№1

$\vec{\xi} \sim R(0,\theta) \quad \theta > 0$ вер. модель

$\vec{x}_n$ выборка

1)

$\tilde{\theta}_1 = 2\bar{x} = 2\frac{1}{n}\sum_{i=1}^{n} x_i$

$M\tilde{\theta}_1 = M\left[\frac{2}{n}\sum x_i\right] = \frac{2}{n}\sum M x_i = \frac{2}{n} n \frac{\theta}{2} = \theta$

$M\xi = \frac{\theta}{2}$

несмещ.

$D\tilde{\theta}_1 = D\frac{2}{n}\sum x_i = \frac{4}{n^2}\sum D x_i = \frac{4}{n^2} n D\xi = \frac{4}{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$

состоят.

2) $\tilde{\theta}_2 = x_{min}$ $\qquad \min(\xi_1 \ldots \xi_n) \sim \underbrace{1-(1-F(y))^n}_{F}$

$M\tilde{\theta}_2 = M[x_{min}]$ $\qquad \rho(y) = n(1-F(y))^{n-1} p(y) =$

$= n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}(0,\theta)$

$M[x_{min}] = \int_0^{\theta} y\, n \frac{1}{\theta}\left(1-\frac{y}{\theta}\right)^{n-1} dy = \langle\langle t = \frac{y}{\theta} \rangle\rangle =$

$= \int_0^1 t\,\theta n (1-t)^{n-1} dt = n\theta B(2,n) =$

$= n\theta \frac{\Gamma(2)\Gamma(n)}{\Gamma(n+2)} = n\theta\frac{\Gamma(2)\Gamma(n)}{(n+1)n\Gamma(n)} = \frac{\theta}{n+1}$

$\boxed{B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx}$

смещен.

не явл. ас. несмещ.

Исправим: $\widetilde{\theta}_2' = (n+1)\, x_{min}$ - несмещ.

$$M[x_{min}^2] = \int_0^{\theta} y^2 n \left(1-\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}\, dy =$$

$$= \int_0^1 \theta^2 t^2 n (1-t)^{n-1}\, dt = \theta^2 n\, B(3,n) =$$

$$= \frac{2\theta^2}{(n+2)(n+1)}$$

$$D[x_{min}] = \frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2} = \frac{\theta^2[(n+1)^{\cancel{2}}2 - n - 2]}{(n+1)^2(n+2)} =$$

$$= \frac{\theta^2 n}{(n+1)^2(n+2)}$$

$$D[\theta_2'] = D[(n+1)\, x_{min}] = \frac{\theta^2 n}{n+2} \underset{n\to\infty}{\not\to} 0$$

проверка: опр.: $\widetilde{\theta}_2' \xrightarrow{P} \theta$

$\theta_2' = (n+1)\, x_{min}$, $\forall \varepsilon > 0 \quad P(|\theta_2' - \theta| \geq \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$

$$P(|\theta_2' - \theta| \geq \varepsilon) \geq P(\theta_2' \geq \theta + \varepsilon) =$$

$$= P\left(x_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = P(x_1 \geq \ldots, x_2 \geq \ldots, x_n \geq \ldots) =$$

$$= \prod_{i=1}^{n} P\left(x_i \geq \frac{\theta+\varepsilon}{n+1}\right) = \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n =$$

$$= \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \underset{n\to\infty}{\longrightarrow} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$$

несост.

3) $\tilde{\theta}_3 = x_{max}$; $x_i = F(y)$, $x_{max} = (F(y))^n$

$\cancel{p_{x_{max}}}$ $p_{x_{max}}(y) = n(F(y))^{n-1} p(y)(0, \theta)$

$$M[\tilde{\theta}_3] = \int_0^\theta y n \left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} dy = \frac{n}{\theta^n}\int_0^\theta y^n dy = \frac{n\theta}{n+1}$$

смещ., асс. несмещ.

$\theta_3' = \frac{n+1}{n} x_{max}$ – несмещ.

$$M[\tilde{\theta}_3^2] = \int_0^\theta y^2 n \left(\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta} dy = \frac{n}{\theta^n}\int_0^\theta y^{n+1} dy =$$

$$= \frac{\theta^2 n}{n+2}$$

$$D[\tilde{\theta}_3] = \frac{\theta^2 n}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = \theta^2 \frac{n}{(n+2)(n+1)^2}$$

$$D[\theta_3'] = \frac{(n+1)^2}{n^2} \frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0$$

состоятельна

4) $\tilde{\theta}_4 = x_{min} + x_{max}$

$$M[\tilde{\theta}_4] = M[x_{min}] + M[x_{min}] = \frac{\theta}{n+1} + \frac{n\theta}{n+1} = \theta$$

несмещ.

$$D[\tilde{\theta}_4] = D[x_{min}] + D[x_{max}] + 2\operatorname{cov}(x_{max}, x_{min})$$

$$\operatorname{cov}(x_{max}, x_{min}) = M[x_{min} x_{max}] - M[x_{min}] \cdot M[x_{min}]$$

$$\rho(y, z) = n(n-1)\left(F(z) - F(y)\right)^{n-2} \rho(z)\rho(y)$$

$$M[x_{min}\, x_{max}] = \iint y z \cdot n(n-1)\left(\frac{z}{\theta} - \frac{y}{\theta}\right)^{n-2}$$

$$\frac{1}{\theta}\frac{1}{\theta}\, dz\, dy = \int_0^{\theta} dz \int_0^{z} y z (n-1) n \left(\frac{z}{\theta} - \frac{y}{\theta}\right)^{n-2}$$

$$\frac{1}{\theta^2} dy = \ll t = \frac{y}{z} \gg = \frac{n(n-1)}{\theta^n} \int_0^{\theta} dz \int_0^{1} t z^2 \cdot$$

$$(z - tz)^{n-2} dt = \frac{n(n-1)}{\theta^n} \int_0^{\theta} z^{n+1} dz \int_0^{1} t(1-t)^{n-2} dt =$$

$$= \frac{n(n-1)}{\theta^n} \int_0^{\theta} z^{n+1} \frac{1}{n(n-1)} dz = \frac{\theta^{n+2}}{(n+2)\theta^n} = \frac{\theta^2}{n+2}$$

$$cov(x_{min}, x_{max}) = \frac{\theta^2}{n+2} - \frac{\theta}{n+1} \cdot \frac{n\theta}{n+1} = \frac{\theta^2}{(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_4] = \frac{\theta^2 n}{(n+1)^2(n+2)} + \frac{\theta^2 n}{(n+1)^2(n+2)} + \frac{2\theta^2}{(n+1)^2(n+2)}$$

$$= \frac{2\theta^2}{(n+2)(n+1)} \xrightarrow[n \to \infty]{} 0 \quad \text{состоятельная}$$

5) $\cancel{M[\tilde{\theta}_5] = M x}$ $\quad \tilde{\theta}_5 = x_1 + \dfrac{\sum\limits_{k=2}^{n} x_k}{(n-1)}$

$$M[\tilde{\theta}_5] = M[x_1] + \frac{n}{n-1} + \frac{n}{n-1} M[x_2] =$$

$$= \frac{\theta}{2}\left(1 + \frac{n-1}{n-1}\right) = \theta \qquad x_i \sim R(0;\theta)$$

$$D[\tilde{\theta}_5] = D[x_1] + \frac{n-1}{(n-1)^2} D[x_2] = \frac{\theta^2}{12}\left(1 + \frac{1}{n-1}\right) \underset{n\to\infty}{\not\to} 0$$

$$x_1 + \frac{1}{n-1}\sum_{m=2}^{n} x_m \xrightarrow{P} x_1 + \frac{\theta}{2}$$ – не явл. состоят.

Эффективность: $\theta_2, \theta_3, \theta_5$ – вычисл. не исследуем $\tilde{\theta}_1, \tilde{\theta}_3', \tilde{\theta}_4$: $D\tilde{\theta}_1 = \frac{\theta^2}{3n}$, $D\tilde{\theta}_3' = \frac{\theta^2}{n(n+2)}$

$$D\tilde{\theta}_4 = \frac{2\theta^2}{(n+1)(n+2)}$$

$$\frac{\theta_1^2}{3n} > \frac{\theta_3^2}{n(n+2)}$$ для $n > 1$

$$\frac{\theta_3'^2}{n(n+2)} < \frac{2\theta_4^2}{(n+1)(n+2)}$$ для $n > 1$ $\quad \tilde{\theta}_3'$ лучшая

№ 2

а) $p = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \gamma = \frac{\tilde{\mu}_3}{\tilde{\mu}_2^{3/2}}$

$$\alpha_1 = \int_0^\infty x e^{-x} dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1$$

$$\mu_2 = \int_0^\infty (x-1)^2 e^{-x} dx = \int_0^\infty x^2 e^{-x} dx - 2\int_0^\infty x e^{-x} dx + \int_0^\infty e^{-x} dx = 2 - 2 + 1 = 1$$

$$\mu_3 = \int_0^{\infty} (x-1)^3 e^{-x} dx = \int_0^{\infty} x^3 e^{-x} dx - 3\int_0^{\infty} x^2 e^{-x} dx$$

$$+ 3\int_0^{\infty} x e^{-x} dx - \int_0^{\infty} e^{-x} dx = 6 - 3\cdot 2 + 3\cdot 1 - 1 = 2$$

$$\gamma = \frac{\mu_3}{(\mu_2)^{3/2}} = 2$$

c) Ядерные оценки плотности

$$h = 2{,}344 \frac{S}{\sqrt[5]{n}}, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\mu_2 = \frac{1}{n}\sum (x_i - \bar{x})^n; \quad \mu_2 n = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$S^2 = \frac{\mu_2 n}{n-1} \qquad S = \sqrt{\frac{\mu_2 n}{n-1}}$$

$$\tilde{p}(z) = \frac{1}{nh}\sum_{i=1}^{n} q\left(\frac{z-x_i}{h}\right) = \frac{1}{nh}\sum_{i=1}^{n}\left(\frac{3}{4}\left(1-\left(\frac{z-x_i}{h}\right)^2\right)\right)$$

$$\tilde{p}(z) = \frac{3}{4nh}\sum_{i=1}^{n}\left(1-\left(\frac{z-x_i}{h}\right)^2\right)$$

d) ЦПТ: $\sqrt{n}\dfrac{\tilde{\alpha}_k - \alpha_k}{\sqrt{\tilde{\alpha}_{2k} - \tilde{\alpha}_k^2}} \sim N(0,1)$

$$\tilde{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \tilde{\alpha}_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$$\alpha_1 = \int_0^{+\infty} x e^{-x} dx \qquad (\tilde{\alpha}_1 - \alpha_1) \sim \frac{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{\sqrt{n}} N(0,1)$$

$$\tilde{\alpha}_1 - \alpha_1 \rightarrow N\left(0, \frac{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{n}\right); \quad \alpha_1 \sim N\left(\tilde{\alpha}_1, \frac{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{n}\right)$$

№3

$$p(x) = \begin{cases} \frac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \theta > 0$$

$n = 3$ $\quad \bar{x}_n = (x_1, \dots, x_n)$ $\quad F(x) = 1 - e^{-\frac{x}{\theta}}$

$\bar{x}_3 = (x_1, x_2, x_3)$

~~$\tilde{\theta}_1 = \bar{x}$~~

$$\tilde{\theta}_1 = \bar{x} \qquad \bar{\theta}_2 = \frac{x_{min} + x_{max}}{2} \qquad \hat{\theta}_3 = x_{(2)}$$

$$\xi \sim p(x) \qquad M[\xi] = \int_0^{\infty} x \cdot \frac{e^{-\frac{x}{\theta}}}{\theta} dx = \frac{1}{\theta} \int_0^{\infty} x e^{-\frac{1}{\theta}} dx = \frac{1}{\theta} \cdot \theta^2 = \theta$$

$$M[\xi^2] = \frac{1}{\theta} \int_0^{\infty} x^2 e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \cdot 2 \cdot \theta^3 = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M[\xi]^2 = \theta^2$$

a) $M[\tilde{\theta}_1] = M[\frac{1}{n} \sum x_i] = \frac{1}{n} M[\sum x_i] = \frac{1}{n} \cdot n M[\xi] = \theta$

несмещ

$M[\tilde{\theta}_3]$: $\tilde{\theta}_3 = x_{(2)}$

$$\tilde{x}_k \sim n\, C_{n-1}^{k-1} \frac{e^{-\frac{x}{\theta}}}{\theta}\left(1-e^{-\frac{x}{\theta}}\right)^{n-k}$$

$k=2$ $\quad \tilde{x}_{(2)} \sim n\, C_{n-1}^{1} \frac{e^{-\frac{x}{\theta}}}{\theta}\left(1-e^{-\frac{x}{\theta}}\right)^{1}\left(e^{-\frac{x}{\theta}}\right)^{n-2} =$

$$= \frac{n(n-1)}{\theta}\left(1-e^{-\frac{x}{\theta}}\right)\left(e^{-\frac{x}{\theta}}\right)^{n-1}$$

$$M[\tilde{\theta}_3] = \frac{n(n-1)}{\theta}\int_0^{\infty} x\left(1-e^{-\frac{x}{\theta}}\right)e^{-\frac{x}{\theta}(n-1)}\,dx =$$

$$= \frac{n(n-1)}{\theta}\left(\int_0^{\infty} x e^{-\frac{x}{\theta}(n-1)}\,dx + \int_0^{\infty} x e^{-\frac{x}{\theta}n}\,dx\right) =$$

$$= \frac{n(n-1)}{\theta}\left(\frac{\theta^2}{(n+1)^2} - \frac{\theta^2}{n^2}\right) = \frac{n\theta}{n-1} - \frac{(n-1)\theta}{n} = \frac{2n-1}{n(n-1)}\theta$$

смещ.

$\tilde{\theta}_3' = \frac{n(n-1)}{2n-1}x_{(2)}$ - несмещ