I. V. Savelyev

PHYSICS

A General Course

ELECTRICITY
& MAGNETISM

WAVES

OPTICS

Mir Publishers
Moscow

II

I. V. SAVELYEV

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(In three volumes)

VOLUME II

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MIR PUBLISHERS
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Translated from Russian by G. Leib

First published 1980
Revised from the 1978 Russian edition
Second printing 1985
Third printing 1989

Printed in the Union of Soviet Socialist Republics

ISBN 5-03-000902-7, 1978
ISBN 5-03-000900-0, 1980

PREFACE

The main content of the present volume is the science of electromagnetism and the science of waves (elastic, electromagnetic, and light).

The International System of Units (SI) has been used throughout the book, although the reader is simultaneously acquainted with the Gaussian system. In addition to a list of symbols, the appendices at the end of the book give the units of electrical and magnetic quantities in the SI and in the Gaussian system of units, and also compare the form of the basic formulas of electromagnetism in both systems.

The course is the result of twenty five year's work in the Department of General Physics of the Moscow Institute of Engineering Physics. I am grateful to my colleagues and friends for their helpful discussions, criticism and advice in the course of the preparation of the book.

The present course is intended above all for higher technical schools with an extended syllabus in physics. The material has been arranged, however, so that the book can be used as a teaching aid for higher technical schools with an ordinary syllabus simply by omitting some sections.

Igor Savelyev

Moscow, November, 1979

Contents

PART I

ELECTRICITY AND MAGNETISM

Chapter 1
ELECTRIC FIELD IN A VACUUM

1.1. Electric Charge

All bodies in nature are capable of becoming electrified, *i.e.*, acquiring an electric charge. The presence of such a charge manifests itself in that a charged body interacts with other charged bodies. Two kinds of electric charges exist. They are conventionally called positive and negative. Like charges repel each other, and unlike charges attract each other.

An electric charge is an integral part of certain elementary particles[1]. The charge of all elementary particles (if it is not absent) is identical in magnitude. It can be called an **elementary charge**. We shall use the symbol e to denote a positive elementary charge.

The elementary particles include, in particular, the electron (carrying the negative charge $-e$), the proton (carrying the positive charge $+e$), and the neutron (carrying no charge). These particles are the bricks which the atoms and molecules of any substance are built of, therefore all bodies contain electric charges. The particles carrying charges of different signs are usually present in a body in equal numbers and are distributed over it with the same density. The algebraic sum of the charges in any elementary volume of the body equals zero in this case, and each such volume (as well as the body as a whole) will be neutral. If in some way or other we create a surplus of particles of one sign in a body (and, correspondingly, a shortage of particles of the opposite sign), the body will be charged. It is also possible, without changing the total number of positive and negative particles, to cause them to be redistributed in a body so that one part of it has a surplus of charges of one sign and the other part a surplus of charges of the opposite sign.

[1]Elementary particles are defined as such microparticles whose internal structure at the present level of development of physics cannot be conceived as a combination of other particles.

This can be done by bringing a charged body close to an uncharged metal one.

Since a charge q is formed by a plurality of elementary charges, it is an integral multiple of e:

$$q = \pm Ne. \tag{1.1}$$

An elementary charge is so small, however, that macroscopic charges may be considered to have continuously changing magnitudes.

If a physical quantity can take on only definite discrete values, it is said to be quantized. The fact expressed by Eq. (1.1) signifies that an electric charge is quantized.

The magnitude of a charge measured in different inertial reference frames will be found to be the same. Hence, an electric charge is relativistically invariant. It thus follows that the magnitude of a charge does not depend on whether the charge is moving or at rest.

Electric charges can vanish and appear again. Two elementary charges of opposite signs always appear or vanish simultaneously, however. For example, an electron and a positron (a positive electron) meeting each other annihilate, *i.e.*, transform into neutral gamma-photons. This is attended by vanishing of the charges $-e$ and $+e$. In the course of the process called the birth of a pair, a gamma-photon getting into the field of an atomic nucleus transforms into a pair of particles—an electron and a positron. This process causes the charges $-e$ and $+e$ to appear.

Thus, the total charge of an electrically isolated system[2] cannot change. This statement forms the **law of electric charge conservation**.

We must note that the law of electric charge conservation is associated very closely with the relativistic invariance of a charge. Indeed, if the magnitude of a charge depended on its velocity, then by bringing charges of one sign into motion we would change the total charge of the relevant isolated system.

1.2. Coulomb's Law

The law obeyed by the force of interaction of point charges was established experimentally in 1785 by the French physicist Charles A. de Coulomb (1736-1806). A **point charge** is defined as a charged body whose dimensions may be disregarded in comparison with the distances from this body to other bodies carrying an electric charge.

Using a torsion balance (Fig. 1.1) similar to that employed by H. Cavendish to determine the gravitational constant (see Vol. I, Sec. 6.1), Coulomb measured the force of interaction of two charged spheres depending on the magnitude of the

[2] A system is referred to as electrically isolated if no charged particles can penetrate through the surface confining it.

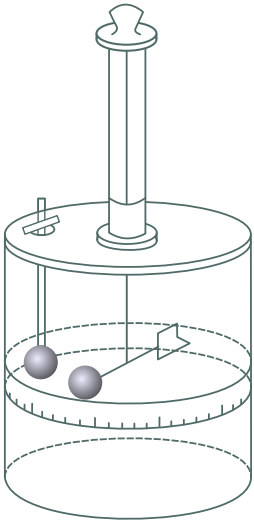


Fig. 1.1

charges on them and on the distance between them. He proceeded from the fact that when a charged metal sphere was touched by an identical uncharged sphere, the charge would be distributed equally between the two spheres.

As a result of his experiments, Coulomb arrived at the conclusion that *the force of interaction between two stationary point charges is proportional to the magnitude of each of them and inversely proportional to the square of the distance between them*. The direction of the force coincides with the straight line connecting the charges.

It must be noted that the direction of the force of interaction along the straight line connecting the point charges follows from considerations of symmetry. An empty space is assumed to be homogeneous and isotropic. Consequently, the only direction distinguished in the space by stationary point charges introduced into it is that from one charge to the other. Assume that the force $\boldsymbol{F}$ acting on the charge q_i (Fig. 1.2) makes the angle α with the direction from q_1 to q_2, and that α differs from 0 or π. But owing to axial symmetry, there are no grounds to set the force $\boldsymbol{F}$ aside from the multitude of forces of other directions making the same angle α with the axis q_1-q_2 (the directions of these forces form a cone with a cone angle of 2α). The difficulty appearing as a result of this vanishes when α equals 0 or π.

Coulomb's law can be expressed by the formula

$$\boldsymbol{F}_{12} = -k\frac{q_1 q_2}{r^2}\,\hat{\boldsymbol{e}}_{12}. \tag{1.2}$$

Here, k is a proportionality constant assumed to be positive, q_1 and q_2 are magnitudes of the interacting charges, r is the distance between the charges, $\hat{\boldsymbol{e}}_{12}$ is the unit

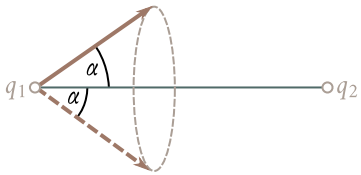


Fig. 1.2

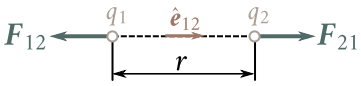


Fig. 1.3

vector directed from the charge q_1 to q_2 and $\boldsymbol{F}_{12}$ is the force acting on the charge q_1 (Fig. 1.3; the figure corresponds to the case of like charges).

The force $\boldsymbol{F}_{21}$ differs from $\boldsymbol{F}_{12}$ in its sign:

$$\boldsymbol{F}_{21} = k\frac{q_1 q_2}{r^2}\,\hat{\boldsymbol{e}}_{12}. \tag{1.3}$$

The magnitude of the interaction force, which is the same for both charges, can be written in the form

$$F = k\frac{|q_1 q_2|}{r^2}. \tag{1.4}$$

Experiments show that the force of interaction between two given charges does not change if other charges are placed near them. Assume that we have the charge q_a and, in addition, N other charges $q_1, q_2, \ldots, q_N$. It can be seen from the above that the resultant force $\boldsymbol{F}$ with which all the N charges q_i act on q_a is

$$\boldsymbol{F} = \sum_{i=1}^{N} \boldsymbol{F}_{\mathrm{a},\,i} \tag{1.5}$$

where $\boldsymbol{F}_{\mathrm{a},\,i}$ is the force with which the charge q_i acts on q_a in the absence of the other $N-1$ charges.

The fact expressed by Eq. (1.5) permits us to calculate the force of interaction between charges concentrated on bodies of finite dimensions, knowing the law of interaction between point charges. For this purpose, we must divide each charge into so small charges dq that they can be considered as point ones, use Eq. (1.2) to calculate the force of interaction between the charges dq taken in pairs, and then perform vector summation of these forces. Mathematically, this procedure coincides completely with the calculation of the force of gravitational attraction between bodies of finite dimensions (see Vol. I, Sec. 6.1).

All experimental facts available lead to the conclusion that Coulomb's law holds for distances from 10^{-15} m to at least several kilometres. There are grounds to presume that for distances smaller than 10^{-16} m the law stops being correct. For very great distances, there are no experimental confirmations of Coulomb's law. But there are also no reasons to expect that this law stops being obeyed with very great distances between charges.

1.3. Systems of Units

We can make the proportionality constant in Eq. (1.2) equal unity by properly choosing the unit of charge (the units for F and r were established in mechanics). The relevant unit of charge (when F and r are measured in cgs units) is called the **absolute electrostatic unit** of charge (cgse_q). It is the magnitude of a charge that interacts with a force of 1 dyn in a vacuum with an equal charge at a distance of 1 cm from it.

Careful measurements (they are described in Sec. **??**) showed that an elementary charge is

$$e = 4.80 \times 10^{-10}\ \text{cgse}_q. \tag{1.6}$$

Adopting the units of length, mass, time, and charge as the basic ones, we can construct a system of units of electrical and magnetic quantities. The system based on the centimetre, gramme, second, and the cgse_q unit is called the **absolute electrostatic system of units** (the cgse system). It is founded on Coulomb's law, *i.e.*, the law of interaction between charges at rest. On a later page, we shall become acquainted with the **absolute electromagnetic system of units** (the cgsm system) based on the law of interaction between conductors carrying an electric current. The Gaussian system in which the units of electrical quantities coincide with those of the cgse system, and of magnetic quantities with those of the cgsm system, is also an absolute system.

Equation (1.4) in the cgse system becomes

$$F = \frac{|q_1 q_2|}{r^2}. \tag{1.7}$$

This equation is correct if the charges are in a vacuum. It has to be determined more accurately for charges in a medium (see Sec. 2.8).

USSR State Standard GOST 9867-61, which came into force on January 1, 1963, prescribes the preferable use of the International System of Units (SI). The basic units of this system are the metre, kilogramme, second, ampere, kelvin, candela, and mole. The SI unit of force is the newton (N) equal to 10^5 dynes.

In establishing the units of electrical and magnetic quantities, the SI system, like the cgsm one, proceeds from the law of interaction of current-carrying conductors instead of charges. Consequently, the proportionality constant in the equation of Coulomb's law is a quantity with a dimension and differing from unity.

The SI unit of charge is the coulomb (C). It has been found experimentally that

$$1\ \text{C} = 2.998 \times 10^9 \approx 3 \times 10^9\ \text{cgse}_q. \tag{1.8}$$

To form an idea of the magnitude of a charge of 1 C, let us calculate the force with which two point charges of 1 C each would interact with each other if they

were 1 m apart. By Eq. (1.7)

$$F = \frac{3 \times 10^9 \times 3 \times 10^9}{100^2} \text{ cgse}_F = 9 \times 10^{14} \text{ dyn} = 9 \times 10^9 \text{ N} \approx 10^9 \text{ kgf}. \quad (1.9)$$

An elementary charge expressed in coulombs is

$$e = 1.60 \times 10^{-19} \text{ C}. \quad (1.10)$$

1.4. Rationalized Form of Writing Formulas

Many formulas of electrodynamics when written in the cgs systems (in particular, in the Gaussian one) include as factors 4π and the so-called electromagnetic constant c equal to the speed of light in a vacuum. To eliminate these factors in the formulas that are most important in practice, the proportionality constant in Coulomb's law is taken equal to $1/4\pi\varepsilon_0$. The equation of the law for charges in a vacuum will thus become

$$F = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{r^2}. \quad (1.11)$$

The other formulas change accordingly. This modified way of writing formulas is called **rationalized**. Systems of units constructed with the use of rationalized formulas are also called **rationalized**. They include the SI system.

The quantity ε_0 is called the **electric constant**. It has the dimension of capacitance divided by length. It is accordingly expressed in units called the farad per metre. To find the numerical value of ε_0, we shall introduce the values of the quantities corresponding to the case of two charges of 1 C each and 1 m apart into Eq. (1.11). By Eq. (1.9), the force of interaction in this case is 9×10^9 N. Using this value of the force, and also $q_1 = q_2 = 1$ C and $r = 1$ m in Eq. (1.11), we get

$$9 \times 10^9 = \frac{1}{4\pi\varepsilon_0} \frac{|1 \times 1|}{1^2}$$

whence

$$\varepsilon_0 = \frac{1}{4\pi \times 9 \times 10^9} = 0.885 \times 10^{-11} \text{ F m}^{-1}. \quad (1.12)$$

The Gaussian system of units was widely used and is continuing to be used in physical publications. We therefore consider it essential to acquaint our reader with both the SI and the Gaussian system. We shall set out the material in the SI units showing at the same time how the formulas look in the Gaussian system. The fundamental formulas of electrodynamics written in the SI and the Gaussian system are compared in Appendix. **??**.

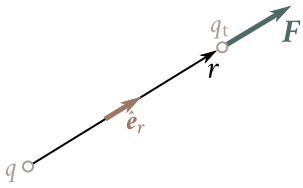


Fig. 1.4

1.5. Electric Field. Field Strength

Charges at rest interact through an electric field[3]. A charge alters the properties of the space surrounding it—it sets up an electric field in it. This field manifests itself in that an electric charge placed at a point of it experiences the action of a force. Hence, to see whether there is an electric field at a given place, we must place a charged body (in the following we shall say simply a charge for brevity) at it and determine whether or not it experiences the action of an electric force. We can evidently assess the "strength" of the field according to the magnitude of the force exerted on the given charge.

Thus, to detect and study an electric field, we must use a "test" charge. For the force acting on our test charge to characterize the field "at the given point", the test charge must be a point one. Otherwise, the force acting on the charge will characterize the properties of the field averaged over the volume occupied by the body that carries the test charge.

Let us study the field set up by the stationary point charge q with the aid of the point test charge q_t. We place the test charge at a point whose position relative to the charge q is determined by the position vector $\boldsymbol{r}$ (Fig. 1.4). We see that the test charge experiences the force

$$\boldsymbol{F} = q_t \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\boldsymbol{e}}_r \right) \tag{1.13}$$

[see Eqs. (1.3) and (1.11)]. Here $\hat{\boldsymbol{e}}_r$ is the unit vector of the position vector $\boldsymbol{r}$.

A glance at Eq. (1.13) shows that the force acting on our test charge depends not only on the quantities determining the field (on q and $\boldsymbol{r}$), but also on the magnitude of the test charge q_t. If we take different test charges q_t', q_t'', etc., then the forces $\boldsymbol{F}'$, $\boldsymbol{F}''$, etc. which they experience at the given point of the field will be different. We can see from Eq. (1.13), however, that the ratio F/q_t for all the test charges will be the same and depend only on the values of q and $\boldsymbol{r}$ determining the field at the given point. It is therefore natural to adopt this ratio as the quantity characterizing an

[3] We shall see in Sec. **??** that when considering moving charges, their interaction in addition to an electric field is due to a magnetic field.

electric field:

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q_\mathrm{t}}. \tag{1.14}$$

This vector quantity is called the **electric field strength** (or **intensity**) at a given point (*i.e.*, at the point where the test charge q_t experiences the action of the force $\boldsymbol{F}$).

According to Eq. (1.14), the electric field strength numerically equals the force acting on a unit point charge at the given point of the field. The direction of the vector $\boldsymbol{E}$ coincides with that of the force acting on a positive charge.

It must be noted that Eq. (1.14) also holds when the test charge is negative ($q_\mathrm{t} < 0$). In this case, the vectors $\boldsymbol{E}$ and $\boldsymbol{F}$ have opposite directions.

We have arrived at the concept of electric field strength when studying the field of a stationary point charge. Definition (1.14), however, also covers the case of a field set up by any collection of stationary charges, but here the following clarification is needed. The arrangement of the charges setting up the field being studied may change under the action of the test charge. This will happen, for example, when the charges producing the field are on a conductor and can freely move within its limits. Therefore, to avoid appreciable alterations in the field being studied, a sufficiently small test charge must be taken.

It follows from Eqs. (1.13) and (1.14) that the field strength of a point charge varies directly with the magnitude of the charge q and inversely with the square of the distance r from the charge to the given point of the field:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{\boldsymbol{e}}_r. \tag{1.15}$$

The vector $\boldsymbol{E}$ is directed along the radial straight line passing through the charge and the given point of the field, from the charge if the latter is positive and toward the charge if it is negative.

In the Gaussian system, the equation for the field strength of a point charge in a vacuum has the form

$$\boldsymbol{E} = \frac{q}{r^2}\,\hat{\boldsymbol{e}}_r. \tag{1.16}$$

The unit of electric field strength is the strength at a point where unit force (1 N in the SI and 1 dyn in the Gaussian system) acts on unit charge (1 C in the SI and 1 cgse_q in the Gaussian system). This unit has no special name in the Gaussian system. The SI unit of electric field strength is called the volt per metre ($\mathrm{V\,m^{-1}}$) [see Eq. (1.44)].

According to Eq. (1.15), a charge of 1 C produces the following field strength in a

vacuum at a distance of 1 m from this charge:

$$E = \frac{1}{4\pi\left(1/4\pi \times 9 \times 10^9\right)} \frac{1}{1^2} = 9 \times 10^9 \,\mathrm{V\,m^{-1}}.$$

This strength in the Gaussian system is

$$E = \frac{q}{r^2} = \frac{3 \times 10^9}{100^2} = 3 \times 10^5 \ \mathrm{cgse}_E.$$

Comparing these two results, we find that

$$1 \ \mathrm{cgse}_E = 3 \times 10^4 \,\mathrm{V\,m^{-1}}. \tag{1.17}$$

According to Eq. (1.14), the force exerted on a test charge is

$$\boldsymbol{F} = q_\mathrm{t}\boldsymbol{E}.$$

It is obvious that any point charge q[4] at a point of a field with the strength $\boldsymbol{E}$ will experience the force

$$\boldsymbol{F} = q\boldsymbol{E}. \tag{1.18}$$

If the charge q is positive, the direction of the force coincides with that of the vector $\boldsymbol{E}$. If q is negative, the vectors $\boldsymbol{F}$ and $\boldsymbol{E}$ are directed oppositely.

We mentioned in Sec. 1.2 that the force with which a system of charges acts on a charge not belonging to the system equals the vector sum of the forces which each of the charges of the system exerts separately on the given charge [see Eq. (1.15)]. Hence it follows that *the field strength of a system of charges equals the vector sum of the field strengths that would be produced by each of the charges of the system separately*:

$$\boldsymbol{E} = \sum_i \boldsymbol{E}_i. \tag{1.19}$$

This statement is called the **principle of electric field superposition**.

The superposition principle allows us to calculate the field strength of any system of charges. By dividing extended charges into sufficiently small fractions $\mathrm{d}q$, we can reduce any system of charges to a collection of point charges. We calculate the contribution of each of such charges to the resultant field by Eq. (1.15).

An electric field can be described by indicating the magnitude and direction of the vector $\boldsymbol{E}$ for each of its points. The combination of these vectors forms the field of the electric field strength vector (compare with the field of the velocity vector, Vol. I, Sec. 9.1). The velocity vector field can be represented very illustratively with the aid of flow lines. Similarly, an electric field can be described with the aid of strength lines, which we shall call for short $\boldsymbol{E}$ lines or field lines. These lines are drawn so that a tangent to them at every point coincides with the direction of the

[4] In Eq. (1.15), q stands for the charge setting up the field. In Eq. (1.18), q stands for the charge experiencing the force $\boldsymbol{F}$ at a point of strength $\boldsymbol{E}$.

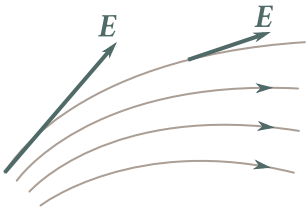


Fig. 1.5

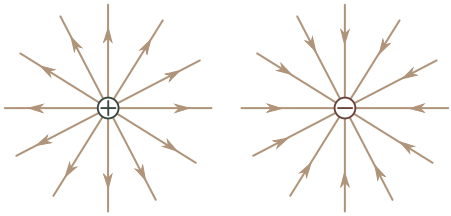


Fig. 1.6

vector $\boldsymbol{E}$. The density of the lines is selected so that their number passing through a unit area at right angles to the lines equals the numerical value of the vector $\boldsymbol{E}$. Hence, the pattern of field lines permits us to assess the direction and magnitude of the vector $\boldsymbol{E}$ at various points of space (Fig. 1.5).

The $\boldsymbol{E}$ lines of a point charge field are a collection of radial straight lines directed away from the charge if it is positive and toward it if it is negative (Fig. 1.6). One end of each line is at the charge, and the other extends to infinity. Indeed, the total number of lines intersecting a spherical surface of arbitrary radius r will equal the product of the density of the lines and the surface area of the sphere $4\pi r^2$. We have assumed that the density of the lines numerically equals $E = (1/4\pi\varepsilon_0)(q/r^2)$. Hence, the number of lines is $(1/4\pi\varepsilon_0)(q/r^2)4\pi r^2 = q/\varepsilon_0$. This result signifies that the number of lines at any distance from a charge will be the same. It thus follows hat the lines do not begin and do not terminate anywhere except tor the charge. Beginning at the charge, they extend to infinity (the charge is positive), or arriving from infinity, they terminate at the charge (the latter is negative). This property of the $\boldsymbol{E}$ lines is common for all electrostatic fields, *i.e.*, fields set up by any system of stationary charges: the field lines can begin or terminate only at charges or extend to infinity.

1.6. Potential

Let us consider the field produced by a stationary point charge q. At any point of this field, the point charge q' experiences the force

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}\hat{\boldsymbol{e}}_r = F(r)\hat{\boldsymbol{e}}_r. \tag{1.20}$$

Here $F(r)$ is the magnitude of the force $\boldsymbol{F}$, and $\hat{\boldsymbol{e}}_r$ is the unit vector of the position vector $\boldsymbol{r}$ determining the position of the charge q' relative to the charge q.

The force (1.20) is a central one (see Vol. I, Sec. 3.4). A central field of forces is conservative. Consequently, the work done by the forces of the field on the charge q' when it is moved from one point to another does not depend on the path. This

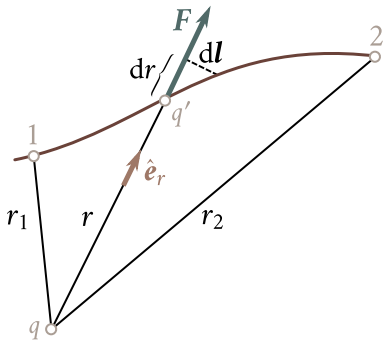


Fig. 1.7

work is

$$A_{12} = \int_1^2 F(r)\hat{\boldsymbol{e}}_r \, \mathrm{d}\boldsymbol{l} \tag{1.21}$$

where $\mathrm{d}\boldsymbol{l}$ is the elementary displacement of the charge q'. Inspection of Fig. 1.7 shows that the scalar product $\hat{\boldsymbol{e}}_r \, \mathrm{d}\boldsymbol{l}$ equals the increment of the magnitude of the position vector $\boldsymbol{r}$, *i.e.*, $\mathrm{d}r$. Equation (1.21) can therefore be written in the form

$$A_{12} = \int_1^2 F(r) \, \mathrm{d}r$$

[compare with Eq. (3.24) of Vol. I]. Introduction of the expression for $F(r)$ yields

$$A_{12} = \frac{qq'}{4\pi\varepsilon_0} \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r^2} = \frac{1}{4\pi\varepsilon_0} \left(\frac{qq'}{r_1} - \frac{qq'}{r_2} \right). \tag{1.22}$$

The work of the forces of a conservative field can be represented as a decrement of the potential energy:

$$A_{12} = W_{\mathrm{p},1} - W_{\mathrm{p},2}. \tag{1.23}$$

A comparison of Eqs. (1.22) and (1.23) leads to the following expression for the potential energy of the charge q' in the field of the charge q:

$$W_{\mathrm{p}} = \frac{1}{4\pi\varepsilon_0} \frac{qq'}{r_2} + \text{constant}.$$

The value of the constant in the expression for the potential energy is usually chosen so that when the charge moves away to infinity (*i.e.*, when $r = \infty$), the potential energy vanishes. When this condition is observed, we get

$$W_{\mathrm{p}} = \frac{1}{4\pi\varepsilon_0} \frac{qq'}{r_2}. \tag{1.24}$$

Let us use the charge q' as a test charge for studying the field. By Eq. (1.24), the potential energy which the test charge has depends not only on its magnitude q',

but also on the quantities q and r determining the field. Thus, we can use this energy to describe the field just like we used the force acting on the test charge for this purpose.

Different test charges q_t', q_t'', etc. will have different energies W_p', W_p'', etc. at the same point of a field. But the ratio W_p/q_t will be the same for all the charges [see Eq. (1.24)]. The quantity

$$\varphi = \frac{W_p}{q_t} \tag{1.25}$$

is called the **field potential** at a given point and is used together with the field strength $\boldsymbol{E}$ to describe electric fields.

It can be seen from Eq. (1.25) that the potential numerically equals the potential energy which a unit positive charge would have at the given point of the field. Substituting for the potential energy in Eq. (1.25) its value from (1.24), we get the following expression for the potential of a point charge:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \tag{1.26}$$

In the Gaussian system, the potential of the field of a point charge in a vacuum is determined by the formula

$$\varphi = \frac{q}{r}. \tag{1.27}$$

Let us consider the field produced by a system of N point charges $q_1, q_2, \ldots, q_N$. Let $r_1, r_2, \ldots, r_N$ be the distances from each of the charges to the given point of the field. The work done by the forces of this field on the charge q' will equal the algebraic sum of the work done by the forces set up by each of the charges separately:

$$A_{12} = \sum_{i=1}^{N} A_i.$$

By Eq. (1.22), each work A_i equals

$$A_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_i q'}{r_{i,1}} - \frac{q_i q'}{r_{i,2}}\right)$$

where $r_{i,1}$ is the distance from the charge q_i to the initial position of the charge q', and $r_{i,2}$ is the distance from q_i to the final position of the charge q'. Hence,

$$A_{i2} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i q'}{r_{i,1}} - \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i q'}{r_{i,2}}.$$

Comparing this equation with Eq. (1.23), we get the following expression for the

potential energy of the charge q' in the field of a system of charges:

$$W_{\mathrm{p}} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i q'}{r_i}$$

from which it can be seen that

$$\varphi = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i}. \tag{1.28}$$

Comparing this formula with Eq. (1.26), we arrive at the conclusion that *the potential of the field produced by a system of charge equals the algebraic sum of the potentials produced by each of the charges separately*. Whereas the field strengths are added vectorially in the superposition of fields, the potentials are added algebraically. This is why it is usually much simpler to calculate the potentials than the electric field strengths.

Examination of Eq. (1.25) shows that the charge q at a point of a field with the potential φ has the potential energy

$$W_{\mathrm{p}} = q\varphi. \tag{1.29}$$

Hence, the work of the field forces on the charge q can be expressed through the potential difference:

$$A_{12} = W_{\mathrm{p},1} - W_{\mathrm{p},2} = q\,(\varphi_1 - \varphi_2)\,. \tag{1.30}$$

Thus, the work done on a charge by the forces of a field equals the product of the magnitude of the charge and the difference between the potentials at the initial and final points (*i.e.*, the potential decrement).

If the charge q is removed from a point having the potential φ to infinity (where by convention the potential vanishes), then the work of the field forces will be

$$A_{\infty} = q\varphi. \tag{1.31}$$

Here, it follows that the potential numerically equals the work done by the forces of a field on a unit positive charge when the latter is removed from the given point to infinity. Work of the same magnitude must be done against the electric field forces to move a unit positive charge from infinity to the given point of a field.

Equation (1.31) can be used to establish the units of potential. The unit of potential is taken equal to the potential at a point of a field when work equal to unity is required to move unit positive charge from infinity to this point. The SI unit of potential called the volt (V) is taken equal to the potential at a point when work of 1 joule has to be done to move a charge of 1 coulomb from infinity to this point:

$$1\,\mathrm{J} = 1\,\mathrm{C} \times 1\,\mathrm{V}, \quad \text{thus,} \quad 1\,\mathrm{V} = \frac{1\,\mathrm{J}}{1\,\mathrm{C}}. \tag{1.32}$$

The absolute electrostatic unit of potential (cgse_{φ}) is taken equal to the potential

at a point when work of 1 erg has to be done to move a charge of 1 cgse_q from infinity to this point. Expressing 1 J and 1 C in Eq. (1.32) through cgse units, we shall find the relation between the volt and the cgse potential unit:

$$1\,\text{V} = \frac{1\,\text{J}}{1\,\text{C}} = \frac{10^7\,\text{erg}}{3\times 10^9\,\text{cgse}_q} = \frac{1}{300}\,\text{cgse}_\varphi. \tag{1.33}$$

Thus, 1 cgse_φ equals 300 V.

A unit of energy and work called the **electron-volt** (eV) is frequently used in physics. An electron-volt is defined as the work done by the forces of a field on a charge equal to that of an electron (*i.e.*, on the elementary charge e) when it passes through a potential difference of 1 V:

$$1\,\text{eV} = 1.60\times 10^{-19}\,\text{C}\times 1\,\text{V} = 1.60\times 10^{-19}\,\text{J} = 1.60\times 10^{-12}\,\text{erg}. \tag{1.34}$$

Multiple units of the electron-volt are also used:

$$1\,\text{keV (kiloelectron-volt)} = 10^3\,\text{eV},$$
$$1\,\text{MeV (megaelectron-volt)} = 10^6\,\text{eV},$$
$$1\,\text{GeV (gigaelectron-volt)} = 10^9\,\text{eV}.$$

1.7. Interaction Energy of a System of Charges

Equation (1.24) can be considered as the mutual potential energy of the charges q and q'. Using the symbols q_1 and q_2 for these charges, we get the following formula for their interaction energy:

$$W_\text{p} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}. \tag{1.35}$$

The symbol r_{12} stands for the distance between the charges.

Let us consider a system consisting of N point charges $q_1, q_2, \ldots, q_N$. We showed in Sec. 3.6 of Vol. I that the energy of interaction of such a system equals the sum of the energies of interaction of the charges taken in pairs:

$$W_\text{p} = \frac{1}{2}\sum_{(i\neq k)} W_{\text{p},ik}(r_{ik}) \tag{1.36}$$

[see Eq. (3.60) of Vol. I].

According to Eq. (1.35)

$$W_{\text{p},ik} = \frac{1}{4\pi\varepsilon_0}\frac{q_i q_k}{r_{ik}}.$$

Using this equation in (1.36), we find that

$$W_\text{p} = \frac{1}{2}\sum_{(i\neq k)} \frac{1}{4\pi\varepsilon_0}\frac{q_i q_k}{r_{ik}}. \tag{1.37}$$

In the Gaussian system, the factor $1/(4\pi\varepsilon_0)$ is absent in this equation.

In Eq. (1.37), summation is performed over the subscripts i and k. Both subscripts pass independently through all the values from 1 to N. Addends for which the value of the subscript i coincides with that of k are not taken into consideration. Let us write Eq. (1.37) as follows:

$$W_{\mathrm{p}} = \frac{1}{2}\sum_{i=1}^{N} q_i \sum_{\substack{i=1\\(i\neq k)}}^{N} \frac{1}{4\pi\varepsilon_0}\frac{q_k}{r_{ik}}. \tag{1.38}$$

The expression

$$\varphi_i = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{i=1\\(i\neq k)}}^{N} \frac{q_k}{r_{ik}}$$

is the potential produced by all the charges except q_i at the point where the charge q_i is. With this in view, we get the following formula for the interaction energy:

$$W_{\mathrm{p}} = \frac{1}{2}\sum_{i=1}^{N} q_i\varphi_i. \tag{1.39}$$

1.8. Relation Between Electric Field Strength and Potential

An electric field can be described either with the aid of the vector quantity $\boldsymbol{E}$, or with the aid of the scalar quantity φ. There must evidently be a definite relation between these quantities. If we bear in mind that $\boldsymbol{E}$ is proportional to the force acting on a charge and φ to the potential energy of the charge, it is easy to see that this relation must be similar to that between the potential energy and the force.

The force $\boldsymbol{F}$ is related to the potential energy by the expression

$$\boldsymbol{F} = -\nabla W_{\mathrm{p}} \tag{1.40}$$

[see Eq. (3.32) of Vol. I]. For a charged particle in an electrostatic field, we have $\boldsymbol{F} = q\boldsymbol{E}$ and $W_{\mathrm{p}} = q\varphi$. Introducing these values into Eq. (1.40), we find that

$$q\boldsymbol{E} = -\nabla(q\varphi).$$

The constant q can be put outside the gradient sign. Doing this and then cancelling q, we arrive at the formula

$$\boldsymbol{E} = -\nabla\varphi \tag{1.41}$$

establishing the relation between the field strength and potential.

Taking into account the definition of the gradient [see Eq. (3.31) of Vol. 1], we

can write that

$$\boldsymbol{E} = -\frac{\partial\varphi}{\partial x}\hat{\boldsymbol{e}}_x - \frac{\partial\varphi}{\partial y}\hat{\boldsymbol{e}}_y - \frac{\partial\varphi}{\partial z}\hat{\boldsymbol{e}}_z. \tag{1.42}$$

Hence, Eq. (1.41) has the following form in projections onto the coordinate axes:

$$E_x = -\frac{\partial\varphi}{\partial x}, \quad E_y = -\frac{\partial\varphi}{\partial y}, \quad E_z = -\frac{\partial\varphi}{\partial z}. \tag{1.43}$$

Similarly, the projection of the vector $\boldsymbol{E}$ onto an arbitrary direction l equals the derivative of φ with respect to l taken with the opposite sign, *i.e.*, the rate of diminishing of the potential when moving along the direction l:

$$E_l = -\frac{\partial\varphi}{\partial l}. \tag{1.44}$$

It is easy to see that Eq. (1.44) is correct by choosing l as one of the coordinate axes and taking Eq. (1.43) into account.

Let us explain Eq. (1.41) using as an example the field of a point charge. The potential of this field is expressed by Eq. (1.26). Passing over to Cartesian coordinates, we get the expression

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{(x^2+y^2+z^2)^{1/2}}.$$

The partial derivative of this function with respect to x is

$$\frac{\partial\varphi}{\partial x} = -\frac{1}{4\pi\varepsilon_0}\frac{qx}{(x^2+y^2+z^2)^{3/2}} = -\frac{1}{4\pi\varepsilon_0}\frac{qx}{r^3}.$$

Similarly,

$$\frac{\partial\varphi}{\partial y} = -\frac{1}{4\pi\varepsilon_0}\frac{qy}{r^3}, \quad \frac{\partial\varphi}{\partial y} = -\frac{1}{4\pi\varepsilon_0}\frac{qz}{r^3}.$$

Using the found values of the derivatives in Eq. (1.42), we arrive at the expression

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\left(x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z\right)}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{q\boldsymbol{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{e}}_r$$

that coincides with Eq. (1.15).

Equation (1.41) allows us to find the field strength at every point from the known values of φ. We can also solve the reverse problem, *i.e.*, find the potential difference between two arbitrary points of a field according to the given values of $\boldsymbol{E}$. For this purpose, we shall take advantage of the circumstance that the work done by the forces of a field on the charge q when it is moved from point 1 to point 2 can be calculated as

$$A_{12} = \int_1^2 q\boldsymbol{E}\,\mathrm{d}\boldsymbol{l}.$$

At the same time in accordance with Eq. (1.30), this work can be written as

$$A_{12} = q\,(\varphi_1 - \varphi_2)\,.$$

Equating these two expressions and cancelling q, we obtain

$$\varphi_1 - \varphi_2 = \int_1^2 \boldsymbol{E}\,\mathrm{d}\boldsymbol{l}. \tag{1.45}$$

The integral can be taken along any line joining points 1 and 2 because the work of the field forces is independent of the path. For circumvention along a closed contour, $\varphi_1 = \varphi_2$, and Eq. (1.45) becomes

$$\oint \boldsymbol{E}\,\mathrm{d}\boldsymbol{l} = 0 \tag{1.46}$$

(the circle on the integral sign indicates that integration is performed over a closed contour). It must be noted that this relation holds only for an electrostatic field. We shall see on a later page that the field of moving charges (*i.e.*, a field changing with time) is not a potential one. Therefore, condition (1.46) is not observed for it.

An imaginary surface all of whose points have the same potential is called an equipotential surface. Its equation has the form

$$\varphi(x, y, z) = \text{constant}.$$

The potential does not change in movement along an equipotential surface over the distance $\mathrm{d}l$ ($\mathrm{d}\varphi = 0$). Hence, according to Eq. (1.44), the tangential component of the vector $\boldsymbol{E}$ to the surface equals zero. We thus conclude that the vector $\boldsymbol{E}$ at every point is directed along a normal to the equipotential surface passing through the given point. Bearing in mind that the vector $\boldsymbol{E}$ is directed along a tangent to an $\boldsymbol{E}$ line, we can easily see that the field lines at every point are orthogonal to the equipotential surfaces.

An equipotential surface can be drawn through any point of a field. Consequently, we can construct an infinitely great number of such surfaces. They are conventionally drawn so that the potential difference for two adjacent surfaces is the same everywhere. Thus, the density of the equipotential surfaces allows us to assess the magnitude of the field strength. Indeed, the denser are the equipotential surfaces, the more rapidly does the potential change when moving along a normal to the surface. Hence, $\nabla\varphi$ is greater at the given place, and, therefore, $\boldsymbol{E}$ is greater too.

Figure 1.8 shows equipotential surfaces (more exactly, their intersections with the plane of the drawing) for the field of a point charge. In accordance with the nature of the dependence of E on r, equipotential surfaces become the denser, the nearer we approach a charge.

Equipotential surfaces for a homogeneous field are a collection of equispaced

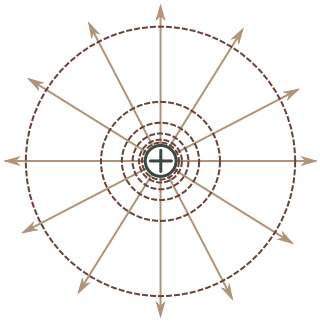


Fig. 1.8

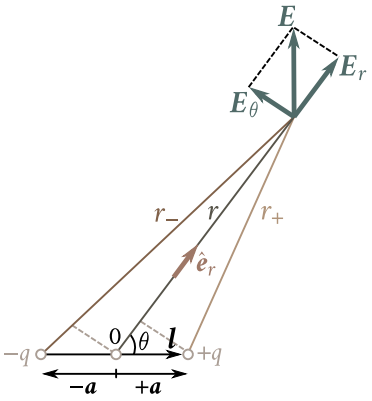


Fig. 1.9

planes at right angles to the direction of the field.

1.9. Dipole

An **electric dipole** is defined as a system of two point charges $+q$ and $-q$ identical in value and opposite in sign, the distance between which is much smaller than that to the points at which the field of the system is being determined. The straight line passing through both charges is called the **dipole axis**.

Let us first calculate the potential and then the field strength of a dipole. This field has axial symmetry. Therefore, the pattern of the field in any plane passing through the dipole axis will be the same, the vector $\boldsymbol{E}$ being in this plane. The position of a point relative to the dipole will be characterized with the aid of the position vector $\boldsymbol{r}$ or with the aid of the polar coordinates r and θ (Fig. 1.9). We shall introduce the vector $\boldsymbol{l}$ passing from the negative charge to the positive one. The position of the charge $+q$ relative to the centre of the dipole is determined by the vector $\boldsymbol{a}$, and of the charge $-q$ by the vector $-\boldsymbol{a}$. It is obvious that $\boldsymbol{l} = 2\boldsymbol{a}$. We shall designate the distances to a given point from the charges $+q$ and $-q$ by r_+ and r_-, respectively.

Owing to the smallness of a in comparison with r, we can assume approximately that

$$\begin{aligned} r_+ &= r - a\cos\theta = r - \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_r, \\ r_- &= r + a\cos\theta = r + \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_r. \end{aligned} \tag{1.47}$$

The potential at a point determined by the position vector $\boldsymbol{r}$ is

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q(r_- - r_+)}{r_+ r_-}.$$

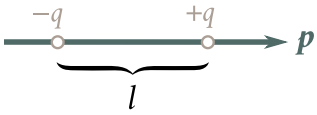


Fig. 1.10

The product r_+r_- can be replaced with r^2. The difference $r_- - r_+$ according to Eqs. (1.47), is $2(\boldsymbol{a}\cdot\hat{\boldsymbol{e}}_r) = \boldsymbol{l}\cdot\hat{\boldsymbol{e}}_r$. Hence,

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\boldsymbol{l}\cdot\hat{\boldsymbol{e}}_r)}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{(\boldsymbol{p}\cdot\hat{\boldsymbol{e}}_r)}{r^2} \tag{1.48}$$

where

$$\boldsymbol{p} = q\boldsymbol{l} \tag{1.49}$$

is a characteristic of a dipole called its **electric moment**. The vector $\boldsymbol{p}$ is directed along the dipole axis from the negative charge to the positive one (Fig. 1.10).

A glance at Eq. (1.48) shows that the field of a dipole is determined by its electric moment $\boldsymbol{p}$. We shall see below that the behaviour of a dipole in an external electric field is also determined by its electric moment $\boldsymbol{p}$. A comparison with Eq. (1.26) shows that the potential of a dipole field diminishes with the distance more rapidly (as $1/r^2$) than the potential of a point charge field (which changes in proportion to $1/r$).

It can be seen from Fig. 1.9 that $\boldsymbol{p}\cdot\hat{\boldsymbol{e}}_r = p\cos\theta$. Therefore, Eq. (1.48) can be written as follows:

$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}. \tag{1.50}$$

To find the field strength of a dipole, let us calculate the projections of the vector $\boldsymbol{E}$ onto two mutually perpendicular directions by Eq. (1.44). One of them is determined by the motion of a point due to the change in the distance r (with θ fixed), the other by the motion of the point due to the change in the angle θ (with r fixed, see Fig. 1.9). The first projection is obtained by differentiation of Eq. (1.50) with respect to r:

$$E_r = -\frac{\partial\varphi}{\partial r} = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}. \tag{1.51}$$

We shall find the second projection (let us designate it by E_θ) by taking the ratio of the increment of the potential φ obtained when the angle θ grows by $\mathrm{d}\theta$ to the distance $r\,\mathrm{d}\theta$ over which the end of the segment r moves (in this case the quantity $\mathrm{d}l$ in Eq. (1.44) equals $r\,\mathrm{d}\theta$]. Thus,

$$E_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta}.$$

Introducing the value of the derivative of function (1.50) with respect to θ we get

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}. \tag{1.52}$$

The sum of the squares of Eqs. (1.51) and (1.52) gives the square of the vector $\boldsymbol{E}$ (see Fig. 1.9):

$$E^2 = E_r^2 + E_\theta^2 = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \left(\frac{p}{r^3}\right)^2 \left(4\cos^2\theta + \sin^2\theta\right)$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \left(\frac{p}{r^3}\right)^2 \left(1 + 3\cos^2\theta\right).$$

Hence

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\left(1 + 3\cos^2\theta\right)^{1/2}. \tag{1.53}$$

Assuming in Eq. (1.53) that $\theta = 0$, we get the strength on the dipole axis:

$$E_\parallel = \frac{1}{4\pi\varepsilon_0}\frac{2p}{r^3}. \tag{1.54}$$

The vector $\boldsymbol{E}_\parallel$ is directed along the dipole axis. This is in agreement with the axial symmetry of the problem. Examination of Eq. (1.51) shows that $E_r > 0$ when $\theta = 0$, and $E_r < 0$ when $\theta = \pi$. This signifies that in any case the vector $\boldsymbol{E}_\parallel$ has a direction coinciding with that from $-q$ to $+q$ (*i.e.*, with the direction of $\boldsymbol{p}$). Equation (1.54) can therefore be written in the vector form:

$$\boldsymbol{E}_\parallel = \frac{1}{4\pi\varepsilon_0}\frac{2\boldsymbol{p}}{r^3}. \tag{1.55}$$

Assuming in Eq. (1.53) that $\theta = \pi/2$, we get the field strength on the straight line passing through the centre of the dipole and perpendicular to its axis:

$$E_\perp = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}. \tag{1.56}$$

By Eq. (1.51), when $\theta = \pi/2$, the projection E_r equals zero. Hence, the vector $\boldsymbol{E}_\perp$ is parallel to the dipole axis. It follows from Eq. (1.52) that when $\theta = \pi/2$, the projection E_θ is positive. This signifies that the vector $\boldsymbol{E}_\perp$ is directed toward the growth of the angle θ, *i.e.*, antiparallel to the vector $\boldsymbol{p}$.

The field strength of a dipole is characterized by the circumstance that it diminishes with the distance from the dipole in proportion to $1/r^3$, *i.e.*, more rapidly than the field strength of a point charge (which diminishes in proportion to $1/r^2$).

Figure 1.11 shows $\boldsymbol{E}$ lines (the solid lines) and equipotential surfaces (the dash lines) of the field of a dipole. According to Eq. (1.50), when $\theta = \pi/2$, the potential vanishes for all the r's. Thus, all the points of a plane at right angles to the dipole axis and passing through its middle have a zero potential. This could have been predicted because the distances from the charges $+q$ and $-q$ to any point of this

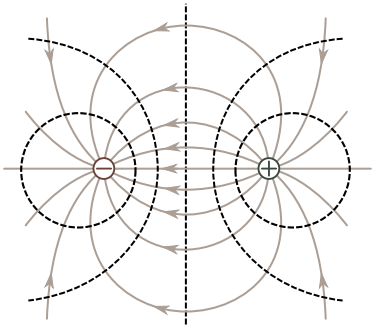


Fig. 1.11

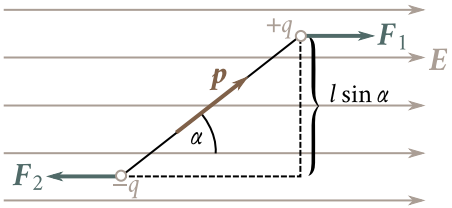


Fig. 1.12

plane are identical.

Now let us turn to the behaviour of a dipole in an external electric field. If a dipole is placed in a homogeneous electric field, the charges $+q$ and $-q$ forming the dipole will be under the action of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ equal in magnitude, but opposite in direction (Fig. 1.12). These forces form a couple whose arm is $l \sin \alpha$, *i.e.*, depends on the orientation of the dipole relative to the field. The magnitude of each of the forces is qE. Multiplying it by the arm, we get the magnitude of the torque acting on a dipole:

$$T = qEl \sin \alpha = pE \sin \alpha \tag{1.57}$$

(p is the electric moment of the dipole). It is easy to see that Eq. (1.57) can be written in the vector form

$$\boldsymbol{T} = \boldsymbol{p} \times \boldsymbol{E}. \tag{1.58}$$

The torque (1.58) tends to turn a dipole so that its electric moment $\boldsymbol{p}$ is in the direction of the field.

Let us find the potential energy belonging to a dipole in an external electric field. By Eq. (1.29), this energy is

$$W_\mathrm{p} = q\varphi_+ - q\varphi_- = q(\varphi_+ - \varphi_-). \tag{1.59}$$

Here φ_+ and φ_- are the values of the potential of the external field at the points where the charges $+q$ and $-q$ are placed.

The potential of a homogeneous field diminishes linearly in the direction of the vector $\boldsymbol{E}$. Assuming that the x-axis is this direction (Fig. 1.13), we can write that $E = E_x = -\mathrm{d}\varphi/\mathrm{d}x$. A glance at Fig. 1.13 shows that the difference $\varphi_+ - \varphi_-$ equals the increment of the potential on the segment $\Delta x = l \cos \alpha$:

$$\varphi_+ - \varphi_- = \frac{\mathrm{d}\varphi}{\mathrm{d}x} l \cos \alpha = -El \cos \alpha.$$

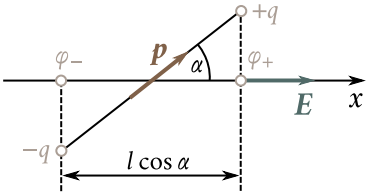


Fig. 1.13

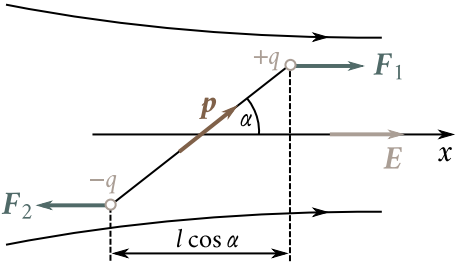


Fig. 1.14

Introducing this value into Eq. (1.59), we find that

$$W_{\mathrm{p}} = -qEl\cos\alpha = -pE\cos\alpha. \tag{1.60}$$

Here α is the angle between the vectors $\boldsymbol{p}$ and $\boldsymbol{E}$. We can therefore write Eq. (1.60) in the form

$$W_{\mathrm{p}} = -\boldsymbol{p}\cdot\boldsymbol{E}. \tag{1.61}$$

We must note that this expression takes no account of the energy of interaction of the charges $+q$ and $-q$ forming a dipole.

We have obtained Eq. (1.61) assuming for simplicity's sake that the field is homogeneous. This equation also holds, however, for an inhomogeneous field.

Let us consider a dipole in an inhomogeneous field that is symmetrical relative to the x-axis[5]. Let the centre of the dipole be on this axis, the dipole electric moment making with the axis an angle α, differing from $\pi/2$ (Fig. 1.14). In this case, the forces acting on the dipole charges are not identical in magnitude. Therefore, apart from the rotational moment (torque), the dipole will experience a force tending to move it in the direction of the x-axis. To find the value of this force, we shall use Eq. (1.40), according to which

$$F_x = -\frac{\partial W_{\mathrm{p}}}{\partial x}, \quad F_y = -\frac{\partial W_{\mathrm{p}}}{\partial y}, \quad F_z = -\frac{\partial W_{\mathrm{p}}}{\partial z}.$$

In view of Eq. (1.60), we can written

$$W_{\mathrm{p}}(x, y, z) = -pE(x, y, z)\cos\alpha$$

(we consider the orientation of the dipole relative to the vector $\boldsymbol{E}$ to be constant, α = constant).

For points on the x-axis, the derivatives of E with respect to y and z are zero. Accordingly, $\partial W_{\mathrm{p}}/\partial y = \partial W_{\mathrm{p}}/\partial z = 0$. Thus, only the force component F_x differs

[5]A particular case of such a field is that of a point charge if we take a straight line passing through the charge as the x-axis.

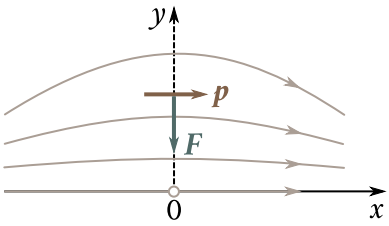


Fig. 1.15

from zero. It is

$$F_x = -\frac{\partial W_p}{\partial x} = p\frac{\partial E}{\partial x}\cos\alpha. \tag{1.62}$$

This result can be obtained if we take account of the fact that the field strength at the points where the charges $+q$ and $-q$ are (see Fig. 1.14) differs by the amount $(\partial E/\partial x)l\cos\alpha$. Accordingly, the difference between the forces acting on the charges is $q(\partial E/\partial x)l\cos\alpha$, which coincides with Eq. (1.62).

When α is less than $\pi/2$, the value of F_x determined by Eq. (1.62) is positive. This signifies that under the action of the force the dipole is pulled into the region of a stronger field (see Fig. 1.14). When α is greater than $\pi/2$, the dipole is pushed out of the field.

In the case shown in Fig. 1.15, only the derivative $\partial E/\partial y$ differs from zero for points on the y-axis. Therefore, the force acting on the dipole is determined by the component

$$F_y = -\frac{\partial W_p}{\partial y} = p\frac{\partial E}{\partial y}, \quad (\cos\alpha = 1).$$

The derivative $\partial E/\partial y$ is negative. Consequently, the force is directed as shown in the figure. Thus, in this case too, the dipole is pulled into the field.

We shall note that like $-\partial W_p/\partial x$ gives the projection of the force acting on the system onto the x-axis, so does the derivative of Eq. (1.60) with respect to α taken with the opposite sign give the projection of the torque onto the α-"axis": $T_\alpha = -pE\sin\alpha$. The minus sign was obtained because the α-"axis" and the torque T are directed oppositely (see Fig. 1.12).

1.10. Field of a System of Charges at Great Distances

Let us take a system of N charges $q_1, q_2, \ldots, q_N$ in a volume having linear dimensions of the order of l, and study the field set up by this system at distances r that are great in comparison with l ($r > l$). We take the origin of coordinates 0 inside the volume occupied by the system and shall determine the positions of the charges with the

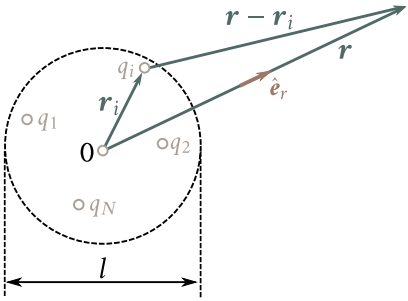


Fig. 1.16

aid of the position vectors $\boldsymbol{r}_i$, (Fig. 1.16; to simplify the figure, we have shown only the position vector of the i-th charge).

The potential at the point determined by the position vector $\boldsymbol{r}$ is

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|}. \tag{1.63}$$

Owing to the smallness of r_i in comparison with r, we can assume that

$$|\boldsymbol{r}-\boldsymbol{r}_i| = r - \boldsymbol{r}_i\cdot\hat{\boldsymbol{e}}_r = r\left(1-\frac{\boldsymbol{r}_i\cdot\hat{\boldsymbol{e}}_r}{r}\right)$$

[compare with Eqs. (1.47)]. Introduction of this expression into Eq. (1.63) yields

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r}\left[\frac{1}{1-(\boldsymbol{r}_i\cdot\hat{\boldsymbol{e}}_r/r)}\right]. \tag{1.64}$$

Using the formula

$$\frac{1}{1-x} \approx 1+x$$

which holds when $x \ll 1$, we can transform Eq. (1.64) as follows:

$$\begin{aligned}\varphi(\boldsymbol{r}) &= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r}\left(1+\frac{\boldsymbol{r}_i\cdot\hat{\boldsymbol{e}}_r}{r}\right)\\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\sum_{i=1}^{N}q_i + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(\sum_{i=1}^{N}q_i\boldsymbol{r}_i\right)\cdot\hat{\boldsymbol{e}}_r.\end{aligned} \tag{1.65}$$

The first term of the expression obtained is the potential of the field of a point charge having the value $q = \sum_i q_i$ [compare with Eq. (1.26)]. The second term has the same form as the expression determining the potential of a dipole field, the part of the electric moment of the dipole being played by the quantity

$$\boldsymbol{p} = \sum_{i=1}^{N} q_i\boldsymbol{r}_i. \tag{1.66}$$

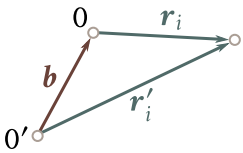
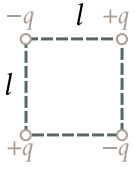
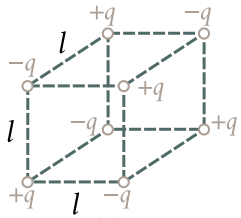


Fig. 1.17



(a)



(b)

Fig. 1.18

This quantity is called the **dipole electric moment** of a system of charges. It is easy to verify that for a dipole Eq. (1.66) transforms into the expression $\boldsymbol{p} = q\boldsymbol{l}$ which we are already familiar with.

If the total charge of a system is zero ($\sum_i q_i = 0$), the value of the dipole moment does not depend on our choice of the origin of coordinates. To convince ourselves that this is true, let us take two arbitrary origins of coordinates 0 and 0′ (Fig. 1.17). The position vectors of the i-th charge conducted from these points are related as follows:

$$\boldsymbol{r}'_i = \boldsymbol{b} + \boldsymbol{r}_i \tag{1.67}$$

(what the vector $\boldsymbol{b}$ is clear from the figure). With account taken of Eq. (1.67), the dipole moment in the system with the origin 0′ is

$$\boldsymbol{p}' = \sum_i q_i \boldsymbol{r}_i = \sum_i q_i(\boldsymbol{b} + \boldsymbol{r}_i) = \boldsymbol{b}\sum_i q_i + \sum_i q_i \boldsymbol{r}_i.$$

The first addend equals zero (because $\sum_i q_i = 0$). The second one is $\boldsymbol{p}$—the dipole moment in a coordinate system with its origin at 0. We have thus obtained that $\boldsymbol{p}' = \boldsymbol{p}$.

Equation (1.65) is in essence the first two terms of the series expansion of function (1.63) by powers of r_i/r. When $\sum_i q_i \neq 0$, the first term of Eq. (1.65) makes the main contribution to the potential (the second term diminishes in proportion to $1/r^2$ and is therefore much smaller than the first one). For an electrically neutral system ($\sum_i q_i = 0$), the first term equals zero, and the potential is determined mainly by the second term of Eq. (1.65). This is how matters stand, in particular, for the field of a dipole.

For the system of charges depicted in Fig. 1.18a and called a **quadrupole**, both $\sum_i q_i$ and $\boldsymbol{p}$ equal zero so that Eq. (1.65) gives a zero value of the potential. Actually, however, the field of a quadrupole, although it is much weaker than that of a dipole (with the same values of q and l), differs from zero. The potential of the field set up by a quadrupole is determined mainly by the third term of the expansion that is proportional to $1/r^3$. To obtain this term, we must take into consideration quantities

of the order of $(r_i/r)^2$ which we disregarded in deriving Eq. (1.65). For the system of charges shown in Fig. 1.18b and called an **octupole**, the third term of the expansion also equals zero. The potential of the field of such a system is determined by the fourth term of the expansion, which is proportional to $1/r^4$.

It must be noted that the quantity equal to $\sum_i q_i$ in the numerator of the first term of Eq. (1.65) is called a **monopole** or a **zero-order multipole**, a dipole is also called a **first-order multipole**, a quadrupole is called a **second-order multipole**, and so on.

Thus, in the general case, the field of a system of charges at great distances can be represented as the superposition of fields set up by multipoles of different orders—a monopole, dipole, quadrupole, octupole, etc.

1.11. A Description of the Properties of Vector Fields

To continue our study of the electric field, we must acquaint ourselves with the mathematical tools used to describe the properties of vector fields. These tools are called **vector analysis**. In the present section, we shall treat the fundamental concepts and selected formulas of vector analysis, and also prove its two main theorems—the Ostrogradsky-Gauss theorem (sometimes called Gauss's divergence theorem) and Stokes's theorem.

The quantities used in vector analysis can be best illustrated for the field of the velocity vector of a flowing liquid. We shall therefore introduce these quantities while dealing with the flow of an ideal incompressible liquid, and then extend the results obtained to vector fields of any nature.

We are already acquainted with one of the concepts of vector analysis. This is the **gradient**, used to characterize scalar fields. If the value of the scalar quantity $\varphi = \varphi(x, y, z)$ is compared with every point P having the coordinates x, y, z, we say that the scalar field of φ has been set. The gradient of the quantity φ is defined as the vector

$$\operatorname{grad} \varphi = \frac{\partial \varphi}{\partial x} \hat{\boldsymbol{e}}_x + \frac{\partial \varphi}{\partial y} \hat{\boldsymbol{e}}_y + \frac{\partial \varphi}{\partial z} \hat{\boldsymbol{e}}_z. \tag{1.68}$$

The increment of the function φ upon displacement over the length $\mathrm{d}\boldsymbol{l} = \hat{\boldsymbol{e}}_x\,\mathrm{d}x + \hat{\boldsymbol{e}}_y\,\mathrm{d}y + \hat{\boldsymbol{e}}_z\,\mathrm{d}z$ is

$$\mathrm{d}\varphi = \frac{\partial \varphi}{\partial x}\,\mathrm{d}x + \frac{\partial \varphi}{\partial y}\,\mathrm{d}y + \frac{\partial \varphi}{\partial z}\,\mathrm{d}z$$

which can be written in the form

$$\mathrm{d}\varphi = \operatorname{grad} \varphi \cdot \mathrm{d}\boldsymbol{l}. \tag{1.69}$$

Now we shall go over to establishing the characteristics of vector fields.

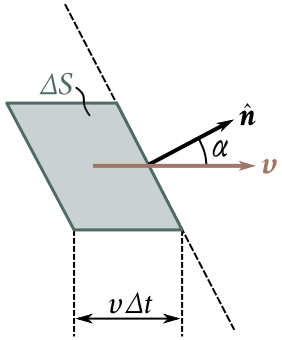


Fig. 1.19

Vector flux. Assume that the flow of a liquid is characterized by the field of the velocity vector. The volume of liquid flowing in unit time through an imaginary surface S is called the flux of the liquid through this surface. To find the flux, let us divide the surface into elementary sections of the size ΔS. It can be seen from Fig. 1.19 that during the time Δt a volume of liquid equal to

$$\Delta V = (\Delta S \cos\alpha) v \Delta t$$

will pass through section ΔS. Dividing this volume by the time Δt, we shall find the flux through surface ΔS:

$$\Delta\Phi = \frac{\Delta V}{\Delta t} = \Delta S v \cos\alpha.$$

Passing over to differentials, we find that

$$d\Phi = (v\cos\alpha)\, dS. \tag{1.70}$$

Equation (1.70) can be written in two other ways. First, if we take into account that $v\cos\alpha$ gives the projection of the velocity vector onto the normal $\hat{\boldsymbol{e}}_n$ to area dS, we can write Eq. (1.70) in the form

$$d\Phi = v_n\, dS. \tag{1.71}$$

Second, we can introduce the vector $d\boldsymbol{S}$ whose magnitude equals that of area dS, while its direction coincides with the direction of a normal $\hat{\boldsymbol{n}}$ to the area:

$$d\boldsymbol{S} = dS\,\hat{\boldsymbol{n}}.$$

Since the direction of the vector $\hat{\boldsymbol{n}}$ is chosen arbitrarily (it can be directed to either side of the area), then $d\boldsymbol{S}$ is not a true vector, but is a pseudo vector. The angle α in Eq. (1.70) is the angle between the vectors $\boldsymbol{v}$ and $d\boldsymbol{S}$. Hence, this equation can be written in the form

$$d\Phi = \boldsymbol{v}\cdot d\boldsymbol{S}. \tag{1.72}$$

By summating the fluxes through all the elementary areas into which we have

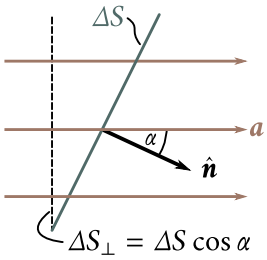


Fig. 1.20

divided surface S, we get the flux of the liquid through S:

$$\Phi_v = \int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_S v_n \, \mathrm{d}S. \tag{1.73}$$

A similar expression written for an arbitrary vector field $\boldsymbol{a}$, *i.e.*, the quantity

$$\Phi_a = \int_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \int_S a_n \, \mathrm{d}S \tag{1.74}$$

is called the **flux of the vector a through surface** S. In accordance with this definition, the flux of a liquid can be called the flux of the vector $\boldsymbol{v}$ through the relevant surface [see Eq. (1.73)].

The flux of a vector is an algebraic quantity. Its sign depends on the choice of the direction of a normal to the elementary areas into which surface S is divided in calculating the flux. Reversal of the direction of the normal changes the sign of a_n and, therefore, the sign of the quantity (1.74). The customary practice for closed surfaces is calculation of the flux "emerging outward" from the region enclosed by the surface. Accordingly, in the following we shall always implicate that $\hat{\boldsymbol{\nu}}$ is an outward normal.

We can give an illustrative geometrical interpretation of the vector flux. For this purpose, we shall represent a vector field by a system of lines $\boldsymbol{a}$ constructed so that the density of the lines at every point is numerically equal to the magnitude of the vector $\boldsymbol{a}$ at the same point of the field (compare with the rule for constructing the lines of the vector $\boldsymbol{E}$ set out at the end of Sec. 1.5). Let us find the number ΔN of intersections of the field lines with the imaginary area ΔS. A glance at Fig. 1.20 shows that this number equals the density of the lines (*i.e.*, a) multiplied by $\Delta S_\perp = \Delta S \cos\alpha$:

$$\Delta N \,(=)\, a\Delta S \cos\alpha = a_n \Delta S.$$

We are speaking only about the numerical equality between ΔN and $a_n \Delta S$. This is why the equality sign is confined in parentheses. According to Eq. (1.74), the expression $a_n \Delta S$ is $\Delta\Phi$—the flux of the vector a through area ΔS. Thus,

$$\Delta N \,(=)\, \Delta\Phi_a. \tag{1.75}$$

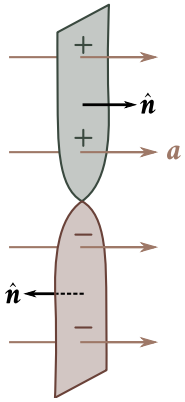


Fig. 1.21

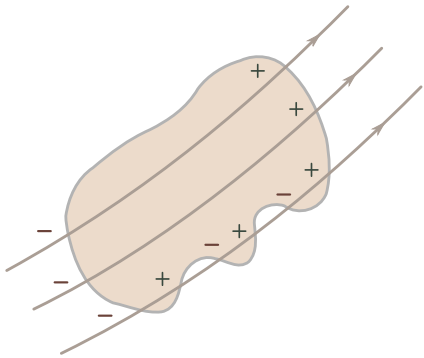


Fig. 1.22

For the sign of ΔN to coincide with that of $\Delta\Phi_a$, we must consider those intersections to be positive for which the angle α between the positive direction of a field line and a normal to the area is acute. The intersection should be considered negative if the angle α is obtuse. For the area shown in Fig. 1.20, all three intersections are positive: $\Delta N = +3$ ($\Delta\Phi_a$ in this case is also positive because $a_n > 0$). If the direction of the normal in Fig. 1.20 is reversed, the intersections will become negative ($\Delta N = -3$), and the flux $\Delta\Phi_a$ will also be negative.

Summation of Eq. (1.75) over the finite imaginary surface S yields

$$\Delta\Phi_a \,(=)\, \sum \Delta N = N_+ - N_- \tag{1.76}$$

where N_+ and N_- are the total number of positive and negative intersections of the field lines with surface S, respectively.

The reader may be puzzled by the circumstance that since the flux, as a rule, is expressed by a fractional number, the number of intersections of the field lines with a surface compared with the flux will also be fractional. Do not be confused by this, however. Field lines are a purely conditional image deprived of a physical meaning.

Let us take an imaginary surface in the form of a strip of paper whose bottom part is twisted relative to the top one through the angle π (Fig. 1.21). The direction of a normal must be chosen identically for the entire surface. Hence, if in the top part of the strip a positive normal is directed to the right, then in the bottom part a normal will be directed to the left. Accordingly, the intersections of the field lines depicted in Fig. 1.21 with the top half of the surface must be considered positive, and with the bottom half, negative.

An outward normal is considered to be positive for a closed surface (Fig. 1.22). Therefore, the intersections corresponding to outward protrusion of the lines (in this case the angle α is acute) must be taken with the plus sign, and the ones appearing

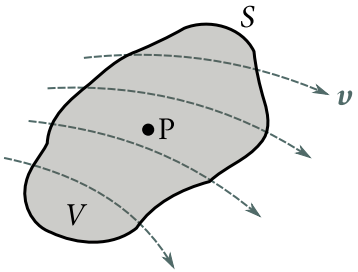


Fig. 1.23

when the lines enter the surface (in this case the angle α is obtuse) must be taken with the minus sign.

Inspection of Fig. 1.22 shows that when the field lines enter a closed surface continuously, each line when intersecting the surface enters it and emerges from it the same number of times. As a result, the flux of the corresponding vector through this surface equals zero. It is easy to see that if field lines end inside a surface, the vector flux through the closed surface will numerically equal the difference between the number of lines beginning inside the surface (N_{beg}) and the number of lines terminating inside the surface (N_{term}):

$$\Phi_a \,(=)\, N_{\text{beg}} - N_{\text{term}}. \tag{1.77}$$

The sign of the flux depends on which of these numbers is greater. When N_{beg} is equal to N_{term}, the flux equals zero.

Divergence. Assume that we are given the field of the velocity vector of an incompressible continuous liquid. Let us take an imaginary closed surface S in the vicinity of point P (Fig. 1.23). If in the volume confined by this surface no liquid appears and no liquid vanishes, then the flux flowing outward through the surface will evidently equal zero. A liquid flux Φ_v other than zero will indicate that there are liquid sources or sinks inside the surface, *i.e.*, points at which the liquid enters the volume (sources) or emerges from it (sinks). The magnitude of the flux determines the total algebraic power of the sources and sinks[6]. When the sources predominate over the sinks, the magnitude of the flux will be positive, and when the sinks predominate, negative.

The quotient obtained when dividing the flux Φ_v by the volume which it flows out from, *i.e.*,

$$\frac{\Phi_v}{V} \tag{1.78}$$

[6]The power of a source (sink) is defined as the volume of liquid discharged (absorbed) in unit time. A sink can be considered as a source with a negative power.

gives the average unit power of the sources confined in the volume V. In the limit when V tends to zero, *i.e.*, when the volume V contracts to point P, expression (1.78) gives the true unit power of the sources at point P, which is called the **divergence** of the vector $\boldsymbol{v}$ (it is designated by div $\boldsymbol{v}$). Thus, by definition,

$$\operatorname{div} \boldsymbol{v} = \lim_{V \to \mathrm{P}} \frac{\Phi_v}{V}.$$

The divergence of any vector $\boldsymbol{a}$ is determined in a similar way:

$$\operatorname{div} \boldsymbol{a} = \lim_{V \to \mathrm{P}} \frac{\Phi_v}{V} = \lim_{V \to \mathrm{P}} \oint \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S}. \tag{1.79}$$

The integral is taken over arbitrary closed surface S surrounding point P[7]; V is the volume confined by this surface. Since the transition $V \to \mathrm{P}$ is being performed upon which S tends to zero, we can assume that Eq. (1.79) cannot depend on the shape of the surface. This assumption is confirmed by strict calculations.

Let us surround point P with a spherical surface of an extremely small radius r (Fig. 1.24). Owing to the smallness of r, the volume V enclosed by the sphere will also be very small. We can therefore consider with a high degree of accuracy that the value of div $\boldsymbol{a}$ within the limits of the volume V is constant[8]. In this case, we can write in accordance with Eq. (1.79) that

$$\Phi_a \approx (\operatorname{div} \boldsymbol{a}) V$$

where Φ_a is the flux of the vector a through the surface surrounding the volume V. By Eq. (1.77), Φ_a equals N_{beg}, the number of lines of a beginning inside V if div $\boldsymbol{a}$ at point P is positive, or N_{term}, the number of lines of a terminating inside V if div $\boldsymbol{a}$ at point P is negative.

It follows from the above that the lines of the vector $\boldsymbol{a}$ begin in the closest vicinity of a point with a positive divergence. The field lines "diverge" from this point; the latter is the "source" of the field (Fig. 1.24a). On the other hand, in the vicinity of a point with a negative divergence, the lines of the vector $\boldsymbol{a}$ terminate. The field lines "converge" toward this point; the latter is the "sink" of the field (Fig. 1.24b). The greater the absolute value of div $\boldsymbol{a}$, the bigger is the number of lines that begin or terminate in the vicinity of the given point.

It can be seen from definition (1.79) that the divergence is a scalar function of the coordinates determining the positions of points in space (briefly—a point function). Definition (1.79) is the most general one that is independent of the kind of coordinate system used.

Let us find an expression for the divergence in a Cartesian coordinate system.

[7] The circle on the integral sign signifies that integration is performed over a closed surface.

[8] It is assumed that the value of div $\boldsymbol{a}$ changes continuously, without any jumps, when passing from one point of a field to another.

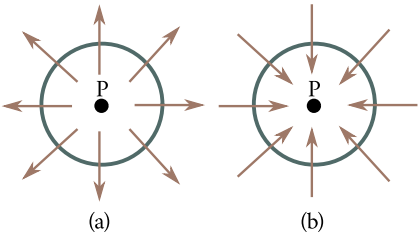


Fig. 1.24

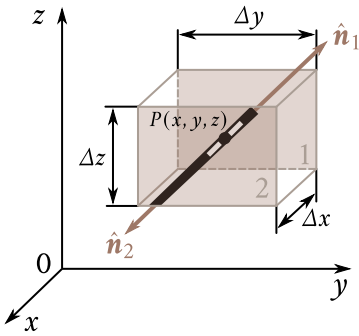


Fig. 1.25

We shall consider a small volume in the form of a parallelepiped with ribs parallel to the coordinate axes in the vicinity of point $P(x, y, z)$ (Fig. 1.25). The vector flux through the surface of the parallelepiped is formed from the fluxes passing through each of the six faces separately.

Let us find the flux through the pair of faces perpendicular to the x-axis (in Fig. 1.25 these faces are designated by shaded areas and by the numbers 1 and 2). The outward normal $\hat{\boldsymbol{n}}_2$ to face 2 coincides with the direction of the x-axis. Hence, for points of this face, $a_{n_2} = a_x$. The outward normal $\hat{\boldsymbol{n}}_1$ to face 1 is directed oppositely to the x-axis. Therefore, for points on this face, $a_{n_2} = -a_x$. The flux through face 2 can be written in the form

$$a_{x,2}\Delta y\Delta z$$

where $a_{x,2}$ is the value of a_x averaged over face 2. The flux through face 1 is

$$-a_{x,1}\Delta y\Delta z$$

where $a_{x,1}$ is the average value of a_x for face 1. The total flux through faces 1 and 2 is determined by the expression

$$(a_{x,2} - a_{x,1})\Delta y\Delta z. \tag{1.80}$$

The difference $a_{x,2} - a_{x,1}$ is the increment of the average (over a face) value of a_x upon a displacement along the x-axis by Δx. Owing to the smallness of the parallelepiped (we remind our reader that we shall let its dimensions shrink to zero), this increment can be written in the form $(\partial a_x/\partial x)\Delta x$, where the value $\partial a_x/\partial x$ is taken at point P[9]. Therefore, Eq. (1.80) becomes

$$\frac{\partial a_x}{\partial x}\Delta x\Delta y\Delta z = \frac{\partial a_x}{\partial x}\Delta V.$$

Similar reasoning allows us to obtain the following expressions for the fluxes

[9]The inaccuracy which we tolerate here vanishes when the volume shrinks to point P in the limit transition.

through the pairs of faces perpendicular to the y- and z-axes:

$$\frac{\partial a_y}{\partial y}\Delta x\Delta y\Delta z = \frac{\partial a_y}{\partial y}\Delta V, \qquad \frac{\partial a_z}{\partial z}\Delta x\Delta y\Delta z = \frac{\partial a_z}{\partial z}\Delta V.$$

Thus, the total flux through the entire close surface is determined by the expression

$$\Phi_a = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)\Delta V.$$

Dividing this expression by ΔV, we shall find the divergence of the vector a at point $P(x, y, z)$:

$$\operatorname{div} \boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}. \tag{1.81}$$

The Ostrogradsky-Gauss Theorem. If we know the divergence of the vector $\boldsymbol{a}$ at every point of space, we can calculate the flux of this vector through any closed surface of finite dimensions. Let us first do this for the flux of the vector $\boldsymbol{v}$ (a liquid flux). The product of $\operatorname{div} \boldsymbol{v}$ and $\mathrm{d}V$ gives the power of the sources of the liquid confined within the volume $\mathrm{d}V$. The sum of such products, *i.e.*, $\int (\operatorname{div} \boldsymbol{v})\, \mathrm{d}V$, gives the total algebraic power of the sources confined in the volume V over which integration is performed. Owing to incompressibility of the liquid, the total power of the sources must equal the liquid flux emerging through surface S enclosing the volume V. We thus arrive at the equation

$$\oint_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_V (\operatorname{div} \boldsymbol{v})\, \mathrm{d}V.$$

A similar equation holds for a vector field of any nature:

$$\oint_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \int_V (\operatorname{div} \boldsymbol{a})\, \mathrm{d}V. \tag{1.82}$$

This relation is called the **Ostrogradsky-Gauss** theorem. The integral in the left-hand side of the equation is calculated over an arbitrary closed surface S, and the integral in the right-hand side over the volume V enclosed by this surface.

Circulation. Let us revert to the flow of an ideal incompressible liquid. Imagine a closed line—the contour Γ. Assume that in some way or other we have instantaneously frozen the liquid in the entire volume except for a very thin closed channel of constant cross section including the contour Γ (Fig. 1.26). Depending on the nature of the velocity vector field, the liquid in the channel formed will either be stationary or move along the contour (circulate) in one of the two possible directions. Let us take the quantity equal to the product of the velocity of the liquid in the channel and the length of the contour l as a measure of this motion. This

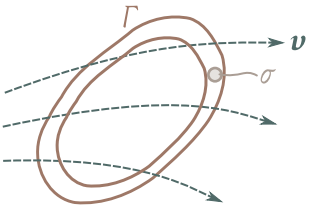


Fig. 1.26

quantity is called the **circulation** of the vector $\boldsymbol{v}$ around the contour Γ. Thus,

$$\text{circulation of } \boldsymbol{v} \text{ around } \Gamma = vl$$

(since we assumed that the channel has a constant cross section, the magnitude of the velocity, v, is a constant).

At the moment when the walls freeze, the velocity component perpendicular to a wall will be eliminated in each of the liquid particles, and only the velocity component tangent to the contour will remain, *i.e.*, v_l. The momentum $\mathrm{d}\boldsymbol{p}_l$, is associated with this component. The magnitude of the momentum for a liquid particle contained within a segment of the channel of length $\mathrm{d}l$ is $\rho\sigma v_l\,\mathrm{d}l$ (ρ is the density of the liquid, and σ is the cross-sectional area of the channel). Since the liquid is ideal, the action of the walls can change only the direction of the vector $\mathrm{d}\boldsymbol{p}_l$, but not its magnitude. The interaction between the liquid particles will cause a redistribution of the momentum between them that will level out the velocities of all the particles. The algebraic sum of the tangential components of the momenta cannot change: the momentum acquired by one of the interacting particles equals the momentum lost by the second particle. This signifies that

$$\rho\sigma vl = \oint_\Gamma \rho\sigma v_l\,\mathrm{d}l$$

where v is the circulation velocity, and v_l is the tangential component of the liquid's velocity in the volume $\sigma\,\mathrm{d}l$ at the moment of time preceding the freezing of the channel walls. Cancelling $\rho\sigma$, we get

$$\text{circulation of } \boldsymbol{v} \text{ around } \Gamma = vl = \oint_\Gamma v_l\,\mathrm{d}l.$$

The circulation of any vector $\boldsymbol{a}$ around an arbitrary closed contour Γ is determined in a similar way:

$$\text{circulation of } \boldsymbol{a} \text{ around } \Gamma = \oint_\Gamma \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} = \oint_\Gamma a_l\,\mathrm{d}l. \tag{1.83}$$

It may seem that for the circulation to be other than zero the vector lines must be closed or at least bent in some way or other in the direction of circumventing the contour. It is easy to see that this assumption is wrong. Let us consider the laminar

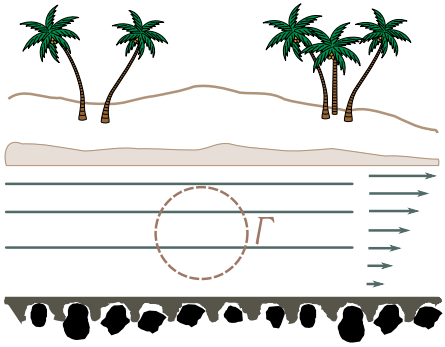


Fig. 1.27

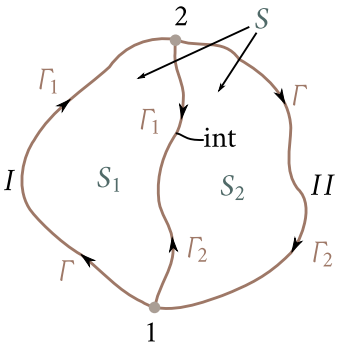


Fig. 1.28

flow of water in a river. The velocity of the water directly at the river bottom is zero and grows as we approach the surface of the water (Fig. 1.27). The streamlines (lines of the vector $\boldsymbol{v}$) are straight. Notwithstanding this fact, the circulation of the vector $\boldsymbol{v}$ around the contour depicted by the dash line obviously differs from zero. On the other hand, in a field with curved lines, the circulation may equal zero.

Circulation has the property of additivity. This signifies that the sum of the circulations around contours Γ_1 and Γ_2 enclosing neighboring surfaces S_1 and S_2 (Fig. 1.28) equals the circulation around contour Γ enclosing surface S, which is the sum of surfaces S_1 and S_2. Indeed, the circulation C_1 around the contour bounding surface S_1 can be represented as the sum of the integrals

$$C_1 = \oint_{\Gamma_1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_{1,(I)}^{2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} + \int_{2,(\mathrm{int.})}^{1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.84}$$

The first integral is taken over section I of the outer contour, the second over the interface between surfaces S_1 and S_2 in direction 2-1.

Similarly, the circulation C_2 around the contour enclosing surface S_2 is

$$C_2 = \oint_{\Gamma_2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_{2,(II)}^{1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} + \int_{1,(\mathrm{int.})}^{2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.85}$$

The first integral is taken over section II of the outer contour, the second over the interface between surfaces S_1 and S_2 in direction 1-2.

The circulation around the contour bounding total surface S can be represented in the form

$$C = \oint_{\Gamma} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_{1,(I)}^{2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} + \int_{2,(II)}^{1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.86}$$

The second addends in Eqs. (1.84) and (1.85) differ only in their sign. Therefore, the sum of these expressions will equal Eq. (1.86). Thus,

$$C = C_1 + C_2. \tag{1.87}$$

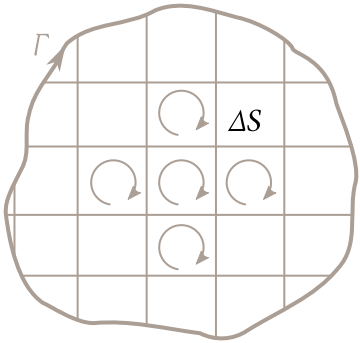


Fig. 1.29

Equation (1.87) which we have proved does not depend on the shape of the surfaces and holds for any number of addends. Hence, if we divide an arbitrary open surface S into a great number of elementary surfaces ΔS[10] (Fig. 1.29), then the circulation around the contour enclosing S can be written as the sum of the elementary circulations ΔC around the contours enclosing the ΔS's:

$$C = \sum_i \Delta C_i. \tag{1.88}$$

Curl. The additivity of the circulation permits us to introduce the concept of unit circulation, *i.e.*, consider the ratio of the circulation C to the magnitude of surface S around which the circulation "flows". When surface S is finite, the ratio C/S gives the mean value of the unit circulation. This value characterizes the properties of a field averaged over surface S. To obtain the characteristic of the field at point P, we must reduce the dimensions of the surface, making it shrink to point P. The ratio C/S tends to a limit that characterizes the properties of the field at point P.

Thus, let us take an imaginary contour Γ in a plane passing through point P, and consider the expression

$$\lim_{S \to P} \frac{C_a}{S} \tag{1.89}$$

where C_a is the circulation of the vector $\boldsymbol{a}$ around the contour Γ and S is the surface area enclosed by the contour.

Limit (1.89) calculated for an arbitrarily oriented plane cannot be an exhaustive characteristic of the field at point P because the magnitude of this limit depends on the orientation of the contour in space in addition to the properties of the field

[10] In the figure, the elementary surfaces are depicted in the form of rectangles. Actually, their shape may be absolutely arbitrary.

at point P. This orientation can be given by the direction of a positive normal $\hat{\boldsymbol{n}}$ to the plane of the contour (a positive normal is one that is associated with the direction of circumvention of the contour in integration by the right-hand screw rule). In determining limit (1.89) at the same point P for different directions $\hat{\boldsymbol{n}}$, we shall obtain different values. For opposite directions, these values will differ only in their sign (reversal of the direction $\hat{\boldsymbol{n}}$ is equivalent to reversing the direction of circumvention of the contour in integration, which only causes a change in the sign of the circulation). For a certain direction of the normal, the magnitude of expression (1.89) at the given point will be maximum.

Thus, quantity (1.89) behaves like the projection of a vector onto the direction of a normal to the plane of the contour around which the circulation is taken. The maximum value of quantity (1.89) determines the magnitude of this vector, and the direction of the positive normal $\hat{\boldsymbol{n}}$ at which the maximum is reached gives the direction of the vector. This vector is called the **curl** of the vector $\boldsymbol{a}$. Its symbol is curl $\boldsymbol{a}$. Using this notation, we can write expression (1.89) in the form

$$(\operatorname{curl} \boldsymbol{a})_n = \lim_{S \to \mathrm{P}} \frac{C_a}{S} = \lim_{S \to \mathrm{P}} \frac{1}{S} \oint_S \boldsymbol{a}\, \mathrm{d}\boldsymbol{l}. \tag{1.90}$$

We can obtain a graphical picture of the curl of the vector $\boldsymbol{v}$ by imagining a small and light fan impeller placed at the given point of a flowing liquid (Fig. 1.30). At the spots where the curl differs from zero, the impeller will rotate, its velocity being the higher, the greater in value is the projection of the curl onto the impeller axis.

Equation (1.90) defines the vector curl $\boldsymbol{a}$. This definition is a most general one that does not depend on the kind of coordinate system used. To find expressions for the projections of the vector curl $\boldsymbol{a}$ onto the axes of a Cartesian coordinate system, we must determine the values of quantity (1.90) for such orientations of area S for which the normal $\hat{\boldsymbol{n}}$ to the area coincides with one of the axes x, y, z. If, for example, we direct $\hat{\boldsymbol{n}}$ along the x-axis, then (1.90) becomes $(\operatorname{curl} \boldsymbol{a})_x$. Contour Γ in this case is arranged in a plane parallel to the coordinate plane yz. Let us take this contour in the form of a rectangle with the sides Δy and Δz (Fig. 1.31, the x-axis is directed toward us in this figure; the direction of circumvention indicated in the figure is associated with the direction of the x-axis by the right-hand screw rule). Section 1 of the contour is opposite in direction to the z-axis. Therefore, a_l on this section coincides with $-a_z$. Similar reasoning shows that a_l on sections 2, 3, and 4 equals a_y, a_z, and $-a_y$, respectively. Hence, the circulation can be written in the form

$$\left(a_{z,3} - a_{z,1}\right) \Delta z - \left(a_{y,4} - a_{y,2}\right) \Delta y \tag{1.91}$$

where $a_{z,3}$ and $a_{z,1}$ are the average values of a_z on sections 3 and 1, respectively, and $a_{y,4}$ and $a_{y,2}$ are the average values of a_y on sections 4 and 2.

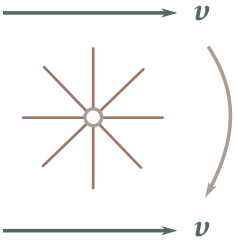


Fig. 1.30

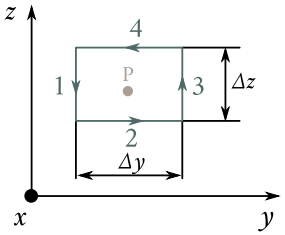


Fig. 1.31

The difference $a_{z,3}-a_{z,1}$ is the increment of the average value of a_z on the section Δz when this section is displaced in the direction of the y-axis by Δy. Owing to the smallness of Δy and Δz, this increment can be represented in the form $(\partial a_z/\partial y)\Delta y$, where the value of $\partial a_z/\partial y$ is taken for point P[11]. Similarly, the difference $a_{y,4} - a_{y,2}$ can be represented in the form $(\partial a_y/\partial z)\Delta z$. Using these expressions in Eq. (1.91) and putting the common factor outside the parentheses, we get the following expression for the circulation:

$$\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\Delta y\,\Delta z = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\Delta S$$

where ΔS is the area of the contour. Dividing the circulation by ΔS, we find the expression for the projection of curl $\boldsymbol{a}$ onto the x-axis:

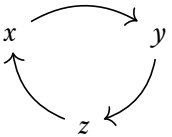
$$(\operatorname{curl}\boldsymbol{a})_x = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}. \tag{1.92}$$

We can find by similar reasoning that

$$(\operatorname{curl}\boldsymbol{a})_y = \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \tag{1.93}$$

$$(\operatorname{curl}\boldsymbol{a})_z = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}. \tag{1.94}$$

It is easy to see that any of the equations (1.92)-(1.94) can be obtained from the preceding one [Eq. (1.94) should be considered as the preceding one for Eq. (1.94)] by the so-called cyclic transposition of the coordinates, *i.e.*, by replacing the coordinates according to the scheme



Thus, the curl of the vector $\boldsymbol{a}$ is determined in the Cartesian coordinate system

[11]The inaccuracy which we tolerate here vanishes when the contour shrinks to point P in the limit transition.

by the following expression:

$$\operatorname{curl} \boldsymbol{a} = \hat{\boldsymbol{e}}_x \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \hat{\boldsymbol{e}}_y \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) + \hat{\boldsymbol{e}}_z \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right). \tag{1.95}$$

Below we shall indicate a more elegant way of writing this expression.

Stokes' Theorem. Knowing the curl of the vector $\boldsymbol{a}$ at every point of surface S (not necessarily plane), we can calculate the circulation of this vector around contour Γ enclosing S (the contour may also not be plane). For this purpose, we divide the surface into very small elements ΔS. Owing to their smallness, these elements can be considered as plane. Therefore in accordance with Eq. (1.90), the circulation of the vector $\boldsymbol{a}$ around the contour bounding ΔS can be written in the form

$$\Delta C \approx (\operatorname{curl} \boldsymbol{a})_n \Delta S = \operatorname{curl} \boldsymbol{a} \cdot \Delta \boldsymbol{S} \tag{1.96}$$

where $\hat{\boldsymbol{n}}$ is a positive normal to surface element ΔS.

In accordance with Eq. (1.88), summation of expression (1.96) over all the ΔS's yields the circulation of the vector $\boldsymbol{a}$ around contour Γ enclosing S:

$$C = \sum \Delta C \approx \sum \operatorname{curl} \boldsymbol{a} \cdot \Delta \boldsymbol{S}.$$

Performing a limit transition in which all the ΔS's shrink to zero (their number grows unlimitedly), we arrive at the equation

$$\oint_\Gamma \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\operatorname{curl} \boldsymbol{a}) \cdot \Delta \boldsymbol{S}. \tag{1.97}$$

Equation (1.97) is called **Stokes' theorem**. Its meaning is that the circulation of the vector $\boldsymbol{a}$ around an arbitrary contour Γ equals the flux of the vector curl $\boldsymbol{a}$ through the arbitrary surface S surrounded by the given contour.

The Del Operator. Writing of the formulas of vector analysis is simplified quite considerably if we introduce a vector differential operator designated by the symbol ∇ (nabla or del) and called the **del operator** or the **Hamiltonian operator**. This operator denotes a vector with the components $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$. Consequently,

$$\nabla = \hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}. \tag{1.98}$$

This vector has no meaning by itself. It acquires a meaning in combination with the scalar or vector function by which it is symbolically multiplied. Thus, if we multiply the vector ∇ by the scalar φ we obtain the vector

$$\nabla \varphi = \hat{\boldsymbol{e}}_x \frac{\partial \varphi}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial \varphi}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial \varphi}{\partial z} \tag{1.99}$$

which is the gradient of the function φ [see Eq. (1.68)].

The scalar product of the vectors ∇ and $\boldsymbol{a}$ gives the scalar

$$\nabla \cdot \boldsymbol{a} = \nabla_x a_x + \nabla_y a_y + \nabla_z a_z \tag{1.100}$$

which we can see to be the divergence of the vector $\boldsymbol{a}$ [see Eq. (1.81)].

Finally, the vector product of the vectors ∇ and $\boldsymbol{a}$ gives a vector with the components $(\nabla \times \boldsymbol{a})_x = \nabla_y a_z - \nabla_z a_y = \partial a_z/\partial y - \partial a_y/\partial z$, etc., that coincide with the components of curl $\boldsymbol{a}$ [see Eqs. (1.92)-(1.94)]. Hence, using the writing of a vector product with the aid of a determinant, we have

$$\operatorname{curl} \boldsymbol{a} = \nabla \times \boldsymbol{a} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}. \tag{1.101}$$

Thus, there are two ways of denoting the gradient, divergence, and curl:

$$\nabla\varphi \equiv \operatorname{grad} \varphi, \quad \nabla \cdot \boldsymbol{a} \equiv \operatorname{div} \boldsymbol{a}, \quad \nabla \times \boldsymbol{a} \equiv \operatorname{curl} \boldsymbol{a}.$$

The use of the del symbol has a number of advantages. We shall therefore use such symbols in the following. One must accustom oneself to identify the symbol $\nabla\varphi$ with the words "gradient of phi" (*i.e.*, to say not "del phi", but "gradient of phi"), the symbol $\nabla \cdot \boldsymbol{a}$ with the words "divergence of a" and, finally, the symbol $\nabla \times \boldsymbol{a}$ with the words "curl of a".

When using the vector ∇, one must remember that it is a differential operator acting on all the functions to the right of it. Consequently, in transforming expressions including ∇, one must take into consideration both the rules of vector algebra and those of differential calculus. For example, the derivative of the product of the functions φ and ψ is

$$(\varphi\psi)' = \varphi'\psi + \varphi\psi'.$$

Accordingly,

$$\operatorname{grad}(\varphi\psi) = \nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi = \psi \operatorname{grad} \varphi + \varphi \operatorname{grad} \psi. \tag{1.102}$$

Similarly,

$$\operatorname{div}(\varphi\boldsymbol{a}) = \nabla \cdot (\varphi\boldsymbol{a}) = \boldsymbol{a} \cdot (\nabla\varphi) + \varphi(\nabla \cdot \boldsymbol{a}). \tag{1.103}$$

The gradient of a function φ is a vector function. Therefore, the divergence and curl operations can be performed with it:

$$\begin{aligned} \operatorname{div} \operatorname{grad} \varphi = \nabla \cdot \nabla\varphi = (\nabla \cdot \nabla)\varphi = \left(\nabla_x^2 + \nabla_y^2 + \nabla_z^2\right)\varphi \\ = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \Delta\varphi \end{aligned} \tag{1.104}$$

(Δ is the Laplacian operator)

$$\operatorname{curl} \operatorname{grad} \varphi = \nabla \times (\nabla\varphi) = (\nabla \times \nabla)\varphi \tag{1.105}$$

(we remind our reader that the vector product of a vector and itself is zero).

Let us apply the divergence and curl operations to the function curl $\boldsymbol{a}$:

$$\operatorname{div}\operatorname{curl}\boldsymbol{a} = \nabla\cdot\nabla\times\boldsymbol{a} = 0 \tag{1.106}$$

(a scalar triple product equals the volume of a parallelepiped constructed on the vectors being multiplied (see Vol. I, p. 22); if two of these vectors coincide, the volume of the parallelepiped equals zero):

$$\operatorname{curl}\operatorname{curl}\boldsymbol{a} = \nabla\times(\nabla\times\boldsymbol{a}) = \nabla(\nabla\cdot\boldsymbol{a}) - (\nabla\cdot\nabla)\boldsymbol{a} = \operatorname{grad}\operatorname{div}\boldsymbol{a} - \Delta\boldsymbol{a} \tag{1.107}$$

[we have used Eq. (1.35) of Vol. I, namely, $\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$].

Equation (1.106) signifies that the field of a curl has no sources. Hence, the lines of the vector curl $\boldsymbol{a}$ have neither a beginning nor an end. It is exactly for this reason that the flux of a curl through any surface S resting on the given contour Γ is the same [see Eq. (1.97)].

We shall note in concluding that when the del operator is used, Eqs. (1.82) and (1.97) can be given the form

$$\oint_S \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} = \oint_V \nabla\cdot\boldsymbol{a}\,\mathrm{d}V, \qquad \text{(the Ostrogradsky-Gauss theorem)} \tag{1.108}$$

$$\oint_\Gamma \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} = \int_S (\nabla\times\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S}. \qquad \text{(Stokes' theorem)} \tag{1.109}$$

1.12. Circulation and Curl of an Electrostatic Field

We established in Sec. 1.6 that the forces acting on the charge q in an electrostatic field are conservative. Hence, the work of these forces on any closed path Γ is zero:

$$A = \oint_\Gamma q\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = 0.$$

Cancelling q, we get

$$\oint_\Gamma \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = 0 \tag{1.110}$$

(compare with Eq. (1.46)].

The integral in the left-hand side of Eq. (1.110) is the circulation of the vector $\boldsymbol{E}$ around contour Γ [see expression (1.80)]. Thus, *an electrostatic field is characterized by the fact that the circulation of the strength (intensity) vector of this field around any closed contour equals zero*.

Let us take an arbitrary surface S resting on contour Γ for which the circulation is calculated (Fig. 1.32). According to Stokes's theorem [see Eq. (1.109)], the integral of curl $\boldsymbol{E}$ taken over this surface equals the circulation of the vector $\boldsymbol{E}$ around contour

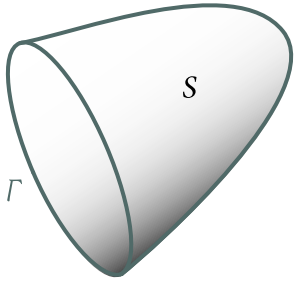


Fig. 1.32

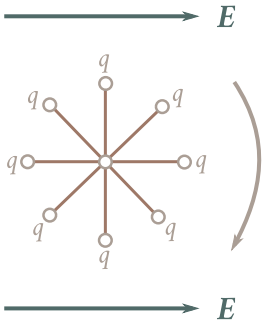


Fig. 1.33

Γ:

$$\int_S (\nabla \times \boldsymbol{E}) \cdot \mathrm{d}\boldsymbol{S} = \oint_\Gamma \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.111}$$

Since the circulation equals zero, we arrive at the conclusion that

$$\int_S (\nabla \times \boldsymbol{E}) \cdot \mathrm{d}\boldsymbol{S} = 0.$$

This condition must be observed for any surface S resting on arbitrary contour Γ. This is possible only if the curl of the vector $\boldsymbol{E}$ at every point of the field equals zero:

$$\nabla \times \boldsymbol{E} = 0. \tag{1.112}$$

By analogy with the fan impeller shown in Fig. 1.25, let us imagine an electrical "impeller" in the form of a light hub with spokes whose ends carry identical positive charges q (Fig. 1.33; the entire arrangement must be small in size). At the points of an electric field where curl $\boldsymbol{E}$ differs from zero, such an impeller would rotate with an acceleration that is the greater, the larger is the projection of the curl onto the impeller axis. For an electrostatic field, such an imaginary arrangement would not rotate with any orientation of its axis.

Thus, a feature of an electrostatic field is that it is a non-circuital one. We established in the preceding section that the curl of the gradient of a scalar function equals zero [see expression (1.96)]. Therefore, the equality to zero of curl $\boldsymbol{E}$ at every point of a field makes it possible to represent $\boldsymbol{E}$ in the form of the gradient of a scalar function φ called the potential. We have already considered this representation in Sec. 1.8 [see Eq. (1.41); the minus sign in this equation was taken from physical considerations].

We can immediately conclude from the need to observe condition (1.110) that the existence of an electrostatic field of the kind shown in Fig. 1.34 is impossible. Indeed, for such a field, the circulation around the contour shown by the dash line

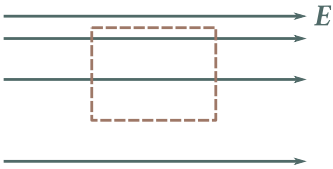


Fig. 1.34

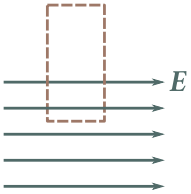


Fig. 1.35

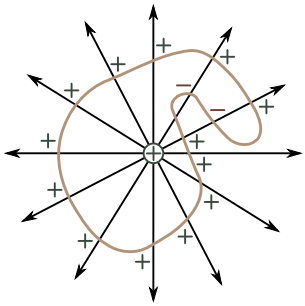


Fig. 1.36

would differ from zero, which contradicts condition (1.110). It is also impossible for a field differing from zero in a restricted volume to be homogeneous throughout this volume (Fig. 1.35). In this case, the circulation around the contour shown by the dash line would differ from zero.

1.13. Gauss's Theorem

We established in the preceding section what the curl of an electrostatic field equals. Now let us find the divergence of a field. For this purpose, we shall consider the field of a point charge q and calculate the flux of the vector $\boldsymbol{E}$ through closed surface S surrounding the charge (Fig. 1.36). We showed in Sec. 1.5 that the number of lines of the vector $\boldsymbol{E}$ beginning at a point charge $+q$ or terminating at a charge $-q$ numerically equals q/ε_0.

By Eq. (1.77), the flux of the vector $\boldsymbol{E}$ through any closed surface equals the number of lines coming out, *i.e.*, beginning on the charge, if it is positive, and the number of lines entering the surface, *i.e.*, terminating on the charge, if it is negative. Taking into account that the number of lines beginning or terminating at a point charge numerically equals q/ε_0 (see Sec. 1.5), we can write that

$$\Phi_E = \frac{q}{\varepsilon_0}. \tag{1.113}$$

The sign of the flux coincides with that of the charge q. The dimensions of both sides of Eq. (1.113) are identical.

Now let us assume that a closed surface surrounds N point charges $q_1, q_2, \ldots, q_N$. On the basis of the superposition principle, the strength $\boldsymbol{E}$ of the field set up by all the charges equals the sum of the strengths $\boldsymbol{E}_i$ set up by each charge separately: $\boldsymbol{E} = \sum_i \boldsymbol{E}_i$. Hence,

$$\Phi_E = \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \oint_S \left(\sum_i \boldsymbol{E}_i \right) \cdot \mathrm{d}\boldsymbol{S} = \sum_i \oint_S \boldsymbol{E}_i \cdot \mathrm{d}\boldsymbol{S}.$$

Each of the integrals inside the sum sign equals q_i/ε_0. Therefore,

$$\Phi_E = \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0} \sum_{i=1}^{N} q_i. \tag{1.114}$$

The statement we have proved is called **Gauss's theorem**. According to it, *the flux of an electric field strength vector through a closed surface equals the algebraic sum of the charges enclosed by this surface divided by* ε_0.

When considering fields set up by macroscopic charges (*i.e.*, charges formed by an enormous number of elementary charges), the discrete structure of these charges is disregarded, and they are considered to be distributed in space continuously with a finite density everywhere. The **volume density of a charge** ρ is determined by analogy with the density of a mass as the ratio of the charge $\mathrm{d}q$ to the infinitely small (physically) volume $\mathrm{d}V$ containing this charge:

$$\rho = \frac{\mathrm{d}q}{\mathrm{d}V}. \tag{1.115}$$

In the given case by an infinitely small (physically) volume, we must understand a volume which on the one hand is sufficiently small for the density within its limits to be considered identical, and on the other is sufficiently great for the discreteness of the charge not to manifest itself.

Knowing the charge density at every point of space, we can find the total charge surrounded by closed surface S. For this purpose, we must calculate the integral of ρ with respect to the volume enclosed by the surface:

$$\sum_i q_i = \int_V \rho \, \mathrm{d}V.$$

Thus, Eq. (1.114) can be written in the form

$$\oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0} \int_V \rho \, \mathrm{d}V. \tag{1.116}$$

Replacing the surface integral with a volume one in accordance with Eq. (1.108), we have

$$\int_V \nabla \cdot \boldsymbol{E} \, \mathrm{d}V = \frac{1}{\varepsilon_0} \int_V \rho \, \mathrm{d}V.$$

The relation which we have arrived at must be observed for any arbitrarily chosen volume V. This is possible only if the values of the integrands for every point of space are the same. Hence, the divergence of the vector $\boldsymbol{E}$ is associated with the density of the charge at the same point by the equation

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0} \rho. \tag{1.117}$$

This equation expresses Gauss's theorem in the differential form.

For a flowing liquid, $\nabla \cdot \boldsymbol{v}$ gives the unit power of the sources of the liquid at a given point. By analogy, charges are said to be sources of an electric field.

1.14. Calculating Fields with the Aid of Gauss's Theorem

Gauss's theorem permits us in a number of cases to find the strength of a field in a much simpler way than by using Eq. (1.15) for the field strength of a point charge and the field superposition principle. We shall demonstrate the possibilities of Gauss's theorem by employing a few examples that will be useful for our further exposition. Before starting on our way, we shall introduce the concepts of surface and linear charge densities.

If a charge is concentrated in a thin surface layer of the body carrying the charge, the distribution of the charge in space can be characterized by the surface density σ, which is determined by the expression

$$\sigma = \frac{\mathrm{d}q}{\mathrm{d}S}. \tag{1.118}$$

Here $\mathrm{d}q$ is the charge contained in the layer of area $\mathrm{d}S$. By $\mathrm{d}S$ is meant an infinitely small (physically) section of the surface.

If a charge is distributed over the volume or surface of a cylindrical body (uniformly in each section), the linear charge density is used, *i.e.*,

$$\lambda = \frac{\mathrm{d}q}{\mathrm{d}l} \tag{1.119}$$

where $\mathrm{d}l$ is the length of an infinitely small (physically) segment of the cylinder, and $\mathrm{d}q$ is the charge concentrated on this segment.

Field of an Infinite Homogeneously Charged Plane. Assume that the surface charge density at all points of a plane is identical and equal to σ; for definiteness we shall consider the charge to be positive. It follows from considerations of symmetry that the field strength at any point is directed at right angles to the plane. Indeed, since the plane is infinite and charged homogeneously, there is no reason why the vector $\boldsymbol{E}$ should deflect to a side from a normal to the plane. It is further evident that at points symmetrical relative to the plane, the field strength is identical in magnitude and opposite in direction.

Let us imagine mentally a cylindrical surface with generatrices perpendicular to the plane and bases of a size ΔS arranged symmetrically relative to the plane (Fig. 1.37). Owing to symmetry, we have $E' = E'' = E$. We shall apply Gauss's theorem to the surface. The flux through the side part of the surface will be absent because E_n at each point of it is zero. For the bases, E_n coincides with E. Hence, the total flux through the surface is $2E\Delta S$. The surface encloses the charge $\sigma \Delta S$.

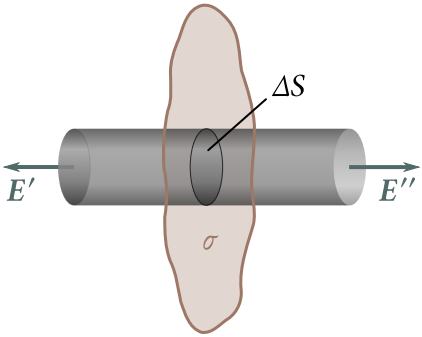


Fig. 1.37

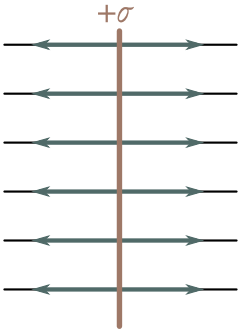


Fig. 1.38

According to Gauss's theorem, the condition must be observed that

$$2E\Delta S = \frac{\sigma \Delta S}{\varepsilon_0}$$

whence

$$E = \frac{\sigma}{2\varepsilon_0}. \tag{1.120}$$

The result we have obtained does not depend on the length of the cylinder. This signifies that at any distances from the plane, the field strength is identical in magnitude. The field lines are shown in Fig. 1.38. For a negatively charged plane, the result will be the same except for the reversal of the direction of the vector $\boldsymbol{E}$ and the field lines.

If we take a plane of finite dimensions, for instance a charged thin plate[12], then the result obtained above will hold only for points, the distance to which from the edge of the plate considerably exceeds the distance from the plate itself (in Fig. 1.39 the region containing such points is outlined by a dash line). At points at an increasing distance from the plane or approaching its edges, the field will differ more and more from that of an infinitely charged plane. It is easy to imagine the nature of the field at great distances if we take into account that at distances considerably exceeding the dimensions of the plate, the field it sets up can be treated as that of a point charge.

Field of Two Uniformly Charged Planes. The field of two parallel infinite planes carrying opposite charges with a constant surface density σ identical in magnitude can be found by superposition of the fields produced by each plane

[12] For a plate, by σ in Eq. (1.120) should be understood the charge concentrated on 1 m^2 of the plate over its entire thickness. In metal bodies, the charge is distributed over the external surface. Therefore by σ we should understand the double value of the charge density on the surfaces surrounding the metal plate.

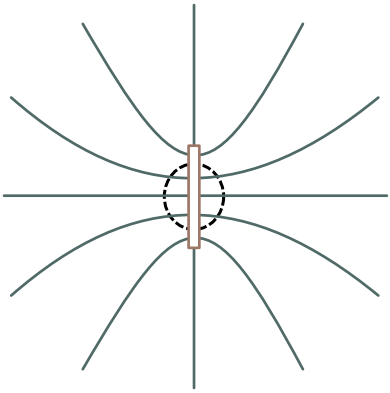


Fig. 1.39

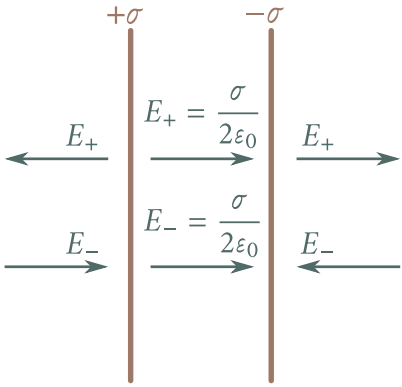


Fig. 1.40

separately (Fig. 1.40). In the region between the planes, the fields being added have the same direction, so that the resultant field strength is

$$E = \frac{\sigma}{\varepsilon_0}. \tag{1.121}$$

Outside the volume bounded by the planes, the fields being added have opposite directions so that the resultant field strength equals zero.

Thus, the field is concentrated between the planes. The field strength at all points of this region is identical in value and in direction; consequently, the field is homogeneous. The field lines are a collection of parallel equispaced straight lines.

The result we have obtained also holds approximately for planes of finite dimensions if the distance between them is much smaller than their linear dimensions (a parallel-plate capacitor). In this case, appreciable deviations of the field from homogeneity are observed only near the edges of the plates (Fig. 1.41).

Field of an Infinite Charged Cylinder. Assume that the field is produced by an infinite cylindrical surface of radius R whose charge has a constant surface density σ. Considerations of symmetry show that the field strength at any point must be directed along a radial line perpendicular to the cylinder axis, and that the magnitude of the strength can depend only on the distance r from the cylinder axis. Let us mentally imagine a coaxial closed cylindrical surface of radius r and height h with a charged surface (Fig. 1.42). For the bases of the cylinder, we have $E_n = 0$, for the side surface $E_n = E(r)$ (the charge is assumed to be positive). Hence, the flux of the vector $\boldsymbol{E}$ through the surface being considered is $E(r) \times 2\pi rh$. If $r > R$, the charge $q = \lambda h$ (where λ is the linear charge density) will get into the surface. Applying Gauss's theorem, we find that

$$E(r) \times 2\pi rh = \frac{2\lambda}{\varepsilon_0}.$$

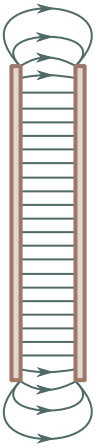


Fig. 1.41

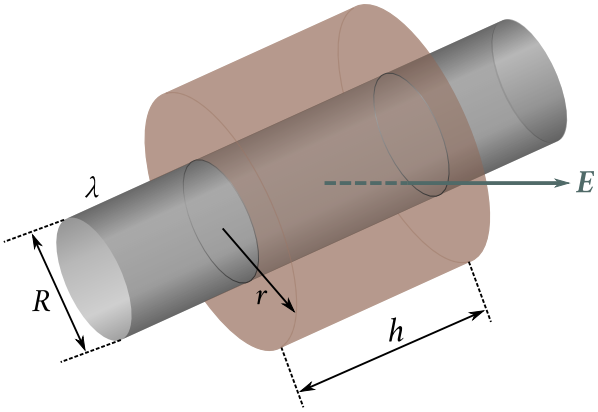


Fig. 1.42

Hence,

$$E(r) = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} \quad (r \geqslant R). \tag{1.122}$$

If $r < R$, the closed surface being considered contains no charges inside, owing to which $E(r) = 0$.

Thus, there is no field inside a uniformly charged cylindrical surface of infinite length. The field strength outside the surface is determined by the linear charge density λ and the distance r from the cylinder axis.

The field of a negatively charged cylinder differs from that of a positively charged one only in the direction of the vector $\boldsymbol{E}$. A glance at Eq. (1.122) shows that by reducing the cylinder radius R (with a constant linear charge density λ), we can obtain a field with a very great strength near the surface of the cylinder.

Introducing $\lambda = 2\pi R\sigma$ into Eq. (1.122) and assuming that $r = R$, we get the following value for the field strength in direct proximity to the surface of a cylinder:

$$E(R) = \frac{\sigma}{\varepsilon_0}. \tag{1.123}$$

The superposition principle makes it simple to find the field of two coaxial cylindrical surfaces carrying a linear charge density λ of the same magnitude, but of opposite signs (Fig. 1.43). There is no field inside the smaller and outside the larger cylinders. The field strength in the gap between the cylinders is determined by Eq. (1.122). This also holds for cylindrical surfaces of a finite length if the gap between the surfaces is much smaller than their length (a cylindrical capacitor). Appreciable deviations from the field of surfaces of an infinite length will be observed only near the edges of the cylinders.

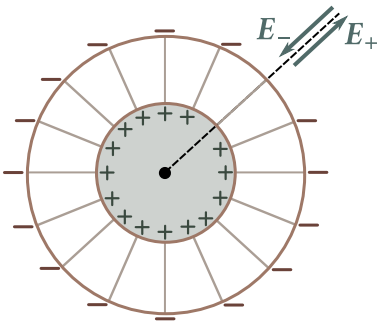


Fig. 1.43

Field of a Charged Spherical Surface. The field produced by a spherical surface of radius R whose charge has a constant surface density σ will obviously be a centrally symmetrical one. This signifies that the direction of the vector $\boldsymbol{E}$ at any point passes through the centre of the sphere, while the magnitude of the field strength is a function of the distance r from the centre of the sphere. Let us imagine a surface of radius r that is concentric with the charged sphere. For all points of this surface, $E_n = E(r)$. If $r > R$, the entire charge q distributed over the sphere will be inside the surface. Hence,

$$E(r) \times 4\pi r^2 = \frac{q}{\varepsilon_0}$$

whence

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \quad (r \geqslant R) \tag{1.124}$$

A spherical surface of radius r less than R will contain no charges, owing to which for $r < R$ we get $E(r) = 0$.

Thus, there is no field inside a spherical surface whose charge has a constant surface density σ. Outside this surface, the field is identical with that of a point charge of the same magnitude at the centre of the sphere.

Using the superposition principle, it is easy to show that the field of two concentric spherical surfaces (a spherical capacitor) carrying charges $+q$ and $-q$ that are identical in magnitude and opposite in sign is concentrated in the gap between the surfaces, the magnitude of the field strength in the gap being determined by Eq. (1.124).

Field of a Volume-Charged Sphere. Assume that a sphere of radius R has a charge with a constant volume density ρ. The field in this case has central symmetry. It is easy to see that the same result is obtained for the field outside the sphere [see Eq. (1.124)] as for a sphere with a surface charge. The result will be different for points inside the sphere, however. A spherical surface of radius r ($r < R$) contains a

charge equal to $\rho \times 4\pi r^3/3$. Therefore, Gauss's theorem for such a surface will be written as follows:

$$E(r) \times 4\pi r^2 = \frac{1}{\varepsilon_0}\rho\frac{4}{3}\pi r^3.$$

Hence, substituting $q/(4\pi R^3/3)$ for ρ, we get

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}r. \quad (r \leqslant R) \tag{1.125}$$

Thus, the field strength inside a sphere grows linearly with the distance r from the centre of the sphere. Outside the sphere, the field strength diminishes according to the same law as for the field of a point charge.

Chapter 2

ELECTRIC FIELD IN DIELECTRICS

2.1. Polar and Non-Polar Molecules

Dielectrics (or insulators) are defined as substances not capable of conducting an electric current. Ideal insulators do not exist in nature. All substances, even if to a negligible extent, conduct an electric current. But substances called conductors conduct a current from 10^{15} to 10^{20} times better than substances called dielectrics.

If a dielectric is introduced into an electric field, then the field and the dielectric itself undergo appreciable changes. To understand why this happens, we must take into account that atoms and molecules contain positively charged nuclei and negatively charged electrons.

A molecule is a system with a total charge of zero. The linear dimensions of this system are very small, of the order of a few angstroms (the angstrom—Å—is a unit of length equal to 10^{-10} m that is very convenient in atomic physics). We established in Sec. 1.10 that the field set up by such a system is determined by the magnitude and orientation of the dipole electric moment

$$\boldsymbol{p} = \sum_i q_i \boldsymbol{r}_i \tag{2.1}$$

(summation is performed both over the electrons and over the nuclei). True, the electrons in a molecule are in motion, so that this moment constantly changes. The velocities of the electrons are so high, however, that the mean value of the moment (2.1) is detected in practice. For this reason in the following by the dipole moment of a molecule, we shall mean the quantity

$$\boldsymbol{p} = \sum_i q_i \langle \boldsymbol{r}_i \rangle \tag{2.2}$$

(for nuclei, $\boldsymbol{r}_i$ is simply taken as $\langle \boldsymbol{r}_i \rangle$ in this sum). In other words, we shall consider that the electrons are at rest relative to the nuclei at certain points obtained by

averaging the positions of the electrons in time.

The behaviour of a molecule in an external electric field is also determined by its dipole moment. We can verify this by calculating the potential energy of a molecule in an external electric field. Selecting the origin of coordinates inside the molecule and taking advantage of the smallness of $\langle \boldsymbol{r}_i \rangle$, let us write the potential at the point where the i-th charge is in the form

$$\varphi_i = \varphi + \nabla\varphi \cdot \langle \boldsymbol{r}_i \rangle$$

where φ is the potential at the origin of coordinates [see Eq. (1.69)]. Hence,

$$W_\mathrm{p} = \sum_i q_i \varphi_i = \sum_i q_i \left(\varphi + \nabla\varphi \cdot \langle \boldsymbol{r}_i \rangle\right) = \varphi \sum_i q_i + \nabla\varphi \sum_i q_i \langle \boldsymbol{r}_i \rangle .$$

Taking into account that $\sum_i q_i = 0$ and substituting $-\boldsymbol{E}$ for $\nabla\varphi$, we get

$$W_\mathrm{p} = -\boldsymbol{E} \sum_i q_i \langle \boldsymbol{r}_i \rangle = -\boldsymbol{p}\cdot\boldsymbol{E} = -pE\cos\alpha.$$

Differentiating this expression with respect to α, we get Eq. (1.57) for the rotational moment; differentiating with respect to x, we arrive at the force (1.62).

Thus, a molecule is equivalent to a dipole both with respect to the field it sets up and with respect to the forces it experiences in an external field. The positive charge of this dipole equals the total charge of the nuclei and is at the "centre of gravity" of the positive charges; the negative charge equals the total charge of the electrons and is at the "centre of gravity" of the negative charges.

In symmetrical molecules (such as H_2,O_2, N_2), the centres of gravity of the positive and negative charges coincide in the absence of an external electric field. Such molecules have no intrinsic dipole moment and are called **non-polar**. In asymmetrical molecules (such as CO, NH, HCl), the centres of gravity of the charges of opposite signs are displaced relative to each other. In this case, the molecules have an intrinsic dipole moment and are called **polar**.

Under the action of an external electric field, the charges in a non-polar molecule become displaced relative to one another, the positive ones in the direction of the field, the negative ones against the field. As a result, the molecule acquires a dipole moment whose magnitude, as shown by experiments, is proportional to the field strength (intensity). In the rationalized system, the constant of proportionality is written in the form $\varepsilon_0\beta$, where ε_0 is the electric constant, and β is a quantity called the **polarizability of a molecule**. Since the directions of $\boldsymbol{p}$ and $\boldsymbol{E}$ coincide, we can write that

$$\boldsymbol{p} = \beta\varepsilon_0\boldsymbol{E}. \tag{2.3}$$

The dipole moment has a dimension of $[q]\mathrm{L}$. By Eq. (1.15), the dimension of $\varepsilon_0\boldsymbol{E}$ is $[q]\mathrm{L}^{-2}$. Hence, the polarizability of a molecule β has the dimension L^3.

The process of polarization of a non-polar molecule proceeds as if the positive and negative charges of the molecule were bound to one another by elastic forces. A non-polar molecule is, therefore said, to behave in an external field like an elastic dipole.

The action of an external field on a polar molecule consists mainly in tending to rotate the molecule so that its dipole moment is arranged in the direction of the field. An external field does not virtually affect the magnitude of a dipole moment. Consequently, a polar molecule behaves in an external field like a rigid dipole.

2.2. Polarization of Dielectrics

In the absence of an external electric field, the dipole moments of the molecules of a dielectric usually either equal zero (non-polar molecules) or are distributed in space by directions chaotically (polar molecules). In both cases, the total dipole moment of a dielectric equals zero[1].

A dielectric becomes polarized under the action of an external field. This signifies that the resultant dipole moment of the dielectric becomes other than zero. It is quite natural to take the dipole moment of a unit volume as the quantity characterizing the degree of polarization. If the field or the dielectric (or both) are not homogeneous, the degrees of polarization at different points of the dielectric will differ. To characterize the polarization at a given point, we must separate an infinitely small (physically) volume ΔV containing this point, find the sum $\sum_{\Delta V} \boldsymbol{p}$ of the moments of the molecules confined in this volume, and take the ratio

$$\boldsymbol{P} = \frac{\sum\limits_{\Delta V} \boldsymbol{p}}{\Delta V}. \tag{2.4}$$

The vector quantity $\boldsymbol{P}$ defined by Eq. (2.4) is called the **polarization of a dielectric**.

The dipole moment $\boldsymbol{p}$ has the dimension $[q]\mathrm{L}$. Consequently, the dimension of $\boldsymbol{P}$ is $[q]\mathrm{L}^{-2}$, *i.e.*, it coincides with the dimension of $\varepsilon_0 \boldsymbol{E}$ [see Eq. (1.15)].

The polarization of isotropic dielectrics of any kind is associated with the field strength at the same point by the simple relation

$$\boldsymbol{P} = \chi \varepsilon_0 \boldsymbol{E} \tag{2.5}$$

where χ is a quantity independent of $\boldsymbol{E}$ called the **electric susceptibility of a dielectric**[2]. It was indicated above that the dimensions of $\boldsymbol{P}$ and $\varepsilon_0 \boldsymbol{E}$ are identical. Hence, χ is a dimensionless quantity.

[1] In Sec. 2.9, we shall acquaint ourselves with substances that can have a dipole moment in the absence of an external field.

[2] In anisotropic dielectrics, the directions of $\boldsymbol{P}$ and $\boldsymbol{E}$, generally speaking, do not coincide. In this

In the Gaussian system of units, Eq. (2.5) has the form

$$\boldsymbol{P} = \chi \boldsymbol{E}. \tag{2.6}$$

For dielectrics built of non-polar molecules, Eq. (2.5) issues from the following simple considerations. The volume ΔV contains a number of molecules equal to $n\Delta V$, where n is the number of molecules per unit volume. Each of the moments $\boldsymbol{p}$ is determined in this case by Eq. (2.3). Hence,

$$\sum \Delta V \boldsymbol{p} = n\Delta V \beta \varepsilon_0 \boldsymbol{E}.$$

Dividing this expression by ΔV, we get the polarization $\boldsymbol{P} = n\beta\varepsilon_0 \boldsymbol{E}$. Finally, introducing the symbol $\chi = n\beta$, we arrive at Eq. (2.5).

For dielectrics built of polar molecules, the orienting action of the external field is counteracted by the thermal motion of the molecules tending to scatter their dipole moments in all directions. As a result, a certain preferred orientation of the dipole moments of the molecules sets in in the direction of the field. The relevant statistical calculations, which agree with experimental data, show that the polarization is proportional to the field strength, *i.e.*, leads to Eq. (2.5). The electric susceptibility of such dielectrics varies inversely with the absolute temperature.

In ionic crystals, the separate molecules lose their individuality. An entire crystal is, as it were, a single giant molecule. The lattice of an ionic crystal can be considered as two lattices inserted into each other, one of which is formed by the positive, and the other by the negative ions. When an external field acts on the crystal ions, both lattices are displaced relative to each other, which leads to polarization of the dielectric. The polarization in this case too is associated with the field strength by Eq. (2.5). We must note that the linear relation between $\boldsymbol{E}$ and $\boldsymbol{P}$ described by Eq. (2.5) may be applied only to not too strong fields [a similar remark relates to Eq. (2.3)].

2.3. The Field Inside a Dielectric

The charges in the molecules of a dielectric are called **bound**. The action of a field can only cause bound charges to be displaced slightly from their equilibrium

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case, the relation between $\boldsymbol{P}$ and $\boldsymbol{E}$ is described by the equations

$$\begin{aligned} P_x &= \varepsilon \left(\chi_{xx} E_x + \chi_{xy} E_y + \chi_{xz} E_z\right), \\ P_y &= \varepsilon \left(\chi_{yx} E_x + \chi_{yy} E_y + \chi_{yz} E_z\right), \\ P_z &= \varepsilon \left(\chi_{zx} E_x + \chi_{zy} E_y + \chi_{zz} E_z\right). \end{aligned}$$

The combination of the nine quantities χ_{ij} forms a symmetrical tensor of rank two called the **tensor of the dielectric susceptibility** [compare with Eqs. (5.30) of Vol. I). This tensor characterizes the electrical properties of an anisotropic dielectric.

positions; they cannot leave the molecule containing them.

Following the example of L. Landau and E. Lifshitz[3], we shall call charges that, although they are within the boundaries of a dielectric, are not inside its molecules, and also charges outside a dielectric, extraneous ones[4].

The field in a dielectric is the superposition of the field $\boldsymbol{E}_{\text{extr}}$ produced by the extraneous charges, and the field $\boldsymbol{E}_{\text{bound}}$ of the bound charges. The resultant field is called **microscopic** (or **true**):

$$\boldsymbol{E}_{\text{micro}} = \boldsymbol{E}_{\text{extr}} + \boldsymbol{E}_{\text{bound}}. \tag{2.7}$$

The microscopic field changes greatly within the limits of the intermolecular distances. Owing to the motion of the bound charges, the field $\boldsymbol{E}_{\text{micro}}$ also changes with time. These changes are not detected in a macroscopic-consideration. Therefore, a field is characterized by the quantity (2.7) averaged over an infinitely small (physically) volume, *i.e.*,

$$\boldsymbol{E} = \langle \boldsymbol{E}_{\text{micro}} \rangle = \langle \boldsymbol{E}_{\text{extr}} \rangle + \langle \boldsymbol{E}_{\text{bound}} \rangle .$$

In the following, we shall designate the averaged field of the extraneous charges by $\boldsymbol{E}_0$, and the averaged field of the bound charges by $\boldsymbol{E}'$. Accordingly, we shall define a macroscopic field as the quantity

$$\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}'. \tag{2.8}$$

The polarization $\boldsymbol{P}$ is a macroscopic quantity. Therefore, $\boldsymbol{E}$ in Eq. (2.5) should be understood as the strength determined by Eq. (2.8).

In the absence of dielectrics (*i.e.*, in a "vacuum"), the macroscopic field is

$$\boldsymbol{E} = \boldsymbol{E}_0 = \langle \boldsymbol{E}_{\text{extr}} \rangle .$$

It is exactly this quantity that is understood to be $\boldsymbol{E}$ in Eq. (1.117).

If the extraneous charges are stationary, the field determined by Eq. (2.8) has the same properties as an electrostatic field in a vacuum. In particular, it can be characterized with the aid of the potential φ related to the field strength (2.8) by Eqs. (1.41) and (1.45).

2.4. Space and Surface Bound Charges

When a dielectric is not polarized, the volume density ρ' and the surface density σ' of the bound charges equal zero. Polarization causes the surface density, and in some cases also the volume density of the bound charges to become different from

[3] See L. D. Landau and E. M. Lifshitz. Elektrodinamika sploshnykh sred (Electrodynamics of Continuous Media). Moscow, Gostekhizdat (1957), p. 57.

[4] It is customary practice to call such charges **free**. This name is extremely unsuccessful, however, because in a number of cases extraneous charges are not at all free.

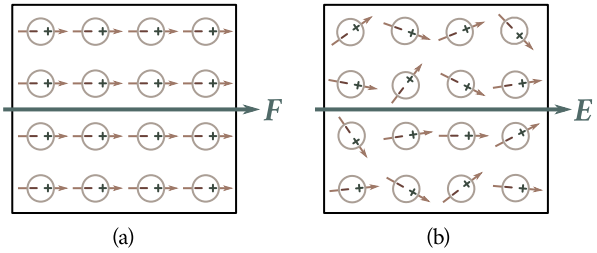

Fig. 2.1

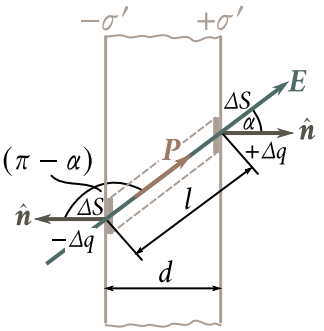

Fig. 2.2

zero.

Figure 2.1 shows schematically a polarized dielectric with nonpolar (a) and polar (b) molecules. Inspection of the figure shows that the polarization is attended by the appearance of a surplus of bound charges of one sign in the thin surface layer of the dielectric. If the normal component of the field strength $\boldsymbol{E}$ for the given section of the surface is other than zero, then under the action of the field, charges of one sign will move away inward, and of the other sign will emerge.

There is a simple relation between the polarization $\boldsymbol{P}$ and the surface density of the bound charges σ'. To find it, let us consider an infinite plane-parallel plate of a homogeneous dielectric placed in a homogeneous electric field (Fig. 2.2). Let us mentally separate an elementary volume in the plate in the form of a very thin cylinder with generatrices parallel to $\boldsymbol{E}$ in the dielectric, and with bases of area ΔS coinciding with the surfaces of the plate. The magnitude of this volume is

$$\Delta V = l \Delta S \cos \alpha$$

where l is the distance between the bases of the cylinder and α the angle between the vector $\boldsymbol{E}$ and an outward normal to the positively charged surface of the dielectric.

The volume ΔV has a dipole electric moment of the magnitude

$$P \Delta V = P l \Delta S \cos \alpha$$

(P is the magnitude of the polarization).

From the macroscopic viewpoint, the volume being considered is equivalent to a dipole formed by the $+\sigma' \Delta S$ and $-\sigma' \Delta S$ with a spacing of l. Therefore, its electric moment can be written in the form $\sigma' \Delta S l$. Equating the two expressions for the electric moment, we get

$$P l \Delta S \cos \alpha = \sigma' \Delta S l.$$

Hence, we get the required relation between σ' and $\boldsymbol{P}$:

$$\sigma' = P \cos \alpha = P_{\mathrm{n}} \tag{2.9}$$

where P_n is the projection of the polarization onto an outward normal to the relevant surface. For the right-hand surface in Fig. 2.2, we have $P_n > 0$, accordingly, σ' for it is positive; for the left-hand surface $P_n < 0$, accordingly, σ' for it is negative.

Expressing $\boldsymbol{P}$ through χ and $\boldsymbol{E}$ by means of Eq. (2.5), we arrive at the formula

$$\sigma' = \chi \varepsilon_0 E_n \tag{2.10}$$

where E_n is the normal component of the field strength inside the dielectric. According to Eq. (2.10), at the places where the field lines emerge from the dielectric ($E_n > 0$), positive bound charges come up to the surface, while where the field lines enter the dielectric ($E_n < 0$), negative surface charges appear.

Equations (2.9) and (2.10) also hold in the most general case when an inhomogeneous dielectric of an arbitrary shape is in an inhomogeneous electric field. By P_n and E_n in this case, we must understand the normal component of the relevant vector taken in direct proximity to the surface element for which σ' is being determined.

Now let us turn to finding the volume density of the bound charges appearing inside an inhomogeneous dielectric. Let us consider an imaginary small area ΔS (Fig. 2.3) in an inhomogeneous isotropic dielectric with non-polar molecules. Assume that a unit volume of the dielectric has n identical particles with a charge of $+e$ and n identical particles with a charge of $-e$. In close proximity to area ΔS, the electric field and the dielectric can be considered homogeneous. Therefore, when the field is switched on, all the positive charges near ΔS will be displaced over the same distance l_1 in the direction of $\boldsymbol{E}$, and all the negative charges will be displaced in the opposite direction over the same distance l_2 (see Fig. 2.3). A certain number of charges of one sign (positive if $\alpha < n/2$ and negative if $\alpha > n/2$) will pass through area ΔS in the direction of a normal to it, and a certain number of charges of the opposite sign (negative if $\alpha < n/2$ and positive if $\alpha < n/2$) in the direction opposite to $\hat{\boldsymbol{n}}$. Area ΔS will be intersected by all the charges $+e$ that were at a distance of not over $l_1 \cos\alpha$ from it before the field was switched on, *i.e.*, by all the $+e$'s in an oblique cylinder of volume $l_1 \Delta S \cos\alpha$. The number of these charges is $nl_1 \Delta S \cos\alpha$, while the charge they carry in the direction of a normal to the area is $enl_1 \Delta S \cos\alpha$ (when $\alpha > \pi/2$, the charge carried in the direction of the normal as a result of displacement of the charges $+e$ will be negative). Similarly, area ΔS will be intersected by all the charges $-e$ in the volume $l_2 \Delta S \cos\alpha$. These charges will carry a charge of $enl_2 \Delta S \cos\alpha$ in the direction of a normal to the area (inspection of Fig. 2.3 shows that when $\alpha < \pi/2$, the charges $-e$ will carry the charge $-enl_2 \Delta S \cos\alpha$ through ΔS in the direction opposite to $\hat{\boldsymbol{n}}$, which is equivalent to carrying the charge $enl_2 \Delta S \cos\alpha$ in the direction of $\hat{\boldsymbol{n}}$).

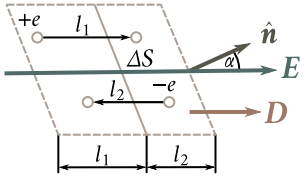

Fig. 2.3

Thus, when the field is switched on, the charge

$$\Delta q' = enl_1 \Delta S \cos\alpha + enl_2 \Delta S \cos\alpha = en(l_1 + l_2)\Delta S \cos\alpha$$

is carried through area ΔS in the direction of a normal to it. The sum $l_1 + l_2$ is the distance l over which the positive and negative bound charges are displaced toward one another in the dielectric. As a result of this displacement, each pair of charges acquires the dipole moment $p = el = e(l_1 + l_2)$. The number of such pairs in a unit volume is n. Consequently, the product $e(l_1 + l_2)n = eln = p$ gives the magnitude of the polarization P. Thus, the charge passing through area ΔS in the direction of a normal to it when the field is switched on is [see Eq. (2.9)]

$$\Delta q' = P\Delta S \cos\alpha.$$

Since the dielectric is isotropic, the directions of the vectors $\boldsymbol{E}$ and $\boldsymbol{P}$ coincide (see Fig. 2.3). Consequently, α is the angle between the vectors $\boldsymbol{P}$ and $\hat{\boldsymbol{n}}$, and in this connection we can write

$$\Delta q' = (\boldsymbol{P} \cdot \hat{\boldsymbol{n}})\, \Delta S.$$

Passing over from deltas to differentials, we get

$$\mathrm{d}q = (\boldsymbol{P} \cdot \hat{\boldsymbol{n}})\, \mathrm{d}S = \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S}.$$

We have found the bound charge $\mathrm{d}q'$ that passes through elementary area $\mathrm{d}S$ in the direction of a normal to it when the field is switched on; $\boldsymbol{P}$ is the polarization set up under the action of the field at the location of area $\mathrm{d}S$.

Let us imagine closed surface S inside the dielectric. When the field is switched on, a bound charge q' will intersect this surface and emerge from it. This charge is

$$q'_{\mathrm{em}} = \oint_S \mathrm{d}q' = \oint_S \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S}$$

(we have agreed to take the outward normal to area $\mathrm{d}S$ for closed surfaces). As a result, a surplus bound charge will appear in the volume enclosed by surface S. Its value is

$$q'_{\mathrm{sur}} = -q'_{\mathrm{em}} = -\oint_S \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S} = -\Phi_{\mathrm{P}} \tag{2.11}$$

(Φ_{P} is the flux of the vector $\boldsymbol{P}$ through surface S).

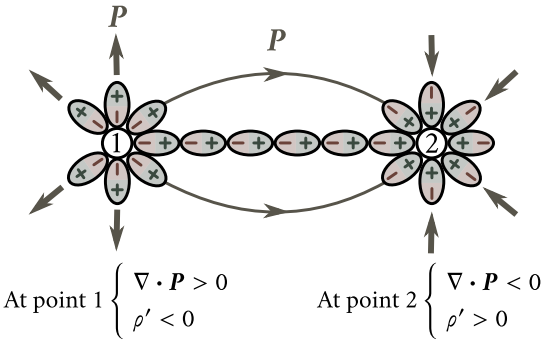

Fig. 2.4

Introducing the volume density of the bound charges ρ', we can write

$$q'_{\text{sur}} = \int_V \rho' \, \mathrm{d}V$$

(the integral is taken over the volume enclosed by surface S). We thus arrive at the formula

$$\int_V \rho' \, \mathrm{d}V = - \oint_S \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S}.$$

Let us transform the surface integral according to the Ostrogradsky-Gauss theorem [see Eq. (1.108)). The result is

$$\int_V \rho' \, \mathrm{d}V = - \int_V \nabla \cdot \boldsymbol{P} \, \mathrm{d}V.$$

This equation must be observed for any arbitrarily chosen volume V. This is possible only if the following equation is observed at every point of the dielectric:

$$\rho' = -\nabla \cdot \boldsymbol{P}. \tag{2.12}$$

Consequently, the density of bound charges equals the divergence of the polarization $\boldsymbol{P}$ taken with the opposite sign.

We obtained Eq. (2.12) when considering a dielectric with non-polar molecules. This equation also holds, however, for dielectrics with polar molecules.

Equation (2.12) can be given a graphical interpretation. Points with a positive $\nabla \cdot \boldsymbol{P}$ are sources of the field of the vector $\boldsymbol{P}$, and the lines of $\boldsymbol{P}$ diverge from them (Fig. 2.4). Points with a negative $\nabla \cdot \boldsymbol{P}$ are sinks of the field of the vector $\boldsymbol{P}$, and the lines of $\boldsymbol{P}$ converge at them. In polarization of the dielectric, the positive bound charges are displaced in the direction of the vector $\boldsymbol{P}$, *i.e.*, in the direction of the lines $\boldsymbol{P}$; the negative bound charges are displaced in the opposite direction (in the figure the bound charges belonging to separate molecules are encircled by ovals). As a result, a surplus of negative bound charges is formed at places with a positive

$\nabla \cdot \boldsymbol{P}$, and a surplus of positive bound charges at places with a negative $\nabla \cdot \boldsymbol{P}$.

Bound charges differ from extraneous ones only in that they cannot leave the confines of the molecules which they are in. Otherwise, they have the same properties as all other charges. In particular, they are sources of an electric field. Therefore, when the density of the bound charges ρ' differs from zero, Eq. (1.117) must be written in the form

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}(\rho + \rho'). \tag{2.13}$$

Here ρ is the density of the extraneous charges.

Let us introduce Eq. (2.5) for $\boldsymbol{P}$ into Eq. (2.12) and use Eq. (1.103). The result is

$$\rho' = -\nabla \cdot (\chi \varepsilon_0 \boldsymbol{E}) = -\varepsilon_0 \nabla \cdot (\chi \boldsymbol{E}) = -\varepsilon_0 [\boldsymbol{E} \cdot \nabla \chi + \chi \nabla \cdot \boldsymbol{E}].$$

Substituting for $\nabla \cdot \boldsymbol{E}$ its value from Eq. (2.13), we arrive at the equation

$$\rho' = -\varepsilon_0 (\boldsymbol{E} \cdot \nabla \chi) - \chi \rho - \chi \rho'.$$

Hence,

$$\rho' = -\left(\frac{1}{1+\chi}\right)[\varepsilon_0 (\boldsymbol{E} \cdot \nabla \chi) + \chi \rho]. \tag{2.14}$$

We can see from Eq. (2.14) that the volume density of bound charges can differ from zero in two cases: (1) if a dielectric is not homogeneous ($\nabla \chi \neq 0$), and (2) if at a given place in a dielectric the density of the extraneous charges is other than zero ($\rho \neq 0$).

When there are no extraneous charges in a dielectric, the volume density of the bound charges is

$$\rho' = -\left(\frac{\varepsilon_0}{1+\chi}\right)(\boldsymbol{E} \cdot \nabla \chi). \tag{2.15}$$

2.5. Electric Displacement Vector

We noted in the preceding section that not only extraneous, but also bound charges are sources of a field. Accordingly,

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}(\rho + \rho') \tag{2.16}$$

[see Eq. (2.13)].

Equation (2.16) is of virtually no use for finding the vector $\boldsymbol{E}$ because it expresses the properties of the unknown quantity $\boldsymbol{E}$ through bound charges, which in turn are determined by the unknown quantity $\boldsymbol{E}$ [see Eqs. (2.10) and (2.14)].

Calculation of the fields is often simplified if we introduce an auxiliary quantity whose sources are only extraneous charges ρ. To establish what this quantity looks

like, let us introduce Eq. (2.12) for ρ' into Eq. (2.16):

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}(\rho - \nabla \cdot \boldsymbol{P})$$

whence it follows that

$$\nabla \cdot (\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho \tag{2.17}$$

(we have put ε_0 inside the del symbol). The expression in parentheses in Eq. (2.17) is the required quantity. It is designated by the symbol $\boldsymbol{D}$ and is called the electric displacement (or electric induction).

Thus, the **electric displacement** is a quantity determined by the relation

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}. \tag{2.18}$$

Inserting Eq. (2.5) for $\boldsymbol{P}$, we get

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \chi \varepsilon_0 \boldsymbol{E} = \varepsilon_0 (1 + \chi) \boldsymbol{E}. \tag{2.19}$$

The dimensionless quantity

$$\varepsilon = 1 + \chi \tag{2.20}$$

is called the **relative permittivity** or simply the **permittivity** of a medium[5]. Thus, Eq. (2.19) can be written in the form

$$\boldsymbol{D} = \varepsilon_0 \varepsilon \boldsymbol{E}. \tag{2.21}$$

According to Eq. (2.21), the vector $\boldsymbol{D}$ is proportional to the vector $\boldsymbol{E}$. We remind our reader that we are dealing with isotropic dielectrics. In anisotropic dielectrics, the vectors $\boldsymbol{E}$ and $\boldsymbol{D}$, generally speaking, are not collinear.

In accordance with Eqs. Eq. (1.15) and Eq. (2.21), the electric displacement of the field of a point charge in a vacuum is

$$\boldsymbol{D} = \frac{1}{4\pi} \frac{q}{r^2} \hat{\boldsymbol{e}}_r. \tag{2.22}$$

The unit of electric displacement is the coulomb per square metre ($\mathrm{C\,m^{-2}}$).

Equation (2.17) can be written as

$$\nabla \cdot \boldsymbol{D} = \rho. \tag{2.23}$$

Integration of this equation over the arbitrary volume V yields

$$\int_V \nabla \cdot \boldsymbol{D} \, \mathrm{d}V = \int_V \rho \, \mathrm{d}V.$$

Let us transform the left-hand side according to the Ostrogradsky-Gauss theorem

[5]The so-called absolute permittivity of a medium $\varepsilon_\mathrm{a} = \varepsilon_0 \varepsilon$ is introduced in electrical engineering. This quantity is deprived of a physical meaning, however, and we shall not use it.

[see Eq. (1.108)]:

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho \, \mathrm{d}V. \tag{2.24}$$

The quantity on the left-hand side is Φ_D—the flux of the vector $\boldsymbol{D}$ through closed surface S, while that on the right-hand side is the sum of the extraneous charges $\sum_i q_i$ enclosed by this surface. Hence, Eq. (2.24) can be written in the form

$$\Phi_D = \sum_i q_i. \tag{2.25}$$

Equations (2.24) and (2.25) express Gauss's theorem for the vector $\boldsymbol{D}$: *the flux of the electric displacement through a closed surface equals the algebraic sum of the extraneous charges enclosed by this surface*.

In a vacuum, $\boldsymbol{P} = 0$, so that the quantity $\boldsymbol{D}$ determined by Eq. (2.18) transforms into $\varepsilon_0 \boldsymbol{E}$, and Eqs. (2.24) and (2.25) transform into Eqs. (1.114) and (1.116).

The unit of the flux of the electric displacement vector is the coulomb. By Eq. (2.25), a charge of 1 C sets up a displacement flux of 1 C through the surface surrounding it.

The field of the vector $\boldsymbol{D}$ can be depicted with the aid of electric displacement lines (we shall call them displacement lines for brevity's sake). Their direction and density are determined in exactly the same way as for the lines of the vector $\boldsymbol{E}$ (see Sec. 1.5). The lines of the vector $\boldsymbol{E}$ can begin and terminate at both extraneous and bound charges. The sources of the field of the vector $\boldsymbol{D}$ are only extraneous charges. Hence, displacement lines can begin or terminate only at extraneous charges. These lines pass without interruption through points at which bound charges are placed.

The electric induction[6] in the Gaussian system is determined by the expression

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P}. \tag{2.26}$$

Substituting for $\boldsymbol{P}$ in this equation its value from Eq. (2.6), we get

$$\boldsymbol{D} = (1 + 4\pi\chi)\boldsymbol{E}. \tag{2.27}$$

The quantity

$$\varepsilon = 1 + 4\pi\chi. \tag{2.28}$$

is called the **permittivity**. Introducing this quantity into Eq. (2.27) we get

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}. \tag{2.29}$$

In the Gaussian system, the electric induction in a vacuum coincides with the field strength $\boldsymbol{E}$. Consequently, the electric induction of the field of a point charge in a vacuum is determined by Eq. (1.16).

By Eq. (2.22) the electric displacement set up by a charge of 1 C at a distance of

[6] The term "electric displacement" is not applied to quantity (2.27).

1 m is

$$D = \frac{1}{4\pi}\frac{q}{r^2} = \frac{1}{4\pi \times 1^2} = \frac{1}{4\pi}\text{C m}^{-2}.$$

In the Gaussian system, the electric induction in this case is

$$D = \frac{q}{r^2} = \frac{3 \times 10^9}{10^4} = 3 \times 10^5 \ \text{cgse}_D.$$

Thus,

$$1\,\text{C m}^{-2} = 4\pi \times 3 \times 10^5 \ \text{cgse}_D.$$

In the Gaussian system, the expressions of Gauss's theorem have the form

$$\oint_S \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = 4\pi \int_V \rho \, \mathrm{d}V \tag{2.30}$$

$$\Phi_D = 4\pi \sum_i q_i. \tag{2.31}$$

According to Eq. (2.31), a charge of 1 C sets up a flux of the electric induction vector of $4\pi q = 4\pi \times 3 \times 10^9$ cgse_{Φ_D}. The following relation thus exists between the units of flux of the vector $\boldsymbol{D}$:

$$1\,\text{C} = 4\pi \times 3 \times 10^9 \ \text{cgse}_{\Phi_D}.$$

2.6. Examples of Calculating the Field in Dielectrics

We shall consider several examples of fields in dielectrics to reveal the meaning of the quantities $\boldsymbol{D}$ and ε.

Field Inside a Flat Plate. Let us consider two infinite parallel oppositely charged planes. Let the field they produce in a vacuum be characterized by the strength $\boldsymbol{E}_0$ and the displacement $\boldsymbol{D}_0 = \varepsilon_0 \boldsymbol{E}_0$. Let us introduce into this field a plate of a homogeneous isotropic dielectric and arrange it as shown in Fig. 2.5. The dielectric becomes polarized under the action of the field, and bound charges of density σ' will appear on its surfaces. These charges will set up a homogeneous field inside the plate whose strength by Eq. (1.121) is $E' = \sigma'/\varepsilon_0$. In the given case, $E' = 0$ outside the dielectric.

The field strength E_0 is σ/ε_0. Both fields are directed toward each other, hence, inside the dielectric we have

$$E = E_0 - E' = E_0 - \frac{\sigma'}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\left(\right)\sigma - \sigma'\right). \tag{2.32}$$

Outside the dielectric, $E = E_0$.

The polarization of the dielectric is due to field (2.32). The latter is perpendicular to the surfaces of the plate. Hence, $E_\text{n} = E$, and in accordance with Eq. (2.10),

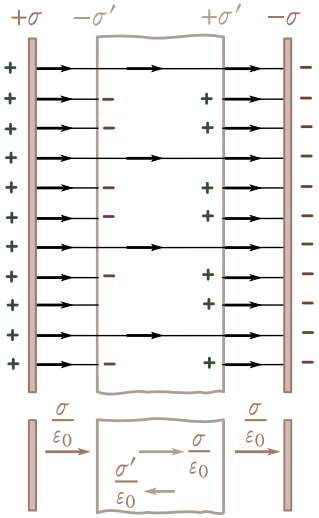

Fig. 2.5

$\sigma' = \chi\varepsilon_0 E$. Using this value in Eq. (2.32), we get

$$E = E_0 - \chi E$$

whence

$$E = \frac{E_0}{1+\chi} = \frac{E_0}{\varepsilon}. \tag{2.33}$$

Thus, in the given case, the permittivity ε shows how many times the field in a dielectric weakens.

Multiplying Eq. (2.33) by $\varepsilon_0\varepsilon$, we get the electric displacement inside the plate:

$$D = \varepsilon_0\varepsilon E = \varepsilon_0 E_0 D_0. \tag{2.34}$$

Hence, the electric displacement inside the plate coincides with that of the external field D_0. Substituting σ/ε_0 for E_0 in Eq. (2.34), we find

$$D = \sigma. \tag{2.35}$$

To find σ', let us express E and E_0 in Eq. (2.33) through the charge densities:

$$\frac{1}{\varepsilon_0}\left(\left(\right)\sigma - \sigma'\right) = \frac{\sigma}{\varepsilon_0\varepsilon}$$

whence

$$\sigma' = \frac{\varepsilon - 1}{\varepsilon}\sigma. \tag{2.36}$$

Figure 2.5 has been drawn assuming that $\varepsilon = 3$. Accordingly, the density of the field lines in the dielectric is one-third of that outside the plate. The lines are equally spaced because the field is homogeneous. In the given case, σ' can be found

without resorting to Eq. (2.36). Indeed, since the field intensity inside the plate is one-third of that outside it, then of three field lines beginning (or terminating) on extraneous charges, two must terminate (or begin respectively) on bound charges. It thus follows that the density of the bound charges must be two-thirds that of the extraneous charges.

In the Gaussian system, the field strength E' produced by the bound charges σ' is $4\pi\sigma'$. Therefore, Eq. (2.32) becomes

$$E = E_0 - E' = E_0 - 4\pi\sigma'. \tag{2.37}$$

The surface density σ' is associated with the field strength E by the equation $\sigma' = \chi E_n$. We can thus write that

$$E = E_0 - 4\pi\chi E$$

whence

$$E = \frac{E_0}{1 + 4\pi\chi} = \frac{E_0}{\varepsilon}.$$

Thus, the permittivity ε, like its counterpart ε in the SI, shows how many times the field inside a dielectric weakens. Therefore, the values of ε in the SI and the Gaussian system coincide. Hence, taking into account Eqs. (2.20) and (2.28), we conclude that the susceptibilities in the Gaussian system (χ_{Gs}) and in the SI (χ_{SI}) differ from each other by the factor 4π:

$$\chi_{SI} = 4\pi\chi_{Gs}. \tag{2.38}$$

Field Inside a Spherical Layer. Let us surround a charged sphere of radius R with a concentric spherical layer of a homogeneous isotropic dielectric (Fig. 2.6). The bound charge q_1' distributed with the density σ_1' will appear on the internal surface of the layer ($q_1' = 4\pi R_1^2\sigma_1'$), and the charge q_2' distributed with the density σ_2' will appear on its external surface ($q_2' = 4\pi R_2^2\sigma_2'$). The sign of the charge q_2' coincides with that of the charge q of the sphere, while q_1' has the opposite sign. The charges q_1' and q_2' set up a field at a distance r exceeding R_1 and R_2, respectively, that coincides with the field of a point charge of the same magnitude [see Eq. (1.124)]. The charges q_1' and q_2' produce no field inside the surfaces over which they are distributed. Hence, the field strength E' inside a dielectric is

$$E' = \frac{1}{4\pi\varepsilon_0}\frac{q_1'}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R_1^2\sigma_1'}{r^2} = \frac{1}{\varepsilon_0}\frac{R_1^2\sigma_1'}{r^2}$$

and is opposite in direction to the field strength E_0. The resultant field in a dielectric is

$$E(r) = E_0 - E' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} - \frac{1}{\varepsilon_0}\frac{R_1^2\sigma_1'}{r^2}. \tag{2.39}$$

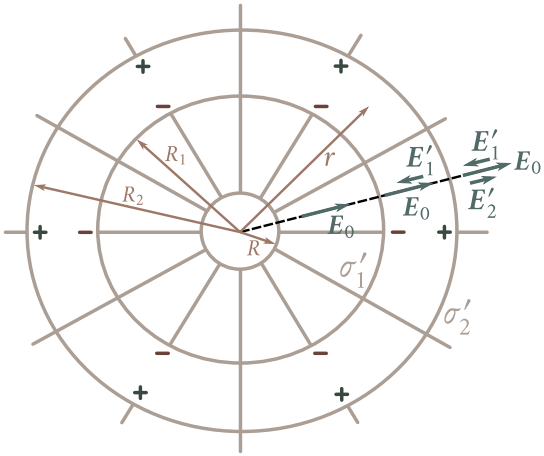

Fig. 2.6

It diminishes in proportion to $1/r^2$. We can therefore state that

$$\frac{E(R_1)}{E(r)} = \frac{r^2}{R_1^2} \quad \Rightarrow \quad E(R_1) = E(r)\frac{r^2}{R_1^2}$$

where $E(R_1)$ is the field strength in a dielectric in direct proximity to the internal surface of the layer. It is exactly this strength that determines the quantity σ_1':

$$\sigma_1' = \chi\varepsilon_0 E(R_1) = \chi\varepsilon_0 E(r)\frac{r^2}{R_1^2} \tag{2.40}$$

(at each point of the surface $|E_n| = E$).

Introducing Eq. (2.40) into Eq. (2.39), we get

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} - \frac{1}{\varepsilon_0}\frac{R_1^2\chi\varepsilon_0 E(r)r^2}{r^2R_1^2} = E_0(r) - \chi E(r).$$

From this equation, we find that inside a dielectric $E = E_0/\varepsilon$, and, consequently, $D = \varepsilon_0 E_0$ [compare with Eqs. (2.33) and (2.34)].

The field inside a dielectric changes in proportion to $1/r^2$. Therefore, the relation $\sigma_1' : \sigma_2' = R_1 : R_2$ holds. Hence, it follows that $q_1' = q_2'$. Consequently, the fields set up by these charges at distances exceeding R_2 mutually destroy each other so that outside the spherical layer $E' = 0$ and $E = E_0$.

Assuming that $R_1 = R$ and $R_2 = \infty$, we arrive at the case of a charged sphere immersed in an infinite homogeneous and isotropic dielectric. The field strength outside such a sphere is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{\varepsilon r^2}. \tag{2.41}$$

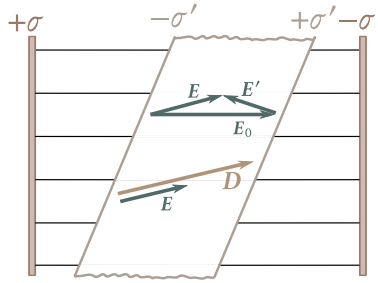

Fig. 2.7

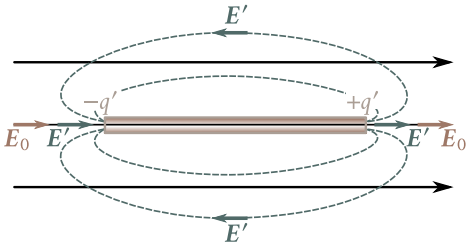

Fig. 2.8

The strength of the field set up in an infinite dielectric by a point charge will be the same.

Both examples considered above are characterized by the fact that the dielectric was homogeneous and isotropic, and the surfaces enclosing it coincided with the equipotential surfaces of the field of extraneous charges. The result we have obtained in these cases is a general one. *If a homogeneous and isotropic dielectric completely fills the volume enclosed by equipotential surfaces of the field of extraneous charges, then the electric displacement vector coincides with the vector of the field strength of the extraneous charges multiplied by* ε_0*, and, therefore, the field strength inside the dielectric is* $1/\varepsilon$ *of that of the field strength of the extraneous charges.*

If the above conditions are not observed, the vectors $\boldsymbol{D}$ and $\varepsilon_0\boldsymbol{E}$ do not coincide. Figure 2.7 shows the field in the plate of a dielectric. The plate is skewed relative to the planes carrying extraneous charges. The vector $\boldsymbol{E}'$ is perpendicular to the faces of the plate, therefore, $\boldsymbol{E}$ and $\boldsymbol{E}_0$ are not collinear. The vector $\boldsymbol{D}$ is directed the same as $\boldsymbol{E}$, consequently, $\boldsymbol{D}$ and $\varepsilon_0\boldsymbol{E}_0$ do not coincide in direction. We can show that they also fail to coincide in magnitude. In the examples considered above owing to the specially selected shape of the dielectric, the field $\boldsymbol{E}'$ differed from zero only inside the dielectric. In the general case, $\boldsymbol{E}'$ may differ from zero outside the dielectric too. Let us place a rod made of a dielectric into an initially homogeneous field (Fig. 2.8). Owing to polarization, bound charges of opposite signs are formed on the ends of the rod. Their field outside the rod is equivalent to the field of a dipole (the lines of $\boldsymbol{E}'$ are dash ones in the figure). It is easy to see that the resultant field $\boldsymbol{E}$ near the ends of the rod is greater than the field $\boldsymbol{E}_0$.

2.7. Conditions on the Interface Between Two Dielectrics

Near the interface between two dielectrics, the vectors $\boldsymbol{E}$ and $\boldsymbol{D}$ must comply with definite boundary conditions following from the relations (1.112) and (2.23):

$$\nabla \times \boldsymbol{E} = 0, \quad \nabla \cdot \boldsymbol{D} = \rho.$$

Let us consider the interface between two dielectrics with the permittivities ε_1 and ε_2 (Fig. 2.9). We choose an arbitrarily directed x-axis on this surface. We take a small rectangular contour of length a and width b that is partly in the first dielectric and partly in the second one. The x-axis passes through the middle of the sides b.

Assume that a field has been set up in the first dielectric whose strength is $\boldsymbol{E}_1$, and in the second one whose strength is $\boldsymbol{E}_2$. Since $\nabla \times \boldsymbol{E} = 0$, the circulation of the vector $\boldsymbol{E}$ around the contour we have chosen must equal zero [see Eq. (1.110)]. With small dimensions of the contour and the direction of circumvention shown in Fig. 2.9, the circulation of the vector $\boldsymbol{E}$ can be written in the form

$$\oint E_l \, \mathrm{d}l = E_{1,x} a - E_{2,x} a + \langle E_b \rangle \, 2b \tag{2.42}$$

where $\langle E_b \rangle$ is the mean value of E_l on sections of the contour perpendicular to the interface. Equating this expression to zero, we arrive at the equation

$$\left(E_{1,x} - E_{2,x}\right) a = \langle E_b \rangle \, 2b.$$

In the limit, when the width b of the contour tends to zero, we get

$$E_{1,x} = E_{2,x}. \tag{2.43}$$

The values of the projections of the vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ onto the x-axis are taken in direct proximity to the interface between the boundary of the dielectrics.

Equation (2.43) is obeyed when the x-axis is selected arbitrarily. It is only essential that this axis be in the plane of the interface between the dielectrics. Inspection of Eq. (2.43) shows that with such a selection of the x-axis when $E_{1,x} = 0$, the projection of $E_{2,x} = 0$ will also equal zero. This signifies that the vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ at two close points taken at opposite sides of the interface are in the same plane as a normal to the interface. Let us represent each of the vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ in the form of the sum of the normal and tangential components:

$$\boldsymbol{E}_1 = \boldsymbol{E}_{1,n} + \boldsymbol{E}_{1,\tau}, \quad \boldsymbol{E}_2 = \boldsymbol{E}_{2,n} + \boldsymbol{E}_{2,\tau}.$$

In accordance with Eq. (2.43)

$$E_{1,\tau} = E_{2,\tau}. \tag{2.44}$$

Here $\boldsymbol{E}_{i,\tau}$ is the projection of the vector $\boldsymbol{E}_i$ onto the unit vector $\hat{\boldsymbol{\tau}}$ directed along the line of intersection of the dielectric interface with the plane containing the vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$.

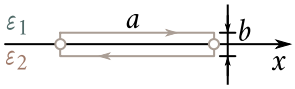

Fig. 2.9

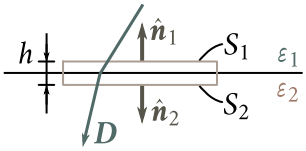

Fig. 2.10

Substituting in accordance with Eq. (2.21) the projections of the vector $\boldsymbol{D}$ divided by $\varepsilon_0\varepsilon$ for the projections of the vector $\boldsymbol{E}$, we get the proportion

$$\frac{D_{1,\tau}}{\varepsilon_0\varepsilon_1} = \frac{D_{2,\tau}}{\varepsilon_0\varepsilon_2}$$

whence it follows that

$$\frac{D_{1,\tau}}{D_{2,\tau}} = \frac{\varepsilon_1}{\varepsilon_2}. \tag{2.45}$$

Now let us take an imaginary cylindrical surface of height h on the interface between the dielectrics (Fig. 2.10). Base S_1 is in the first dielectric, and base S_2 in the second. Both bases are identical in size ($S_1 = S_2 = S$) and are so small that within the limits of each of them the field may be considered homogeneous. Let us apply Gauss's theorem [see Eq. (2.25)] to this surface. If there are no extraneous charges on the interface between the dielectrics, the right-hand side in Eq. (2.25) equals zero. Hence, $\Phi_D = 0$.

The flux through base S_1 is $D_{1,n}S$, where $D_{1,n}$ is the projection of the vector $\boldsymbol{D}$ in the first dielectric onto the normal $\hat{\boldsymbol{n}}_1$. Similarly, the flux through base S_2 is $D_{2,n}S$, where $D_{2,n}$ is the projection of the vector $\boldsymbol{D}$ in the second dielectric onto the normal $\hat{\boldsymbol{n}}_2$. The flux through the side surface can be written in the form $\langle D\rangle_n S_{\text{side}}$, where $\langle D\rangle_n$ is the value of D_n averaged over the entire side surface, and S_{side} is the magnitude of this surface. We can thus write that

$$\Phi_D = D_{1,n}S + D_{2,n}S + \langle D\rangle_n S_{\text{side}} = 0. \tag{2.46}$$

If the altitude h of the cylinder is made to tend to zero, then S_{side} will also tend to zero. Hence, in the limit, we get

$$D_{1,n} = -D_{2,n}.$$

Here $D_{i,n}$ is the projection onto $\hat{\boldsymbol{n}}_i$ of the vector $\boldsymbol{D}$ the i-th dielectric in direct proximity to its interface with the other dielectric. The signs of the projections are different because the normals $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ to the bases of the cylinder have opposite directions. If we project $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ onto the same normal, we get the condition

$$D_{1,n} = D_{2,n}. \tag{2.47}$$

Using Eq. (2.21) to replace the projections of $\boldsymbol{D}$ with the corresponding projec-

tions of the vector $\boldsymbol{E}$ multiplied by $\varepsilon_0\varepsilon$, we get the relation

$$\varepsilon_0\varepsilon_1 E_{1,n} = \varepsilon_0\varepsilon_2 E_{2,n}$$

whence

$$\frac{E_{1,n}}{E_{2,n}} = \frac{\varepsilon_2}{\varepsilon_1}. \tag{2.48}$$

The results we have obtained signify that when passing through the interface between two dielectrics, the normal component of the vector $\boldsymbol{D}$ and the tangential component of the vector $\boldsymbol{E}$ change continuously. The tangential component of the vector $\boldsymbol{D}$ and the normal component of the vector $\boldsymbol{E}$, however, are disrupted when passing through the interface.

Equations (2.44), (2.45), (2.47), and (2.48) determine the conditions which the vectors $\boldsymbol{E}$ and $\boldsymbol{D}$ must comply with on the interface between two dielectrics (if there are no extraneous charges on this interface). We have obtained these equations for an electrostatic field. They also hold, however, for fields varying with time (see Sec. **??**).

The conditions we have found also hold for the interface between a dielectric and a vacuum. In this case, one of the permittivities must be taken equal to unity.

We must note that condition (2.47) can be obtained on the basis of the fact that the displacement lines pass through the interface between two dielectrics without being interrupted (Fig. 2.11). According to the rule for drawing these lines, the number of lines arriving at area ΔS from the first dielectric is $D_1\Delta S_1 = D_1\Delta S\cos\alpha_1$. Similarly, the number of lines emerging from area ΔS into the second dielectric is $D_2\Delta S_2 = D_2\Delta S\cos\alpha_2$. If the lines are not interrupted at the interface, both these numbers must be the same:

$$D_1\Delta S\cos\alpha_1 = D_2\Delta S\cos\alpha_2.$$

Cancelling ΔS and taking into account that the product $D\cos\alpha$ gives the value of the normal component of the vector $\boldsymbol{D}$, we arrive at condition (2.47).

The displacement lines are bent (refracted) on the interface between dielectrics, owing to which the angle α between a normal to the interface and the line $\boldsymbol{D}$ changes. Inspection of Fig. 2.12 shows that

$$\tan\alpha_1 : \tan\alpha_2 = \frac{D_{1,\tau}}{D_{1,n}} : \frac{D_{2,\tau}}{D_{2,n}}$$

whence with account taken of Eqs. (2.45) and (2.47), we get the law of displacement line refraction:

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\varepsilon_1}{\varepsilon_2}. \tag{2.49}$$

When displacement lines pass into a dielectric with a lower permittivity ε, the angle

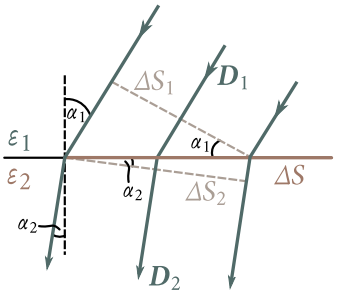

Fig. 2.11

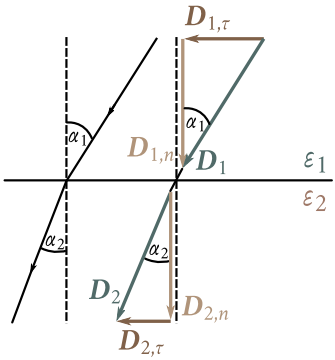

Fig. 2.12

made by them with a normal diminishes, hence, the lines are spaced farther apart; when the lines pass into a dielectric with a higher permittivity ε, on the contrary, they become closer together.

2.8. Forces Acting on a Charge in a Dielectric

If we introduce into an electric field in a vacuum a charged body of such small dimensions that the external field within the body can be considered homogeneous, then the body will experience the force

$$\boldsymbol{F} = q\boldsymbol{E}. \tag{2.50}$$

To place a charged body in a field set up in a dielectric, a cavity must be made in the latter. In a fluid dielectric, the body itself forms the cavity by displacing the dielectric from the volume it occupies. The field inside the cavity $\boldsymbol{E}_{\text{cav}}$ will differ from that in a continuous dielectric. Thus, we cannot calculate the force exerted on a charged body placed in a cavity as the product of the charge q and the field strength $\boldsymbol{E}$ in the dielectric before the body was introduced into it.

When calculating the force acting on a charged body in a fluid dielectric, we must take another circumstance into account. Mechanical tension is set up on the boundary with the body in the dielectric. This sets up an additional mechanical force $\boldsymbol{F}_{\text{ten}}$ acting on the body.

Thus, the force acting on a charged body in a dielectric, generally speaking, cannot be determined by Eq. (2.50), and it is usually a very complicated task to calculate it. These calculations give an interesting result for a fluid dielectric. The resultant of the electric force $q\boldsymbol{E}_{\text{cav}}$ and the mechanical force $\boldsymbol{F}_{\text{ten}}$ is found to be exactly equal to $q\boldsymbol{E}$, where $\boldsymbol{E}$ is the field strength in the continuous dielectric

$$\boldsymbol{F} = q\boldsymbol{E}_{\text{cav}} + \boldsymbol{F}_{\text{ten}} = q\boldsymbol{E}. \tag{2.51}$$

The strength of the field produced in a homogeneous infinitely extending dielectric by a point charge is determined by Eq. (2.49). Hence, we get the following expression for the forces of interaction of two point charges immersed in a homogeneous infinitely extending dielectric:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\varepsilon r^2}. \tag{2.52}$$

This formula expresses Coulomb's law for charges in a dielectric. It holds only for fluid dielectrics.

Some authors characterize Eq. (2.52) as "the most general expression of Coulomb's law". In this connection, we shall cite Richard P. Feynman: "Many older books on electricity start with the 'fundamental' law that the force between two charges is…[Eq. (2.52) is given]…, a point of view which is thoroughly unsatisfactory. For one thing, it is not true in general; it is true only for a world filled with a liquid. Secondly, it depends on the fact that ε is a constant which is only approximately true for most real materials"[7].

We shall not treat questions relating to the forces acting on a charge inside a cavity made in a solid dielectric.

2.9. Ferroelectrics

There is a group of substances that can have the property of spontaneous polarization in the absence of an external field. They are called **ferroelectrics**. This phenomenon was first discovered for Rochelle salt, and the first detailed investigation of the electrical properties of this salt was carried out by the Soviet physicists I. Kurchatov and P. Kobeko.

Ferroelectrics differ from the other dielectrics in a number of features:

1. Whereas the permittivity ε of ordinary dielectrics is only several units, reaching as an exception several scores (for example, for water $\varepsilon = 81$), the permittivity of ferroelectrics may be of the order of several thousands.

2. The dependence of P on E is not linear (see branch 1 of the curve shown in Fig. 2.13). Hence, the permittivity depends on the field strength.

3. When the field changes, the values of the polarization P (and, therefore, of the displacement D too) lag behind the field strength E. As a result, P and D are determined not only by the value of E at the given moment, but also by the preceding values of E, *i.e.*, they depend on the preceding history of the dielectric. This phenomenon is called **hysteresis** (from the Greek word "husterein"—to come

[7] R. P. Feynman, R. B. Leighton, M. Sands. The Feynman Lectures on Physics. Vol. II. Reading, Mass., Addison-Wesley (1965), p. 10-8.

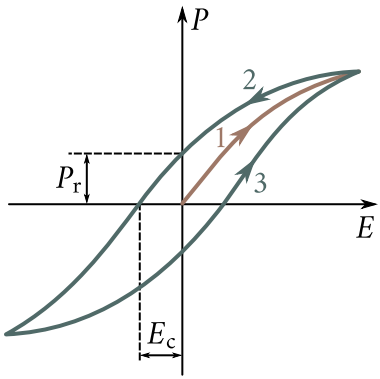

Fig. 2.13

late, be behind). Upon cyclic changes of the field, the dependence of P on E follows the curve shown in Fig. 2.13 and called a **hysteresis loop**. When the field is initially switched on, the polarization grows with E according to branch 1 of the curve. Diminishing of P takes place along branch 2. When E vanishes, the substance retains a value of the polarization P_r called the **residual polarization**. The polarization vanishes only under the action of an oppositely directed field E_c. This value of the field strength is called the **coercive force**. Upon a further change in E, branch 3 of the hysteresis loop is obtained, and so on.

The behaviour of the polarization of ferroelectrics is similar to that of the magnetization of ferromagnetics (see Sec. **??**), and this is the origin of their name.

Only crystalline substances having no centre of symmetry can be ferroelectrics. For example, the crystals of Rochelle salt belong to the rhombic system (see Sec. 13.2 of Vol. I). The interaction of the particles in a ferroelectric crystal leads to the fact that their dipole moments line up spontaneously parallel to one another. In exclusive cases, the identical orientation of the dipole moments extends to the entire crystal. Ordinarily, however, regions appear in a crystal in whose confines the dipole moments are parallel to one another, but the directions of polarization in different regions are different. Thus, the resultant moment of an entire crystal may equal zero. The regions of spontaneous polarization are also called **domains**. Under the action of an external field, the moments of the domains rotate as a single whole, arranging themselves in the direction of the field.

Every ferroelectric has a temperature at which the substance loses its unusual properties and becomes a normal dielectric. This temperature is called the **Curie point**. Rochelle salt has two Curie points, namely, −15 °C and 22 °C, and it behaves like a ferroelectric only in the interval between these two temperatures. Its electrical properties are conventional at temperatures below −15 °C and above 22 °C.

Chapter 3

CONDUCTORS IN AN ELECTRIC FIELD

3.1. Equilibrium of Charges on a Conductor

The carriers of a charge in a conductor are capable of moving under the action of a vanishingly small force. Therefore, the following conditions must be observed for the equilibrium of charges on a conductor:

1. The strength of the field everywhere inside the conductor must be zero:

$$\boldsymbol{E} = 0. \tag{3.1}$$

   In acccordance with Eq. (1.41), this signifies that the potential inside the conductor must be constant (φ = constant).

2. The strength of the field on the surface of the conductor must be directed along a normal to the surface at every point:

$$\boldsymbol{E} = \boldsymbol{E}_n. \tag{3.2}$$

   Consequently, when the charges are in equilibrium, the surface of the conductor will be an equipotential one.

If a charge q is imparted to a conducting body, the charge will be distributed so as to observe conditions of equilibrium. Let us imagine an arbitrary closed surface completely confined in a body. When the charges are in equilibrium, there is no field at every point inside the conductor; therefore, the flux of the electric displacement vector through the surface vanishes. According to Gauss's theorem, the sum of the charges inside the surface will also equal zero. This holds for a surface of any dimensions arbitrarily arranged inside a conductor. Hence, in equilibrium, there can be no surplus charges anywhere inside a conductor—they will all be distributed over the surface of the conductor with a certain density σ.

Since there are no surplus charges in a conductor in the state of equilibrium,

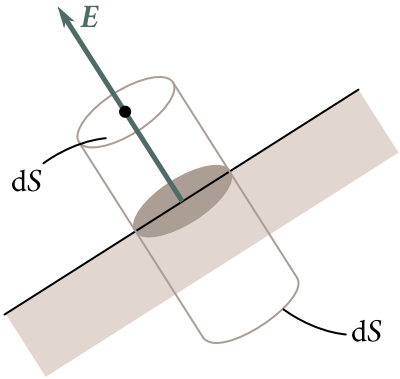

Fig. 3.1

the removal of substance from a volume taken inside the conductor will have no effect whatsoever on the equilibrium arrangement of the charges. Thus, a surplus charge will be distributed on a hollow conductor in the same way as on a solid one, *i.e.*, along its external surface. No surplus charges can be located on the surface of a cavity in the state of equilibrium. This conclusion also follows from the fact that the like elementary charges forming the given charge q mutually repel one another and, consequently, tend to take up positions at the farthest distance apart.

Imagine a small cylindrical surface formed by normals to the surface of a conductor and bases of the magnitude $\mathrm{d}S$, one of which is inside and the other outside the conductor (Fig. 3.1). The flux of the electric displacement vector through the inner part of the surface equals zero because $\boldsymbol{E}$ and, consequently, $\boldsymbol{D}$ vanish inside the conductor. Outside the conductor in direct proximity to it, the field strength $\boldsymbol{E}$ is directed along a normal to the surface. Hence, for the side surface of the cylinder protruding outward, $D_n = 0$, and for the outside base $D_n = D$ (the outside base is assumed to be very close to the surface of the conductor). Hence, the displacement flux through the surface being considered is $D\,\mathrm{d}S$, where D is the value of the displacement in direct proximity to the surface of the conductor. The cylinder contains an extraneous charge $\sigma\,\mathrm{d}S$ (σ is the charge density at the given spot on the surface of the conductor).

Applying Gauss's theorem, we get $D\,\mathrm{d}S = \sigma\,\mathrm{d}S$, *i.e.*, $D = \sigma$. We thus see that the strength of the field near the surface of the conductor is

$$E = \frac{\sigma}{\varepsilon_0 \varepsilon} \tag{3.3}$$

where ε is the permittivity of the medium surrounding the conductor [compare with Eq. (1.123) obtained for the case when $\varepsilon = 1$].

Let us consider the field produced by the charged conductor shown in Fig. 3.2.

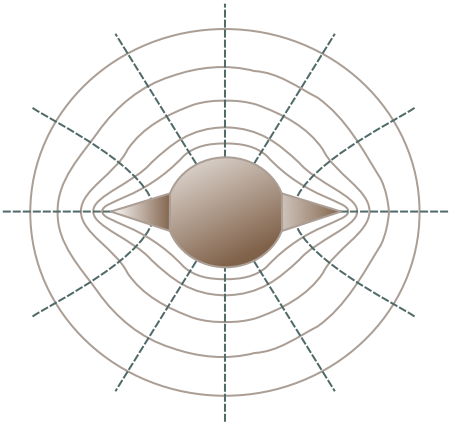

Fig. 3.2

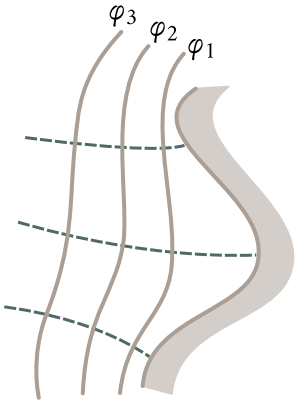

Fig. 3.3

At great distances from the conductor, equipotential surfaces have the shape of a sphere that is characteristic of a point charge (owing to the lack of space, a spherical surface is shown in the figure at a small distance from the conductor; the dash lines are field lines). As we approach the conductor, the equipotential surfaces become more and more similar to the surface of the conductor, which is an equipotential one. Near the projections, the equipotential surfaces are denser, hence, the field strength is also greater here. It thus follows that the density of the charges on the projections is especially great [see Eq. (3.3)]. We can arrive at the same conclusion by taking into account that owing to their mutual repulsion, charges tend to take up positions as far as possible from one another.

Near depressions in a conductor, the equipotential surfaces have a lower density (see Fig. 3.3). Accordingly, the field strength and the density of the charges at these spots will he smaller. In general, the density of charges with a given potential of a conductor is determined by the curvature of the surface—it grows with an increase in the positive curvature (convexity) and diminishes with an increase in the negative curvature (concavity). The density of charges is especially high on sharp points. Consequently, the field strength near such points may be so great that the gas molecules surrounding the conductor become ionized. Ions of the sign opposite to that of q are attracted to the conductor and neutralize its charge. Ions of the same sign as q begin to move away from the conductor, carrying along neutral molecules of the gas. The result is a noticeable motion of the gas called an electric wind. The charge of the conductor diminishes, it flows off the point, as it were, and is carried away by the wind. This phenomenon is therefore called emanation of a charge from a point.

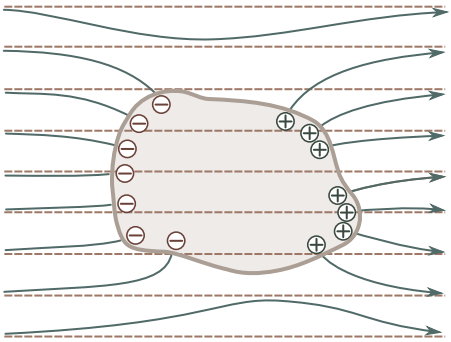

Fig. 3.4

3.2. A Conductor in an External Electric Field

When an uncharged conductor is introduced into an electric field, the charge carriers come into motion: the positive ones in the direction of the vector $\boldsymbol{E}$, the negative ones in the opposite direction. As a result, charges of opposite signs called **induced charges** appear at the ends of the conductor (Fig. 3.4, the dash lines depict the external field lines). The field of these charges is directed oppositely to the external field. Hence, the accumulation of charges at the ends of a conductor leads to weakening of the field in it. The charge carriers will be redistributed until conditions (3.1) and (3.2) are observed, *i.e.*, until the strength of the field inside the conductor vanishes and the field lines outside the conductor are perpendicular to its surface (see Fig. 3.4). Thus, a neutral conductor introduced into an electric field disrupts part of the field lines—they terminate on the negative induced charges and begin again on the positive ones.

The induced charges distribute themselves over the outer surface of a conductor. If a conductor contains a cavity, then upon equilibrium distribution of the induced charges, the field inside it vanishes. Electrostatic shielding is based on this phenomenon. If an instrument is to be protected from the action of external fields, it is surrounded by a conducting screen. The external field is compensated inside the screen by the induced charges appearing on its surface. Such a screen also functions quite well if it is made not solid, but in the form of a dense network.

3.3. Capacitance

A charge q imparted to a conductor distributes itself over its surface so that the strength of the field inside the conductor vanishes. Such a distribution is the only possible one. Therefore, if we impart to a conductor already carrying the charge

q another charge of the same magnitude, then the second charge must distribute itself over the conductor in exactly the same way as the first one. Otherwise, the charge will set up in the conductor a field differing from zero. We must note that this holds only for a conductor remote from other bodies (an isolated conductor). If other bodies are near the conductor, the imparting to the latter of a new portion of charge will produce either a change in the polarization of these bodies or a change in the induced charges on them. As a result, similarity in the distribution of different portions of the charge will be violated.

Thus, charges differing in magnitude distribute themselves on an isolated conductor in a similar way (the ratio of the densities of the charge at two arbitrary points on the surface of the conductor with any magnitude of the charge will be the same). It thus follows that the potential of an isolated conductor is proportional to the charge on it. Indeed, an increase in the charge a certain number of times leads to an increase in the strength of the field at every point of the space surrounding the conductor the same number of times. Accordingly, the work needed for transferring a unit charge from infinity to the surface of a conductor, *i.e.*, the potential of the conductor, grows the same number of times. Thus, for an isolated conductor

$$q = C\varphi. \tag{3.4}$$

The constant of proportionality C between the potential and the charge is called the **capacitance**. From Eq. (3.4), we get

$$C = \frac{q}{\varphi}. \tag{3.5}$$

In accordance with Eq. (3.5), the capacitance numerically equals the charge which when imparted to a conductor increases its potential by unity.

Let us calculate the potential of a charged sphere of radius R. The potential difference and the field strength are related by Eq. (1.45). We can therefore find the potential of the sphere φ by integrating Eq. (2.41) over r from R to ∞ (we assume that the potential at infinity equals zero):

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int_0^\infty \frac{q}{\varepsilon r^2}\,\mathrm{d}r = \frac{1}{4\pi\varepsilon_0}\frac{q}{\varepsilon R}. \tag{3.6}$$

Comparing Eqs. (3.5) and (3.6), we find that the capacitance of an isolated sphere of radius R immersed in a homogeneous infinite dielectric of permittivity ε is

$$C = 4\pi\varepsilon_0\varepsilon R. \tag{3.7}$$

The unit of capacitance is the capacitance of a conductor whose potential changes by 1 V when a charge of 1 C is imparted to it. This unit of capacitance is called the **farad** (F). In the Gaussian system, the formula for the capacitance of an

isolated sphere has the form

$$C = \varepsilon R. \tag{3.8}$$

Since ε is a dimensionless quantity, the capacitance determined by Eq. (3.8) has the dimension of length. The unit of capacitance is the capacitance of an isolated sphere with a radius of 1 cm in a vacuum. This unit of capacitance is called the **centimetre**. According to Eq. (3.5),

$$1\ \text{F} = \frac{1\ \text{C}}{1\ \text{V}} = \frac{3 \times 10^9\ \text{cgse}_C}{1/300} = 9 \times 10^{11}\ \text{cm}. \tag{3.9}$$

An isolated sphere having a radius of 9×10^{11} cm, *i.e.*, a radius 1500 times greater than that of the Earth, would have a capacitance of 1 F. We can thus see that the farad is a very great unit. For this reason, submultiples of a farad are used in practice—the millifarad (mF), the microfarad (μF), the nanofarad (nF), and the picofarad (pF) (see Vol. I, Table 3.1).

3.4. Capacitors

Isolated conductors have a small capacitance. Even a sphere of the Earth's size has a capacitance of only 700 μF. Devices are needed in practice, however, that with a low potential relative to the surrounding bodies would accumulate charges of an appreciable magnitude (*i.e.*, would have a high charge "capacity"). Such devices, called **capacitors**, are based on the fact that the capacitance of a conductor grows when other bodies are brought close to it. This is due to the circumstance that under the action of the field set up by the charged conductor, induced (on a conductor) or bound (on a dielectric) charges appear on the body brought up to it. Charges of the sign opposite to that of the charge q of the conductor will be closer to the conductor than charges of the same sign as q and, consequently, will have a greater influence on its potential. Therefore, when a body is brought close to a charged conductor, the potential of the latter diminishes in absolute value. According to Eq. (3.5), this signifies an increase in the capacitance of the conductor.

Capacitors are made in the form of two conductors placed close to each other. The conductors forming a capacitor are called its **plates**. To prevent external bodies from influencing the capacitance of a capacitor, the plates are shaped and arranged relative to each other so that the field set up by the charges accumulating on them is concentrated inside the capacitor. This condition is satisfied (see Sec. 1.14) by two plates arranged close to each other, two coaxial cylinders, and two concentric spheres. Accordingly, parallel-plate (plane), cylindrical, and spherical capacitors are encountered. Since the field is confined inside a capacitor, the electric displacement lines begin on one plate and terminate on the other. Consequently, the extraneous

charges produced on the plates have the same magnitude and are opposite in sign.

The basic characteristic of a capacitor is its capacitance, by which is meant a quantity proportional to the charge q and inversely proportional to the potential difference between the plates:

$$C = \frac{q}{\varphi_1 - \varphi_2}. \tag{3.10}$$

The potential difference $\varphi_1 - \varphi_2$ is called the **voltage** across the relevant points[1]. We shall use the symbol U to designate the voltage. Hence, Eq. (3.10) can be written as follows:

$$C = \frac{q}{U}. \tag{3.11}$$

Here, U is the voltage across the plates.

The capacitance of capacitors is measured in the same units as that of isolated conductors (see the preceding section).

The magnitude of the capacitance is determined by the geometry of the capacitor (the shape and dimensions of the plates and their separation distance), and also by the dielectric properties of the medium filling the space between the plates. Let us find the equation for the capacitance of a parallel-plate capacitor. If the area of a plate is S and the charge on it is q, then the strength of the field between the plates is

$$E = \frac{\sigma}{\varepsilon_0 \varepsilon} = \frac{q}{\varepsilon_0 \varepsilon S}$$

[see Eqs. (1.121) and (2.33); ε is the permittivity of the medium filling the gap between the plates].

In accordance with Eq. (1.45), the potential difference between the plates is

$$\varphi_1 - \varphi_2 = Ed = \frac{qd}{\varepsilon_0 \varepsilon S}.$$

Hence, for the capacitance of a parallel-plate capacitor, we get

$$C = \frac{\varepsilon_0 \varepsilon S}{d} \tag{3.12}$$

where S is the area of a plate, d is the separation distance of the plates, and ε is the permittivity of the substance filling the gap.

It must be noted that the accuracy of determining the capacitance of a real parallel-plate capacitor by Eq. (3.12) is the greater, the smaller is the separation distance d in comparison with the linear dimensions of the plates.

It can be seen from Eq. (3.12) that the dimension of the electric constant ε_0 equals the dimension of capacitance divided by that of length. Accordingly, ε_0 is measured in farads per metre [see Eq. (1.12)].

[1] A more general definition of the quantity called voltage will be given in Sec. **??** [see Eq. (**??**)].

If we disregard the dispersion of the field near the plate edges, we can easily obtain the following equation for the capacitance of a cylindrical capacitor:

$$C = \frac{2\pi\varepsilon_0\varepsilon l}{\ln\left(\frac{R_2}{R_1}\right)} \tag{3.13}$$

where l is length of the capacitor, R_1 and R_2 the radii of the internal and external plates.

The accuracy of determining the capacitance of a real capacitor by Eq. (3.13) is the greater, the smaller is the separation distance of the plates $d = R_2 - R_1$ in comparison with l and R_1.

The capacitance of a spherical capacitor is

$$C = 4\pi\varepsilon_0\varepsilon\left(\frac{R_1R_2}{R_2 - R_1}\right) \tag{3.14}$$

where R_1 and R_2 are the radii of the internal and external plates.

Apart from the capacitance, every capacitor is characterized by the maximum voltage $U_{\max}$ that may be applied across its plates without the danger of a breakdown. When this voltage is exceeded, a spark jumps across the space between the plates. The result is destruction of the dielectric and failure of the capacitor.