Stark-Zeeman Effect

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1 Introduction

This article gives details on Stark effect and Stark-Zeeman effect for the Balmer-Alpha transitions. A derivation of the perturbed energies, transition rates, and Stokes parameters is given. These are useful in plasma applications for diagnostics based on the Motional Stark Effect. The last section gives an overview of the changes implemented in the FIDASIM code for the Stark-Zeeman splitting and Stokes parameters.

2 Hamiltonian

The Hamiltonian for the hydrogen atom in an E-field and B-field is given by,

$$H = H_0 + H_E + H_B \tag{1}$$

Where, H_0 is the Hydrogen atom's unperturbed Hamiltonian (possibly including fine structure although I have ignored it so far). Also,

$$H_E = e|\mathbf{E}|x = \frac{\epsilon x}{3a_0} \tag{2}$$

$$H_B = \frac{e|\mathbf{B}|}{2m_e}(L_z + 2S_z) = \frac{\gamma}{\hbar}(L_z + 2S_z) \tag{3}$$

are the perturbations due to the external fields, assuming that the $\vec{E}$ and $\vec{B}$ fields are perpendicular.

3 Pure Stark effect for n=2

First we want to get matrix elements of the perturbed part of the Hamiltonian. Using the n,l,m,m_s basis these matrix elements are given by,

$$\langle n, l, m| \frac{\epsilon x}{3a_0} |n', l', m'\rangle \tag{4}$$

The unperturbed hydrogen wave-functions can be looked up in most quantum mechanics books and are given by,

$$\psi_{n,l,m} = [(\frac{2}{na_0})^3\frac{(n-l-1)!}{2n(n+l)!}]^{1/2}\exp\left(-\frac{r}{na_0}\right)(\frac{2r}{na_0})^l L_{n+l}^{2l+1}(\frac{2r}{na_0})Y_{lm}(\theta,\phi) \quad (5)$$

Here, $L_{n+l}^{2l+1}(x)$ are the Laguerre polynomials and $Y_{lm}(\theta,\phi)$ are the spherical harmonics. Note that $l \in \{0,1,...n-1\}$ and $m \in \{-l,-l+1,...,l-1,l\}$. Using this, the matrix elements can be written in integral form as,

$$\langle n,l,m|\frac{\epsilon x}{3a_0}|n',l',m'\rangle = \int \psi^*_{n,l,m}(\frac{\epsilon}{3a_0}r\cos\theta\sin\phi)\psi_{n',l',m'}d^3\vec{r} \quad (6)$$

Using mathematica to do the calculations, the matrix form of the perturbed Hamiltonian for the n=2 states is determined to be,

$$\begin{pmatrix} 0 & -\frac{\epsilon}{\sqrt{2}} & 0 & \frac{\epsilon}{\sqrt{2}} \\ -\frac{\epsilon}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\epsilon}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix} \quad (7)$$

Here, the first column is for $|2,0,0\rangle$, the second for $|2,1,-1\rangle$, the third for $|2,1,0\rangle$, and fourth for $|2,1,1\rangle$. Similarly for the rows. The perturbed energies are then simply the Eigenvalues of this matrix. The eigenvalues are given by,

$$\{-\epsilon,\epsilon,0,0\} \quad (8)$$

The Eigen-vectors of the matrix are the states corresponding to each energy level.

4 Pure Stark effect for n=3

Following the same process as for n=2, I get the perturbed matrix elements of the Hamiltonian for n=3,

$$\begin{pmatrix} 0 & -\sqrt{3}\epsilon & 0 & \sqrt{3}\epsilon & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3}\epsilon & 0 & 0 & 0 & -\frac{3\epsilon}{2} & 0 & \frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{3\epsilon}{2\sqrt{2}} & 0 & \frac{3\epsilon}{2\sqrt{2}} & 0 \\ \sqrt{3}\epsilon & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & \frac{3\epsilon}{2} \\ 0 & -\frac{3\epsilon}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3\epsilon}{2\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3\epsilon}{2\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3\epsilon}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (9)$$

And the eigenvalues(perturbed energies) are,

$$\left\{-3\epsilon, 3\epsilon, -\frac{3\epsilon}{2}, -\frac{3\epsilon}{2}, \frac{3\epsilon}{2}, \frac{3\epsilon}{2}, 0, 0, 0\right\} \quad (10)$$

5 Stark-Zeeman Effect

Now we add the third Hamiltonian term, $H_B = \frac{e|\mathbf{B}|}{2m_e}(L_z + 2S_z) = \frac{\gamma}{\hbar}(L_z + 2S_z)$. The matrix elements in the $|n, l, m, m_s\rangle$ basis are very easy to calculate.

$$\langle n, l, m, m_s|\frac{\gamma}{\hbar}(L_z + 2S_z)|n, l', m', m_s'\rangle = \gamma(m\delta_{m,m'} + 2m_s\delta_{m_s,m_s'}) \tag{11}$$

Since the spin doesn't change in the transitions we are concerned with, the contribution of the spin to the perturbation in the energy will be the same for both n=2 and n=3. Because of this the spectrum will not shift due to the spin portion of the Zeeman effect. So to simplify future calculations this part will be dropped and the effective matrix elements will be,

$$\langle n, l, m|\frac{\gamma}{\hbar}(L_z)|n, l', m'\rangle = \gamma m\delta_{m,m'} \tag{12}$$

So for n=2 the full matrix is,

$$\begin{pmatrix} 0 & -\frac{\epsilon}{\sqrt{2}} & 0 & \frac{\epsilon}{\sqrt{2}} \\ -\frac{\epsilon}{\sqrt{2}} & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\epsilon}{\sqrt{2}} & 0 & 0 & \gamma \end{pmatrix} \tag{13}$$

with eigenvalues,

$$\left\{0, 0, -\sqrt{\gamma^2 + \epsilon^2}, \sqrt{\gamma^2 + \epsilon^2}\right\} \tag{14}$$

For n=3 the matrix is,

$$\begin{pmatrix}
0 & -\sqrt{3}\epsilon & 0 & \sqrt{3}\epsilon & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{3}\epsilon & -\gamma & 0 & 0 & -\frac{3\epsilon}{2} & 0 & \frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3\epsilon}{2\sqrt{2}} & 0 & \frac{3\epsilon}{2\sqrt{2}} & 0 \\
\sqrt{3}\epsilon & 0 & 0 & \gamma & 0 & 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & \frac{3\epsilon}{2} \\
0 & -\frac{3\epsilon}{2} & 0 & 0 & -2\gamma & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3\epsilon}{2\sqrt{2}} & 0 & 0 & -\gamma & 0 & 0 & 0 \\
0 & \frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & -\frac{1}{2}\sqrt{\frac{3}{2}}\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3\epsilon}{2\sqrt{2}} & 0 & 0 & 0 & 0 & \gamma & 0 \\
0 & 0 & 0 & \frac{3\epsilon}{2} & 0 & 0 & 0 & 0 & 2\gamma
\end{pmatrix} \tag{15}$$

With eigenvalues,

$$\left\{0, 0, 0, -\sqrt{4\gamma^2 + 9\epsilon^2}, -\frac{1}{2}\sqrt{4\gamma^2 + 9\epsilon^2}, -\frac{1}{2}\sqrt{4\gamma^2 + 9\epsilon^2}, \frac{1}{2}\sqrt{4\gamma^2 + 9\epsilon^2}, \frac{1}{2}\sqrt{4\gamma^2 + 9\epsilon^2}, \sqrt{4\gamma^2 + 9\epsilon^2}\right\} \tag{16}$$

Alex Thorman's paper [2] has a nice diagram using the variables $q_0 = \sqrt{\gamma^2 + \epsilon^2}$ and $q_1 = \sqrt{4\gamma^2 + 9\epsilon^2}$ that summarizes the results derived so far. It is shown in Figure 1.

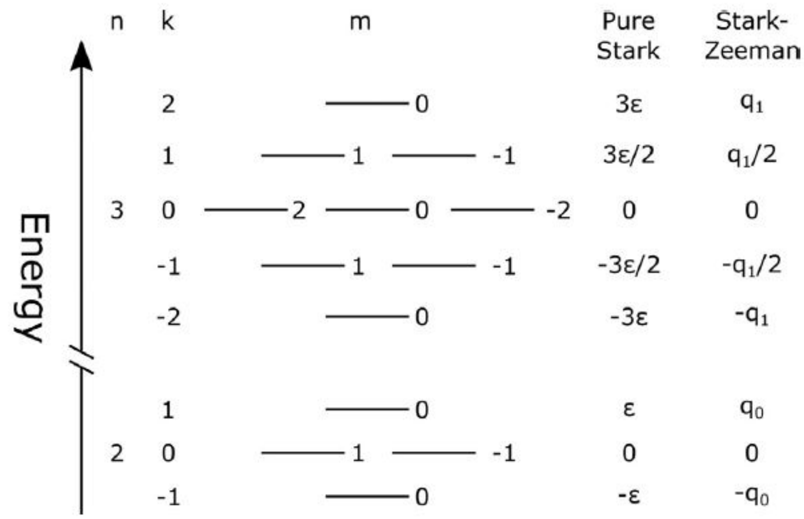

Figure 1: Energy and degeneracy of the $n = 2$ and $n = 3$ levels of the pure Stark states and Stark-Zeeman states. In the Stark-Zeeman case, the degeneracy only exists when the electric and magnetic fields are orthogonal. From [2].

6 Wavelength shifts

The wavelength of a transition from energy E_i to E_j can be determined using the formula,

$$E_i - E_j = \frac{hc}{\lambda} \tag{17}$$

In the Stark-Zeeman case of n=3 to n=2, this becomes,

$$\left(\frac{E_1}{3^2} + k_1 q_1/2\right) - \left(\frac{E_1}{2^2} + k_0 q_0\right) = \frac{hc}{\lambda} \tag{18}$$

Where E_1 is the binding energy of Hydrogen. Rearranging this equation and using λ_0 for the unperturbed wavelength gives the following formula for the wavelength,

$$\lambda = \frac{2ch\lambda_0}{2ch - 2k_0\lambda_0 q_0 + k_1\lambda_0 q_1} \tag{19}$$

In the Motional Stark Effect the electric field is given by $\vec{E} = \vec{v} \times \vec{B}$. In this case, the above formula can give the wavelength as only a function of $|\vec{B}|$ and vp where vp is the velocity perpendicular to the B field. After Solving for λ and Taylor expanding, the wavelength is given by,

$$\lambda_0 + \frac{Be\lambda_0^2\left(k_2\sqrt{36a_0^2\mathrm{vp}^2m_e^2 + \hbar^2} - k_1\sqrt{81a_0^2\mathrm{vp}^2m_e^2 + \hbar^2}\right)}{2chm_e} + O\left(B^2\right) \tag{20}$$

7 Transitions

The transition rate between two states in the dipole approximation is given by,

$$A_{ij} = |\langle\psi_i|\vec{r}|\psi_j\rangle|^2 \frac{\omega_{ij}^3 e^2}{3\pi\epsilon_0\hbar c^3} \tag{21}$$

Where ω_{ij} is equal to $\frac{E_i - E_j}{\hbar}$. This formula shows that calculating the relative intensities of the transitions is essentially equivalent to calculating the $\langle\psi_i|\vec{r}|\psi_j\rangle$ matrix elements. Calculating these in the $|n,l,m\rangle$ basis is easy in Mathematica using the wave-functions in equation 5. However these aren't the right states for the stark zeeman effect. The good linear combinations are given by the normalized eigenvectors of matrices in equation 7 and 9. This change of basis can be done like so,

$$\langle\psi_i|\vec{r}|\psi_j\rangle = \sum_{n,l,m,n',l',m'} \langle\psi_i|\psi_{n,l,m}\rangle \, \langle\psi_{n,l,m}|\vec{r}|\psi_{n',l',m'}\rangle \, \langle\psi_{n',l',m'}|\psi_j\rangle \tag{22}$$

Note that this summation can be simplified since the n and m quantum numbers are common to both the $|n,l,m\rangle$ states and the stark states. The result for transitions from n=3 to n=2 is in the table below,

$\|n,k,m\rangle_C$	$\|2,1,0\rangle_C$	$\|2,-1,0\rangle_C$	$\|2,0,1\rangle_C$	$\|2,0,-1\rangle_C$
$\|3,2,0\rangle_C$	$(1681)\pi_4$	$(1)\pi_8$	$(18)\sigma_6^+$	$(18)\sigma_6^-$
$\|3,-2,0\rangle_C$	$(1)\pi_{-8}$	$(1681)\pi_{-4}$	$(18)\sigma_{-6}^+$	$(18)\sigma_{-6}^-$
$\|3,0,0\rangle_C$	$(729)\pi_{-2}$	$(729)\pi_2$	$(882)\sigma_0^+$	$(882)\sigma_0^-$
$\|3,0,2\rangle_C$	–	–	$(4608)\sigma_0^-$	–
$\|3,0,-2\rangle_C$	–	–	–	$(4608)\sigma_0^+$
$\|3,1,1\rangle_C$	$(1936)\sigma_1^-$	$(16)\sigma_5^-$	$(1152)\pi_3$	–
$\|3,1,-1\rangle_C$	$(1936)\sigma_1^+$	$(16)\sigma_5^+$	–	$(1152)\pi_3$
$\|3,-1,1\rangle_C$	$(16)\sigma_{-5}^-$	$(1936)\sigma_{-1}^-$	$(1152)\pi_{-3}$	–
$\|3,-1,-1\rangle_C$	$(16)\sigma_{-5}^+$	$(1936)\sigma_{-1}^+$	–	$(1152)\pi_{-3}$

Figure 2: Table from Alex Thorman's paper[2]. Normalization $|r_{ij}|^2 = 2^{14}3^6 a_0^2(Intensity)/5^{14}$ was used.

8 Polarization & Stokes Parameters

The polarization of the emitted light is dependent on the change in the m quantum number in the transition. If $\Delta m = 0$, the light will be linearly polarized along the direction of $\vec{E}$ (π lines). For $\Delta m = \pm 1$, the light will be circularly polarized perpendicular to $\vec{E}$ (σ lines).

The observed intensities will be the product of the transition probability and a factor depending on the angle between $\vec{E}$ and the observation direction in addition to the polarization, θ. The observed intensities are given by,

$$I_\pi^{obs} = I_\pi \sin^2\theta \tag{23}$$

$$I_\sigma^{obs} = I_\sigma \frac{1+\cos^2\theta}{2} \tag{24}$$

These factors match the angular distribution of dipole radiation in classical electrodynamics and can be understood classically. See Jackson page 437.

Stokes parameters are a way of parametrizing the polarization state of light. The following derivation of the stokes parameters for the Stark Effect mostly follows the derivation by Edward Collett [1]. Completely polarized light can be written in the form $\psi = c_1\phi_1 + c_2\phi_2$ where ϕ_1 and ϕ_2 are orthogonal polarization states. The stokes parameters are then defined using the complex amplitudes c_1 and c_2 to be,

$$s_0 = c_1c_1^* + c_2c_2^* \tag{25}$$

$$s_1 = c_1c_1^* - c_2c_2^* \tag{26}$$

$$s_2 = c_1c_2^* + c_2c_1^* \tag{27}$$

$$s_3 = i(c_1c_2^* - c_2c_1^*) \tag{28}$$

The electric field of an accelerating non-relativistic electric charge is given by,

$$\vec{E} = (-e/c^2R)[\vec{n} \times (\vec{n} \times \dot{\vec{v}})] \tag{29}$$

where $\vec{n} = \vec{R}/R$ is the unit vector to observation point and $\dot{\vec{v}}$ is the acceleration. In spherical coordinates, this can be rewritten as,

$$\vec{E} = (-e/c^2R)[(\ddot{x}\cos\theta - \ddot{z}\sin\theta)\vec{e_\theta} - \ddot{y}\vec{e_\phi}] \tag{30}$$

Here I have assumed that I can set $\phi = 0$ using symmetry about the axis of the external $\vec{E}$ field. The complex amplitude coefficients are then,

$$c_\theta = (-e/c^2R)(\ddot{x}\cos\theta - \ddot{z}\sin\theta) \tag{31}$$

$$c_\phi = (-e/c^2R)\ddot{y} \tag{32}$$

Using the model of an oscillating dipole and putting these in a quantum mechanical form, the time derivatives can be replaced by $-\omega_{ij}^2$. So we have,

$$c_\theta = (e\omega_{ij}^2/c^2R)(x\cos\theta - z\sin\theta) \tag{33}$$

$$c_\phi = (e\omega_{ij}^2/c^2R)y \tag{34}$$

These complex amplitudes can then be calculated using the $\langle\psi_i|\vec{r}|\psi_j\rangle$ matrix elements. Ignoring constant factors, the π transitions have $\vec{r} = (0, 0, 1)$ and the $\sigma^\pm$ transitions have $\vec{r} = (\pm 1, -i, 0)$. Plugging these in to 33 and 34 allows us to write the form of the stokes parameters, ignoring constant terms, as,

$$\vec{S}_\pi = \begin{pmatrix} \sin^2\theta \\ -\sin^2\theta \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

$$\vec{S}_{\sigma^\pm} = \frac{1}{2}\begin{pmatrix} 1+\cos^2\theta \\ \sin^2\theta \\ 0 \\ \pm 2\cos\theta \end{pmatrix} \tag{36}$$

Looking at equation 36, since the σ^+ and σ^- lines have the same transition probabilities, the sum will give $s_4 = 0$ for the σ transitions assuming that the upper state populations are statistically occupied. For a discussion of the effect of unequal upper states see Alex Thorman's paper [2]. One thing to note about this derivation is that the assumption that ϕ can be set to zero may not be true in the Stark-Zeeman case if the $\vec{B}$ field is too large since the $\vec{B}$ field will break the azimuthal symmetry. These stokes vectors should only be valid in the limit $\gamma/\epsilon \to 0$.

9 First order corrections to Stokes parameters for Stark-Zeeman effect.

The Zeeman effect breaks the symmetry that was assumed in Sec. 4 leading to changes in the stokes parameters even when upper state populations are statistically occupied[3]. To derive a correction to the stokes parameters, we can use the eigenvectors of the Stark-Zeeman Hamiltonian in equations 13 and 15 to calculate the dipole moments.

To demonstrate how to derive this correction, the correction to the Lyman-Alpha stokes parameters is presented. The Stark-Zeeman corrections to the hamiltonian with an electric field in the z-direction and a B-field in the x-direction is,

$$\begin{pmatrix} 0 & -\epsilon & 0 & 0 \\ -\epsilon & 0 & \frac{\gamma}{\sqrt{2}} & \frac{\gamma}{\sqrt{2}} \\ 0 & \frac{\gamma}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\gamma}{\sqrt{2}} & 0 & 0 \end{pmatrix} \tag{37}$$

This matrix has eigenvectors,

$$|2,0,-1\rangle = \frac{x}{\sqrt{x^2+2}}|2,0,0\rangle + \frac{\sqrt{2}}{\sqrt{x^2+2}}|2,1,-1\rangle$$

$$|2,0,1\rangle = \frac{x}{\sqrt{x^2+2}}|2,0,0\rangle + \frac{\sqrt{2}}{\sqrt{x^2+2}}|2,1,1\rangle$$

$$|2,-1,0\rangle = -\frac{1}{\sqrt{2}\sqrt{1+x^2}}|2,0,0\rangle - \frac{1}{\sqrt{2}}|2,1,0\rangle + \frac{1}{2\sqrt{\frac{1}{x^2}+1}}|2,1,1\rangle + \frac{1}{2\sqrt{\frac{1}{x^2}+1}}|2,1,-1\rangle$$

$$|2,1,0\rangle = -\frac{1}{\sqrt{2+2x^2}}|2,0,0\rangle + \frac{1}{\sqrt{2}}|2,1,0\rangle + \frac{1}{2\sqrt{\frac{1}{x^2}+1}}|2,1,1\rangle + \frac{1}{2\sqrt{\frac{1}{x^2}+1}}|2,1,-1\rangle$$

where the left side of the above equations have the $|n,k,m\rangle$ states and the right side has the $|n,l,m\rangle$ states. To simplify the expressions, I defined $x = \gamma/\epsilon$.

These states correspond to the eigenvalues,

$$\left\{0, 0, -\sqrt{\gamma^2+\epsilon^2}, \sqrt{\gamma^2+\epsilon^2}\right\} \tag{38}$$

Using these states, the dipole moments for the transitions can be calculated from equation 22. The dipole moments are,

$$\vec{r}_{\sigma^+} = \langle 2,0,-1|\vec{r}|1,0,0\rangle = \left(\frac{128a_0}{243\sqrt{\frac{x^2}{2}+1}}, \frac{128ia_0}{243\sqrt{\frac{x^2}{2}+1}}, 0\right) \tag{39}$$

$$\vec{r}_{\sigma^-} = \langle 2,0,1|\vec{r}|1,0,0\rangle = \left(-\frac{128a_0}{243\sqrt{\frac{x^2}{2}+1}}, \frac{128ia_0}{243\sqrt{\frac{x^2}{2}+1}}, 0\right) \tag{40}$$

$$\vec{r}_{\pi_-} = \langle 2,-1,0|\vec{r}|1,0,0\rangle = \left(0, \frac{128ia_0}{243\sqrt{\frac{1}{x^2}+1}}, -\frac{128a_0}{243}\right) \tag{41}$$

$$\vec{r}_{\pi_+} = \langle 2,1,0|\vec{r}|1,0,0\rangle = \left(0, \frac{128ia_0}{243\sqrt{\frac{1}{x^2}+1}}, \frac{128a_0}{243}\right) \tag{42}$$

By considering the limit that $x \to 0$ it is clear that the first two transitions are the σ transitions and the next two are π transitions. By going back to equation 29, and expanding in a taylor series to first order in x, the normalized stokes parameters can be calculated to be,

$$\vec{S}_{\sigma^\pm} = \frac{1}{2}\begin{pmatrix} 1+\cos^2\theta \\ \sin^2\theta \\ 0 \\ \pm 2\cos\theta \end{pmatrix} \tag{43}$$

$$\vec{S}_{\pi_\pm} = \begin{pmatrix} \sin^2\theta \\ -\sin^2\theta \\ 0 \\ \pm\frac{2\gamma}{\epsilon}\cos\phi\sin\theta \end{pmatrix} \tag{44}$$

Here I used π_+ for the transition from $|2,1,0\rangle$ and π_- for the transition from $|2,-1,0\rangle$. I can recognize that since the magnetic field is in the x-direction that $\cos\phi\sin\theta = \hat{\mathbf{i}}\cdot\hat{\mathbf{B}}$ where $\hat{\mathbf{i}}$ is the observation direction.

The Balmer-Alpha lines also have a similar correction to the s_3 parameter that can be derived in a similar way to the Lyman-Alpha corrections. The form of the correction is,

$$s_3 = 2(rate)(factor)(\hat{\mathbf{i}}\cdot\hat{\mathbf{B}}) \tag{45}$$

For the Balmer-Alpha lines, the factors and rates are summarized in table 1.

Transition	Energy Shift	Rate	Factor
σ_0	0	5490	0
$\sigma_{\pm 1}$	$\pm(\frac{q_1}{2} - q_0)$	1936	$\mp\frac{\gamma}{3\epsilon}$
$\pi_{\pm 2}$	$\pm q_0$	729	$\mp\frac{17\gamma}{81\epsilon}$
$\pi_{\pm 3}$	$\pm\frac{q_1}{2}$	2304	$\mp\frac{\gamma}{2\epsilon}$
$\pi_{\pm 4}$	$\pm(q_1 - q_0)$	1681	$\mp\frac{97\gamma}{123\epsilon}$
$\sigma_{\pm 5}$	$\pm(\frac{q_1}{2} + q_0)$	16	$\pm\frac{5\gamma}{3\epsilon}$
$\sigma_{\pm 6}$	$\pm q_1$	18	$\pm\frac{4\gamma}{3\epsilon}$
$\sigma_{\pm 7}$	$\pm(q_1 + q_0)$	1	$\pm\frac{\gamma}{3\epsilon}$

Table 1: Table for next order correction to Balmer-alpha stokes parameters. Values taken from Alex Thorman's Thesis [3]

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10 FIDASIM Implementation

In FIDASIM, the Stark and Zeeman corrections to the wavelengths have been implemented using the formula below.

$$\lambda = \frac{2ch\lambda_0}{2ch - 2k_0\lambda_0 q_0 + k_1\lambda_0 q_1}$$

$$q_0 = \sqrt{\gamma^2 + \epsilon^2}$$

$$q_1 = \sqrt{4\gamma^2 + 9\epsilon^2}$$

$$\gamma = \frac{e\hbar}{2m_e}\left|\vec{B}\right|$$

$$\epsilon = 3ea_0\left|\vec{E}\right|$$

Here, k_0 go from -1 to 1 and k_1 go from -2 to 2 to specify the different transitions. An assumption was made here that the $\vec{E}$ and $\vec{B}$ fields are perpendicular. This assumption is satisfied in the Motional Stark Effect exactly but an external $\vec{E}$ might violate this assumption. However, typically the $\vec{v} \times \vec{B}$ portion of the $\vec{E}$ field is much larger and this assumption is a reasonable approximation.

The transition rates, which determine the relative intensities of the spectral lines, are unchanged from the pure stark effects using the assumption that the stark effect is dominant. The stokes parameters are dependent on the observation direction and are given, to first order in γ/ϵ by,

$$\vec{S}_\sigma = \begin{pmatrix} 1+\cos^2\theta \\ \sin^2\theta \\ 0 \\ 2(factor)(\hat{\mathbf{i}}\cdot\hat{\mathbf{B}}) \end{pmatrix}$$

$$\vec{S}_\pi = \begin{pmatrix} \sin^2\theta \\ -\sin^2\theta \\ 0 \\ 2(factor)(\hat{\mathbf{i}}\cdot\hat{\mathbf{B}}) \end{pmatrix}$$

Where $\hat{\mathbf{i}}$ is the observation direction, θ is the angle between the $\vec{E}$ field and $\vec{i}$ and $\hat{\mathbf{B}}$ is the unit vector for the magnetic field direction. The (factor) term for the first order correction can be taken from table 1. These $\vec{S}$ are multiplied by the intensity in FIDASIM so that the first stoke parameter is equal to the observed intensity.

As an example, below are plots created in FIDASIM of the beam emission from a 81.1 keV beam (30LT) for DIII-D discharge #179571 at 920 ms, a 2.0 T shot. The input geometry is shown in figure 3. The output stokes parameters

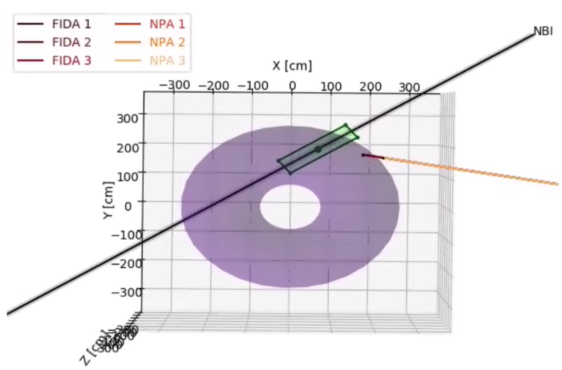

Figure 3: The black line labeled 'NBI' is the 81.1 keV beam. The red/orange lines are the observation line of sight.

calculated by FIDASIM are shown in figure 4 with stark lines split and with stark lines summed in figure 5.

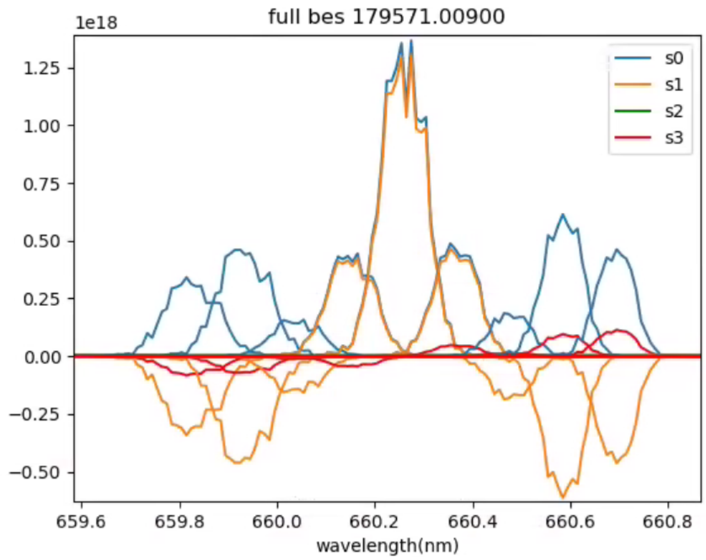

Figure 4: 81.1 keV full component beam emission stokes parameters split into stark lines.

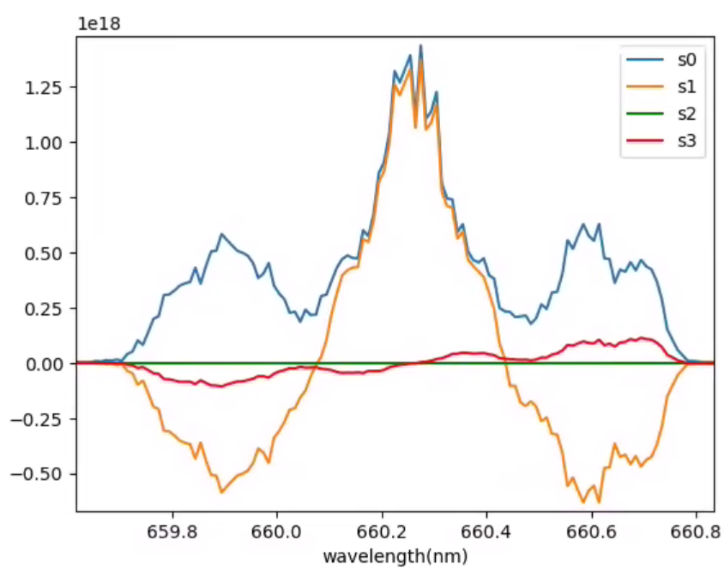

Figure 5: 81.1 keV full component beam emission stokes parameters.

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