FAST FOURIER TRANSFORM

COE150

FAST FOURIER TRANSFORM (FFT)

A Fast Fourier Transform (FFT) is an algorithm that calculates the discrete Fourier transform (DFT) of some sequence.

The aim of FFT is to have an efficient algorithm for evaluating the DFT and to reduce the number of mathematical operations in solving the DFT.

DISCRETE FOURIER TRANSFORM

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

Computational complexity:

Complex multiplications: $\boldsymbol{N(N) = N^2}$

Complex additions: $\boldsymbol{N(N - 1) = N^2 - N}$

DECIMATION-IN-TIME ($\boldsymbol{N}$ IS EVEN)

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$X[k] = \sum_{n=even}^{N-1} x[n] W_N^{nk} + \sum_{n=odd}^{N-1} x[n] W_N^{nk}$$

DECIMATION-IN-TIME ($\boldsymbol{N}$ IS EVEN)

Let $n = 2r$,

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_N^{(2r+1)k}$$

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\,(W_N^2)^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1](W_N^2)^{rk}$$

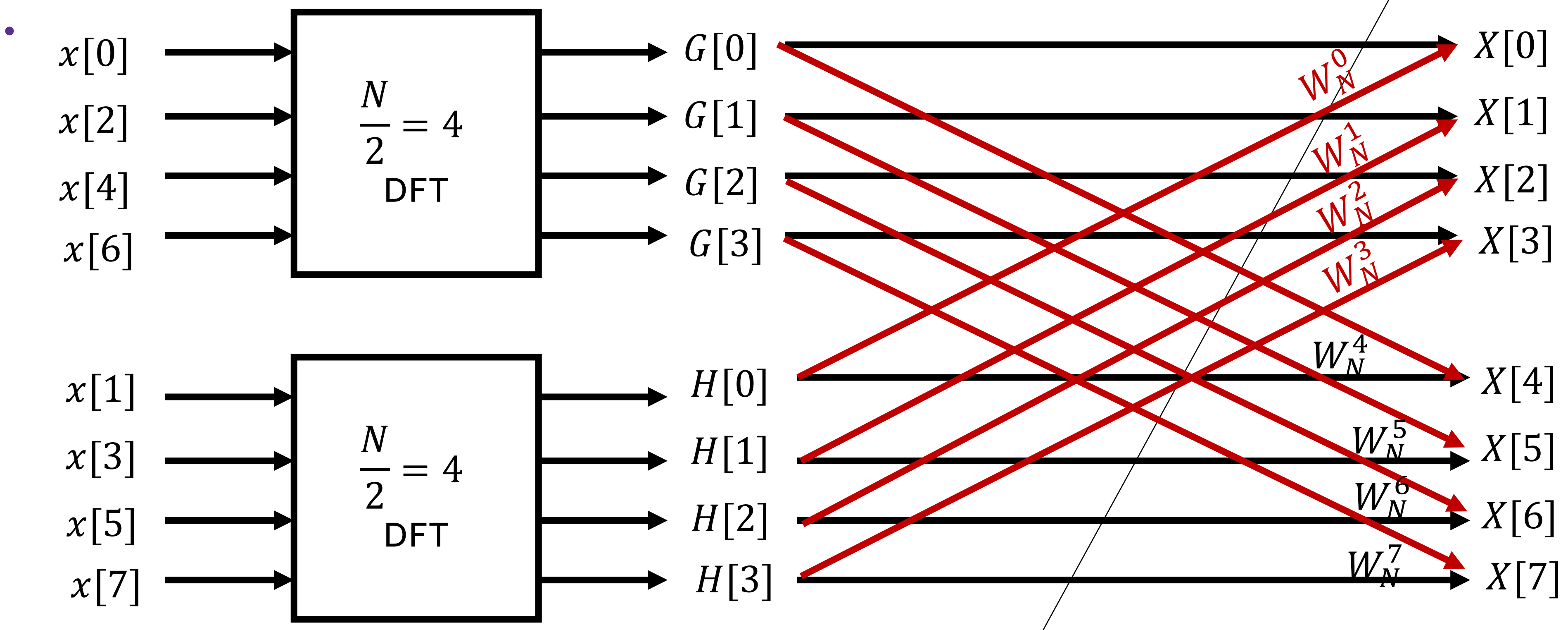
DECIMATION-IN-TIME ($\boldsymbol{N}$ IS EVEN)

$$X[k] = \sum_{r=0}^{\frac{N}{2}-1} x[2r]W_{N/2}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]W_{N/2}^{rk}$$

$$X[k] = G[k] + W_N^k H[k]$$

DECIMATION-IN-TIME ($\boldsymbol{N}$ IS EVEN)

N=8,



RADIX-2 FFT

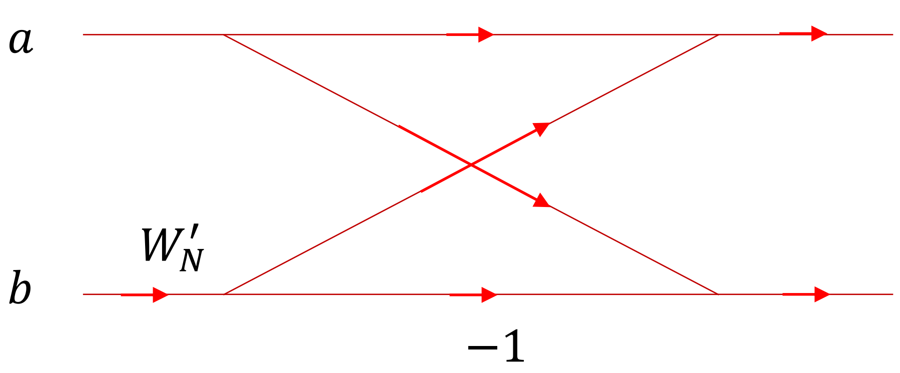
The radix-2 FFT algorithms are used for data vectors of lengths $N = 2^K$.

Two Types of Radix-2 FFT:

- Decimation in Time (DIT)
- Decimation in Frequency (DIF)

DECIMATION IN TIME (DIT) RADIX 2 FFT

Basic Butterfly Computation

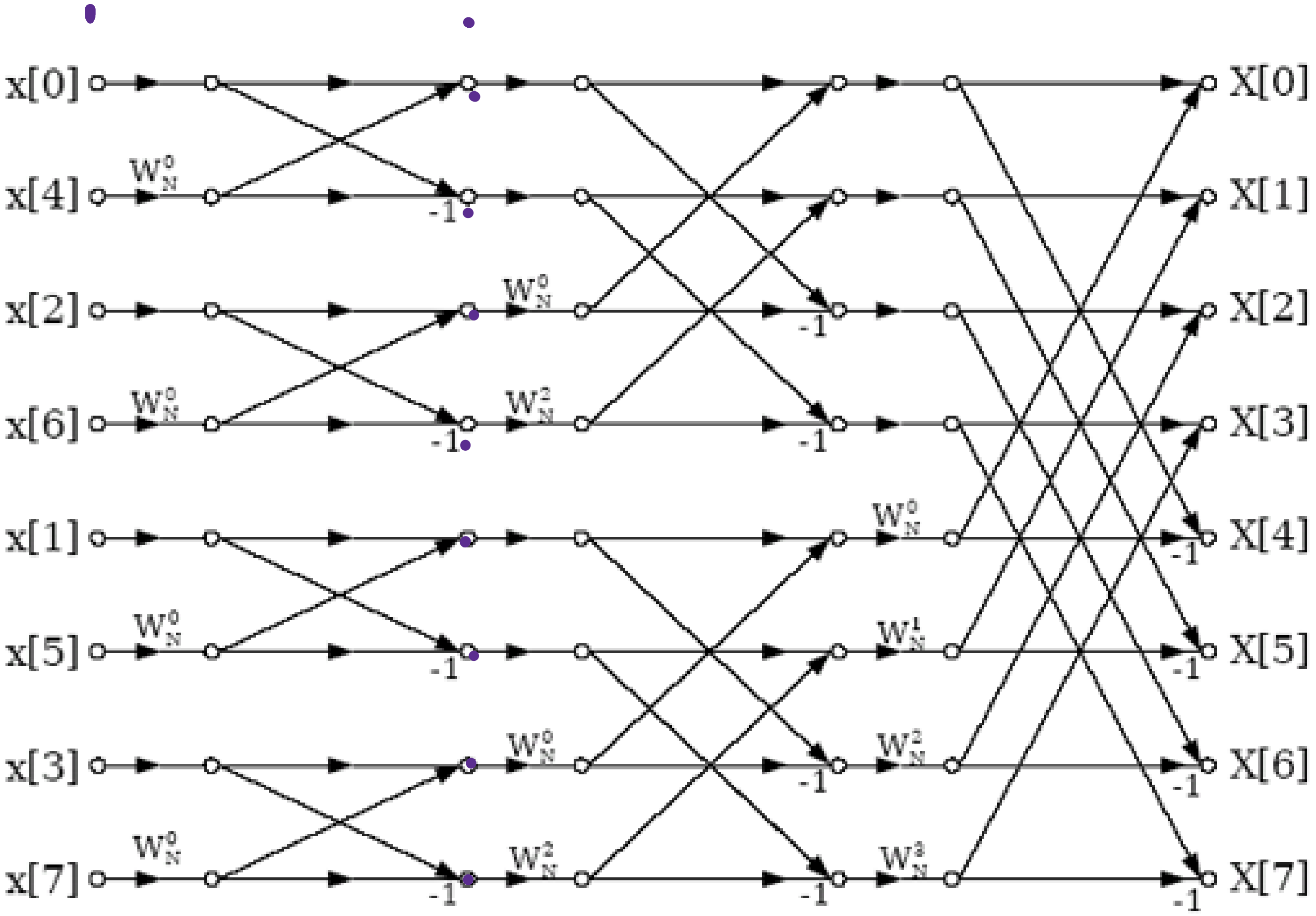


$$A = a + W_N^{\prime} b$$

$$B = a - W_N^{\prime} b$$

DECIMATION IN TIME (DIT) RADIX 2 FFT

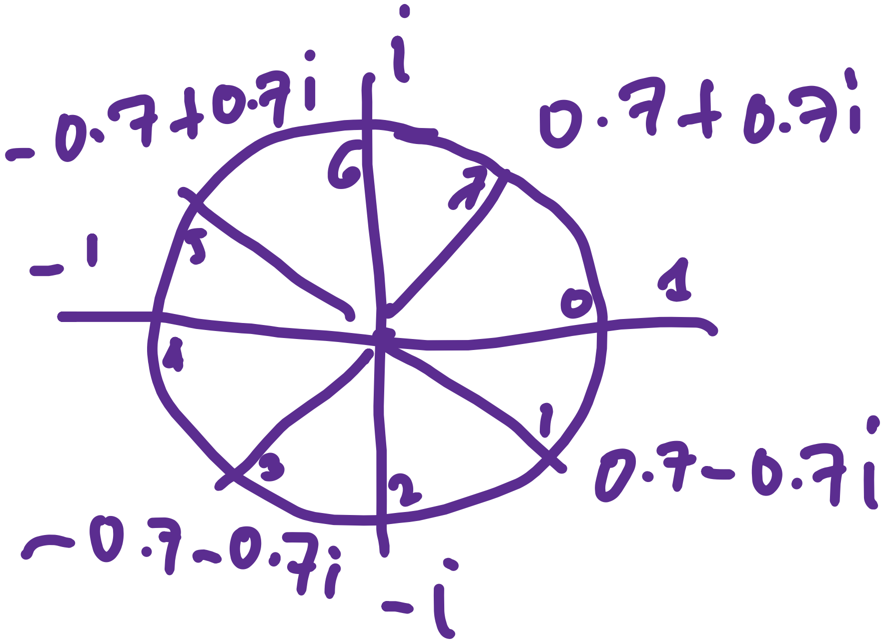
For N=8,



DECIMATION IN TIME (DIT) RADIX 2 FFT

Example:

$x(n) = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0,0,0,0\}$. Find $X(k)$.



DECIMATION IN TIME (DIT) RADIX 2 FFT

First stage:

[1] $\frac{1}{2} + (1)0 = \frac{1}{2}$

[2] $\frac{1}{2} - (1)0 = \frac{1}{2}$

[3] $\frac{1}{2} + (1)0 = \frac{1}{2}$

[4] $\frac{1}{2} - (1)0 = \frac{1}{2}$

[5] $\frac{1}{2} + (1)0 = \frac{1}{2}$

[6] $\frac{1}{2} - (1)0 = \frac{1}{2}$

[7] $\frac{1}{2} + (1)0 = \frac{1}{2}$

[8] $\frac{1}{2} - (1)0 = \frac{1}{2}$

$$\{\mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}}\}$$

DECIMATION IN TIME (DIT) RADIX 2 FFT

Second stage:

[1] $\frac{1}{2}+(1)\frac{1}{2}=1$

[2] $\frac{1}{2}+(-i)\frac{1}{2}=\frac{1}{2}-\frac{1}{2}i$

[3] $\frac{1}{2}-(1)\frac{1}{2}=0$

[4] $\frac{1}{2}-(-i)\frac{1}{2}=\frac{1}{2}+\frac{1}{2}i$

[5] $\frac{1}{2}+(1)\frac{1}{2}=1$

[6] $\frac{1}{2}+(-i)\frac{1}{2}=\frac{1}{2}-\frac{1}{2}i$

[7] $\frac{1}{2}-(1)\frac{1}{2}=0$

[8] $\frac{1}{2}-(-i)\frac{1}{2}=\frac{1}{2}+\frac{1}{2}i$

$$\{\mathbf{1},\ \frac{1}{2}-\frac{1}{2}i,\ 0,\ \frac{1}{2}+\frac{1}{2}i,\ 1,\ \frac{1}{2}-\frac{1}{2}i,\ 0,\ \frac{1}{2}+\frac{1}{2}i\}$$

DECIMATION IN TIME (DIT) RADIX 2 FFT

Final stage:

$\mathrm{X}(0) = 1 + (1)1 = 2$

$X(1) = \left(\frac{1}{2} - \frac{1}{2}i\right) + (0.7 - 0.7i)\left(\frac{1}{2} - \frac{1}{2}i\right) = 0.5 - 1.2i$

$X(2) = 0 + (-i)(0) = 0$

$X(3) = \left(\frac{1}{2} + \frac{1}{2}i\right) + (-0.7 - 0.7i)\left(\frac{1}{2} + \frac{1}{2}i\right) = 0.5 - 0.2i$

$X(4) = 1 - (1)(1) = 0$

$X(5) = \left(\frac{1}{2} - \frac{1}{2}i\right) - (0.7 - 0.7i)\left(\frac{1}{2} - \frac{1}{2}i\right) = 0.5 + 0.2i$

$X(6) = 0 - (-i)(0) = 0$

$X(7) = \left(\frac{1}{2} + \frac{1}{2}i\right) - (-0.7 - 0.7i)\left(\frac{1}{2} + \frac{1}{2}i\right) = 0.5 + 1.2i$

$$\mathrm{X}(\boldsymbol{k}) = \{2, 0.5 - 1.2i, \quad 0, 0.5 - 0.2i, \quad 0, \quad 0.5 + 0.2i, \quad 0, \quad 0.5 + 1.2i\}$$

DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)W_N^{kn} + \sum_{n=N/2}^{N-1} x(n)W_N^{kn}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n)W_N^{kn} + W_N^{Nk/2} \sum_{n=0}^{\frac{N}{2}-1} x(n + \frac{N}{2})W_N^{kn}$$

DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

Since $W_N^{Nk/2} = (-1)^k$,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right] W_N^{kN}$$

where $k = 0, 1, \ldots, N-1$

DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

Split(decimate) $X(k)$ into even and odd,

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{kn}$$

and

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \{\left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^{n}\} W_{N/2}^{kn}$$

where $k = 0, 1, \ldots, N-1$

DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

If we define the $\frac{N}{2} - point$ sequence $g_1(n)$ and $g_2(n)$,

$$g_1(n) = x(n) + x\left(n + \frac{N}{2}\right)$$

$$g2(n) = [x(n) - x\left(n + \frac{N}{2}\right)]W_N^n$$

where $n = 0, 1, \ldots, N - 1$

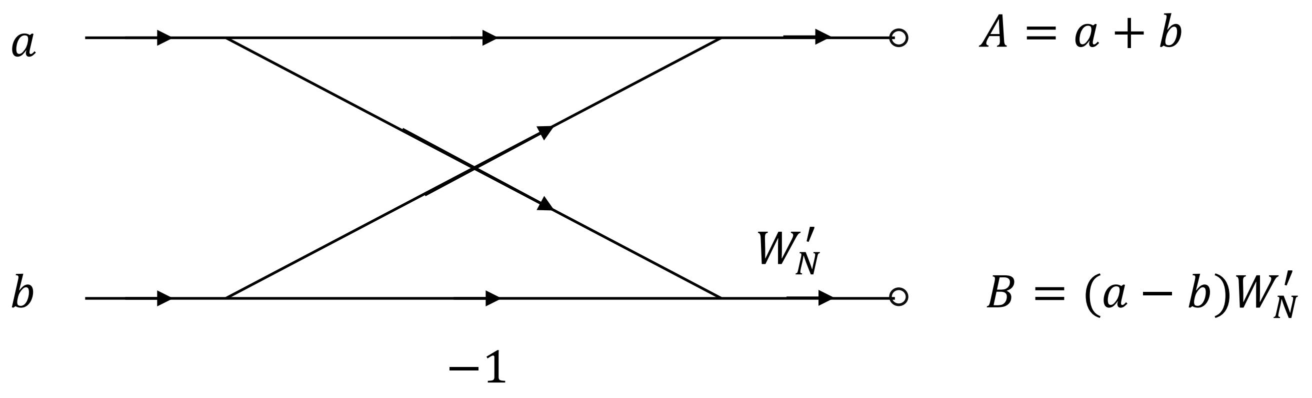
then,

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)W_{N/2}^{kn}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n)W_{N/2}^{kn}$$

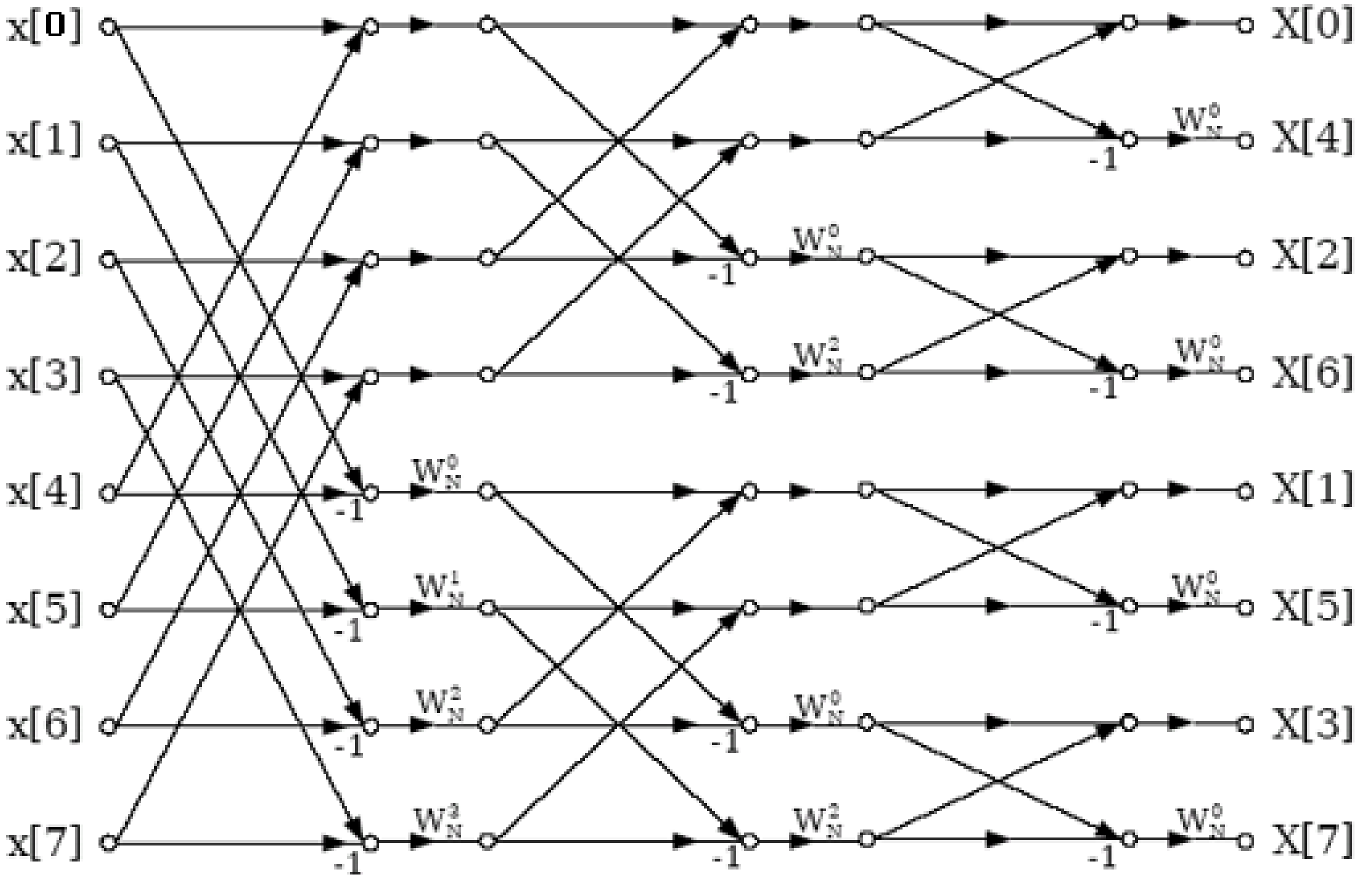
DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

Basic Butterfly Computation



DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

For N=8,



DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

Example:

$x(n) = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0,0,0,0\}$. Find $X(k)$.

DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

First stage:

[1]	$\frac{1}{2} + 0 = \frac{1}{2}$	[5]	$(\frac{1}{2} - 0)(1) = \frac{1}{2}$
[2]	$\frac{1}{2} + 0 = \frac{1}{2}$	[6]	$(\frac{1}{2} - 0)(0.7 - 0.7i) = \frac{1}{2}(0.7 - 0.7i)$
[3]	$\frac{1}{2} + 0 = \frac{1}{2}$	[7]	$\left(\frac{1}{2} - 0\right)(-i) = -\frac{1}{2}i$
[4]	$\frac{1}{2} + 0 = \frac{1}{2}$	[8]	$(\frac{1}{2} - 0)(-0.7 - 0.7i) = \frac{1}{2}(-0.7 - 0.7i)$

$$\{\mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \mathbf{\tfrac{1}{2}},\ \tfrac{1}{2}(0.7 - 0.7i), -\tfrac{1}{2}i,\ \mathbf{\tfrac{1}{2}}(-0.7 - 0.7i)\}$$

DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

Second stage:

[1] $\frac{1}{2}+\frac{1}{2}=1$

[2] $\frac{1}{2}+\frac{1}{2}=1$

[3] $\left(\frac{1}{2}-\frac{1}{2}\right)(1)=0$

[4] $(\frac{1}{2}-\frac{1}{2})(-i)=0$

[5] $\frac{1}{2}+(-\frac{1}{2}i)=\frac{1}{2}-\frac{1}{2}i$

[6] $\frac{1}{2}(0.7-0.7i)+\frac{1}{2}(-0.7-0.7i)=-0.7i$

[7] $\left(\frac{1}{2}-\left(-\frac{1}{2}i\right)\right)(1)=\frac{1}{2}+\frac{1}{2}i$

[8] $\left(\frac{1}{2}(0.7-0.7i)-\frac{1}{2}(-0.7-0.7i)\right)(-i)=-0.7i$

$\{\mathbf{1},\ 1,\ 0,\ 0,\ \frac{1}{2}-\frac{1}{2}i,\ -0.7i,\ \frac{1}{2}+\frac{1}{2}i,\ \frac{1}{2}+-0.7i\}$

DECIMATION IN FREQUENCY (DIF) RADIX 2 FFT

Final stage:

$$\mathrm{X}(0) = 1 + 1 = 2 \qquad X(1) = \left(\frac{1}{2} - \frac{1}{2}i\right) + (-0.7i) = 0$$

$$X(4) = 1 - 1 = 0 \qquad X(5) = \left(\left(\frac{1}{2} - \frac{1}{2}i\right) - (-0.7i)\right) = 0.5 + 0.2i$$

$$X(2) = 0 + 0 = 0 \qquad X(3) = \left(\frac{1}{2} + \frac{1}{2}i\right) + (-0.7i) = 0.5 - 0.2i$$

$$X(6) = 0 - 0 = 0 \qquad X(7) = \left(\frac{1}{2} + \frac{1}{2}i\right) - (-0.7i\ ) = 0.5 + 1.2i$$

$$\mathrm{X}(k) = \{2, 0.5 - 1.2i,\ \ 0, 0.5 - 0.2i,\ \ 0,\ \ 0.5 + 0.2i,\ \ 0,\ \ 0.5 + 1.2i\}$$

COMPUTATION COMPLEXITY

N	Direct Computation		DIT/DIF RADIX-2 FFT		IMPROVEMENT IN PROCESSING SPEED FOR MULTIPLICATION
	Complex Multiplication N^2	Complex Addition $N^2 - N$	Complex Multiplication $\frac{N}{2} log_2 N$	Complex Addition $N log_2 N$	
8	64	52	12	24	5.3 times
16	256	240	32	64	8 times
256	65536	65280	1024	2048	64 times

ACTIVITY #5

1. EVEN DIT – 6 samples
2. DIT RADIX 2 FFT
3. DIF RADIX 2 FFT