I. V. Savelyev

PHYSICS

A General Course

ELECTRICITY
& MAGNETISM

WAVES

OPTICS

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II

I. V. SAVELYEV

PHYSICS

A GENERAL COURSE

(In three volumes)

VOLUME II

ELECTRICITY
AND MAGNETISM

WAVES

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PREFACE

The main content of the present volume is the science of electromagnetism and the science of waves (elastic, electromagnetic, and light).

The International System of Units (SI) has been used throughout the book, although the reader is simultaneously acquainted with the Gaussian system. In addition to a list of symbols, the appendices at the end of the book give the units of electrical and magnetic quantities in the SI and in the Gaussian system of units, and also compare the form of the basic formulas of electromagnetism in both systems.

The course is the result of twenty five year's work in the Department of General Physics of the Moscow Institute of Engineering Physics. I am grateful to my colleagues and friends for their helpful discussions, criticism and advice in the course of the preparation of the book.

The present course is intended above all for higher technical schools with an extended syllabus in physics. The material has been arranged, however, so that the book can be used as a teaching aid for higher technical schools with an ordinary syllabus simply by omitting some sections.

Igor Savelyev

Moscow, November, 1979

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PART I

ELECTRICITY AND MAGNETISM

Chapter 1

ELECTRIC FIELD IN A VACUUM

1.1. Electric Charge

All bodies in nature are capable of becoming electrified, *i.e.*, acquiring an electric charge. The presence of such a charge manifests itself in that a charged body interacts with other charged bodies. Two kinds of electric charges exist. They are conventionally called positive and negative. Like charges repel each other, and unlike charges attract each other.

An electric charge is an integral part of certain elementary particles[1]. The charge of all elementary particles (if it is not absent) is identical in magnitude. It can be called an **elementary charge**. We shall use the symbol e to denote a positive elementary charge.

The elementary particles include, in particular, the electron (carrying the negative charge $-e$), the proton (carrying the positive charge $+e$), and the neutron (carrying no charge). These particles are the bricks which the atoms and molecules of any substance are built of, therefore all bodies contain electric charges. The particles carrying charges of different signs are usually present in a body in equal numbers and are distributed over it with the same density. The algebraic sum of the charges in any elementary volume of the body equals zero in this case, and each such volume (as well as the body as a whole) will be neutral. If in some way or other we create a surplus of particles of one sign in a body (and, correspondingly, a shortage of particles of the opposite sign), the body will be charged. It is also possible, without changing the total number of positive and negative particles, to cause them to be redistributed in a body so that one part of it has a surplus of charges of one sign and the other part a surplus of charges of the opposite sign. This can be

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[1]Elementary particles are defined as such microparticles whose internal structure at the present level of development of physics cannot be conceived as a combination of other particles.

done by bringing a charged body close to an uncharged metal one.

Since a charge q is formed by a plurality of elementary charges, it is an integral multiple of e:

$$q = \pm Ne. \qquad (1.1)$$

An elementary charge is so small, however, that macroscopic charges may be considered to have continuously changing magnitudes.

If a physical quantity can take on only definite discrete values, it is said to be quantized. The fact expressed by Eq. (1.1) signifies that an electric charge is quantized.

The magnitude of a charge measured in different inertial reference frames will be found to be the same. Hence, an electric charge is relativistically invariant. It thus follows that the magnitude of a charge does not depend on whether the charge is moving or at rest.

Electric charges can vanish and appear again. Two elementary charges of opposite signs always appear or vanish simultaneously, however. For example, an electron and a positron (a positive electron) meeting each other annihilate, *i.e.*, transform into neutral gamma-photons. This is attended by vanishing of the charges $-e$ and $+e$. In the course of the process called the birth of a pair, a gamma-photon getting into the field of an atomic nucleus transforms into a pair of particles—an electron and a positron. This process causes the charges $-e$ and $+e$ to appear.

Thus, the total charge of an electrically isolated system[2] cannot change. This statement forms the **law of electric charge conservation**.

We must note that the law of electric charge conservation is associated very closely with the relativistic invariance of a charge. Indeed, if the magnitude of a charge depended on its velocity, then by bringing charges of one sign into motion we would change the total charge of the relevant isolated system.

1.2. Coulomb's Law

The law obeyed by the force of interaction of point charges was established experimentally in 1785 by the French physicist Charles A. de Coulomb (1736-1806). A **point charge** is defined as a charged body whose dimensions may be disregarded in comparison with the distances from this body to other bodies carrying an electric charge.

Using a torsion balance (Fig. 1.1) similar to that employed by H. Cavendish to determine the gravitational constant (see Vol. I, Sec. 6.1), Coulomb measured the

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[2]A system is referred to as electrically isolated if no charged particles can penetrate through the surface confining it.
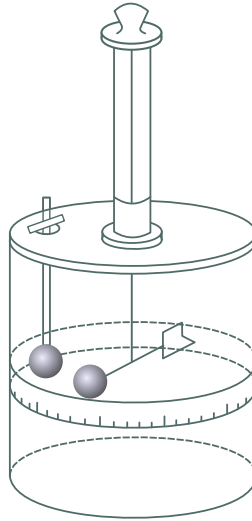
Fig. 1.1

force of interaction of two charged spheres depending on the magnitude of the charges on them and on the distance between them. He proceeded from the fact that when a charged metal sphere was touched by an identical uncharged sphere, the charge would be distributed equally between the two spheres.

As a result of his experiments, Coulomb arrived at the conclusion that *the force of interaction between two stationary point charges is proportional to the magnitude of each of them and inversely proportional to the square of the distance between them.* The direction of the force coincides with the straight line connecting the charges.

It must be noted that the direction of the force of interaction along the straight line connecting the point charges follows from considerations of symmetry. An empty space is assumed to be homogeneous and isotropic. Consequently, the only direction distinguished in the space by stationary point charges introduced into it is that from one charge to the other. Assume that the force F acting on the charge q_i (Fig. 1.2) makes the angle α with the direction from q_1 to q_2, and that α differs from 0 or π. But owing to axial symmetry, there are no grounds to set the force F aside from the multitude of forces of other directions making the same angle α with the axis q_1-q_2 (the directions of these forces form a cone with a cone angle of 2α). The difficulty appearing as a result of this vanishes when α equals 0 or π.

Coulomb's law can be expressed by the formula

$$F_{12} = -k\frac{q_1 q_2}{r^2}\,\hat{e}_{12}. \qquad (1.2)$$

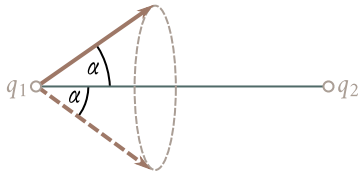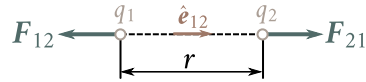Here, k is a proportionality constant assumed to be positive, q_1 and q_2 are magni-

Fig. 1.2

Fig. 1.3

tudes of the interacting charges, r is the distance between the charges, $\hat{\boldsymbol{e}}_{12}$ is the unit vector directed from the charge q_1 to q_2 and $\boldsymbol{F}_{12}$ is the force acting on the charge q_1 (Fig. 1.3; the figure corresponds to the case of like charges).

The force $\boldsymbol{F}_{21}$ differs from $\boldsymbol{F}_{12}$ in its sign:

$$\boldsymbol{F}_{21} = k\frac{q_1 q_2}{r^2}\,\hat{\boldsymbol{e}}_{12}. \tag{1.3}$$

The magnitude of the interaction force, which is the same for both charges, can be written in the form

$$F = k\frac{|q_1 q_2|}{r^2}. \tag{1.4}$$

Experiments show that the force of interaction between two given charges does not change if other charges are placed near them. Assume that we have the charge q_a and, in addition, N other charges $q_1, q_2, \ldots, q_N$. It can be seen from the above that the resultant force $\boldsymbol{F}$ with which all the N charges q_i act on q_a is

$$\boldsymbol{F} = \sum_{i=1}^{N} \boldsymbol{F}_{a,\,i} \tag{1.5}$$

where $\boldsymbol{F}_{a,\,i}$ is the force with which the charge q_i acts on q_a in the absence of the other $N - 1$ charges.

The fact expressed by Eq. (1.5) permits us to calculate the force of interaction between charges concentrated on bodies of finite dimensions, knowing the law of interaction between point charges. For this purpose, we must divide each charge into so small charges dq that they can be considered as point ones, use Eq. (1.2) to calculate the force of interaction between the charges dq taken in pairs, and then perform vector summation of these forces. Mathematically, this procedure coincides completely with the calculation of the force of gravitational attraction between bodies of finite dimensions (see Vol. I, Sec. 6.1).

All experimental facts available lead to the conclusion that Coulomb's law holds for distances from 10^{-15} m to at least several kilometres. There are grounds to presume that for distances smaller than 10^{-16} m the law stops being correct. For very great distances, there are no experimental confirmations of Coulomb's law. But there are also no reasons to expect that this law stops being obeyed with very

great distances between charges.

1.3. Systems of Units

We can make the proportionality constant in Eq. (1.2) equal unity by properly choosing the unit of charge (the units for F and r were established in mechanics). The relevant unit of charge (when F and r are measured in cgs units) is called the **absolute electrostatic unit** of charge (cgse_q). It is the magnitude of a charge that interacts with a force of 1 dyn in a vacuum with an equal charge at a distance of 1 cm from it.

Careful measurements (they are described in Sec. **??**) showed that an elementary charge is

$$e = 4.80 \times 10^{-10} \ \text{cgse}_q. \tag{1.6}$$

Adopting the units of length, mass, time, and charge as the basic ones, we can construct a system of units of electrical and magnetic quantities. The system based on the centimetre, gramme, second, and the cgse_q unit is called the **absolute electrostatic system of units** (the cgse system). It is founded on Coulomb's law, *i.e.*, the law of interaction between charges at rest. On a later page, we shall become acquainted with the **absolute electromagnetic system of units** (the cgsm system) based on the law of interaction between conductors carrying an electric current. The Gaussian system in which the units of electrical quantities coincide with those of the cgse system, and of magnetic quantities with those of the cgsm system, is also an absolute system.

Equation (1.4) in the cgse system becomes

$$F = \frac{|q_1 q_2|}{r^2}. \tag{1.7}$$

This equation is correct if the charges are in a vacuum. It has to be determined more accurately for charges in a medium (see Sec. **??**).

USSR State Standard GOST 9867-61, which came into force on January 1, 1963, prescribes the preferable use of the International System of Units (SI). The basic units of this system are the metre, kilogramme, second, ampere, kelvin, candela, and mole. The SI unit of force is the newton (N) equal to 10^5 dynes.

In establishing the units of electrical and magnetic quantities, the SI system, like the cgsm one, proceeds from the law of interaction of current-carrying conductors instead of charges. Consequently, the proportionality constant in the equation of Coulomb's law is a quantity with a dimension and differing from unity.

The SI unit of charge is the coulomb (C). It has been found experimentally that

$$1 \, \text{C} = 2.998 \times 10^9 \approx 3 \times 10^9 \ \text{cgse}_q. \tag{1.8}$$

To form an idea of the magnitude of a charge of 1 C, let us calculate the force with which two point charges of 1 C each would interact with each other if they were 1 m apart. By Eq. (1.7)

$$F = \frac{3 \times 10^9 \times 3 \times 10^9}{100^2} \; \text{cgse}_F = 9 \times 10^{14} \, \text{dyn} = 9 \times 10^9 \, \text{N} \approx 10^9 \, \text{kgf.} \quad (1.9)$$

An elementary charge expressed in coulombs is

$$e = 1.60 \times 10^{-19} \, \text{C.} \tag{1.10}$$

1.4. Rationalized Form of Writing Formulas

Many formulas of electrodynamics when written in the cgs systems (in particular, in the Gaussian one) include as factors 4π and the so-called electromagnetic constant c equal to the speed of light in a vacuum. To eliminate these factors in the formulas that are most important in practice, the proportionality constant in Coulomb's law is taken equal to $1/4\pi\varepsilon_0$. The equation of the law for charges in a vacuum will thus become

$$F = \frac{1}{4\pi\varepsilon_0} \frac{|q_1 q_2|}{r^2}. \tag{1.11}$$

The other formulas change accordingly. This modified way of writing formulas is called **rationalized**. Systems of units constructed with the use of rationalized formulas are also called **rationalized**. They include the SI system.

The quantity ε_0 is called the **electric constant**. It has the dimension of capacitance divided by length. It is accordingly expressed in units called the farad per metre. To find the numerical value of ε_0, we shall introduce the values of the quantities corresponding to the case of two charges of 1 C each and 1 m apart into Eq. (1.11). By Eq. (1.9), the force of interaction in this case is 9×10^9 N. Using this value of the force, and also $q_1 = q_2 = 1$ C and $r = 1$ m in Eq. (1.11), we get

$$9 \times 10^9 = \frac{1}{4\pi\varepsilon_0} \frac{|1 \times 1|}{1^2}$$

whence

$$\varepsilon_0 = \frac{1}{4\pi \times 9 \times 10^9} = 0.885 \times 10^{-11} \, \text{F m}^{-1}. \tag{1.12}$$

The Gaussian system of units was widely used and is continuing to be used in physical publications. We therefore consider it essential to acquaint our reader with both the SI and the Gaussian system. We shall set out the material in the SI units showing at the same time how the formulas look in the Gaussian system. The fundamental formulas of electrodynamics written in the SI and the Gaussian system are compared in Appendix. **??**.
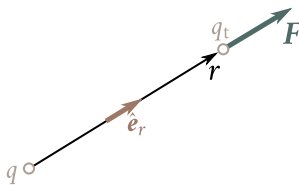
Fig. 1.4

1.5. Electric Field. Field Strength

Charges at rest interact through an electric field[3]. A charge alters the properties of the space surrounding it—it sets up an electric field in it. This field manifests itself in that an electric charge placed at a point of it experiences the action of a force. Hence, to see whether there is an electric field at a given place, we must place a charged body (in the following we shall say simply a charge for brevity) at it and determine whether or not it experiences the action of an electric force. We can evidently assess the "strength" of the field according to the magnitude of the force exerted on the given charge.

Thus, to detect and study an electric field, we must use a "test" charge. For the force acting on our test charge to characterize the field "at the given point", the test charge must be a point one. Otherwise, the force acting on the charge will characterize the properties of the field averaged over the volume occupied by the body that carries the test charge.

Let us study the field set up by the stationary point charge q with the aid of the point test charge q_t. We place the test charge at a point whose position relative to the charge q is determined by the position vector $\boldsymbol{r}$ (Fig. 1.4). We see that the test charge experiences the force

$$\boldsymbol{F} = q_t \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\boldsymbol{e}}_r \right) \tag{1.13}$$

[see Eqs. (1.3) and (1.11)]. Here $\hat{\boldsymbol{e}}_r$ is the unit vector of the position vector $\boldsymbol{r}$.

A glance at Eq. (1.13) shows that the force acting on our test charge depends not only on the quantities determining the field (on q and $\boldsymbol{r}$), but also on the magnitude of the test charge q_t. If we take different test charges q_t', q_t'', etc., then the forces $\boldsymbol{F}'$, $\boldsymbol{F}''$, etc. which they experience at the given point of the field will be different. We can see from Eq. (1.13), however, that the ratio F/q_t for all the test charges will be the same and depend only on the values of q and $\boldsymbol{r}$ determining the field at the given point. It is therefore natural to adopt this ratio as the quantity characterizing

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[3]We shall see in Sec. **??** that when considering moving charges, their interaction in addition to an electric field is due to a magnetic field.

an electric field:

$$E = \frac{F}{q_t}. \tag{1.14}$$

This vector quantity is called the **electric field strength** (or **intensity**) at a given point (*i.e.,* at the point where the test charge q_t experiences the action of the force **F**).

According to Eq. (1.14), the electric field strength numerically equals the force acting on a unit point charge at the given point of the field. The direction of the vector **E** coincides with that of the force acting on a positive charge.

It must be noted that Eq. (1.14) also holds when the test charge is negative ($q_t <$ 0). In this case, the vectors **E** and **F** have opposite directions.

We have arrived at the concept of electric field strength when studying the field of a stationary point charge. Definition (1.14), however, also covers the case of a field set up by any collection of stationary charges, but here the following clarification is needed. The arrangement of the charges setting up the field being studied may change under the action of the test charge. This will happen, for example, when the charges producing the field are on a conductor and can freely move within its limits. Therefore, to avoid appreciable alterations in the field being studied, a sufficiently small test charge must be taken.

It follows from Eqs. (1.13) and (1.14) that the field strength of a point charge varies directly with the magnitude of the charge q and inversely with the square of the distance r from the charge to the given point of the field:

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{e}_r. \tag{1.15}$$

The vector **E** is directed along the radial straight line passing through the charge and the given point of the field, from the charge if the latter is positive and toward the charge if it is negative.

In the Gaussian system, the equation for the field strength of a point charge in a vacuum has the form

$$E = \frac{q}{r^2} \hat{e}_r. \tag{1.16}$$

The unit of electric field strength is the strength at a point where unit force (1 N in the SI and 1 dyn in the Gaussian system) acts on unit charge (1 C in the SI and 1 cgse$_q$ in the Gaussian system). This unit has no special name in the Gaussian system. The SI unit of electric field strength is called the volt per metre (V m^{-1}) [see Eq. (1.44)].

According to Eq. (1.15), a charge of 1 C produces the following field strength in

a vacuum at a distance of 1 m from this charge:

$$E = \frac{1}{4\pi \left(1/4\pi \times 9 \times 10^9\right)} \frac{1}{1^2} = 9 \times 10^9 \, \text{V m}^{-1}.$$

This strength in the Gaussian system is

$$E = \frac{q}{r^2} = \frac{3 \times 10^9}{100^2} = 3 \times 10^5 \, \text{cgse}_E.$$

Comparing these two results, we find that

$$1 \, \text{cgse}_E = 3 \times 10^4 \, \text{V m}^{-1}. \tag{1.17}$$

According to Eq. (1.14), the force exerted on a test charge is

$$\boldsymbol{F} = q_t \boldsymbol{E}.$$

It is obvious that any point charge q[4] at a point of a field with the strength $\boldsymbol{E}$ will experience the force

$$\boldsymbol{F} = q\boldsymbol{E}. \tag{1.18}$$

If the charge q is positive, the direction of the force coincides with that of the vector $\boldsymbol{E}$. If q is negative, the vectors $\boldsymbol{F}$ and $\boldsymbol{E}$ are directed oppositely.

We mentioned in Sec. 1.2 that the force with which a system of charges acts on a charge not belonging to the system equals the vector sum of the forces which each of the charges of the system exerts separately on the given charge [see Eq. (1.15)]. Hence it follows that *the field strength of a system of charges equals the vector sum of the field strengths that would be produced by each of the charges of the system separately*:

$$\boldsymbol{E} = \sum_i \boldsymbol{E}_i. \tag{1.19}$$

This statement is called the **principle of electric field superposition**.

The superposition principle allows us to calculate the field strength of any system of charges. By dividing extended charges into sufficiently small fractions dq, we can reduce any system of charges to a collection of point charges. We calculate the contribution of each of such charges to the resultant field by Eq. (1.15).

An electric field can be described by indicating the magnitude and direction of the vector $\boldsymbol{E}$ for each of its points. The combination of these vectors forms the field of the electric field strength vector (compare with the field of the velocity vector, Vol. I, Sec. 9.1). The velocity vector field can be represented very illustratively with the aid of flow lines. Similarly, an electric field can be described with the aid of strength lines, which we shall call for short $\boldsymbol{E}$ lines or field lines. These lines are drawn so that a tangent to them at every point coincides with the direction of the

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[4]In Eq. (1.15), q stands for the charge setting up the field. In Eq. (1.18), q stands for the charge experiencing the force $\boldsymbol{F}$ at a point of strength $\boldsymbol{E}$.
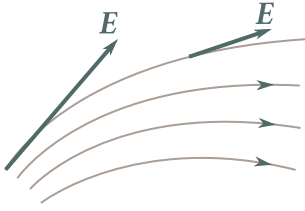
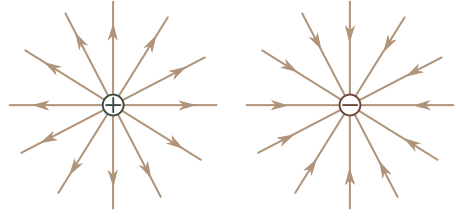Fig. 1.5   Fig. 1.6

vector E. The density of the lines is selected so that their number passing through a unit area at right angles to the lines equals the numerical value of the vector E. Hence, the pattern of field lines permits us to assess the direction and magnitude of the vector E at various points of space (Fig. 1.5).

The E lines of a point charge field are a collection of radial straight lines directed away from the charge if it is positive and toward it if it is negative (Fig. 1.6). One end of each line is at the charge, and the other extends to infinity. Indeed, the total number of lines intersecting a spherical surface of arbitrary radius r will equal the product of the density of the lines and the surface area of the sphere $4\pi r^2$. We have assumed that the density of the lines numerically equals $E = (1/4\pi\varepsilon_0)(q/r^2)$. Hence, the number of lines is $(1/4\pi\varepsilon_0)(q/r^2)4\pi r^2 = q/\varepsilon_0$. This result signifies that the number of lines at any distance from a charge will be the same. It thus follows hat the lines do not begin and do not terminate anywhere except tor the charge. Beginning at the charge, they extend to infinity (the charge is positive), or arriving from infinity, they terminate at the charge (the latter is negative). This property of the E lines is common for all electrostatic fields, *i.e.*, fields set up by any system of stationary charges: the field lines can begin or terminate only at charges or extend to infinity.

1.6. Potential

Let us consider the field produced by a stationary point charge q. At any point of this field, the point charge q' experiences the force

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}\,\hat{\boldsymbol{e}}_r = F(r)\hat{\boldsymbol{e}}_r. \tag{1.20}$$

Here $F(r)$ is the magnitude of the force $\boldsymbol{F}$, and $\hat{\boldsymbol{e}}_r$ is the unit vector of the position vector $\boldsymbol{r}$ determining the position of the charge q' relative to the charge q.

The force (1.20) is a central one (see Vol. I, Sec. 3.4). A central field of forces is conservative. Consequently, the work done by the forces of the field on the charge q' when it is moved from one point to another does not depend on the path. This
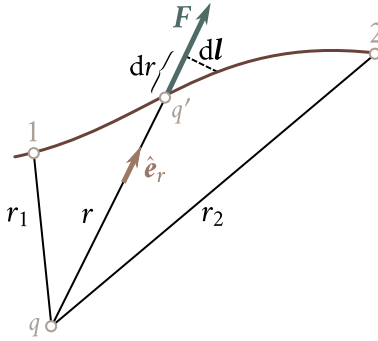
Fig. 1.7

work is

$$A_{12} = \int_1^2 F(r)\hat{e}_r\, d\boldsymbol{l} \tag{1.21}$$

where $d\boldsymbol{l}$ is the elementary displacement of the charge q'. Inspection of Fig. 1.7 shows that the scalar product $\hat{e}_r\, d\boldsymbol{l}$ equals the increment of the magnitude of the position vector $\boldsymbol{r}$, i.e., dr. Equation (1.21) can therefore be written in the form

$$A_{12} = \int_1^2 F(r)\, dr$$

[compare with Eq. (3.24) of Vol. I]. Introduction of the expression for $F(r)$ yields

$$A_{12} = \frac{qq'}{4\pi\varepsilon_0} \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{1}{4\pi\varepsilon_0}\left(\frac{qq'}{r_1} - \frac{qq'}{r_2}\right). \tag{1.22}$$

The work of the forces of a conservative field can be represented as a decrement of the potential energy:

$$A_{12} = W_{\mathrm{p},1} - W_{\mathrm{p},2}. \tag{1.23}$$

A comparison of Eqs. (1.22) and (1.23) leads to the following expression for the potential energy of the charge q' in the field of the charge q:

$$W_{\mathrm{p}} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r_2} + \text{constant}.$$

The value of the constant in the expression for the potential energy is usually chosen so that when the charge moves away to infinity (i.e., when $r = \infty$), the potential energy vanishes. When this condition is observed, we get

$$W_{\mathrm{p}} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r_2}. \tag{1.24}$$

Let us use the charge q' as a test charge for studying the field. By Eq. (1.24), the potential energy which the test charge has depends not only on its magnitude

q', but also on the quantities q and r determining the field. Thus, we can use this energy to describe the field just like we used the force acting on the test charge for this purpose.

Different test charges q'_t, q''_t, etc. will have different energies W'_p, W''_p, etc. at the same point of a field. But the ratio W_p/q_t will be the same for all the charges [see Eq. (1.24)]. The quantity

$$\varphi = \frac{W_p}{q_t} \qquad (1.25)$$

is called the **field potential** at a given point and is used together with the field strength E to describe electric fields.

It can be seen from Eq. (1.25) that the potential numerically equals the potential energy which a unit positive charge would have at the given point of the field. Substituting for the potential energy in Eq. (1.25) its value from (1.24), we get the following expression for the potential of a point charge:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \qquad (1.26)$$

In the Gaussian system, the potential of the field of a point charge in a vacuum is determined by the formula

$$\varphi = \frac{q}{r}. \qquad (1.27)$$

Let us consider the field produced by a system of N point charges $q_1, q_2, \ldots, q_N$. Let $r_1, r_2, \ldots, r_N$ be the distances from each of the charges to the given point of the field. The work done by the forces of this field on the charge q' will equal the algebraic sum of the work done by the forces set up by each of the charges separately:

$$A_{12} = \sum_{i=1}^{N} A_i.$$

By Eq. (1.22), each work A_i equals

$$A_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_i q'}{r_{i,1}} - \frac{q_i q'}{r_{i,2}}\right)$$

where $r_{i,1}$ is the distance from the charge q_i to the initial position of the charge q', and $r_{i,2}$ is the distance from q_i to the final position of the charge q'. Hence,

$$A_{i2} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i q'}{r_{i,1}} - \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i q'}{r_{i,2}}.$$

Comparing this equation with Eq. (1.23), we get the following expression for the

potential energy of the charge q' in the field of a system of charges:

$$W_{\mathrm{p}} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i q'}{r_i}$$

from which it can be seen that

$$\varphi = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i}. \tag{1.28}$$

Comparing this formula with Eq. (1.26), we arrive at the conclusion that *the potential of the field produced by a system of charge equals the algebraic sum of the potentials produced by each of the charges separately.* Whereas the field strengths are added vectorially in the superposition of fields, the potentials are added algebraically. This is why it is usually much simpler to calculate the potentials than the electric field strengths.

Examination of Eq. (1.25) shows that the charge q at a point of a field with the potential φ has the potential energy

$$W_{\mathrm{p}} = q\varphi. \tag{1.29}$$

Hence, the work of the field forces on the charge q can be expressed through the potential difference:

$$A_{12} = W_{\mathrm{p},1} - W_{\mathrm{p},2} = q\,(\varphi_1 - \varphi_2). \tag{1.30}$$

Thus, the work done on a charge by the forces of a field equals the product of the magnitude of the charge and the difference between the potentials at the initial and final points (*i.e.*, the potential decrement).

If the charge q is removed from a point having the potential φ to infinity (where by convention the potential vanishes), then the work of the field forces will be

$$A_\infty = q\varphi. \tag{1.31}$$

Here, it follows that the potential numerically equals the work done by the forces of a field on a unit positive charge when the latter is removed from the given point to infinity. Work of the same magnitude must be done against the electric field forces to move a unit positive charge from infinity to the given point of a field.

Equation (1.31) can be used to establish the units of potential. The unit of potential is taken equal to the potential at a point of a field when work equal to unity is required to move unit positive charge from infinity to this point. The SI unit of potential called the volt (V) is taken equal to the potential at a point when work of 1 joule has to be done to move a charge of 1 coulomb from infinity to this point:

$$1\,\mathrm{J} = 1\,\mathrm{C} \times 1\,\mathrm{V}, \quad \text{thus,} \quad 1\,\mathrm{V} = \frac{1\,\mathrm{J}}{1\,\mathrm{C}}. \tag{1.32}$$

The absolute electrostatic unit of potential (cgse_φ) is taken equal to the poten-

tial at a point when work of 1 erg has to be done to move a charge of 1 cgse_q from infinity to this point. Expressing 1 J and 1 C in Eq. (1.32) through cgse units, we shall find the relation between the volt and the cgse potential unit:

$$1\,\text{V} = \frac{1\,\text{J}}{1\,\text{C}} = \frac{10^7\,\text{erg}}{3 \times 10^9\,\text{cgse}_q} = \frac{1}{300}\,\text{cgse}_\varphi. \tag{1.33}$$

Thus, 1 cgse_φ equals 300 V.

A unit of energy and work called the **electron-volt** (eV) is frequently used in physics. An electron-volt is defined as the work done by the forces of a field on a charge equal to that of an electron (*i.e.*, on the elementary charge e) when it passes through a potential difference of 1 V:

$$1\,\text{eV} = 1.60 \times 10^{-19}\,\text{C} \times 1\,\text{V} = 1.60 \times 10^{-19}\,\text{J} = 1.60 \times 10^{-12}\,\text{erg}. \tag{1.34}$$

Multiple units of the electron-volt are also used:

$$1\,\text{keV (kiloelectron-volt)} = 10^3\,\text{eV},$$
$$1\,\text{MeV (megaelectron-volt)} = 10^6\,\text{eV},$$
$$1\,\text{GeV (gigaelectron-volt)} = 10^9\,\text{eV}.$$

1.7. Interaction Energy of a System of Charges

Equation (1.24) can be considered as the mutual potential energy of the charges q and q'. Using the symbols q_1 and q_2 for these charges, we get the following formula for their interaction energy:

$$W_\text{p} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}. \tag{1.35}$$

The symbol r_{12} stands for the distance between the charges.

Let us consider a system consisting of N point charges $q_1, q_2, \ldots, q_N$. We showed in Sec. 3.6 of Vol. I that the energy of interaction of such a system equals the sum of the energies of interaction of the charges taken in pairs:

$$W_\text{p} = \frac{1}{2}\sum_{(i \neq k)} W_{\text{p},ik}(r_{ik}) \tag{1.36}$$

[see Eq. (3.60) of Vol. I].

According to Eq. (1.35)

$$W_{\text{p},ik} = \frac{1}{4\pi\varepsilon_0}\frac{q_i q_k}{r_{ik}}.$$

Using this equation in (1.36), we find that

$$W_\text{p} = \frac{1}{2}\sum_{(i \neq k)} \frac{1}{4\pi\varepsilon_0}\frac{q_i q_k}{r_{ik}}. \tag{1.37}$$

In the Gaussian system, the factor $1/(4\pi\varepsilon_0)$ is absent in this equation.

In Eq. (1.37), summation is performed over the subscripts i and k. Both subscripts pass independently through all the values from 1 to N. Addends for which the value of the subscript i coincides with that of k are not taken into consideration. Let us write Eq. (1.37) as follows:

$$W_{\mathrm{p}} = \frac{1}{2} \sum_{i=1}^{N} q_i \sum_{\substack{i=1 \\ (i\neq k)}}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_k}{r_{ik}}. \tag{1.38}$$

The expression

$$\varphi_i = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{i=1 \\ (i\neq k)}}^{N} \frac{q_k}{r_{ik}}$$

is the potential produced by all the charges except q_i at the point where the charge q_i is. With this in view, we get the following formula for the interaction energy:

$$W_{\mathrm{p}} = \frac{1}{2} \sum_{i=1}^{N} q_i \varphi_i. \tag{1.39}$$

1.8. Relation Between Electric Field Strength and Potential

An electric field can be described either with the aid of the vector quantity $\boldsymbol{E}$, or with the aid of the scalar quantity φ. There must evidently be a definite relation between these quantities. If we bear in mind that $\boldsymbol{E}$ is proportional to the force acting on a charge and φ to the potential energy of the charge, it is easy to see that this relation must be similar to that between the potential energy and the force.

The force $\boldsymbol{F}$ is related to the potential energy by the expression

$$\boldsymbol{F} = -\nabla W_{\mathrm{p}} \tag{1.40}$$

[see Eq. (3.32) of Vol. I]. For a charged particle in an electrostatic field, we have $\boldsymbol{F} = q\boldsymbol{E}$ and $W_{\mathrm{p}} = q\varphi$. Introducing these values into Eq. (1.40), we find that

$$q\boldsymbol{E} = -\nabla(q\varphi).$$

The constant q can be put outside the gradient sign. Doing this and then cancelling q, we arrive at the formula

$$\boldsymbol{E} = -\nabla\varphi \tag{1.41}$$

establishing the relation between the field strength and potential.

Taking into account the definition of the gradient [see Eq. (3.31) of Vol. 1], we

can write that

$$E = -\frac{\partial \varphi}{\partial x}\hat{e}_x - \frac{\partial \varphi}{\partial y}\hat{e}_y - \frac{\partial \varphi}{\partial z}\hat{e}_z. \tag{1.42}$$

Hence, Eq. (1.41) has the following form in projections onto the coordinate axes:

$$E_x = -\frac{\partial \varphi}{\partial x}, \quad E_y = -\frac{\partial \varphi}{\partial y}, \quad E_z = -\frac{\partial \varphi}{\partial z}. \tag{1.43}$$

Similarly, the projection of the vector E onto an arbitrary direction l equals the derivative of φ with respect to l taken with the opposite sign, *i.e.*, the rate of diminishing of the potential when moving along the direction l:

$$E_l = -\frac{\partial \varphi}{\partial l}. \tag{1.44}$$

It is easy to see that Eq. (1.44) is correct by choosing l as one of the coordinate axes and taking Eq. (1.43) into account.

Let us explain Eq. (1.41) using as an example the field of a point charge. The potential of this field is expressed by Eq. (1.26). Passing over to Cartesian coordinates, we get the expression

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{(x^2 + y^2 + z^2)^{1/2}}.$$

The partial derivative of this function with respect to x is

$$\frac{\partial \varphi}{\partial x} = -\frac{1}{4\pi\varepsilon_0}\frac{qx}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{1}{4\pi\varepsilon_0}\frac{qx}{r^3}.$$

Similarly,

$$\frac{\partial \varphi}{\partial y} = -\frac{1}{4\pi\varepsilon_0}\frac{qy}{r^3}, \quad \frac{\partial \varphi}{\partial y} = -\frac{1}{4\pi\varepsilon_0}\frac{qz}{r^3}.$$

Using the found values of the derivatives in Eq. (1.42), we arrive at the expression

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q\left(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z\right)}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{q\boldsymbol{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{e}_r$$

that coincides with Eq. (1.15).

Equation (1.41) allows us to find the field strength at every point from the known values of φ. We can also solve the reverse problem, *i.e.*, find the potential difference between two arbitrary points of a field according to the given values of E. For this purpose, we shall take advantage of the circumstance that the work done by the forces of a field on the charge q when it is moved from point 1 to point 2 can be calculated as

$$A_{12} = \int_1^2 q\boldsymbol{E}\, d\boldsymbol{l}.$$

At the same time in accordance with Eq. (1.30), this work can be written as

$$A_{12} = q\,(\varphi_1 - \varphi_2).$$

Equating these two expressions and cancelling q, we obtain

$$\varphi_1 - \varphi_2 = \int_1^2 \mathbf{E}\,\mathrm{d}\mathbf{l}. \qquad (1.45)$$

The integral can be taken along any line joining points 1 and 2 because the work of the field forces is independent of the path. For circumvention along a closed contour, $\varphi_1 = \varphi_2$, and Eq. (1.45) becomes

$$\oint \mathbf{E}\,\mathrm{d}\mathbf{l} = 0 \qquad (1.46)$$

(the circle on the integral sign indicates that integration is performed over a closed contour). It must be noted that this relation holds only for an electrostatic field. We shall see on a later page that the field of moving charges (*i.e.*, a field changing with time) is not a potential one. Therefore, condition (1.46) is not observed for it.

An imaginary surface all of whose points have the same potential is called an equipotential surface. Its equation has the form

$$\varphi(x, y, z) = \text{constant}.$$

The potential does not change in movement along an equipotential surface over the distance $\mathrm{d}l$ ($\mathrm{d}\varphi = 0$). Hence, according to Eq. (1.44), the tangential component of the vector $\mathbf{E}$ to the surface equals zero. We thus conclude that the vector $\mathbf{E}$ at every point is directed along a normal to the equipotential surface passing through the given point. Bearing in mind that the vector $\mathbf{E}$ is directed along a tangent to an $\mathbf{E}$ line, we can easily see that the field lines at every point are orthogonal to the equipotential surfaces.

An equipotential surface can be drawn through any point of a field. Consequently, we can construct an infinitely great number of such surfaces. They are conventionally drawn so that the potential difference for two adjacent surfaces is the same everywhere. Thus, the density of the equipotential surfaces allows us to assess the magnitude of the field strength. Indeed, the denser are the equipotential surfaces, the more rapidly does the potential change when moving along a normal to the surface. Hence, $\nabla\varphi$ is greater at the given place, and, therefore, $\mathbf{E}$ is greater too.

Figure 1.8 shows equipotential surfaces (more exactly, their intersections with the plane of the drawing) for the field of a point charge. In accordance with the nature of the dependence of E on r, equipotential surfaces become the denser, the nearer we approach a charge.

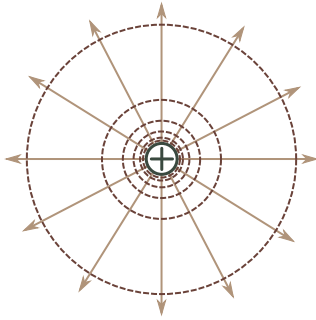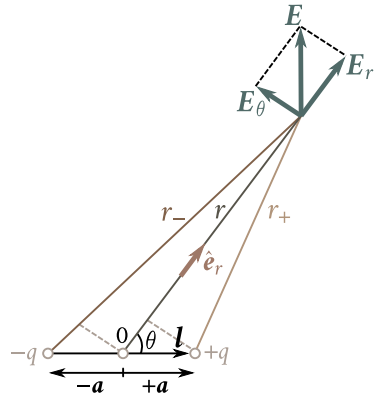Equipotential surfaces for a homogeneous field are a collection of equispaced

Fig. 1.8   Fig. 1.9

planes at right angles to the direction of the field.

1.9. Dipole

An **electric dipole** is defined as a system of two point charges $+q$ and $-q$ identical in value and opposite in sign, the distance between which is much smaller than that to the points at which the field of the system is being determined. The straight line passing through both charges is called the **dipole axis**.

Let us first calculate the potential and then the field strength of a dipole. This field has axial symmetry. Therefore, the pattern of the field in any plane passing through the dipole axis will be the same, the vector $\boldsymbol{E}$ being in this plane. The position of a point relative to the dipole will be characterized with the aid of the position vector $\boldsymbol{r}$ or with the aid of the polar coordinates r and θ (Fig. 1.9). We shall introduce the vector $\boldsymbol{l}$ passing from the negative charge to the positive one. The position of the charge $+q$ relative to the centre of the dipole is determined by the vector $\boldsymbol{a}$, and of the charge $-q$ by the vector $-\boldsymbol{a}$. It is obvious that $\boldsymbol{l} = 2\boldsymbol{a}$. We shall designate the distances to a given point from the charges $+q$ and $-q$ by r_+ and r_-, respectively.

Owing to the smallness of a in comparison with r, we can assume approximately that

$$r_+ = r - a\cos\theta = r - \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_r,$$
$$r_- = r + a\cos\theta = r + \boldsymbol{a}\cdot\hat{\boldsymbol{e}}_r.$$
(1.47)

The potential at a point determined by the position vector $\boldsymbol{r}$ is

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) = \frac{1}{4\pi\varepsilon_0}\frac{q(r_- - r_+)}{r_+ r_-}.$$
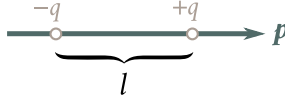
Fig. 1.10

The product r_+r_- can be replaced with r^2. The difference $r_- - r_+$ according to Eqs. (1.47), is $2(\boldsymbol{a}\cdot\hat{\boldsymbol{e}}_r) = \boldsymbol{l}\cdot\hat{\boldsymbol{e}}_r$. Hence,

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\boldsymbol{l}\cdot\hat{\boldsymbol{e}}_r)}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{(\boldsymbol{p}\cdot\hat{\boldsymbol{e}}_r)}{r^2} \tag{1.48}$$

where

$$\boldsymbol{p} = q\boldsymbol{l} \tag{1.49}$$

is a characteristic of a dipole called its **electric moment**. The vector $\boldsymbol{p}$ is directed along the dipole axis from the negative charge to the positive one (Fig. 1.10).

A glance at Eq. (1.48) shows that the field of a dipole is determined by its electric moment $\boldsymbol{p}$. We shall see below that the behaviour of a dipole in an external electric field is also determined by its electric moment $\boldsymbol{p}$. A comparison with Eq. (1.26) shows that the potential of a dipole field diminishes with the distance more rapidly (as $1/r^2$) than the potential of a point charge field (which changes in proportion to $1/r$).

It can be seen from Fig. 1.9 that $\boldsymbol{p}\cdot\hat{\boldsymbol{e}}_r = p\cos\theta$. Therefore, Eq. (1.48) can be written as follows:

$$\varphi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}. \tag{1.50}$$

To find the field strength of a dipole, let us calculate the projections of the vector $\boldsymbol{E}$ onto two mutually perpendicular directions by Eq. (1.44). One of them is determined by the motion of a point due to the change in the distance r (with θ fixed), the other by the motion of the point due to the change in the angle θ (with r fixed, see Fig. 1.9). The first projection is obtained by differentiation of Eq. (1.50) with respect to r:

$$E_r = -\frac{\partial\varphi}{\partial r} = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}. \tag{1.51}$$

We shall find the second projection (let us designate it by E_θ) by taking the ratio of the increment of the potential φ obtained when the angle θ grows by $d\theta$ to the distance $r\,d\theta$ over which the end of the segment r moves (in this case the quantity dl in Eq. (1.44) equals $r\,d\theta$]. Thus,

$$E_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta}.$$

Introducing the value of the derivative of function (1.50) with respect to θ we get

$$E_\theta = \frac{1}{4\pi\varepsilon_0} \frac{p \sin\theta}{r^3}. \tag{1.52}$$

The sum of the squares of Eqs. (1.51) and (1.52) gives the square of the vector $\boldsymbol{E}$ (see Fig. 1.9):

$$E^2 = E_r^2 + E_\theta^2 = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \left(\frac{p}{r^3}\right)^2 \left(4\cos^2\theta + \sin^2\theta\right)$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \left(\frac{p}{r^3}\right)^2 \left(1 + 3\cos^2\theta\right).$$

Hence

$$E = \frac{1}{4\pi\varepsilon_0} \frac{p}{r^3} \left(1 + 3\cos^2\theta\right)^{1/2}. \tag{1.53}$$

Assuming in Eq. (1.53) that $\theta = 0$, we get the strength on the dipole axis:

$$E_\| = \frac{1}{4\pi\varepsilon_0} \frac{2p}{r^3}. \tag{1.54}$$

The vector $\boldsymbol{E}_\|$ is directed along the dipole axis. This is in agreement with the axial symmetry of the problem. Examination of Eq. (1.51) shows that $E_r > 0$ when $\theta = 0$, and $E_r < 0$ when $\theta = \pi$. This signifies that in any case the vector $\boldsymbol{E}_\|$ has a direction coinciding with that from $-q$ to $+q$ (i.e., with the direction of $\boldsymbol{p}$). Equation (1.54) can therefore be written in the vector form:

$$\boldsymbol{E}_\| = \frac{1}{4\pi\varepsilon_0} \frac{2\boldsymbol{p}}{r^3}. \tag{1.55}$$

Assuming in Eq. (1.53) that $\theta = \pi/2$, we get the field strength on the straight line passing through the centre of the dipole and perpendicular to its axis:

$$E_\perp = \frac{1}{4\pi\varepsilon_0} \frac{p}{r^3}. \tag{1.56}$$

By Eq. (1.51), when $\theta = \pi/2$, the projection E_r equals zero. Hence, the vector $\boldsymbol{E}_\perp$ is parallel to the dipole axis. It follows from Eq. (1.52) that when $\theta = \pi/2$, the projection E_θ is positive. This signifies that the vector $\boldsymbol{E}_\perp$ is directed toward the growth of the angle θ, i.e., antiparallel to the vector $\boldsymbol{p}$.

The field strength of a dipole is characterized by the circumstance that it diminishes with the distance from the dipole in proportion to $1/r^3$, i.e., more rapidly than the field strength of a point charge (which diminishes in proportion to $1/r^2$).

Figure 1.11 shows $\boldsymbol{E}$ lines (the solid lines) and equipotential surfaces (the dash lines) of the field of a dipole. According to Eq. (1.50), when $\theta = \pi/2$, the potential vanishes for all the r's. Thus, all the points of a plane at right angles to the dipole axis and passing through its middle have a zero potential. This could have been predicted because the distances from the charges $+q$ and $-q$ to any point of this
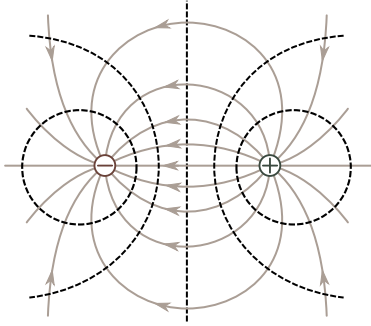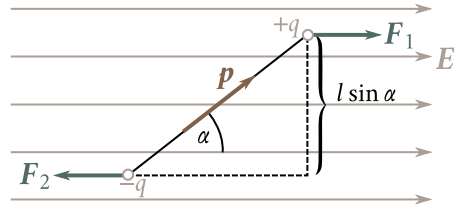
Fig. 1.11

Fig. 1.12

plane are identical.

Now let us turn to the behaviour of a dipole in an external electric field. If a dipole is placed in a homogeneous electric field, the charges $+q$ and $-q$ forming the dipole will be under the action of the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ equal in magnitude, but opposite in direction (Fig. 1.12). These forces form a couple whose arm is $l \sin \alpha$, i.e., depends on the orientation of the dipole relative to the field. The magnitude of each of the forces is qE. Multiplying it by the arm, we get the magnitude of the torque acting on a dipole:

$$T = qEl \sin \alpha = pE \sin \alpha \qquad (1.57)$$

(p is the electric moment of the dipole). It is easy to see that Eq. (1.57) can be written in the vector form

$$\boldsymbol{T} = \boldsymbol{p} \times \boldsymbol{E}. \qquad (1.58)$$

The torque (1.58) tends to turn a dipole so that its electric moment $\boldsymbol{p}$ is in the direction of the field.

Let us find the potential energy belonging to a dipole in an external electric field. By Eq. (1.29), this energy is

$$W_\mathrm{p} = q\varphi_+ - q\varphi_- = q(\varphi_+ - \varphi_-). \qquad (1.59)$$

Here φ_+ and φ_- are the values of the potential of the external field at the points where the charges $+q$ and $-q$ are placed.

The potential of a homogeneous field diminishes linearly in the direction of the vector $\boldsymbol{E}$. Assuming that the x-axis is this direction (Fig. 1.13), we can write that $E = E_x = -\mathrm{d}\varphi/\mathrm{d}x$. A glance at Fig. 1.13 shows that the difference $\varphi_+ - \varphi_-$ equals the increment of the potential on the segment $\Delta x = l \cos \alpha$:

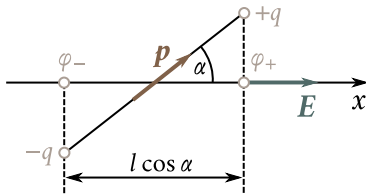$$\varphi_+ - \varphi_- = \frac{\mathrm{d}\varphi}{\mathrm{d}x} l \cos \alpha = -El \cos \alpha.$$
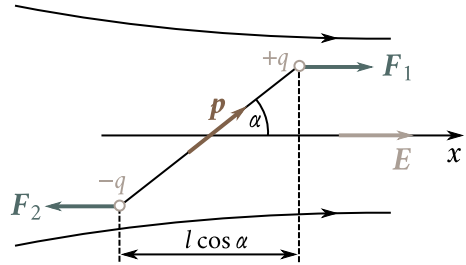
Fig. 1.13

Fig. 1.14

Introducing this value into Eq. (1.59), we find that

$$W_\text{p} = -qEl \cos \alpha = -pE \cos \alpha. \tag{1.60}$$

Here α is the angle between the vectors $\boldsymbol{p}$ and $\boldsymbol{E}$. We can therefore write Eq. (1.60) in the form

$$W_\text{p} = -\boldsymbol{p} \cdot \boldsymbol{E}. \tag{1.61}$$

We must note that this expression takes no account of the energy of interaction of the charges $+q$ and $-q$ forming a dipole.

We have obtained Eq. (1.61) assuming for simplicity's sake that the field is homogeneous. This equation also holds, however, for an inhomogeneous field.

Let us consider a dipole in an inhomogeneous field that is symmetrical relative to the x-axis[5]. Let the centre of the dipole be on this axis, the dipole electric moment making with the axis an angle α, differing from $\pi/2$ (Fig. 1.14). In this case, the forces acting on the dipole charges are not identical in magnitude. Therefore, apart from the rotational moment (torque), the dipole will experience a force tending to move it in the direction of the x-axis. To find the value of this force, we shall use Eq. (1.40), according to which

$$F_x = -\frac{\partial W_\text{p}}{\partial x}, \quad F_y = -\frac{\partial W_\text{p}}{\partial y}, \quad F_z = -\frac{\partial W_\text{p}}{\partial z}.$$

In view of Eq. (1.60), we can written

$$W_\text{p}(x, y, z) = -pE(x, y, z) \cos \alpha$$

(we consider the orientation of the dipole relative to the vector $\boldsymbol{E}$ to be constant, $\alpha = $ constant).

For points on the x-axis, the derivatives of E with respect to y and z are zero. Accordingly, $\partial W_\text{p}/\partial y = \partial W_\text{p}/\partial z = 0$. Thus, only the force component F_x differs

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[5]A particular case of such a field is that of a point charge if we take a straight line passing through the charge as the x-axis.
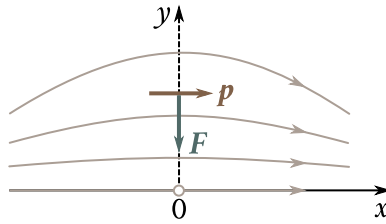
Fig. 1.15

from zero. It is

$$F_x = -\frac{\partial W_\mathrm{p}}{\partial x} = p\frac{\partial E}{\partial x}\cos\alpha. \tag{1.62}$$

This result can be obtained if we take account of the fact that the field strength at the points where the charges $+q$ and $-q$ are (see Fig. 1.14) differs by the amount $(\partial E/\partial x)l\cos\alpha$. Accordingly, the difference between the forces acting on the charges is $q(\partial E/\partial x)l\cos\alpha$, which coincides with Eq. (1.62).

When α is less than $\pi/2$, the value of F_x determined by Eq. (1.62) is positive. This signifies that under the action of the force the dipole is pulled into the region of a stronger field (see Fig. 1.14). When α is greater than $\pi/2$, the dipole is pushed out of the field.

In the case shown in Fig. 1.15, only the derivative $\partial E/\partial y$ differs from zero for points on the y-axis. Therefore, the force acting on the dipole is determined by the component

$$F_y = -\frac{\partial W_\mathrm{p}}{\partial y} = p\frac{\partial E}{\partial y}, \quad (\cos\alpha = 1).$$

The derivative $\partial E/\partial y$ is negative. Consequently, the force is directed as shown in the figure. Thus, in this case too, the dipole is pulled into the field.

We shall note that like $-\partial W_\mathrm{p}/\partial x$ gives the projection of the force acting on the system onto the x-axis, so does the derivative of Eq. (1.60) with respect to α taken with the opposite sign give the projection of the torque onto the α-"axis": $T_\alpha = -pE\sin\alpha$. The minus sign was obtained because the α-"axis" and the torque T are directed oppositely (see Fig. 1.12).

1.10. Field of a System of Charges at Great Distances

Let us take a system of N charges $q_1, q_2, \ldots, q_N$ in a volume having linear dimensions of the order of l, and study the field set up by this system at distances r that are great in comparison with l ($r > l$). We take the origin of coordinates 0 inside the volume occupied by the system and shall determine the positions of the
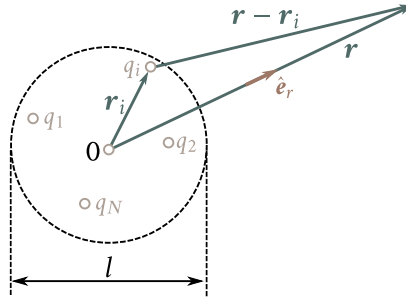
Fig. 1.16

charges with the aid of the position vectors r_i, (Fig. 1.16; to simplify the figure, we have shown only the position vector of the i-th charge).

The potential at the point determined by the position vector r is

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|r - r_i|}. \tag{1.63}$$

Owing to the smallness of r_i in comparison with r, we can assume that

$$|r - r_i| = r - r_i \cdot \hat{e}_r = r\left(1 - \frac{r_i \cdot \hat{e}_r}{r}\right)$$

[compare with Eqs. (1.47)]. Introduction of this expression into Eq. (1.63) yields

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r} \left[\frac{1}{1 - (r_i \cdot \hat{e}_r/r)}\right]. \tag{1.64}$$

Using the formula

$$\frac{1}{1 - x} \approx 1 + x$$

which holds when $x \ll 1$, we can transform Eq. (1.64) as follows:

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r} \left(1 + \frac{r_i \cdot \hat{e}_r}{r}\right)$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \sum_{i=1}^{N} q_i + \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \left(\sum_{i=1}^{N} q_i r_i\right) \cdot \hat{e}_r. \tag{1.65}$$

The first term of the expression obtained is the potential of the field of a point charge having the value $q = \sum_i q_i$ [compare with Eq. (1.26)]. The second term has the same form as the expression determining the potential of a dipole field, the part of the electric moment of the dipole being played by the quantity
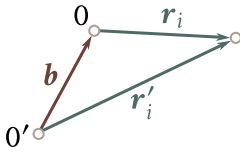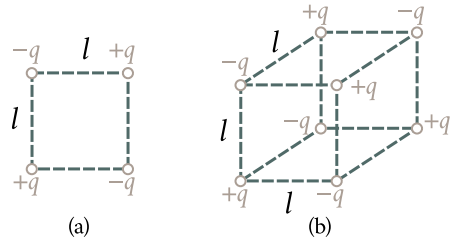
$$p = \sum_{i=1}^{N} q_i r_i. \tag{1.66}$$

Fig. 1.17

Fig. 1.18

This quantity is called the **dipole electric moment** of a system of charges. It is easy to verify that for a dipole Eq. (1.66) transforms into the expression $p = ql$ which we are already familiar with.

If the total charge of a system is zero ($\sum_i q_i = 0$), the value of the dipole moment does not depend on our choice of the origin of coordinates. To convince ourselves that this is true, let us take two arbitrary origins of coordinates 0 and 0′ (Fig. 1.17). The position vectors of the i-th charge conducted from these points are related as follows:

$$r'_i = b + r_i \qquad (1.67)$$

(what the vector b is clear from the figure). With account taken of Eq. (1.67), the dipole moment in the system with the origin 0′ is

$$p' = \sum_i q_i r_i = \sum_i q_i (b + r_i) = b \sum_i q_i + \sum_i q_i r_i.$$

The first addend equals zero (because $\sum_i q_i = 0$). The second one is p—the dipole moment in a coordinate system with its origin at 0. We have thus obtained that $p' = p$.

Equation (1.65) is in essence the first two terms of the series expansion of function (1.63) by powers of r_i/r. When $\sum_i q_i \neq 0$, the first term of Eq. (1.65) makes the main contribution to the potential (the second term diminishes in proportion to $1/r^2$ and is therefore much smaller than the first one). For an electrically neutral system ($\sum_i q_i = 0$), the first term equals zero, and the potential is determined mainly by the second term of Eq. (1.65). This is how matters stand, in particular, for the field of a dipole.

For the system of charges depicted in Fig. 1.18a and called a **quadrupole**, both $\sum_i q_i$ and p equal zero so that Eq. (1.65) gives a zero value of the potential. Actually, however, the field of a quadrupole, although it is much weaker than that of a dipole (with the same values of q and l), differs from zero. The potential of the field set up by a quadrupole is determined mainly by the third term of the expansion that is proportional to $1/r^3$. To obtain this term, we must take into consideration

quantities of the order of $(r_i/r)^2$ which we disregarded in deriving Eq. (1.65). For the system of charges shown in Fig. 1.18b and called an **octupole**, the third term of the expansion also equals zero. The potential of the field of such a system is determined by the fourth term of the expansion, which is proportional to $1/r^4$.

It must be noted that the quantity equal to $\sum_i q_i$ in the numerator of the first term of Eq. (1.65) is called a **monopole** or a **zero-order multipole**, a dipole is also called a **first-order multipole**, a quadrupole is called a **second-order multipole**, and so on.

Thus, in the general case, the field of a system of charges at great distances can be represented as the superposition of fields set up by multipoles of different orders—a monopole, dipole, quadrupole, octupole, etc.

1.11. A Description of the Properties of Vector Fields

To continue our study of the electric field, we must acquaint ourselves with the mathematical tools used to describe the properties of vector fields. These tools are called **vector analysis**. In the present section, we shall treat the fundamental concepts and selected formulas of vector analysis, and also prove its two main theorems—the Ostrogradsky-Gauss theorem (sometimes called Gauss's divergence theorem) and Stokes's theorem.

The quantities used in vector analysis can be best illustrated for the field of the velocity vector of a flowing liquid. We shall therefore introduce these quantities while dealing with the flow of an ideal incompressible liquid, and then extend the results obtained to vector fields of any nature.

We are already acquainted with one of the concepts of vector analysis. This is the **gradient**, used to characterize scalar fields. If the value of the scalar quantity $\varphi = \varphi(x, y, z)$ is compared with every point P having the coordinates x, y, z, we say that the scalar field of φ has been set. The gradient of the quantity φ is defined as the vector

$$\text{grad } \varphi = \frac{\partial \varphi}{\partial x} \hat{\boldsymbol{e}}_x + \frac{\partial \varphi}{\partial y} \hat{\boldsymbol{e}}_y + \frac{\partial \varphi}{\partial z} \hat{\boldsymbol{e}}_z. \tag{1.68}$$

The increment of the function φ upon displacement over the length $d\boldsymbol{l} = \hat{\boldsymbol{e}}_x \, dx + \hat{\boldsymbol{e}}_y \, dy + \hat{\boldsymbol{e}}_z \, dz$ is

$$d\varphi = \frac{\partial \varphi}{\partial x} \, dx + \frac{\partial \varphi}{\partial y} \, dy + \frac{\partial \varphi}{\partial z} \, dz$$

which can be written in the form

$$d\varphi = \text{grad } \varphi \cdot d\boldsymbol{l}. \tag{1.69}$$

Now we shall go over to establishing the characteristics of vector fields.
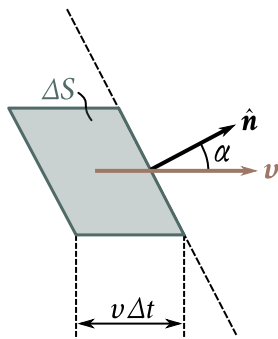
Fig. 1.19

Vector flux. Assume that the flow of a liquid is characterized by the field of the velocity vector. The volume of liquid flowing in unit time through an imaginary surface S is called the flux of the liquid through this surface. To find the flux, let us divide the surface into elementary sections of the size ΔS. It can be seen from Fig. 1.19 that during the time Δt a volume of liquid equal to

$$\Delta V = (\Delta S \cos \alpha) v \Delta t$$

will pass through section ΔS. Dividing this volume by the time Δt, we shall find the flux through surface ΔS:

$$\Delta \Phi = \frac{\Delta V}{\Delta t} = \Delta S v \cos \alpha.$$

Passing over to differentials, we find that

$$d\Phi = (v \cos \alpha)\, dS. \tag{1.70}$$

Equation (1.70) can be written in two other ways. First, if we take into account that $v \cos \alpha$ gives the projection of the velocity vector onto the normal $\hat{\boldsymbol{e}}_n$ to area dS, we can write Eq. (1.70) in the form

$$d\Phi = v_n\, dS. \tag{1.71}$$

Second, we can introduce the vector $d\boldsymbol{S}$ whose magnitude equals that of area dS, while its direction coincides with the direction of a normal $\hat{\boldsymbol{n}}$ to the area:

$$d\boldsymbol{S} = dS\, \hat{\boldsymbol{n}}.$$

Since the direction of the vector $\hat{\boldsymbol{n}}$ is chosen arbitrarily (it can be directed to either side of the area), then $d\boldsymbol{S}$ is not a true vector, but is a pseudo vector. The angle α in Eq. (1.70) is the angle between the vectors $\boldsymbol{v}$ and $d\boldsymbol{S}$. Hence, this equation can be written in the form

$$d\Phi = \boldsymbol{v} \cdot d\boldsymbol{S}. \tag{1.72}$$

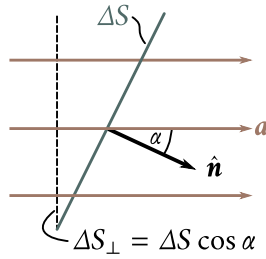By summating the fluxes through all the elementary areas into which we have

$$\Delta S_\perp = \Delta S \cos \alpha$$

Fig. 1.20

divided surface S, we get the flux of the liquid through S:

$$\Phi_v = \int_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_S v_n \, \mathrm{d}S. \tag{1.73}$$

A similar expression written for an arbitrary vector field $\boldsymbol{a}$, *i.e.*, the quantity

$$\Phi_a = \int_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \int_S a_n \, \mathrm{d}S \tag{1.74}$$

is called the **flux of the vector a through surface** S. In accordance with this definition, the flux of a liquid can be called the flux of the vector $\boldsymbol{v}$ through the relevant surface [see Eq. (1.73)].

The flux of a vector is an algebraic quantity. Its sign depends on the choice of the direction of a normal to the elementary areas into which surface S is divided in calculating the flux. Reversal of the direction of the normal changes the sign of a_n and, therefore, the sign of the quantity (1.74). The customary practice for closed surfaces is calculation of the flux "emerging outward" from the region enclosed by the surface. Accordingly, in the following we shall always implicate that $\hat{\boldsymbol{v}}$ is an outward normal.

We can give an illustrative geometrical interpretation of the vector flux. For this purpose, we shall represent a vector field by a system of lines $\boldsymbol{a}$ constructed so that the density of the lines at every point is numerically equal to the magnitude of the vector $\boldsymbol{a}$ at the same point of the field (compare with the rule for constructing the lines of the vector $\boldsymbol{E}$ set out at the end of Sec. 1.5). Let us find the number ΔN of intersections of the field lines with the imaginary area ΔS. A glance at Fig. 1.20 shows that this number equals the density of the lines (*i.e.*, a) multiplied by $\Delta S_\perp = \Delta S \cos \alpha$:

$$\Delta N \ (=) \ a \Delta S \cos \alpha = a_n \Delta S.$$

We are speaking only about the numerical equality between ΔN and $a_n \Delta S$. This is why the equality sign is confined in parentheses. According to Eq. (1.74), the expression $a_n \Delta S$ is $\Delta \Phi$—the flux of the vector a through area ΔS. Thus,

$$\Delta N \ (=) \ \Delta \Phi_a. \tag{1.75}$$

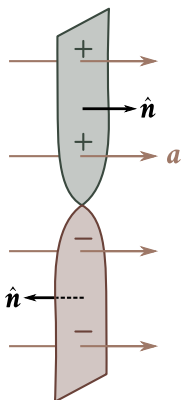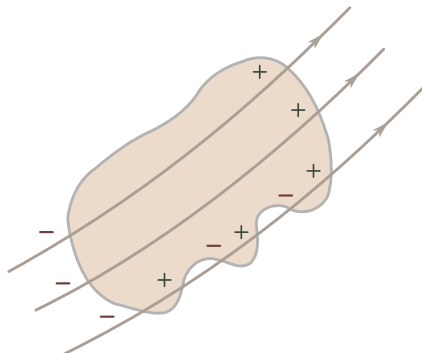**Fig. 1.21**   **Fig. 1.22**

For the sign of ΔN to coincide with that of $\Delta \Phi_a$, we must consider those intersections to be positive for which the angle α between the positive direction of a field line and a normal to the area is acute. The intersection should be considered negative if the angle α is obtuse. For the area shown in Fig. 1.20, all three intersections are positive: $\Delta N = +3$ ($\Delta \Phi_a$ in this case is also positive because $a_n > 0$). If the direction of the normal in Fig. 1.20 is reversed, the intersections will become negative ($\Delta N = -3$), and the flux $\Delta \Phi_a$ will also be negative.

Summation of Eq. (1.75) over the finite imaginary surface S yields

$$\Delta \Phi_a \ (=) \sum \Delta N = N_+ - N_- \tag{1.76}$$

where N_+ and N_- are the total number of positive and negative intersections of the field lines with surface S, respectively.

The reader may be puzzled by the circumstance that since the flux, as a rule, is expressed by a fractional number, the number of intersections of the field lines with a surface compared with the flux will also be fractional. Do not be confused by this, however. Field lines are a purely conditional image deprived of a physical meaning.

Let us take an imaginary surface in the form of a strip of paper whose bottom part is twisted relative to the top one through the angle π (Fig. 1.21). The direction of a normal must be chosen identically for the entire surface. Hence, if in the top part of the strip a positive normal is directed to the right, then in the bottom part a normal will be directed to the left. Accordingly, the intersections of the field lines depicted in Fig. 1.21 with the top half of the surface must be considered positive, and with the bottom half, negative.

An outward normal is considered to be positive for a closed surface (Fig. 1.22). Therefore, the intersections corresponding to outward protrusion of the lines (in
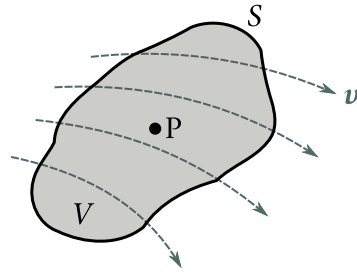
Fig. 1.23

this case the angle α is acute) must be taken with the plus sign, and the ones appearing when the lines enter the surface (in this case the angle α is obtuse) must be taken with the minus sign.

Inspection of Fig. 1.22 shows that when the field lines enter a closed surface continuously, each line when intersecting the surface enters it and emerges from it the same number of times. As a result, the flux of the corresponding vector through this surface equals zero. It is easy to see that if field lines end inside a surface, the vector flux through the closed surface will numerically equal the difference between the number of lines beginning inside the surface (N_{beg}) and the number of lines terminating inside the surface (N_{term}):

$$\Phi_a \ (=) \ N_{\text{beg}} - N_{\text{term}}. \tag{1.77}$$

The sign of the flux depends on which of these numbers is greater. When N_{beg} is equal to N_{term}, the flux equals zero.

Divergence. Assume that we are given the field of the velocity vector of an incompressible continuous liquid. Let us take an imaginary closed surface S in the vicinity of point P (Fig. 1.23). If in the volume confined by this surface no liquid appears and no liquid vanishes, then the flux flowing outward through the surface will evidently equal zero. A liquid flux Φ_v other than zero will indicate that there are liquid sources or sinks inside the surface, *i.e.*, points at which the liquid enters the volume (sources) or emerges from it (sinks). The magnitude of the flux determines the total algebraic power of the sources and sinks[6]. When the sources predominate over the sinks, the magnitude of the flux will be positive, and when the sinks predominate, negative.

The quotient obtained when dividing the flux Φ_v by the volume which it flows out from, *i.e.*,

$$\frac{\Phi_v}{V} \tag{1.78}$$

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[6]The power of a source (sink) is defined as the volume of liquid discharged (absorbed) in unit time. A sink can be considered as a source with a negative power.

gives the average unit power of the sources confined in the volume V. In the limit when V tends to zero, *i.e.*, when the volume V contracts to point P, expression (1.78) gives the true unit power of the sources at point P, which is called the **divergence** of the vector $\boldsymbol{v}$ (it is designated by div $\boldsymbol{v}$). Thus, by definition,

$$\text{div } \boldsymbol{v} = \lim_{V \to P} \frac{\Phi_v}{V}.$$

The divergence of any vector $\boldsymbol{a}$ is determined in a similar way:

$$\text{div } \boldsymbol{a} = \lim_{V \to P} \frac{\Phi_v}{V} = \lim_{V \to P} \oint \boldsymbol{a} \cdot d\boldsymbol{S}. \tag{1.79}$$

The integral is taken over arbitrary closed surface S surrounding point P[7]; V is the volume confined by this surface. Since the transition $V \to$P is being performed upon which S tends to zero, we can assume that Eq. (1.79) cannot depend on the shape of the surface. This assumption is confirmed by strict calculations.

Let us surround point P with a spherical surface of an extremely small radius r (Fig. 1.24). Owing to the smallness of r, the volume V enclosed by the sphere will also be very small. We can therefore consider with a high degree of accuracy that the value of div $\boldsymbol{a}$ within the limits of the volume V is constant[8]. In this case, we can write in accordance with Eq. (1.79) that

$$\Phi_a \approx (\text{div } \boldsymbol{a})V$$

where Φ_a is the flux of the vector a through the surface surrounding the volume V. By Eq. (1.77), Φ_a equals N_{beg}, the number of lines of a beginning inside V if div $\boldsymbol{a}$ at point P is positive, or N_{term}, the number of lines of a terminating inside V if div $\boldsymbol{a}$ at point P is negative.

It follows from the above that the lines of the vector $\boldsymbol{a}$ begin in the closest vicinity of a point with a positive divergence. The field lines "diverge" from this point; the latter is the "source" of the field (Fig. 1.24a). On the other hand, in the vicinity of a point with a negative divergence, the lines of the vector $\boldsymbol{a}$ terminate. The field lines "converge" toward this point; the latter is the "sink" of the field (Fig. 1.24b). The greater the absolute value of div $\boldsymbol{a}$, the bigger is the number of lines that begin or terminate in the vicinity of the given point.

It can be seen from definition (1.79) that the divergence is a scalar function of the coordinates determining the positions of points in space (briefly—a point function). Definition (1.79) is the most general one that is independent of the kind of coordinate system used.

Let us find an expression for the divergence in a Cartesian coordinate system.

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[7]The circle on the integral sign signifies that integration is performed over a closed surface.

[8]It is assumed that the value of div $\boldsymbol{a}$ changes continuously, without any jumps, when passing from one point of a field to another.
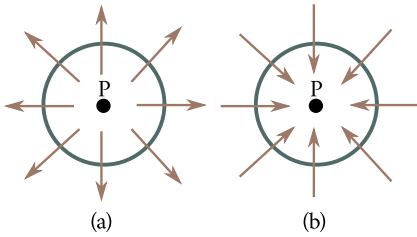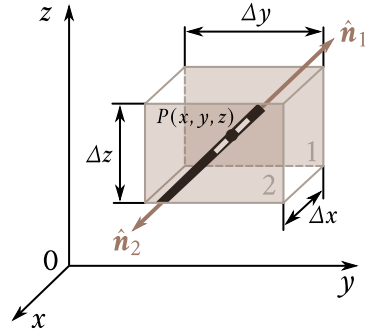
Fig. 1.24

Fig. 1.25

We shall consider a small volume in the form of a parallelepiped with ribs parallel to the coordinate axes in the vicinity of point $P(x, y, z)$ (Fig. 1.25). The vector flux through the surface of the parallelepiped is formed from the fluxes passing through each of the six faces separately.

Let us find the flux through the pair of faces perpendicular to the x-axis (in Fig. 1.25 these faces are designated by shaded areas and by the numbers 1 and 2). The outward normal $\hat{n}_2$ to face 2 coincides with the direction of the x-axis. Hence, for points of this face, $a_{n_2} = a_x$. The outward normal $\hat{n}_1$ to face 1 is directed oppositely to the x-axis. Therefore, for points on this face, $a_{n_2} = -a_x$. The flux through face 2 can be written in the form

$$a_{x,2} \Delta y \Delta z$$

where $a_{x,2}$ is the value of a_x averaged over face 2. The flux through face 1 is

$$-a_{x,1} \Delta y \Delta z$$

where $a_{x,1}$ is the average value of a_x for face 1. The total flux through faces 1 and 2 is determined by the expression

$$(a_{x,2} - a_{x,1}) \Delta y \Delta z. \tag{1.80}$$

The difference $a_{x,2} - a_{x,1}$ is the increment of the average (over a face) value of a_x upon a displacement along the x-axis by Δx. Owing to the smallness of the parallelepiped (we remind our reader that we shall let its dimensions shrink to zero), this increment can be written in the form $(\partial a_x / \partial x) \Delta x$, where the value $\partial a_x / \partial x$ is taken at point P[9]. Therefore, Eq. (1.80) becomes

$$\frac{\partial a_x}{\partial x} \Delta x \Delta y \Delta z = \frac{\partial a_x}{\partial x} \Delta V.$$

Similar reasoning allows us to obtain the following expressions for the fluxes through

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[9]The inaccuracy which we tolerate here vanishes when the volume shrinks to point P in the limit transition.

the pairs of faces perpendicular to the y- and z-axes:

$$\frac{\partial a_y}{\partial y} \Delta x \Delta y \Delta z = \frac{\partial a_y}{\partial y} \Delta V, \quad \frac{\partial a_z}{\partial z} \Delta x \Delta y \Delta z = \frac{\partial a_z}{\partial z} \Delta V.$$

Thus, the total flux through the entire close surface is determined by the expression

$$\Phi_a = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) \Delta V.$$

Dividing this expression by ΔV, we shall find the divergence of the vector a at point $P(x, y, z)$:

$$\operatorname{div} \boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}. \tag{1.81}$$

The Ostrogradsky-Gauss Theorem. If we know the divergence of the vector $\boldsymbol{a}$ at every point of space, we can calculate the flux of this vector through any closed surface of finite dimensions. Let us first do this for the flux of the vector $\boldsymbol{v}$ (a liquid flux). The product of div $\boldsymbol{v}$ and dV gives the power of the sources of the liquid confined within the volume dV. The sum of such products, *i.e.*, $\int (\operatorname{div} \boldsymbol{v}) \, \mathrm{d}V$, gives the total algebraic power of the sources confined in the volume V over which integration is performed. Owing to incompressibility of the liquid, the total power of the sources must equal the liquid flux emerging through surface S enclosing the volume V. We thus arrive at the equation

$$\oint_S \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_V (\operatorname{div} \boldsymbol{v}) \, \mathrm{d}V.$$

A similar equation holds for a vector field of any nature:

$$\oint_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \int_V (\operatorname{div} \boldsymbol{a}) \, \mathrm{d}V. \tag{1.82}$$

This relation is called the **Ostrogradsky-Gauss** theorem. The integral in the left-hand side of the equation is calculated over an arbitrary closed surface S, and the integral in the right-hand side over the volume V enclosed by this surface.

Circulation. Let us revert to the flow of an ideal incompressible liquid. Imagine a closed line—the contour Γ. Assume that in some way or other we have instantaneously frozen the liquid in the entire volume except for a very thin closed channel of constant cross section including the contour Γ (Fig. 1.26). Depending on the nature of the velocity vector field, the liquid in the channel formed will either be stationary or move along the contour (circulate) in one of the two possible directions. Let us take the quantity equal to the product of the velocity of the liquid in the channel and the length of the contour l as a measure of this motion. This
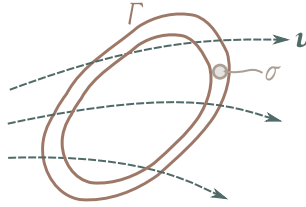
Fig. 1.26

quantity is called the **circulation** of the vector $\boldsymbol{v}$ around the contour Γ. Thus,

circulation of $\boldsymbol{v}$ around $\Gamma = vl$

(since we assumed that the channel has a constant cross section, the magnitude of the velocity, v, is a constant).

At the moment when the walls freeze, the velocity component perpendicular to a wall will be eliminated in each of the liquid particles, and only the velocity component tangent to the contour will remain, *i.e.*, v_l. The momentum $d\boldsymbol{p}_l$, is associated with this component. The magnitude of the momentum for a liquid particle contained within a segment of the channel of length dl is $\rho \sigma v_l$ dl (ρ is the density of the liquid, and σ is the cross-sectional area of the channel). Since the liquid is ideal, the action of the walls can change only the direction of the vector $d\boldsymbol{p}_l$, but not its magnitude. The interaction between the liquid particles will cause a redistribution of the momentum between them that will level out the velocities of all the particles. The algebraic sum of the tangential components of the momenta cannot change: the momentum acquired by one of the interacting particles equals the momentum lost by the second particle. This signifies that

$$\rho \sigma v l = \oint_{\Gamma} \rho \sigma v_l \, dl$$

where v is the circulation velocity, and v_l is the tangential component of the liquid's velocity in the volume σ dl at the moment of time preceding the freezing of the channel walls. Cancelling $\rho \sigma$, we get

circulation of $\boldsymbol{v}$ around $\Gamma = vl = \oint_{\Gamma} v_l \, dl.$

The circulation of any vector $\boldsymbol{a}$ around an arbitrary closed contour Γ is determined in a similar way:

$$\text{circulation of } \boldsymbol{a} \text{ around } \Gamma = \oint_{\Gamma} \boldsymbol{a} \cdot d\boldsymbol{l} = \oint_{\Gamma} a_l \, dl. \tag{1.83}$$

It may seem that for the circulation to be other than zero the vector lines must be closed or at least bent in some way or other in the direction of circumventing the contour. It is easy to see that this assumption is wrong. Let us consider the laminar
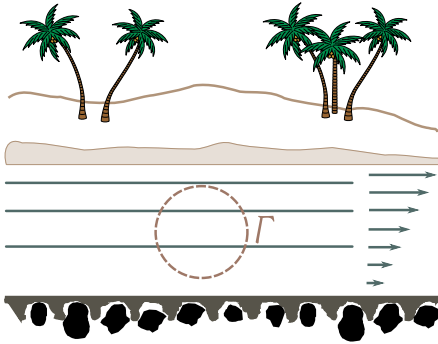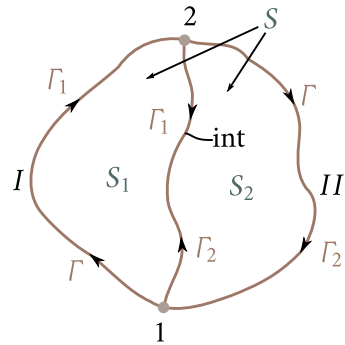
Fig. 1.27

Fig. 1.28

flow of water in a river. The velocity of the water directly at the river bottom is zero and grows as we approach the surface of the water (Fig. 1.27). The streamlines (lines of the vector $\boldsymbol{v}$) are straight. Notwithstanding this fact, the circulation of the vector $\boldsymbol{v}$ around the contour depicted by the dash line obviously differs from zero. On the other hand, in a field with curved lines, the circulation may equal zero.

Circulation has the property of additivity. This signifies that the sum of the circulations around contours Γ_1 and Γ_2 enclosing neighboring surfaces S_1 and S_2 (Fig. 1.28) equals the circulation around contour Γ enclosing surface S, which is the sum of surfaces S_1 and S_2. Indeed, the circulation C_1 around the contour bounding surface S_1 can be represented as the sum of the integrals

$$C_1 = \oint_{\Gamma_1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_{1,(I)}^{2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} + \int_{2,(\mathrm{int.})}^{1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.84}$$

The first integral is taken over section I of the outer contour, the second over the interface between surfaces S_1 and S_2 in direction 2-1.

Similarly, the circulation C_2 around the contour enclosing surface S_2 is

$$C_2 = \oint_{\Gamma_2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_{2,(II)}^{1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} + \int_{1,(\mathrm{int.})}^{2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.85}$$

The first integral is taken over section II of the outer contour, the second over the interface between surfaces S_1 and S_2 in direction 1-2.

The circulation around the contour bounding total surface S can be represented in the form

$$C = \oint_{\Gamma} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_{1,(I)}^{2} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} + \int_{2,(II)}^{1} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.86}$$

The second addends in Eqs. (1.84) and (1.85) differ only in their sign. Therefore, the sum of these expressions will equal Eq. (1.86). Thus,
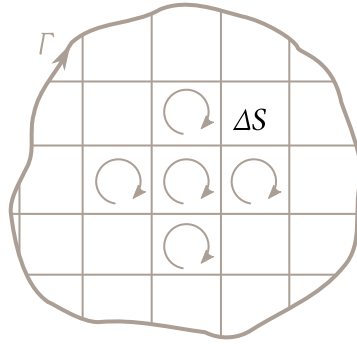
$$C = C_1 + C_2. \tag{1.87}$$

Fig. 1.29

Equation (1.87) which we have proved does not depend on the shape of the surfaces and holds for any number of addends. Hence, if we divide an arbitrary open surface S into a great number of elementary surfaces ΔS[10] (Fig. 1.29), then the circulation around the contour enclosing S can be written as the sum of the elementary circulations ΔC around the contours enclosing the ΔS's:

$$C = \sum_i \Delta C_i. \tag{1.88}$$

Curl. The additivity of the circulation permits us to introduce the concept of unit circulation, *i.e.*, consider the ratio of the circulation C to the magnitude of surface S around which the circulation "flows". When surface S is finite, the ratio C/S gives the mean value of the unit circulation. This value characterizes the properties of a field averaged over surface S. To obtain the characteristic of the field at point P, we must reduce the dimensions of the surface, making it shrink to point P. The ratio C/S tends to a limit that characterizes the properties of the field at point P.

Thus, let us take an imaginary contour Γ in a plane passing through point P, and consider the expression

$$\lim_{S \to P} \frac{C_a}{S} \tag{1.89}$$

where C_a is the circulation of the vector $\boldsymbol{a}$ around the contour Γ and S is the surface area enclosed by the contour.

Limit (1.89) calculated for an arbitrarily oriented plane cannot be an exhaustive characteristic of the field at point P because the magnitude of this limit depends on the orientation of the contour in space in addition to the properties of the field

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[10] In the figure, the elementary surfaces are depicted in the form of rectangles. Actually, their shape may be absolutely arbitrary.

at point P. This orientation can be given by the direction of a positive normal $\hat{\boldsymbol{n}}$ to the plane of the contour (a positive normal is one that is associated with the direction of circumvention of the contour in integration by the right-hand screw rule). In determining limit (1.89) at the same point P for different directions $\hat{\boldsymbol{n}}$, we shall obtain different values. For opposite directions, these values will differ only in their sign (reversal of the direction $\hat{\boldsymbol{n}}$ is equivalent to reversing the direction of circumvention of the contour in integration, which only causes a change in the sign of the circulation). For a certain direction of the normal, the magnitude of expression (1.89) at the given point will be maximum.

Thus, quantity (1.89) behaves like the projection of a vector onto the direction of a normal to the plane of the contour around which the circulation is taken. The maximum value of quantity (1.89) determines the magnitude of this vector, and the direction of the positive normal $\hat{\boldsymbol{n}}$ at which the maximum is reached gives the direction of the vector. This vector is called the **curl** of the vector $\boldsymbol{a}$. Its symbol is curl $\boldsymbol{a}$. Using this notation, we can write expression (1.89) in the form

$$(\text{curl } \boldsymbol{a})_n = \lim_{S \to P} \frac{C_a}{S} = \lim_{S \to P} \frac{1}{S} \oint_S \boldsymbol{a} \, d\boldsymbol{l}. \tag{1.90}$$

We can obtain a graphical picture of the curl of the vector $\boldsymbol{v}$ by imagining a small and light fan impeller placed at the given point of a flowing liquid (Fig. 1.30). At the spots where the curl differs from zero, the impeller will rotate, its velocity being the higher, the greater in value is the projection of the curl onto the impeller axis.

Equation (1.90) defines the vector curl $\boldsymbol{a}$. This definition is a most general one that does not depend on the kind of coordinate system used. To find expressions for the projections of the vector curl $\boldsymbol{a}$ onto the axes of a Cartesian coordinate system, we must determine the values of quantity (1.90) for such orientations of area S for which the normal $\hat{\boldsymbol{n}}$ to the area coincides with one of the axes x, y, z. If, for example, we direct $\hat{\boldsymbol{n}}$ along the x-axis, then (1.90) becomes $(\text{curl } \boldsymbol{a})_x$. Contour Γ in this case is arranged in a plane parallel to the coordinate plane yz. Let us take this contour in the form of a rectangle with the sides Δy and Δz (Fig. 1.31, the x-axis is directed toward us in this figure; the direction of circumvention indicated in the figure is associated with the direction of the x-axis by the right-hand screw rule). Section 1 of the contour is opposite in direction to the z-axis. Therefore, a_l on this section coincides with $-a_z$. Similar reasoning shows that a_l on sections 2, 3, and 4 equals a_y, a_z, and $-a_y$, respectively. Hence, the circulation can be written in the form

$$\left(a_{z,3} - a_{z,1}\right) \Delta z - \left(a_{y,4} - a_{y,2}\right) \Delta y \tag{1.91}$$

where $a_{z,3}$ and $a_{z,1}$ are the average values of a_z on sections 3 and 1, respectively,
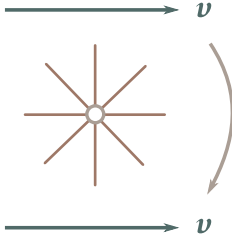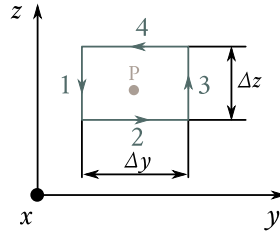
Fig. 1.30

Fig. 1.31

and $a_{y,4}$ and $a_{y,2}$ are the average values of a_y on sections 4 and 2.

The difference $a_{z,3} - a_{z,1}$ is the increment of the average value of a_z on the section Δz when this section is displaced in the direction of the y-axis by Δy. Owing to the smallness of Δy and Δz, this increment can be represented in the form $(\partial a_z/\partial y)\Delta y$, where the value of $\partial a_z/\partial y$ is taken for point P[11]. Similarly, the difference $a_{y,4} - a_{y,2}$ can be represented in the form $(\partial a_y/\partial z)\Delta z$. Using these expressions in Eq. (1.91) and putting the common factor outside the parentheses, we get the following expression for the circulation:

$$\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\Delta y\,\Delta z = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\Delta S$$

where ΔS is the area of the contour. Dividing the circulation by ΔS, we find the expression for the projection of curl $\boldsymbol{a}$ onto the x-axis:
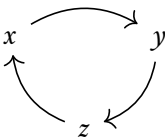
$$(\text{curl } \boldsymbol{a})_x = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}. \tag{1.92}$$

We can find by similar reasoning that

$$(\text{curl } \boldsymbol{a})_y = \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \tag{1.93}$$

$$(\text{curl } \boldsymbol{a})_z = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}. \tag{1.94}$$

It is easy to see that any of the equations (1.92)-(1.94) can be obtained from the preceding one [Eq. (1.94) should be considered as the preceding one for Eq. (1.94)] by the so-called cyclic transposition of the coordinates, *i.e.*, by replacing the coordinates according to the scheme

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[11]The inaccuracy which we tolerate here vanishes when the contour shrinks to point P in the limit transition.

Thus, the curl of the vector $\boldsymbol{a}$ is determined in the Cartesian coordinate system by the following expression:

$$\operatorname{curl} \boldsymbol{a} = \hat{\boldsymbol{e}}_x \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + \hat{\boldsymbol{e}}_y \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x} \right) + \hat{\boldsymbol{e}}_z \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right). \tag{1.95}$$

Below we shall indicate a more elegant way of writing this expression.

Stokes' Theorem. Knowing the curl of the vector $\boldsymbol{a}$ at every point of surface S (not necessarily plane), we can calculate the circulation of this vector around contour Γ enclosing S (the contour may also not be plane). For this purpose, we divide the surface into very small elements ΔS. Owing to their smallness, these elements can be considered as plane. Therefore in accordance with Eq. (1.90), the circulation of the vector $\boldsymbol{a}$ around the contour bounding ΔS can be written in the form

$$\Delta C \approx (\operatorname{curl} \boldsymbol{a})_n \Delta S = \operatorname{curl} \boldsymbol{a} \cdot \Delta \boldsymbol{S} \tag{1.96}$$

where $\hat{\boldsymbol{n}}$ is a positive normal to surface element ΔS.

In accordance with Eq. (1.88), summation of expression (1.96) over all the ΔS's yields the circulation of the vector $\boldsymbol{a}$ around contour Γ enclosing S:

$$C = \sum \Delta C \approx \sum \operatorname{curl} \boldsymbol{a} \cdot \Delta \boldsymbol{S}.$$

Performing a limit transition in which all the ΔS's shrink to zero (their number grows unlimitedly), we arrive at the equation

$$\oint_\Gamma \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\operatorname{curl} \boldsymbol{a}) \cdot \Delta \boldsymbol{S}. \tag{1.97}$$

Equation (1.97) is called **Stokes' theorem**. Its meaning is that the circulation of the vector $\boldsymbol{a}$ around an arbitrary contour Γ equals the flux of the vector curl $\boldsymbol{a}$ through the arbitrary surface S surrounded by the given contour.

The Del Operator. Writing of the formulas of vector analysis is simplified quite considerably if we introduce a vector differential operator designated by the symbol ∇ (nabla or del) and called the **del operator** or the **Hamiltonian operator**. This operator denotes a vector with the components $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$. Consequently,

$$\nabla = \hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}. \tag{1.98}$$

This vector has no meaning by itself. It acquires a meaning in combination with the scalar or vector function by which it is symbolically multiplied. Thus, if we multiply the vector ∇ by the scalar φ we obtain the vector

$$\nabla \varphi = \hat{\boldsymbol{e}}_x \frac{\partial \varphi}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial \varphi}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial \varphi}{\partial z} \tag{1.99}$$

which is the gradient of the function φ [see Eq. (1.68)].

The scalar product of the vectors ∇ and $\boldsymbol{a}$ gives the scalar

$$\nabla \cdot \boldsymbol{a} = \nabla_x a_x + \nabla_y a_y + \nabla_z a_z \tag{1.100}$$

which we can see to be the divergence of the vector $\boldsymbol{a}$ [see Eq. (1.81)].

Finally, the vector product of the vectors ∇ and $\boldsymbol{a}$ gives a vector with the components $(\nabla \times \boldsymbol{a})_x = \nabla_y a_z - \nabla_z a_y = \partial a_z / \partial y - \partial a_y / \partial z$, etc., that coincide with the components of curl $\boldsymbol{a}$ [see Eqs. (1.92)-(1.94)]. Hence, using the writing of a vector product with the aid of a determinant, we have

$$\text{curl } \boldsymbol{a} = \nabla \times \boldsymbol{a} = \begin{vmatrix} \hat{\boldsymbol{e}}_x & \hat{\boldsymbol{e}}_y & \hat{\boldsymbol{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}. \tag{1.101}$$

Thus, there are two ways of denoting the gradient, divergence, and curl:

$$\nabla \varphi \equiv \text{grad } \varphi, \quad \nabla \cdot \boldsymbol{a} \equiv \text{div } \boldsymbol{a}, \quad \nabla \times \boldsymbol{a} \equiv \text{curl } \boldsymbol{a}.$$

The use of the del symbol has a number of advantages. We shall therefore use such symbols in the following. One must accustom oneself to identify the symbol $\nabla \varphi$ with the words "gradient of phi" (*i.e.*, to say not "del phi", but "gradient of phi"), the symbol $\nabla \cdot \boldsymbol{a}$ with the words "divergence of a" and, finally, the symbol $\nabla \times \boldsymbol{a}$ with the words "curl of a".

When using the vector ∇, one must remember that it is a differential operator acting on all the functions to the right of it. Consequently, in transforming expressions including ∇, one must take into consideration both the rules of vector algebra and those of differential calculus. For example, the derivative of the product of the functions φ and ψ is

$$(\varphi \psi)' = \varphi' \psi + \varphi \psi'.$$

Accordingly,

$$\text{grad } (\varphi \psi) = \nabla(\varphi \psi) = \psi \nabla \varphi + \varphi \nabla \psi = \psi \text{ grad } \varphi + \varphi \text{ grad } \psi. \tag{1.102}$$

Similarly,

$$\text{div } (\varphi \boldsymbol{a}) = \nabla \cdot (\varphi \boldsymbol{a}) = \boldsymbol{a} \cdot (\nabla \varphi) + \varphi (\nabla \cdot \boldsymbol{a}). \tag{1.103}$$

The gradient of a function φ is a vector function. Therefore, the divergence and curl operations can be performed with it:

$$\text{div grad } \varphi = \nabla \cdot \nabla \varphi = (\nabla \cdot \nabla)\varphi = \left(\nabla_x^2 + \nabla_y^2 + \nabla_z^2 \right) \varphi$$

$$= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \Delta \varphi \tag{1.104}$$

(Δ is the Laplacian operator)

$$\text{curl grad } \varphi = \nabla \times (\nabla \varphi) = (\nabla \times \nabla)\varphi \tag{1.105}$$

(we remind our reader that the vector product of a vector and itself is zero).

Let us apply the divergence and curl operations to the function curl $\boldsymbol{a}$:

$$\text{div curl } \boldsymbol{a} = \nabla \cdot \nabla \times \boldsymbol{a} = 0 \tag{1.106}$$

(a scalar triple product equals the volume of a parallelepiped constructed on the vectors being multiplied (see Vol. I, p. 22); if two of these vectors coincide, the volume of the parallelepiped equals zero):

$$\text{curl curl } \boldsymbol{a} = \nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - (\nabla \cdot \nabla)\boldsymbol{a} = \text{grad div } \boldsymbol{a} - \Delta\boldsymbol{a} \tag{1.107}$$

[we have used Eq. (1.35) of Vol. I, namely, $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$].

Equation (1.106) signifies that the field of a curl has no sources. Hence, the lines of the vector curl $\boldsymbol{a}$ have neither a beginning nor an end. It is exactly for this reason that the flux of a curl through any surface S resting on the given contour Γ is the same [see Eq. (1.97)).

We shall note in concluding that when the del operator is used, Eqs. (1.82) and (1.97) can be given the form

$$\oint_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \oint_V \nabla \cdot \boldsymbol{a} \, \mathrm{d}V, \qquad \text{(the Ostrogradsky-Gauss theorem)} \tag{1.108}$$

$$\oint_\Gamma \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\nabla \times \boldsymbol{a}) \cdot \mathrm{d}\boldsymbol{S}. \quad \text{(Stokes' theorem)} \tag{1.109}$$

1.12. Circulation and Curl of an Electrostatic Field

We established in Sec. 1.6 that the forces acting on the charge q in an electrostatic field are conservative. Hence, the work of these forces on any closed path Γ is zero:

$$A = \oint_\Gamma q\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0.$$

Cancelling q, we get

$$\oint_\Gamma \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0 \tag{1.110}$$

(compare with Eq. (1.46)].

The integral in the left-hand side of Eq. (1.110) is the circulation of the vector $\boldsymbol{E}$ around contour Γ [see expression (1.80)]. Thus, *an electrostatic field is characterized by the fact that the circulation of the strength (intensity) vector of this field around any closed contour equals zero.*

Let us take an arbitrary surface S resting on contour Γ for which the circulation is calculated (Fig. 1.32). According to Stokes's theorem [see Eq. (1.109)], the integral of curl $\boldsymbol{E}$ taken over this surface equals the circulation of the vector $\boldsymbol{E}$ around contour

Fig. 1.32
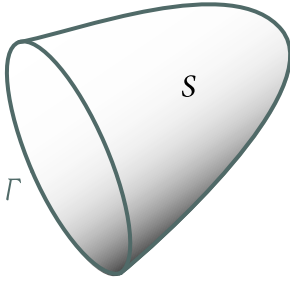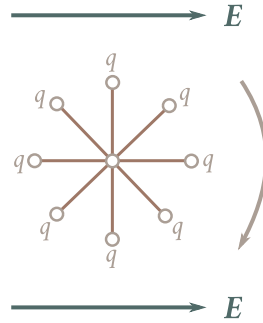

Fig. 1.33

Γ:

$$\int_S (\nabla \times E) \cdot dS = \oint_\Gamma E \cdot dl. \tag{1.111}$$

Since the circulation equals zero, we arrive at the conclusion that

$$\int_S (\nabla \times E) \cdot dS = 0.$$

This condition must be observed for any surface S resting on arbitrary contour Γ. This is possible only if the curl of the vector E at every point of the field equals zero:

$$\nabla \times E = 0. \tag{1.112}$$

By analogy with the fan impeller shown in Fig. 1.25, let us imagine an electrical "impeller" in the form of a light hub with spokes whose ends carry identical positive charges q (Fig. 1.33; the entire arrangement must be small in size). At the points of an electric field where curl E differs from zero, such an impeller would rotate with an acceleration that is the greater, the larger is the projection of the curl onto the impeller axis. For an electrostatic field, such an imaginary arrangement would not rotate with any orientation of its axis.

Thus, a feature of an electrostatic field is that it is a non-circuital one. We established in the preceding section that the curl of the gradient of a scalar function equals zero [see expression (1.96)]. Therefore, the equality to zero of curl E at every point of a field makes it possible to represent E in the form of the gradient of a scalar function φ called the potential. We have already considered this representation in Sec. 1.8 [see Eq. (1.41); the minus sign in this equation was taken from physical considerations].

We can immediately conclude from the need to observe condition (1.110) that the existence of an electrostatic field of the kind shown in Fig. 1.34 is impossible. Indeed, for such a field, the circulation around the contour shown by the dash line
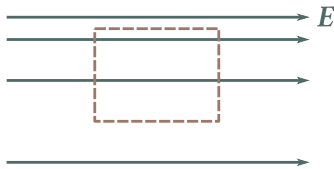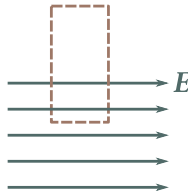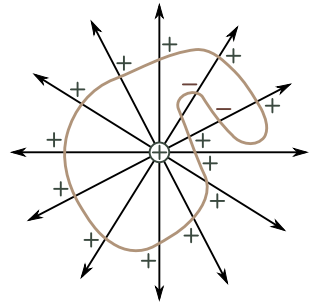
Fig. 1.34   Fig. 1.35   Fig. 1.36

would differ from zero, which contradicts condition (1.110). It is also impossible for a field differing from zero in a restricted volume to be homogeneous throughout this volume (Fig. 1.35). In this case, the circulation around the contour shown by the dash line would differ from zero.

1.13. Gauss's Theorem

We established in the preceding section what the curl of an electrostatic field equals. Now let us find the divergence of a field. For this purpose, we shall consider the field of a point charge q and calculate the flux of the vector E through closed surface S surrounding the charge (Fig. 1.36). We showed in Sec. 1.5 that the number of lines of the vector E beginning at a point charge $+q$ or terminating at a charge $-q$ numerically equals q/ε_0.

By Eq. (1.77), the flux of the vector E through any closed surface equals the number of lines coming out, *i.e.*, beginning on the charge, if it is positive, and the number of lines entering the surface, *i.e.*, terminating on the charge, if it is negative. Taking into account that the number of lines beginning or terminating at a point charge numerically equals q/ε_0 (see Sec. 1.5), we can write that

$$\Phi_E = \frac{q}{\varepsilon_0}. \tag{1.113}$$

The sign of the flux coincides with that of the charge q. The dimensions of both sides of Eq. (1.113) are identical.

Now let us assume that a closed surface surrounds N point charges $q_1, q_2, \ldots, q_N$. On the basis of the superposition principle, the strength E of the field set up by all the charges equals the sum of the strengths E_i set up by each charge separately: $E = \sum_i E_i$. Hence,

$$\Phi_E = \oint_S E \cdot dS = \oint_S \left(\sum_i E_i \right) \cdot dS = \sum_i \oint_S E_i \cdot dS.$$

Each of the integrals inside the sum sign equals q_i/ε_0. Therefore,

$$\Phi_E = \oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0} \sum_{i=1}^{N} q_i. \tag{1.114}$$

The statement we have proved is called **Gauss's theorem**. According to it, *the flux of an electric field strength vector through a closed surface equals the algebraic sum of the charges enclosed by this surface divided by ε_0*.

When considering fields set up by macroscopic charges (*i.e.*, charges formed by an enormous number of elementary charges), the discrete structure of these charges is disregarded, and they are considered to be distributed in space continuously with a finite density everywhere. The **volume density of a charge** ρ is determined by analogy with the density of a mass as the ratio of the charge dq to the infinitely small (physically) volume dV containing this charge:

$$\rho = \frac{dq}{dV}. \tag{1.115}$$

In the given case by an infinitely small (physically) volume, we must understand a volume which on the one hand is sufficiently small for the density within its limits to be considered identical, and on the other is sufficiently great for the discreteness of the charge not to manifest itself.

Knowing the charge density at every point of space, we can find the total charge surrounded by closed surface S. For this purpose, we must calculate the integral of ρ with respect to the volume enclosed by the surface:

$$\sum_i q_i = \int_V \rho \, dV.$$

Thus, Eq. (1.114) can be written in the form

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV. \tag{1.116}$$

Replacing the surface integral with a volume one in accordance with Eq. (1.108), we have

$$\int_V \nabla \cdot \boldsymbol{E} \, dV = \frac{1}{\varepsilon_0} \int_V \rho \, dV.$$

The relation which we have arrived at must be observed for any arbitrarily chosen volume V. This is possible only if the values of the integrands for every point of space are the same. Hence, the divergence of the vector $\boldsymbol{E}$ is associated with the density of the charge at the same point by the equation

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0} \rho. \tag{1.117}$$

This equation expresses Gauss's theorem in the differential form.

For a flowing liquid, $\nabla \cdot \boldsymbol{v}$ gives the unit power of the sources of the liquid at a given point. By analogy, charges are said to be sources of an electric field.

1.14. Calculating Fields with the Aid of Gauss's Theorem

Gauss's theorem permits us in a number of cases to find the strength of a field in a much simpler way than by using Eq. (1.15) for the field strength of a point charge and the field superposition principle. We shall demonstrate the possibilities of Gauss's theorem by employing a few examples that will be useful for our further exposition. Before starting on our way, we shall introduce the concepts of surface and linear charge densities.

If a charge is concentrated in a thin surface layer of the body carrying the charge, the distribution of the charge in space can be characterized by the surface density σ, which is determined by the expression

$$\sigma = \frac{dq}{dS}. \tag{1.118}$$

Here dq is the charge contained in the layer of area dS. By dS is meant an infinitely small (physically) section of the surface.

If a charge is distributed over the volume or surface of a cylindrical body (uniformly in each section), the linear charge density is used, *i.e.*,

$$\lambda = \frac{dq}{dl} \tag{1.119}$$

where dl is the length of an infinitely small (physically) segment of the cylinder, and dq is the charge concentrated on this segment.

Field of an Infinite Homogeneously Charged Plane. Assume that the surface charge density at all points of a plane is identical and equal to σ; for definiteness we shall consider the charge to be positive. It follows from considerations of symmetry that the field strength at any point is directed at right angles to the plane. Indeed, since the plane is infinite and charged homogeneously, there is no reason why the vector $\boldsymbol{E}$ should deflect to a side from a normal to the plane. It is further evident that at points symmetrical relative to the plane, the field strength is identical in magnitude and opposite in direction.

Let us imagine mentally a cylindrical surface with generatrices perpendicular to the plane and bases of a size ΔS arranged symmetrically relative to the plane (Fig. 1.37). Owing to symmetry, we have $E' = E'' = E$. We shall apply Gauss's theorem to the surface. The flux through the side part of the surface will be absent because E_n at each point of it is zero. For the bases, E_n coincides with E. Hence, the total flux through the surface is $2E\Delta S$. The surface encloses the charge $\sigma \Delta S$.
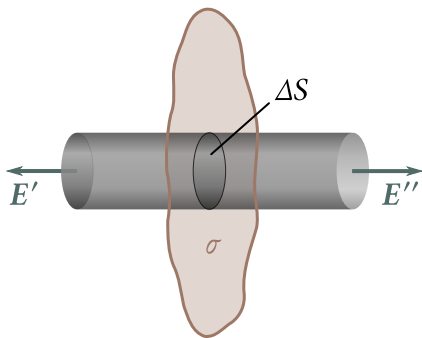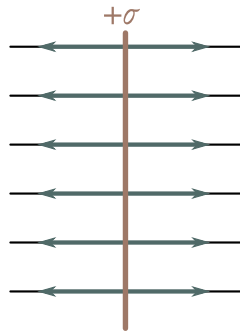
Fig. 1.37

Fig. 1.38

According to Gauss's theorem, the condition must be observed that

$$2E\Delta S = \frac{\sigma \Delta S}{\varepsilon_0}$$

whence

$$E = \frac{\sigma}{2\varepsilon_0}. \qquad (1.120)$$

The result we have obtained does not depend on the length of the cylinder. This signifies that at any distances from the plane, the field strength is identical in magnitude. The field lines are shown in Fig. 1.38. For a negatively charged plane, the result will be the same except for the reversal of the direction of the vector E and the field lines.

If we take a plane of finite dimensions, for instance a charged thin plate[12], then the result obtained above will hold only for points, the distance to which from the edge of the plate considerably exceeds the distance from the plate itself (in Fig. 1.39 the region containing such points is outlined by a dash line). At points at an increasing distance from the plane or approaching its edges, the field will differ more and more from that of an infinitely charged plane. It is easy to imagine the nature of the field at great distances if we take into account that at distances considerably exceeding the dimensions of the plate, the field it sets up can be treated as that of a point charge.

Field of Two Uniformly Charged Planes. The field of two parallel infinite planes carrying opposite charges with a constant surface density σ identical in magnitude can be found by superposition of the fields produced by each plane

---

[12]For a plate, by σ in Eq. (1.120) should be understood the charge concentrated on 1 m^2 of the plate over its entire thickness. In metal bodies, the charge is distributed over the external surface. Therefore by σ we should understand the double value of the charge density on the surfaces surrounding the metal plate.
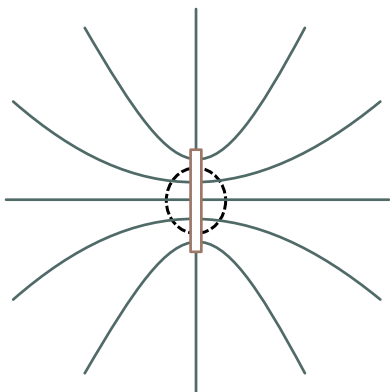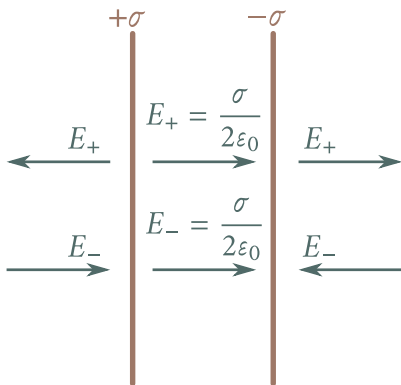
Fig. 1.39

Fig. 1.40

separately (Fig. 1.40). In the region between the planes, the fields being added have the same direction, so that the resultant field strength is

$$E = \frac{\sigma}{\varepsilon_0}. \tag{1.121}$$

Outside the volume bounded by the planes, the fields being added have opposite directions so that the resultant field strength equals zero.

Thus, the field is concentrated between the planes. The field strength at all points of this region is identical in value and in direction; consequently, the field is homogeneous. The field lines are a collection of parallel equispaced straight lines.

The result we have obtained also holds approximately for planes of finite dimensions if the distance between them is much smaller than their linear dimensions (a parallel-plate capacitor). In this case, appreciable deviations of the field from homogeneity are observed only near the edges of the plates (Fig. 1.41).

Field of an Infinite Charged Cylinder. Assume that the field is produced by an infinite cylindrical surface of radius R whose charge has a constant surface density σ. Considerations of symmetry show that the field strength at any point must be directed along a radial line perpendicular to the cylinder axis, and that the magnitude of the strength can depend only on the distance r from the cylinder axis. Let us mentally imagine a coaxial closed cylindrical surface of radius r and height h with a charged surface (Fig. 1.42). For the bases of the cylinder, we have $E_n = 0$, for the side surface $E_n = E(r)$ (the charge is assumed to be positive). Hence, the flux of the vector $\boldsymbol{E}$ through the surface being considered is $E(r) \times 2\pi rh$. If $r > R$, the charge $q = \lambda h$ (where λ is the linear charge density) will get into the surface. Applying Gauss's theorem, we find that

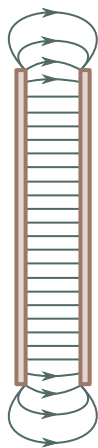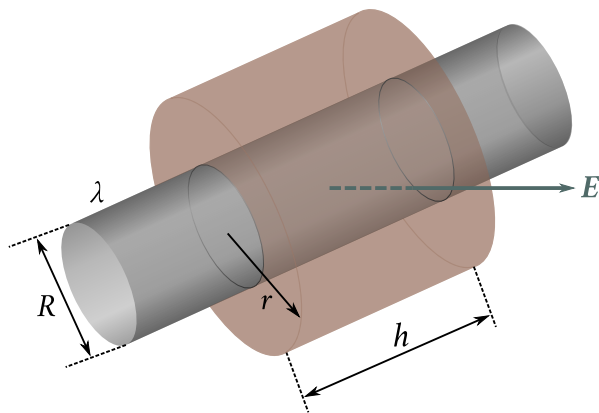$$E(r) \times 2\pi rh = \frac{2\lambda}{\varepsilon_0}.$$

Fig. 1.41

Fig. 1.42

Hence,

$$E(r) = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r} \quad (r \geqslant R). \tag{1.122}$$

If $r < R$, the closed surface being considered contains no charges inside, owing to which $E(r) = 0$.

Thus, there is no field inside a uniformly charged cylindrical surface of infinite length. The field strength outside the surface is determined by the linear charge density λ and the distance r from the cylinder axis.

The field of a negatively charged cylinder differs from that of a positively charged one only in the direction of the vector E. A glance at Eq. (1.122) shows that by reducing the cylinder radius R (with a constant linear charge density λ), we can obtain a field with a very great strength near the surface of the cylinder.

Introducing $\lambda = 2\pi R\sigma$ into Eq. (1.122) and assuming that $r = R$, we get the following value for the field strength in direct proximity to the surface of a cylinder:

$$E(R) = \frac{\sigma}{\varepsilon_0}. \tag{1.123}$$

The superposition principle makes it simple to find the field of two coaxial cylindrical surfaces carrying a linear charge density λ of the same magnitude, but of opposite signs (Fig. 1.43). There is no field inside the smaller and outside the larger cylinders. The field strength in the gap between the cylinders is determined by Eq. (1.122). This also holds for cylindrical surfaces of a finite length if the gap between the surfaces is much smaller than their length (a cylindrical capacitor). Appreciable deviations from the field of surfaces of an infinite length will be observed only near the edges of the cylinders.
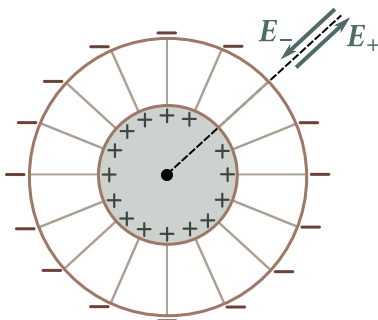
Fig. 1.43

Field of a Charged Spherical Surface. The field produced by a spherical surface of radius R whose charge has a constant surface density σ will obviously be a centrally symmetrical one. This signifies that the direction of the vector $\boldsymbol{E}$ at any point passes through the centre of the sphere, while the magnitude of the field strength is a function of the distance r from the centre of the sphere. Let us imagine a surface of radius r that is concentric with the charged sphere. For all points of this surface, $E_n = E(r)$. If $r > R$, the entire charge q distributed over the sphere will be inside the surface. Hence,

$$E(r) \times 4\pi r^2 = \frac{q}{\varepsilon_0}$$

whence

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \quad (r \geqslant R) \tag{1.124}$$

A spherical surface of radius r less than R will contain no charges, owing to which for $r < R$ we get $E(r) = 0$.

Thus, there is no field inside a spherical surface whose charge has a constant surface density σ. Outside this surface, the field is identical with that of a point charge of the same magnitude at the centre of the sphere.

Using the superposition principle, it is easy to show that the field of two concentric spherical surfaces (a spherical capacitor) carrying charges $+q$ and $-q$ that are identical in magnitude and opposite in sign is concentrated in the gap between the surfaces, the magnitude of the field strength in the gap being determined by Eq. (1.124).

Field of a Volume-Charged Sphere. Assume that a sphere of radius R has a charge with a constant volume density ρ. The field in this case has central symmetry. It is easy to see that the same result is obtained for the field outside the sphere [see Eq. (1.124)] as for a sphere with a surface charge. The result will be different for points inside the sphere, however. A spherical surface of radius r $(r < R)$ contains

a charge equal to $\rho \times 4\pi r^3/3$. Therefore, Gauss's theorem for such a surface will be written as follows:

$$E(r) \times 4\pi r^2 = \frac{1}{\varepsilon_0}\rho\frac{4}{3}\pi r^3.$$

Hence, substituting $q/(4\pi R^3/3)$ for ρ, we get

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^3}r. \quad (r \leqslant R) \tag{1.125}$$

Thus, the field strength inside a sphere grows linearly with the distance r from the centre of the sphere. Outside the sphere, the field strength diminishes according to the same law as for the field of a point charge.