Beam-Target Reaction Rates

$$\langle \sigma v \rangle = \iint \sigma(E_{rel}) ||\mathbf{v} - \mathbf{v}'|| \delta(\mathbf{v}' - \mathbf{v}_B) \left[\frac{m_T}{2\pi kT}\right]^{\frac{3}{2}} e^{-\frac{m_T}{2kT}(\mathbf{v}\cdot\mathbf{v})} d\mathbf{v}'\, d\mathbf{v} \tag{1}$$

Let $\mathbf{v}_B$ be in the $\hat{z}$ direction and calculate the first integral

$$\int \sigma(E_{rel}) \sqrt{v_x^2 + v_y^2 + (v_z - v_B)^2} \left[\frac{m_T}{2\pi kT}\right]^{\frac{3}{2}} e^{-\frac{m_T}{2kT}(\mathbf{v}\cdot\mathbf{v})} d\mathbf{v} \tag{2}$$

Let $v_x = v_r cos(\theta)$ and $v_y = v_r sin(\theta)$ the integral takes the form

$$\left[\frac{m_T}{2\pi kT}\right]^{\frac{3}{2}} \iint \int_0^{2\pi} \sigma(E_{rel}) \sqrt{v_r^2 + (v_z - v_B)^2} e^{-\frac{m_T}{2kT}(v_r^2 + v_z^2)} v_r d\theta\, dv_r\, dv_z \tag{3}$$

Integrating over θ yields

$$\frac{2}{\sqrt{\pi}} \left[\frac{m_T}{2kT}\right]^{\frac{3}{2}} \iint \sigma(E_{rel}) \sqrt{v_r^2 + (v_z - v_B)^2} e^{-\frac{m_T}{2kT}(v_r^2 + v_z^2)} v_r\, dv_r\, dv_z \tag{4}$$

Let $v_z = \sqrt{\frac{2kT}{m_T}} u_z$ and $v_r = \sqrt{\frac{2kT}{m_T}} u_r$

$$\frac{2}{\sqrt{\pi}} \left[\frac{m_T}{2kT}\right]^{\frac{3}{2}} \iint \sigma(E_{rel}) \sqrt{\frac{2kT}{m_T} u_r^2 + \left(\sqrt{\frac{2kT}{m_T}} u_z - v_B\right)^2} e^{-(u_r^2 + u_z^2)} \left[\frac{2kT}{m_T}\right]^{\frac{3}{2}} u_r\, du_r\, du_z \tag{5}$$

Simplifying

$$\langle \sigma v \rangle = \frac{2}{\sqrt{\pi}} \sqrt{\frac{2kT}{m_T}} \iint \sigma(E_{rel}) \sqrt{u_r^2 + \left(u_z - \sqrt{\frac{m_T}{2kT}} v_B\right)^2} e^{-(u_r^2 + u_z^2)} u_r\, du_r\, du_z \tag{6}$$

The velocity of the beam ion is given by

$$v_B = \sqrt{\frac{2E_B}{m_B}} \tag{7}$$

This plugging this into equation 6 gives

$$\langle \sigma v \rangle = \frac{2}{\sqrt{\pi}} \sqrt{\frac{2kT}{m_T}} \iint \sigma(E_{rel}) \sqrt{u_r^2 + \left(u_z - \sqrt{\frac{E_B m_T}{m_B kT}}\right)^2} e^{-(u_r^2 + u_z^2)} u_r\, du_r\, du_z \tag{8}$$

The relative energy, E_{rel} is given by

$$E_{rel} = \frac{1}{2}\mu\left(v_r^2 + (v_z - v_B)^2\right) \tag{9}$$

where μ is the reduced mass. In terms of the transformed velocity u the relative energy is

$$E_{rel} = \mu\frac{kT}{m_T}\left(u_r^2 + \left(u_z - \sqrt{\frac{E_B m_T}{m_B kT}}\right)^2\right) \tag{10}$$

Mass Independent Formulation

Letting $\overline{E} = E_B/m_B$ and $\overline{T} = kT/m_T$ the reaction rates can be expressed as

$$\langle\sigma v\rangle = \frac{2}{\sqrt{\pi}}\sqrt{2\overline{T}}\iint \sigma(E_{rel})\sqrt{u_r^2 + \left(u_z - \sqrt{\overline{E}/\overline{T}}\right)^2}\, e^{-(u_r^2+u_z^2)}u_r\,du_r\,du_z \tag{11}$$

where E_{rel} is equal to

$$E_{rel} = \mu\overline{T}\left(u_r^2 + \left(u_z - \sqrt{\overline{E}/\overline{T}}\right)^2\right) \tag{12}$$

and

$$\mu = \frac{m_T\,m_B}{m_T + m_B} = m_p\left(\frac{a_T\,a_B}{a_T + a_B}\right) \tag{13}$$

If the cross sections are a function of energy/mass then we can combine the above equations to form

$$\langle\sigma v\rangle = \frac{2}{\sqrt{\pi}}\sqrt{2\overline{T}}\iint \sigma\left(\overline{T}u_{rel}^2\right)\, u_{rel}\, e^{-(u_r^2+u_z^2)}u_r\,du_r\,du_z \tag{14}$$

where

$$u_{rel} = \sqrt{u_r^2 + \left(u_z - \sqrt{\overline{E}/\overline{T}}\right)^2} \tag{15}$$