Fourier Series

FOURIER TRANSFORM

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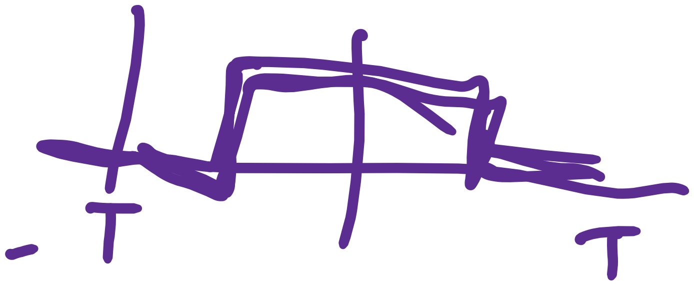
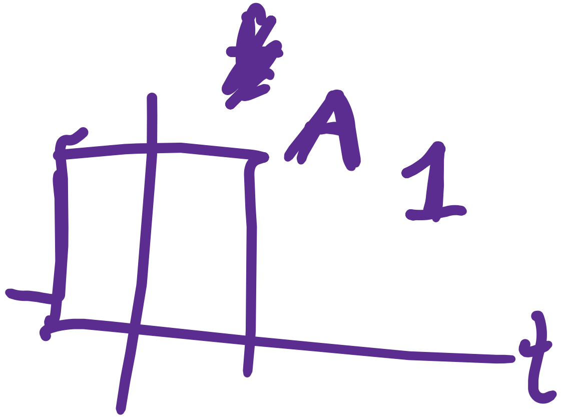


FOURIER TRANSFORM

A mathematical method to convert a function in the amplitude vs time domain to the amplitude vs frequency domain for non-periodic functions.

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$



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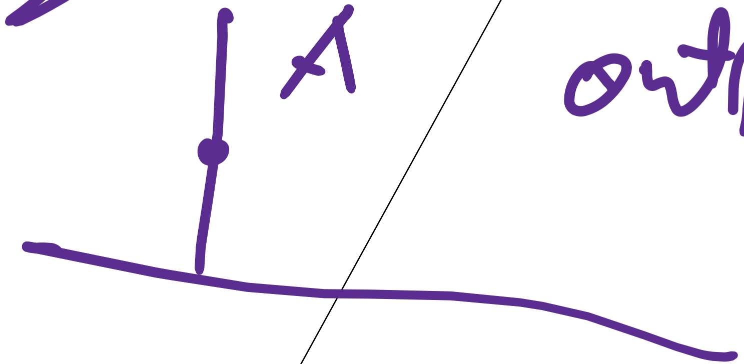
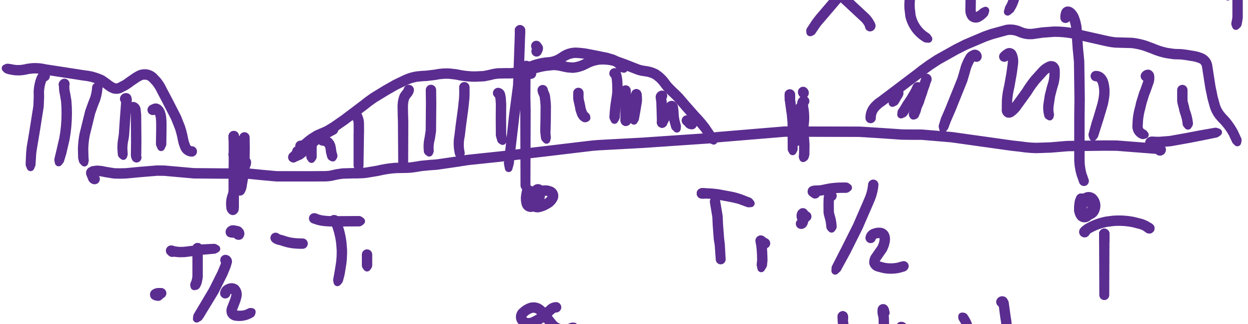


output Fourier transform

X(t)

X(t)

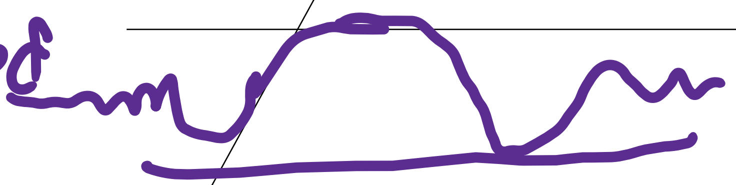


$$\omega_0 = \frac{2\pi}{T} \qquad \frac{1}{T} = \frac{\omega_0}{2\pi}$$

T-period

Fourier Series

periodic



sinc function

$$\tilde{x}(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

FS

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)^{-j\omega t} dt$$

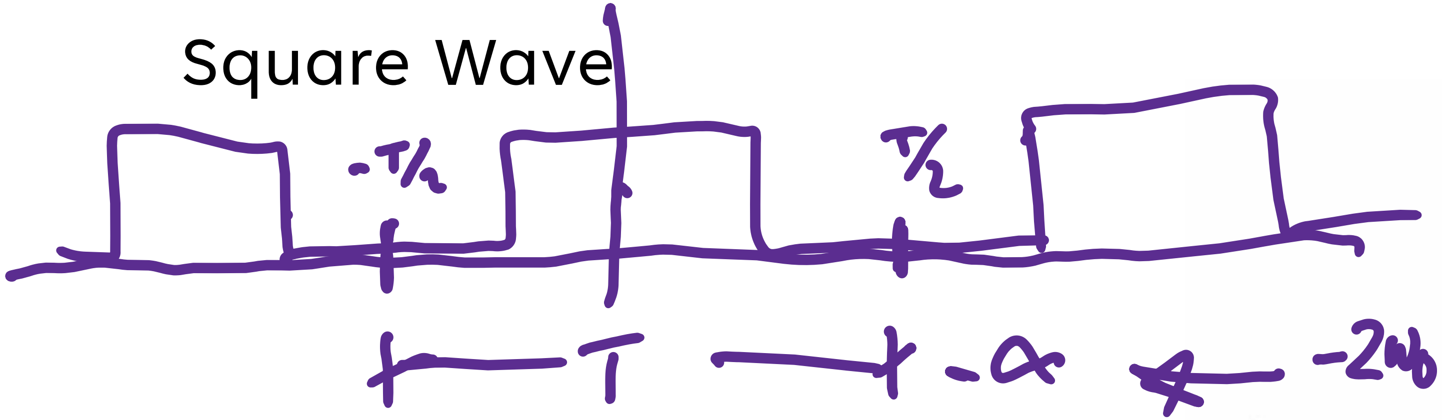
$$\tilde{x}(t) = \sum \frac{1}{T} X(k\omega_0) e^{jk\omega_0 t}$$

$$= \sum \frac{\omega_0}{2\pi} X(k\omega_0) e^{jk\omega_0 t}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum [X(k\omega_0) e^{jk\omega_0 t}]\omega$$

FOURIER TRANSFORM

Square Wave



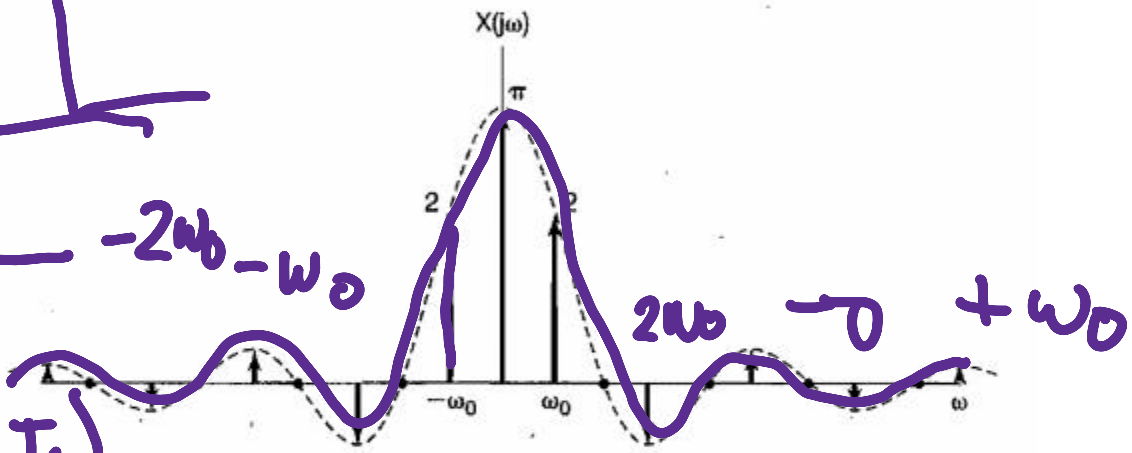


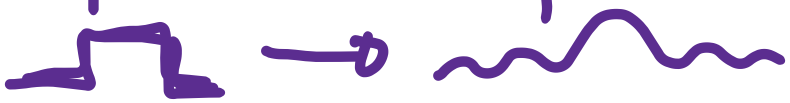
Figure 4.12 Fourier transform of a symmetric periodic square wave.

$$a_k = \frac{\sin k\pi/2}{k\pi/2} = \frac{\sin(k\omega_0 T_1)}{k\omega_0 T}$$

$a_k =$

INVERSE FOURIER TRANSFORM

Fourier transform



pulse → sinc

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

time domain

$\frac{\sin x}{x}$

frequency

inverse fourier transform

freq. →

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$t \rightarrow$ frequency

$x(t)$

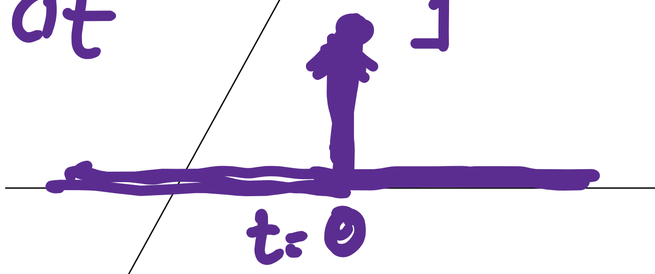
periodic signal → Fourier Series

aperiodic signal → Fourier transform

$\operatorname{sinc} x = \frac{\sin x}{x}$

$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$

$x(t) = \delta(t)$

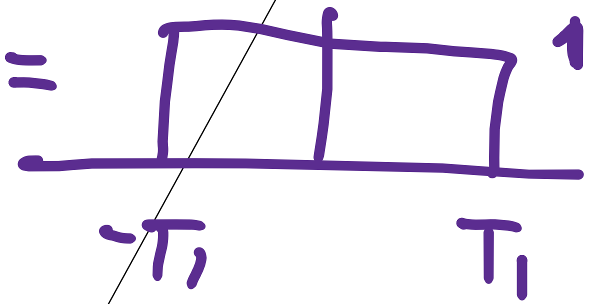


$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$

$e^{-j\omega(0)}\, dt$

$\int \delta(t)\,(1)\, dt = \int \delta\, dt = 1$

$x(t) =$



$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$

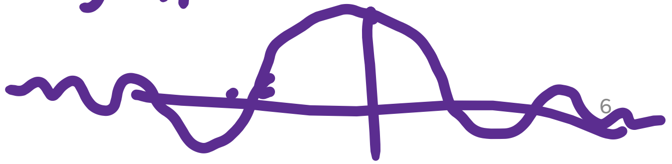
$= \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$

$= -\frac{1}{j\omega} \int_{-T_1}^{T_1} e^{-j\omega t} \cdot -j\omega\, dt$

$= -\frac{1}{j\omega}\, e^{-j\omega t} \Big|_{-T_1}^{T_1}$

$= 2 - \frac{1}{j\omega} \left[\frac{e^{-j\omega T_1} - e^{j\omega T_1}}{2} \right]$

$= \frac{2 \sin \omega T_1}{\omega T_1} (T_1) = 2 \operatorname{sinc} \omega T_1$



PROPERTIES OF FOURIER TRANSFORM

Linearity

$ax(t) + by(t) \Rightarrow aX(\omega) + bY(\omega)$

- **The property of linearity:**

$$\mathcal{F}\{ax(t)+by(t)\} = aX(j\omega)+bY(j\omega) \quad \Rightarrow \quad ax(t)+by(t) \leftrightarrow aX(j\omega)+bY(j\omega)$$

Proof:

$$\begin{aligned}\mathcal{F}\{ax(t)+by(t)\} &= \frac{1}{T}\int_{-\infty}^{\infty}\{ax(t)+by(t)\}e^{-j\omega t}\,dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} ax(t)\, e^{-j\omega t}\,dt + \frac{1}{T}\int_{-\infty}^{\infty} by(t)\, e^{-j\omega t}\,dt \\ &= a\left\{\frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt\right\} + b\left\{\frac{1}{T}\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\,dt\right\} \\ &= aX(j\omega)+bY(j\omega)\end{aligned}$$

$X(\omega)$

$X(jw)$

fourier transform

freq

PROPERTIES OF FOURIER TRANSFORM

x(at)

- **Time Scaling:**

$$x(at) \leftrightarrow \frac{1}{|a|} X(\frac{j\omega}{a})$$

Proof:

$$\mathcal{F}\{x(at)\} = \frac{1}{T} \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

λ

assume $a > 0$, make a change of variables : $\lambda = at$, which implies $t = \lambda / a$, and $dt = (1/a)d\lambda$

$$\mathcal{F}\{ax(t)\} = \frac{1}{T} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega(\frac{\lambda}{a})} (\frac{1}{a}) d\lambda$$

$$= (\frac{1}{a}) \left\{ \frac{1}{T} \int_{-\infty}^{\infty} x(\lambda)\, e^{-j(\omega/a)\lambda} d\lambda \right\}$$

$$= (\frac{1}{a}) X(\underline{\frac{j\omega}{a}})$$

PROPERTIES OF FOURIER TRANSFORM

$x(t) \to X(\omega)$

$x(-t) \to X(-\omega)$

- **Time Reversal:**

  $$x(-t) \leftrightarrow X(-j\omega)$$

  Proof:

  $$\mathcal{F}\{x(-t)\} = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)\Bigg|_{a=-1} = X(-j\omega)$$

  frequency domain

  We can also note that for real-valued signals:

  $x^*(j\omega) = X(-j\omega)$

  $$X(-j\omega) = |X(-j\omega)| \angle X(-j\omega)$$
  $$= |X(j\omega)| \angle X(-j\omega) = X^*(j\omega) \quad \text{(complex conjugate)}$$

- **Time reversal is equivalent to conjugation in the frequency domain.**

PROPERTIES OF FOURIER TRANSFORM

- **Multiplication by a power of *t*:**

$$t^n x(t) \leftrightarrow (j)^n \frac{d^n}{d\omega^n} X(j\omega)$$

Proof:

$$X(j\omega) = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

differentiate with respect to ω :

$$\frac{dX(j\omega)}{d\omega} = \frac{1}{T}\int_{-\infty}^{\infty} (-jt)\; x(t)\; e^{-j\omega t}\, dt$$

multiply by j :

$$j\,\frac{dX(j\omega)}{d\omega} = (j)\frac{1}{T}\int_{-\infty}^{\infty} (-jt)\; x(t)\; e^{-j\omega t}\, dt$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} (t)\; x(t)\; e^{-j\omega t}\, dt = \mathcal{F}\{t\; x(t)\}$$

PROPERTIES OF FOURIER TRANSFORM

- **Multiplication by a complex exponential:**

$$x(t)e^{j\omega t} \leftrightarrow X(j(\omega - \omega_0)) \quad \textit{for any real number } \omega_0$$

Proof:

$$\mathcal{F}\left\{x(t)e^{j\omega_0 t}\right\} = \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}\; e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\,dt$$

$$= X(j(\omega - \omega_0))$$

PROPERTIES OF FOURIER TRANSFORM

- **Convolution in the time domain:**

$$x(t) * h(t) \leftrightarrow X(j\omega)H(j\omega)$$

- **Proof:**

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

$$\mathcal{F}\{x(t) * h(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\lambda)\left[\int_{-\infty}^{\infty} h(t-\lambda)e^{-j\omega t}dt\right]d\lambda$$

$$\text{change of variables}: \gamma = t-\lambda \Rightarrow d\gamma = dt$$

$$= \int_{-\infty}^{\infty} x(\lambda)\left[\int_{-\infty}^{\infty} h(\gamma)e^{-j\omega(\gamma+\lambda)}d\gamma\right]d\lambda$$

dft →

dtft →

discrete form

TABLE 3.1 Properties of the Fourier Transform

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)\ a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega)\ n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)\quad \omega_0$ real
Multiplication by $\sin(\omega_0 t)$	$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega)\ n = 1, 2, \ldots$
Integration in the time domain	$\int_{-\infty}^{t} x(\lambda)\,d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{X(\omega)}V(\omega)\,d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2\,d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

$\delta(t)$

$x(t-c)$

$X(\omega e^{-j\omega c})$

$\leftrightarrow$

discrete

$x[n]$