Z TRANSFORM

COE150

INTRODUCTION

The z-transform of a sequence x[n] is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

discrete

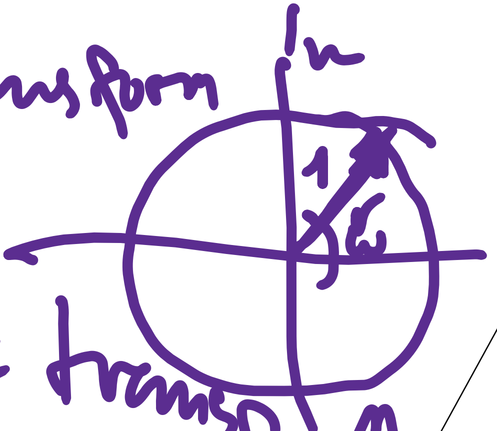
unit circle

$z = re^{-j\omega}$

r - radius

ω phase

bidirectional z transform



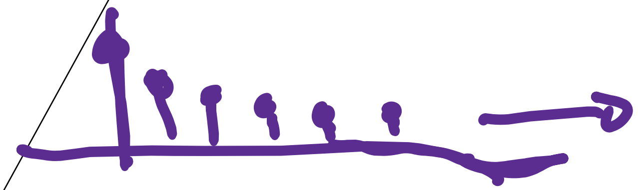
FT

unidirectional z transform

$$X[z] = \sum_{n=0}^{+\infty} x[n] z^{-n}$$

REGION OF CONVERGENCE

The region of convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value.



The ROC is the region which: Z transform $< \infty$

t z

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

Region of converg

$x[n] z^{-n} \propto$

REGION OF CONVERGENCE

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$< \infty$

Find the ROC:

1. $x[n] = \{1, 2, 5, 7, 0, 1\} = 1z^0 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$

   ROC: entire z-plane except z=0

2. $x[n] = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\} = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$

   ROC: entire z-plane except z=0 and z=∞

3. $x[n] = \delta(n) \xleftrightarrow{z} = 1$

   ROC: entire z-plane

4. $x[n] = \delta(n-k), k > 0 \longleftrightarrow z^{-k}$

   ROC entire z plane except z=0

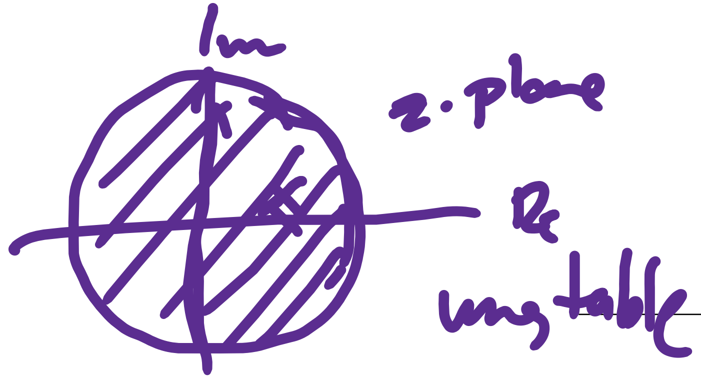
5. $x[n] = \delta(n+k), k > 0$

   $z^{-(-k)} = z^k$

   ROC: entire plane except z=∞

$\frac{1}{0}$

PROPERTIES OF ROC



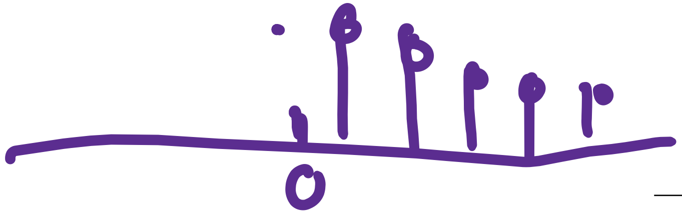
inf.

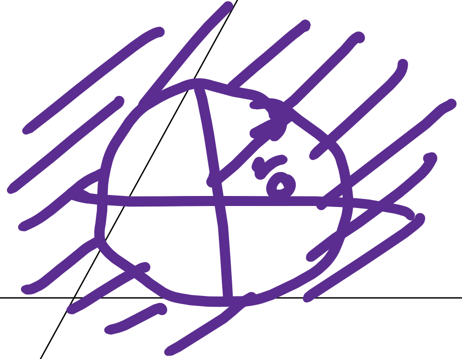
x poles/o zeros

$\frac{z-1}{z+1}$ pole: -1 zero: 1

- The ROC is an annular ring in the z-plane centered about the origin (which is equivalent to a vertical strip in the s-plane).
- The ROC does not contain any poles (similar to the Laplace transform).
- If x[n] is of finite duration, then the ROC is the entire z-plane except possibly $z = 0$ and/or $z = \infty$:

PROPERTIES OF ROC





- If x[n] is a right-sided sequence, and if $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ are also in the ROC.

- If x[n] is a left-sided sequence, and if $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| < r_0$ are also in the ROC.
- If x[n] is a two-sided sequence, and if $|z| = r_0$ is in the ROC, then the ROC consists of a ring in the z-plane including $|z| = r_0$.

EXAMPLE

Determine the z-transform of the signal

$$x(n) = \left(\frac{1}{2}\right)^n u(n)$$

$n \geq 0$

$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n}$

$x[n] = [\frac{1}{2}^0, \frac{1}{2}^1, \frac{1}{2}^2, \frac{1}{2}^3, \ldots \ldots \frac{1}{2}^n \ldots]$

$X[z] = [1, \frac{1}{2} z^{-1}, \frac{1}{2}^2 z^{-2}, \frac{1}{2}^3 z^{-3}, \ldots \frac{1}{2}^n z^{-n} \ldots]$

$= \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n}$

$= \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n \overset{PS}{=} \frac{1}{1 - \frac{1}{2} z^{-1}}$

$1 + A + A^2 + A^3 \ldots A^\infty = \frac{1}{1-A}$

ROC: $\{|\frac{1}{2} z^{-1}| < 1\}$ 2

$|z^{-1}| < 2$

$|\frac{1}{z}| < 2$

$\underline{|z| > \frac{1}{2}}$

EXAMPLE

Determine the z-transform of

$$x[n] = a^n u[n] \begin{cases} a^n, n \geq 0 \\ 0, n < 0 \end{cases}$$

$$X[z] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}}$$

$$|az^{-1}| < 1$$

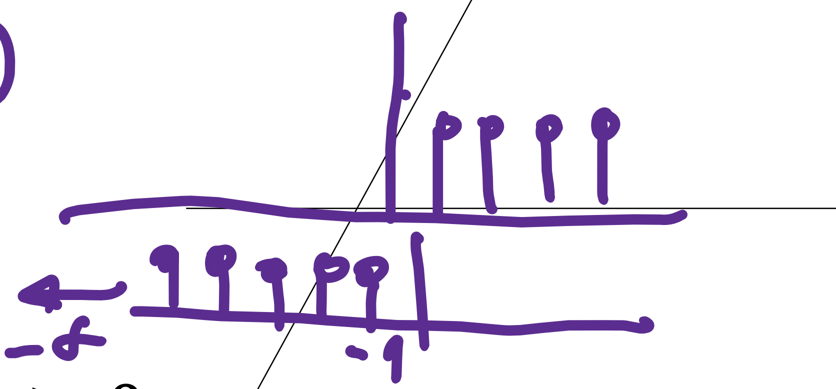
$$|a| < |z|$$

$$|z| > |a| \Rightarrow \text{ROC}$$

EXAMPLE

$A(1+A+A^2+A^3 \dots) - (n+1)$

$A\left(\frac{1}{1-A}\right) = \frac{A}{1-A}$



Determine the z-transform of

$$x[n] = -a^n u[-n-1] = \begin{cases} 0, n \geq 0 \\ -a^n, n \leq -1 \end{cases}$$

$$X[z] = \sum_{n=-\infty}^{-1} (-a^n) z^{-n}$$

$l = -n$

$$= -\sum_{l=1}^{\infty} (a^{-l}) z^{l}$$

$$= -\sum_{l=1}^{\infty} (a^{-1}z)^{l}$$

PS $X[z] = -\frac{a^{-1}z}{1-a^{-1}z}\left(\frac{az^{-1}}{az^{-1}}\right)$

$$= -\frac{1}{az^{-1}-1}$$

$$X[z] = \frac{1}{1-az^{-1}}$$

ROC: $|az^{-1}| < 1$; $|z| < |a|$

EXAMPLE

ROC@case 2: $|a| < |z| < |b|$

$$X[z] = \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}} = \frac{b-a}{a+b-z-abz^{-1}}$$

Determine the z-transform of

$$x(n) = a^n u(n) + b^n u(-n-1)$$

$$X[z] = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{l=1}^{\infty} (b^{-1}z)^l$$

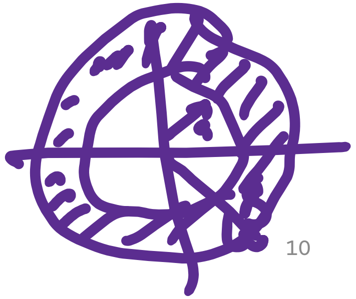
ROC

a	b
$\lvert az^{-1}\rvert < 1$	$\lvert b^{-1}z\rvert < 1$
$\lvert z\rvert > \lvert a\rvert$	$\lvert z\rvert < \lvert b\rvert$

case 1: $|b| < |a|$

X[z] does not exist

case 2: $|b| > |a|$



EXAMPLE

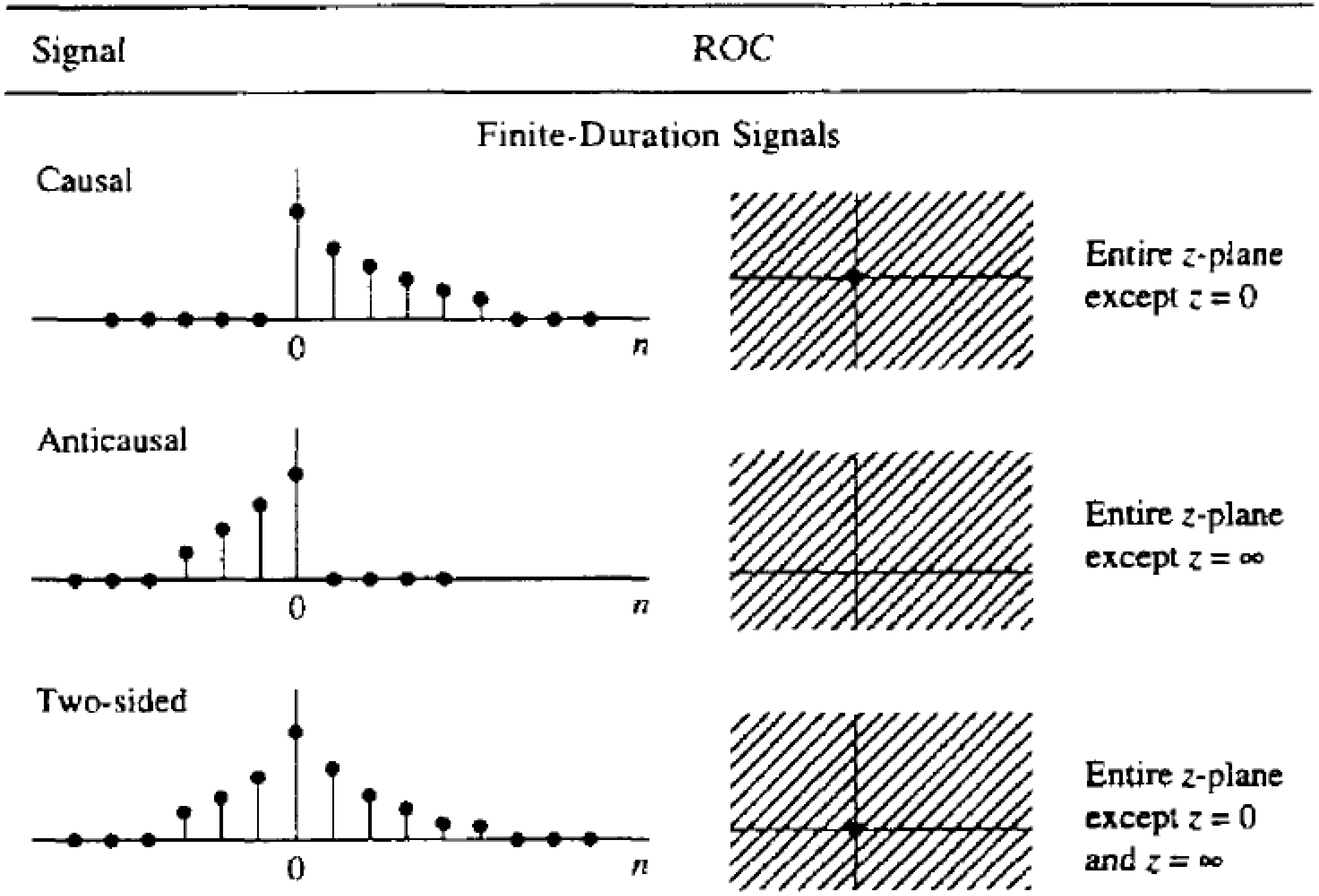
Activity

Determine the z-transform of

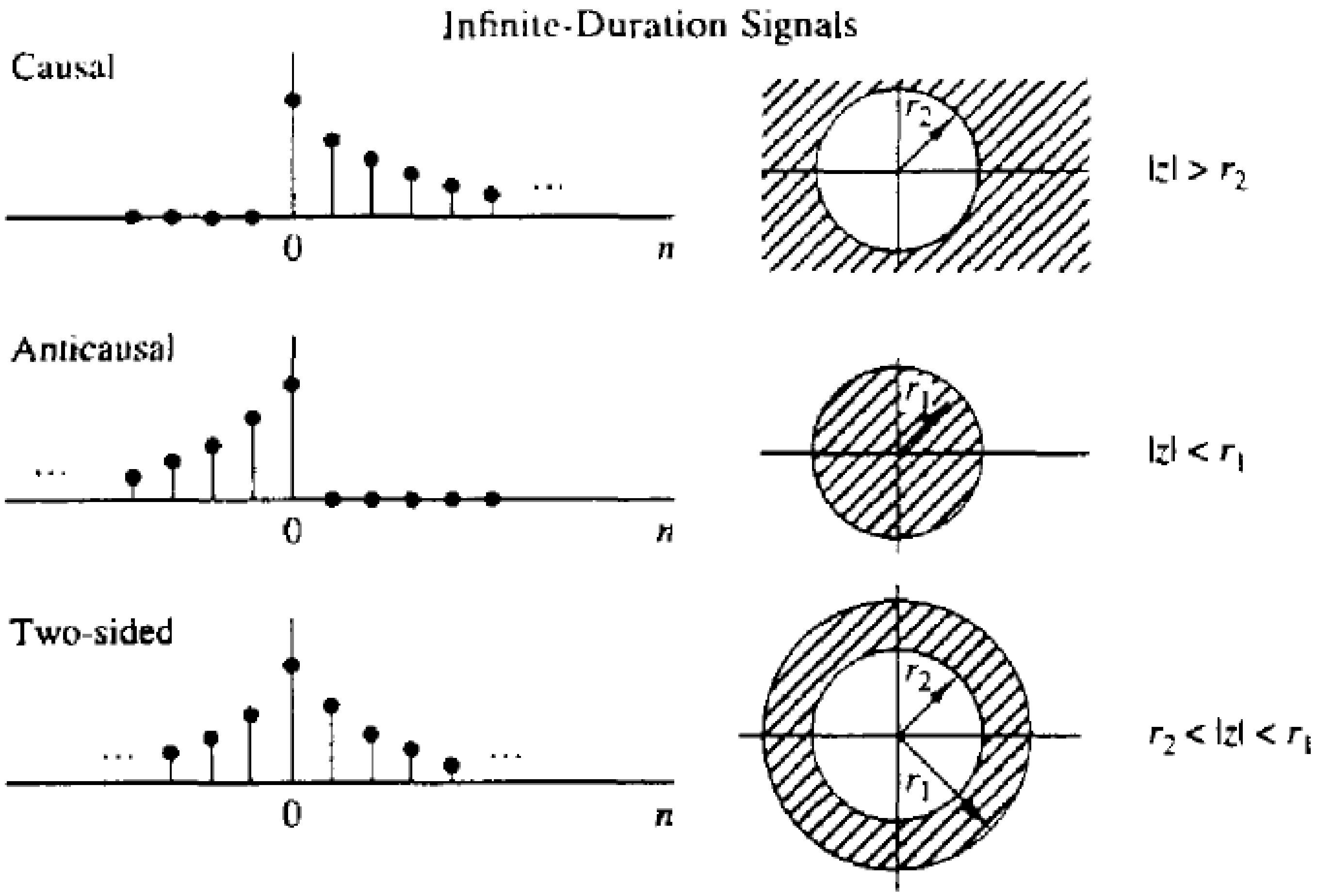
$$x[n] = b^{|n|}, \qquad b > 0$$

$$x[n] = b^n u[n] + b^{-n} u[-n]$$

CHARACTERISTIC FAMILIES OF SIGNAL WITH THEIR CORRESPONDING ROC



CHARACTERISTIC FAMILIES OF SIGNAL WITH THEIR CORRESPONDING ROC



PROPERTIES OF Z TRANSFORM

- **Linearity**

  If

$$x_1[n] \leftrightarrow X_1[z]$$
$$x_2[n] \leftrightarrow X_2[z]$$

  Then,

$$\mathrm{x}[\mathrm{n}] = ax_1[n] + bx_2[n] \leftrightarrow X[z] = aX_1[z] + bX_2[n]$$

- **Time-shift**

  If

$$x[n] \leftrightarrow X[z]$$

  Then,

$$x[n-k] \leftrightarrow z^{-k}X[z]$$

PROPERTIES OF Z TRANSFORM

- **Scaling in the Z domain**

If

$$x[n] \leftrightarrow X[z] \qquad \text{ROC: } r_1 < |z| < r_2$$

Then,

$$a^n x[n] \leftrightarrow X(a^{-1}z) \qquad \text{ROC: } |a|r_1 < |z| < |a|r_2$$

PROPERTIES OF Z TRANSFORM

- **Time Reversal**

If

$$x[n] \leftrightarrow X[z] \qquad \text{ROC: } r_1 < |z| < r_2$$

Then,

$$x[-n] \leftrightarrow X(z^{-1}) \qquad \text{ROC: } \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

PROPERTIES OF Z TRANSFORM

- **Differentiation in the Z domain**

  If

$$x[n] \leftrightarrow X[z]$$

  Then,

$$nx[n] \leftrightarrow -z\frac{dX[z]}{dz}$$

PROPERTIES OF Z TRANSFORM

- **Convolution of two sequences**

  If

$$x_1[n] \leftrightarrow X_1[z]$$
$$x_2[n] \leftrightarrow X_2[z]$$

  Then,

$$x[n] = x_1[n] * x_2[n] \leftrightarrow X[z] = X_1[z]X_2[z]$$

PROPERTIES OF Z TRANSFORM

- **Correlation of two sequences**

If

$$x_1[n] \leftrightarrow X_1[z]$$
$$x_2[n] \leftrightarrow X_2[z]$$

Then,

$$r_{x_1x_2}[l] = \sum_{-\infty}^{\infty} x_1[n]x_2[n-l]\, X[z] \leftrightarrow R_{x_1x_2}[z] = X_1[z]X_2[z^{-l}]$$

PROPERTIES OF Z TRANSFORM

- **Multiplication of two sequences**

If

$$x_1[n] \leftrightarrow X_1[z]$$
$$x_2[n] \leftrightarrow X_2[z]$$

Then

$$x[n] = x_1[n]x_2[n] \leftrightarrow X[z] = \frac{1}{2\pi j}\oint X_1[v]X_2[\frac{z}{v}]\, v^{-1}dv$$

$$ROC\colon\ r_{1l}r_{2l} < |z| < r_{1u}r_{2u}$$

PROPERTIES OF Z TRANSFORM

- **Parseval's Relation**

If

$x_1[n]$ and $x_2[n]$ are complex value sequences

Then,

$$\sum_{-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint X_1[v]X_2^*[\frac{1}{v^*}]\, v^{-1}dv$$

PROPERTIES OF Z TRANSFORM

- **Initial Value Theorem**

If

$x[n]$ is causal [i.e. $x[n] = 0$ for $n < 0$].

Then,

$$x(0) = \lim_{z \to \infty} X(z)$$

PROPERTIES OF Z TRANSFORM SUMMARY

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC: $r_2 < \lvert z\rvert < r_1$ ROC_1 ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\lvert a\rvert r_2 < \lvert z\rvert < \lvert a\rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z\rvert < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \lvert z\rvert < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \lvert z\rvert < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

Z TRANSFORM TABLE

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\|z\| > r$

ACTIVITY 3

FRIDAY 10:30 AM

Determine the Z transform and ROC of the ff. x[n]:

Friday Quiz #2

1. $x[n] = \{3, 0, 0, 0, 0, \underset{\uparrow}{6}, 1, -4\}$

2. $x[n] = n^2 u[n]$

3. $x[n] = \begin{cases} \left(\frac{1}{3}\right)^n, & n \geq 0 \\ \left(\frac{1}{2}\right)^{-n}, & n < 0 \end{cases}$

4. $x[n] = b^{|n|}, \qquad b > 0$

$x[n] = b^n u[n] + b^{-n} u[-n]$