DISCRETE FOURIER TRANSFORM

COE150

Fourier transform

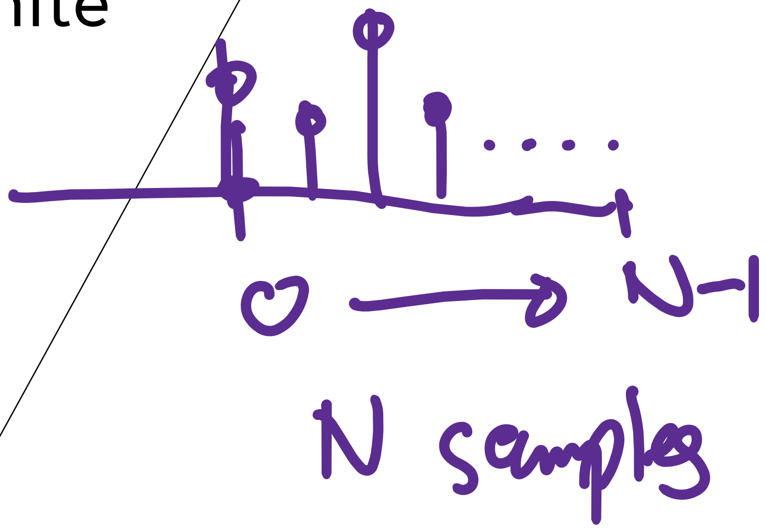
CTS

Discrete

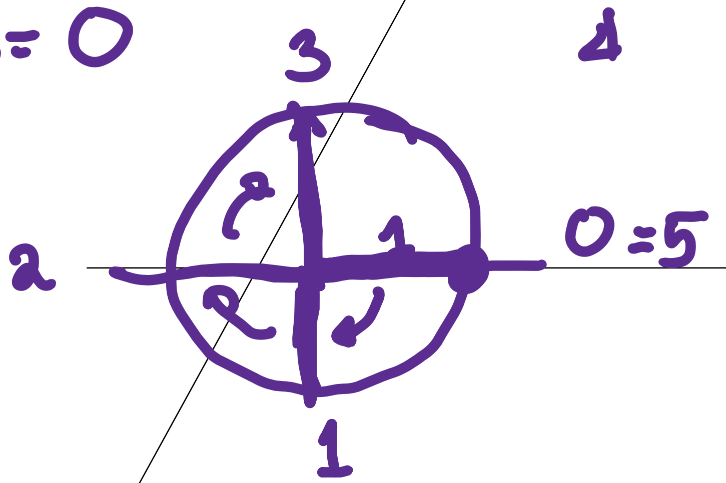
DISCRETE FOURIER TRANSFORM

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for signals known only at N instants by sample times T (i.e., at finite sequence of data).

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}$$



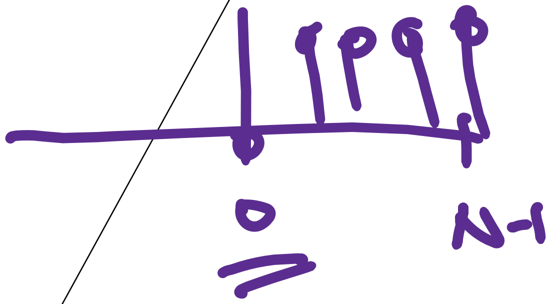
$kn = kn \% N$ $7 \% 5 = 2$ $10 \% 5 = 0$



DISCRETE FOURIER TRANSFORM

The discrete Fourier transform of a length N signal $x[n]$, $n = 0, 1, \ldots, N - 1$ is given by

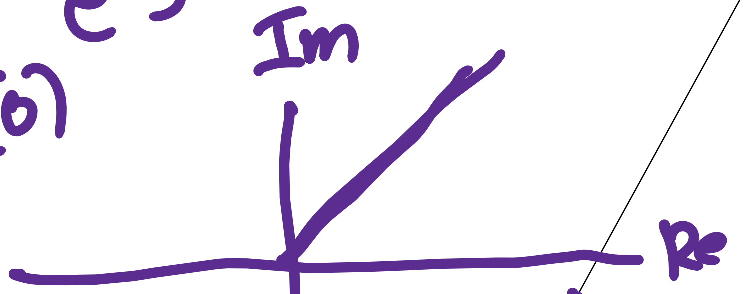
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$



$n = 7$

$n = 10 = 0$

$e^{-j\omega t}$

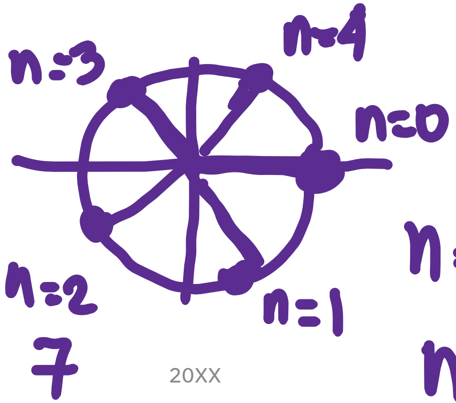


$k=1$ $n=0$: $e^{-j(2\pi/5)1(0)}$

$= 1$

N= 4 sample

N= 5 samples



$n=1$ $e^{-j(2\pi/5)1(1)} = 0.31 - 0.95i$

$n=2$ $e^{-j(2\pi/5)1(2)} = -0.81 - 0.58i$

$n=3$ $e^{-j(2\pi/5)(1)(3)} = -0.81 + 0.58i$

$n=4$ $e^{-j(2\pi/5)(1)(4)} = 0.30 + 0.95i$

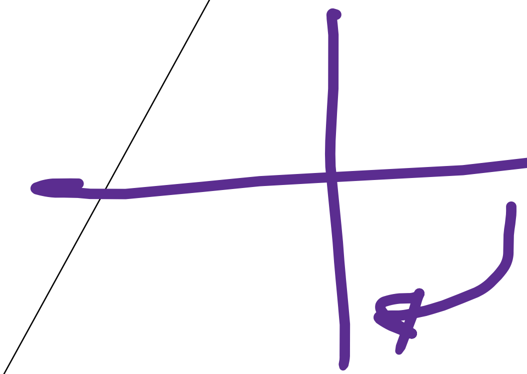
DISCRETE FOURIER TRANSFORM

When dealing with the DFT, it is common to define the complex quantity

$$W_N = e^{-j(2\pi/N)}$$

$$\sum_{n=0}^{N-1} x[n] \boxed{e^{-j(2\pi/N)}}^{kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$



clockwise

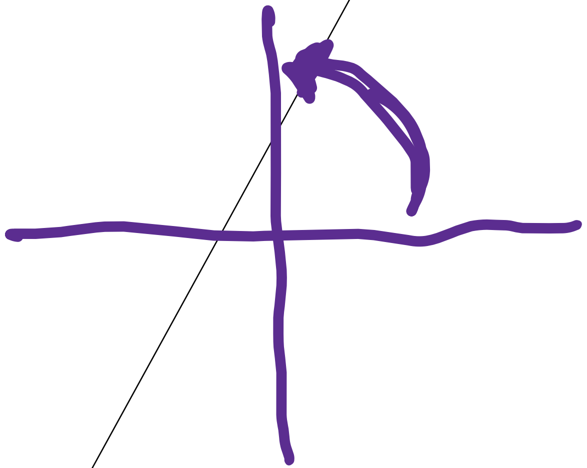
kn = kn % N

INVERSE DFT

For inverse DFT,

j +

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}.$$



$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

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DFT

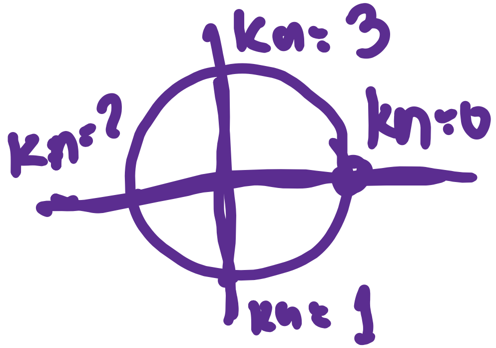
An important property of the DFT is that it is cyclic, with period N

$$X[k+rN] = \sum_{n=0}^{N-1} x[n] W_N^{(k+rN)n} = \sum_{n=0}^{N-1} x[n] W_N^{kn} (\cancel{W_N^N})^{rn}$$

$$= \sum_{n=0}^{N-1} x[n] W_N^{kn} = X[k],$$

Linearity

DFT



$H[3] = 1W_4^{3(0)} + 3W_4^{3(1)} + (-1)W_4^{3(2)} + -2W_4^{3(3)} = 1 + 3i + 1 \cancel{3} + 2i$

$= 2 + 5i$

Compute the DFT of the following two sequences:

a. $h[n] = \{1,3,-1,-2\}$

b. $x[n] = \{1,2,0,-1\}$

$H[2] = 1W_4^{2(0)} + 3W_4^{2(1)} + (-1)W_4^{2(2)} + (-2)W_4^{2(3)}$

$= 1(1) + 3(-1) + (-1)(1) + (-2)(-1)$

$H[2] = 1 - 3 - 1 + 2$

$H[2] = -1$

$$H[k] = \sum_{n=0}^{N-1} h[n] W_N^{kn} \nearrow e^{-j\frac{2\pi}{N}kn}$$

$H[0] = 1e^{-j\frac{2\pi}{4}0(0)} \; 3e^{-j\frac{2\pi}{4}0(1)} - 1e^{-j\frac{2\pi}{4}0(2)} - 2e^{-j\frac{2\pi}{4}0(3)}$

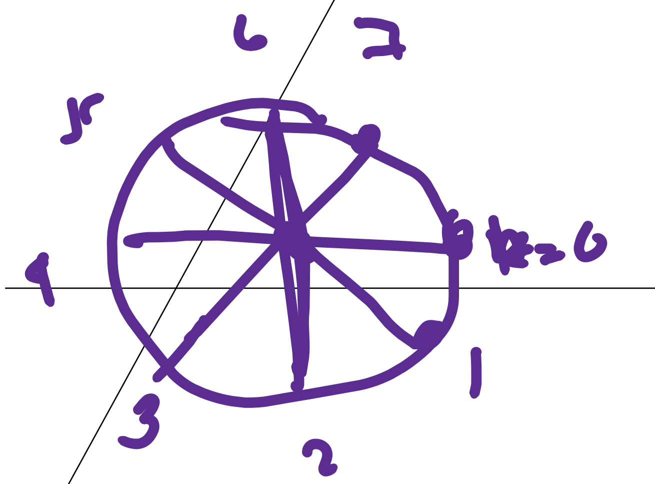
$= 1 + 3 - 1 - 2 = 1$

$H[1] = 1e^{-j\frac{2\pi}{4}1(0)} \; 3e^{-j\frac{2\pi}{4}1(1)} - 1e^{-j\frac{2\pi}{4}1(2)} - 2e^{-j\frac{2\pi}{4}1(3)}$

$= 1 + 3(-i) + (-1)(-1) + (-2)(i)$

$= 2 - 5i$

MATRIX REPRESENTATION OF DFT



Let us write the variables involved in matrix form:

$$X(k) = \sum_{n=0}^{N-1} x[n].W_N^{kn} \quad for \quad k = 0,1,2,\cdots,N-1$$

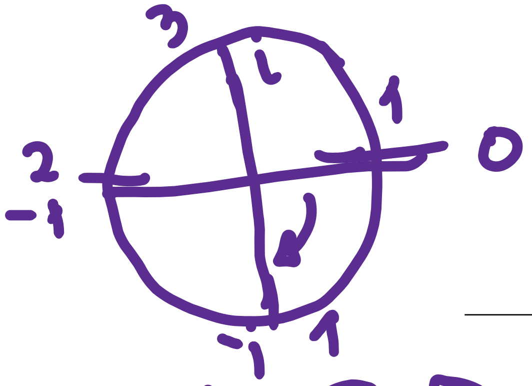
Then the weight matrix is simply:

$$W = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & & \cdots & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 6 \\ 1 & 3 & 6 & 9 \\ \cdot & & & \end{bmatrix}$$

MATRIX REPRESENTATION OF DFT

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ . \\ . \\ . \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & . & . & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & . & . & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & . & . & W_N^{2(N-1)} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & . & . & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ . \\ . \\ . \\ x[N-1] \end{bmatrix}$$

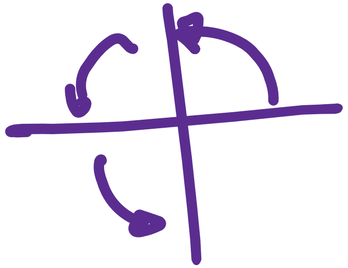


$h[n] = \{1,3,-1,-2\}$ $N=4$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-5i \\ -1 \\ 2+5i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1(1) + 1(3) + 1(-1) + 1(-2) \\ 1(1) - i(3) - 1(-1) + i(-2) \\ 1(1) - 1(3) \quad 1(-1) - 1(-2) \\ 1(1) \quad i(3) - 1(-1) - i(-2) \end{bmatrix}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$



$$X[k] = \{1,\ 2-5i,\ -1,\ 2+5i\}$$

$N = 4$ samples

$$x[0] = \frac{1}{4}\left\{ 1W_4^{-0(0)} + (2-5i)W_4^{-1(0)} + (-1)W_4^{-2(0)} + (2+5i)W_4^{-3(0)} \right\}$$

$$= \frac{1}{4}\left\{ \cancel{1} + (2-\cancel{5i}) + (\cancel{-1}) + (2+\cancel{5i}) \right\}$$

$$= \frac{1}{4}\{4\} = 1$$

$$x[1] = \frac{1}{4}\left\{ 1W_4^{-0(1)} \quad (2-5i)W_4^{-1(1)} \quad (-1)W_4^{-2(1)} \quad (2+5i)W_4^{-3(1)} \right\}$$

$$= \frac{1}{4}\left\{ 1 + (2-5i)i + (-1)(-1) + (2+5i)(-i) \right. = 3\cancel{x}$$

$$x[2] = \frac{1}{4}\left[1W_4^{-0(2)} + (2-5i)W_4^{-1(2)} \quad (-1)W_4^{-2(2)} \quad (2+5i)W_4^{-3(2)} \right.$$

$$= -1$$

$$x[3] = \text{——}//$$

DFT ↔ FFT

IDFT

4 samples

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

Matrix

$$x[n] = \frac{1}{N} W X[k]$$

$$W = \begin{vmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^0 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^0 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{vmatrix}$$

$X[k]: 1,\ 2-5i,\ -1,\ 2+5i$

$$x[n] = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 2-5i \\ -1 \\ 2+5i \end{bmatrix} =$$

FFT fast fourier transform

Friday 10:30 AM

ACTIVITY

$$x[n] = \{1,2,0,-1,3\}$$

1. DFT mathematically.
2. DFT matrix
3. Check with IDFT mathematically
4. Check with IDFT matrix.