№4

$\xi \sim R(\theta, 2\theta) \quad p(x, \theta) = \frac{1}{\theta} \; (\theta, 2\theta)$

а) $\vec{X}_n$

$$M\xi = \int_{\theta}^{2\theta} x \frac{1}{\theta} dx = \frac{x^2}{2\theta}\Big|_{\theta}^{2\theta} = \frac{4\theta^2 - \theta^2}{2\theta} = \frac{3}{2}\theta$$

$$M\xi^2 = \int_{\theta}^{2\theta} x^2 \frac{1}{\theta} dx = \frac{x^3}{3\theta}\Big|_{\theta}^{2\theta} = \frac{8\theta^3 - \theta^3}{3\theta} = \frac{7\theta^2}{3}$$

$$D\xi = M\xi^2 - (M\xi)^2 = \frac{7\theta^2}{3} - \frac{9}{4}\theta^2 = \frac{\theta^2}{12}$$

ОММ

$$\alpha_1 = \int_{\theta}^{2\theta} x \cdot \frac{1}{\theta} dx = \frac{3}{2}\theta = M\xi = \bar{x} \Rightarrow \tilde{\theta}_1 = \frac{2}{3}\bar{x}$$

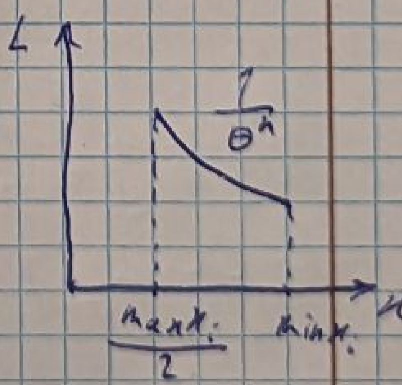
ОМП

$$L(\theta) = \begin{cases} \frac{1}{\theta^n}, & x_i \in [\theta, 2\theta] \\ 0, & x_i \notin [\theta, 2\theta] \end{cases} = \begin{cases} \frac{1}{\theta^n}, & \theta \le x_{min} \le x_{max} \le 2\theta \\ 0 \end{cases} =$$

$L = \frac{1}{\theta^n} \quad (\min x_i \geq \theta, \max x_i \leq 2\theta)$

$\sup L$ при $\theta = \frac{\max x_i}{2}$

$\hat{\theta}_2 = \frac{\max x_i}{2}$



б) 1) $\tilde{\theta}_1 = \frac{2\tilde{\alpha}_1}{3}$ - ОММ

$$M[\tilde{\theta}_1] = M\left[\frac{2\tilde{\alpha}_1}{3}\right] = \frac{2}{3} M\left[\frac{1}{n}\sum x_i\right] =$$

$$= \frac{2}{3} M[\xi] = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta$$ - несмещ

$$D[\tilde{\theta}_1] = \frac{4}{9} D[\tilde{\alpha}_1] = \frac{4}{9} D\left[\frac{1}{n}\sum x_i\right] = \frac{4}{9n} D\xi =$$

$$= \frac{4}{9n}\cdot\frac{1}{12}\theta^2 = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0$$ состоятельна

2) $\tilde{\theta}_2 = \frac{x_{max}}{2}$ - ОМП

$$M[\tilde{\theta}_2] = \frac{1}{2} M[x_{max}] = \frac{1}{2}\int_{\theta}^{2\theta} x\, n (x-\theta)^{n-1}\cdot\frac{1}{\theta^n}\, dx =$$

$$= \frac{1}{2}\frac{n}{\theta^n}\int_{\theta}^{2\theta} x(x-\theta)^{n-1} dx = \frac{1}{2} n\theta \int_{1}^{2} x(x-1)^{n-1} dx =$$

$$= \frac{1}{2} n\theta \int_0^1 y^{n-1}(y+1)\, dy = \frac{2n+1}{2n+2}\theta$$ смещена

$$\tilde{\theta}_2' = \frac{2n+2}{2n+1}\hat{\theta}_2 = \frac{n+1}{2n+1} x_{max}$$ - несмещ

$$D[\tilde{\theta}_2] = \frac{1}{4} D[x_{max}] = \frac{n\theta^2}{4(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_2'] = \left(\frac{2n+2}{2n+1}\right)^2 \frac{n\theta^2}{4(n+2)(n+1)^2} = \frac{n\theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$ - сост.

3) $\tilde{\theta}_3 = \frac{1}{5}(x_{min} + 2x_{max})$

$M[\tilde{\theta}_3] = \frac{1}{5}M[x_{min}] + \frac{2}{5}M[x_{max}] =$

$= \frac{1}{5}\left(n\theta\left(\frac{2}{n} - \frac{1}{n+1}\right)\right) + \frac{2}{5}\left(n\theta\left(\frac{1}{n+1} + \frac{1}{n}\right)\right) =$

$= \theta\frac{5n+4}{5n+5}$ смещ. $\to \tilde{\theta}_3' = \frac{5n+5}{5n+4}\tilde{\theta}_3 = \frac{n+1}{5n+4}(x_{min} + 2x_{max})$

$D[\tilde{\theta}_3] = D\left[\frac{x_{min}}{5}\right] + D\left[\frac{2x_{max}}{5}\right] + 2\,cov\, x_{(1)} x_{(n)}$

$D[x_{min}] = D[x_{max}] = \frac{n\theta^2}{(n+2)(n+1)^2}$

$cov(x_{(1)}, x_{(n)}) = M[x_{(1)} x_{(n)}] - Mx_{(1)} Mx_{(n)}$

$M[x_{(1)} x_{(n)}] = \iint uv f_{1n}(u,v)\,du\,dv$

$f_{1n} = \begin{cases} n(n-1) f(x) f(y) (F(y) - F(x))^{n-2}, & x, y \in [\theta, 2\theta] \\ 0, & \end{cases} =$

$= \begin{cases} n(n-1)\frac{1}{\theta^n}(y-x)^{n-2}, & x, y \in [\theta, 2\theta] \\ 0 & \end{cases}$

$M[x_{(1)} x_{(n)}] = n(n-1)\frac{1}{\theta^n}\int\int_{\theta}^{2\theta} uv(v-u)^{n-2}\,du\,dv =$

$= \frac{n(n-1)}{\theta^n}\int_{\theta}^{2\theta} u\left[\int_u^{2\theta} v(v-u)^{n-2}\,dv\right]du = \frac{n(n-1)}{\theta^n}\int_{\theta}^{2\theta} u\left[\frac{(2\theta-u)^n}{n} + \frac{u(2\theta-u)^{n-1}}{n-1}\right]du = \frac{n(n-1)}{\theta^n}\left[\frac{\theta^{n+1}}{n+1} + \ldots\right] = \theta^2\left(2 + \frac{1}{n+2}\right)$

$$\operatorname{cov}(x_{(1)}, x_{(n)}) = \theta^2\left(2 + \frac{1}{n+2}\right) - \left(n\theta\left(\frac{1}{n+1} + \frac{1}{n}\right)\right) \cdot$$

$$\cdot\left(n\theta\left(\frac{2}{n} - \frac{1}{n+1}\right)\right) = \theta^2\left(\frac{1}{n+2} - \frac{n}{n+1} + \frac{n^2}{(n+1)^2}\right) = \frac{\theta^2(2n-1)}{(n+2)(n+1)^2}$$

$$D[\theta_3] = \frac{1}{25}\frac{n\theta^2}{(n+2)(n+1)^2} + \frac{4}{25}\frac{n\theta^2}{(n+2)(n+1)^2} +$$

$$+ 2\cdot\frac{2}{25}\frac{(2n-1)\theta^2}{(n+2)(n+1)^2} = \frac{(13n-4)\theta^2}{25(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_3'] = \left(\frac{5n+5}{5n+4}\right)^2 D[\tilde{\theta}_3] = \frac{(13n-4)\theta^2}{(n+2)(5n+4)^2} \xrightarrow[n\to\infty]{} 0$$

состоят.

c) $\tilde{\theta}_1 = \frac{2\bar{x}}{3}$: $D[\tilde{\theta}_1] = \frac{1}{27}\frac{\theta^2}{n}$

$\tilde{\theta}_2' = \frac{n+1}{2n+1}x_{max}$: $D[\tilde{\theta}_2'] = \frac{n\theta^2}{(n+2)(2n+1)^2}$

$\tilde{\theta}_3' = \frac{n+1}{5n+4}(x_{min} + 2x_{max})$: $D[\tilde{\theta}_3'] = \frac{(13n-4)\theta^2}{(n+2)(5n+4)^2}$

при $n \geq 4$ – эффективна $\tilde{\theta}_2$

d) $x_i \sim R(\theta, 2\theta)$

$y_i = \frac{x_i}{\theta} - 1 \sim R(0,1)$ $\quad y_n = \frac{x_n}{\theta} - 1$

$F_y(y) = y^n\ [0,1]$ $\quad p(y) = ny^{n-1}\ [0,1]$

$$P\left(t_1 < \frac{x_n}{\theta} - 1 < t_2\right) = P\left(q_{\frac{1-\beta}{2}} < \frac{x_n}{\theta} - 1 < q_{\frac{1+\beta}{2}}\right) \Rightarrow$$

$$P\left(\frac{x_n}{t_2+1} < \theta < \frac{x_n}{t_1+1}\right) \geq \beta$$

$t_1: \ P(0 < y_n < t_1) = F_y(t_1) - F_y(0) = t_1^n - 0 = \frac{1-\beta}{2}$

$$t_1 = \sqrt[n]{\frac{1-\beta}{2}}$$

$t_2: \ t_2 = \sqrt[n]{\frac{1+\beta}{2}}$

$$P_\theta\left(\frac{x_n}{\sqrt[n]{\frac{1+\beta}{2}}+1} < \theta < \frac{x_n}{\sqrt[n]{\frac{1-\beta}{2}}+1}\right) \geq \beta$$

c) OMM

$\tilde{\theta} = \frac{2}{3}\bar{x}$

$$\frac{f(\tilde{\alpha}) - f(\alpha)}{\sigma(\alpha)}\sqrt{n} \sim N(0,1)$$

$f(\alpha) = \frac{2}{3}\alpha_1 = \theta \qquad f'(\alpha) = \frac{2}{3} \qquad k_{11} = \alpha_2 - \alpha_1^2$

$\sigma(\alpha) = \sqrt{\frac{2}{3}(\alpha_2 - \alpha_1^2)\frac{2}{3}} = \sqrt{\frac{4}{9}(\alpha_2 - \alpha_1^2)}$

$$\frac{\tilde{\theta} - \theta}{\sqrt{\frac{4}{9}(\tilde{\alpha}_2 - \tilde{\alpha}_1^2)}}\sqrt{n} \sim N(0,1)$$

$$-1{,}96 < \frac{\tilde{\theta} - \theta}{\sqrt{\frac{4}{9}(\tilde{\alpha}_2 - \tilde{\alpha}_1^2)}} < 1{,}96$$

$$-1{,}96 \cdot \frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2} + \tilde{\theta} < \theta < 1{,}96 \cdot \frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2} + \tilde{\theta}$$

№5

$$p(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases} \quad \theta > 1$$

а) $L(\theta) = \prod p(x_i, \theta) = \prod_{i=1}^{n} \left(\frac{\theta-1}{x_i^\theta} \right) = (\theta-1)^n \prod (x_i)^{-\theta}$

если все $x_i \geq 1$ $\Leftrightarrow$ $x_{min} \geq 1$

$$\ln L(\theta) = n \ln(\theta-1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta-1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \tilde{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\left.\frac{\partial^2 \ln L}{\partial \theta^2}\right|_{\theta=\tilde{\theta}} = \left.-\frac{n}{(\theta-1)^2}\right|_{\theta=\tilde{\theta}} = -\left(\sum_{i=1}^{n} \ln x_i\right)^2 \cdot \frac{1}{n} = 0 \Rightarrow \max$$

б) $\frac{f(\tilde{\theta}) - f(\theta)}{\sigma} \sqrt{n} \sim N(0,1)$ $\quad \sigma = \sqrt{\nabla^T f(\theta) I^{-1} \nabla f}$

$f(\theta) = x_{med}$: $F(x_{med}) = \int_{-\infty}^{x_{med}} \frac{\theta-1}{t^\theta} dt = x_{med}^{1-\theta} = \frac{1}{2}$ $\Rightarrow$ $x_{med} = 2^{\frac{1}{\theta-1}}$

$$f'(\theta) = 2^{\frac{1}{\theta-1}} \cdot \ln 2 \cdot \frac{-1}{(1-\theta)^2} = -\ln 2 \cdot 2^{\frac{1}{\theta-1}} \cdot \frac{1}{(\theta-1)^2}$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right]$$

$p = \frac{\theta-1}{x^\theta}$ $\quad$ | $\ln p = \ln(\theta-1) - \theta \ln x$ | $\frac{\partial \ln p}{\partial \theta} = \frac{1}{\theta-1} - \ln x$

$$I(\theta) = \int_1^{+\infty} \frac{\theta-1}{x^\theta} \left(\frac{1}{\theta-1} - \ln x\right)^2 dx = \frac{1}{(\theta-1)^2}$$

$$\frac{2^{\frac{1}{\tilde\theta-1}}-2^{\frac{1}{\theta-1}}}{\ln 2\cdot 2^{\frac{1}{\tilde\theta-1}}\cdot\frac{1}{(\tilde\theta-1)^2}\cdot(\tilde\theta-1)}\sqrt{n}\rightsquigarrow N(0,1)$$

$$u_{\frac{1-\beta}{2}} < \frac{(\tilde\theta-1)\sqrt{n}}{\ln 2}\left(1-(x_{med})\,2^{-\frac{1}{\theta-1}}\right) < u_{\frac{1+\beta}{2}}$$

$$2^{\frac{1}{\theta-1}}$$

$$\frac{\ln 2}{(\tilde\theta-1)\sqrt{n}}u_{\frac{1-\beta}{2}} < 1-x_{med}\,2^{-\frac{1}{\theta-1}} < \frac{\ln}{(\tilde\theta-1)\sqrt{n}}u_{\frac{1+\beta}{2}}$$

$$2^{\frac{1}{\tilde\theta-1}}\left(1-\frac{\ln 2}{(\tilde\theta-1)\sqrt{n}}u_{\frac{1+\beta}{2}}\right) < x_{med} < 2^{\frac{1}{\theta-1}}\left(1-\frac{\ln 2}{(\tilde\theta-1)\sqrt{n}}u_{\frac{1-\beta}{2}}\right)$$

c) $p(y)=\begin{cases} e^{1-y}, & y\geq 1\\ 0, & y<1\end{cases}$

$\theta\sim\begin{cases} e^{1-\theta}, & \theta\geq 1\\ 0, & \theta<1\end{cases}$

$P(\theta|\vec{x}_n)=CLP(\theta)$

$\ln p(\theta|\vec{x}_n)=\ln C+\ln L+\ln p(\theta)\to\max$

$L=\frac{(\theta-1)^n}{\prod_{i=1}^{n} x_i^{\theta}},\ x_i\geq 1$

$\ln p(\theta|\vec{x}_n)=\ln C+n\ln(\theta-1)-\theta\sum\ln x_i+1-\theta\to\max$

$$\frac{\partial \ln p(\theta|\vec{x}_n)}{\partial\theta} = \frac{n}{\theta-1} - 1 - \sum \ln x_i = 0 \Rightarrow \frac{n}{\theta-1} = 1 + \sum \ln x_i$$

$$\Downarrow$$

$$\tilde{\theta} = 1 + \frac{n}{1+\sum \ln x_i}$$

Доверительный инт:

$$p(\theta|\vec{x}_n) = C \cdot e^{1-\theta} \frac{(\theta-1)^n}{(\prod x_i)^\theta}$$

$$\int_1^{+\infty} e^{1-\theta} \frac{(\theta-1)^n \cdot C}{(\prod x_i)^\theta} d\theta = 1$$

$$\int_1^{q_1} p(\theta|\vec{x}_n) d\theta = 0{,}025 \Rightarrow q_1 = 5{,}75$$

$$\int_{q_2}^{+\infty} p(\theta|\vec{x}_n) d\theta = 0{,}025 \Rightarrow q_2 = 8{,}05$$

d) $\frac{(\tilde{\theta}-\theta)}{\sqrt{I^{-1}(\theta)}} \sqrt{n} \sim N(0,1)$

$$\frac{\tilde{\theta}-\theta}{\tilde{\theta}-1} \sqrt{n} \sim N(0,1)$$

$$U_{\frac{1-\beta}{2}} < (\tilde{\theta}-\theta)\frac{\sqrt{n}}{\tilde{\theta}-1} < U_{\frac{1+\beta}{2}}$$

$$\tilde{\theta} - \frac{\tilde{\theta}-1}{\sqrt{n}} U_{\frac{1+\beta}{2}} < \theta < \tilde{\theta} - \frac{\tilde{\theta}-1}{\sqrt{n}} U_{\frac{1-\beta}{2}}, \quad \tilde{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$