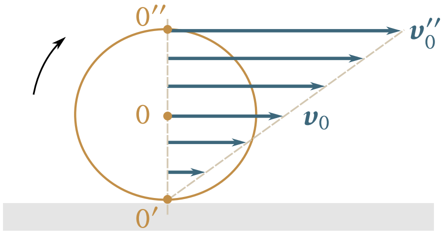

Fig. 5.1

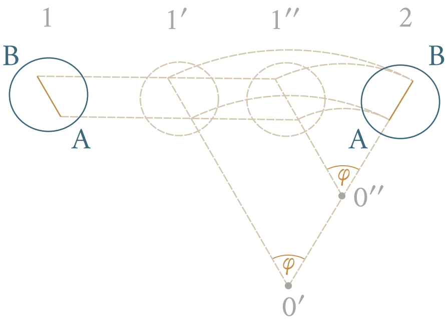

Fig. 5.2

"rotational" one $\mathrm{d}\boldsymbol{s}_{\mathrm{rot}}$:

$$\mathrm{d}\boldsymbol{s} = \mathrm{d}\boldsymbol{s}_{\mathrm{tr}} + \mathrm{d}\boldsymbol{s}_{\mathrm{rot}}$$

where $\mathrm{d}\boldsymbol{s}_{\mathrm{tr}}$ is the same for all the points of the body. This resolution of the displacement $\mathrm{d}\boldsymbol{s}$, as we have seen, can be performed in different ways. In each of them, the rotational displacement $\mathrm{d}\boldsymbol{s}_{\mathrm{rot}}$ is performed by rotation of the body through the same angle $\mathrm{d}\varphi$ (but relative to different axes), whereas $\mathrm{d}\boldsymbol{s}_{\mathrm{tr}}$ and $\mathrm{d}\boldsymbol{s}_{\mathrm{rot}}$ are different.

Dividing $\mathrm{d}\boldsymbol{s}$ by the corresponding time interval $\mathrm{d}t$, we get the velocity of a point:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{s}_{\mathrm{tr}}}{t} + \frac{\mathrm{d}\boldsymbol{s}_{\mathrm{rot}}}{t} = \boldsymbol{v}_0 + \boldsymbol{v}'$$

where $\boldsymbol{v}_0$ is the velocity of translation, which is the same for all the points of a body, $\boldsymbol{v}'$ is the velocity due to rotation, which is different for different points of the body.

Thus, the plane motion of a rigid body can be represented as the sum of two motions—translation with the velocity $\boldsymbol{v}_0$ and rotation with the angular velocity $\boldsymbol{\omega}$ (the vector $\boldsymbol{\omega}$ in Fig. 5.1 is directed at right angles to the plane of the drawing, beyond it). Such a representation of complex motion can be accomplished in many ways differing in the values of $\boldsymbol{v}_0$ and $\boldsymbol{v}'$, but corresponding to the same angular velocity $\boldsymbol{\omega}$. For example, the motion of a cylinder rolling without slipping along a plane (Fig. 5.1) can be represented either as translation with the velocity $\boldsymbol{v}_0$ and simultaneous rotation with the angular velocity $\boldsymbol{\omega}$ about the axis 0, or as translation with the velocity $\boldsymbol{v}_0'' = 2\boldsymbol{v}_0$ and rotation with the same angular velocity $\boldsymbol{\omega}$ about the axis 0″, or, finally, as only rotation, again with the same angular velocity $\boldsymbol{\omega}$ about the axis 0′.

Assuming that the reference frame relative to which we are considering the complex motion of a rigid body is stationary, the motion of the body can be represented as rotation with the angular velocity $\boldsymbol{\omega}$ in a reference frame moving translationally with the velocity $\boldsymbol{v}_0$ relative to the stationary frame.

The linear velocity $\boldsymbol{v}'$ of a point with the position vector $\boldsymbol{r}$ due to rotation of a rigid body is $\boldsymbol{v}' = \boldsymbol{\omega} \times \boldsymbol{r}$ [see Eq. (1.100)]. Consequently, the velocity of this point in complex motion can be represented in the form

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}. \tag{5.1}$$

An elementary displacement of a rigid body in plane motion can always be represented as rotation about an axis called the **instantaneous axis of rotation**. This axis may be either inside the body or outside it. The position of the instantaneous axis of rotation relative to a fixed reference frame and relative to the body itself, generally speaking, changes with time. For a rolling cylinder (Fig. 5.2), the instantaneous axis $0'$ coincides with the line of contact of the cylinder with the plane. When the cylinder rolls, the instantaneous axis moves both along the plane (*i.e.*, relative to a fixed reference frame) and along the surface of the cylinder.

The velocities of all the points of the body for each moment of time can be considered as due to rotation about the corresponding instantaneous axis. Consequently, plane motion of a rigid body can be considered as a number of consecutive elementary rotations about instantaneous axes.

In non-planar motion, an elementary displacement of a body can be represented as rotation about an instantaneous axis only if the vectors $\boldsymbol{v}_0$ and $\boldsymbol{\omega}$ are mutually perpendicular. If the angle between these vectors differs from $\pi/2$, the motion of the body at each moment of time will be the superposition of two motions—rotation about a certain axis, and translation along this axis.

5.2. Motion of the Centre of Mass of a Body

By dividing a body into elementary masses m_i we can represent it as a system of point particles whose mutual arrangement remains unchanged. Any of these elementary masses may be acted upon both by internal forces due to its interaction with other elementary masses of the body being considered, and by external forces. For example, if a body is in the field of the Earth's gravitational forces, each elementary mass of the body m_i will experience an external force equal to $m\boldsymbol{g}$.

Let us write the equation of Newton's second law for each elementary mass:

$$m_i \boldsymbol{a}_i = \boldsymbol{f}_i + \boldsymbol{F}_i \tag{5.2}$$

where $\boldsymbol{f}_i$ is the resultant of all the internal forces, and $\boldsymbol{F}_i$ the resultant of all the external forces applied to the given elementary mass. Summation of Eqs. (5.2) for all the elementary masses yields

$$\sum_i m_i \boldsymbol{a}_i = \sum_i \boldsymbol{f}_i + \sum_i \boldsymbol{F}_i. \tag{5.3}$$

The sum of all the internal forces acting in a system, however, equals zero. Hence, Eq. (5.3) can be simplified as follows:

$$\sum_i m_i \boldsymbol{a}_i = \sum_i \boldsymbol{F}_i. \tag{5.4}$$

Here the resultant of all the external forces acting on the body is in the right-hand side.

The sum in the left-hand side of Eq. (5.4) can be replaced with the product of the mass of the body m and the acceleration of its centre of mass (centre of inertia) $\boldsymbol{a}_C$. Indeed, according to Eq. (3.91), we have

$$\sum_i m_i \boldsymbol{r}_i = m\boldsymbol{r}_C.$$

Differentiating this relation twice with respect to time and taking into account that $\ddot{\boldsymbol{r}}_i = \boldsymbol{a}_i$, and $\ddot{\boldsymbol{r}}_C = \boldsymbol{a}_C$, we can write

$$\sum_i m_i \boldsymbol{a}_i = m\boldsymbol{a}_C. \tag{5.5}$$

Comparing Eqs. (5.4) and (5.5), we arrive at the equation

$$m\boldsymbol{a}_C = \sum \boldsymbol{F}_{\text{ext}} \tag{5.6}$$

which signifies that *the centre of mass of a rigid body moves in the same way as a point particle of a mass equal to that of the body would move under the action of all the forces applied to the body.*

Equation (5.6) permits us to find the motion of the centre of mass of a rigid body if we know the mass of the body and the forces acting on it. For translation, this equation will determine the acceleration not only of the centre of mass, but also of any other point of the body.

5.3. Rotation of a Body about a Fixed Axis

Let us consider a rigid body that can rotate about a fixed vertical axis (Fig. 5.3). We shall confine the axis in bearings to prevent its displacements in space. The flange Fl resting on the lower bearing prevents motion of the axis in a vertical direction.

A perfectly rigid body can be considered as a system of particles (point particles) with constant distances between them. Equation (3.118), *i.e.*,

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum \boldsymbol{M}_{\text{ext}}$$

holds for any system of particles, including a rigid body. In the latter case, $\boldsymbol{L}$ is the angular momentum of the body. The right-hand side of Eq. (3.118) is the sum of the moments of the external forces acting on the body.

Let us take point 0 on the axis of rotation and characterize the position of the particles forming the body with the aid of position vectors $\boldsymbol{r}$ drawn from this point

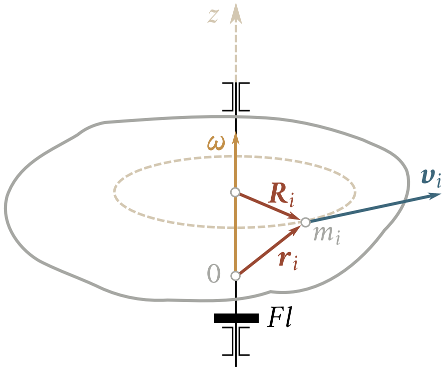

Fig. 5.3

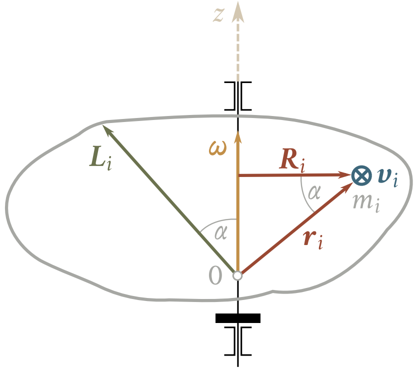

Fig. 5.4

(Fig. 5.3 depicts the i-th particle of mass m_1). According to Eq. (3.105), the angular momentum of the i-th particle relative to point 0 is

$$\boldsymbol{L}_i = \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = m_i \boldsymbol{r}_i \times \boldsymbol{v}_i. \tag{5.7}$$

The vectors $\boldsymbol{r}_i$ and $\boldsymbol{v}_i$ are mutually perpendicular for all the particles of the body. Therefore, the magnitude of the vector $\boldsymbol{L}_i$ [Eq. (5.7)] is

$$L_i = m_i r_i v_i = m_i r_i \omega R_i \tag{5.8}$$

[see Eq. (1.99)]. The direction of the vector $\boldsymbol{L}_i$ is shown in Fig. 5.4. It must be noted that the "length" of the vector $\boldsymbol{L}_i$, according to Eq. (5.8), is proportional to the velocity of rotation of the body $\boldsymbol{\omega}$. The direction of the vector $\boldsymbol{L}_i$, however, is independent of $\boldsymbol{\omega}$. The vector $\boldsymbol{L}_i$ is in a plane passing through the axis of rotation and the particle m_i and is perpendicular to $\boldsymbol{r}_i$.

The projection of the vector $\boldsymbol{L}_i$ onto the axis of rotation (the z-axis), as can be seen from Fig. 5.4, is [see Eq. (5.8)]

$$L_{zi} = L_i \cos\alpha = m_i r_i \omega R_i \cos\alpha = m_i (r_i \cos\alpha) R_i \omega = m_i R_i^2 \omega. \tag{5.9}$$

It is not difficult to see that for a homogeneous[1] body which is symmetrical relative to the axis of rotation (for a homogeneous body of revolution), the directions of the total angular momentum (equal to $\sum_i \boldsymbol{L}_i$) and of $\boldsymbol{\omega}$ along the axis of rotation are the same (Fig. 5.5). Indeed, in this case, the body can be divided into pairs of symmetrically arranged particles of equal mass (two pairs of particles are shown in the figure—m_i-m_i' and m_k-m_k'). The sum of the angular momenta of each pair (in the figure $\boldsymbol{L}_i + \boldsymbol{L}_i'$ and $\boldsymbol{L}_k + \boldsymbol{L}_k'$) is directed along the vector $\boldsymbol{\omega}$. Hence, the total angular momentum $\boldsymbol{L}$ will also coincide in direction with $\boldsymbol{\omega}$. The magnitude of the vector $\boldsymbol{L}$ in this case equals the sum of the projections of the momenta $\boldsymbol{L}_i$

[1]In mechanics, a body is defined as homogeneous when its density is the same throughout the entire volume (see Sec. 5.4).

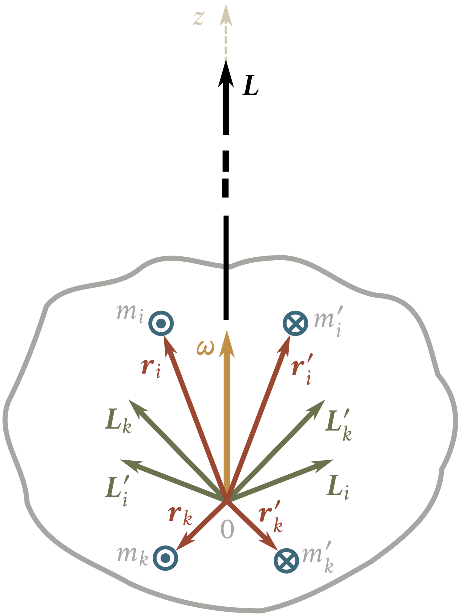

Fig. 5.5

onto the z-axis. Taking Eq. (5.9) into account, we get the following expression for the magnitude of the angular momentum of a body:

$$L = \sum_i L_{zi} = \omega \sum_i m_i R_i^2 = I\omega. \tag{5.10}$$

The quantity I equal to the sum of the products of the elementary masses and the squares of their distances from a certain axis is called the **rotational inertia** or the **moment of inertia of the body** relative to the given axis:

$$I = \sum_i m_i R_i^2. \tag{5.11}$$

Summation is performed over all the elementary masses m_i into which the body was mentally divided.

With a view to the fact that the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ have identical directions, we can write Eq. (5.10) as follows:

$$\boldsymbol{L} = I\boldsymbol{\omega}. \tag{5.12}$$

We remind our reader that we have obtained this relation for a homogeneous body rotating about an axis of symmetry. In the general case, as we shall see below, Eq. (5.12) is not obeyed.

For an asymmetrical (or non-homogeneous) body, the angular momentum $\boldsymbol{L}$, generally speaking, does not coincide in direction with the vector $\boldsymbol{\omega}$. The dash line

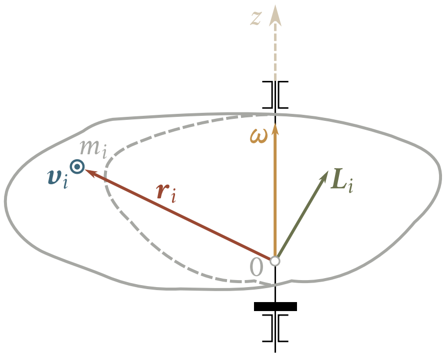

Fig. 5.6

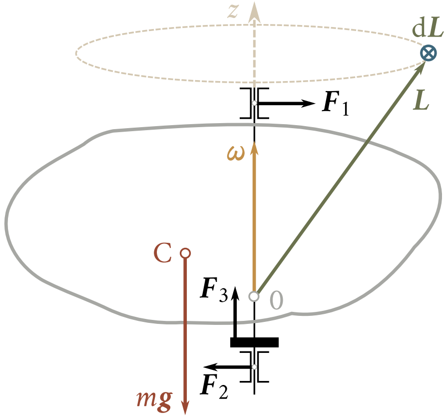

Fig. 5.7

in Fig. 5.6 shows the part of an asymmetrical homogeneous body that is symmetrical relative to the axis of rotation. The total angular momentum of this part, as we have established above, is directed along $\boldsymbol{\omega}$. The momentum $\boldsymbol{L}_i$ of each particle not belonging to the symmetrical part deviates to the right from the axis of rotation (in a plane figure). Consequently, the total angular momentum of the entire body will also deviate to the right (Fig. 5.7). Upon rotation of the body, the vector $\boldsymbol{L}$ rotates together with it, describing a cone. During the time dt, the vector $\boldsymbol{L}$ receives the increment d$\boldsymbol{L}$, which according to Eq. (3.118) equals

$$\mathrm{d}\boldsymbol{L} = \left(\sum \boldsymbol{M}_{\mathrm{ext}}\right) \mathrm{d}t. \tag{5.13}$$

If the vector $\boldsymbol{L}$ does not change in magnitude, then the vector d$\boldsymbol{L}$ is directed beyond the drawing (Fig. 5.7). The vector $\sum \boldsymbol{M}_{\mathrm{ext}}$ has the same direction. In the example we are treating, the moments of the external forces include (1) the moment of the force of gravity $m\boldsymbol{g}$ directed toward us—we shall call it negative (this force is applied to the centre of mass of the body C), (2) the positive moments of the forces of lateral pressure of the bearings on the axis (the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$), and (3) the positive moment of the force of pressure of the bearing shoulder on the flange $\boldsymbol{F}_3$. We assume that friction forces are absent, otherwise the vector $\boldsymbol{L}$ would not be constant in magnitude, and d$\boldsymbol{L}$ would not be perpendicular to $\boldsymbol{L}$.

The angular momentum relative to the axis of rotation [see Eq. (3.108)] for any (homogeneous or non-homogeneous, symmetrical or asymmetrical) body is

$$L_z = \sum_i L_{zi} = \sum_i m_i R_i^2 \omega = I\omega \tag{5.14}$$

(see Eqs. (5.9) and (5.11)]. It must be stressed that unlike Eq. (5.12), Eq. (5.14) is always correct.

Equation (3.119) states that

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = \sum M_{z,\mathrm{ext}}.$$

Introducing into this expression Eq. (5.14) for L_z, we get

$$I\alpha_z = \sum M_{z,\mathrm{ext}} \tag{5.15}$$

where $\alpha_z = \dot{\omega}$ is the projection of the angular acceleration onto the z-axis (we are considering rotation about a fixed axis, therefore the vector $\boldsymbol{\omega}$ can change only in magnitude). Equation (5.15) is similar to the equation $m\boldsymbol{a} = \boldsymbol{F}$. The part of the mass is played by the moment of inertia, that of the linear acceleration by the angular acceleration, and, finally, the part of the resultant force is played by the total moment of the external forces.

In the above example, the moments of all the external forces are perpendicular to the axis of rotation. Hence, their projections onto the z-axis equal zero. Accordingly, the angular velocity $\boldsymbol{\omega}$ remains constant, which is what should be expected in the absence of friction.

We must point out that in the rotation of a homogeneous symmetrical body, forces of lateral pressure of the bearings on the axis (the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in Fig. 5.7) do not appear. In the absence of the force of gravity, we could remove the bearings—the axis would retain its position in space without them. An axis whose position in space remains constant when bodies rotate about it in the absence of external forces is called a **free axis** of a body.

It is possible to prove that for a body of any shape and with an arbitrary arrangement of its mass there are three mutually perpendicular axes passing through the centre of mass of the body that can be free axes. They are called the **principal axes** of inertia of the body.

In a homogeneous parallelepiped (Fig. 5.8), the principal axes of inertia are obviously the axes O_1O_1, O_2O_2, and O_3O_3 passing through the centres of opposite faces.

In bodies possessing axial symmetry (for example, in a homogeneous[2] cylinder), the axis of symmetry is one of the principal axes of inertia. Any two mutually perpendicular axes in a plane at right angles to the axis of symmetry and passing through the centre of mass of the body can be the other two principal axes (Fig. 5.9). Thus, in such a body only one of the principal axes of inertia is fixed.

In a body with central symmetry, *i.e.*, in a sphere whose density depends only on the distance from its centre, any three mutually perpendicular axes passing through the centre of mass are the principal axes of inertia. Consequently, none of

[2]It is sufficient that the density of the body in each cross section be a function only of the distance from the axis of symmetry.

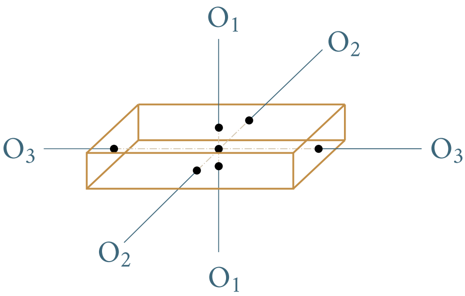

Fig. 5.8

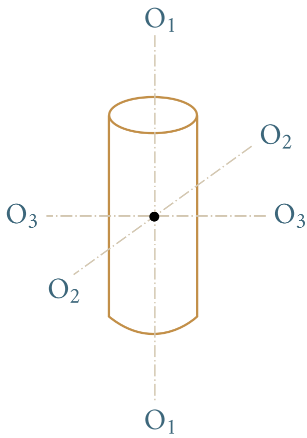

Fig. 5.9

the principal axes of inertia is fixed.

The moments of inertia relative to the principal axes are called the **principal moments of inertia** of a body. In the general case, these moments differ: $I_1 \neq I_2 \neq I_3$. For a body with axial symmetry, two of the principal moments of inertia are the same, while the third one, generally speaking, differs from them: $I_1 = I_2 \neq I_3$. And, finally, for a body with central symmetry, all three principal moments of inertia are the same: $I_1 = I_2 = I_3$.

Not only a homogeneous sphere, but also, for instance, a homogeneous cube has equal values of the principal moments of inertia. In the general case, such equality may be observed for bodies of an absolutely arbitrary shape when their mass is properly distributed. All such bodies are called **spherical tops**. Their feature is that any axis passing through their centre of mass has the properties of a free axis, and, consequently, none of the principal axes is fixed, as for a sphere. All spherical tops behave the same when they rotate in identical conditions.

Bodies for which $I_1 = I_2 \neq I_3$ behave like homogeneous bodies of revolution. They are called **symmetrical tops**. Finally, bodies for which $I_1 = I_2 = I_3$ are called **asymmetrical tops**.

If a body rotates in conditions when there is no external action, then only rotation about the principal axes corresponding to the maximum and minimum values of the moment of inertia is stable. Rotation about an axis corresponding to an intermediate value of the moment will be unstable. This signifies that the forces appearing upon the slightest deviation of the axis of rotation from this principal axis act in a direction causing the magnitude of this deviation to grow. When the axis of rotation deviates from a stable axis, the forces produced return the body to rotation about the corresponding principal axis.

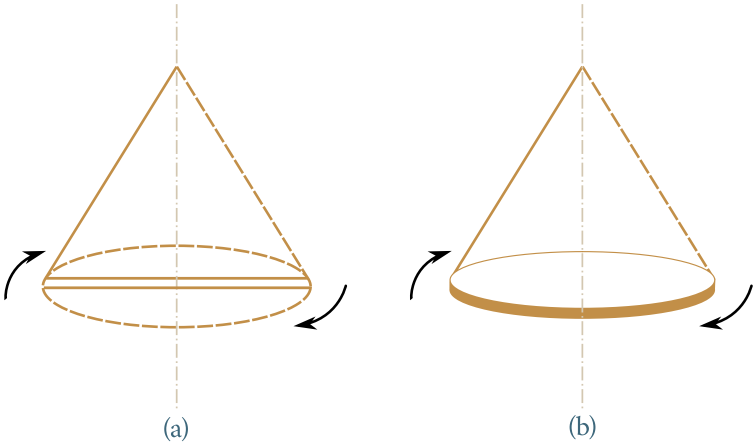

Fig. 5.10

We can convince ourselves that what has been said above is true by tossing a body having the shape of a parallelepiped (for example, a match box) and simultaneously bringing it into rotation[3]. We shall see that the body when falling can rotate stably about axes passing through the biggest or smallest faces. Attempts to toss the body so that it rotates about an axis passing through the faces of an intermediate size will be unsuccessful.

If an external force is exerted, for instance, by the string on which a rotating body is suspended, then only rotation about the principal axis corresponding to the maximum value of the moment of inertia will be stable. This is why a thin rod suspended by means of a string fastened to its end when brought into rapid rotation will in the long run rotate about an axis normal to it passing through its centre (Fig. 5.10a). A disk suspended by means of a string fastened to its edge (Fig. 5.10b) behaves in a similar way.

Up to now, we have treated bodies with a constant distribution of their mass. Now let us assume that a rigid body can lose for a certain time its property of a constant arrangement of its parts, and within this time redistribution of the body's mass occurs that results in the moment of inertia changing from I_1 to I_2. If such a redistribution occurs in conditions when $\sum \boldsymbol{M}_{\text{ext}} = 0$, then in accordance with the law of conservation of angular momentum the following equation must be observed:

$$I_1\omega_1 = I_2\omega_2 \tag{5.16}$$

where ω_1 is the initial, and ω_2 is the final value of the angular velocity of the body.

[3]The action of the force of gravity in this case is not significant. It only causes the body to fall in addition to its rotation.

Thus, a change in the moment of inertia leads to a corresponding change in the angular velocity. This explains why a spinning figure skater (or a man on a rotating platform) begins to rotate more slowly when he stretches his arms out, and gains speed when he presses his arms against his body.

5.4. Moment of Inertia

From the definition of the moment of inertia[4] [see Eq. (5.11)]

$$I = \sum_i \Delta m_i R_i^2$$

we can see that it is an additive quantity. This signifies that the moment of inertia of a body equals the sum of the moments of inertia of its parts.

We introduced the concept of the moment of inertia when dealing with the rotation of a rigid body. It must be borne in mind, however, that this quantity exists irrespective of rotation. Every body, regardless of whether it is rotating or at rest, has a definite moment of inertia relative to any axis, just like a body has a mass regardless of whether it is moving or at rest.

The distribution of the mass within a body can be characterized with the aid of a quantity called the density. If a body is homogeneous, *i.e.*, its properties are the same at all of its points, then the density is defined as the quantity

$$\rho = \frac{m}{V} \tag{5.17}$$

where m and V are the mass and volume of the body, respectively. Thus, the density of a homogeneous body is the mass of a unit of its volume.

For a body with an unevenly distributed mass, Eq. (5.17) gives the average density. The density at a given point is determined in this case as follows:

$$\rho = \lim_{\Delta v \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d}m}{\mathrm{d}V}. \tag{5.18}$$

In this expression, Δm is the mass contained in the volume ΔV, which in the limit transition contracts to the point at which the density is being determined.

The limit transition in Eq. (5.18) must not be understood in the sense that ΔV contracts literally to a point. If such a meaning is implied, we would get a greatly differing result for two virtually coinciding points, one of which is at the nucleus of an atom, while the other is at a space between nuclei (the density for the first point would be enormous, and for the second one it would be zero). Therefore, ΔV should be diminished until we get an infinitely small volume from the physi-

[4]In this section, it is expedient to use the symbol Δm_i instead of m_i for the elementary mass of a body.

cal viewpoint. We understand this to mean such a volume which on the one hand is small enough for the macroscopic (*i.e.*, belonging to a great complex of atoms) properties within its limits to be considered identical, and on the other hand is sufficiently great to prevent discreteness (discontinuity) of the substance from manifesting itself.

By Eq. (5.18), the elementary mass Δm_i equals the product of the density of a body ρ_i at a given point and the corresponding elementary volume ΔV_i:

$$\Delta m_i = \rho_i \Delta V_i.$$

Consequently, the moment of inertia can be written in the form

$$I = \sum_i \rho_i R_i^2 \Delta V_i. \tag{5.19}$$

If the density of a body is constant, it can be put outside the sum:

$$I = \rho \sum_i R_i^2 \Delta V_i. \tag{5.20}$$

Equations (5.19) and (5.20) are approximate. Their accuracy grows with diminishing elementary volumes ΔV_i and the elementary masses Δm_i corresponding to them. Hence, the task of finding the moments of inertia consists in integration:

$$I = \int R^2 \, \mathrm{d}m = \int \rho R^2 \, \mathrm{d}V. \tag{5.21}$$

The integrals in Eq. (5.21) are taken over the entire volume of the body. The quantities ρ and R in these integrals are position functions, *i.e.*, for example, functions of the Cartesian coordinates x, y, and z.

As an example, let us find the moment of inertia of a homogeneous disk relative to an axis perpendicular to the plane of the disk and passing through its centre (Fig. 5.11). Let us divide the disk into annular layers of thickness dR. All the points of one layer will be at the same distance R from the axis. The volume of such a layer is

$$\mathrm{d}V = 2\pi b R \, \mathrm{d}R$$

where b is the thickness of the disk.

Since the disk is homogeneous, its density at all its points is the same, and ρ in Eq. (5.21) can be put outside the integral:

$$I = \rho \int R^2 \, \mathrm{d}V = \rho \int_0^{R_0} R^2 2\pi b R \, \mathrm{d}R$$

where R_0 is the radius of the disk. Let us put the constant factor $2\pi b$ outside the integral:

$$I = 2\pi b \rho \int_0^{R_0} R^3 \, \mathrm{d}R = 2\pi b \rho \frac{R_0^4}{4}.$$

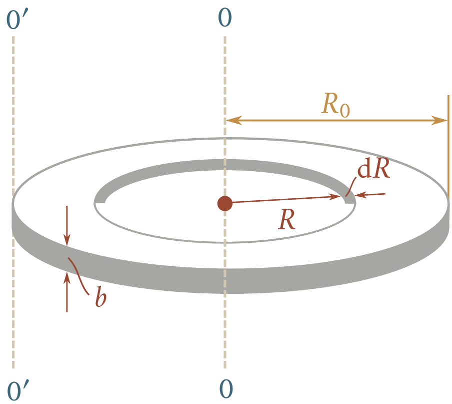

Fig. 5.11

Finally, introducing the mass of the disk m equal to the product of the density ρ and the volume of the disk $b\pi R_0^2$, we get

$$I = \frac{mR_0^2}{2}. \tag{5.22}$$

The finding of the moment of inertia in the above example was simplified quite considerably owing to the fact that the body was homogeneous and symmetrical, and we sought the moment of inertia relative to an axis of symmetry. If we wanted to find the moment of inertia of the disk relative, for example, to the axis 0′0′ perpendicular to the disk and passing through its edge (see Fig. 5.11), the calculations would evidently be much more complicated. The finding of the moment of inertia is considerably simplified in such cases if we use the Steiner or parallel axis theorem, which is formulated as follows: *the moment of inertia I relative to an arbitrary axis equals the moment of inertia I_C relative to an axis parallel to the given one and passing through the body's centre of mass plus the product of the body's mass m and the square of the distance b between the axes*:

$$I = I_C = mb^2. \tag{5.23}$$

According to the parallel axis theorem, the moment of inertia of the disk relative to the axis 0′0′ equals the moment of inertia relative to the axis passing through the centre of the disk, which we have found [Eq. (5.22)] plus mR_0^2 (the distance between the axes 0′0′ and 00 equals the radius of the disk R_0):

$$I = \frac{mR_0^2}{2} + mR_0^2 = \frac{3}{2}mR_0^2.$$

Thus, the parallel axis theorem in essence reduces the calculation of the moment of inertia relative to an arbitrary axis to the calculation of the moment of inertia relative to an axis passing through the centre of mass of the body.

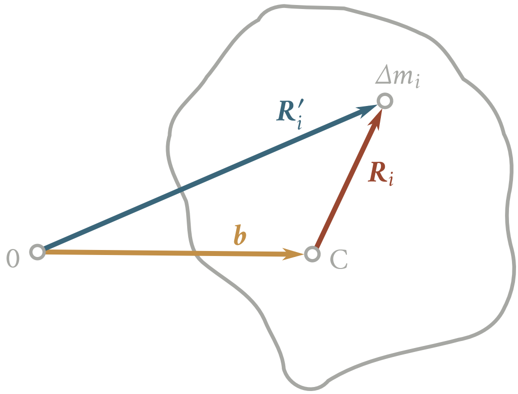

Fig. 5.12

To prove the parallel axis theorem, let us consider axis C passing through the centre of mass of a body and axis 0 parallel to it and at a distance b from axis C (Fig. 5.12, both axes are perpendicular to the plane of the drawing). Let $\boldsymbol{R}_i$ be a vector perpendicular to axis C and drawn from the axis to the elementary mass Δm_i and $\Delta \boldsymbol{R}_i$ be a similar vector drawn from axis 0. We shall also introduce the vector $\boldsymbol{b}$ perpendicular to the axes and connecting the corresponding points of axes 0 and C. For any pair of points opposite each other, this vector has the same value (equal to the distance b between the axes) and the same direction. The following relation holds between the vectors listed above:

$$\boldsymbol{R}_i' = \boldsymbol{b} + \boldsymbol{R}_i.$$

The square of the distance to the elementary mass Δm_i from axis C is $R_i^2 = \boldsymbol{R}^2$, and from axis 0 is

$$R_i'^2 = (\boldsymbol{b} + \boldsymbol{R}_i)^2 = b^2 + 2\boldsymbol{b}\cdot\boldsymbol{R}_i + R_i^2.$$

With a view to the above expression, the moment of inertia of the body relative to axis 0 can be written in the form

$$I = \sum_i \Delta m_i R_i'^2 = b^2 \sum_i \Delta m_i + 2\boldsymbol{b} \sum_i \Delta m_i \boldsymbol{R}_i + \sum_i \Delta m_i R_i^2 \qquad (5.24)$$

(we have put the constant factors outside the sum). The last term in this expression is the moment of inertia of the body relative to axis C. Let us designate it I_C. The sum of the elementary masses gives the mass of the body m. The sum $\sum_i \Delta m_i \boldsymbol{R}_i$ equals the product of the mass of the body and the vector $\boldsymbol{R}$ drawn from axis C to the centre of mass of the body. Since the centre of mass is on axis C, this vector $\boldsymbol{R}$ and, consequently, the second term in Eq. (5.24) vanish. We thus arrive at the conclusion that

$$I = mb^2 + I_\mathrm{C}$$

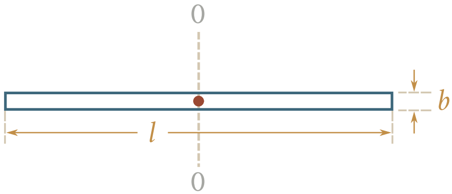

Fig. 5.13

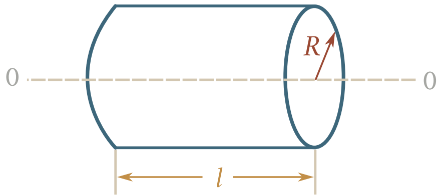

Fig. 5.14

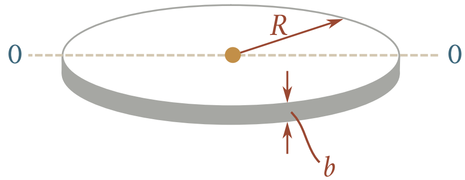

Fig. 5.15

Q.E.D [see Eq. (5.23)].

In concluding, we shall give the values of the moments of inertia for selected bodies (the latter are assumed to be homogeneous, m is the mass of the body).

1. The body is a thin long rod with a cross section of any shape. The maximum cross-sectional dimension b of the rod is many times smaller than its length l ($b \sim l$). The moment of inertia relative to an axis perpendicular to the rod and passing through its middle (Fig. 5.13) is

$$I = \frac{1}{12} m l^2. \tag{5.25}$$

2. For a disk or cylinder with any ratio of R to l (Fig. 5.14), the moment of inertia relative to an axis coinciding with the geometrical axis of the cylinder is

$$I = \frac{1}{2} m R^2. \tag{5.26}$$

3. The body is a thin disk. The thickness of the disk b is many times smaller than the radius of the disk R ($b \sim R$). The moment of inertia relative to an axis coinciding with the diameter of the disk (Fig. 5.15) is

$$I = \frac{1}{4} m R^2. \tag{5.27}$$

4. The moment of inertia of a sphere of radius R relative to an axis passing through its centre is

$$I = \frac{2}{5} m R^2. \tag{5.28}$$

5.5. Concept of Inertia Tensor

We established in Sec. 5.3 that for a homogeneous body rotating about an axis of symmetry, the relation between the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ has a very simple form [Eq. (5.12)]

$$\boldsymbol{L} = I\boldsymbol{\omega}$$

or

$$L_x = I\omega_x, \quad L_y = I\omega_y, \quad L_z = I\omega_z. \tag{5.29}$$

The explanation is that for such a body the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ are collinear. In the general case, however, the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ make an angle differing from zero (see Fig. 5.7), so that the relation between them cannot be expressed by Eq. (5.12).

Let us try to find a way of relating the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ analytically in the most general case. We shall proceed from the fact that the magnitudes of $\boldsymbol{L}$ and $\boldsymbol{\omega}$ are proportional to each other. Indeed, according to Eq. (5.8), the magnitudes of the elementary vectors $\boldsymbol{L}_i$ are proportional to the magnitude of $\boldsymbol{\omega}$. Hence, the magnitude of the sum of these vectors is also proportional to $\boldsymbol{\omega}$. It is easy to understand that such proportionality is obtained when each component of the vector $\boldsymbol{L}$ depends linearly on the components of the vector $\boldsymbol{\omega}$:

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z. \end{aligned} \tag{5.30}$$

Here the quantities I_{xx}, I_{xy}, etc. are proportionality constants having the dimension of the moment of inertia [compare with Eq. (5.29)]. When $\boldsymbol{\omega}$ increases a certain number of times, each of the components $\omega_x, \omega_y, \omega_z$, and accordingly each of the components L_x, L_y, L_z grows the same number of times, as, consequently, does the vector $\boldsymbol{L}$ itself.

The mutual orientation of the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ is determined by the values of the proportionality constants. Assume, for example, that $I_{xx} = I_{yy} = I_{zz}$, and the remaining constants equal zero. In this case, Eqs. (5.30) transform into Eqs. (5.29), *i.e.*, the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ will be collinear. Now let us assume that the vector $\boldsymbol{\omega}$ is directed along the z-axis, and the constants I_{xz}, I_{yz}, I_{zz} differ from zero. In this case $\omega_z = \omega, \omega_x = \omega_y = 0$. Substitution of these values in Eqs. (5.30) yields

$$L_x = I_{xz}\omega \neq 0, \quad L_y = I_{yz}\omega \neq 0, \quad L_z = I_{zz}\omega \neq 0.$$

All three components of the vector $\boldsymbol{L}$ differ from zero. Hence, the vector $\boldsymbol{L}$ makes a certain angle with the vector $\boldsymbol{\omega}$ directed along the z-axis.

It follows from the above that in the most general case the relation between the angular momentum and the angular velocity of a body can be expressed with the aid of Eqs. (5.30). Similar equations can be written for any vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ whose magnitudes are proportional to each other:

$$\begin{aligned} b_x &= T_{xx}a_x + T_{xy}a_y + T_{xz}a_z \\ b_y &= T_{yx}a_x + T_{yy}a_y + T_{yz}a_z \\ b_z &= T_{zx}a_x + T_{zy}a_y + T_{zz}a_z. \end{aligned} \tag{5.31}$$

These three equations can be written compactly in the form of a single expression:

$$b_i = \sum_{k=x,y,z} T_{ik}a_k. \tag{5.32}$$

Assuming that $i = x$ and performing summation with the subscript k sequentially having the values x, y, z, we get the first of the equations (5.31), assuming that $i = y$, we get the second equation, etc.

The combination of the nine quantities $T_{xx}, T_{xy}, \ldots, T_{zz}$ is called a **tensor of rank two**[5], and the operation expressed by Eqs. (5.31) is called multiplication of the vector $\boldsymbol{a}$ by the tensor T. Such multiplication produces a new vector $\boldsymbol{b}$.

It is customary practice to write a tensor in the form of a square table[6]:

$$T = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} \tag{5.33}$$

(we can write the subscripts 1, 2, 3 instead of x, y, z). The quantities $T_{xx}, T_{xy}, \ldots$ are defined as the components of the tensor. The components T_{xx}, T_{yy}, T_{zz} along the diagonal of matrix (5.33) are called diagonal ones. The values of the components depend on the choice of the coordinate axes onto which the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are projected (the components of these vectors also depend on the choice of the axes).

A comparison of Eqs. (5.30) and (5.31) shows that the constants in Eqs.(5.30) are the components of a tensor of rank two:

$$I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}. \tag{5.34}$$

It is called the **inertia tensor** of a body. This tensor characterizes the inertia properties of a body in rotation.

To find the values of the components of the inertia tensor, we shall proceed

[5]A tensor of rank two is defined as a combination of the nine quantities $T_{xx}, T_{xy}, \ldots, T_{zz}$ that transform according to definite rules upon rotations of the coordinate axes.

[6]More commonly known as matrix form –Ed.

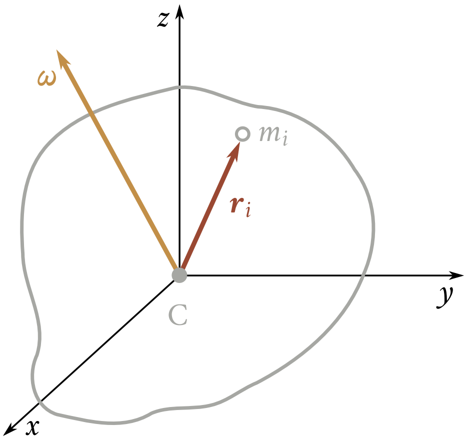

Fig. 5.16

from the definition of the angular momentum of a body:

$$\boldsymbol{L} = \sum_i m_i [\boldsymbol{r}_i \times \boldsymbol{v}_i] \tag{5.35}$$

[see Eq. (5.7)]. We shall plot the vectors $\boldsymbol{r}_i$ from the centre of mass of a body (Fig. 5.16). Let us substitute the vector product $\boldsymbol{\omega} \times \boldsymbol{r}_i$ for the velocity $\boldsymbol{v}_i$ in Eq. (5.35) [see Eq. (1.100)]. We get

$$\boldsymbol{L} = \sum_i m_i [\boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)].$$

We shall now use Eq. (1.35):

$$\boldsymbol{L} = \sum_i m_i [\boldsymbol{\omega}(\boldsymbol{r}_i \cdot \boldsymbol{r}_i) - \boldsymbol{r}_i(\boldsymbol{r}_i \cdot \boldsymbol{\omega})]. \tag{5.36}$$

We remind our reader that summation is conducted of all the elementary masses into which we have mentally divided the body.

Let us associate a Cartesian system of coordinates with the body[7] (see Fig. 5.16) and write the scalar products figuring in Eq. (5.16) through the components of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}_i$ along the axes of this system [see Eq. (1.23)]. We place the origin of coordinates at the centre of mass of the body C (it must be remembered that we plotted the vectors $\boldsymbol{r}_i$ from this point). Taking into account that $r_{xi} = x_i, r_{yi} = y_i, r_{zi} = z_i$, we get

$$\boldsymbol{L} = \sum_i m_i [\boldsymbol{\omega}(x_i^2 + y_i^2 + z_i^2) - \boldsymbol{r}_i(x_i\omega_x + y_i\omega_y + z_i\omega_z)]. \tag{5.37}$$

[7]It must be stressed that the axes of this system are rigidly associated with the body and rotate together with it.

Let us find the projection of this vector onto the x-axis:

$$L_x = \sum_i m_i[\omega_x(x_i^2 + y_i^2 + z_i^2) - x_i(x_i\omega_x + y_i\omega_y + z_i\omega_z)]$$
$$= \omega_x \sum_i m_i(y_i^2 + z_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i. \tag{5.38}$$

In a similar way, we find the projections of the vector $\boldsymbol{L}$ onto the axes y and z:

$$L_y = -\omega_x \sum_i m_i y_i x_i + \omega_y \sum_i m_i(x_i^2 + z_i^2) - \omega_z \sum_i m_i y_i z_i \tag{5.39}$$

$$L_z = -\omega_x \sum_i m_i z_i x_i - \omega_y \sum_i m_i z_i y_i + \omega_z \sum_i m_i(x_i^2 + y_i^2). \tag{5.40}$$

A comparison of the expressions obtained with Eqs. (5.30) allows us to find the values of the components of the inertia tensor. Let us write these values at once in the form of a matrix:

$$I = \begin{pmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(x_i^2 + z_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2 + y_i^2) \end{pmatrix}. \tag{5.41}$$

The diagonal components of the tensor are the moments of inertia relative to the corresponding coordinate axes considered in the preceding section. These components are called **axial moments of inertia**. The non-diagonal components are called **centrifugal moments of inertia**. It must be noted that the non-diagonal components of the tensor (5.41) comply with the condition that $I_{xy} = I_{yx}, I_{xz} = I_{zx}, I_{yz} = I_{zy}$. A tensor complying with such a condition is called **symmetrical**.

In practice, the inertia tensor components are computed with the aid of integration. For example, the component I_{xx} is determined by the formula

$$I_{xx} = \int \rho(x, y, z)(y^2 + z^2)\, \mathrm{d}V$$

where $\rho(x, y, z)$ is the density, and $\mathrm{d}V$ is the elementary volume. Integration is performed over the entire volume of the body.

Let us find the components of the inertia tensor for a homogeneous rectangular parallelepiped. We select the coordinate axes as shown in Fig. 5.17. The origin of coordinates coincides with the centre of mass of the body C. To calculate the axial moment of inertia I_{zz} we divide our parallelepiped into columns with a base area of $\mathrm{d}x\,\mathrm{d}y$. All the elements of such a column have identical values of the coordinates x and y. The volume of a column is $2c\,\mathrm{d}x\,\mathrm{d}y$, and its mass $\mathrm{d}m$ is $\rho 2c\,\mathrm{d}x\,\mathrm{d}y$. Therefore,

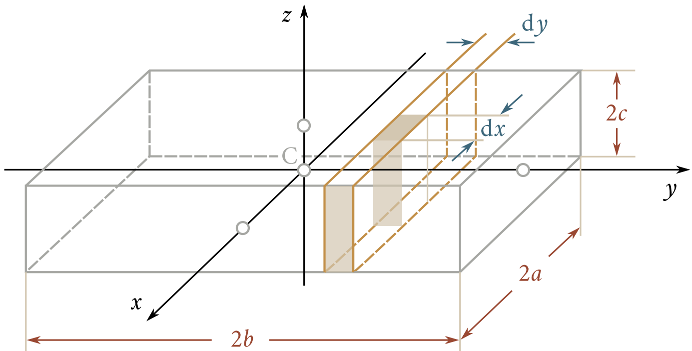

Fig. 5.17

the contribution of the column to I_{zz} is determined by the expression

$$dI_{zz,\text{column}} = 2\rho c(x^2 + y^2)dx\,dy.$$

Integration of this expression with respect to x gives the contribution to I_{zz} of the layer of length $2a$, width $2c$, and thickness dy shown in Fig. 5.17:

$$\begin{aligned} dI_{zz,\text{layer}} &= \int_{-a}^{+a} 2\rho c(x^2 + y^2)\,dx\,dy \\ &= 2\rho c\,dy \int_{-a}^{+a} x^2\,dx + 2\rho c y^2\,dy \int_{-a}^{+a} dx \\ &= \left(\frac{4}{3}\rho c a^3 + 4\rho c a y^2\right) dy \end{aligned} \tag{5.42}$$

(the density ρ does not depend on the coordinates x, y, and z because the body is homogeneous).

Finally, integrating Eq. (5.42) with respect to y, we get I_{zz} for the entire parallelepiped of mass m:

$$\begin{aligned} I_{zz} &= \int_{-b}^{+b} \left(\frac{4}{3}\rho c a^3 + 4\rho c a y^2\right) dy = \frac{4}{3}\rho c a^3 \int_{-b}^{+b} dy + 4\rho c a \int_{-b}^{+b} y^2\,dy \\ &= \frac{8}{3}\rho c a^3 b + \frac{8}{3}\rho c a b^3 = \frac{1}{3}\rho(2a)(2b)(2c)(a^2 + b^2) = \frac{1}{3}m(a^2 + b^2). \end{aligned}$$

Similar calculations give $I_{xx} = m(b^2 + c^2)/3$, and $I_{yy} = m(a^2 + c^2)/3$.

Now let us calculate one of the centrifugal moments, for instance I_{xy}. The contribution to this moment of a column with the base $dx\,dy$ is

$$dI_{xy,\text{column}} = -\rho x y 2c\,dx\,dy$$

and the contribution of a layer is

$$dI_{xy,\text{layer}} = -2\rho c y\,dx \int_{-a}^{+a} x\,dy = 0.$$

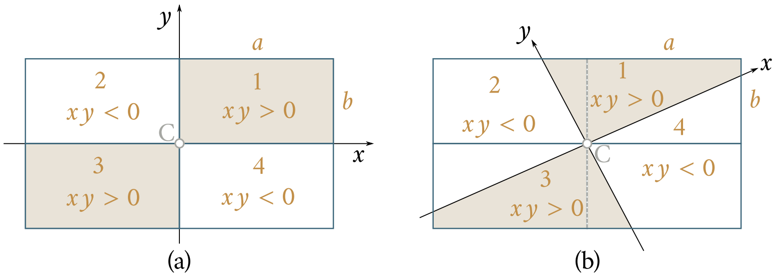

Fig. 5.18

Accordingly, the moment of the entire parallelepiped equals zero. A similar result is also obtained for the other centrifugal moments. Thus, when we choose the coordinate axes as shown in Fig. 5.17, the inertia tensor of a homogeneous rectangular parallelepiped has the form

$$I = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \tag{5.43}$$

(we have retained only one of the two identical subscripts for the diagonal components).

We obtained such a result because we choose the principal axes of inertia (see Sec. 5.3) of the parallelepiped as the coordinate axes. Upon a different choice of the coordinate axes, the centrifugal moments of inertia will differ from zero. The following reasoning will convince us that this is true. When we choose the axes as shown in Fig. 5.18a, the areas of rectangles 1, 2, 3, and 4 are the same. On two of them, the product xy is positive, and on two negative. As a result, the integral of xy taken over the entire area vanishes. When we choose the axes as shown in Fig. 5.18b, the areas of the shaded figures 1 and 3 are less than those of the unshaded figures 2 and 4 (because $a > b$). Therefore, the integral of xy taken over the entire area will differ from zero. Accordingly, the centrifugal moment I_{xy} also differs from zero.

The result obtained is common for all bodies regardless of their shape and mass distribution. If we take the principal axes of inertia of a body as the coordinate axes, the inertia tensor has the form given by Eq. (5.43). The quantities I_x, I_y, I_z [but not I_{xx}, I_{yy}, I_{zz} in Eq. (5.34); upon rotation of the coordinate axes all the tensor components change, the diagonal ones included] are called the **principal moments of inertia** of a body. It must be underlined that the axial moments calculated not about arbitrary axes, but about the principal ones, are called the principal moments of inertia.

The principal axes of inertia are mutually perpendicular and intersect at the centre of mass of a body. In the general case (when $I_x \neq I_y \neq I_z$), we can choose these axes in a single way. For a spherical top (*i.e.*, a body for which $I_x = I_y = I_z$, see Sec. 5.3), the position of the principal axes is absolutely indeterminate. For a symmetrical top ($I_x = I_y \neq I_z$), only the z-axis is fixed, the other two axes being indeterminate.

Assume that a body rotates about one of its principal axes of inertia, say about the z-axis. Selecting the principal axes as the coordinate ones, we have $\omega_z = \omega$, $\omega_x = \omega_y = 0$. Since the inertia tensor has the form of Eq. (5.43) when the coordinate axes are chosen in this way, Eqs. (5.30) give the following values of the components of the angular momentum of a body:

$$L_x = L_y = 0, \quad L_z = I_z\omega.$$

Consequently, the vector $\boldsymbol{L}$ has the same direction as $\boldsymbol{\omega}$. The same result is obtained for rotation of a body about the other principal axes. In all these cases, we arrive at Eq. (5.12):

$$\boldsymbol{L} = I\boldsymbol{\omega}$$

where I is the corresponding principal moment of inertia of the body. In Sec. 5.3, we obtained Eq. (5.12) for a homogeneous body rotating about its axis of symmetry. Now we have established that this equation holds when an arbitrary body rotates about one of its principal axes of inertia.

In conclusion, let us determine when the equation $\dot{\boldsymbol{L}} = \boldsymbol{M}$ [see Eq. (3.118)], which is always correct, can be written in the form

$$I\boldsymbol{\alpha} = \boldsymbol{M}. \tag{5.44}$$

We may do this first of all when a body rotates about a principal axis, and the moment of the forces $\boldsymbol{M}$ is directed along this axis. Indeed, in this case, the moment $\boldsymbol{M}$ produces the increment $\mathrm{d}\boldsymbol{L}$ that is collinear with $\boldsymbol{L}$ ($\mathrm{d}\boldsymbol{L} = \boldsymbol{M}\,\mathrm{d}t$). Hence, rotation constantly takes place about a principal axis so that the relation $\boldsymbol{L} = I\boldsymbol{\omega}$ is never violated. In this case, however, Eq. (5.44) gives nothing new in comparison with the formula

$$I\alpha_z = M_z. \tag{5.45}$$

Here z is the axis of rotation.

When $\boldsymbol{M}$ is not collinear with $\boldsymbol{L}$ (for example, when $\boldsymbol{M}$ is perpendicular to $\boldsymbol{L}$, the axis of rotation moves relative to the body with time. Consequently, even provided that the relation $\boldsymbol{L} = I\boldsymbol{\omega}$ is obeyed at the initial moment, this relation stops being obeyed with time, and Eq. (5.44) loses its meaning. The displacement of the axis of rotation relative to the body is of no significance only when the body is a spherical top. For such a top, any axis is a principal one and has the same value of the moment of inertia I. Therefore, Eq. (5.44) holds for any mutual direction of the vectors $\boldsymbol{M}$ and $\boldsymbol{\omega}$.

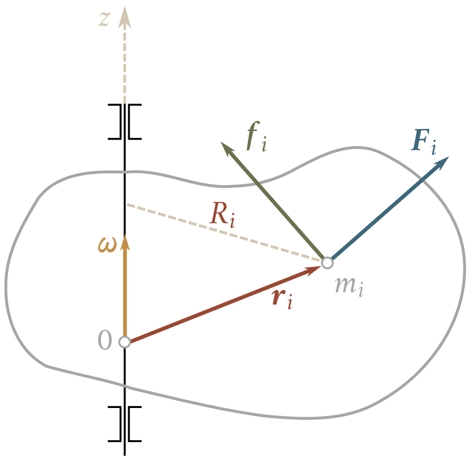

Fig. 5.19

5.6. Kinetic Energy of a Rotating Body

Let us begin with a consideration of the rotation of a body about a fixed axis, which we shall call the z-axis (Fig. 5.19). The linear velocity of the elementary mass m_i is $v_i = \omega R_i$ where R_i is the distance from the mass m_i to the z-axis. Consequently, we get the following expression for the kinetic energy of the i-th elementary mass:

$$E_{k,i} = \frac{m_i v_i^2}{2} = \frac{1}{2} m_i \omega^2 R_i^2.$$

The kinetic energy of a body is composed of the kinetic energies of its parts:

$$E_k = \sum_i E_{k,i} = \frac{1}{2}\omega^2 \sum_i m_i R_i^2.$$

The sum in the right-hand side of this equation is the moment of inertia of the body I_z relative to the axis of rotation. The kinetic energy of a body rotating about a fixed axis thus equals

$$E_k = \frac{1}{2} I_z \omega^2. \tag{5.46}$$

Assume that the mass m_i experiences[8] the internal force $\boldsymbol{f}_i$, and the external force $\boldsymbol{F}_i$ (see FFig. 5.19). According to Eq. (3.16), these forces do the following work during the time dt:

$$\mathrm{d}A_i = \boldsymbol{f}_i \cdot \boldsymbol{v}_i \,\mathrm{d}t + \boldsymbol{F}_i \cdot \boldsymbol{v}_i \,\mathrm{d}t = \boldsymbol{f}_i \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i)\,\mathrm{d}t + \boldsymbol{F}_i \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i)\,\mathrm{d}t.$$

Performing a cyclic transposition of the multipliers in the scalar triple products

[8] The resultant force $\boldsymbol{f}_i + \boldsymbol{F}_i$ is in a plane perpendicular to the axis of rotation.

[see Eq. (1.34)] we get

$$\mathrm{d}A_i = \boldsymbol{\omega} \cdot (\boldsymbol{r}_i \times \boldsymbol{f}_i)\,\mathrm{d}t + \boldsymbol{\omega} \cdot (\boldsymbol{r}_i \times \boldsymbol{F}_i)\,\mathrm{d}t = \boldsymbol{\omega}\cdot\boldsymbol{M}_{\mathrm{int},i}\,\mathrm{d}t + \boldsymbol{\omega}\cdot\boldsymbol{M}_i\,\mathrm{d}t \tag{5.47}$$

where $\boldsymbol{M}_{\mathrm{int},i}$ is the moment of an internal force relative to point 0, and $\boldsymbol{M}_i$ is the similar moment of an external force.

Summation of Eq. (5.47) for all the elementary masses yields the elementary work done on the body during the time $\mathrm{d}t$:

$$\mathrm{d}A = \sum_i \mathrm{d}A_i = \omega\left(\sum_i \boldsymbol{M}_{\mathrm{int},i}\right)\mathrm{d}t + \omega\left(\sum_i \boldsymbol{M}_i\right)\mathrm{d}t.$$

The sum of the moments of the internal forces equals zero [see Eq. (3.117)]. Consequently, designating the total moment of the external forces by $\boldsymbol{M}$, we get the expression

$$\mathrm{d}A = \boldsymbol{\omega}\cdot\boldsymbol{M}\,\mathrm{d}t = \omega M_z\,\mathrm{d}t \tag{5.48}$$

[we have used Eq. (1.21), taking into account that $M_\omega = M_z$]. Finally, since $\omega\,\mathrm{d}t$ is the angle $\mathrm{d}\varphi$ through which the body turns during the time $\mathrm{d}t$, we have

$$\mathrm{d}A = M_z\,\mathrm{d}\varphi. \tag{5.49}$$

The sign of the work depends on that of M_z, *i.e.*, on the sign of the projection of the vector $\boldsymbol{M}$ onto the direction of the vector $\boldsymbol{\omega}$.

Thus, internal forces do no work when a body rotates, the work of the external forces is determined by Eq. (5.49). We can arrive at Eq. (5.49) by taking advantage of the fact that the work done by all the forces applied to a body goes to increase its kinetic energy [see Eq. (3.11)]. Differentiating both sides of Eq. (5.46), we obtain

$$\mathrm{d}E_{\mathrm{k}} = I_z\omega\,\mathrm{d}\omega = I_z\omega\dot{\omega}\,\mathrm{d}t.$$

According to Eq. (5.15), $I_z\dot{\omega} = M_z$, and the product $\omega\,\mathrm{d}t$ equals $\mathrm{d}\varphi$. Hence, substituting $\mathrm{d}A$ for $\mathrm{d}E_{\mathrm{k}}$ we arrive at Eq. (5.49).

Table 5.1 compares the formulas of mechanics of rotation with similar formulas of mechanics of translation (mechanics of a particle). This comparison shows that in all cases of rotation the part of mass is played by the moment of inertia, the part of force by the moment of a force, the part of momentum by the angular momentum, and so on.

We obtained Eq. (5.46) for the case when a body rotates about a stationary axis fixed in the body. Now let us assume that a body rotates arbitrarily relative to a fixed point coinciding with its centre of mass. We shall rigidly associate a Cartesian system of coordinates with the body and place its origin at the centre of mass. The velocity of the i-th elementary mass is $\boldsymbol{v}_i = \boldsymbol{\omega}\times\boldsymbol{r}_i$. Consequently, we can write the following expression for the kinetic energy of the body:

$$E_{\mathrm{k}} = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i(\boldsymbol{\omega}\times\boldsymbol{r}_i)^2 = \frac{1}{2}\sum_i m_i\omega^2 r_i^2 \sin^2\varphi_i$$

where φ_i is the angle between the vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}_i$. Substituting $1 - \cos^2\varphi_i$ for $\sin^2\varphi_i$, and taking into account that $\omega r_i \cos\varphi = \boldsymbol{\omega}\cdot\boldsymbol{r}_i$, we have

$$E_{\mathrm{k}} = \frac{1}{2}\sum_i m_i[\boldsymbol{\omega}^2\cdot\boldsymbol{r}_i^2 - (\boldsymbol{\omega}\cdot\boldsymbol{r}_i)^2]^2.$$

Let us write out the scalar products through the projections of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{r}_i$ onto the axes of the coordinate system associated with the body:

$$\begin{aligned}
E_{\mathrm{k}} &= \frac{1}{2}\sum_i m_i\Big[(\omega_x^2+\omega_y^2+\omega_z^2)(x_i^2+y_i^2+z_i^2) \\
&\qquad -(\omega_x x_i+\omega_y y_i+\omega_z z_i)(\omega_x x_i+\omega_y y_i+\omega_z z_i)\Big] \\
&= \frac{1}{2}\sum_i m_i\Big[(\omega_x^2+\omega_y^2+\omega_z^2)(x_i^2+y_i^2+z_i^2) \\
&\qquad -\omega_x^2x_i^2-\omega_x\omega_y x_i y_i-\omega_x\omega_z x_i z_i-\omega_y\omega_x y_i x_i-\omega_y^2 y_i^2 \\
&\qquad -\omega_y\omega_z y_i z_i-\omega_z\omega_x z_i x_i-\omega_z\omega_y z_i y_i-\omega_z^2 z_i^2\Big].
\end{aligned}$$

Finally, combining addends with identical products of the angular velocity components and putting these products outside the sums, we get

Table 5.1

Translation	**Rotation**
$\boldsymbol{v} =$ linear velocity	$\boldsymbol{\omega} =$ angular velocity
$\boldsymbol{a} = \dot{\boldsymbol{v}} =$ linear acceleration	$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} =$ angular acceleration
$m =$ mass	$I_z =$ moment of inertia
$\boldsymbol{p} = m\boldsymbol{v} =$ momentum	$L_z = I_z\omega =$ angular momentum
$\boldsymbol{F} =$ force	$\boldsymbol{M}$ or $M_z =$ moment of force
$\dot{\boldsymbol{p}} = \boldsymbol{F}$	$\dot{\boldsymbol{L}} = \boldsymbol{M}$
$m\boldsymbol{a} = \boldsymbol{F}$	$I\alpha_z = M_z$
$E_{\mathrm{k}} = \frac{1}{2}mv^2$	$E_{\mathrm{k}} = \frac{1}{2}I\omega^2$ (for a fixed axis of rotation)
$\mathrm{d}A = F_s\,\mathrm{d}s$	$\mathrm{d}A = M_z\,\mathrm{d}\varphi$

$$E_{\mathrm{k}} = \frac{1}{2}\Bigg[(\omega_x^2 \sum_i m_i(y_i^2 + z_i^2) + \omega_y^2 \sum_i m_i(x_i^2 + z_i^2) + \omega_z^2 \sum_i m_i(x_i^2 + y_i^2)$$
$$-\omega_x\omega_y \sum_i m_i x_i y_i - \omega_x\omega_z \sum_i m_i x_i z_i - \omega_y\omega_x \sum_i m_i y_i x_i$$
$$-\omega_y\omega_z \sum_i m_i y_i z_i - \omega_z\omega_x \sum_i m_i z_i x_i - \omega_z\omega_y \sum_i m_i z_i y_i\Bigg].$$

The sums by which the products of the angular velocity components are multiplied are the components of the inertia tensor [see Eq. (5.41)]. Hence, we have arrived at the equation

$$E_{\mathrm{k}} = \frac{1}{2}\big[I_{xx}\omega_x^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yx}\omega_y\omega_x$$
$$+I_{yy}\omega_y^2 + I_{yz}\omega_y\omega_z + I_{zx}\omega_z\omega_x + I_{zy}\omega_z\omega_y + I_{zz}\omega_z^2\big]. \tag{5.50}$$

This equation can be written in the form

$$E_{\mathrm{k}} = \frac{1}{2}\sum_{i,k=x,y,z} I_{ik}\omega_i\omega_k. \tag{5.51}$$

In summation, the subscripts i and k are sequentially given the values x, y, z independently of each other.

If the axes of a coordinate system associated with a body are chosen so that they coincide with the principal axes of inertia of the body, the centrifugal moments of inertia will vanish, and Eq. (5.50) will become simplified as follows:

$$E_{\mathrm{k}} = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2). \tag{5.52}$$

Here I_x, I_y, I_z are the principal moments of inertia of the body. For a spherical top, these moments have the identical value I so that Eq. (5.52) becomes $E_{\mathrm{k}} = I\omega^2/2$ [compare with Eq. (5.46)]. When an arbitrary body rotates about one of the principal axes of inertia, say the z-axis, we have $\omega_z = \omega$, $\omega_x\omega_y = 0$, and Eq. (5.52) transforms into Eq. (5.46). Thus, the kinetic energy of a rotating body equals half the product of the moment of inertia and the square of the angular velocity in three cases: (1) for a body rotating about a fixed axis, (2) for a body rotating about one of the principal axes of inertia, and (3) for a spherical top. In all other cases, the kinetic energy is determined by more complicated equations (5.50) or (5.52).

5.7. Kinetic Energy of a Body in Plane Motion

The plane motion of a body can be represented as the superposition of two motions—translation with a velocity $\boldsymbol{v}_0$ and rotation about the relevant axis with the angular velocity $\boldsymbol{\omega}$ (see Sec. 5.1). By Eq. (5.1), the velocity of the i-th elementary mass of a body is

$$\boldsymbol{v}_i = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}_i$$

where $\boldsymbol{v}_0$ is the velocity of a certain point 0 of the body, and $\boldsymbol{r}_i$ is the position vector determining the position of the elementary mass with respect to point 0.

The kinetic energy of the i-th elementary mass is

$$E_{k,i} = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i(\boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}_i)^2.$$

Squaring the expression in parenthesis, we get

$$E_{k,i} = \frac{1}{2}m_i\left[v_0^2 + 2\boldsymbol{v}_0 \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) + (\boldsymbol{\omega} \times \boldsymbol{r}_i)^2\right].$$

The vector product of $\boldsymbol{\omega}$ and $\boldsymbol{r}_i$ has a magnitude equal to ωR_i, where R_i is the distance to the mass m_i from the axis of rotation [see Fig. 1.33 and the text preceding Eq. (1.100)]. Consequently, the third addend in the brackets equals $\omega^2 R_i^2$. Let us perform a cyclic transposition of the multipliers in the second addend [see Eq. (1.34)]. As a result, we obtain

$$E_{k,i} = \frac{1}{2}m_i\left[v_0^2 + 2(\boldsymbol{v}_0 \times \boldsymbol{\omega}) \cdot \boldsymbol{r}_i + \omega^2 R_i^2\right].$$

To obtain the kinetic energy of a body, we find the sum of Eq. (**??**) for all the elementary masses, putting the constant factors outside the sum:

$$E_k = \frac{1}{2}v_0^2 \sum_i m_i + (\boldsymbol{v}_0 \times \boldsymbol{\omega}) \cdot \sum_i m_i \boldsymbol{r}_i + \frac{1}{2}\omega^2 \sum_i m_i R_i^2.$$

The sum of the elementary masses $\sum_i m_i$ is the mass of the body m. The expression $\sum_i m_i \boldsymbol{r}_i$ is the product of the mass of the body and the position vector $\boldsymbol{r}_C$ of the centre of mass of the body. Finally, $\sum_i m_i R_i^2$ is the moment of inertia of the body I_0 relative to an axis passing through point 0. We can therefore write that

$$E_k = \frac{1}{2}mv_0^2 + m\boldsymbol{r}_C \cdot (\boldsymbol{v}_0 \times \boldsymbol{\omega}) + \frac{1}{2}I_0\omega^2. \tag{5.53}$$

If we take the centre of mass of the body as point 0, the position vector $\boldsymbol{r}_C$ will equal zero, and the second addend will vanish. Consequently, designating by $\boldsymbol{v}_C$ the velocity of the centre of mass, and by I_C the moment of inertia of the body relative to an axis passing through point C, we get the following expression for the kinetic energy of the body:

$$E_k = \frac{1}{2}mv_C^2 + \frac{1}{2}I_C\omega^2. \tag{5.54}$$

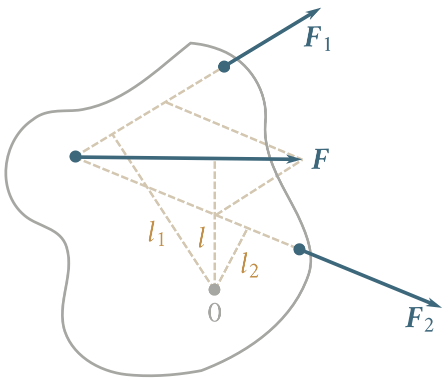

Fig. 5.20

Thus, the kinetic energy of the body in plane motion consists of the energy of translation with a velocity equalling that of the centre of mass and the energy of rotation about an axis passing through the centre of mass of the body.

5.8. Application of the Laws of Dynamics of a Body

The motion of a rigid body is described by two equations ((5.6) and (3.118)) that have already been given in previous sections:

$$m\boldsymbol{a}_C = \sum \boldsymbol{F}_{\text{ext}}$$
$$\dot{\boldsymbol{L}} = \sum \boldsymbol{M}_{\text{ext}}.$$

The motion of a body is thus determined by the external forces and the moments of these forces acting on it.

The moments of the forces may be taken relative to any point that is stationary or moving without acceleration. If we took the moment of the external forces relative to a point moving with acceleration, we would in essence write Eq. (3.118) in a non-inertial reference frame. In this case, we must take into consideration the forces of inertia and their moments apart from the external forces due to the interaction of the given body with other bodies.

The points of application of the forces acting on a body may be transferred along the lines of action of the forces because neither the sum of the forces nor their moments will change when this is done (when a force is transferred along the line of its action, the moment arm relative to any point remains unchanged). This permits us to replace several forces with a single one equivalent to them in its action on a body. For example, the two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ in one plane (Fig. 5.20) may be replaced with the force $\boldsymbol{F}$ equivalent to them. The point of application of the latter may also be chosen arbitrarily on the direction of its action.

A combination of parallel forces acting on a body may be replaced with their resultant equal to the sum of all the forces and applied to a point of the body such that its moment equals the sum of the moments of the separate forces.

Let us find the resultant of the forces of gravity. These forces are applied to all the elements of a body, the force $m_i\boldsymbol{g}$ acting on the elementary mass m_i. The sum of these forces is $\boldsymbol{P} = m\boldsymbol{g}$, where $m = \sum_i m_i$ is the mass of the body. The total moment of the forces of gravity relative to a certain point 0 is

$$\boldsymbol{M} = \sum_i \boldsymbol{r}_i \times (m_i\boldsymbol{g})$$

where $\boldsymbol{r}_i$ is the position vector determining the position of the mass m_i with respect to point 0. Transferring the scalar multiplier m_i from the second member of the product to the first one and then putting the common factor $\boldsymbol{g}$ outside the sum, we get

$$\boldsymbol{M} = \left(\sum_i m_i\boldsymbol{r}_i\right) \times \boldsymbol{g}.$$

The sum in parentheses equals the product of the mass of the body and the position vector $\boldsymbol{r}_\text{C}$ of the centre of mass C. Hence,

$$\boldsymbol{M} = (m\boldsymbol{r}) \times \boldsymbol{g} = \boldsymbol{r}_\text{C} \times (m\boldsymbol{g}) = \boldsymbol{r}_\text{C} \times \boldsymbol{P}. \tag{5.55}$$

Thus, the total moment of the forces of gravity relative to an arbitrary point 0 coincides with the moment of the force $m\boldsymbol{g}$ applied to point C. Thus, the resultant of the forces of gravity equals $\boldsymbol{P} = m\boldsymbol{g}$ and is applied to the centre of mass of the body. We must note that this holds only when the field of the forces of gravity is homogeneous within the body [in deriving Eq. (5.55) we considered that $\boldsymbol{g}$ = constant].

It follows from Eq. (5.55) that the moment of the forces of gravity relative to the centre of mass equals zero (in this case $\boldsymbol{r}_\text{C} = 0$). The point relative to which the moment of the forces of gravity equals zero is called the **centre of gravity** of the body. Thus, when the field of gravity forces is homogeneous within a body, the centre of gravity coincides with the centre of mass.

For a homogeneous gravitational field, the forces of gravity applied to different elementary masses have an identical direction and are proportional to m_i. The forces of inertia produced in a non-inertial reference frame moving in a straight line relative to inertial frames have the same property. Indeed, in this case, the forces of inertia applied to the elementary masses m_i equal $-m_i\boldsymbol{a}_0$, where $\boldsymbol{a}_0$ is the acceleration of the non-inertial frame [see Eq. (4.2)]. By repeating the reasoning that led us to Eq. (5.55) (here $-m_i\boldsymbol{a}_0$ must be substituted for $m\boldsymbol{g}$), we can show that the resultant of the inertia forces equals $-m\boldsymbol{a}_0$ and is applied to the centre of mass of the body. It must be stressed that this holds only for reference frames moving in a straight line.

The moment of the inertia forces relative to the centre of mass equals zero (in a frame with translational motion). Therefore, when compiling Eq. (3.118) for the moments taken relative to the centre of mass, the forces of inertia do not have to be taken into consideration.

Let us find the conditions of equilibrium of a rigid body. A body can remain in a state of rest if nothing causes the appearance of translation or rotation. According to Eqs. (5.6) and (3.118), two conditions are essential and sufficient in this case:

(1) the sum of all the external forces applied to a body must equal zero:

$$\sum \boldsymbol{F}_{\text{ext}} = 0. \tag{5.56}$$

(2) the resultant moment of the external forces relative to any point must equal zero:

$$\sum \boldsymbol{M}_{\text{ext}} = 0. \tag{5.57}$$

When condition (5.56) is obeyed, from the equality to zero of the sum of the moments for one point 0 we get the equality to zero of the sum of the moments relative to any other point $0'$. Indeed, assume that for a certain point 0 we have

$$\sum_i \boldsymbol{M}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i = 0. \tag{5.58}$$

Let us take another point $0'$ whose position relative to 0 is determined by the vector $\boldsymbol{b}$. Examination of Fig. 5.21 shows that $\boldsymbol{r}'_i = \boldsymbol{r}_i - \boldsymbol{b}$. Consequently, the sum of the moments relative to point $0'$ is

$$\sum_i \boldsymbol{M}'_i = \sum_i \boldsymbol{r}'_i \times \boldsymbol{F}_i = \sum_i (\boldsymbol{r}_i - \boldsymbol{b}) \times \boldsymbol{F}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i - \sum_i \boldsymbol{b} \times \boldsymbol{F}_i.$$

According to Eq. (5.58), the first sum equals zero. Factoring out the constant quantity $\boldsymbol{b}$ in the second sum, we get the expression $-(\boldsymbol{b} \times \sum_i \boldsymbol{F}_i)$ which in view of Eq. (5.56) also vanishes. Thus, from Eq. (5.56) and condition (5.58) for point 0, we get condition (5.58) for point $0'$.

It must be noted that the vector condition (5.57) is equivalent to three scalar ones:

$$\sum M_{x,\text{ext}} = 0, \quad \sum M_{y,\text{ext}} = 0, \quad \sum M_{z,\text{ext}} = 0. \tag{5.59}$$

Thus, the conditions of equilibrium of a rigid body are determined by Eqs. (5.56) and (5.57), or by Eqs. (5.56) and (5.59).

In conclusion, let us consider an example of the application of the laws of dynamics of a rigid body. Assume that a homogeneous cylinder of radius R and mass m rolls down an inclined plane (Fig. 5.22) without slipping. The angle of inclination of the plane is β and its height is h ($h \sim R$). The initial velocity of the cylinder is

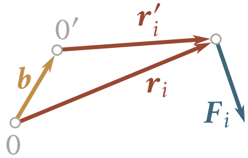

Fig. 5.21

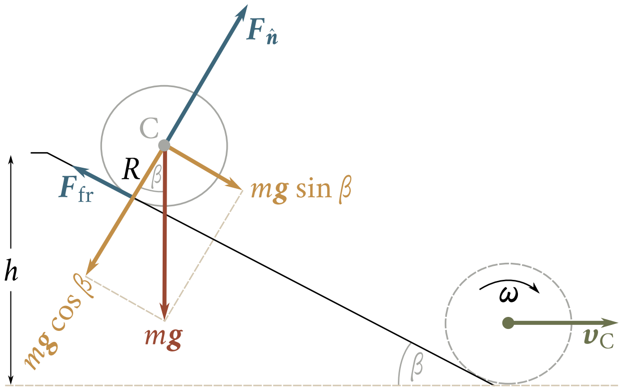

Fig. 5.22

zero. We are to find the velocity of the centre of mass and the angular velocity of the cylinder at the moment when it reaches the horizontal section. We shall give two variants of the solution.

First Variant. The cylinder will move under the action of three forces—the force $\boldsymbol{P} = m\boldsymbol{g}$, the force of friction $\boldsymbol{F}_{\mathrm{fr}}$, and the force of normal pressure $\boldsymbol{F}_{\hat{\boldsymbol{n}}}$ (see Sec. 2.12. The acceleration of the cylinder in the direction of a normal to the plane is zero. Consequently, the magnitude of the force of normal pressure equals the normal component of the force $\boldsymbol{P}$ having the magnitude $mg \cos\beta$.

Friction appears between the cylinder and the plane at the points of their contact. In the absence of slipping, these points of the cylinder are stationary (they form an instantaneous axis of rotation). Hence, the force of friction we are dealing with is a static force of friction. We know from See. 2.10 that the static force of friction can range from zero to the maximum value F_0 that is determined by the product of the coefficient of friction and the force of normal pressure pressing the contacting bodies against each other ($F_0 = fmg \cos\beta$). In the case under consideration, the force of friction takes on a value such that slipping will be absent. Slipping will be absent when the cylinder rolls along the plane provided that the linear velocity of the points of contact vanishes. This will occur, in turn, if the velocity of the centre of mass v_{C} at each moment of time equals the angular velocity of rotation of the cylinder ω multiplied by the radius of the cylinder R:

$$v_{\mathrm{C}} = \omega R. \tag{5.60}$$

The acceleration of the centre of mass a_{C} will accordingly equal the angular acceleration α multiplied by R:

$$a_{\mathrm{C}} = \alpha R. \tag{5.61}$$

If the force of friction needed to obey conditions (5.60) and (5.61) does not exceed the maximum value F_0, then the cylinder will roll down the plane without slipping. Otherwise rolling without slipping is impossible.

Equation (5.6) in the given case has the form

$$m\boldsymbol{a}_C = m\boldsymbol{g} + \boldsymbol{F}_{\text{fr}} + \boldsymbol{F}_{\hat{\boldsymbol{n}}}.$$

Projecting it onto the direction of motion, we get

$$ma_C = mg\sin\beta - F_{\text{fr}}. \tag{5.62}$$

For a homogeneous cylinder rotating about an axis of symmetry, $\boldsymbol{L} = I\boldsymbol{\omega}$. Therefore, Eq. (3.118) can be written in the form

$$I\alpha = \sum M_z \tag{5.63}$$

where z is the axis of the cylinder [see Eq. (5.51)]. In Eq. (5.63) written relative to the axis of the cylinder, only the moment of the force of friction will differ from zero. The remaining forces including the resultant of the forces of inertia are directed through the axis of the cylinder. As a result, their moments relative to this axis equal zero. Thus, Eq. (5.63) will be written as follows:

$$I\alpha = R\,F_{\text{fr}}. \tag{5.64}$$

Here I is the moment of inertia of the cylinder relative to its axis equal to $mR^2/2$.

Equations (5.62) and (5.64) contain three unknown quantities, F_{fr}, a_C and α. The last two of them are related by Eq. (5.61) resulting from the absence of friction. By solving the system of equations (5.61), (5.62), and (5.64), we shall find (with account of the fact that $I = mR^2/2$) the values of the required quantities:

$$F_{\text{fr}} = \frac{1}{3}mg\sin\beta, \tag{5.65}$$

$$a_C = \frac{2}{3}g\sin\beta, \tag{5.66}$$

$$\alpha = \frac{2}{3}\left(\frac{g}{R}\right)\sin\beta. \tag{5.67}$$

Now that we know the value of the static force of friction needed for rolling down of the cylinder without slipping, we can find the condition at which this rolling is possible. For the cylinder to roll down without slipping, the force (5.65) must not exceed the maximum value of the static force of friction F_0 equal to $fmg\cos\beta$:

$$\frac{1}{3}mg\sin\beta \leqslant mg\cos\beta$$

whence

$$\tan\beta \leqslant 3f.$$

Consequently, if the slope ($\tan\beta$) of the plane exceeds the triple value of the static coefficient of friction between the cylinder and the plane, rolling down cannot occur without slipping.

From the constancy of a_C [see Eq. (5.66)] it follows that the centre of mass of the cylinder moves with uniform acceleration. During the time t_r that it rolls down, the cylinder travels the distance $h/\sin\beta$. In uniformly accelerated motion, the distance, acceleration, and time are related by the equation $s = at^2/2$. Introducing the value of s, we get

$$\frac{h}{\sin\beta} = \frac{1}{2}a_C t_r^2$$

whence, introducing the value of a_C from Eq. (5.66), we have

$$t_r = \frac{1}{\sin\beta}\left(\frac{3h}{g}\right)^{1/2}.$$

This time, like a_C, does not depend on the mass and radius of the cylinder[9]. It is determined only by the angle of inclination of the plane β and the difference between the levels of its edges h.

The velocity of the centre of mass when the cylinder reaches the horizontal section will be

$$v_C = a_C t_r = \left(\frac{4}{3}gh\right)^{1/2}$$

and the angular velocity of the cylinder will be

$$\omega = \alpha t_r = \frac{1}{R}\left(\frac{4}{3}gh\right)^{1/2}.$$

We must note that the static force of friction does no work on the cylinder because the points of the cylinder which this force is applied to are stationary at each moment of time [see Eq. (3.16)].

We find for the horizontal plane ($\beta = 0$) by Eqs. (5.66) and (5.67) that the cylinder will travel without acceleration if it is first imparted a certain translational velocity and the corresponding (such that no slipping occurs) angular velocity. The motion will actually be retarded. This is due to the force of rolling friction which is directed so that its moment reduces the angular velocity ω, while the force itself produces a corresponding (again such that no slipping will appear) retardation of the centre of mass. The force of rolling friction does negative work on a rolling body.

In solving the problem on the rolling of a cylinder down an inclined plane, we disregarded rolling friction.

Second Variant. Since the force of friction does no work (we disregard rolling friction), the total energy of the cylinder remains constant. At the initial moment,

[9]This holds only for a homogeneous solid cylinder.

the kinetic energy is zero, and the potential energy is mgh. At the bottom of the inclined plane, the potential energy vanishes but a kinetic energy appears equal to [see Eq. (5.54)]:

$$E_k = \frac{mv_C^2}{2} + \frac{I_C\omega^2}{2}.$$

Since slipping is absent, v_C and ω are related by the expression $v_C = \omega R$. Introducing $\omega = v_C/R$ and $I_C = mR^2/2$ into the expression for the kinetic energy, we get

$$E_k = \frac{mv_C^2}{2} + \frac{mv_C^2}{4} = \frac{3}{4}mv_C^2.$$

The total energy at the beginning and end of rolling down the inclined plane must be the same:

$$\frac{3}{4}m\,v_C^2 = mgh$$

whence

$$v_C = \left(\frac{4}{3}gh\right)^{1/2}$$

and the angular velocity is

$$\omega = \frac{v_C}{R} = \frac{1}{R}\left(\frac{4}{3}gh\right)^{1/2}.$$

Pay attention to how much simpler the second variant of solution is than the first one.

5.9. Gyroscopes

A gyroscope (or top) is a massive symmetrical body rotating with a great velocity about an axis of symmetry. We shall call this axis the axis of the gyroscope. It is one of the principal axes of inertia. Therefore, if it does not turn in space, the angular momentum is $\boldsymbol{L} = I\boldsymbol{\omega}$, where I is the moment of inertia relative to the gyroscope axis. Let us now assume that the gyroscope axis rotates with a certain velocity $\boldsymbol{\omega}'$. In this case, the resultant rotation of the gyroscope occurs about an axis not coinciding with an axis of symmetry, and the direction of the vector $\boldsymbol{L}$ does not coincide with that of the gyroscope axis. If the angular velocity ω' of the axis is negligibly small in comparison with the angular velocity ω of the gyroscope itself, however ($\omega' \ll \omega$), then we may assume that the vector $\boldsymbol{L}$ is approximately equal to $I\boldsymbol{\omega}$ and is directed along the gyroscope axis. In this condition, rotation of the vector $\boldsymbol{L}$ and rotation of the gyroscope axis will be equivalent. We shall assume in the following that the condition $\omega' \ll \omega$ is obeyed.

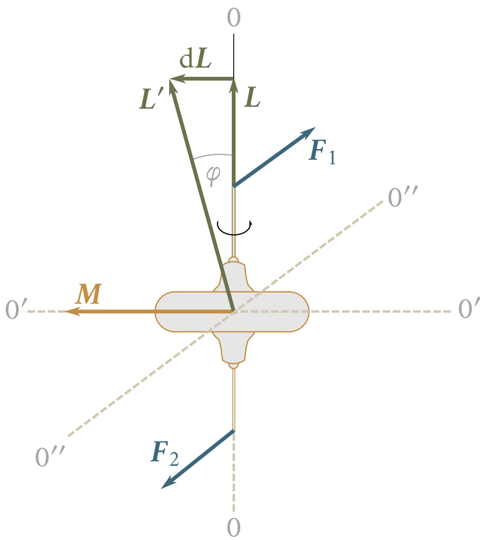

Fig. 5.23

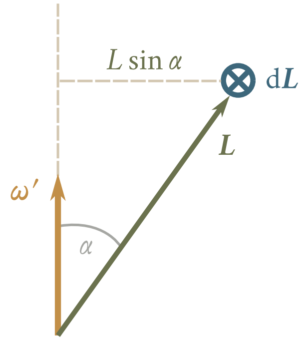

Fig. 5.24

When an attempt is made to turn the gyroscope axis, a distinctive phenomenon is observed called the gyroscopic effect: under the action of forces that ought to cause rotation of the gyroscope axis 00 about straight line 0′0′ (Fig. 5.23), the axis turns about straight line 0″0″ (axis 00 and straight line 0′0′ are in the plane of the drawing, and straight line 0″0″ and the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are at right angles to this plane). The behaviour of the gyroscope, which seems unnatural at first sight, completely conforms with the laws of rotational dynamics. Indeed, the moment of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is directed along straight line 0′0′. During the time dt, the angular momentum of the gyroscope $\boldsymbol{L}$ receives the increment $\mathrm{d}\boldsymbol{L} = \boldsymbol{M}\,\mathrm{d}t$, which has the same direction as $\boldsymbol{M}$. After the time dt elapses, the angular momentum of the gyroscope will equal the resultant $\boldsymbol{L}' = \boldsymbol{L} + \mathrm{d}\boldsymbol{L}$ in the plane of the figure. The direction of the vector $\boldsymbol{L}'$ coincides with the new direction of the gyroscope axis. Thus, the latter will turn about straight line 0″0″ through a certain angle dφ. It can be seen from Fig. 5.23 that $\mathrm{d}\varphi = |\mathrm{d}\boldsymbol{L}|/L = M\,\mathrm{d}t/L$. Hence, it follows that the gyroscope axis turned to its new position with the angular velocity $\omega' = \mathrm{d}\varphi/\mathrm{d}t = M/L$. Let us write this relation in the form $M = \omega' L$. The vectors $\boldsymbol{M}$, $\boldsymbol{L}$ and $\boldsymbol{\omega}'$ are mutually perpendicular (the vector $\boldsymbol{\omega}'$ is directed along straight line 0″0″ toward us). The relation between them can therefore be written in the form

$$\boldsymbol{M} = \boldsymbol{\omega}' \times \boldsymbol{L}. \tag{5.68}$$

We have obtained this equation for the case when the vectors $\boldsymbol{\omega}'$ and $\boldsymbol{L}$ are mutually perpendicular. It also holds, however, in the most general case. A glance at Fig. 5.24 shows that when the gyroscope axis turns about the vector $\boldsymbol{\omega}'$ through the angle

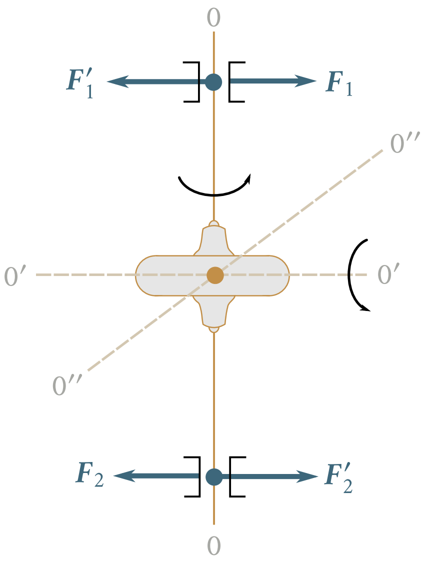

Fig. 5.25

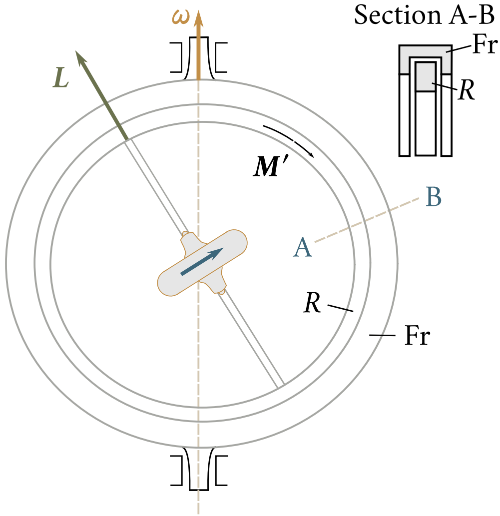

Fig. 5.26

$d\varphi$ the vector $\boldsymbol{L}$ receives an increment whose magnitude is $|d\boldsymbol{L}| = L \sin\alpha \, d\varphi$. At the same time $|d\boldsymbol{L}| = M \, dt$. Thus, $L \sin\alpha \, d\varphi = M \, dt$, whence $M = \omega' L \sin\alpha$. It is easy to see with the aid of Fig. 5.24 that in this case Eq. (5.68) holds (the vectors $\boldsymbol{\omega}'$ and $\boldsymbol{L}$ are in the plane of the figure, the vector $d\boldsymbol{L}$ is directed beyond the drawing and is therefore depicted by a circle with a cross in it). We remind our reader that Eq. (5.68) is correct only if $\omega' \ll \omega$.

When attempts are made to cause the axis of a gyroscope to turn in a given way, the so-called gyroscopic forces are set up owing to the **gyroscopic effect**. These forces act on the bearings in which the gyroscope axis rotates. For example, if gyroscope axis 00 is forcibly turned about straight line 0′0′ (Fig. 5.25), the gyroscope axis tends to turn about straight line 0″0″. To prevent this rotation, the forces $\boldsymbol{F}_1'$ and $\boldsymbol{F}_2'$ acting from the side of the bearings must be applied to the gyroscope axis. According to Newton's third law, the gyroscope axis will act on the bearings with the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, and the latter are exactly the gyroscopic forces. Upon forced turning of the gyroscope axis with the angular velocity $\boldsymbol{\omega}'$, the moment of the forces with which the bearings act on the axis is determined by Eq. (5.68). The moment of the gyroscopic forces with which the axis acts on the bearings is

$$\boldsymbol{M}' = \boldsymbol{L}' \times \boldsymbol{\omega}. \tag{5.69}$$

Let us assume that the axis of a gyroscope is fixed in ring R that can freely turn in frame Fr (Fig. 5.26). Let us turn the frame about an axis in its plane with the angular velocity $\boldsymbol{\omega}'$. In this case, as we have found out, a moment of gyroscopic

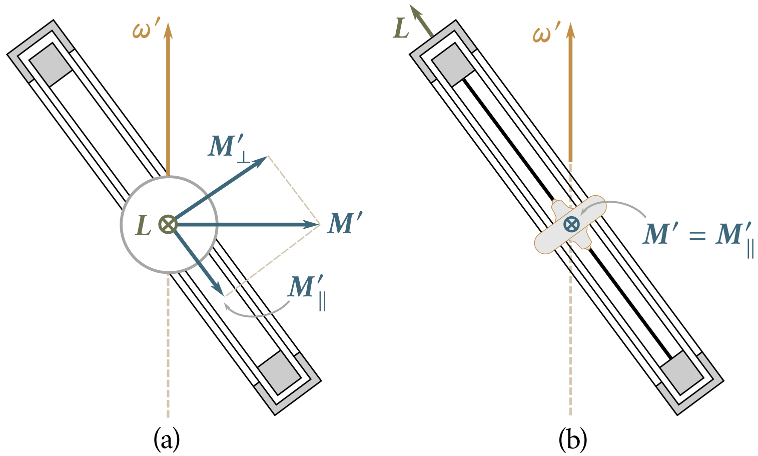

Fig. 5.27

forces determined by Eq. (5.69) is produced that acts on the ring. This moment will cause the ring to turn in the frame in the direction indicated by the arrow until the gyroscope axis becomes arranged in the direction of the axis of rotation of the frame and the moment (5.69) vanishes. The direction of rotation of the gyroscope itself and the direction in which the frame turns will coincide. When $\boldsymbol{L}$ and $\boldsymbol{\omega}'$ are directed oppositely, the moment (5.69) also vanishes. The corresponding position of the gyroscope axis, however, will be unstable—upon the slightest deviation of the angle between $\boldsymbol{L}$ and $\boldsymbol{\omega}'$ from 180°, the moment $\boldsymbol{M}'$ will be set up that will turn the axis until this angle becomes equal to zero.

Now let us assume that the frame turns with the angular velocity $\boldsymbol{\omega}'$ about an axis not in its plane (Fig. 5.27). In the position of the ring at which the angular momentum of the gyroscope $\boldsymbol{L}$ is perpendicular to $\boldsymbol{\omega}'$ (Fig. 5.27a), the vector $\boldsymbol{M}'$ has the direction shown in the figure. The component $\boldsymbol{M}'_{\perp}$ of this vector causes the ring to turn in the frame, and as a result the angle between the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}'$ will diminish. The component $\boldsymbol{M}'_{\parallel}$ tends to misalign the ring relative to the frame. When the ring occupies a position such that the angle between the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}'$ takes on the smallest possible value (Fig. 5.27b), the component $\boldsymbol{M}'_{\perp}$ will vanish because in this case the moment of the gyroscopic forces $\boldsymbol{M}'$ is in the plane of the ring; this moment cannot produce rotation of the ring in the frame. Thus, under the action of gyroscopic forces, the ring occupies such a position in the frame in which the angle between the gyroscope axis and the axis of rotation of the frame is minimum.

An instrument called the gyrocompass (gyroscopic compass) is based on the behaviour of a gyroscope described above. This instrument is a gyroscope whose

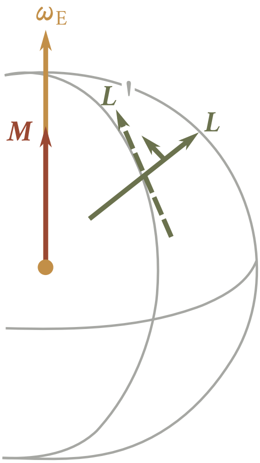

Fig. 5.28

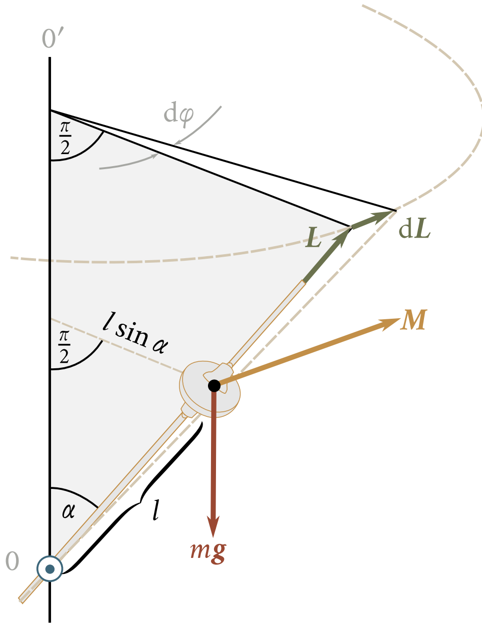

Fig. 5.29

axis can freely turn in a horizontal plane. The Earth's daily rotation causes the axis of the gyrocompass to arrange itself in a position such that the angle between this axis and the Earth's axis of rotation will be minimum (Fig. 5.28). In this position, the axis of the gyrocompass will be in a meridian plane and, consequently, line up in a north-south direction. A gyrocompass advantageously differs from its magnetic pointer counterpart in that no corrections have to be introduced into its readings for the so-called magnetic declination (the angle between the magnetic and the geographic meridians). Another advantage is that no measures have to be taken to compensate for the action on the pointer of ferromagnetic objects near the compass (for example, the steel hull of a ship).

Assume that the axis of a gyroscope can freely turn about point 0 (Fig. 5.29). Let us consider the behaviour of such a gyroscope in the field of forces of gravity. The magnitude of the moment of the forces applied to the gyroscope is

$$M = mgl \sin \alpha \tag{5.70}$$

where m is the mass of the gyroscope, l is the distance from point 0 to the centre of mass of the gyroscope and α is the angle made by the gyroscope axis with a vertical line.

The vector $\boldsymbol{M}$ is perpendicular to the vertical plane passing through the gyroscope axis (this plane is shaded in Fig. 5.29).

Under the action of the moment $\boldsymbol{M}$, the angular momentum $\boldsymbol{L}$ changes during the time $\mathrm{d}t$ by the increment $\mathrm{d}\boldsymbol{L} = \boldsymbol{M}\,\mathrm{d}t$ perpendicular to the vector $\boldsymbol{L}$. The amount by which the vector $\boldsymbol{L}$ changes upon receiving the increment $\mathrm{d}\boldsymbol{L}$ corresponds to turning of the gyroscope axis such that the angle a does not change. The vertical plane passing through the gyroscope axis turns through the angle $\mathrm{d}\varphi$.

The vector $\boldsymbol{M}$ turns through the same angle in a horizontal plane. As a result, when the time $\mathrm{d}t$ elapses, the vectors $\boldsymbol{L}$ and $\boldsymbol{M}$ will have the same mutual arrangement as at the initial moment.

During the next moment $\mathrm{d}t$, the vector $\boldsymbol{L}$ again receives the increment $\mathrm{d}\boldsymbol{L}$ that is perpendicular to the new direction of the vector $\boldsymbol{L}$ setting in after the preceding elementary turn, etc. As a result, the gyroscope axis will rotate about the vertical axis passing through point 0 with the angular velocity ω'. It will describe a cone with an apex angle of 2α (compare with Fig. 5.24). (When $\alpha = \pi/2$, the cone degenerates into a plane.) The vector $\boldsymbol{L}$ will change only in direction. Its magnitude will be constant because the elementary increments $\mathrm{d}\boldsymbol{L}$ will always be perpendicular to the vector $\boldsymbol{L}$[10].

Thus, in the field of forces of gravity, the axis of a gyroscope with a fixed point rotates about a vertical line, describing a cone. Such motion of a gyroscope is called precession. We can find the angular velocity ω' of precession if we take into account that by Eq. (5.68) $M = \omega' L \sin\alpha$. Equating this value to Eq. (5.70), we get $\omega' L \sin\alpha = mgl \sin\alpha$, whence

$$\omega' = \frac{mgl}{L} = \frac{mgl}{I\omega}. \tag{5.71}$$

It follows from Eq. (5.71) that the velocity of precession does not depend on the angle of inclination of the gyroscope axis with respect to a vertical line (on the angle α).

We have considered the approximate theory of the gyroscope. According to the strict theory, rotation of the axis about a vertical line is accompanied by oscillations of the axis in a vertical plane. The latter are attended by changes in the angle α ranging from α_1 to α_2. This wobbling of the axis is called **nutation**. Depending on the initial conditions, the end of a gyroscope axis draws one of the curves depicted in Fig. 5.30 on an imaginary spherical surface. If, for example, after positioning the axis at the angle α_1, we make the gyroscope rotate and then release the axis gently, the latter will first lower while rotating about the vertical line. After reaching the angle α_2, the axis will begin to rise, and so on (this case is shown in Fig. 5.30b).

[10]We can find similar behaviour in the velocity vector when a point moves uniformly along a circle. The vector $\boldsymbol{v}$ receives the increment $\mathrm{d}\boldsymbol{v} = \boldsymbol{a}_{\hat{n}}\,\mathrm{d}t$ ($\boldsymbol{a}_{\hat{n}}$ = constant) during the time $\mathrm{d}t$. As a result, the direction of the vector $\boldsymbol{v}$ changes, while its magnitude remains constant.

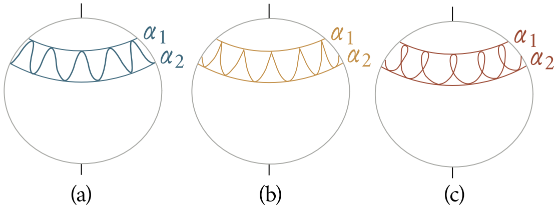

Fig. 5.30

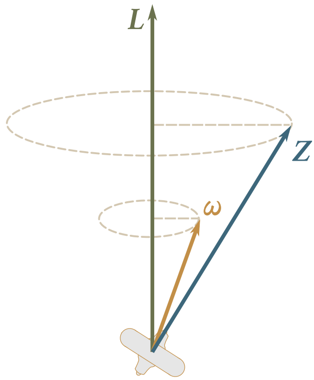

Fig. 5.31

By imparting an initial impetus of a quite definite magnitude and direction to a gyroscope, we can achieve precession of its axis without nutation. Such precession is defined as **regular**. The amplitude of nutation diminishes with an increasing gyroscope velocity of rotation. Nutation is also absorbed by friction in the support. This is why nutation is often unnoticeable in practice. Precession that is only approximately regular is called **pseudoregular**.

If point 0 is at the centre of mass of a gyroscope (see Fig. 5.29), the moment of the force of gravity becomes equal to zero, and we get the so-called free symmetrical top. Owing to the law of conservation, the angular momentum of such a top will change neither in magnitude nor in direction. If we rotate the top about its axis of symmetry, the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ will have the same direction which remains constant for an infinitely long time. If, however, the top is rotated about an axis not coinciding with any of its principal axes of inertia, the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ will not coincide (Fig. 5.31). The relevant calculations give us the following results. The vector $\boldsymbol{\omega}$ remains constant in magnitude and precesses about the direction of the vector $\boldsymbol{L}$ describing a cone. At the same time, the axis of symmetry z of the top precesses, the vectors $\boldsymbol{L}$ and $\boldsymbol{\omega}$ and the z-axis constantly being in one plane. The top rotates about the z-axis with the angular velocity $\omega_z = L_z/I_z$, where L_z is the projection of the vector $\boldsymbol{L}$ onto the z-axis, and I_z is the moment of inertia of the top relative to this axis. The angular velocity of precession is $\omega_{\text{pr}} = L/I$, where I is the identical value of the moments of inertia I_x and I_y.

Chapter 6

UNIVERSAL GRAVITATION

6.1. Law of Universal Gravitation

All bodies in nature mutually attract one another. The law which this attraction obeys was established by Newton and is called the **law of universal gravitation**. This law states: *the force with which two point particles attract each other is proportional to their masses and inversely proportional to the square of the distance between them*:

$$F = G\frac{m_1 m_2}{r^2}. \tag{6.1}$$

Here G is a constant of proportionality called the gravitational constant. The force is directed along the straight line passing through the interacting particles (Fig. 6.1).

The force with which the second particle attracts the first one can be written in the vector form as follows:

$$\boldsymbol{F}_{12} = G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{e}}_{12}. \tag{6.2}$$

The symbol $\hat{\boldsymbol{e}}_{12}$ stands for a unit vector directed from the first particle to the second one (see Fig. 6.1). Substituting the vector $\hat{\boldsymbol{e}}_{21}$ for the vector $\hat{\boldsymbol{e}}_{12}$ in Eq. (6.2), we get the force $\boldsymbol{F}_{21}$ acting on the second particle.

To find the force of interaction of extended bodies, they must be divided into elementary masses Δm, each of which can be assumed to be a point particle (Fig. 6.2). According to Eq. (6.2), the i-th elementary mass of body 1 is attracted to the k-th elementary mass of body 2 with the force

$$\boldsymbol{F}_{ik} = G\frac{\Delta m_i \Delta m_k}{r_{ik}^2}\hat{\boldsymbol{e}}_{ik} \tag{6.3}$$

where r_{ik} is the distance between the elementary masses.

Summation of Eq. (6.3) over all the values of the subscript k gives the force

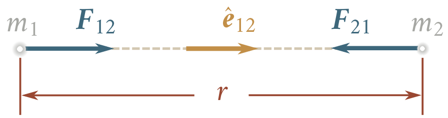

Fig. 6.1

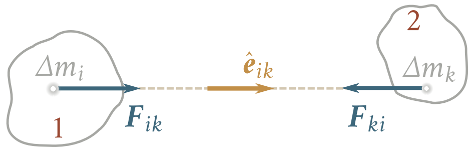

Fig. 6.2

exerted by body 2 on the elementary mass Δm_i, belonging to body 1:

$$\boldsymbol{F}_{i2} = \sum_k G \frac{\Delta m_i \Delta m_k}{r_{ik}^2} \hat{\boldsymbol{e}}_{ik}. \tag{6.4}$$

Finally, summation of Eq. (6.4) over all the values of the subscript i, *i.e.*, summation of the forces applied to all the elementary masses of the first body gives the force exerted by body 2 on body 1:

$$\boldsymbol{F}_{12} = \sum_i \sum_k G \frac{\Delta m_i \Delta m_k}{r_{ik}^2} \hat{\boldsymbol{e}}_{ik}. \tag{6.5}$$

Summation is performed over all the values of the subscripts i and k. Consequently, if body 1 is divided into N_1, and body 2 into N_2 elementary masses, then the sum (6.5) will contain $N_1 N_2$ addends.

In practice, the summation according to Eq. (6.5) consists in integration and, generally speaking, is a very complicated mathematical problem. If the interacting bodies are homogeneous and have a regular shape, the calculations are greatly simplified. In particular, when the interacting bodies are homogeneous[1] spheres, calculation by Eq. (6.5) leads to Eq. (6.2), m_1 and m_2 now being the masses of the spheres, r the distance between their centres, and $\hat{\boldsymbol{e}}_{12}$ a unit vector directed from the centre of the first sphere to that of the second one. The spheres thus interact like point particles of masses equal to those of the spheres and situated at their centres.

If one of the bodies is a homogeneous sphere of a very great radius (for example, the Earth), while the second body can be considered as a point particle, then their interaction is described by Eq. (6.2) in which r stands for the distance from the centre of the sphere to the particle (this statement will be proved in the following section).

The dimension of the gravitational constant in accordance with Eq. (6.1) is

$$[G] = \frac{[F][r^2]}{[m^2]} = \frac{(ML/T^2)L^2}{M^2} = L^3 M^{-1} T^{-2}.$$

The numerical value of G was determined by measuring the force with which two

[1]It is sufficient for the distribution of the mass within the limits of each sphere to have central symmetry, *i.e.*, for the density to be a function only of the distance from the centre of the sphere.

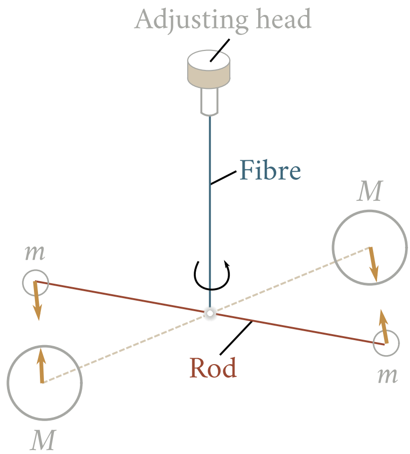

Fig. 6.3

bodies of known mass attract each other. Great difficulties appear in such measurements because the forces of attraction are extremely small for bodies whose masses can be measured directly. For example, two bodies each having a mass of 100 kg and one metre apart interact with a force of the order of 10^{-6} N, *i.e.*, about 10^{-4} gf.

The first successful attempt to determine G was its measurement carried out by Henry Cavendish (1731-1810) in 1798. He used the very sensitive torsion balance method (Fig. 6.3). Two lead spheres m (each of mass 0.729 kg) fastened to the ends of a light rod were placed near symmetrically arranged spheres M (each of mass 158 kg). The rod was suspended on an elastic torsion fibre. Twisting of the latter was measured, and its magnitude showed the force of attraction between the spheres. The top end of the fibre was fastened in an adjusting head whose turning made it possible to change the distance between the spheres m and M. The value

$$G = 6.670 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}\ \ (\text{or } \mathrm{N\,m^2\,kg^{-2}})$$

is considered to be the most accurate of all the values determined in different ways.

If in Eq. (6.1) we assume that m_1, m_2, and r equal unity, then the force numerically equals G. Thus, two spheres each having a mass of 1 kg whose centres are 1 m apart attract each other with a force of 6.670×10^{-11} N.

6.2. Gravitational Field

Gravitational interaction is carried out through a gravitational field. Every body changes the properties of the space surrounding it—it sets up a gravitational field in this space. The field manifests itself in that another body placed in it experiences a force. The "intensity" of a gravitational field can obviously be assessed according to the magnitude of the force acting at a given point on a body of unit mass. Accordingly, the quantity

$$\boldsymbol{g}' = \frac{\boldsymbol{F}}{m} \tag{6.6}$$

is called the **gravitational intensity**, or the **gravitational field vector**. In Eq. (6.6), $\boldsymbol{F}$ is the gravitational force acting on a point particle of mass m at a given point of the field.

The dimension of $\boldsymbol{g}'$ coincides with that of acceleration. The intensity of the gravitational field near the Earth's surface equals the acceleration of free fall $\boldsymbol{g}$ (with an accuracy up to the correction due to the Earth's rotation, see Sec. 4.2).

It is easy to conclude from Eq. (6.2) that the intensity of the field set up by a point particle of mass m is

$$\boldsymbol{g}' = -G\frac{m}{r^2}\hat{\boldsymbol{e}}_r \tag{6.7}$$

where $\hat{\boldsymbol{e}}_r$ is the unit vector of the position vector drawn from the particle to the given point of the field, and r is the magnitude of this position vector.

Assume that a gravitational field is produced by a point particle of mass m fixed at the origin of coordinates. Hence, the following force will act on a particle of mass m' at a point with the position vector $\boldsymbol{r}$:

$$\boldsymbol{F} = \boldsymbol{g}'m' = -G\frac{mm'}{r^2}\hat{\boldsymbol{e}}_r \tag{6.8}$$

[compare with Eq. (3.120)]. We showed in Sec. 3.13 that the potential energy of the particle m' is determined in this case by the equation

$$E_{\mathrm{p}} = -G\frac{mm'}{r^2} \tag{6.9}$$

(the potential energy is assumed to vanish when $r \to \infty$). Equation (6.9) can also be interpreted as the mutual potential energy of the point particles m and m'.

Inspection of Eq. (6.9) shows that to each point of the field produced by the particle m there corresponds a definite value of the potential energy which the particle m' has in this field. Consequently, the field can be characterized by the potential energy which a particle of mass $m' = 1$ has at the given point. The quantity

$$\varphi = \frac{E_{\mathrm{p}}}{m'} \tag{6.10}$$

is called the **potential of the gravitational field**. In this equation, E_{p} is the potential energy which a point particle of mass m' has at a given point of the field.

Knowing the potential of a field, we can calculate the work done on the particle m' by the forces of the field when moving it from position 1 to position 2.

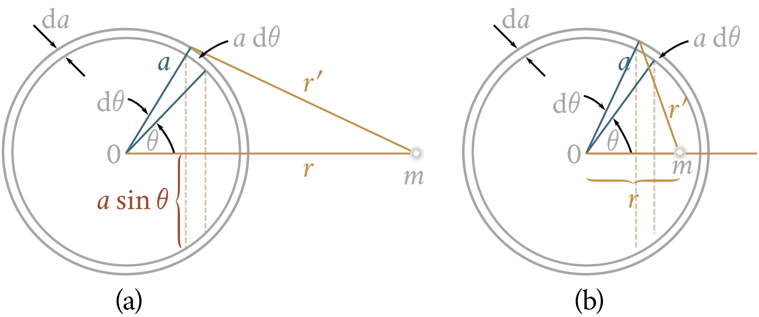

Fig. 6.4

According to Eq. (3.30), this work is

$$A_{12} = E_{p,1} - E_{p,2} = m'(\varphi_1 - \varphi_2). \tag{6.11}$$

According to Eqs. (6.6) and (6.10), the force acting on the particle m' is $\boldsymbol{F} = m'\boldsymbol{g}'$, and the potential energy of this particle is $E_p = m'\varphi$. By Eq. (3.31), we have $\boldsymbol{F} = -\nabla E_p$, *i.e.*, $m'\boldsymbol{g}' = -\nabla(m'\varphi)$. Putting the constant m' outside the gradient sign and then cancelling this constant, we arrive at a relation between the intensity and potential of a gravitational field:

$$\boldsymbol{g}' = -\nabla\varphi. \tag{6.12}$$

Let us find an expression for the mutual potential energy of a homogeneous spherical layer and a point particle of mass m. We shall consider two cases corresponding to the particle being outside and inside the layer, and shall begin with the former one (Fig. 6.4a). Let us separate from the layer a ring whose edges correspond to the values of the angle θ and $\theta + \mathrm{d}\theta$. The radius of this ring is $a \sin\theta$, and its width is a $\mathrm{d}\theta$ (here a is the radius of the layer). Hence, the area of the ring is determined by the expression $2\pi a^2 \sin\theta\, \mathrm{d}\theta$. If the thickness of the layer is $\mathrm{d}a$ and its density is ρ, then the mass of the ring is $2\pi\rho a^2\, \mathrm{d}a\, \sin\theta\, \mathrm{d}\theta$. All the points of the ring are at the same distance r' from m. Consequently, by Eq. (6.9), the mutual potential energy of the ring and the mass m is determined by the expression

$$\mathrm{d}E_p = -G\frac{2\pi\rho a^2\, \mathrm{d}a\, \sin\theta\, \mathrm{d}\theta\, m}{r'}. \tag{6.13}$$

To obtain the potential energy of the entire spherical layer and the mass m, we must integrate Eq. (6.13) with respect to the angle θ within the limits from 0 to π. Here the variable r' varies within the limits from $r-a$ to $r+a$, where r is the distance from the centre of the layer 0 to m. Equation (6.13) contains two related variables, namely, a and r'. We must exclude one of these variables prior to integration. The latter is simplified if we exclude the variable θ. We can obtain the relation between

θ and r' by using the theorem of cosines. Inspection of Fig. 6.4 shows that

$$r'^2 = a^2 + r^2 - 2ar\cos\theta.$$

Differentiation of this expression yields

$$2r'\,\mathrm{d}r' = 2ar\sin\theta\,\mathrm{d}\theta.$$

Hence, $\sin\theta\,\mathrm{d}\theta = (r'/ar)\,\mathrm{d}r'$. Making such a substitution in Eq. (6.13), we get

$$\mathrm{d}E_\mathrm{p} = -G\frac{2\pi\rho a\,\mathrm{d}a\,m\,\mathrm{d}r'}{r}.$$

Integration with respect to r' within the limits from $r'_1 = r - a$ to $r'_2 = r + a$ yields

$$\mathrm{d}E_{\mathrm{p,lay}} = -G\frac{2\pi\rho a\,\mathrm{d}a\,m}{r}\int_{r-a}^{r+a}\mathrm{d}r' = -G\frac{4\pi\rho a^2\,\mathrm{d}a\,m}{r}. \tag{6.14}$$

The expression $4\pi a^2\,\mathrm{d}a$ gives the volume of the layer, and $4\pi\rho a^2\,\mathrm{d}a$ its mass $\mathrm{d}M$. Thus, the mutual potential energy of the sphere layer and the mass m is

$$\mathrm{d}E_{\mathrm{p,lay}} = -G\frac{\mathrm{d}M\,m}{r} \tag{6.15}$$

where r is the distance from the centre of the layer to m.

All our calculations remain the same for the case when the mass is inside the layer (see Fig. 6.4b). Only the integration limits in Eq. (6.14) will differ because r' changes in this case from $r'_1 = r - a$ to $r'_2 = r + a$. Consequently,

$$\begin{aligned}\mathrm{d}E'_{\mathrm{p,lay}} &= -G\frac{2\pi\rho a\,\mathrm{d}a\,m}{r}\int_{a-r}^{a+r}\mathrm{d}r' = -G4\pi\rho a\,\mathrm{d}a\,m\\ &= -G\frac{4\pi\rho a^2\,\mathrm{d}a\,m}{a} = -G\frac{\mathrm{d}M\,m}{a}.\end{aligned} \tag{6.16}$$

Hence, in this case the potential energy is the same for all r's and equals the value obtained in Eq. (6.15) for $r = a$.

Equation (6.15) can be interpreted as the potential energy of the particle m in the field set up by a sphere layer of mass $\mathrm{d}M$. The derivative of this energy with respect to r taken with the opposite sign equals the projection onto the direction r of the force acting on the particle:

$$\mathrm{d}F_r = -\frac{\partial E_\mathrm{p}}{\partial r} = -G\frac{\mathrm{d}M\,m}{r^2}. \tag{6.17}$$

The minus sign indicates that the force is directed toward the diminishing of r, *i.e.*, to the centre of the layer.

It follows from Eq. (6.17) that the sphere layer acts on the particle with the same force that would be exerted on it by a point particle of a mass equal to that of the layer and placed at the centre of the latter.

Equation (6.16) does not depend on the coordinates of a particle. Therefore, the gradient of this function vanishes for all r's less than a. Thus, no force acts on a particle inside the layer. Every element of the layer naturally exerts a certain force on the particle, but the sum of the forces exerted by all the elements of the layer equals zero.

Now let us consider a system consisting of a homogeneous sphere of mass M and a point particle of mass m. Let us divide the sphere into layers of mass $\mathrm{d}M$. Each layer acts on the particle with a force determined by Eq. (6.17). Summation of this expression over all the layers gives the force exerted on the particle by the sphere:

$$F_r = \int \mathrm{d}F_r = -\int G\frac{\mathrm{d}M\, m}{r^2} = -G\frac{Mm}{r^2}. \tag{6.18}$$

The action of the sphere on the particle is equivalent to the action of a point particle of a mass equal to that of the sphere and placed at its centre (see the preceding section).

If we take a sphere with a spherical space inside, then no force will act on a particle in this space.

Summation of Eq. (6.15) over all the layers of a solid or a hollow sphere yields the mutual potential energy of a particle and a sphere:

$$E_\mathrm{p} = -G\frac{Mm}{r^2}. \tag{6.19}$$

Here M is the mass of the sphere, m is the mass of the particle, and r is the distance from the particle to the centre of the sphere.

It follows from Eqs. (6.18) and (6.19) that the gravitational field produced by a homogeneous sphere is equivalent (outside the sphere) to the field produced by a point particle of the same mass at the centre of the sphere.

Let us consider two homogeneous spheres of masses M_1 and M_2. The second sphere experiences the same action on the part of the first one as would be exerted by a point particle of mass M_1 at the centre of the first sphere. According to Newton's third law, the corresponding force is equal in magnitude to the force that the second sphere would exert on the particle M_1. By Eq. (6.18), the magnitude of this force is GM_1M_2/r^2. We have thus proved that homogeneous spheres interact like point particles at their centres.

6.3. The Equivalence Principle

Mass comes up in two different laws—in Newton's second law and in the law of universal gravitation. In the former case, it characterizes the inertial properties of bodies, and in the latter their gravitational properties, *i.e.*, the ability of bodies to attract one another. In this connection, the question arises whether we ought to distinguish the inertial mass m_{in} and the gravitational mass m_g.

This question can be answered only by experiments. Let us consider the free falling of bodies in a heliocentric reference frame. Any body near the Earth's surface experiences a force of attraction to the Earth that by Eq. (6.18) is

$$F = G\frac{m_g M_E}{R_E^2}$$

where m_g is the gravitational mass of a given body, M_E is the gravitational mass of the Earth, and R_E is the radius of the Earth.

This force causes the body to acquire the acceleration a (but not g, see Sec. 4.2) that must equal the force F divided by the inertial mass of the body m_{in}:

$$a = \frac{F}{m_{in}} = G\frac{M_E}{R_E^2}\frac{m_g}{m_{in}}. \tag{6.20}$$

Experiments show that the acceleration a is the same for all bodies (it was shown in Sec. 4.2 that the identical values of a follow from the identical values of g). The factor $G(M_E/R_E^2)$ is also the same for all bodies. Consequently, the ratio m_g/m_{in} is the same for all bodies too. All other experiments in which the difference between the inertial and the gravitational masses could manifest itself lead to a similar result.

We shall describe the experiment of R. Eötvös, which he began in 1887 and continued over 25 years, as an example of such experiments. Eötvös proceeded from the circumstance that a body at rest near the Earth's surface, apart from the reaction of its support, experiences the gravitational force $\boldsymbol{F}_g$ directed toward the Earth's centre and also the centrifugal force of inertia $\boldsymbol{F}_{cf}$ directed perpendicularly to the Earth's axis of rotation (Fig. 6.5—this figure is not drawn to scale—the magnitude of the centrifugal force is two orders smaller than that of the gravitational force, see Sec. 4.2). The gravitational force is proportional to the gravitational mass of a body m_g:

$$\boldsymbol{F}_g = m_g\boldsymbol{g}'$$

($\boldsymbol{g}'$ is the gravitational intensity). The centrifugal force of inertia is proportional to the inertial mass m_{in}. According to Eq. (4.7), its magnitude is determined by the expression

$$F_{cf} = m_{in}\omega^2 R_E \cos\varphi$$

where φ is the latitude of the locality. It follows from Fig. 6.5 that the magnitude of the vertical component of the centrifugal force of inertia is

$$F_{vert} = F_{cf}\cos\varphi = m_{in}\omega^2 R_E\cos^2\varphi = Am_{in}.$$

We have introduced the symbol $A = \omega^2 R_E \cos^2\varphi$. Eötvös ran his experiment at the

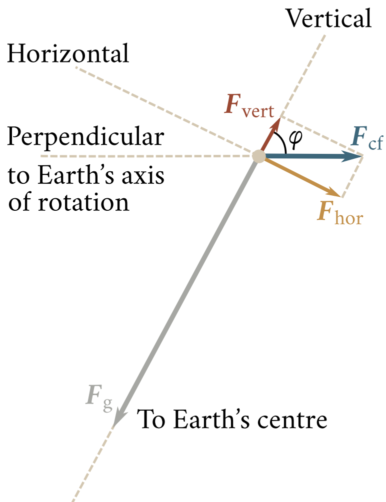

Fig. 6.5

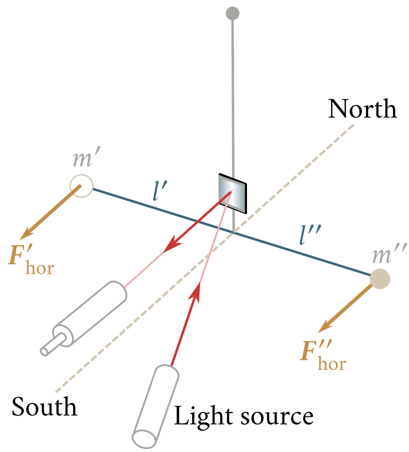

Fig. 6.6

latitude of $\varphi = 45°$. In this case the coefficient A is about one-hundredth of g'.

The magnitude of the horizontal component of the force F_{cf} is

$$F_{\text{hor}} = F_{\text{cf}} \sin\varphi = m_{\text{in}}\omega^2 R_{\text{E}} \cos\varphi \sin\varphi = Bm_{\text{in}}$$

where $B = \omega^2 R_{\text{E}} \cos\varphi \sin\varphi$ (for $\varphi = 45°$, the values of the coefficients A and B coincide).

Eötvös suspended a rod with bodies fastened to its ends on an elastic thread (Fig. 6.6). The bodies were of different materials, but their masses were as equal as possible. A mirror was attached to the bottom part of the thread. The beam from the light source reflected from the mirror struck the cross hairs of a telescope. The arms l' and l'' were selected so that the rod was in equilibrium in the vertical plane. The condition for this equilibrium is as follows:

$$(m'_{\text{g}}g' - m'_{\text{in}}A)l' = (m''_{\text{g}}g' - m''_{\text{in}}A)l''. \tag{6.21}$$

The instrument was arranged with the rod perpendicular to the plane of the meridian (see Fig. 6.6). In this case, the horizontal components of the centrifugal force of inertia set up a twisting moment equal to

$$M_{\text{t}} = m'_{\text{in}}Bl' - m''_{\text{in}}Bl''. \tag{6.22}$$

Eliminating the arm l'' from Eqs. (6.21) and (6.22), we can arrive at the following equation after simple transformations:

$$M_{\text{t}} = m'_{\text{in}}Bl'\left[1 - \frac{(m'_{\text{g}}/m'_{\text{in}})g' - A}{(m''_{\text{g}}/m''_{\text{in}})g' - A}\right].$$

It can be seen from this equation that when the ratio of the gravitational and inertia

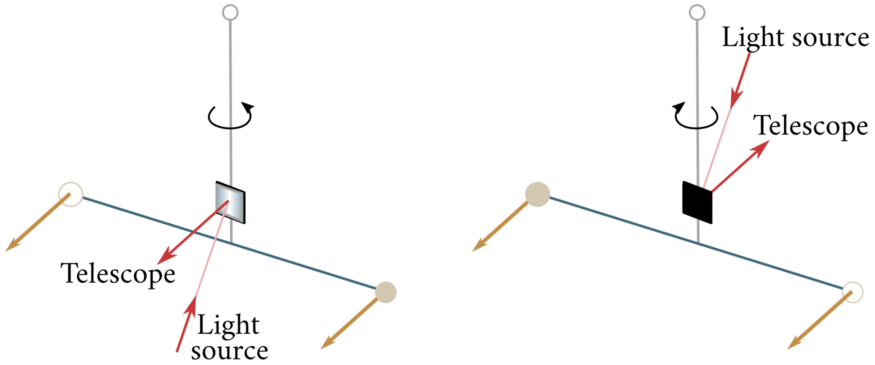

Fig. 6.7

masses is the same for both bodies, the moment twisting the thread must vanish. If the ratio m_g/m_{in} for the first and second bodies is not the same, the twisting moment differs from zero. In this case when the entire instrument is turned through 180°, the twisting moment would reverse its sign and the light spot would move from the cross hairs of the telescope (Fig. 6.7). Eötvös compared eight different bodies (including a wooden one) with a platinum body taken as the standard and discovered no twisting of the thread. This gave him the grounds to state that the ratio m_g/m_{in} for these bodies is identical with an accuracy of 10^{-8}.

In 1961-1964, R. Dicke improved Eötvös's method. He used the Sun's gravitational field and the centrifugal force of inertia due to the Earth's orbital motion for producing the twisting moment. As a result of his measurements, he arrived at the conclusion that the ratio m_g/m_{in} is the same for the studied bodies with an accuracy of 10^{-11}. Finally, in 1971, V. Braginsky and V. Panov obtained the constancy of the ratio with an accuracy up to 10^{-12}.

Thus, all the experimental facts indicate that the inertial and gravitational masses of all bodies are strictly proportional to each other. This signifies that these masses become identical when the units are selected properly. This is why physicists simply speak of mass. Albert Einstein based his general theory of relativity on the gravitational and inertial masses being identical.

We have already noted in Sec. 4.1 that the forces of inertia are similar to gravitational forces—both are proportional to the mass of the body which they are acting on. We have indicated there that if we are in a closed cab, no experiments will help us to establish what the action of the force $m\mathbf{g}$ is due to—whether it is due to the cab moving with the acceleration $-\mathbf{g}$, or to the fact that the stationary cab is near the Earth's surface. This statement forms the content of the so-called **equivalence principle**.

The identical nature of inertial and gravitational masses is the result of the equivalence of forces of inertia and gravitational forces.

It must be noted that from the very beginning we assumed in Eq. (6.1) that the mass coincides with the inertial mass of bodies, and we therefore determined the numerical value of G assuming that m_g/m_{in}. We can thus write Eq. (6.20) in the form

$$a = G\frac{M_E}{R_E^2}. \tag{6.23}$$

Equation (6.23) permits us to determine the mass of the Earth M_E. Use of the measured values of a, R_E and G in it gives the value of 5.98×10^{24} kg for the Earth's mass.

Further, knowing the radius of the Earth's orbit R_{orb} and the time T of one complete revolution of the Earth about the Sun, we can find the Sun's mass M_S. The Earth's acceleration equal to $\omega^2 R_{orb}$ (the angular velocity $\omega = 2\pi/T$) is due to the force with which the Sun attracts the Earth. Hence,

$$M_E\omega^2 R_{orb} = G\frac{M_E M_S}{R_{orb}^2}$$

whence we can calculate the Sun's mass.

The masses of other celestial bodies were determined in a similar way.

6.4. Orbital and Escape Velocities

To travel about the Earth in a circular orbit with a radius differing only slightly from the Earth's radius R_E, a body must have a definite velocity v_1. Its value can be found from the condition of equality of the product of the mass of the body and its acceleration to the force of gravity acting on the body:

$$m\frac{m_1^2}{R_E} = mg.$$

Hence,

$$v_1 = (gR_E)^{1/2}. \tag{6.24}$$

Consequently, for a body to become a satellite of the Earth, it must be given the velocity v_1 called the **tangential** or **orbital velocity** (v_1 is also sometimes called the **first cosmic velocity**). Introduction of the values of g and R_E gives the following value for the orbital velocity:

$$v_1 = (gR_E)^{1/2} = (9.8 \times 6.4 \times 10^6)^{1/2} \approx 8 \times 10^3 \text{ m s}^{-1} = 8 \text{ km s}^{-1}.$$

A body having the velocity v_1 will not fall onto the Earth. This velocity, however, is not sufficient for the body to leave the sphere of the Earth's attraction, *i.e.*, to travel away from the Earth over a distance such that its attraction to the Earth stops playing a significant part. The velocity v_2 required for this purpose is called the **escape velocity** (the **second cosmic velocity**).

To find the escape velocity, we must calculate the work that must be done against the forces of the Earth's attraction for moving a body from the Earth's surface to infinity. When a body moves away, the forces of the Earth's attraction do the following work on it:

$$A' = E_{\mathrm{p,init}} - E_{\mathrm{p,fin}}.$$

According to Eq. (6.19), the initial potential energy is

$$E_{\mathrm{p,init}} = -G\frac{M_{\mathrm{E}}m}{R_{\mathrm{E}}}.$$

and the final potential energy is zero. Thus,

$$A' = -G\frac{M_{\mathrm{E}}m}{R_{\mathrm{E}}}.$$

The work A that must be done against the forces of the Earth's attraction equals the work A' taken with the opposite sign, *i.e.*

$$A = G\frac{M_{\mathrm{E}}m}{R_{\mathrm{E}}}. \tag{6.25}$$

Disregarding the difference between the force of gravity mg and the force of gravitational attraction of a body to the Earth, we can write that

$$mg = G\frac{M_{\mathrm{E}}m}{R_{\mathrm{E}}^2}.$$

Hence,

$$G\frac{M_{\mathrm{E}}m}{R_{\mathrm{E}}} = mgR_{\mathrm{E}}.$$

Consequently, the work (6.25) can be written in the form

$$A = mgR_{\mathrm{E}}. \tag{6.26}$$

A body leaving the Earth does this work at the expense of its store of kinetic energy. For this store of energy to be sufficient for doing the work (6.26), the body must be projected from the Earth's surface with a velocity v not lower than the value v_2 determined by the condition

$$\frac{mv_2^2}{2} = mgR_{\mathrm{E}}$$

whence

$$v_2 = (2gR_{\mathrm{E}})^{1/2}. \tag{6.27}$$

It is exactly the velocity v_2 that is the escape velocity from the Earth, or the second cosmic velocity. A comparison of Eqs. (6.27) and (6.24) shows that this velocity is $\sqrt{2}$ times greater than the orbital one. Multiplying $8\,\mathrm{km\,s^{-1}}$ by $\sqrt{2}$, we get the approximate value of $11\,\mathrm{km\,s^{-1}}$ by v_2.

It must be noted that the required magnitude of the velocity does not depend on the direction in which a body is launched from the Earth. This direction only affects the shape of the trajectory along which the body travels away from the Earth.

To leave the solar system, a body must overcome the forces of attraction to the Sun in addition to the Earth's attraction. The velocity of launching a body from the Earth's surface needed for this purpose is called the **escape velocity from the solar system**, the **space velocity**, or the **third cosmic velocity** v_3. The velocity v_3 depends on the direction of launching. When a body is launched in the direction of orbital motion of the Earth, this velocity is minimum and is about $17\,\mathrm{km\,s^{-1}}$ (in this case the body's velocity relative to the Sun is the sum of its velocity relative to the Earth and the velocity with which the Earth is travelling about the Sun). When a body is launched in a direction opposite to that of the Earth's rotation, $v_3 \approx 73\,\mathrm{km\,s^{-1}}$.

The orbital and escape velocities were reached for the first time in the USSR. On October 4, 1957, the first successful launching of an artificial satellite of the Earth in the history of mankind was carried out in the Soviet Union. A second advance occurred on January 2, 1959. This day saw the launching from Soviet soil of a spaceship that escaped from the sphere of the Earth's attraction and became the first artificial planet of our solar system. On April 12, 1961, the first flight of a man into outer space was accomplished in the Soviet Union. The first Soviet cosmonaut Yuri Gagarin completed a flight around the Earth and landed successfully.

Chapter 7

OSCILLATORY MOTION

7.1. General

Oscillations are defined as processes distinguished by a certain degree of repetition. For example, the swings of a clock pendulum, the vibrations of a string or the leg of a tuning fork, and the voltage across the plates of a capacitor in a radio receiver circuit have this property of repetition.

Depending on the physical nature of the repeating process, we distinguish mechanical, electromagnetic, sound, and other oscillations. In the present chapter, we shall deal with mechanical oscillations.

Oscillations (vibrations) are widespread in nature and engineering. They often have a negative influence. The oscillations of a railway bridge due to the impacts imparted to it by the wheels of a train passing over the rail joints, the vibrations of a ship's hull caused by rotation of the propeller, the vibrations of the wings of an aircraft are all processes that may have catastrophic consequences. The task in such cases is to prevent the setting up of oscillations or at any rate to prevent them from reaching dangerous magnitudes. Oscillatory processes are also at the very foundation of various branches of engineering. For instance, radio engineering owes its very existence to oscillatory processes. Depending on the nature of the action on an oscillating system, we distinguish free (or natural) oscillations, forced oscillations, auto-oscillations, and parametric oscillations.

Free or **natural oscillations** occur in a system left alone after an impetus was imparted to it or it was brought out of the equilibrium position. An example are the oscillations of a ball suspended on a string (a pendulum). To initiate oscillations, we may either push the ball or move it to a side and release it.

In **forced oscillations**, the oscillating system is acted upon by an external periodically changing force. An example here are the oscillations of a bridge set up when people walking in step pass over it.

Auto-oscillations, like forced ones, are attended by the action of external forces on the oscillating system, but the moments of time when these actions are exerted are set by the oscillating system itself—the latter controls the external action. Examples of an auto-oscillating system are clocks in which a pendulum receives pushes at the expense of the energy of a lifted weight or a coiled spring, and these pushes occur when the pendulum passes through its middle position.

In **parametric oscillations**, external action causes periodic changes in a parameter of a system, for instance, in the length of a thread on which an oscillating ball is suspended.

Harmonic oscillations are the simplest ones. These are oscillations when the oscillating quantity (for example, the deflection of a pendulum) changes with time according to a sine or cosine law. This kind of oscillations is especially important for the following reasons: first, oscillations in nature and engineering are often close to harmonic ones in their character, and, second, periodic processes of a different form (with a different time dependence) can be represented as the superposition of several harmonic oscillations.

7.2. Small-Amplitude Oscillations

Let us consider a mechanical system whose position can be set with the aid of a single quantity which we shall designate x. The system is said to have one degree of freedom in such cases. The angle measured from a certain plane, or the distance measured along a given curve, in particular a straight line, etc. may be the quantity x determining the position of the system. The potential energy of the system will be a function of the single variable x: $E_\text{p} = E_\text{p}(x)$. Assume that the system has a position of stable equilibrium. In this position, the function $E_\text{p}(x)$ has a minimum (see Sec. 3.9). We shall measure the coordinate x and the potential energy E_p from the position of equilibrium. Hence $E_\text{p}(0) = 0$.

Let us expand the function $E_\text{p}(x)$ in a power series and consider only small-amplitude oscillations, so that the higher powers of x may be disregarded. According to the Maclaurin theorem

$$E_\text{p}(x) = E_\text{p}(0) + E'_\text{p}(0)\,x + \frac{1}{2}E''_\text{p}(0)\,x^2$$

(owing to the small value of x we disregard the remaining terms). Since $E_\text{p}(x)$ at $x = 0$ has a minimum, then $E'_\text{p}(0)$ equals zero, and $E''_\text{p}(0)$ is positive. In addition, according to our condition, $E_\text{p}(0) = 0$. Let us introduce the symbol $E''_\text{p}(0) = k$

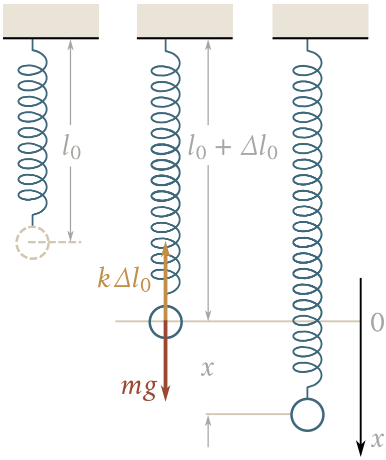

Fig. 7.1

(where $k > 0$). Hence,

$$E_p(x) = \frac{1}{2}kx^2. \tag{7.1}$$

Equation (7.1) is identical with Eq. (3.78) for the potential energy of a deformed spring. Using Eq. (3.32), let us find the force acting on the system:

$$F_x = -\frac{\partial E_p}{\partial x} = -kx. \tag{7.2}$$

This equation gives the projection of the force onto the direction x. In the following, we shall omit the subscript x in designating the force, *i.e.*, we shall write Eq. (7.2) in the form $F = -kx$.

Equation (7.2) is identical with Eq. (2.26) for the elastic force of a deformed spring. This is why forces of the kind shown by Eq. (7.2) regardless of their nature, are called **quasi-elastic**. It is easy to see that a force described by Eq. (7.2) is always directed toward the position of equilibrium. The magnitude of the force is proportional to the deviation of the system from its equilibrium position. A force having such properties is sometimes defined as a **restoring force**.

Let us consider as an example a system consisting of a ball of mass m suspended on a spring whose mass may be ignored in comparison with m (Fig. 7.1). In the equilibrium position, the force mg is balanced by the elastic force $k\Delta l_0$:

$$mg = k\Delta l_0 \tag{7.3}$$

(Δl_0 is the elongation of the spring). We shall characterize the displacement of the ball from its equilibrium position by the coordinate x with the x-axis directed vertically downward and the zero of the axis coinciding with the position of equilib-

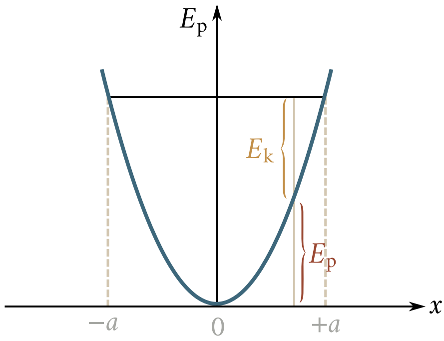

Fig. 7.2

rium of the ball. If we shift the ball to the position characterized by the coordinate x, then the elongation of the spring will become equal to $\Delta l_0 + x$, and the projection of the resultant force onto the x-axis will acquire the value $F = mg - k(\Delta l_0 + x)$. Taking into account condition (7.3), we find that

$$F = -kx. \tag{7.4}$$

Thus, in the example considered, the resultant of the force of gravity and of the elastic force has the nature of a quasi-elastic force.

Let us impart the displacement $x = a$ to the ball and then leave the system alone. Under the action of the quasi-elastic force, the ball will move toward its equilibrium position with the constantly growing velocity $v = \dot{x}$: The potential energy of the system will diminish (Fig. 7.2), but a constantly growing kinetic energy $E_k = m\dot{x}^2/2$ will appear instead (we disregard the mass of the spring).

Arriving at its equilibrium position, the ball continues to move by inertia. This motion will be retarded and will stop when the kinetic energy completely transforms into potential energy, *i.e.*, when the displacement of the ball becomes equal to $-a$. Next the same process will repeat when the ball moves in the reverse direction. If friction is absent in the system, its energy should be conserved, and the ball will move within the limits from $x = a$ to $x = -a$ for an infinitely long time.

The equation of Newton's second law for the ball is

$$m\ddot{x} = -kx. \tag{7.5}$$

Introducing the symbol

$$\omega_0^2 = \frac{k}{m} \tag{7.6}$$

we can transform Eq. (7.5) as follows:

$$\ddot{x} + \omega_0^2 x = 0. \tag{7.7}$$

Since $k/m > 0$, then ω_0 is a real quantity.

Thus, in the absence of forces of friction, motion under the action of a quasi-elastic force is described by the differential equation (7.7).

Any real oscillatory system contains resistance or damping forces whose action leads to diminishing of the energy of the system. If the depletion of the energy is not replenished as a result of the work of external forces, the oscillations will be damped. In the simplest and also the most frequently encountered case, the damping force F^* is proportional to the magnitude of the velocity:

$$F_x^* = -r\dot{x}. \tag{7.8}$$

Here r is a constant called the **resistance coefficient**. The minus sign is due to the circumstance that the force $\boldsymbol{F}^*$ and the velocity $\boldsymbol{v}$ have opposite directions, consequently, their projections onto the x-axis have opposite signs.

The equation of Newton's second law when damping forces are present has the form

$$m\ddot{x} = -kx - r\dot{x}. \tag{7.9}$$

Introducing the notation

$$2\beta = \frac{r}{m} \tag{7.10}$$

and using Eq. (7.6), we can write Eq. (7.9) as follows:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \tag{7.11}$$

This differential equation describes the damping oscillations of a system.

The oscillations described by Eqs. (7.7) and (7.11) are free (or natural): a system brought out of its equilibrium position or having received an impetus performs oscillations when left alone. Now assume that an oscillatory system experiences an external force that changes with time according to a harmonic law:

$$F_x = F_0 \cos \omega t. \tag{7.12}$$

In this case, the equation of Newton's second law has the form

$$m\ddot{x} = -kx - r\dot{x} + F_0 \cos \omega t.$$

Using Eqs. (7.6) and (7.10), let us write this equation as follows:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos \omega t \tag{7.13}$$

where

$$f_0 = \frac{F_0}{m}. \tag{7.14}$$

Equation (7.13) describes forced oscillations.

We have established that when studying various kinds of oscillations we are confronted with the need to solve differential equations of the form

$$\ddot{x} + a\dot{x} + bx = f(t) \tag{7.15}$$

where a and b are constants, and $f(t)$ is a function of t. An equation such as (7.15) is called a **linear differential equation with constant coefficients**. For Eq. (7.7), we have $a = 0$ and $b = \omega_0^2$ and for Eq. (7.11), we have $a = 2\beta$ and $b = \omega_0^2$. In both cases, the function $f(t)$ identically equals zero: $f(t) \equiv 0$. For forced oscillations, $f(t) = f_0 \cos \omega t$.

The solution of Eq. (7.15) is greatly facilitated if we pass over to complex quantities. This is why we shall stop to briefly treat complex numbers and methods of solving linear differential equations with constant coefficients.

7.3. Complex Numbers

The complex number z is defined as a number of the kind

$$z = x + iy \tag{7.16}$$

where x and y are real numbers, and i is imaginary unity ($i^2 = -1$). The number x is called the **real part** of the complex number z. This is written symbolically[1] in the form $x = \Re\{z\}$. The number y is the **imaginary part** of z (symbolically $y = \Im\{z\}$). The number

$$z^* = x - iy \tag{7.17}$$

is called the **complex conjugate** of the number $x + iy$.

The real number x can be depicted by a point on the x-axis. The complex number z can be depicted by a point on a plane with the coordinates x and y (Fig. 7.3). Each point of the plane corresponds to a complex number z. Consequently, a complex number can be given in the form of Eq. (7.16) with the aid of the Cartesian coordinates of the relevant point. The same number, however, can be given with the aid of the polar coordinates ρ and φ. The following relations exist between the two pairs of coordinates:

$$\begin{cases} x = \rho\cos\varphi, & y = \rho\sin\varphi, \\ \rho = \left(x^2 + y^2\right)^{1/2}, & \varphi = \arctan\dfrac{y}{x}. \end{cases} \tag{7.18}$$

The distance from the origin of coordinates to the point depicting the number z is called the **absolute value** or **modulus** of the complex number (its symbol is $|z|$). It is obvious that

$$|z| = \rho = \left(x^2 + y^2\right)^{1/2}. \tag{7.19}$$

[1]Another form of symbolically represent the real and imaginary parts of a complex number is: $\mathrm{Re}\{z\}$ and $\mathrm{Im}\{z\}$.

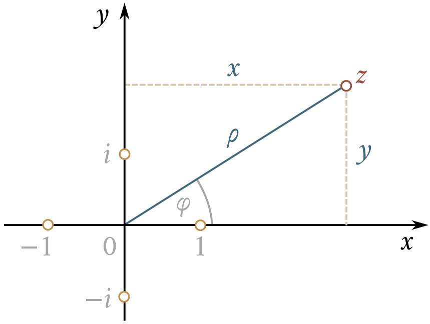

Fig. 7.3

The quantity φ is called the **argument** of the complex number z.

With a view to Eqs. (7.18), we can write a complex number in the trigonometric form:

$$z = \rho(\cos\varphi + i\sin\varphi). \tag{7.20}$$

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are considered to equal each other if their real and imaginary parts are separately equal, namely,

$$z_1 = z_2 \text{ if } x_1 = x_2 \text{ and } y_1 = y_2. \tag{7.21}$$

The moduli of two equal complex numbers are identical, while their arguments can differ only in an addend that is a multiple of 2π:

$$\rho_1 = \rho_2, \quad \varphi_1 = \varphi_2 \pm 2k\pi \tag{7.22}$$

where k is an integer.

Examination of Eqs. (7.16) and (7.17) shows that when $z^* = z$, the imaginary part of z vanishes, *i.e.*, the number z is a pure real number. Thus, the condition for the number z being real can be written in the form

$$z^* = z. \tag{7.23}$$

The relation

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \tag{7.24}$$

is proved in mathematics and is called the **Euler formula**. Substituting $-\varphi$ for φ in this equation and bearing in mind that $\cos(-\varphi) = \cos\varphi$ and $\sin(-\varphi) = -sin\varphi$ we get

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi. \tag{7.25}$$

Let us summate Eqs. (7.24) and (7.25) and solve the resulting equation relative to $\cos\varphi$. We obtain

$$\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right). \tag{7.26}$$

Subtraction of Eq. (7.25) from (7.24) yields $\sin\varphi = \left(e^{i\varphi} + e^{-i\varphi}\right)/2i$.

Equation (7.24) can be used to write a complex number in the exponential form:

$$z = \rho e^{i\varphi} \tag{7.27}$$

[see Eq. (7.20)]. The complex conjugate number in the exponential form is

$$z^* = \rho e^{-i\varphi}. \tag{7.28}$$

In the addition of complex numbers, their real and imaginary parts are added separately:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \tag{7.29}$$

It is convenient to multiply complex numbers by taking them in the exponential form:

$$z = z_1 z_2 = \rho_1 e^{i\varphi_1} \rho_2 e^{i\varphi_2} = \rho_1 \rho_2 e^{i(\varphi_1+\varphi_2}. \tag{7.30}$$

The moduli of the complex numbers in this case are multiplied, and the arguments are added:

$$\rho = \rho_1 \rho_2, \quad \varphi = \varphi_1 + \varphi_2. \tag{7.31}$$

Complex numbers are divided in a similar way:

$$z = \frac{z_1}{z_2} = \frac{\rho_1 e^{i\varphi_1}}{\rho_2 e^{i\varphi_2}} = \frac{\rho_1}{\rho_2} e^{i(\varphi_1-\varphi_2}. \tag{7.32}$$

It is a simple matter to find from Eqs. (7.27) and (7.28) that

$$zz^* = \rho^2 \tag{7.33}$$

(the square of the modulus of a complex number equals the product of this number and its complex conjugate).

7.4. Linear Differential Equations

An equation of the kind

$$\ddot{x} + a\dot{x} + bx = f(t) \tag{7.34}$$

where a and b are constants, and $f(t)$ is a given function of t, is called a **linear differential equation of the second order with constant coefficients**. The constants a and b may also be zero.

If the function $f(t)$ is identically equal to zero [$f(t) \equiv 0$], the equation is called **homogeneous**, otherwise it is called **non-homogeneous**. A homogeneous equation has the form

$$\ddot{x} + a\dot{x} + bx = 0. \tag{7.35}$$

The solution of any second-order differential equation (*i.e.*, with a second derivative as the senior term) contains two arbitrary constants C_1 and C_2. This can be understood in view of the circumstance that a function is determined from its second derivative by double integration. An integration constant appears upon each integration. Let us consider as an example the equation

$$\ddot{x} = 0. \tag{7.36}$$

Integration of this equation yields $\dot{x} = C_1$. Repeated integration results in the function

$$x = C_1 t + C_2. \tag{7.37}$$

It is easy to see that the function (7.37) satisfies Eq. (7.36) with any values of the constants C_1 and C_2.

Assigning definite values to the constants C_1 and C_2, we get the so-called **partial solution** of a differential equation. For example, the function $5t + 3$ is one of the partial solutions of Eq. (7.36).

The multitude of all the partial solutions without any exception is called the **general solution** of a differential equation. The general solution of Eq. (7.36) has the form of Eq. (7.37).

It is proved in the theory of linear differential equations that if x_1 and x_2 are linearly independent[2] solutions of the homogeneous equation (7.35), then the general solution of this equation can be represented in the form

$$x = C_1 x_1 + C_2 x_2 \tag{7.38}$$

where C_1 and C_2 are arbitrary constants.

Assume that $x_n(t, C_1, C_2)$ is the general solution of the non-homogeneous equation (7.34) (the arbitrary constants C_1 and C_2 are parameters in this solution), and $x_n(t)$ is one of the partial solutions of the same equation (it contains no arbitrary constants). We shall introduce the notation

$$x(t, C_1, C_2) = x_n(t, C_1, C_2) - x_n(t).$$

The general solution of the non-homogeneous equation can therefore be written in the form

$$x_n(t, C_1, C_2) = x_n(t) + x(t, C_1, C_2). \tag{7.39}$$

The function (7.39) satisfies Eq. (7.34) at any values of the constants C_1 and C_2. Consequently, we can write the relation

$$\ddot{x}_n(t) + \ddot{x}(t, C_1, C_2) + a\dot{x}_n(t) + a\dot{x}(t, C_1, C_2) + bx_n(t) + bx(t, C_1, C_2) = f(t).$$

Grouping of the addends yields

$$\ddot{x}(t, C_1, C_2) + a\dot{x}(t, C_1, C_2) + bx(t, C_1, C_2) + [\ddot{x}_n(t) + a\dot{x}_n(t) + bx_n(t)] = f(t). \tag{7.40}$$

[2]The functions x_1 and x_2 are called linearly independent if the relation $\alpha_1 x_1 + \alpha_2 x_2 = 0$ is obeyed only when α_1 and α_2 equal zero.

The partial solution $x_n(t)$ also satisfies Eq. (7.34). Consequently, the expression in brackets in the left-hand side of Eq. (7.40) equals the right-hand side of this equation. It thus follows that the function $x(t, C_1, C_2)$ must satisfy the condition

$$\ddot{x}(t, C_1, C_2) + a\dot{x}(t, C_1, C_2) + bx(t, C_1, C_2) = 0$$

i.e., it is the general solution of the homogeneous equation (7.35). We have therefore arrived at a very useful theorem: *the general solution of a non-homogeneous equation equals the sum of the general solution of the corresponding homogeneous equation and a partial solution of the non-homogeneous equation*:

$$x_{\text{gen,non-hom}} = x_{\text{gen,hom}} + x_{\text{part,non-hom}}. \tag{7.41}$$

Linear homogeneous differential equations with constant coefficients are solved using the substitution

$$x(t) = e^{\lambda t} \tag{7.42}$$

where λ is a constant quantity. Differentiation of the function (7.42) yields

$$\dot{x}(t) = \lambda e^{\lambda t}, \quad \ddot{x}(t) = \lambda^2 e^{\lambda t}. \tag{7.43}$$

The introduction of Eqs. (7.42) and (7.43) into (7.35) results in the following equation, after the factor $e^{\lambda t}$ differing from zero is cancelled out:

$$\lambda^2 + a\lambda + b = 0. \tag{7.44}$$

This equation is called a **characteristic** one. Its roots are the values of λ at which the function (7.42) satisfies (7.35).

If the roots of Eq. (7.44) do not coincide ($\lambda_1 \neq \lambda_2$) the functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ will be linearly independent. Consequently, according to Eq. (7.38), the general solution of Eq. (7.35) can be written as follows:

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \tag{7.45}$$

It can be shown that when $\lambda_1 = \lambda_2 = \lambda$ the general solution of Eq. (7.35) is as follows:

$$x = C_1 e^{\lambda t} + C_2 t e^{\lambda t}. \tag{7.46}$$

Assume that the coefficients a and b are real, while the function in the right-hand side of Eq. (7.34) is complex. Writing this function in the form $f(t) + i\varphi(t)$, we arrive at the equation:

$$\ddot{z} + a\dot{z} + bz = f + i\varphi \tag{7.47}$$

(we have used the symbol z to denote the required function). The solution of this equation will evidently be complex. Writing the solution in the form $z(t) = x(t) + iy(t)$, we shall introduce it into Eq. (7.47). The result will be

$$\ddot{x} + i\ddot{y} + a\dot{x} + ai\dot{y} + bx + biy = f + i\varphi. \tag{7.48}$$

When complex numbers are equal to each other, their real and imaginary parts equal each other separately [see Eq. (7.21)]. Hence, Eq. (7.48) breaks up into two separate equations:

$$\ddot{x} + a\dot{x} + bx = f(t), \quad \ddot{y} + a\dot{y} + by = \varphi \tag{7.49}$$

the first of which coincides with Eq. (7.34). This property of Eq. (7.48) allows us to use the following procedure that sometimes facilitates calculations quite significantly. Let us assume that the right-hand side of Eq. (7.34) we are solving is real. By adding an arbitrary imaginary function to it, we reduce the equation to the form of Eq. (7.47). After next finding the complex solution of the equation, we take its real part. It will be the solution of the initial equation [Eq. (7.34)].

7.5. Harmonic Oscillations

Let us consider oscillations described by the equation

$$\ddot{x} + \omega_0^2 x = 0. \tag{7.7 revisited}$$

Such oscillations are performed by a body of mass m experiencing only the quasi-elastic force $F = -kx$. The coefficient of x in Eq. (7.7) has the value

$$\omega_0^2 = \frac{k}{m}.. \tag{7.6 revisited}$$

Using the expression $x = e^{\lambda t}$ [see Eq. (7.42)] in Eq. (7.7), we arrive at the characteristic equation

$$\lambda^2 + \omega_0^2 = 0. \tag{7.50}$$

This equation has imaginary roots: $\lambda_1 = +i\omega_0$ and $\lambda_2 = -i\omega_0$. According to Eq. (7.45), the general solution of Eq. (7.7) has the form

$$x = C_1 ei\omega_0 t + C_2 e{-}i\omega_0 t \tag{7.51}$$

where C_l and C_2 are complex constants.

The function $x(t)$ describing the oscillation must be real. For this end, the coefficients C_l and C_2 in Eq. (7.51) must be selected so as to observe the condition [see Eq. (7.23)]:

$$C_1^* e{-}i\omega_0 t + C_2^* ei\omega_0 t = C_1 ei\omega_0 t + C_2 e{-}i\omega_0 t \tag{7.52}$$

[we have equated expression (7.51) to its complex conjugate]. Equation (7.52) will be obeyed if $C_1 = C_2^*$ (in this case $C_2 = C_1^*$). Let us write the coefficients C_l and C_2 satisfying this condition in the exponential form [see Eq. (7.17)], denoting their modulus by $A/2$ and their argument by α:

$$C_1 = \frac{A}{2}e^{i\alpha}, \quad C_2 = \frac{A}{2}e^{-i\alpha}. \tag{7.53}$$

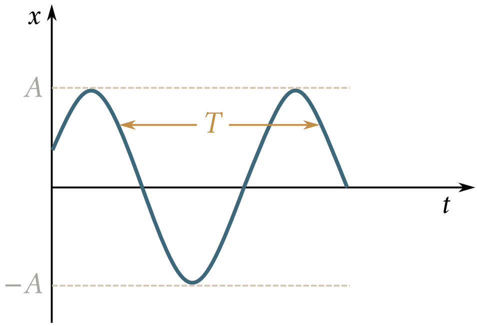

Fig. 7.4

Introduction of these expressions into Eq. (7.51) yields

$$x = \frac{A}{2}\left[e^{i(\omega_0 t+\alpha)} + e^{-i(\omega_0 t+\alpha)}\right] = A\cos(\omega_0 t + \alpha) \tag{7.54}$$

[see Eq. (7.26)]. Thus, the general solution of Eq. (7.49) is

$$x = A\cos(\omega_0 t + \alpha) \tag{7.55}$$

where A and α are arbitrary constants[3].

Thus, the displacement x changes with time according to a cosine law. Consequently, the motion of the system experiencing the action of a force of the kind $F = -kx$ is a harmonic oscillation.

A graph of the harmonic oscillation, *i.e.*, one of the function (7.55), is shown in Fig. 7.4. The time t is laid off along the horizontal axis, and the displacement x along the vertical one. Since a cosine varies from -1 to 1, then the values of x range from $-A$ to A.

The maximum deviation of a system from its equilibrium position is called the **amplitude** of oscillation. The amplitude A is a constant positive quantity. Its value is determined by the magnitude of the initial deviation or push that brought the system out of the equilibrium position.

The cosine argument $(\omega_0 t + \alpha)$ is called the **phase** of oscillation. The constant α is the value of the phase at the moment $t = 0$ and is called the **initial phase** of oscillation. The constant α will change when the moment from which we begin to measure the time is changed. Hence, the value of the initial phase is determined by when we begin to measure the time. Since the value of x does not change when

[3]The solution of Eq. (7.7) can be written in two other ways. Let us transform Eq. (7.55) according to the formula for the cosine of a sum: $x = A(\cos\alpha\cos\omega_0 t - \sin\alpha\sin\omega_0 t)$, and introduce the notation $c_1 = A\cos\alpha$ and $c_2 = -A\sin\alpha$. The function $x(t)$ can therefore be written in the form $x = c_1\cos\omega_0 t + c_2\sin\omega_0 t$, where c_1 and c_2 are arbitrary constants. Finally, using Eq. (7.24), we can write Eq. (7.55) as follows: $x = \Re\left\{Ae^{i(\omega_0 t+\alpha)}\right\}$.

a whole number of 2π's is added to or subtracted from the phase, we can always ensure that the initial phase will be less than π in magnitude. This is why only values of α within the limits from $-\pi$ to $+\pi$ are usually considered.

Since a cosine is a periodic function with the period 2π, different states[4] of a system performing harmonic oscillations repeat during the time interval T in which the phase of oscillation receives an increment equal to 2π (Fig. 7.4). The interval T is called the **period** of oscillation. It can be found from the condition $|\omega_0(t+T)+\alpha| = |\omega_0 t+\alpha| + 2\pi$, whence

$$T = \frac{2\pi}{\omega_0}. \tag{7.56}$$

The number of oscillations in unit time is called the **frequency** of oscillation ν. It is quite evident that the frequency ν is related to the duration of one oscillation T by the expression

$$\nu = \frac{1}{T}. \tag{7.57}$$

The unit of frequency is the frequency of oscillations whose period is 1 s. This unit is called the hertz (Hz). A frequency of 10^3 Hz is called a kilohertz (kHz), and of 10^6 Hz a megahertz (MHz).

It follows from Eq. (7.56) that

$$\omega_0 = \frac{2\pi}{T}. \tag{7.58}$$

Thus, ω_0 is the number of oscillations in 2π seconds. The quantity ω_0 is called the **circular** or **cyclic** frequency. It is related to the conventional frequency ν by the expression

$$\omega_0 = 2\pi\nu. \tag{7.59}$$

Time differentiation of Eq. (7.55) yields an expression for the velocity:

$$v = \dot{x} = -A\omega_0 \sin(\omega_0 t + \alpha) = A\omega_0 \cos\left(\omega_0 t + \alpha + \frac{\pi}{2}\right). \tag{7.60}$$

Examination of Eq. (7.60) shows that the velocity also changes according to a harmonic law, the amplitude of the velocity being $A\omega_0$. It follows from a comparison of Eqs. (7.55) and (7.60) that the phase of the velocity is in advance of that of the displacement by $\pi/2$.

Time differentiation of Eq. (7.60) yields an expression for the acceleration:

$$a = \ddot{x} = -A\omega_0^2 \cos(\omega_0 t + \alpha) = A\omega_0^2 \cos(\omega_0 t + \alpha + \pi). \tag{7.61}$$

It can be seen from Eq. (7.61) that the acceleration and the displacement are opposite in phase. This signifies that when the displacement reaches its maximum positive

[4]We remind our reader that the state of a mechanical system is characterized by the values of the coordinates and the velocities of the bodies forming the system.

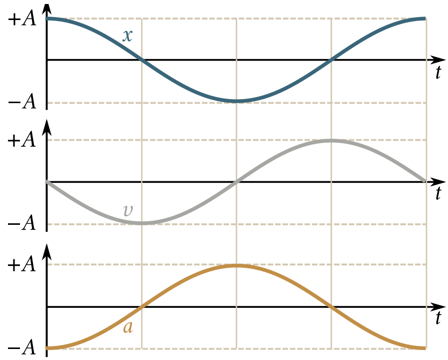

Fig. 7.5

value, the acceleration reaches its maximum negative value, and vice versa.

Figure 7.5 compares graphs for the displacement, velocity, and acceleration.

A particular oscillation is characterized by definite values of the amplitude A and the initial phase α. The values of these quantities for a given oscillation can be determined from the initial conditions, *i.e.*, from the values of the deviation x_0 and the velocity v_0 at the initial moment. Indeed, assuming in Eqs. (7.55) and (7.60) that $t = 0$, we get two equations:

$$x_0 = A\cos\alpha, \qquad v_0 = -A\omega_0 \sin\alpha$$

from which we find that

$$A = \left(x_0^2 + \frac{v_0^2}{\omega_0^2}\right)^{1/2}, \tag{7.62}$$

$$\tan\alpha = -\frac{v_0}{x_0\omega_0}. \tag{7.63}$$

Equation (7.63) is satisfied by two values of α within the interval from $-\pi$ to $+\pi$. That value must be taken which gives the correct signs of cosine and sine.

A quasi-elastic force is conservative. Therefore, the total energy of a harmonic oscillation must remain constant. In the course of oscillations, kinetic energy transforms into potential energy, and vice versa. At the moments of maximum deviation from the equilibrium position, the total energy E consists only of potential energy, which reaches its maximum value $E_{\mathrm{p,max}}$:

$$E = E_{\mathrm{p,max}} = \frac{kA^2}{2}. \tag{7.64}$$

When the system passes through its equilibrium position, the total energy consists only of kinetic energy, which at these moments reaches its maximum value $E_{\mathrm{k,max}}$:

$$E = E_{\mathrm{k,max}} = \frac{mv_{\max}^2}{2} = \frac{mA^2\omega_0^2}{2} \tag{7.65}$$

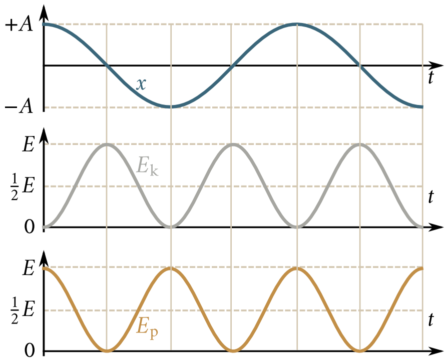

Fig. 7.6

(it was shown above that the velocity amplitude is $A\omega_0$). Equations (7.64) and (7.65) equal each other because by Eq. (7.6) we have $m\omega_0^2 = k$.

Let us see how the kinetic and potential energies of harmonic oscillations change with time. The kinetic energy is [see Eq. (7.60) for $\dot{x}$]

$$E_k = \frac{m\dot{x}^2}{2} = \frac{mA^2\omega_0^2}{2}\sin^2(\omega_0 t + \alpha). \tag{7.66}$$

The potential energy is expressed by the equation

$$E_p = \frac{kx^2}{2} = \frac{kA^2}{2}\cos^2(\omega_0 t + \alpha). \tag{7.67}$$

Adding Eqs. (7.66) and (7.67) and bearing in mind that $m\omega_0^2 = k$, we get a formula for the total energy:

$$E = E_k + E_p = \frac{kA^2}{2} = \frac{mA^2\omega_0^2}{2} \tag{7.68}$$

[compare with Eqs. (7.64) and (7.65)]. Thus, the total energy of a harmonic oscillation is indeed constant.

Using formulas of trigonometry, we can write the expressions for E_k and E_p as follows:

$$E_k = E\sin^2(\omega_0 t + \alpha) = E\left\{\frac{1}{2} - \frac{1}{2}\cos[2(\omega_0 t + \alpha)]\right\} \tag{7.69}$$

$$E_p = E\cos^2(\omega_0 t + \alpha) = E\left\{\frac{1}{2} + \frac{1}{2}\cos[2(\omega_0 t + \alpha)]\right\} \tag{7.70}$$

where E is the total energy of the system. A glance at these equations shows that E_k and E_p change with a frequency of $2\omega_0$, *i.e.*, with a frequency twice that of the harmonic oscillations. Figure 7.6 compares graphs for x, E_k and E_p.

It is known that the mean value of sine square and of cosine square equals one-half. Hence, the mean value of E_k coincides with that of E_p and equals $E/2$.

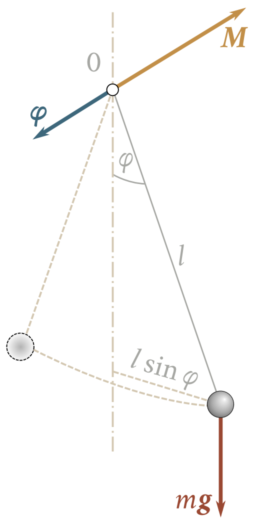

Fig. 7.7

7.6. The Pendulum

Physicists understand a pendulum to be a rigid body performing oscillations about a fixed point or axis under the acting of the force of gravity.

A mathematical or simple pendulum is defined as an idealized system consisting of a weightless and unstretchable string on which a mass concentrated at one point is suspended. A sufficiently close approximation to a simple pendulum is a small heavy sphere suspended on a long thin thread.

We shall characterize the deviation of a pendulum from its equilibrium position by the angle φ made by the thread with a vertical line (Fig. 7.7). Deviation of a pendulum from its equilibrium position is attended by the appearance of a rotational moment (torque) M whose magnitude is $mgl \sin \varphi$ (here m is the mass and l is the length of the pendulum). Its direction is such that it tends to return the pendulum to its equilibrium position, and is similar in this respect to a quasi-elastic force. Therefore, opposite signs must be assigned to the moment M and the angular displacement φ[5], as is done to the displacement and the quasi-elastic force. Hence, the expression for the rotational moment has the form

$$M = -mgl \sin \varphi. \tag{7.71}$$

Let us write an equation for the dynamics of rotation of a pendulum. Denoting

[5]Considering φ as a vector related to the direction of rotation by the right-hand screw rule (this is permissible at small values of φ), the opposite signs of M and φ can be explained by the fact that the vectors $\boldsymbol{M}$ and $\boldsymbol{\varphi}$ have opposite directions (Fig. 7.7).

the angular acceleration by $\ddot{\varphi}$ and taking into account that the moment of inertia of a pendulum is ml^2 we get

$$ml^2\ddot{\varphi} = -mgl\sin\varphi.$$

This equation can be transformed as follows:

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0. \tag{7.72}$$

Let us consider only small-amplitude oscillations. We can thus assume that $\sin\varphi \approx \varphi$. Introducing, in addition, the notation

$$\frac{g}{l} = \omega_0^2 \tag{7.73}$$

we arrive at the equation

$$\ddot{\varphi} + \omega_0^2\varphi = 0 \tag{7.74}$$

similar to Eq. (7.7). Its solution has the form

$$\varphi = A\cos(\omega_0 t + \alpha). \tag{7.75}$$

Consequently, in small-amplitude oscillations, the angular displacement of a simple pendulum changes with time according to a harmonic law.

Equation (7.73) shows that the frequency of oscillations of a simple pendulum depends only on its length and on the acceleration of the force of gravity and does not depend on the mass of the pendulum. Equation (7.56) after (7.73) is introduced into it gives the expression for the period of oscillations of a simple pendulum known from school days:

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2}. \tag{7.76}$$

By solving Eq. (7.72), we can obtain the following formula for the period of oscillations:

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2}\left\{1 + \left(\frac{1}{2}\right)^2\sin^2\frac{A}{2} + \left(\frac{1}{2}\times\frac{3}{4}\right)^2\sin^4\frac{A}{2} + \dots\right\}. \tag{7.77}$$

where A is the amplitude of the oscillations, *i.e.*, the greatest angle through which a pendulum deflects from its equilibrium position.

If an oscillating body cannot be treated as a point particle, the pendulum is called a physical one. When the pendulum deviates from its equilibrium position by the angle φ, a rotational moment (torque) appears that tends to return the pendulum to its equilibrium position. This moment is

$$M = -mgl\sin\varphi \tag{7.78}$$

where m is the mass of the pendulum and l is the distance between the suspension point 0 and the centre of mass C of the pendulum (Fig. 7.8). The minus sign has the

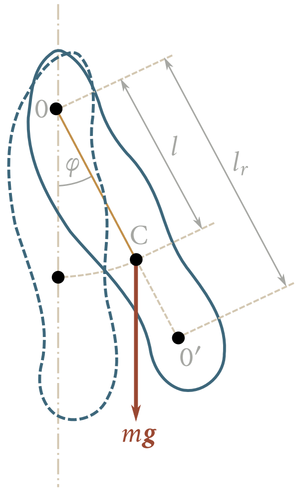

Fig. 7.8

same meaning as in Eq. (7.71).

Denoting the moment of inertia of a pendulum relative to the axis passing through the point of suspension by the symbol I, we can write:

$$I\ddot{\varphi} = -mgl \sin \varphi. \tag{7.79}$$

For small-amplitude oscillations, Eq. (7.79) transforms into Eq. (7.74) that we already know:

$$\ddot{\varphi} + \omega_0^2 \varphi = 0.$$

Here ω_0^2 stands for the following quantity:

$$\omega_0^2 = \frac{mgl}{I}. \tag{7.80}$$

It follows from Eqs. (7.74) and (7.80) that with small displacements from the equilibrium position, a physical pendulum performs harmonic oscillations whose frequency depends on the mass of the pendulum, the moment of inertia of the pendulum relative to the axis of rotation, and the distance between the latter and the centre of mass of the pendulum. According to Eq. (7.80), the period of oscillation of a physical pendulum is determined by the expression

$$T = 2\pi \left(\frac{I}{mgl} \right)^{1/2}. \tag{7.81}$$

A comparison of Eqs. (7.76) and (7.81) shows that a mathematical pendulum of length

$$l_{\mathrm{r}} = \frac{I}{ml} \tag{7.82}$$

will have the same period of oscillations as the given physical pendulum. The quantity (7.82) is called the **reduced length** of the physical pendulum. Thus, the reduced length of a physical pendulum is the length of a simple pendulum whose period of oscillations coincides with that of the given physical pendulum.

The point on the straight line joining the point of suspension to the centre of mass at a distance of the reduced length from the axis of rotation is called the **centre of oscillation** of the physical pendulum (see point $0'$ in Fig. 7.8). It can be shown (we invite our reader to do this as an exercise) that when a pendulum is suspended by its centre of oscillation $0'$, its reduced length and, consequently, its period of oscillations will be the same as initially. Hence, the point of suspension and the centre of oscillation are interchangeable: when the point of suspension is transferred to the centre of oscillation, the previous point of suspension becomes the new centre of oscillation. This property underlies the determination of the acceleration of free fall with the aid of the so-called reversible pendulum. The latter has two parallel knife edges fastened near its ends by which it can be suspended in turn. Heavy weights can be moved along the pendulum and be fastened to it. The weights are adjusted to ensure the pendulum having the same period of oscillations when suspended by any of the knife edges. In this case, the distance between the knife edges will be l_r. By measuring the period of oscillations of the pendulum and knowing l, we can find the acceleration of free fall g by the equation

$$T = 2\pi\left(\frac{l_\mathrm{r}}{g}\right)^{1/2}.$$

7.7. Vector Diagram

The solution of a number of problems, particularly the addition of several oscillations of the same direction (or, which is the same the addition of several harmonic functions) is considerably facilitated and becomes clear if we depict oscillations graphically in the form of vectors in a plane. The result obtained is called a **vector diagram**.

Let us take an axis which we shall denote by the symbol x (Fig. 7.9). From point 0 on the axis we shall lay off a vector of length A making the angle α with the axis. If we rotate this vector with the angular velocity ω_0, then the projection of the end of the vector will move along the x-axis within the limits from $-A$ to $+A$. The coordinate of this projection will change with time according to the law

$$x = A\cos(\omega_0 t + \alpha).$$

Consequently, the projection of the tip of the vector onto the axis will perform a harmonic oscillation with an amplitude equal to the length of the vector, an angular

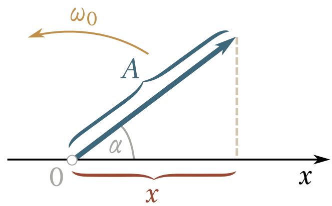

Fig. 7.9

frequency equal to the angular velocity of the vector, and an initial phase equal to the angle formed by the vector with the axis at the initial moment.

It follows from the above that a harmonic oscillation can be given with the aid of a vector whose length equals the amplitude of the oscillation, while the direction of the vector makes an angle with the x-axis that equals the initial phase of the oscillation.

Let us consider the addition of two harmonic oscillations of the same direction and the same frequency. The displacement x of the oscillating body will be the sum of the displacements x_1 and x_2, which can be written as follows:

$$x_1 = A_1 \cos(\omega_0 t + \alpha_1), \quad x_2 = A_2 \cos(\omega_0 t + \alpha_2). \tag{7.83}$$

Let us represent both oscillations with the aid of the vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ (Fig. 7.10). We shall construct the resultant vector $\boldsymbol{A}$ according to the rules of vector addition. It is easy to see that the projection of this vector onto the x-axis equals the sum of the projections of the vectors being added:

$$x = x_1 + x_2.$$

Consequently, the vector $\boldsymbol{A}$ is the resultant oscillation. This vector rotates with the same angular velocity as the vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ so that the resultant motion will be a harmonic oscillation with the frequency ω_0, amplitude A, and the initial phase α. It can be seen from the construction that

$$\begin{aligned} A^2 &= A_1^2 + A_2^2 - 2A_1A_2 \cos[\pi - (\alpha_2 - \alpha_1)] \\ &= A_1^2 + A_2^2 - 2A_1A_2 \cos(\alpha_2 - \alpha_1), \end{aligned} \tag{7.84}$$

$$\tan \alpha = \frac{A_1 \sin \alpha_1 + A_2 \sin \alpha_2}{A_1 \cos \alpha_1 + A_2 \cos \alpha_2}. \tag{7.85}$$

Thus, the representation of harmonic oscillations by means of vectors makes it possible to reduce the addition of several oscillations to the operation of vector addition. This procedure is especially useful in optics, for example, where the light oscillations at a point are determined as the result of the superposition of many oscillations arriving at the given point from different sections of a wavefront.

Equations (7.84) and (7.85) can naturally be obtained by summation of Eqs. (7.83)

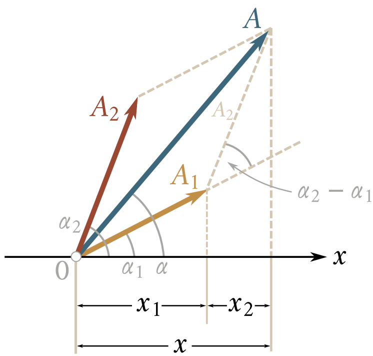

Fig. 7.10

and the corresponding trigonometric transformations. But the way we have used to obtain these equations is distinguished by its great simplicity and clarity.

Let us analyse Eq. (7.84) for the amplitude. If the difference between the phases of both oscillations $\alpha_2 - \alpha_1$ vanishes, the amplitude of the resulting oscillation equals the sum of A_1 and A_2. If the phase difference $\alpha_2 - \alpha_1$ equals $+\pi$ or $-\pi$, *i.e.*, both oscillations are in counterphase, then the amplitude of the resulting oscillation equals $|A_1 - A_2|$.

If the frequencies of the oscillations x_1 and x_2 are not the same, the vectors $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ will rotate with different velocities. In this case, the resultant vector $\boldsymbol{A}$ pulsates in magnitude and rotates with a varying velocity. Consequently, the resultant motion in this case will be a complex oscillating process instead of a harmonic oscillation.

7.8. Beats

Of special interest is the case when two harmonic oscillations of the same direction being added differ only slightly in frequency. We shall now show that the resultant motion in these conditions can be considered as a harmonic oscillation with a pulsating amplitude. Such oscillations are called **beats**.

Let ω stand for the frequency of one of the oscillations and $\omega + \Delta\omega$ for that of the second one. According to our conditions, $\Delta\omega \ll \omega$. We shall assume that the amplitudes of both oscillations are the same and equal A. To avoid unnecessary complications in our formulas, we shall consider that the initial phases of both oscillations equal zero. The equations of the oscillations will thus become

$$x_1 = A\cos\omega t, \qquad x_2 = A\cos(\omega + \Delta\omega)t.$$

By summing these expressions and using the trigonometric formula for the sum of

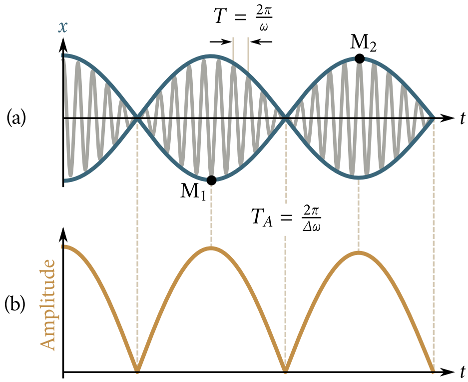

Fig. 7.11

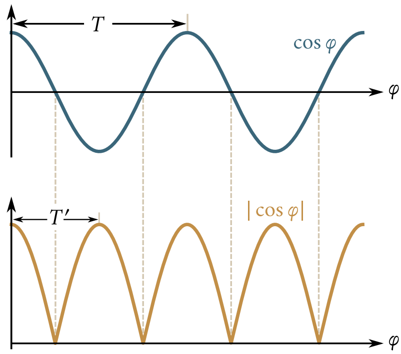

Fig. 7.12

cosines, we get

$$x = x_1 + x_2 = \left[2A \cos\left(\frac{\Delta\omega}{2} t\right)\right] \cos \omega t \tag{7.86}$$

(in the second multiplier we disregard the term $\Delta\omega/2$ in comparison with ω). A graph of the function (7.86) is shown in Fig. 7.11a. The graph has been plotted for $\omega/\Delta\omega = 10$.

The multiplier in parentheses in Eq. (7.86) changes much more slowly than the second multiplier. Owing to the condition $\Delta\omega \ll \omega$, the multiplier in parentheses does not virtually change during the time in which the multiplier $\cos \omega t$ performs several complete oscillations. This gives us the grounds to consider the oscillation (7.86) as a harmonic oscillation of frequency ω whose amplitude changes according to a periodic law. The multiplier in parentheses cannot be an expression of this law because it changes within the limits from $-2A$ to $+2A$ whereas the amplitude by definition is a positive quantity. A graph of the amplitude is shown in Fig. 7.11b. The analytic expression of the amplitude obviously has the form

$$\text{amplitude} = \left| 2A \cos\left(\frac{\Delta\omega}{2} t\right)\right|. \tag{7.87}$$

The function (7.87) is a periodic function with a frequency double that of the expression inside the magnitude sign (see Fig. 7.12 comparing graphs of the cosine and its magnitude), *i.e.*, with a frequency of $\Delta\omega$. Thus, the frequency of pulsations of the amplitude—it is called the frequency of the beats—equals the difference between the frequencies of the oscillations being added.

We must note that the multiplier $2A\cos(\Delta\omega t/2)$ not only determines the amplitude, but also affects the phase of the oscillations. This is manifested, for example, in that the deflections corresponding to adjacent peaks of the amplitude have opposite signs (see points M$_1$ and M$_2$ in Fig. 7.11a).

7.9. Addition of Mutually Perpendicular Oscillations

Assume that a point particle can oscillate both along the x-axis and along the y-axis perpendicular to it. If we induce both oscillations, the particle will move along a certain, generally speaking, curved trajectory whose shape depends on the phase difference between the two oscillations.

Let us choose the beginning of counting time so that the initial phase of the first oscillation equals zero. The equations of the oscillations will therefore be written as follows:

$$x = A\cos\omega t, \qquad y = B\cos(\omega t + \alpha) \tag{7.88}$$

where α is the difference between the phases of the two oscillations.

Equations (7.88) describe the trajectory along which a body participating in both oscillations moves and are given in the parametric form. To obtain an equation of the trajectory in the conventional form, we must exclude the parameter t from Eqs. (7.88). It follows from the first of the Eqs.(7.88) that

$$\cos\omega t = \frac{x}{A}. \tag{7.89}$$

Hence,

$$\sin\omega t = \left(1 - \frac{x^2}{A^2}\right)^{1/2}. \tag{7.90}$$

Now let us expand the cosine in the second of the Eqs. (7.88) according to the formula for the cosine of a sum, using instead of $\cos\omega t$ and $\sin\omega t$ their values from Eqs. (7.89) and (7.90). As a result we get

$$\frac{y}{A} = \frac{x}{A}\cos\alpha - \sin\alpha\left(1 - \frac{x^2}{A^2}\right)^{1/2}.$$

This equation after simple transformations can be given the form

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\alpha = \sin^2\alpha. \tag{7.91}$$

It is known from analytical geometry that Eq. (7.91) is the equation of an ellipse whose axes are oriented arbitrarily relative to the coordinate axes x and y. The orientation of the ellipse and the dimensions of its semiaxes depend in a quite complicated way on the amplitudes A and B and the phase difference α.

Let us study the shape of the trajectory in some particular cases.

1. The phase difference a equals zero. In this case, Eq. (7.91) becomes

$$\left(\frac{x}{A} - \frac{y}{B}\right)^2 = 0$$

whence we get an equation of a straight line:

$$y = \frac{B}{A}x. \tag{7.92}$$

The oscillating particle moves along this straight line, its distance from the origin of coordinates being $r = \sqrt{x^2 + y^2}$. Introducing into this equation the expressions (7.88) for x and y and taking into account that $a = 0$, we get the law of the change in r with time:

$$r = \left(x^2 + y^2\right)^{1/2} \cos \omega t. \tag{7.93}$$

It follows from Eq. (7.93) that the resultant motion is a harmonic oscillation along the straight line (7.92) with the frequency ω and the amplitude $\sqrt{A^2 + B^2}$ (Fig. 7.13).

2. The phase difference a equals $\pm\pi$. Equation (7.91) has the form

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 = 0$$

whence we find that the resultant motion is a harmonic oscillation along a straight line (Fig. 7.14):

$$y = -\frac{B}{A}x.$$

3. When $\alpha = \pm\pi/2$, Eq. (7.91) becomes

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \tag{7.94}$$

i.e., it becomes the equation of an ellipse reduced to the coordinate axes, the semiaxes of the ellipse being equal to the corresponding amplitudes of the oscillations. When the amplitudes A and B are equal, the ellipse degenerates into a circle.

The cases $\alpha = +\pi/2$ and $\alpha = -\pi/2$ differ in the direction of motion along the ellipse or circle. If $\alpha = +\pi/2$, Eqs. (**??**) can be written as follows:

$$x = A \cos \omega t, \qquad y = -B \sin \omega t. \tag{7.95}$$

At the moment $t = 0$, the body is at point 1 (Fig. 7.15). At the following moments, the coordinate x diminishes, while the coordinate y becomes negative. Consequently, the motion is clockwise.

When $\alpha = -\pi/2$, the equations of the oscillations become

$$x = A \cos \omega t, \qquad y = B \sin \omega t. \tag{7.96}$$

Hence we can conclude that the motion is counter-clockwise.

It follows from the above that uniform motion along a circle of radius R with the angular velocity ω can be represented as the sum of two mutually perpendicular

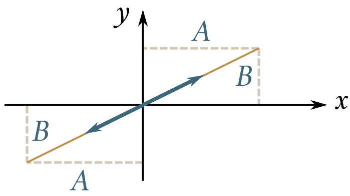

Fig. 7.13

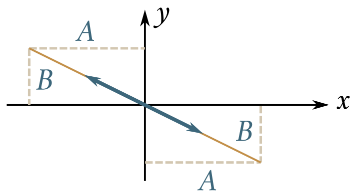

Fig. 7.14

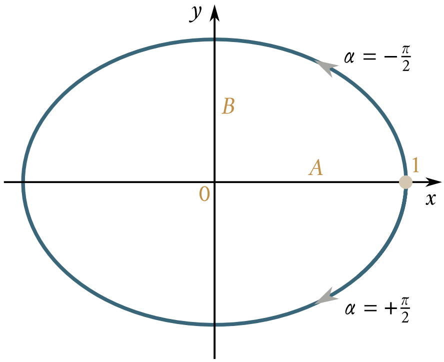

Fig. 7.15

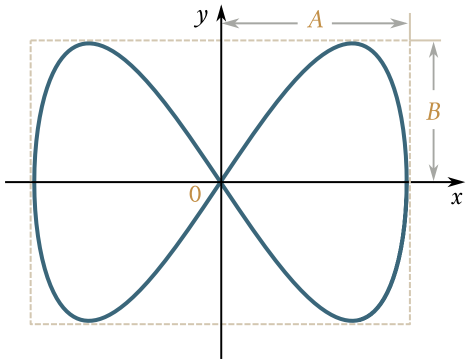

Fig. 7.16

oscillations:

$$x = R\cos\omega t, \quad y = \pm R\sin\omega t \tag{7.97}$$

(the plus sign in the expression for y corresponds to counter-clockwise motion, and the minus sign to clockwise motion).

When the frequencies of mutually perpendicular oscillations differ by a very small value $\Delta\omega$, they can be considered as oscillations of an identical frequency, but with a slowly changing phase difference. Indeed, the equations of the oscillations can be written as follows:

$$x = A\cos\omega t, \quad y = B\cos[\omega t + (\Delta\omega t + \alpha)]$$

and the expression $\Delta\omega t + \alpha$ can be considered as the phase difference slowly changing with time according to a linear law.

The resultant motion in this case occurs along a slowly changing curve that will sequentially take on a shape corresponding to all the values of the phase difference from $-\pi$ to $+\pi$.

If the frequencies of mutually perpendicular oscillations are not identical, then the trajectory of the resultant motion has the shape of rather intricate curves called **Lissajous figures**. Figure 7.16 shows one of the simple trajectories obtained at a frequency ratio of 1:2 and a phase difference of $\pi/2$. The equations of the oscilla-

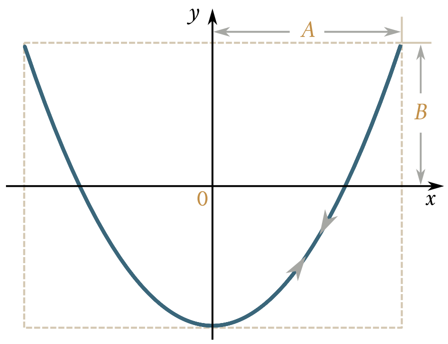

Fig. 7.17

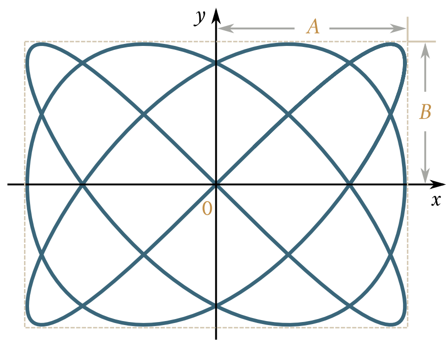

Fig. 7.18

tions have the form

$$x = A\cos\omega t, \qquad y = B\cos\left(\omega t + \frac{\pi}{2}\right).$$

During the time the particle manages to move from one extreme position to the other along the x-axis, it will be able to leave its zero position, reach one extreme position on the y-axis, then the other one, and return to its zero position.

With a frequency ratio of 1:2 and a phase difference of zero, the trajectory degenerates into an open curve (Fig. 7.17) along which the particle moves to and fro.

The closer to unity is the rational fraction expressing the ratio of the frequencies of the oscillations, the more intricate is the Lissajous figure. Figure 7.18 shows as an example a curve for the frequency ratio of 3:4 and the phase difference $\pi/2$.

7.10. Damped Oscillations

Damped oscillations are described by Eq. (7.11):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

where, by Eqs. (7.10) and (7.6),

$$2\beta = \frac{r}{m}, \qquad \omega_0^2 = \frac{k}{m}.$$

Here r is the resistance coefficient, *i.e.*, the coefficient of proportionality between the velocity x and the force of resistance, and k is the quasi-elastic force coefficient. We must note that ω_0 is the frequency with which free oscillations would take place in the absence of resistance of the medium (when $r = 0$). This frequency is called the **natural frequency** of the system.

Introduction of the function $x = e^{\lambda t}$ into Eq. (7.11) leads to the characteristic

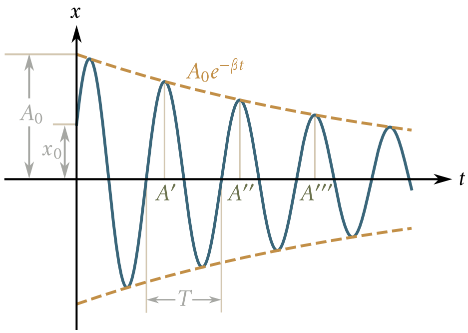

Fig. 7.19

equation

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0. \tag{7.98}$$

The roots of this equation are

$$\lambda_1 = -\beta + \left(\beta^2 - \omega_0^2\right)^{1/2}, \quad \lambda_2 = -\beta - \left(\beta^2 - \omega_0^2\right)^{1/2}. \tag{7.99}$$

When the damping is not too great (at $\beta < \omega_0$) the radicand will be negative. Let us write it in the form $(i\omega)^2$, where ω is a real quantity equal to

$$\omega = \left(\beta^2 - \omega_0^2\right)^{1/2}. \tag{7.100}$$

Here, the roots of the characteristic equation will be as follows:

$$\lambda_1 = -\beta + i\omega, \quad \lambda_2 = -\beta - i\omega. \tag{7.101}$$

By Eq. (7.38), the general solution of Eq. (7.11) will be the function

$$x = C_1e^{(-\beta+i\omega)t} + C_2e^{(-\beta-i\omega)t} = e^{\beta t}\left(C_1e^{i\omega t} + C_2e^{-i\omega t}\right).$$

The expression in parentheses is similar to Eq. (7.51). It can therefore be written in a form similar to Eq. (7.55). Thus, when damping is not too great, the general solution of Eq. (7.11) has the form

$$x = A_0e^{\beta t}\cos(\omega t + \alpha). \tag{7.102}$$

Here A_0 and α are arbitrary constants, and ω is a quantity determined by Eq. (7.100). Figure 7.19 gives a graph of the function (7.102). The dash lines show the limits confining the displacement x of the oscillating particle.

In accordance with the kind of the function (7.102), the motion of the system can be considered as a harmonic oscillation of frequency ω with an amplitude varying according to the law $A(t) = A_0e^{-\beta t}$. The upper dash curve in Fig. 7.19 depicts the function $A(t)$, the quantity A_0 being the amplitude at the initial moment of time. The initial displacement x_0, apart from A_0, also depends on the initial phase α: $x_0 = A_0\cos\alpha$.

The rate of damping of oscillations is determined by the quantity $\beta = r/(2m)$ defined as the **damping factor**. Let us find the time τ during which the amplitude diminishes e times. By definition, $e^{-\beta\tau} = e^{-1}$, whence $\beta\tau = 1$. Consequently, the damping factor is the reciprocal of the time interval during which the amplitude diminishes e times.

According to Eq. (7.56), the period of damped oscillations is

$$T = \frac{2\pi}{\left(\omega_0^2 - \beta^2\right)^{1/2}}. \tag{7.103}$$

When the resistance of the medium is insignificant, the period of oscillations virtually equals $T_0 = 2\pi/\omega_0$. The period of oscillations grows with an increasing damping factor.

The following maximum displacements to either side (for example A', A'', A''', etc. in Fig. 7.19) form a geometrical progression. Indeed, if $A' = A_0 e^{-\beta t}$, then $A'' = A_0 e^{-\beta(t+T)} = A' e^{-\beta T}$, $A''' = A_0 e^{[-\beta(t+2T)]} = A'' e^{-\beta T}$, etc. In general, the ratio of the values of the amplitudes corresponding to moments of time differing by a period is

$$\frac{A(t)}{A(t+T)} = e^{\beta T}.$$

This ratio is called the **damping decrement**, and its logarithm is called the **logarithmic decrement**:

$$\lambda = \ln\left[\frac{A(t)}{A(t+T)}\right] = \beta T \tag{7.104}$$

[do not confuse with the constant λ in Eqs. (7.98) and (7.101)].

To characterize an oscillatory system, the logarithmic decrement λ is usually used. Expressing β through λ and T in accordance with Eq. (7.104), we can write the law of diminishing of the amplitude with time in the form

$$A = A_0 \exp\left(-\frac{\lambda}{T}\, t\right). \tag{7.105}$$

In the interval during which the amplitude diminishes e times, the system manages to complete $N_e = \tau/T$ oscillations. We find from the condition $\exp(-\lambda t/T) = \exp(-1)$ that $\lambda t/T = 1$. Hence the logarithmic decrement is the reciprocal of the number of oscillations completed during the interval in which the amplitude diminishes e times.

An oscillatory system is often also characterized by the quantity

$$Q = \frac{\pi}{\lambda} = \pi N_e \tag{7.106}$$

called the **quality**, or simply the Q, of the system. As can be seen from its definition, the quality is proportional to the number of oscillations N_e performed by the

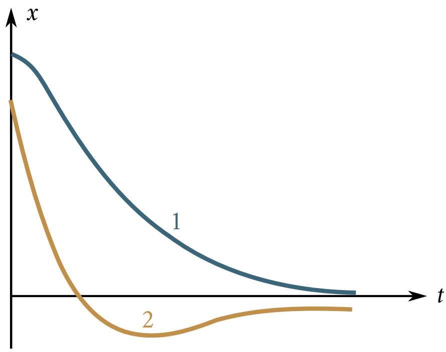

Fig. 7.20

system in the interval τ during which the amplitude of the oscillations diminishes e times.

We established in Sec. 7.5 that the total energy of an oscillating system is proportional to the square of the amplitude [see Eq. (7.68)]. Accordingly, the energy of the system in damped oscillations diminishes with time according to the law

$$E = E_0 e^{-2\beta t} \tag{7.107}$$

(E_0 is the value of the energy at $t = 0$). Time differentiation of this expression gives the rate of growth of the system's energy:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2\beta E_0 e^{-2\beta t} = -2\beta E.$$

By reversing the signs, we find the rate of diminishing of the energy:

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = 2\beta E. \tag{7.108}$$

If the energy changes only slightly during the time equal to a period of oscillations, the reduction of the energy during a period can be found by multiplying Eq. (7.108) by T:

$$-\Delta E = 2\beta T E$$

(we remind our reader that ΔE stands for the increment, and $-\Delta E$ for the decrement of the energy). Finally, taking into consideration Eqs. (7.104) and (7.106), we arrive at the relation

$$\frac{E}{(-\Delta E)} = \frac{Q}{2\pi} \tag{7.109}$$

from which it follows that upon slight damping of oscillations, the quality with an accuracy up to the factor 2π equals the ratio of the energy stored in the system at a given moment to the decrement of this energy during one period of oscillations.

It follows from Eq. (7.103) that a growth in the damping factor is attended by an increase in the period of oscillations. At $\beta = \omega_0$, the period of oscillations becomes infinite, *i.e.*, the motion stops being periodic.

At $\beta > \omega_0$, the roots of the characteristic equation become real [see Eq. (7.99)], and the solution of the differential equation (7.11) is equal to the sum of two exponents:

$$x = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}.$$

Here C_1 and C_2 are real constants whose values depend on the initial conditions (on x_0 and $v_0 = \dot{x}_0$). The motion is therefore aperiodic—a system displaced from its equilibrium position returns to it without performing oscillations. Figure 7.20 shows two possible ways for a system to return to its equilibrium position in aperiodic motion. How the system arrives at its equilibrium position depends on the initial conditions. The motion depicted by curve 2 is obtained when the system begins to move from the position characterized by the displacement x_0 to its equilibrium position with the initial velocity v_0 determined by the condition

$$|v_0| > |x_0| \left[\beta + \left(\beta^2 + \omega_0^2\right)^{1/2}\right]. \tag{7.110}$$

This condition will be obeyed when a system brought out of its equilibrium position is given a sufficiently strong impetus toward it. If after displacing a system from its equilibrium position we release it without an impetus (*i.e.*, with $v_0 = 0$) or impart to it an impetus of insufficient force [such that v_0 is less than the value determined by the condition (7.110)], the motion will occur according to curve 1 in Fig. 7.20.

7.11. Auto-Oscillations

The energy of a system in damped oscillations is used to overcome the resistance of the medium. If this decrease of energy is replenished, the oscillations will become undamped. The energy of a system can be replenished at the expense of impetuses from outside, but they must be imparted to the system in step with its oscillations, otherwise they may weaken the latter and even stop them. An oscillating system can be made to control the external action itself, ensuring agreement between the impetuses imparted to it and its motion. Such a system is called an **auto-oscillating** one, and the undamped oscillations it performs are called **auto-oscillations**.

Let us consider a clock mechanism as an example of an auto-oscillatory system. The clock pendulum is fitted onto the same axis as a bent lever-the anchor (Fig. 7.11). The ends of the anchor carry projections of a special shape called pallets. The toothed escape wheel is acted upon by a chain with a weight or a wound up

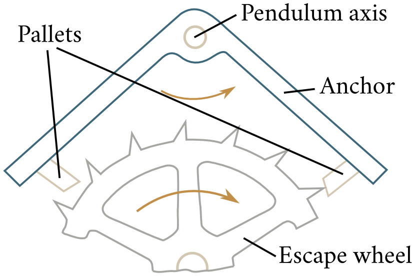

Fig. 7.21

spring that tends to turn it clockwise. During the major part of the time, however, one of the wheel's teeth bears against the side face of a pallet, the latter sliding along the tooth's surface when the pendulum oscillates. Only when the pendulum is near its middle position do the pallets stop being in the way of the teeth, and the escape wheel turns, pushing the anchor by means of a tooth whose tip slides along the chamfered end of a pallet. During a complete cycle of pendulum oscillations (during a period), the escape wheel turns through two teeth, and each of the pallets receives a push. These pushes, performed at the expense of the energy of a lifted weight or a wound up spring, are exactly what replenishes the decrease in the energy of the pendulum due to friction.

7.12. Forced Oscillations

When the driving force changes according to a harmonic law, the oscillations are described by the differential equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t \tag{7.111}$$

[see Eq. (7.13)]. Here β is the damping factor, ω_0 is the natural frequency of the system [see Eqs. (7.6), (7.10)], $f_0 = F_0/m$ (F_0 is the amplitude of the driving force), and ω is the frequency of the force.

Equation (7.111) is a non-homogeneous one. According to the theorem (7.41), the general solution of a non-homogeneous equation equals the sum of the general solution of the corresponding homogeneous equation and the partial solution of the non-homogeneous one. We already know the general solution of a homogeneous equation [see the function (7.102), which is the general solution of Eq. (7.11)]. It has the form

$$x = Ae^{-\beta t}\cos(\omega' t + \alpha) \tag{7.112}$$

where $\omega' = \left(\omega_0^2 - \beta^2\right)^{1/2}$, and A_0 and α are arbitrary constants[6].

It remains for us to find the partial (containing no arbitrary constants) solution of Eq. (7.111). We shall use the procedure described at the end of Sec. 7.4 for this purpose. Let us add the imaginary function $if_0 \sin \omega t$ to the function in the right-hand side of Eq. (7.111). After this, we can write the right-hand side in the form $f_0 \exp(i\omega t)$ [see Eq. (7.24)]. We thus arrive at the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 e^{i\omega t}. \tag{7.113}$$

It is easier to solve this equation than Eq. (7.111) because it is simpler to differentiate and integrate an exponent than trigonometric functions.

We shall try to find the partial solution of Eq. (7.113) in the form

$$\hat{x} = \hat{A} e^{i\omega t} \tag{7.114}$$

where $\hat{A}$ is a complex number. The function (7.114) is also complex, which has been indicated by capping the x. Time differentiation of this function yields

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = i\omega\hat{A}e^{i\omega t}, \qquad \frac{\mathrm{d}^2\hat{x}}{\mathrm{d}t^2} = -\omega^2\hat{A}e^{i\omega t}. \tag{7.115}$$

Introduction of Eqs. (7.114) and (7.115) into Eq. (7.113) and cancelling off the common factor $e^{i\omega t}$ give the algebraic equation

$$-\omega^2\hat{A} + 2i\beta\omega\hat{A} + \omega^2\hat{A} = f_0.$$

Hence,

$$\hat{A} = \frac{f_0}{\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega}. \tag{7.116}$$

We have found the value of $\hat{A}$ at which the function (7.114) satisfies Eq. (7.113). Let us write the complex number in the denominator in the exponential form:

$$\left(\omega_0^2 - \omega^2\right) + 2i\beta\omega = \rho e^{i\varphi}. \tag{7.117}$$

By Eqs. (7.18), we have

$$\rho = \left[\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2\right], \qquad \varphi = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right). \tag{7.118}$$

Substitution of the denominator in Eq. (7.116) in accordance with Eq. (7.117) yields

$$\hat{A} = \frac{f_0}{\rho e^{i\varphi}} = \frac{f_0}{\rho}e^{-i\varphi}.$$

Introduction of this value of $\hat{A}$ into (7.114) gives the partial solution of Eq. (7.113):

$$\hat{x} = \frac{f_0}{\rho}e^{-i\varphi}e^{i\omega t} = \frac{f_0}{\rho}e^{i(\omega t - \varphi)}.$$

[6] The symbol ω without a prime stands for the frequency of the driving force.

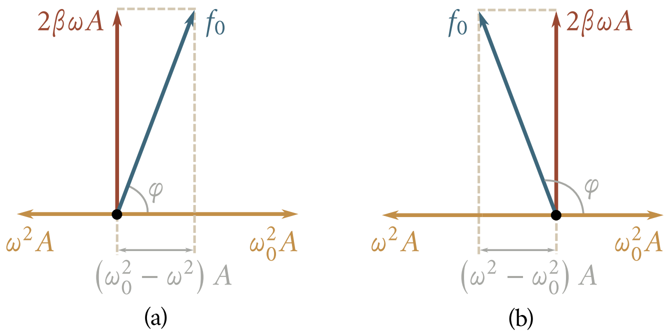

Fig. 7.22

Finally, by taking the real part of this function, we get the partial solution of Eq. (7.111):

$$x = \frac{f_0}{\rho} \cos(\omega t - \varphi).$$

Introduction of the values of f_0, and also of the values of ρ and φ from Eqs. (7.118), gives the final expression

$$x = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2\right]} \cos\left[\omega t - \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)\right]. \tag{7.119}$$

We must note that the function (7.119) contains no arbitrary constants.

Let us obtain a partial solution of Eq. (7.111) in another way with the aid of a vector diagram. We shall assume that the partial solution of Eq. (7.111) has the form

$$x = A\cos(\omega t - \varphi). \tag{7.120}$$

Hence,

$$\dot{x} = -\omega A \sin(\omega t - \varphi) = \omega A \cos\left(\omega t - \varphi + \frac{\pi}{2}\right) \tag{7.121}$$

$$\ddot{x} = \omega^2 A \cos(\omega t - \varphi) = \omega^2 A \cos(\omega t - \varphi + \pi). \tag{7.122}$$

The use of Eqs. (7.120)-(7.122) in Eq. (7.111) yields

$$\omega^2 A \cos(\omega t - \varphi + \pi) + 2\beta\omega A \cos\left(\omega t - \varphi + \frac{\pi}{2}\right) + \omega^2 A \cos(\omega t - \varphi) = f_0 \cos\omega t. \tag{7.123}$$

It follows from Eq. (7.123) that the constants A and φ must have values such that the harmonic function $f_0 \cos\omega t$ will equal the sum of the three harmonic functions in the left-hand side of the equation. If we depict the function $\omega_0^2 A \cos(\omega t - \varphi)$ by a vector of length $\omega_0^2 A$ directed to the right (Fig. 7.22), then the function $2\beta\omega A\cos(\omega t - \varphi + \pi/2)$ will be depicted by a vector of length $2\beta\omega A$ turned counterclockwise relative to the vector $\omega_0^2 A$ through the angle $\pi/2$ (see Sec. 7.7), and the function $\omega^2 A\cos(\omega t - \varphi + \pi)$ by a vector of length $\omega_0^2 A$ turned through the angle π relative to the vector $\omega_0^2 A$. For Eq. (7.123) to be satisfied, the sum of the three

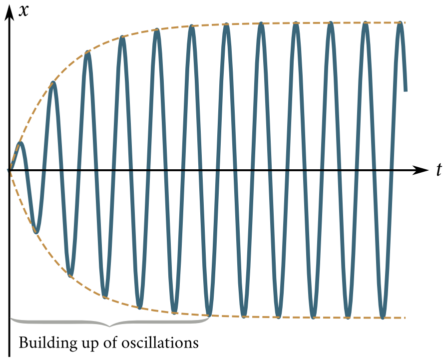

Fig. 7.23

enumerated vectors must coincide with the vector depicting the function $f_0 \cos \omega t$. Inspection of Fig. 7.22a shows that such coincidence is possible only at a value of the amplitude A determined by the condition

$$\left(\omega_0^2 - \omega^2\right)^2 A^2 + 4\beta^2\omega^2 A^2 = f_0^2$$

whence,

$$A = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2\right]} \tag{7.124}$$

(we have replaced f_0 with the ratio F_0/m). Figure 7.22a corresponds to the case when $\omega < \omega_0$. We get the same value of A from Fig. 7.22b corresponding to the case when $\omega > \omega_0$.

Figure 7.22 also allows us to obtain the value of φ showing the lagging in phase of the forced oscillation (7.120) behind the driving force producing it. It can be seen from the figure that

$$\tan \varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{7.125}$$

Using the values of A and φ determined by Eqs. (7.124) and (7.125) in Eq. (7.120), we get the function (7.119).

Function (7.119) when added to Eq. (7.112) gives the general solution of Eq. (7.111) describing the behaviour of the system upon forced oscillations. Addend (7.112) plays an appreciable part only in the initial stage of the process, during the so-called setting in of the oscillations (Fig. 7.23). With the passage of time, owing to the exponential factor $e^{-\beta t}$, the part played by addend (7.112) diminishes to a greater and greater extent, and after sufficient time elapses it may be disregarded, retaining only addend (7.119) in the solution.

Thus, function (7.119) describes steady-state forced oscillations. They are harmonic oscillations with a frequency equal to that of the driving force. The amplitude (7.124) of the forced oscillations is proportional to that of the driving force. The amplitude of a given oscillatory system (determined by ω_0 and β) depends on the frequency of the driving force. Forced oscillations lag in phase behind their driving force; the lagging φ also depends on the frequency of the force [see Eq. (7.125)].

As a result of the amplitude of forced oscillations depending on the frequency of the driving force, the amplitude of the oscillations reaches a maximum value at a definite frequency for the given system. The oscillatory system responds especially to the action of the driving force at this frequency. This phenomenon is called **resonance**, and the corresponding frequency—the **resonance frequency**.

To determine the resonance frequency ω_{res} we must find the maximum of the function (7.124) or, which is the same, the minimum of the expression inside the radical in the denominator. Differentiating this expression with respect to ω and equating it to zero, we get the condition determining ω_{res}:

$$-4\left(\omega_0^2-\omega^2\right)\omega+8\beta^2\omega=0. \tag{7.126}$$

Equation (7.126) has three solutions: $\omega = 0$ and $\omega = \pm\left(\omega_0^2-2\beta^2\right)^{1/2}$. The solution equal to zero corresponds to a maximum of the denominator. Of the remaining two solutions, the negative one must be discarded as being deprived of a physical meaning (the frequency cannot be negative). We thus get a single value for the resonance frequency:

$$\omega_{\text{res}}=\left(\omega_0^2-2\beta^2\right)^{1/2}. \tag{7.127}$$

Using this value of the frequency in Eq. (7.124), we get an expression for the amplitude in resonance:

$$A_{\text{res}}=\frac{F_0/m}{2\beta\left(\omega_0^2-\beta^2\right)^{1/2}}. \tag{7.128}$$

It follows from Eq. (7.128) that the amplitude in resonance would become equal to infinity in the absence of resistance of the medium. By Eq. (7.127), the resonance frequency in such conditions (at $\beta = 0$) coincides with the natural frequency of oscillations of the system ω_0.

The dependence of the amplitude of forced oscillations on the frequency of the driving force (or, which is the same, on the frequency of oscillations) is shown graphically in Fig. 7.24. The separate curves correspond to different values of the parameter β. According to Eqs. (7.127) and (7.128), the peak of a given curve is higher and further to the right with decreasing values of β. The expression for the resonance frequency becomes imaginary upon very great damping (such that $2\beta^2 > \omega_0^2$). This signifies that no resonance is observed in these conditions—the amplitude of forced oscillations monotonously diminishes with increasing fre-

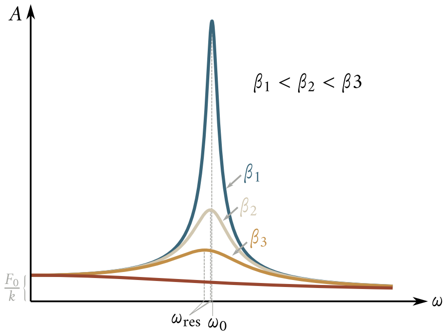

Fig. 7.24

quency (see the lower curve in Fig. 7.24). The curves shown in Fig. 7.24 corresponding to different values of the parameter β are called **resonance curves**.

We can add the following remarks with respect to resonance curves. When ω tends to zero, all the curves arrive at the same limiting value equal to $F_0/(m\omega_0^2)$, *i.e.*, F_0/k, differing from zero. This value is the displacement from the equilibrium position received by the system under the action of a constant force of magnitude F_0. When ω tends to infinity, all the curves asymptotically tend to zero because at a high frequency the force changes its direction so rapidly that the system does not manage to become displaced from its equilibrium position. Finally, we must note that diminishing of β is attended by a greater change in the amplitude with the frequency near resonance and by a sharper "peak".

It follows from Eq. (7.128) that with small damping (*i.e.*, when $\beta \ll \omega_0$), the amplitude in resonance is

$$A_{\text{res}} \approx \frac{F_0/m}{2\beta\omega}.$$

Let us divide this expression by the displacement x_0 from the equilibrium position under the action of the constant force F_0 equal to $F_0/(m\omega_0^2)$. The result is

$$\frac{A_{\text{res}}}{x_0} \approx \frac{\omega_0}{2\beta} = \frac{2\pi}{2\beta T} = \frac{\pi}{\lambda} = Q \tag{7.129}$$

[see Eq. (7.106)]. Thus, the quality Q shows how many times the amplitude at the moment of resonance exceeds the displacement of the system from its equilibrium position under the action of a constant force of the same magnitude as the amplitude of the driving force (this holds only with slight damping).

Inspection of Fig. 7.22 shows that forced oscillations lag in phase behind their driving force; this lagging ranges from 0 to π. The dependence of φ on ω at various values of β is shown in Fig. 7.25. The value $\varphi = \pi/2$ corresponds to the frequency ω_0. The resonance frequency is lower than the natural one [see Eq. (7.127)]. Hence, at

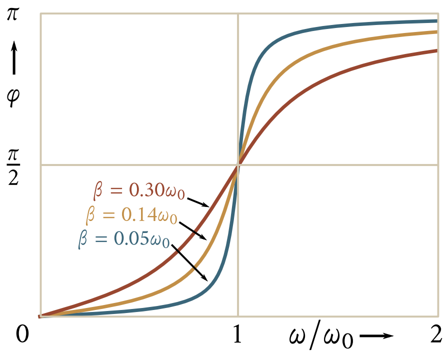

Fig. 7.25

the moment of resonance, $\varphi < \pi/2$. When the damping is insignificant, $\omega_{res} \approx \omega_0$, and we may assume that in resonance $\varphi = \pi/2$.

The phenomenon of resonance must never be forgotten in designing machines and various structures. The natural frequency of oscillations of such equipment and facilities must not be close to the frequency of possible external actions. For example, the natural frequency of vibrations of a ship's hull or an aeroplane's wings must greatly differ from the frequency of the vibrations that might be produced by rotation of the propeller. Otherwise vibrations will appear that may cause a catastrophe. Cases are known when bridges collapsed owing to the marching of columns of soldiers over them. The reason was that the natural frequency of oscillations of the bridge was close to the frequency of the soldier's steps.

The phenomenon of resonance, at the same time, is often very useful, especially in acoustics, radio engineering, etc.

7.13. Parametric Resonance

In the case dealt with in the preceding section, a driving force applied from outside produced a direct displacement of a system from its equilibrium position. Another kind of external action is known to exist by means of which great oscillations can be imparted to a system. This kind of action consists in periodically changing a parameter of the system in step with its oscillations, owing to which the phenomenon is called **parametric resonance**.

Let us take as an example a simple pendulum—a ball on a thread. If we periodically change the length l of the pendulum, increasing it when the pendulum is at its extreme positions and decreasing it when the pendulum is at its middle position (Fig. 7.26), then the pendulum starts swinging violently. The energy of the pendulum here grows at the expense of the work done by the force acting on the thread.

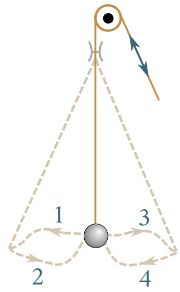

Fig. 7.26

The force tensioning the thread is not constant when the pendulum oscillates: it is smaller at the extreme positions when the velocity vanishes, and is greater at the middle position when the velocity of the pendulum is maximum. Consequently, the negative work of the external force upon elongation of the pendulum is smaller in magnitude than the positive work done upon shortening of the pendulum. As a result, the work done by the external force during a period is greater than zero.

Chapter 8

RELATIVISTIC MECHANICS

8.1. The Special Theory of Relativity

It was noted in Sec. 2.1 that Newtonian mechanics holds only for bodies travelling with speeds that are much lower than the speed of light in a vacuum (this speed is denoted by the symbol c). To describe motion at speeds comparable with c, Albert Einstein advanced relativistic mechanics, *i.e.*, mechanics taking the requirements of the special theory of relativity into account.

The special theory of relativity presented by Einstein in 1905 is a physical theory of space and time[1]. The foundation of this theory is formed by two postulates called **Einstein's principle of relativity** and the **principle of constancy of the speed of light**.

Einstein's principle of relativity is an extension of Galileo' mechanical principle (see Sec. 2.7 to all physical phenomena without any exception. According to this principle, *all laws of nature are the same in all inertial reference frames*. The unchanged form of an equation when the coordinates and time of one reference frame are replaced in it with the coordinates and time of another frame is called the **invariance** of the equation. The principle of relativity can therefore be formulated as follows: *the equations expressing the laws of nature are invariant with respect to transformations of coordinates and time from one inertial reference frame to another*.

The principle of constancy of the *speed of light states that the speed of light in a vacuum is the same in all inertial reference frames and does not depend on the motion of the sources and receivers of light.*[2]

[1]In 1915, Einstein presented the fundamentals of the general theory of relativity, which is the theory of gravitation.

[2]The experiment performed by A. Michelson and E. Morley confirming the validity of this principle will be described in the second volume of our course.

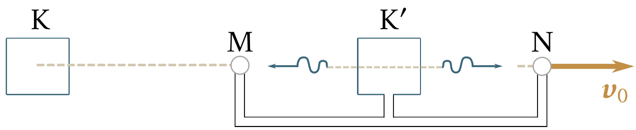

Fig. 8.1

A number of important conclusions relating to the properties of space and time follow from the above postulates. Space and time were considered independently of each other in Newtonian mechanics. Newton considered that absolute space and absolute time exist. He defined absolute space as a container of articles that always remains the same and stationary and that bas no relation to anything external. Newton wrote about time that absolute, true, or mathematical time flows uniformly without any relation to anything external by itself and owing to its internal nature. Accordingly, it was considered absolutely obvious that two events occurring simultaneously in one reference frame will also be simultaneous in all other reference frames. It is easy to see, however, that the latter statement contradicts the principle of the constancy of the speed of light.

Let us take two bodies K and K′ forming inertial reference frames together with their corresponding clocks. Assume that body K′ moves relative to body K with the velocity $\boldsymbol{v}_0$ directed along the straight line passing through the centres of the bodies (Fig. 8.1). Let us put two bodies M and N on this line. The bodies are at equal distances from body K′ and are rigidly joined to it. These bodies move relative to body K with the velocity $\boldsymbol{v}_0$, and are at rest relative to body K′. Let us consider the same process in both frames, namely, the emission of a light signal from the centre of body K′ and its reaching bodies M and N. The speed of light in all directions is the same and equals c. Hence, in the reference frame K′, the signal will reach bodies M and N at the same moment t'.

In the reference frame K, light also propagates in all directions with the speed c. In this frame, M moves toward the light signal. Body N moves in the same direction as the signal. Consequently, the signal reaches M before it reaches N, and therefore $t_{\mathrm{M}} < t_{\mathrm{N}}$. Thus, the events that were simultaneous in the frame K′ will not be simultaneous in the frame K. Hence, it follows that time flows differently in different reference frames.

To describe an event in a reference frame, we must indicate the place and the moment at which it occurs. This task can be coped with if we set up equally spaced coordinate marks in space and put a clock at each mark that will permit us to determine the moment at which the event occurs at the given place. The coordinate marks can be made by transferring a unit scale. Any system performing a periodically repeating process can be used as a clock. To compare the moments at which two events occur at different points of space, we must see that the clocks at these points are synchronized.

It would seem possible to synchronize the clocks by first placing them next to one another, and then, after comparing their readings, by transferring them to the corresponding points of space. Such a method must be rejected, however, because we do not know how the transfer of the clocks from one place to another will affect their running. We must therefore first put the clocks at their relevant places and only then compare their readings. This can be done by sending a light signal from one clock to the others[3]. Assume that a light signal is sent from point A at the moment t_1 (according to the clock at A). The signal is reflected from a mirror at point B and returns to A at the moment t_2. The clock at B should be considered synchronized with the one at A if at the moment when the signal reaches it the clock at B shows the time t equal to $(t_1 + t_2)/2$. This procedure must be performed for all the clocks arranged at the different points of the frame K. The events at A and B will be considered simultaneous in the frame K if the readings of the clocks at A and B corresponding to them coincide.

All the clocks in the frame K′ and in any other inertial reference frame are synchronized in a similar way. The speed of the light signal used for synchronization is the same in all the inertial reference frames. This explains why it is a light signal that is chosen as the signal for clock synchronization. The speed of light was found to be the limit. No signal, no action of one body on another can propagate with a speed exceeding that of light in a vacuum. This is the reason for light having the same speed in a vacuum in all reference frames. According to the principle of relativity, the laws of nature in all inertial frames must be identical. The circumstance that the speed of a signal cannot exceed a limiting value is also a law of nature. Hence, the value of the limiting speed must be the same in all reference frames.

The constancy of the speed of light results in space and time being mutually related, forming a single space-time. This relation can be depicted especially clearly with the aid of an imaginary four-dimensional space along three axes of which the space coordinates x, y, z are laid off, and along the fourth axis the time t, more exactly the time coordinate ct proportional to t and having the same dimension as the space coordinates.

An event (for instance, the decay of a particle) is characterized by the place where it occurred (by the coordinates x, y, z) and by the time t when it occurred. Thus, a point with the coordinates x, y, z, ct corresponds to an event in our imaginary four-dimensional space. This point is called the **world point**. A line called the **world line** corresponds to any particle (even a stationary one) in four-dimensional

[3]The checking of clocks according to radio signals is in essence such synchronization.

space (for a particle at rest it has the form of a straight line parallel to the ct-axis).

Thus, space and time are parts of a single whole. But time differs qualitatively from space. This manifests itself in that our imaginary four-dimensional space differs in its properties from the conventional three-dimensional space. The latter has Euclidean metric. This signifies that the square of the distance Δl between two points equals the sum of the squares of the coordinate differences:

$$\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2.$$

The square of the "distance" between two world points (this distance is called an **interval** and is designated by the symbol Δs) is determined by the equation

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \tag{8.1}$$

(the properties of an interval are treated in Sec. 8.4).

Spaces for which the square of the distance is determined by a formula such as (8.1) are called **pseudo-Euclidean**. The qualitative distinction between time and space manifests itself in that the square of the time coordinate and the squares of the space coordinates enter Eq. (8.1) with different signs.

A distinctive part in the special theory of relativity is played by quantities that are **invariant** with respect to the transformations of the coordinates and time from one inertial reference frame to another (in other words, quantities having the same numerical value in all inertial reference frames). We know one such quantity, namely, the speed of light in a vacuum. We shall show in Sec. 8.4 that the interval defined by Eq. (8.1) is also an invariant.

A distinctive part is also played by equations and relations that are invariant with respect to the transformations indicated above (*i.e.*, having the same form in all inertial reference frames). For example, the relativistic expressions for the momentum and energy are determined so that the laws of conservation of these quantities are not violated when transferring to another inertial reference frame. We shall acquaint ourselves with a number of invariant quantities and relations in our further treatment.

8.2. Lorentz Transformations

Let us consider two inertial reference frames K and K′ (Fig. 8.2). Assume that the frame K′ moves relative to the frame K with the velocity $\boldsymbol{v}_0$[4]. Let us direct the axes

[4]We remind our reader that the name inertial is used to designate a reference frame relative to which a free particle moves without acceleration (see Sec. 2.2). In Sec. 2.7, we showed on the basis of the Galilean transformation that the frame K′ moving relative to the inertial frame K with the constant velocity $\boldsymbol{v}_0$ is also inertial, in turn. In relativistic mechanics, the Galilean transformations have to be replaced with other ones that agree with the principle of the constancy of the speed of light.

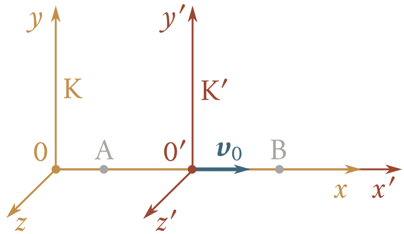

Fig. 8.2

x and x' along the vector $\boldsymbol{v}_0$, and assume that the axes y and z are parallel to the axes y' and z', respectively.

Owing to the principle of relativity, the frames K and K′ have absolutely equal rights. Their only formal distinction is that the x-coordinate of origin 0′ of the frame K′ measured in the frame K changes according to the law

$$x_{0'} = v_0 t \tag{8.2}$$

whereas the x'-coordinate of origin 0 of the frame K measured in the frame K′ changes according to the law

$$x'_0 = -v_0 t'. \tag{8.3}$$

This distinction is due to the fact that we have chosen identical directions of the axes x and x', but the frames K and K′ move in opposite directions relative to each other. Hence, the projection of the relative velocity of the frame K onto the x-axis is $\boldsymbol{v}_0$, and that of the frame K′ onto the x'-axis is $-\boldsymbol{v}_0$.

In non-relativistic mechanics, we used the Galilean transformation (2.9) to pass over from the coordinates and time of one inertial reference frame to the coordinates and time of another inertial frame. The rule of velocity addition $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}_0$ [see Eq. (2.21)] follows from these transformations. This rule contradicts the principle of constancy of the speed of light. Indeed, if in the frame K′ a light signal propagates in the direction of the vector $\boldsymbol{v}_0$ with the velocity c, then according to Eq. (2.21) in the frame K the velocity of the signal will be $c + v_0$, *i.e.*, it will exceed c. Hence, it follows that the Galilean transformations must be replaced with other formulas. It is not difficult to find the latter.

In the most general form, the transformations of the coordinates and time from

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It is clear, however, that no matter what the law of transformation is when passing from the frame K to the frame K′ moving relative to it with the constant velocity $\boldsymbol{v}_0$, if the velocity $\boldsymbol{v}$ of a particle in the frame K is constant, then its velocity $\boldsymbol{v}'$ in the frame K′ will also be constant. Consequently, in relativistic mechanics too, the frame K′ moving with a constant velocity $\boldsymbol{v}_0$ relative to the inertial frame K will also be inertial.

the frame K′ to the frame K are as follows:

$$\begin{cases} x = f_1(x', y', z', t'), \quad y = f_2(x', y', z', t'), \\ z = f_3(x', y', z', t'), \quad t = f_4(x', y', z', t'). \end{cases} \tag{8.4}$$

It follows from the uniformity of time and space that Eqs. (8.4) should be linear, *i.e.*, have the form

$$x = \alpha_1 x' + \alpha_2 y' + \alpha_3 z' + \alpha_4 t' + \alpha_5 \tag{8.5}$$

and so on, where $\alpha_1, \alpha_2, \ldots$ are constants. Accordingly

$$\mathrm{d}x = \alpha_1\,\mathrm{d}x' + \alpha_2\,\mathrm{d}y' + \alpha_3\,\mathrm{d}z' + \alpha_4\,\mathrm{d}t' \tag{8.6}$$

and so on.

Indeed, according to Eqs. (8.4)

$$\mathrm{d}x = \frac{\partial f_1}{\partial x'}\mathrm{d}x' + \frac{\partial f_2}{\partial y'}\mathrm{d}y' + \frac{\partial f_3}{\partial z'}\mathrm{d}z' + \frac{\partial f_4}{\partial t'}\mathrm{d}t' \tag{8.7}$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

If we take the arbitrarily chosen values $\mathrm{d}x', \mathrm{d}y', \mathrm{d}z', \mathrm{d}t'$ for the point x_1', y_1', z_1', t_1', then upon introducing into Eqs. (8.7) the values of the derivatives at the given point, we get a certain value $\mathrm{d}x_1$ for $\mathrm{d}x$. Owing to the uniformity of space and time, however, for any other point x_2', y_2', z_2', t_2' at the same values $\mathrm{d}x', \mathrm{d}y', \mathrm{d}z', \mathrm{d}t'$ we should get the same value for $\mathrm{d}x$ as for the first point, *i.e.*, we should have $\mathrm{d}x_2 = \mathrm{d}x_1$. The same should hold for $\mathrm{d}y$, $\mathrm{d}z$, and $\mathrm{d}t$. Since $\mathrm{d}x', \mathrm{d}y', \mathrm{d}z', \mathrm{d}t'$ were chosen absolutely arbitrarily, this requirement can be observed only if the derivatives of $\partial f_1/\partial x'$, etc. do not depend on the coordinates, *i.e.*, are constants. Hence follows Eq. (8.6), and then also Eq. (8.5).

With the choice of the coordinate axes shown in Fig. 8.2, the plane $y = 0$ coincides with the plane $y' = 0$ and the plane $z = 0$ with the plane $z' = 0$. It thus follows that, for example, the coordinates y and y' must become equal to zero simultaneously regardless of the values of the other coordinates and time. Therefore, y and y' can be related only by expressions of the kind

$$y = \varepsilon y'$$

where ε is a constant. Owing to the frames K and K′ having equal rights, the reverse relation must hold, *i.e.*,

$$y' = \varepsilon y$$

with the same value of the constant ε as in the first case. Multiplication of these two equations yields $\varepsilon^2 = 1$, whence $\varepsilon = \pm 1$. The plus sign corresponds to the axes y and y' having the same directions, and the minus sign to their having opposite directions. Giving the axes the same direction, we get

$$y = y'. \tag{8.8}$$

Similar reasoning yields

$$z = z'. \tag{8.9}$$

Now let us turn to finding the transformations for x and t. It can be seen from Eqs. (8.8) and (8.9) that the values of y and z do not depend on x' and t'. Hence, the values of x' and t' cannot depend on y and z, correspondingly, the values of x and t cannot depend on y' and z'. Thus, x and t can be linear functions of only x' and t'.

The origin of coordinates 0 of the frame K has the coordinate $x = 0$ in the frame K and $x' = -v_0 t'$ in the frame K′ [see Eq. (8.3)]. Consequently, the expression $(x' + v_0 t')$ must vanish simultaneously with the coordinate x. For this to occur, the linear transformation should have the form

$$x = \gamma(x' + v_0 t') \tag{8.10}$$

where γ is a constant.

Similarly, the origin of coordinates 0′ of the frame K′ has the coordinate $x' = 0$ in the frame K′ and $x = v_0 t$ in the frame K [see Eq. (8.2)]. Hence,

$$x' = \gamma(x - v_0 t) \tag{8.11}$$

It follows from the frames K and K′ having equal rights that the constant of proportionality in both cases should be the same.

We shall use the principle of constancy of the speed of light to find the constant γ. Let us begin to count the time in both frames from the moment when their origins of coordinates coincide. Assume that at the moment $t = t' = 0$ a light signal is sent in the direction of the axes x and x' that causes a flash of light to appear on a screen at a point with the coordinate x in the frame K and with the coordinate x' in the frame K′. This event (flash) is described by the coordinate x and the moment t in the frame K, and by the coordinate x' and the moment t' in the frame K′, and

$$x = ct, \qquad x' = ct'.$$

Using these values of x and x' in Eqs. (8.10) and (8.11), we get

$$ct = \gamma(ct' + v_0 t') = \gamma(c + v_0)t',$$
$$ct' = \gamma(ct - v_0 t) = \gamma(c - v_0)t.$$

Multiplication of these two equations yields

$$\gamma = \frac{1}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}. \tag{8.12}$$

Introduction of this value into Eq. (8.10) gives

$$x = \frac{x + v_0 t'}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}. \tag{8.13}$$

Equation (8.13) allows us to find the value of x according to known values of x' and t'. To obtain an equation allowing us to find the value of t according to the known values of x' and t', let us delete the coordinate x from Eqs. (8.10) and (8.11) and solve the resulting expression relative to t. We obtain

$$t = \gamma\left[t' + \frac{x'}{v_0}\left(1 - \frac{1}{\gamma^2}\right)\right].$$

Substituting for γ its value from Eq. (8.12), we have

$$t = \frac{t' + (v_0/c^2)x'}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}. \tag{8.14}$$

The combination of Eqs. (8.8), (8.9), (8.13), and (8.14) is called **Lorentz transformations**. If we use the generally adopted symbol

$$\beta = \frac{v_0}{c} \tag{8.15}$$

then the Lorentz transformations acquire the form

$$x = \frac{x + \beta c t'}{(1 - \beta^2)^{1/2}}, \quad y = y', \quad z = z', \quad t = \frac{t' + (\beta/c)x'}{(1 - \beta^2)^{1/2}}. \tag{8.16}$$

Equations (8.16) allow us to pass over from coordinates and time measured in the frame K′ to those measured in the frame K (in short, to pass over from the frame K′ to the frame K). If we solve Eqs. (8.16) relative to the primed quantities, we get the equations for transformation from the frame K to K′:

$$x' = \frac{x - \beta c t}{(1 - \beta^2)^{1/2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - (\beta/c)x}{(1 - \beta^2)^{1/2}}. \tag{8.17}$$

As should be expected with a view to the equal rights of the frames K and K′, Eqs. (8.17) differ from their counterparts (8.16) only in the sign of β, *i.e.*, of v_0.

It is easy to understand that when $v_0 \ll c$ (*i.e.*, $\beta \ll 1$), the Lorentz transformations become the same as the Galilean ones [see Eqs. (2.19)]. The latter thus retain their importance for speeds that are small in comparison with the speed of light in a vacuum.

When $v_0 > c$, Eqs. (8.16) and (8.17) for x, t, x', and t' become imaginary. This agrees with the fact that motion at a speed exceeding that of light in a vacuum is impossible. It is impossible even to use a reference frame moving with the speed c because when $v_0 = c$, we get zero in the denominators of the equations for x and t.

The Lorentz transformations have an especially simple and symmetrical form if we write them for x and (ct) instead of for x and t, *i.e.*, for quantities of the same dimension. In this case, Eqs. (8.16) have the form

$$x = \frac{x' + \beta(ct)'}{(1-\beta^2)^{1/2}}, \quad y = y', \quad z = z', \quad t = \frac{(ct)' + \beta x'}{(1-\beta^2)^{1/2}}. \tag{8.18}$$

It is simple to memorize Eqs. (8.18) by bearing in mind that the first of them differs from the "obvious" equation $x = x' + v_0 t'$ in containing in its denominator the expression $\left(1-\beta^2\right)^{1/2}$ characteristic of relativistic formulas. The last equation is obtained from the first one if we change the places of x' and ct'.

8.3. Corollaries of the Lorentz Transformations

A number of corollaries follow from the Lorentz transformations that are unusual from the viewpoint of Newtonian mechanics.

Simultaneity of Events in Different Reference Frames. Assume that two events occur simultaneously in the frame K at points with the coordinates x_1 and x_2 and at the moment $t_1 = t_2 = b$. According to the last of the equations (8.17), the moments

$$t_1' = \frac{b - (\beta/c)x_1}{(1-\beta^2)^{1/2}}, \quad t_2' = \frac{b - (\beta/c)x_2}{(1-\beta^2)^{1/2}}$$

will correspond to these events in the frame K′. Examination of these equations shows that if the events occur at different points of space ($x_1 \neq x_2$) in the frame K, then they will not be simultaneous in the frame K′ ($t_1' \neq t_2'$). The sign of the difference $t_2' - t_1'$ is determined by that of the expression $(\beta/c)(x_1 - x_2)$. Consequently, in different frames K′ (with different β's), the difference $t_2' - t_1'$ will vary in magnitude and may differ in sign. This signifies that in some frames event 1 will precede event 2, whereas in others, on the contrary, event 2 will precede event 1. It must be noted that what has been said above relates only to events between which there is no causal relationship. Causally related events (for example, the throwing of a stone and its falling onto the Earth) will not be simultaneous in any reference frame, and in all frames the event that is the cause will precede the effect. This will be treated in greater detail in the following section.

The Length of Bodies in Different Frames. Let us consider a rod arranged along the x'-axis and at rest relative to the reference frame K′ (Fig. 8.3). Its length in this frame is $l_0 = x_2' - x_1'$ where x_1' and x_2' are the coordinates of the rod ends that do not change with the time t'. The rod travels with the velocity $v = v_0$ relative to the frame K. To determine its length in this frame, we must note the coordinates of the rod ends x_1 and x_2 at the same moment $t_1 = t_2 = b$. Their difference $l = x_2 - x_1$ will give the length of the rod measured in the frame K. To find the relationship

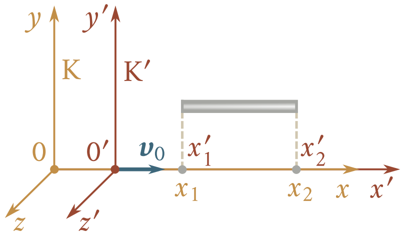

Fig. 8.3

between l_0 and l, we must take the equation of the Lorentz transformations that contains x', x, and t, *i.e.*, the first of the equations (8.17). Substituting $v0/c$ for β in this equation, we obtain

$$x'_1 = \frac{x_1 - v_0 b}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}, \quad x'_2 = \frac{x_2 - v_0 b}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}$$

whence

$$x'_2 - x'_1 = \frac{x_2 - x_1}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}.$$

Using the symbols l and l_0 and also replacing the relative velocity of the reference frames v_0 with the velocity v of the rod the frame K that equals it, we arrive at the expression

$$l = l_0 \left(1 - \frac{v_0^2}{c^2}\right)^{1/2}. \tag{8.19}$$

Thus, the length of the rod l measured in a frame relative to which it is moving is shorter than the length l_0 measured in the frame relative to which the rod is at rest.[5]

If a rod of length $l_0 = x2 - x1$ is at rest relative to the frame K, then to determine its length in the frame K′ we must note the coordinates of its ends x'_1 and x'_2 at the same moment $t'_1 = t'_2 = b$. The difference $l = x'_2 - x'_1$ gives the length of the rod in the frame K′ relative to which it is moving with the velocity v. Using the first of the equations (8.16), we again arrive at Eq. (8.19).

It must be noted that the dimensions of the rod are identical in all the reference frames in the direction of the axes y and z.

Thus, in moving bodies, their dimensions contract in the direction of their motion the greater, the higher is the velocity. This phenomenon is called the **Lorentz** (or **Fitzgerald**) contraction. It is interesting to note that the change in the shape

[5]The length l_0 measured in the frame relative to which the rod is at rest is called the **proper length** of the rod.

of bodies even at velocities comparable with c cannot be detected visually (or in a photograph). The reason is very simple. When observing visually or photographing a body, we register light pulses from different points of the body that reach the retina of our eye or the photographic plate simultaneously. These pulses, however, are not emitted simultaneously. The pulses from the more remote sections were emitted before those from the nearer sections. Thus, if the body is moving, a distorted image of it is formed on the retina of the eye or on the photograph. The relevant calculations show that this distortion will result in compensation of the Lorentz contraction[6] so that the bodies seem to be only turned instead of distorted. Consequently, a spherically shaped body even at high velocities will be perceived visually as a body with a spherical configuration.

Length of Time Between Events. Let us suppose that two events occur at the same point of the frame K′. The coordinate $x_1' = a$ and the moment t_1' correspond to the first event in this frame, and the coordinate $x_2' = a$ and the moment t_2' to the second one. According to the last of the equations (8.16), the moments corresponding to these events in the frame K will be (we have introduced v_0/c instead of β)

$$t_1 = \frac{t_1' + (v_0/c)^2 a}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}, \qquad t_2 = \frac{t_2' + (v_0/c^2) a}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}.$$

Hence,

$$t_2 - t_1 = \frac{t_2' - t_2'}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}}.$$

Introducing the notation $t_2 - t_1 = \Delta t$ and $t_2' - t_1' = \Delta t'$, we get the equation

$$\Delta t = \frac{\Delta t'}{\left[1 - \left(v_0^2/c^2\right)\right]^{1/2}} \tag{8.20}$$

that relates the lengths of time between two events measured in the frames K and K′. We remind our reader that in the frame K′ both events occur at the same point, *i.e.*, $x_1' = x_2'$.

Assume that both events occur with the same particle that is at rest in the frame K′ and is moving relative to the frame K with the velocity $v = v_0$. Therefore, $\Delta t'$ can be interpreted as the length of time measured on a clock that is stationary relative to the particle, or, in other words, measured on a clock that is moving together with the particle (we have in mind motion relative to the frame K). The time measured on a clock moving together with a body is called the **proper time** of this body and

[6]If there were no Lorentz contraction, rapidly moving bodies ought to seem extended in the direction of their motion.

is usually denoted by the symbol τ. Thus, $\Delta t' = \Delta\tau$. We can thus write Eq. (8.20) as follows:

$$\Delta\tau = \Delta t\left[1 - \left(v^2/c^2\right)\right]^{1/2} \tag{8.21}$$

(we have replaced the relative velocity of the reference frames v_0 with the velocity of the particle v equal to it).

Equation (8.21) relates the proper time of a body τ to the time t read on a clock of a reference frame relative to which the body is moving with the velocity v (this clock itself is moving relative to the body with the velocity $-v$).

A glance at Eq. (8.21) shows that the proper time is always smaller than the time measured on a clock moving relative to a body (in the latter case the effect called **time dilation** is observed). We shall show in the following section that the proper time is an invariant (*i.e.*, is identical in all reference frames).

Considering the events occurring with the particle in the frame K, we can define Δt as the length of time measured on a stationary clock, and $\Delta\tau$ as the length of time measured on a clock moving with the velocity v. By Eq. (8.21), we have $\Delta\tau < \Delta t$. We can therefore say that the moving clock runs slower than the clock at rest (it must not be forgotten that in all respects except for their velocity the clocks are absolutely identical).

Equation (8.21) has been directly confirmed experimentally. Cosmic rays contain particles called mu-mesons or muons. These particles are unstable and decay spontaneously into an electron (or positron) and two neutrinos. The mean lifetime of muons measured in conditions when they are stationary (or are moving with a low velocity) is about 2×10^{-6} s. It would seem that even when travelling with the speed of light, muons could cover a distance of only about 600 m. As observations show, however, muons are formed in cosmic rays at an altitude of from 20 km to 30 km, and a considerable number of them manage to reach the Earth's surface. The explanation is that 2×10^{-6} s is the proper lifetime of a muon, *i.e.*, time measured on a clock travelling together with it. The time according to the clock of an observer on the Earth will be much greater [see Eq. (8.21); v of a muon is close to c]. It is therefore not surprising that the observer registers a distance travelled by a muon much greater than 600 m. We must note that from the position of an observer travelling together with a muon, the distance it covers to the Earth's surface contracts to 600 m [see Eq. (8.19)], so that the muon manages to travel this distance in 2×10^{-6} s.

8.4. Interval

We pointed out in Sec. 8.1 that a world point with the coordinates ct, x, y, z can be associated with every event in imaginary four-dimensional space. Let one event have the coordinates ct_1, x_1, y_1, z_1 and another the coordinates ct_2, x_2, y_2, z_2. We shall introduce the notation $t_2 - t_1 = \Delta t$, $x_2 - x_1 = \Delta x$, etc.

We remind our reader that owing to the qualitative distinction between time and space, the square of the difference between the time coordinates $(c\Delta t)^2$ and the squares of the differences between the space coordinates $\Delta x^2, \Delta y^2, \Delta z^2$ enter the expression for the square of the "distance" between events (more exactly, between the world points corresponding to the events) with opposite signs:

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2. \tag{8.22}$$

The quantity Δs determined by this equation is defined as the **interval** between events.

Introducing the distance $\Delta l = \left(\Delta x^2 + \Delta y^2 + \Delta z^2\right)^{1/2}$ between the points of conventional three-dimensional space at which the events being considered occurred, the expression for the interval can be written in the form

$$\Delta s = \left(c^2\Delta t^2 - \Delta l^2\right)^{1/2}. \tag{8.23}$$

It is easy to convince ourselves that the interval between two given events is the same in all inertial reference frames. It is exactly this circumstance that served as the grounds to consider it the analogue of the distance Δl between two points in conventional three-dimensional space (Δl does not change its value when we pass over from one three-dimensional reference frame to another).

Assume that in the reference frame K the square of the interval is determined by Eq. (8.22). The square of the interval between the same events in the frame K′ is

$$\Delta s'^2 = c^2\Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2. \tag{8.24}$$

By Eqs. (8.17)

$$\Delta x' = \frac{\Delta x - \beta c\Delta t}{(1-\beta^2)^{1/2}}, \quad \Delta y' = \Delta y, \quad \Delta z' = \Delta z, \quad \Delta t' = \frac{\Delta t - (\beta/c)\Delta x}{(1-\beta^2)^{1/2}}.$$

Introducing these values into Eq. (8.24), after simple transformations we find that $\Delta s'^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$, *i.e.*, that

$$\Delta s'^2 = \Delta s^2.$$

The interval is thus invariant with respect to a transition from one inertial reference frame to another. We saw in the preceding section that the lengths of time Δt and lengths Δl are not invariant with respect to such a transition. Hence, each of the addends forming the quantity $\Delta s^2 = c^2\Delta t^2 - \Delta l^2$ changes in a transition from one frame to another; the quantity Δs^2 itself remains unchanged.

The interval between two events occurring with a particle is in a simple relation with the length of the proper time between these events. By Eq. (8.21), the length of the proper time $\Delta\tau$ is related to the time Δt measured on a clock of the frame relative to which the particle is travelling with the velocity v by the expression

$$\Delta\tau = \Delta t \left(1 - \frac{v^2}{c^2}\right).$$

Let us transform this equation as follows:

$$\Delta\tau = \frac{1}{c}\left[c^2\Delta t^2 - (v\Delta t)^2\right]^{1/2} = \frac{1}{c}\left(c^2\Delta t^2 - \Delta l^2\right)^{1/2}.$$

Here $\Delta l = v\Delta t$ is the distance travelled by the particle during the time Δt. A comparison with Eq. (8.23) shows that

$$\Delta\tau = \frac{1}{c}\Delta s \tag{8.25}$$

where Δs is the interval between events separated by the time $\Delta\tau$.

It follows from Eq. (8.25) that the length of the proper time is proportional to the interval between events. The interval is an invariant. Consequently, the proper time is also an invariant, *i.e.*, does not depend on the reference frame in which the motion of a given body is being observed.

According to Eq. (8.23), the interval may be real (if $c\Delta t > \Delta l$) or imaginary (if $c\Delta t < \Delta l$). In a particular case, the interval may equal zero (if $c\Delta t = \Delta l$). The last case occurs for events consisting in the emission of a light signal from the point x_1, y_1, z_1 at the moment t_1 and in the arrival of this signal at the point x_2, y_2, z_2 at the moment t_2. Since here $\Delta l = c\Delta t$, the interval between the events equals zero.

Owing to its invariance, an interval that is real (or imaginary) in a reference frame K will be real (or imaginary) in any other inertial frame K′.

For a real interval, we have

$$c^2\Delta t^2 - \Delta l^2 = c^2\Delta t^2 - \Delta l'^2 > 0.$$

It can be seen from this expression that we can find a frame K′ in which $\Delta l' = 0$, *i.e.*, both events will coincide in space. No reference frame exists, however, in which $\Delta t' = 0$ (the interval would become imaginary at such a value of $\Delta t'$). Thus, events separated by a real interval cannot become simultaneous in any reference frame. For this reason, real intervals are called **timelike**.

We must note that events occurring with the same particle (we have in mind a particle with a rest mass differing from zero) can be separated only by a timelike interval. Indeed, the velocity of such a particle v is always lower than c. Hence, the path Δl travelled by the particle is less than $c\Delta t$, whence it follows that $\Delta s^2 > 0$.

According to the last of the equations (8.17), we have

$$\Delta t' = \frac{\Delta t - (\beta/c)\Delta x}{(1-\beta^2)^{1/2}}. \tag{8.26}$$

If Δx and Δx separate events occurring with the same particle, then $\Delta x/\Delta t$ gives the component v_x of the particle's velocity. Therefore, Eq. (8.26) in this condition can be written in the form

$$\Delta t' = \frac{\Delta t - (\beta/c)(\Delta x/\Delta t)\Delta t}{(1-\beta^2)^{1/2}} = \frac{\Delta t}{(1-\beta^2)^{1/2}}\left(1 - \beta\frac{v_x}{c}\right).$$

Since both $\beta = v_0/c$ and v_x/c are less than unity, the quantity in parentheses in the right-hand side of the equation is positive for all frames K′. Hence, it follows that $\Delta t'$ and Δt have the same signs. This signifies that two events occurring with a particle take place in the same sequence in all frames. For example, the birth of a particle in all reference frames precedes its decay.

For an imaginary interval, we have

$$c^2\Delta t^2 - \Delta l^2 = c^2\Delta t'^2 - \Delta'^2 > 0.$$

This shows that we can find a frame K′ in which $\Delta t' = 0$, *i.e.*, both events occur at the same moment t'. No reference frame exists, however, in which we would have $\Delta l' = 0$ (the interval would be real with such a value of $\Delta l'$). Thus, events separated by an imaginary interval cannot coincide in space in any reference frame. For this reason, imaginary intervals are called **spacelike**.

The distance Δl between points at which events separated by a spacelike interval occur exceeds $c\Delta t$. Therefore, these events cannot in any way affect each other, *i.e.*, cannot be causally related to each other (we remind our reader that no actions exist which propagate at a velocity exceeding c).

Causally related events can be separated only by a timelike or a zero interval.

8.5. Transformation and Addition of Velocities

Let us consider the motion of a point particle. The position of the particle in the frame K is determined at each moment t by the coordinates x, y, z. The expressions

$$v_x = \mathrm{d}x/\mathrm{d}t, \quad v_y = \mathrm{d}y/\mathrm{d}t, \quad v_z = \mathrm{d}z/\mathrm{d}t$$

are the projections of the vector of the particle's velocity relative to the frame K onto the axes x, y, z. The position of the particle in the frame K′ is characterized at each moment t' by the coordinates x', y', z'. The projections of the vector of the particle's velocity relative to the frame K′ onto the axes x', y', z' are determined by the expressions

$$v'_x = \mathrm{d}x'/\mathrm{d}t', \quad v'_y = \mathrm{d}y'/\mathrm{d}t', \quad v'_z = \mathrm{d}z'/\mathrm{d}t'.$$

From Eqs. (8.16), we have

$$dx = \frac{dx' + v_0 dt'}{\left(1 - v_0^2/c^2\right)^{1/2}}, \quad dy = dy', \quad dz = dz', \quad dt = \frac{dt' + (v_0/c^2)dx'}{\left(1 - v_0^2/c^2\right)^{1/2}}$$

(we have replaced β with v_0/c). Dividing the first three equations by the fourth one, we get formulas for transformation of the velocities when passing over from one reference frame to another:

$$v_x = \frac{v_x' + v_0}{1 - v_0 v_x'/c^2}, \quad v_y = \frac{v_y'\left(1 - v_0^2/c^2\right)^{1/2}}{1 - v_0 v_x'/c^2}, \quad v_z = \frac{v_z'\left(1 - v_0^2/c^2\right)^{1/2}}{1 - v_0 v_x'/c^2}. \tag{8.27}$$

When $v_0 \ll c$, equations (8.27) become the same as the velocity addition equations (2.20) of classical mechanics.

It is simple to obtain expressions for velocities in the frame K′ through the velocities in the frame K from Eqs. (8.17):

$$v_x' = \frac{v_x - v_0}{1 - v_0 v_x/c^2}, \quad v_y' = \frac{v_y\left(1 - v_0^2/c^2\right)^{1/2}}{1 - v_0 v_x/c^2}, \quad v_z' = \frac{v_z\left(1 - v_0^2/c^2\right)^{1/2}}{1 - v_0 v_x/c^2}. \tag{8.28}$$

These equations differ from equations (8.27) only in the sign before v_0. This result could naturally be predicted.

If a body is travelling parallel to the x-axis, its velocity v relative to the frame K coincides with v_x, and its velocity v' relative to the frame K′ coincides with v_x'. In this case, the law of velocity addition has the form

$$v = \frac{v' + v_0}{1 + v_0 v'/c^2}. \tag{8.29}$$

Assume that the velocity v' equals c. Hence, Eq. (8.29) gives us the following value for v:

$$v = \frac{c + v_0}{1 + v_0 c/c^2} = c.$$

This result is not surprising because the Lorentz transformations (and, consequently, the velocity addition equations too) are based on the assertion that the speed of light is the same in all reference frames. Assuming in Eq. (8.29) that $v' = v_0 = c$, we also get a value of c for v. Thus, if the velocities v' and v_0 being added do not exceed c, then the resultant velocity v also cannot exceed c.

8.6. Relativistic Expression for the Momentum

Newton's equations are invariant with respect to the Galilean transformations (see Sec. 2.7). They are not invariant, however, with respect to the Lorentz transformations. In particular, the law of momentum conservation (see Sec. 3.10) following from Newton's laws is not invariant with respect to the Lorentz transformations.

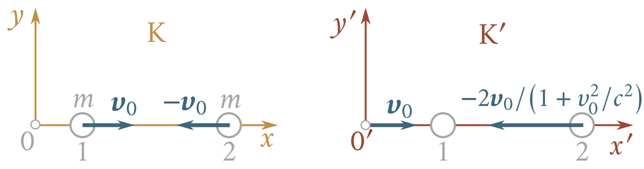

Fig. 8.4

To convince ourselves in the truth of this statement, let us see what a completely inelastic collision of two identical balls of mass m is like in the frames K and K′ (Fig. 8.4).

Assume that the balls are moving toward each other in the frame K along the x-axis with velocities identical in magnitude whose projections onto the x-axis are $v_{x1} = v_0$ and $v_{x2} = -v_0$ (v_0 is the relative velocity of the frames K and K′). In these conditions, the balls will be at rest after colliding: $v_{x1} = v_{x2} = 0$. Thus, the total momentum of the system both before and after the collision equals zero—the momentum in the frame K is conserved.

Let us now consider the same process in the frame K′. Using the first of the equations (8.28), we find for the velocities of the balls before they collide the values $vx1' = 0$ and $vx2' = -2v_0/(1+v_0^2/c^2)$, and for the velocities of the balls after they collide the same value $c'_{x1} = vx2' = -v_0$. Therefore, the total momentum before the collision is $-2mv_0/(1 + v_0^2/c^2)$, and after the collision is $-2mv_0$. If $v_0 \ll c$, the momentum of the system before and after the collision is virtually the same. When the balls are travelling with a great velocity v_0, however, the difference between the initial and the final momenta becomes quite appreciable. Thus, using the Newtonian expression for the momentum, we arrived at the conclusion that the momentum does not seem to be conserved in the frame K′. One of the fundamental laws of mechanics—the law of momentum conservation—is not invariant with respect to the Lorentz transformations in the Newtonian formulation.

It can be shown that the law of momentum conservation will be invariant with respect to the Lorentz transformations at any velocities if we substitute the proper time of a particle τ for the time t in the classical expression

$$\boldsymbol{p} = m\boldsymbol{v} = m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{8.30}$$

Consequently, the relativistic expression for the momentum has the form

$$\boldsymbol{p} = m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}\tau}. \tag{8.31}$$

When $v \ll c$, the length of the proper time of a particle $\mathrm{d}\tau$ does not virtually differ from the length $\mathrm{d}t$ measured according to the clock of the frame in which the motion of the particle is being considered [see Eq. (8.21)]. Hence, Eq. (8.31) transforms into the classical expression (8.30).

Remember that d$\boldsymbol{r}$ in Eq. (8.31) is the displacement of the particle in the reference frame in which the momentum $\boldsymbol{p}$ is determined, whereas the length of time dτ is determined on a clock travelling together with the particle.

We get an expression for the momentum through the time t of the frame of reference relative to which the motion of bodies is being observed. By Eq. (8.21), we have $\mathrm{d}\tau = \mathrm{d}t\left(1 - v^2/c^2\right)^{1/2}$, where v is the velocity of the body. This substitution in Eq. (8.31) yields

$$\boldsymbol{p} = \frac{m}{\left(1 - v^2/c^2\right)^{1/2}} \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$$

or, since $\mathrm{d}\boldsymbol{r}/\mathrm{d}t = \boldsymbol{v}$:

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\left(1 - v^2/c^2\right)^{1/2}}. \tag{8.32}$$

The mass m in Eq. (8.32) is invariant and, consequently, does not depend on the velocity of the body.

It can be seen from Eq. (8.32) that the velocity dependence of the momentum is more complicated than is assumed in Newtonian mechanics. When $v \ll c$, Eq. (8.32) transforms into the Newtonian expression $\boldsymbol{p} = m\boldsymbol{v}$.

We must note that Eq. (8.32) permits the following interpretation to be made, which is gradually losing favour. The momentum, as in Newtonian mechanics, equals the product of the mass of a body and its velocity:

$$\boldsymbol{p} = m_{\mathrm{r}}\boldsymbol{v}. \tag{8.33}$$

The mass of a body, however, is not a constant invariant quantity, but depends on the velocity according to the law

$$m_{\mathrm{r}} = \frac{m}{\left(1 - v^2/c^2\right)^{1/2}}. \tag{8.34}$$

In this interpretation, the invariant mass m is called the **rest mass** (it is often denoted by the symbol m_0). The non-invariant mass m_{r} depending on the velocity is called the **relativistic mass**.

8.7. Relativistic Expression for the Energy

Newton's second law states that the time derivative of the momentum of a particle (point particle) equals the resultant force acting on the particle [see Eq. (2.10)]. The equation of the second law is invariant relative to the Lorentz transformations if by the momentum we understand the quantity (8.32). Hence, the relativistic expression of Newton's second law has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\boldsymbol{v}}{(1-v^2/c^2)^{1/2}}\right] = \boldsymbol{F}. \tag{8.35}$$

It should be borne in mind that the equation $m\boldsymbol{a} = \boldsymbol{F}$ cannot be used in the relativistic case, the acceleration $\boldsymbol{a}$ and the force $\boldsymbol{F}$, generally speaking, being non-collinear.

We shall note that neither the momentum nor the force are invariant quantities. Equations for the transformation of the momentum components when passing over from one inertial reference frame to another will be obtained in the following section. We give the equations for transformation of the force components without deriving them:

$$F_x = \frac{F'_x + (\beta/c)\boldsymbol{F}' \cdot \boldsymbol{v}'}{1 + \beta(v'_x/c)}, \quad F_y = \frac{F'_y\left(1-\beta^2\right)^{1/2}}{1 + \beta(v'_x/c)}, \quad F_z = \frac{F'_z\left(1-\beta^2\right)^{1/2}}{1 + \beta(v'_x/c)} \tag{8.36}$$

(here $\beta = v_0/c$ and $\boldsymbol{v}'$ is the velocity of a particle in the frame K'). If in the frame K′ the force $\boldsymbol{F}'$ acting on a particle is perpendicular to the velocity of the particle $\boldsymbol{v}'$, the scalar product $\boldsymbol{F}' \cdot \boldsymbol{v}'$ equals zero, and the first of the equations (8.36) is simplified as follows

$$F_x = \frac{F'_x}{1 + \beta(v'_x/c)}. \tag{8.37}$$

To find the relativistic expression for the energy, let us proceed in the same way as we did in Sec. 3.2. We shall multiply Eq. (8.35) by the displacement of a particle $\mathrm{d}\boldsymbol{s} = \boldsymbol{v}\,\mathrm{d}t$. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\boldsymbol{v}}{(1-v^2/c^2)^{1/2}}\right]\boldsymbol{v}\,\mathrm{d}t = \boldsymbol{F}\,\mathrm{d}\boldsymbol{s}.$$

The right-hand side of this equation gives the work $\mathrm{d}A$ done on the particle during the time $\mathrm{d}t$. We saw in Sec. 3.2 that the work of the resultant of all the forces is spent on an increment of the kinetic energy of the particle [see Eq. (3.11)]. Consequently, the left-hand side of the equation should be interpreted as the increment of the kinetic energy E_k of the particle during the time $\mathrm{d}t$. Thus,

$$\mathrm{d}E_\mathrm{k} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\boldsymbol{v}}{(1-v^2/c^2)^{1/2}}\right] \cdot \boldsymbol{v}\,\mathrm{d}t = \boldsymbol{v} \cdot \mathrm{d}\left[\frac{m\boldsymbol{v}}{(1-v^2/c^2)^{1/2}}\right].$$

Let us transform the obtained expression, bearing in mind that $\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{v} = \mathrm{d}(\boldsymbol{v}^2/2)$

[see Eq. (1.54)]:

$$\mathrm{d}E_\mathrm{k} = \boldsymbol{v}\cdot\left[\frac{m\,\mathrm{d}\boldsymbol{v}}{(1-v^2/c^2)^{1/2}} + \frac{m\boldsymbol{v}(\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{v}/c^2)}{(1-v^2/c^2)^{3/2}}\right]$$
$$= \frac{m\,\mathrm{d}(v^2/2)}{(1-v^2/c^2)^{3/2}} = \frac{mc^2\mathrm{d}v^2/c^2}{2\,(1-v^2/c^2)^{3/2}} = \mathrm{d}\left[\frac{mc^2}{(1-v^2/c^2)^{1/2}}\right].$$

Integration of this expression yields

$$E_\mathrm{k} = \frac{mc^2}{(1-v^2/c^2)^{1/2}} + \text{constant}. \tag{8.38}$$

According to the meaning of kinetic energy, it must vanish when $v = 0$. We thus get a value of $-mc^2$ for the constant. Hence, the relativistic expression for the kinetic energy of a particle has the form

$$E_\mathrm{k} = \frac{mc^2}{(1-v^2/c^2)^{1/2}} - mc^2 = mc^2\left[\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right]. \tag{8.39}$$

For small velocities ($v \ll c$), Eq. (8.39) can be transformed as follows:

$$E_\mathrm{k} = mc^2\left[\frac{1}{(1-v^2/c^2)^{1/2}} - 1\right] \approx mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = \frac{1}{2}mv^2.$$

We have arrived at the Newtonian expression for the kinetic energy of a particle. This is what should be expected because for velocities much smaller than the speed of light all the equations of relativistic mechanics must transform into the relevant equations of Newtonian mechanics.

Let us consider a free particle (*i.e.*, one that does not experience the action of external forces) travelling with the velocity v. We have learned that this particle has a kinetic energy determined by Eq. (8.39). We have grounds, however (see below), to ascribe the additional energy equal to

$$E_0 = mc^2 \tag{8.40}$$

to a free particle in addition to the kinetic energy (8.39). Thus, the total energy of a free particle is determined by the expression $E = E_\mathrm{k} + E_0 = E_\mathrm{k} + mc^2$. With a view to Eq. (8.39), we find that

$$E = \frac{mc^2}{(1-v^2/c^2)^{1/2}}. \tag{8.41}$$

When $v = 0$, Eq. (8.41) transforms into Eq. (8.40). This is why $E_0 = mc^2$ is called the rest energy. This energy is the internal energy of a particle not associated with its motion as a whole. Equations (8.40) and (8.41) hold not only for an elementary particle, but also for a complicated body consisting of many particles. The energy

E_0 of such a body includes, apart from the rest energies of its particles, the kinetic energy of these particles (due to their motion relative to the body's centre of mass) and the energy of their interaction with one another. The rest energy, like the total[7] energy (8.41), does not include the potential energy of a body in an external force field.

Eliminating the velocity v from Eqs. (8.32) and (8.41) [Eq. (8.32) should be taken in the scalar form], we obtain an expression giving the total energy of a particle through its momentum p:

$$E = c\left(p^2 + m^2c^2\right)^{1/2}. \tag{8.42}$$

When $p \ll mc$, this equation can be written in the form

$$E = mc^2\left[1 + \left(\frac{p}{mc}\right)^2\right] \approx mc^2\left[1 + \left(\frac{1}{2}\frac{p}{mc}\right)^2\right] = mc^2 + \frac{p^2}{2m}. \tag{8.43}$$

The expression obtained differs from Newton's equation for the kinetic energy $E_k = p^2/(2m)$ in the addend mc^2.

It must be noted that the following equation results from a comparison of Eqs. (8.32) and (8.41):

$$\boldsymbol{p} = \frac{E}{c^2}\boldsymbol{v}. \tag{8.44}$$

We shall explain why the energy (8.41), and not only the kinetic energy (8.39), should be ascribed to a free particle. Energy according to its meaning must be a conserved quantity. The relevant treatment shows that when particles collide, the sum (for the particles) of expressions of the form of Eq. (8.41) is conserved, whereas the sum of Eqs. (8.39) is not conserved. It is impossible to comply with the requirement of energy conservation in all inertial reference frames if we do not include the rest energy (8.40) in the total energy.

In addition, we succeed in forming an invariant, *i.e.*, a quantity that does not change in the Lorentz transformations, from Eq. (8.41) for the energy and (8.42) for the momentum. Indeed, it can be seen from Eq. (8.42) that

$$\frac{E^2}{c^2} - p^2 = m^2c^2 = \text{inv} \tag{8.45}$$

(we remind our reader that the mass m and speed c are invariant quantities). Experiments with fast particles confirm the invariance of the quantity in Eq. (8.45). If by E in Eq. (8.45) we understand the kinetic energy (8.39), then Eq. (8.45) will not be

[7]We shall note here that the term "total energy" has a different meaning in relativistic mechanics than in Newtonian mechanics. In the latter, the total energy is defined as the sum of the kinetic and potential energies of a particle. In relativistic mechanics, by the total energy is meant the sum of the kinetic and rest energies of a particle.

invariant.

Let us obtain another expression for the relativistic energy. From Eq. (8.21), we find that

$$\frac{1}{(1 - v^2/c^2)^{1/2}} = \frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{8.46}$$

where $\mathrm{d}t$ is the time that elapses between two events occurring with a particle and measured on a clock of the reference frame relative to which the particle is travelling with the velocity v, while $\mathrm{d}\tau$ is the same time measured on a clock travelling together with the particle (proper time). Using Eq. (8.46) in Eq. (8.41) we get the expression

$$E = mc^2\,\frac{\mathrm{d}t}{\mathrm{d}\tau}. \tag{8.47}$$

We shall use this equation in the following section.

8.8. Transformations of Momentum and Energy

The total energy E and momentum p are not invariants. Indeed, both quantities depend on v, while the latter has different values in different reference frames. Let us see how the energy and momentum transform when we pass over from one reference frame to another.

Consider an elementary displacement of a particle. Assume that in the reference frame K this displacement occurs during the time $\mathrm{d}t$, and its components are $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$. In the frame K′, the same displacement occurs during the time $\mathrm{d}t'$, and its components are $\mathrm{d}x', \mathrm{d}y', \mathrm{d}z'$. According to Eqs. (8.18), the following relations hold between the lengths of time and the components of the displacement:

$$\mathrm{d}x = \frac{\mathrm{d}x' + \beta c\,\mathrm{d}t'}{(1-\beta^2)^{1/2}}, \quad \mathrm{d}y = \mathrm{d}y', \quad \mathrm{d}z = \mathrm{d}z', \quad c\,\mathrm{d}t = \frac{c\,\mathrm{d}t' + \beta\,\mathrm{d}x'}{(1-\beta^2)^{1/2}}.$$

Let us multiply these equations by the mass of the particle m and divide them by the proper time of the particle $\mathrm{d}\tau$ corresponding to the lengths of time $\mathrm{d}t$ and $\mathrm{d}t'$ (it should be remembered that the mass and the proper time are invariant quantities, *i.e.*, have the same value in both frames). As a result, we get

$$\begin{aligned} m\,\frac{\mathrm{d}x}{\mathrm{d}\tau} &= \frac{m(\mathrm{d}x'/\mathrm{d}\tau) + \beta mc(\mathrm{d}t'/\mathrm{d}\tau)}{(1-\beta^2)^{1/2}}, \quad m\,\frac{\mathrm{d}y}{\mathrm{d}\tau} = m\,\frac{\mathrm{d}y'}{\mathrm{d}\tau}, \\ m\,\frac{\mathrm{d}z}{\mathrm{d}\tau} &= m\,\frac{\mathrm{d}z'}{\mathrm{d}\tau}, \quad mc\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{mc(\mathrm{d}t'/\mathrm{d}\tau) + \beta m(\mathrm{d}x'/\mathrm{d}\tau)}{(1-\beta^2)^{1/2}}. \end{aligned} \tag{8.48}$$

By Eq. (8.31), we have $m\,(\mathrm{d}x/\mathrm{d}\tau) = p_x, m\,(\mathrm{d}x'/\mathrm{d}\tau) = p'_x, m\,(\mathrm{d}y/\mathrm{d}\tau) = p_y$, etc. According to Eq. (8.47), we have $mc\,(\mathrm{d}t/\mathrm{d}\tau) = E/c$, and $mc\,(\mathrm{d}t'/\mathrm{d}\tau) = E'/c$. Hence, Eqs. (8.48) can be written in the form

$$p_x = \frac{p'_x + \beta(E'/c)}{(1-\beta^2)^{1/2}}, \quad p_y = p'_y, \quad p_z = p'_z, \quad \frac{E}{c} = \frac{(E'/c) + \beta p'_x}{(1-\beta^2)^{1/2}}. \tag{8.49}$$

We have obtained equations by means of which the momentum and energy of a particle are transformed when we pass over from one inertial reference frame to another. These equations coincide with Eqs. (8.18) used to transform the coordinates and time. To facilitate a comparison, let us write Eqs. (8.18) and (8.49) side by side:

$$\begin{aligned} x &= \frac{x' + \beta(ct')}{(1-\beta^2)^{1/2}}, \quad y = y', \quad z = z', \quad (ct) = \frac{(ct') + \beta x'}{(1-\beta^2)^{1/2}} \\ p_x &= \frac{p'_x + \beta(E'/c)}{(1-\beta^2)^{1/2}}, \quad p_y = p'_y, \quad p_z = p'_z, \quad \frac{E}{c} = \frac{(E'/c) + \beta p'_x}{(1-\beta^2)^{1/2}}. \end{aligned} \tag{8.50}$$

It follows from the comparison that the components of the momentum behave in transformations like coordinates, and the energy like time.

The analogy disclosed by Eqs. (**??**) allows us to present the mathematics of relativistic mechanics in the form of relations between vectors in an imaginary four-dimensional space (four-vectors). We have already noted in Sec. 8.1 that we have to ascribe unusual properties to this space which differ from the properties of the Euclidean space we are accustomed to. In three-dimensional Euclidean space, the quantity

$$\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

is an invariant, *i.e.*, does not change upon rotations of the coordinate axes. Unlike this, the quantity

$$c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{8.51}$$

is not invariant—it is not conserved upon transition from one inertial reference frame to another (such a transition can be imagined as rotation of the axes in four-dimensional space). Hence, the quantity (8.51) does not have the properties of the square of the distance between two world points. We have seen in Sec. 8.4 that Eq. (8.22), *i.e.*,

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

is invariant, and it should be considered as the square of the distance between two points in the four-dimensional space we are interested in[8].

Having given four-dimensional space such properties, we can consider the quantities ct, x, y, z as the components of a four-vector drawn from the origin of

[8]Naturally, we can also consider Euclidean four-dimensional space. The latter is not suitable for the needs of relativistic mechanics, however.

coordinates to the given world point. Accordingly, $c\Delta t$, Δx, Δy, Δz can be considered as components of a four-vector—the displacement from one world point to another. In three-dimensional Euclidean space, other vectors are dealt with (velocity, acceleration, force, etc.) in addition to the position and displacement vectors, and for any vector $\boldsymbol{a}$, the quantity

$$\boldsymbol{a}^2 = a_x^2 + a_y^2 + a_z^2$$

is an invariant. The components of any such vector transform upon rotation of the coordinate axes according to the same equations as the coordinates do.

By analogy with three-dimensional vectors in Euclidean space, we can determine four-dimensional vectors. A four-dimensional vector or four-vector is defined as a combination of the four quantities a_t, a_x, a_y, a_z that transform according to the same equations as ct, x, y, z [see the first line of Eqs. (8.50)]. The "square" of such a vector should be determined as

$$a_t^2 - a_x^2 - a_y^2 - a_z^2. \tag{8.52}$$

Since the components transform in the same way as the coordinates, expression (8.52) is invariant with respect to the Lorentz transformations.

Inspection of Eqs. (8.50) shows that the combination of the quantities

$$E/c,\ p_x,\ p_y,\ p_z \tag{8.53}$$

forms a four-vector. It is called the **energy-momentum vector**. An expression such as (8.52) formed from the components (8.53), as we have established [see Eq. (8.45)], is an invariant:

$$\left(\frac{E}{c}\right)^2 - p_x^2 - p_y^2 - p_z^2 = m^2c^2.$$

8.9. Relation Between Mass and Energy

Using the relativistic mass [see Eq. (8.34)], we can write Eq. (8.41) in the form

$$E = m_{\mathrm{r}}c^2. \tag{8.54}$$

It can be seen from this equation that the energy of a body and its relativistic mass are always proportional to each other. Any change in the energy of a body ΔE (except for a change in the potential energy in an external force field) is attended by a change in the relativistic mass of the body $\Delta m_{\mathrm{r}} = \Delta E/c^2$, and, conversely, any change in the relativistic mass Δm_{r} is attended by a change in the energy of the body

$$\Delta E = c^2\Delta m_{\mathrm{r}}. \tag{8.55}$$

This statement is called the **law of the relation between the relativistic mass**

and energy[9].

The proportionality between the relativistic mass and energy leads to the fact that the statement on the conservation of the total relativistic mass of particles is the statement on the conservation of the total energy using different words. In this connection, it is not customary practice to speak of the law of relativistic mass conservation as of a separate law.

Unlike the relativistic mass, the total rest mass of a system of interacting particles is not conserved. For example, upon an inelastic collision of two particles observed in the frame of their centre of mass, the rest mass of the particle formed is

$$m_\Sigma = m_1 + m_2 + \frac{E_{k,1} + E_{k,2}}{c^2}$$

where m_1 and $E_{k,1}$ are the rest mass and kinetic energy of the first initial particle, and m_2 and $E_{k,2}$ are the relevant quantities of the second particle. Thus,

$$m_\Sigma > m_1 + m_2.$$

In this case, the kinetic energy of the initial particles transformed into the internal energy of the formed particle. As a result, the rest mass of this particle exceeded the sum of the rest masses of the initial particles.

The operation of nuclear power plants is based on the chain reaction of fission of nuclei of uranium ${}_{92}U^{235}$ (or plutonium) when they capture slow neutrons n[10]. Fission occurs in various ways. One of the reactions is

$$ {}_{92}U^{235} + n \longrightarrow {}_{92}U^{236} \longrightarrow {}_{55}Cs^{140} + {}_{37}Rb^{94} + 2n. \tag{8.56}$$

After capturing a neutron, a uranium nucleus decays into a caesium nucleus with the mass number 140 and a rubidium nucleus with the mass number 94. Two neutrons are also emitted. The total rest mass of uranium-235 and a neutron exceeds the total rest mass of the particles in the right-hand side of the reaction formula (8.56) by about 4×10^{-28} kg. The internal energy corresponding to this surplus mass and equal to

$$E = c^2 \Delta m = \left(3 \times 10^8\right)^2 \times 4 \times 10^{-28} \approx 4 \times 10^{-11}\,\mathrm{J}$$

transforms into the kinetic energy of the particles formed (fission fragments) and into the energy of electromagnetic radiation appearing upon fission.

[9] We sometimes speak of the equivalence of mass and energy having in mind their relation and proportionality to each other.

[10] The symbol ${}_{92}U^{235}$ stands for the uranium isotope with a mass number of 235. The nucleus of an atom of this isotope consists of 92 protons and $235 - 92 = 143$ neutrons. The symbol n stands for a neutron.

8.10. Particles with a Zero Rest Mass

Assuming in Eq. (8.42) that m equals zero, we get

$$E = cp. \tag{8.57}$$

This equation agrees with Eq. (8.44) only if $v = c$. Hence it follows that a particle having a zero rest mass always travels with the speed of light. Such particles include a light particle called a **photon**, and also elementary particles called **neutrinos**.

The energy of a photon is determined by the equation

$$E = \hbar\omega \tag{8.58}$$

where $\hbar$ is Plancks' constant h divided by 2π, and ω is the cyclic frequency [see Eq. (7.58)].

According to Eqs. (8.57) and (8.58), a photon has the momentum

$$p = \frac{\hbar\omega}{c}. \tag{8.59}$$

Light is a stream of photons. When light is absorbed or reflected from the surface of a body, a momentum is imparted to the latter. This manifests itself in the form of pressure exerted by the light on the body. P. Lebedev succeeded in discovering and measuring light pressure in 1900. The results of his measurements completely agreed with Eq. (8.59).

According to Einstein's general theory of relativity, any object having the energy E also has the gravitational mass

$$m_{\mathrm{g}} = \frac{E}{c^2}$$

i.e., it should be attracted to other objects. Accordingly, a photon should behave in a gravitational field like a particle of the gravitational mass

$$m_{\mathrm{g}} = \frac{\hbar\omega}{c^2}. \tag{8.60}$$

Particularly, when moving vertically upward near the Earth's surface, a photon must spend part of its energy on doing work against the forces of gravity equal to

$$A = m_{\mathrm{g}}gl = \frac{\hbar\omega gl}{c^2}$$

where l is the distance travelled. Accordingly, the initial energy of a photon equal to $\hbar\omega$ must diminish by

$$\Delta E = \Delta(\hbar\omega) = \frac{\hbar\omega gl}{c^2}.$$

Hence,

$$\Delta\omega = \frac{\omega gl}{c^2}.$$

We thus get the following expression for the relative reduction in the frequency of a photon:

$$\frac{\Delta\omega}{\omega} = \frac{gl}{c^2}. \tag{8.61}$$

The change in the frequency of a photon when propagating vertically was measured in 1959 by the U.S. scientists R. Pound and G. Rebka, Jr. Their result coincided with that calculated by Eq. (8.61) with an accuracy of 15%. We must note that in the conditions of their experiment the relative change in the frequency had a negligibly small value equal to 2×10^{-15}.

The effect of the change in the frequency of light when moving away from a large gravitating mass is called the **gravitational red shift**. The meaning of this term will be disclosed in the third volume of the present course.