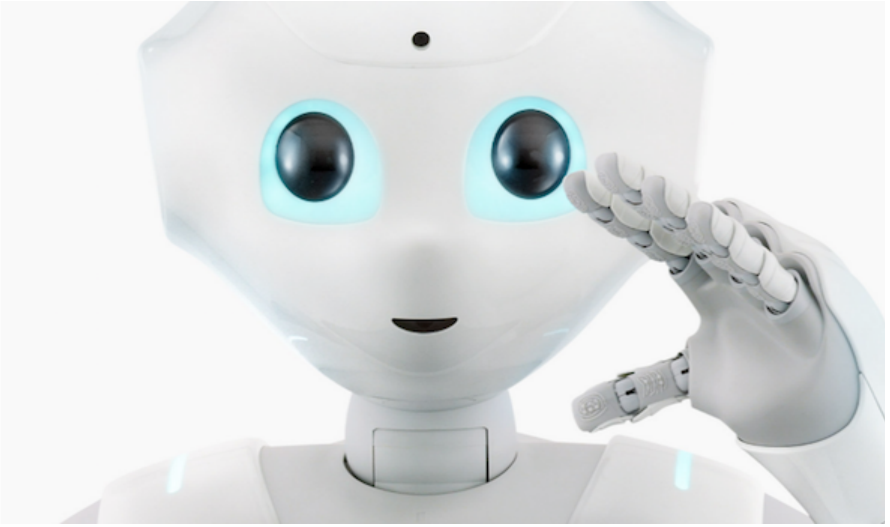
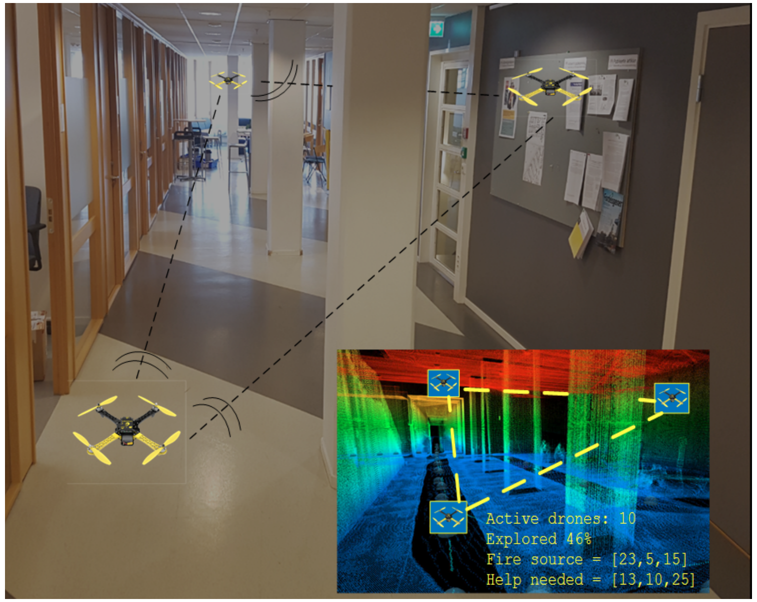




Introduction to Robotics
Robot arm kinematics

OsloMet ELVE3610 Robotteknikk

Outline

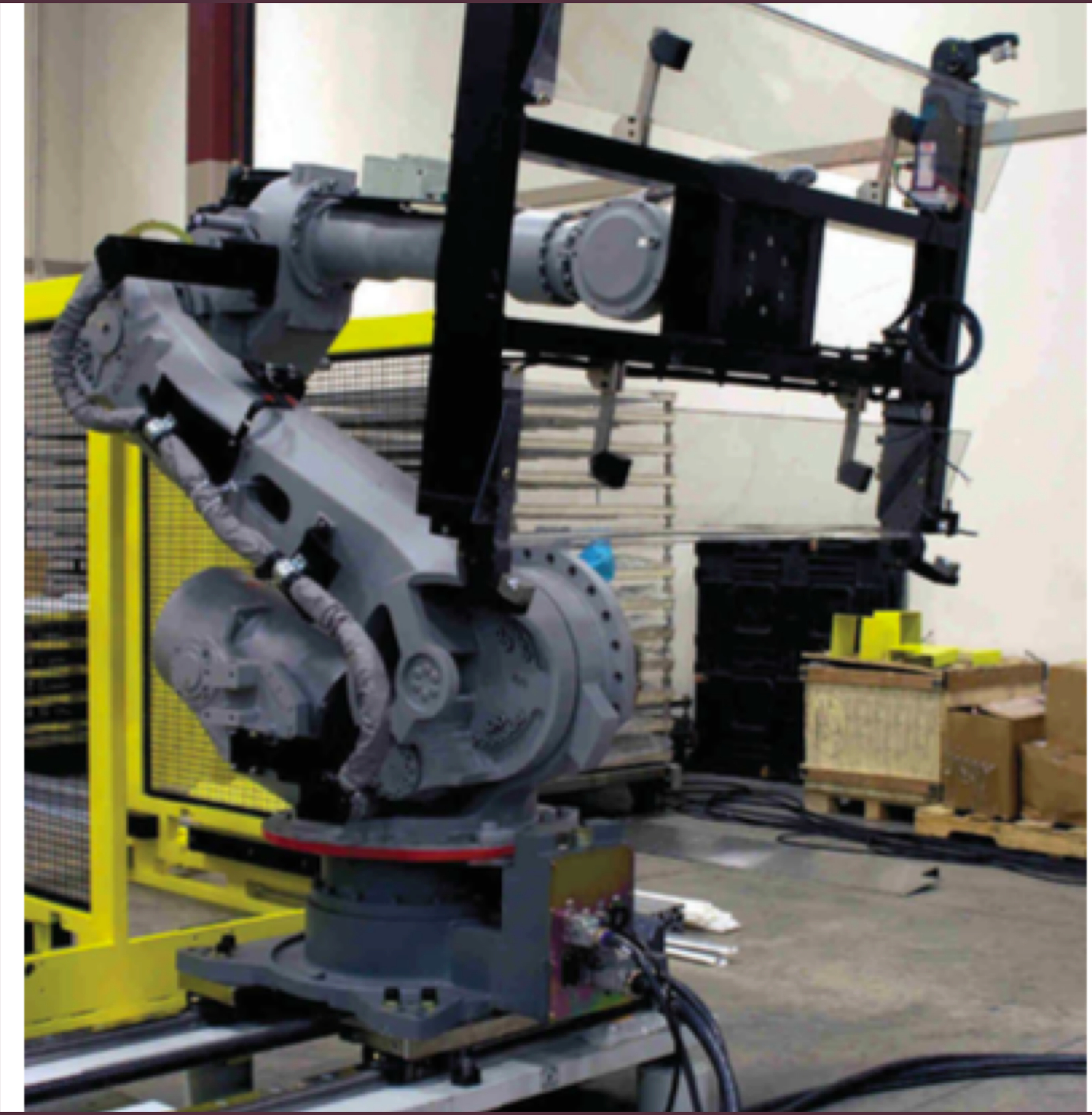
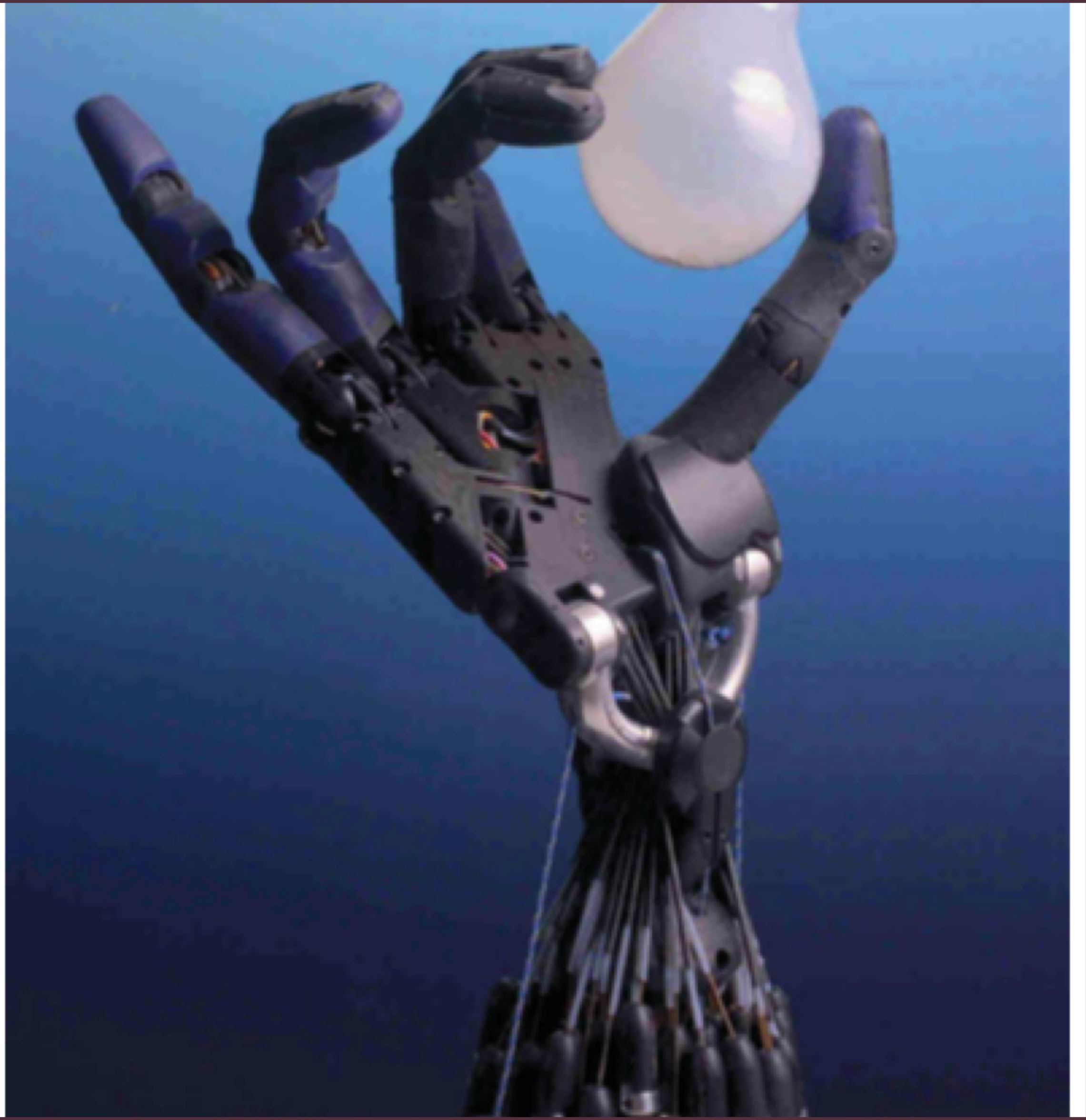
- Kinematic links, types of robots
- Links, joints
- Configuration space, joint space, workspace
- Forward kinematics
- Examples
- DH convention
- Singularities
- Paths in configuration/joint space: animations
- Workspace

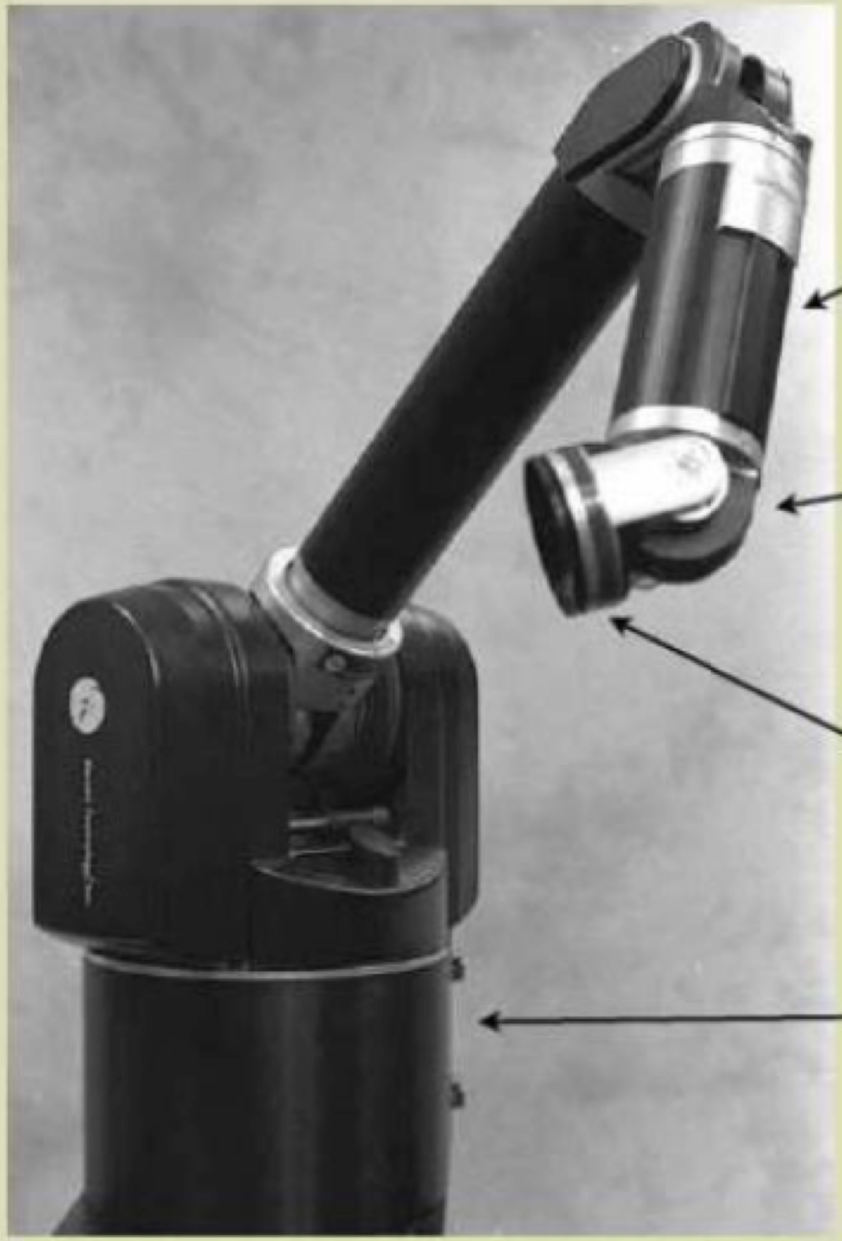
6DOF serial link manipulator arm

4DOF SCARA

Gantry robot

Parallel link robot



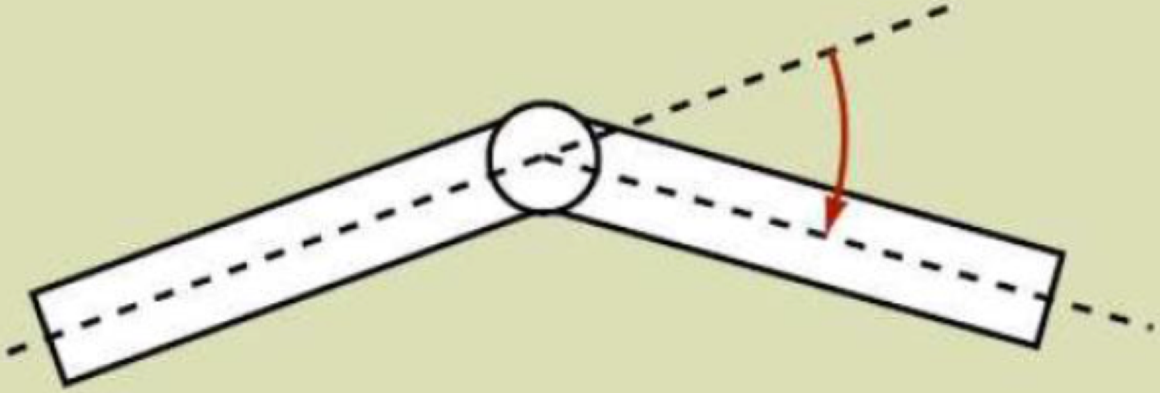
Link : rigid body, 6 degrees of freedom

Joint : connection between two links

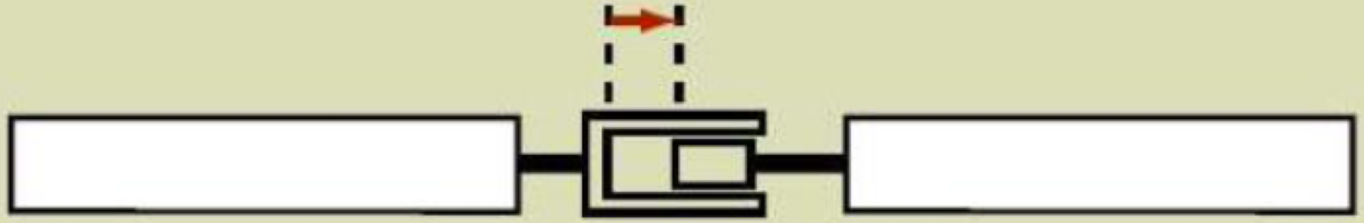
End-effector : interacts with the environment

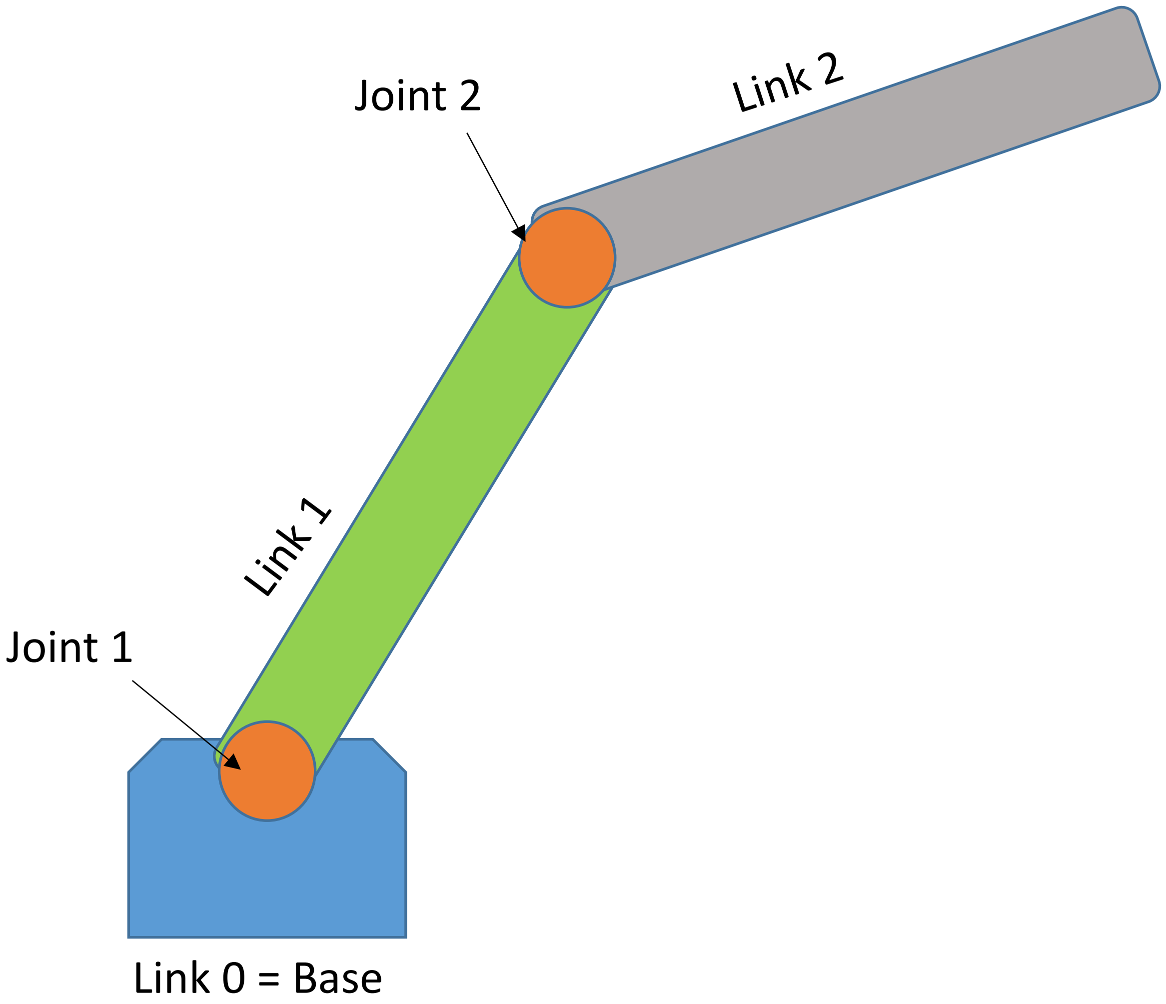
Base : connected to ground

(R)evolute : angular displacement between adjacent links

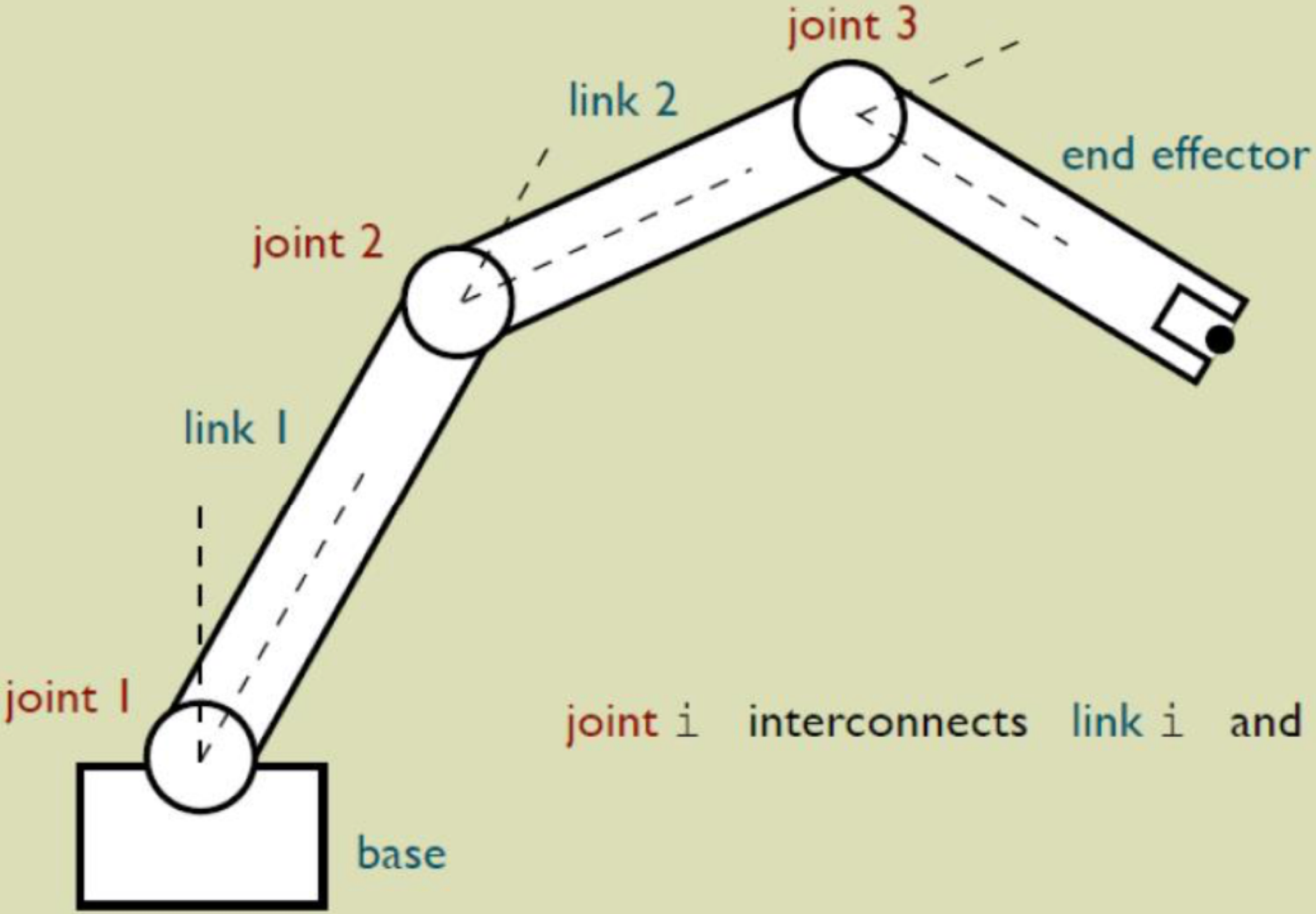


(P)rismatic : linear displacement between adjacent links




Joint 2
Link 2
Link 1
Joint 1
Link 0 = Base

A **kinematic chain** is a system of rigid bodies connected by joints



joint i interconnects link i and link i-1

With the i^{th} joint, we associate a *joint variable*, denoted by q_i. In the case of a revolute joint, q_i is the angle of rotation, and in the case of a prismatic joint, q_i is the joint displacement:

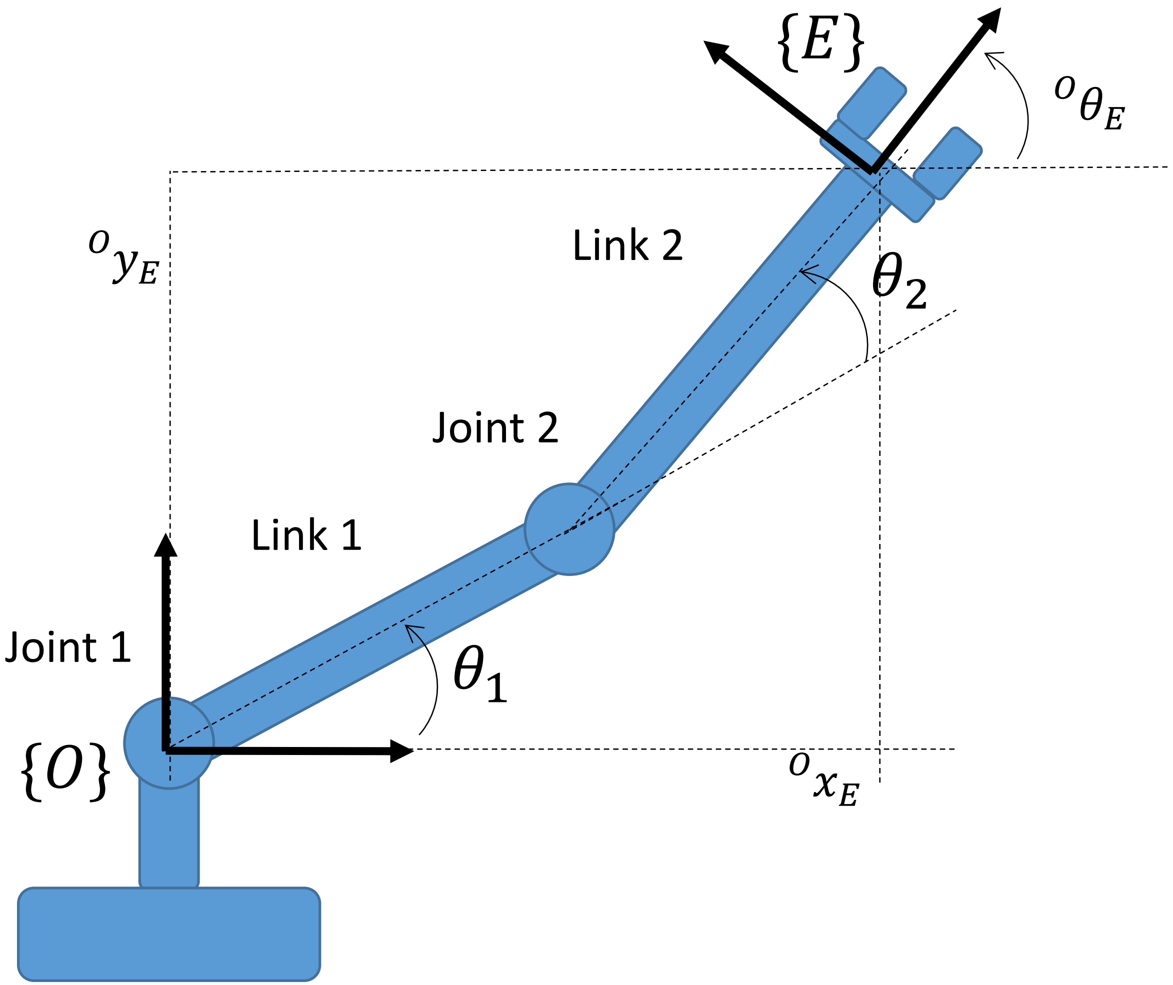
$$q_i = \begin{cases} \theta_i & : \text{ joint i revolute} \\ d_i & : \text{ joint i prismatic} \end{cases} . \tag{3.1}$$

$$^{i-i}T_i = A_i\,(q_i)$$

Homogeneous transformation between frame i-1 and frame i

$$^{i}T_j = \begin{bmatrix} ^{i}R_j & ^{i}t_j \\ 0_{1\times 2} & 1 \end{bmatrix}$$

Joint space and Cartesian space



Joint space

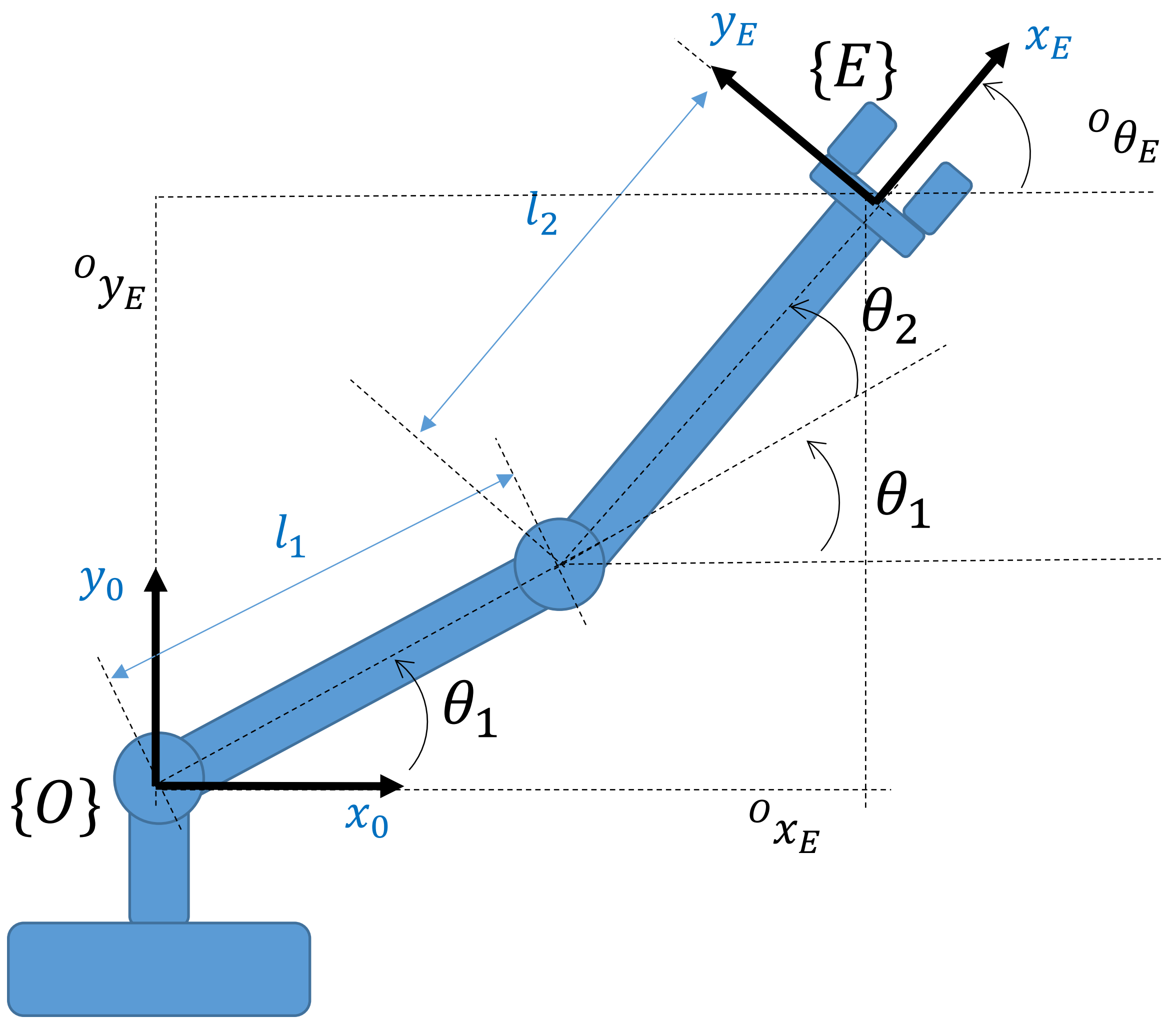
$$\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{S}\times\mathbb{S}$$

$${}^{O}\boldsymbol{\xi}_E = ({}^{O}x_E, {}^{O}x_E, {}^{O}\theta_E) \in SE(3)$$

Cartesian space

End effector pose

Joint space and Cartesian space



Exercise: Detemine end effector pose

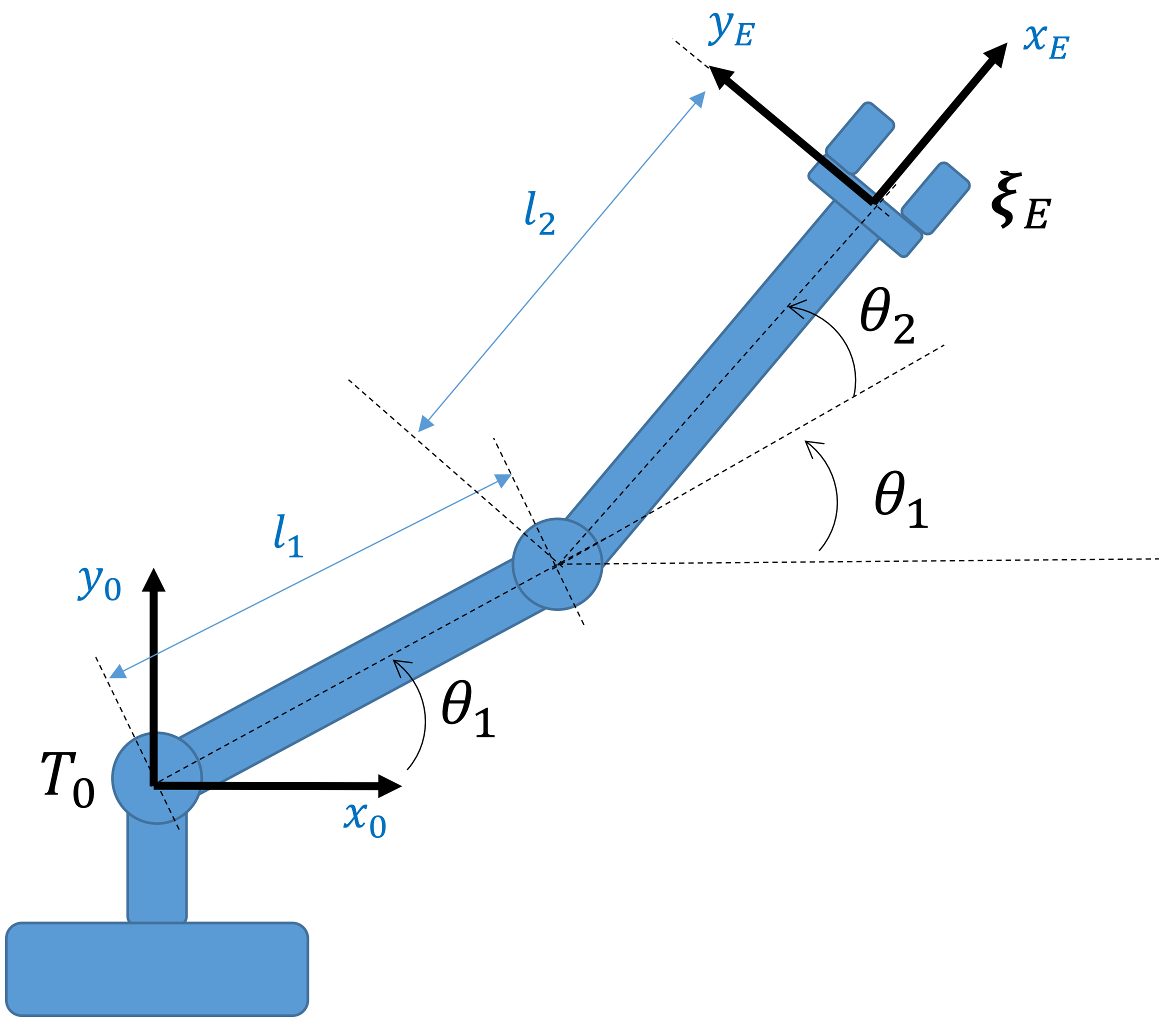
$$ {}^O\boldsymbol{\xi}_E = ({}^O x_E, {}^O y_E, {}^O\theta_E) \in SE(3) $$

$$ {}^O x_E = ? $$

$$ {}^O y_E = ? $$

$$ {}^O\theta_E = ? $$

Joint space and Cartesian space



$$ {}^{O}\xi_E = ({}^{O}x_E, {}^{O}y_E, {}^{O}\theta_E) $$

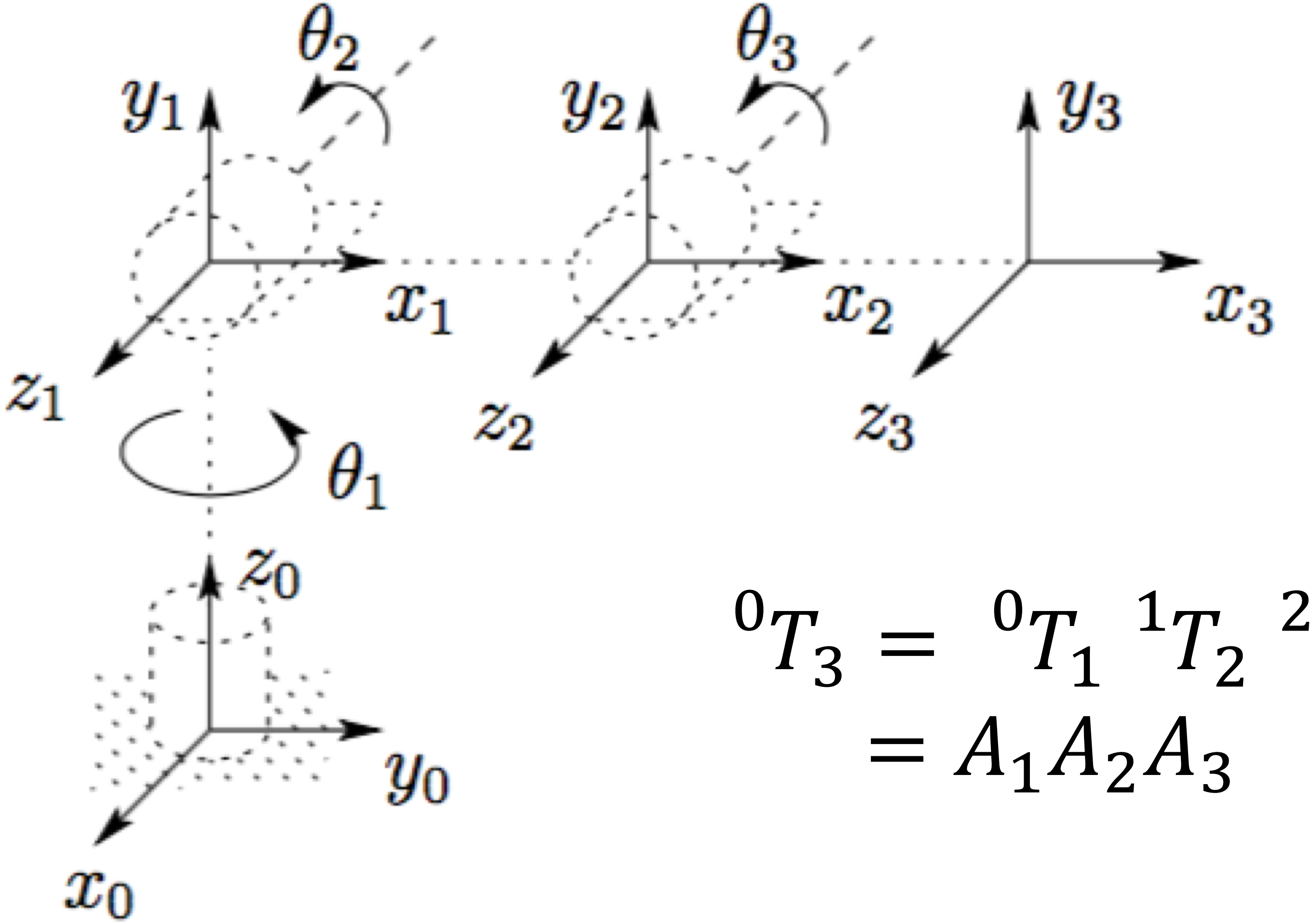
$$ {}^{O}x_E = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) $$

$$ {}^{O}y_E = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) $$

$$ {}^{O}\theta_E = \theta_1 + \theta_2 $$

Forward kinematics

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad \Longrightarrow \quad {}^{\boldsymbol{O}}\boldsymbol{\xi}_E = \begin{bmatrix} {}^{O}x_E \\ {}^{O}y_E \\ {}^{O}\theta_E \end{bmatrix} = \begin{bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{bmatrix}$$



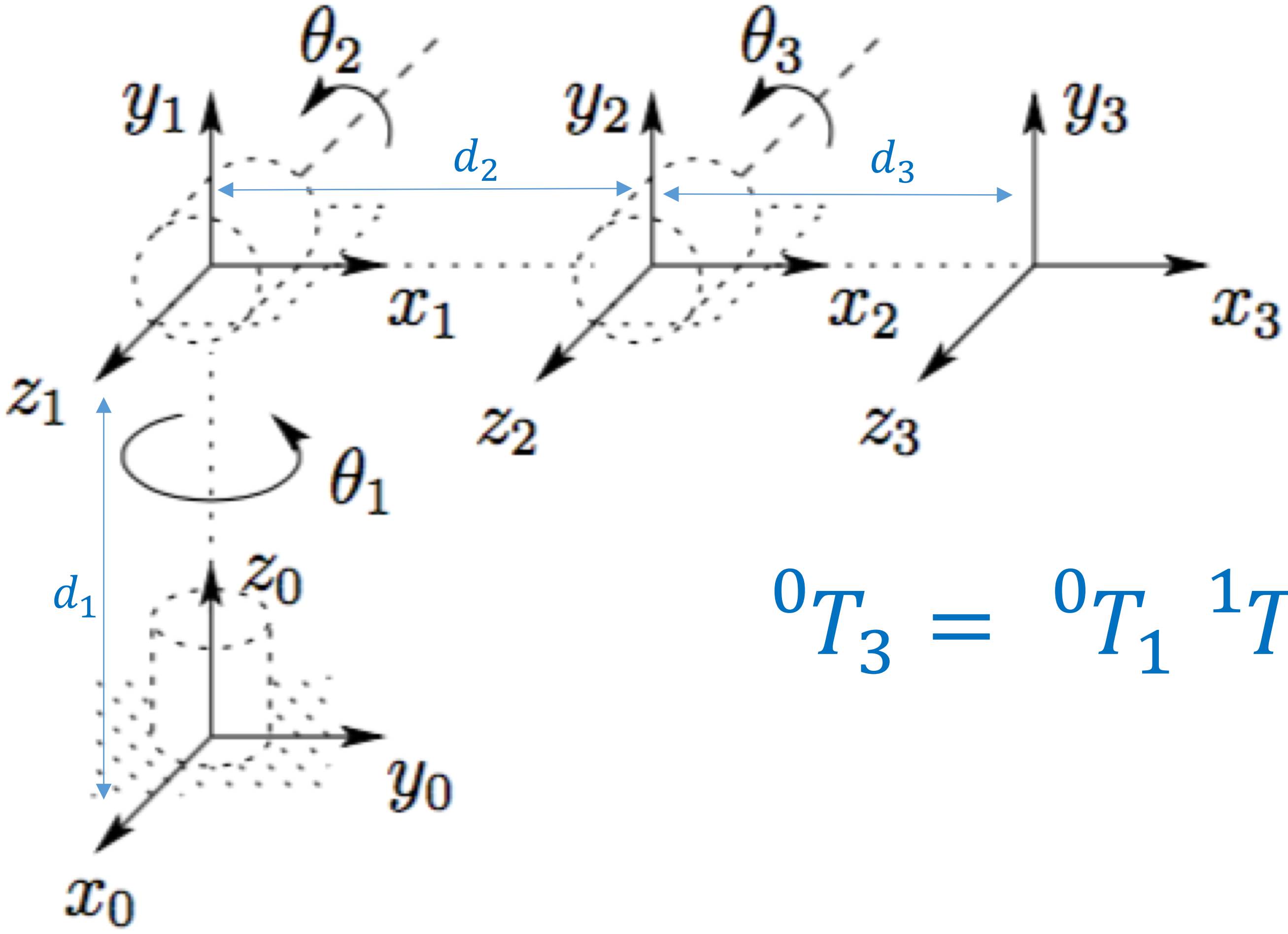
$$
\begin{aligned}
{}^{0}T_3 &= {}^{0}T_1\ {}^{1}T_2\ {}^{2}T_3 \\
&= A_1 A_2 A_3
\end{aligned}
$$

Rotation z-axis
Translation x-axis

$$ {}^{1}T_2 = Rot_{z,\theta_2} Trans_{x,d_2} $$

Rotation z-axis
Translation x-axis

$$ {}^{2}T_3 = Rot_{z,\theta_3} Trans_{x,d_3} $$



Rotation z-axis
Translation z-axis

$$ {}^{0}T_1 = Rot_{z,\theta_1} Trans_{z,d_1} $$

$$ {}^{0}T_3 = {}^{0}T_1 \, {}^{1}T_2 \, {}^{2}T_3 $$

Denavit Hartenberg (DH) Representation

- Z-Axis : Aligned with rotation axis / Prismatic joint direction
- DH1: The axis x_{i+1} is perpendicular to the axis z_i
- DH2: The axis x_{i+1} intersects the axis z_i

Denavit Hartenberg (DH) Representation

Transforms along Z axis	Joint angle	θ_j	the angle between the x_{j-1} and x_j axes about the z_{j-1} axis
	Link offset	d_j	the distance from the origin of frame $j-1$ to the x_j axis along the z_{j-1} axis
Transforms along X axis	Link length	a_j	the distance between the z_{j-1} and z_j axes along the x_j axis; for intersecting axes is parallel to $\hat{z}_{j-1} \times \hat{z}_j$
	Link twist	α_j	the angle from the z_{j-1} axis to the z_j axis about the x_j axis

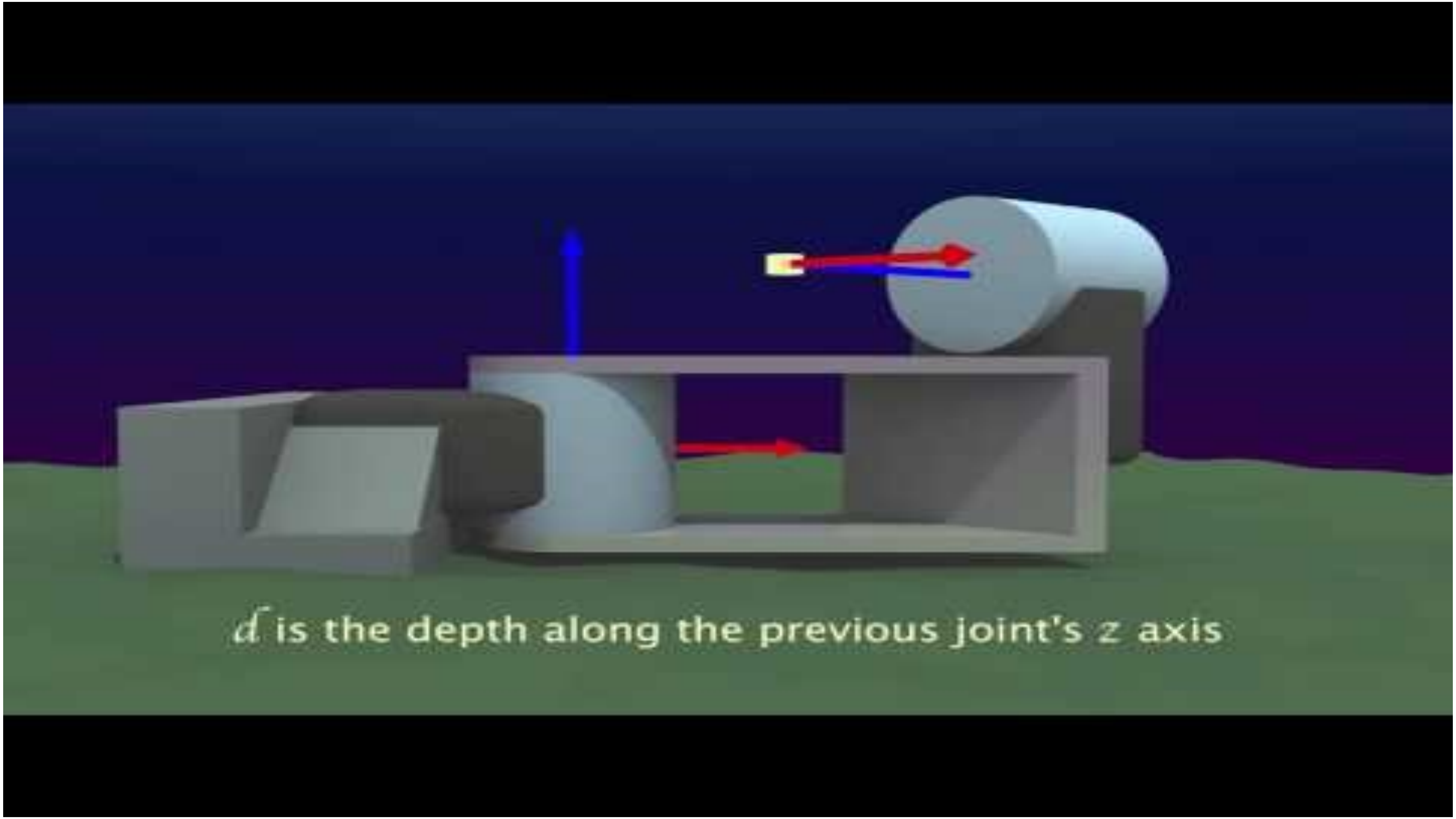
DH uses 4 parameters to define a transform

Denavit Hartenberg (DH) Representation

Joint angle	Link offset	Link length	Link twist
θ_i	d_i	a_i	α_i

$$
\begin{aligned}
A_i &= Rot_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}Rot_{x,\alpha_i} \qquad (3.10)\\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

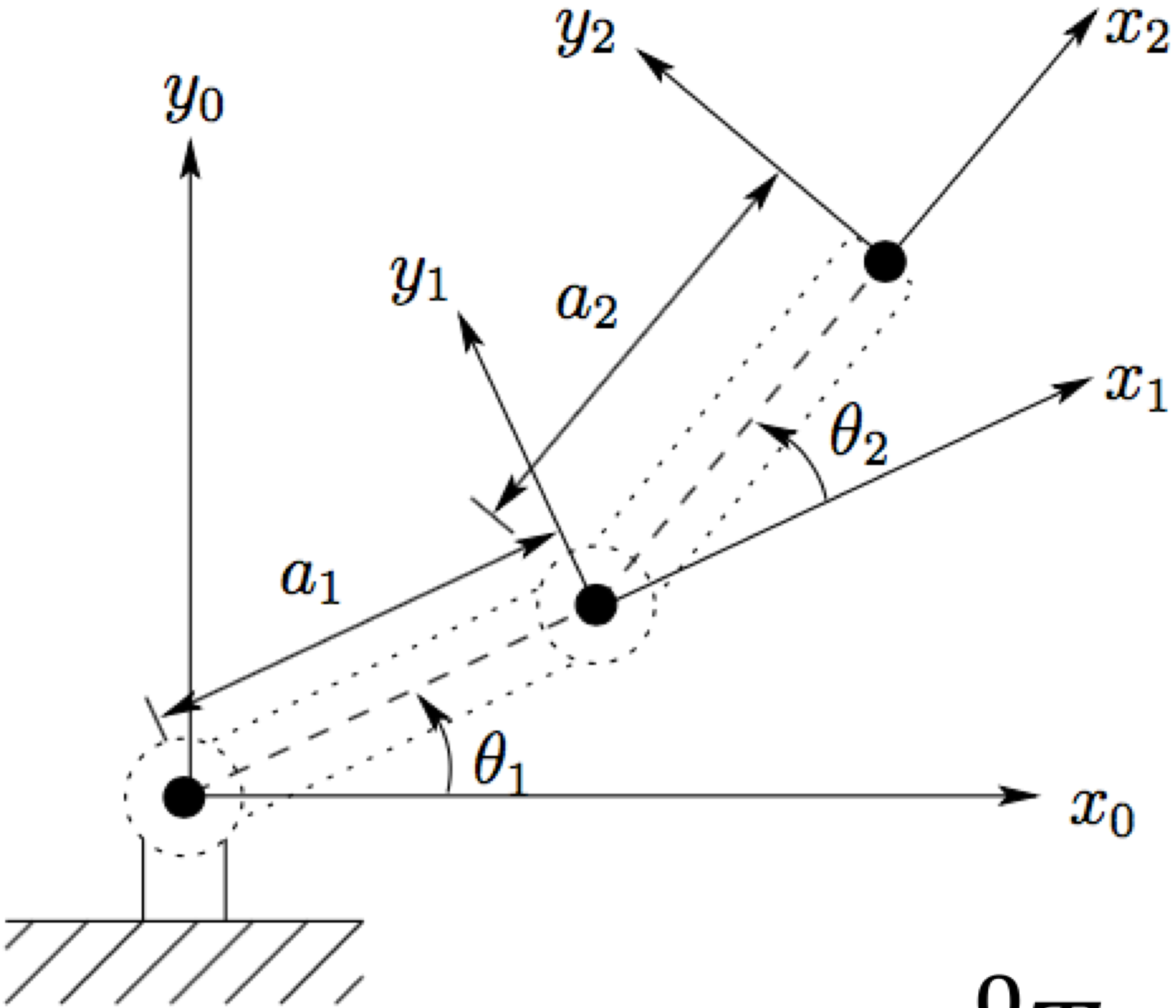
$$
A_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



https://youtu.be/rA9tm0gTln8

$$ {}^{j-1}\mathbf{A}_j(\theta_j, d_j, a_j, \alpha_j) = \boldsymbol{T}_{Rz}(\theta_j)\boldsymbol{T}_z(d_j)\boldsymbol{T}_x(a_j)\boldsymbol{T}_{Rx}(\alpha_j) $$

$$ A_i = \begin{bmatrix} \mathrm{c}_{\theta_i} & -\mathrm{s}_{\theta_i} c_{\alpha_i} & \mathrm{s}_{\theta_i} s_{\alpha_i} & a_i \mathrm{c}_{\theta_i} \\ \mathrm{s}_{\theta_i} & \mathrm{c}_{\theta_i} c_{\alpha_i} & -\mathrm{c}_{\theta_i} s_{\alpha_i} & a_i \mathrm{s}_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

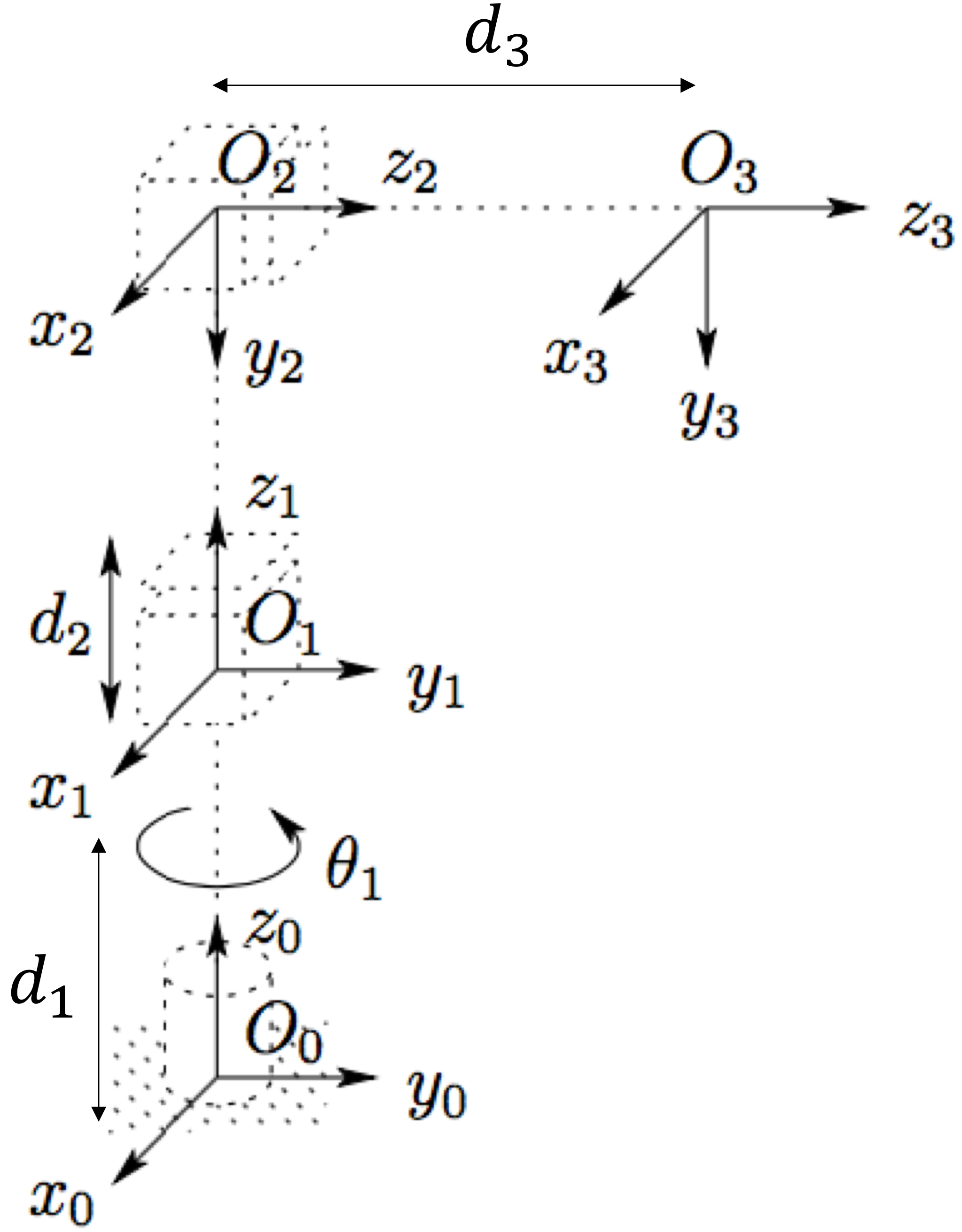


Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

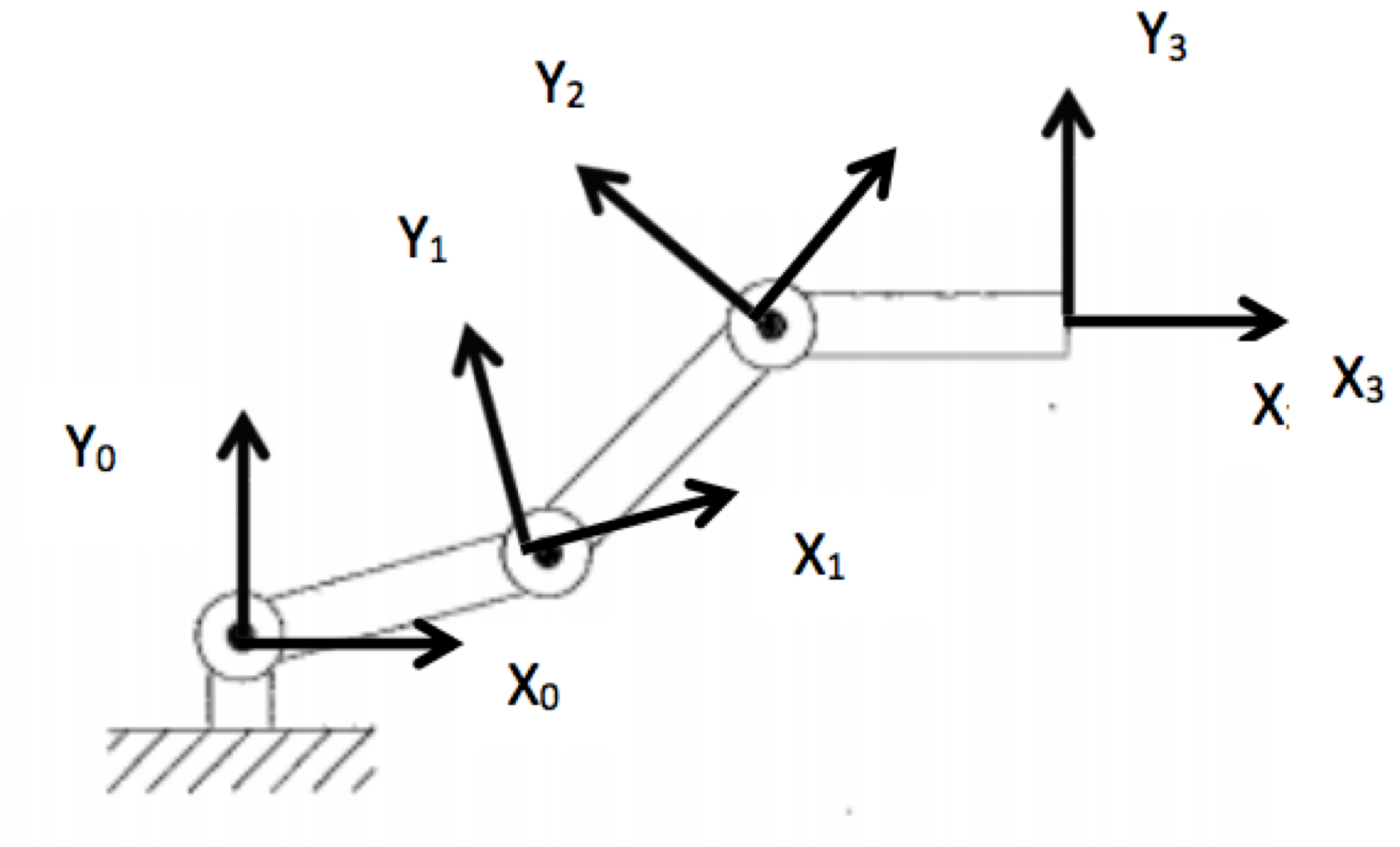
Exercise: Determine ${}^{0}T_{2}$

$$ {}^{0}T_{2} = {}^{0}T_{1}\ {}^{1}T_{2} = A_1 A_2 $$

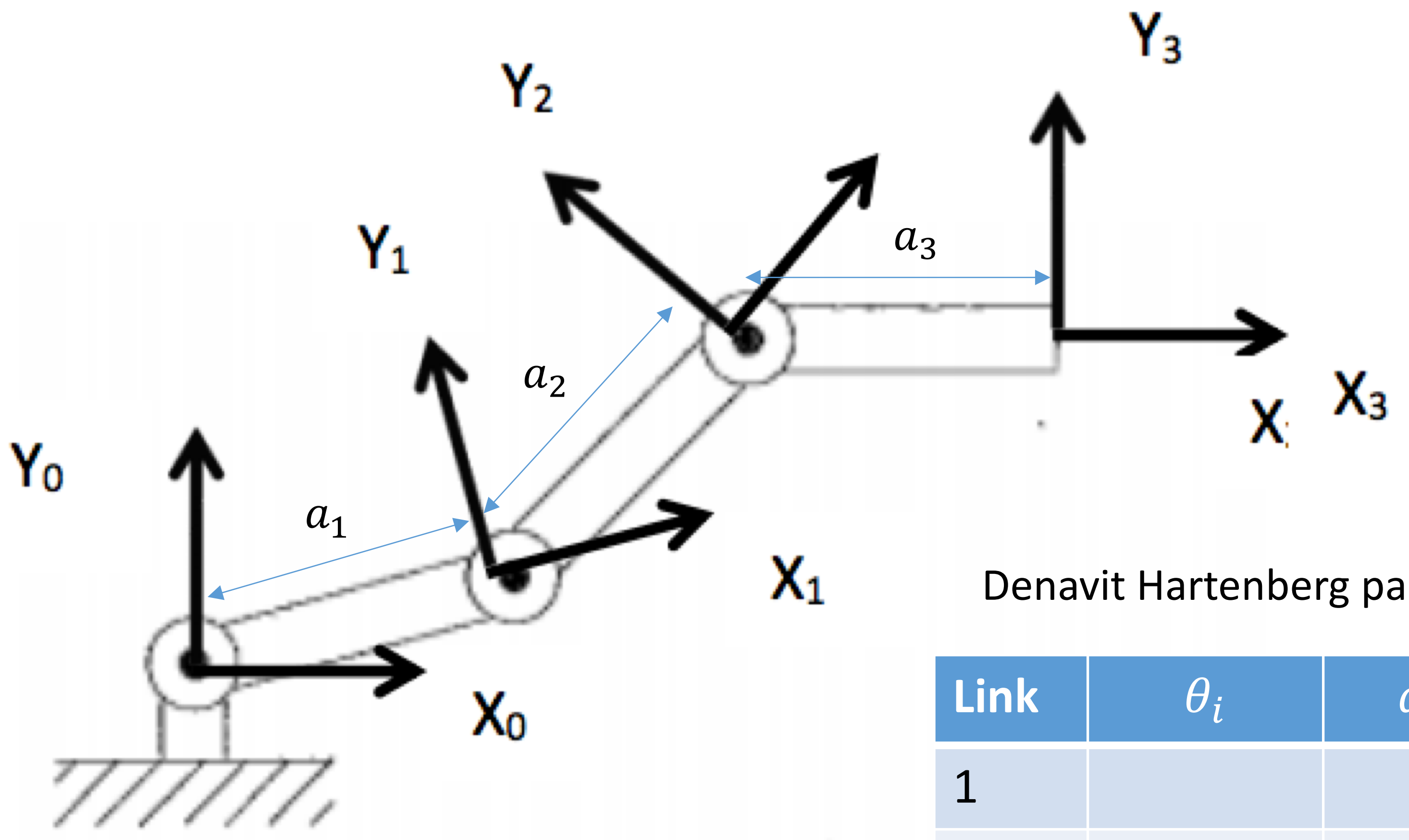
$$ {}^{0}T_{2} = {}^{0}T_{1}\ {}^{1}T_{2} = A_1 A_2 $$



Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1^*
2	0	-90	d_2^*	0
3	0	0	d_3^*	0

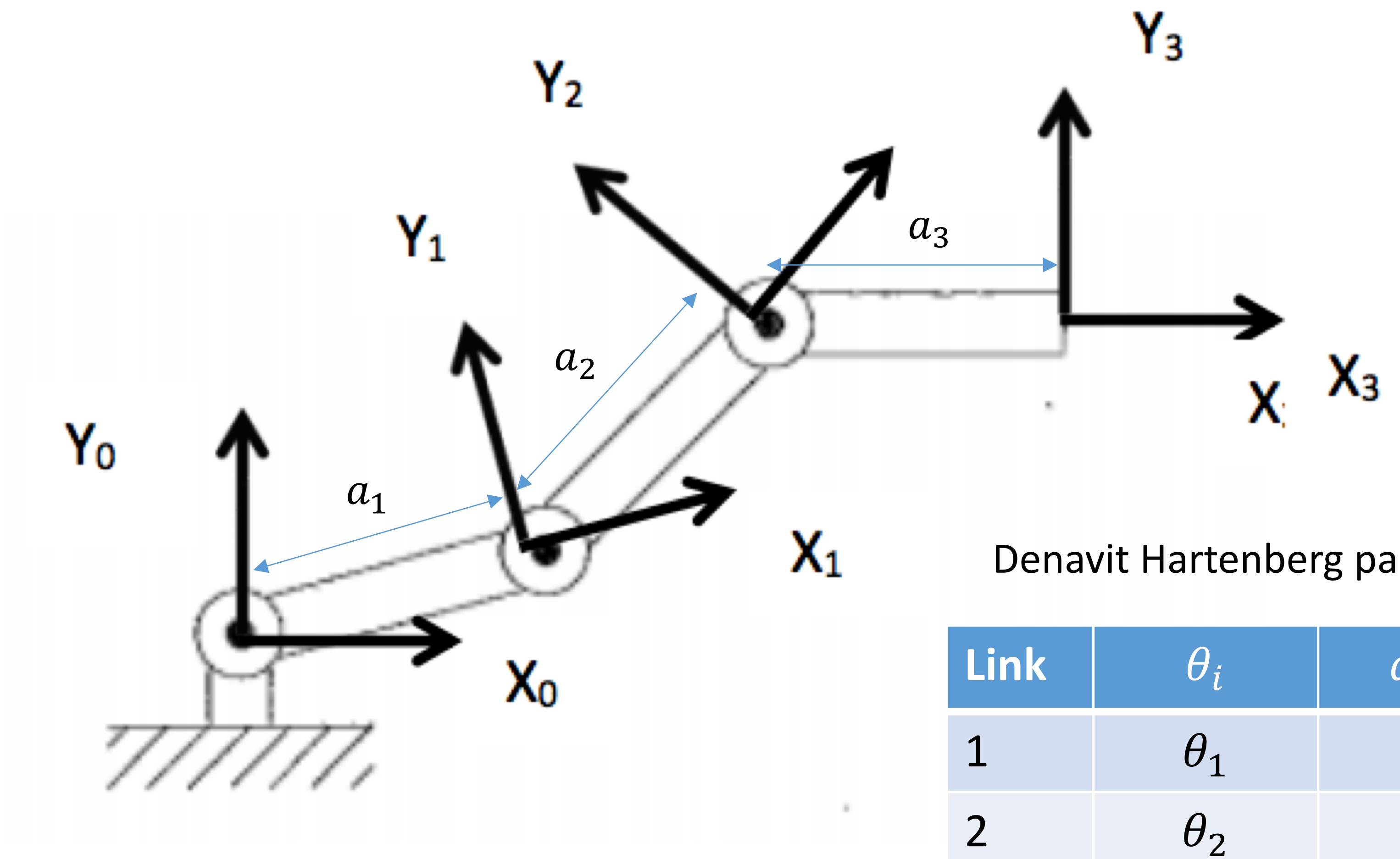


Y_3
Y_2
Y_1
Y_0
X_3
X_1
X_0



Denavit Hartenberg parameters

Link	θ_i	d_i	a_i	α_i
1				
2				
3				



Denavit Hartenberg parameters

Link	θ_i	d_i	a_i	α_i
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0

Denavit Hartenberg parameters

Link	θ_i	d_i	a_i	α_i
1	θ_1	0	a_1	0
2	θ_2	0	a_2	0
3	θ_3	0	a_3	0

$$A_1=\begin{bmatrix} c_{\theta1} & -s_{\theta1} & 0 & a_1c_{\theta1} \\ s_{\theta1} & c_{\theta1} & 0 & a_1s_{\theta1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2=\begin{bmatrix} c_{\theta2} & -s_{\theta2} & 0 & a_2c_{\theta2} \\ s_{\theta2} & c_{\theta2} & 0 & a_2s_{\theta2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3=\begin{bmatrix} c_{\theta3} & -s_{\theta3} & 0 & a_3c_{\theta3} \\ s_{\theta3} & c_{\theta3} & 0 & a_3s_{\theta3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

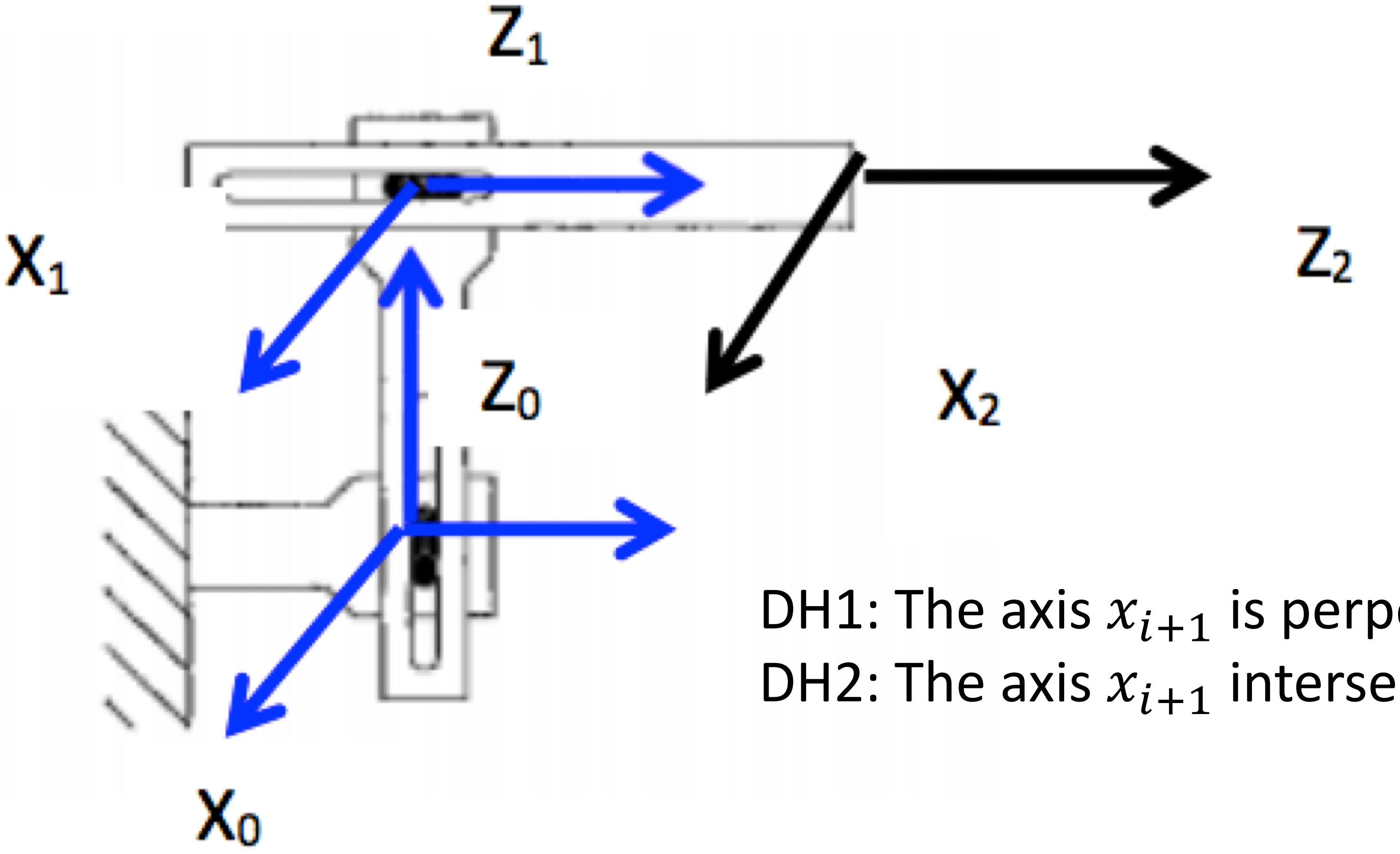
$$^{j-1}\boldsymbol{A}_j=\begin{pmatrix} \cos\theta_j & -\sin\theta_j\cos\alpha_j & \sin\theta_j\sin\alpha_j & a_j\cos\theta_j \\ \sin\theta_j & \cos\theta_j\cos\alpha_j & -\cos\theta_j\sin\alpha_j & a_j\sin\theta_i \\ 0 & \sin\alpha_j & \cos\alpha_j & d_j \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_1=\begin{bmatrix} c_{\theta 1} & -s_{\theta 1} & 0 & a_1 c_{\theta 1} \\ s_{\theta 1} & c_{\theta 1} & 0 & a_1 s_{\theta 1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2=\begin{bmatrix} c_{\theta 2} & -s_{\theta 2} & 0 & a_2 c_{\theta 2} \\ s_{\theta 2} & c_{\theta 2} & 0 & a_2 s_{\theta 2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

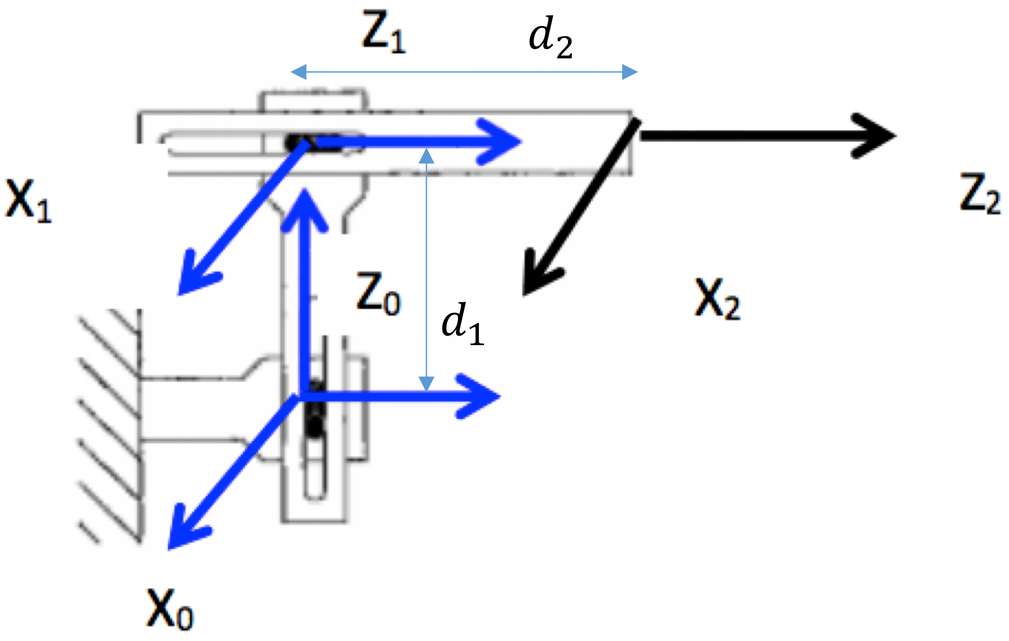
$$A_3=\begin{bmatrix} c_{\theta 3} & -s_{\theta 3} & 0 & a_3 c_{\theta 3} \\ s_{\theta 3} & c_{\theta 3} & 0 & a_3 s_{\theta 3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}T_3 = A_1 A_2 A_3 = \begin{bmatrix} c_{\theta 123} & -s_{\theta 123} & 0 & a_1 c_{\theta 1} + a_2 c_{\theta 12} + a_3 c_{\theta 123} \\ s_{\theta 123} & c_{\theta 123} & 0 & a_1 s_{\theta 1} + a_2 s_{\theta 12} + a_3 s_{\theta 123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



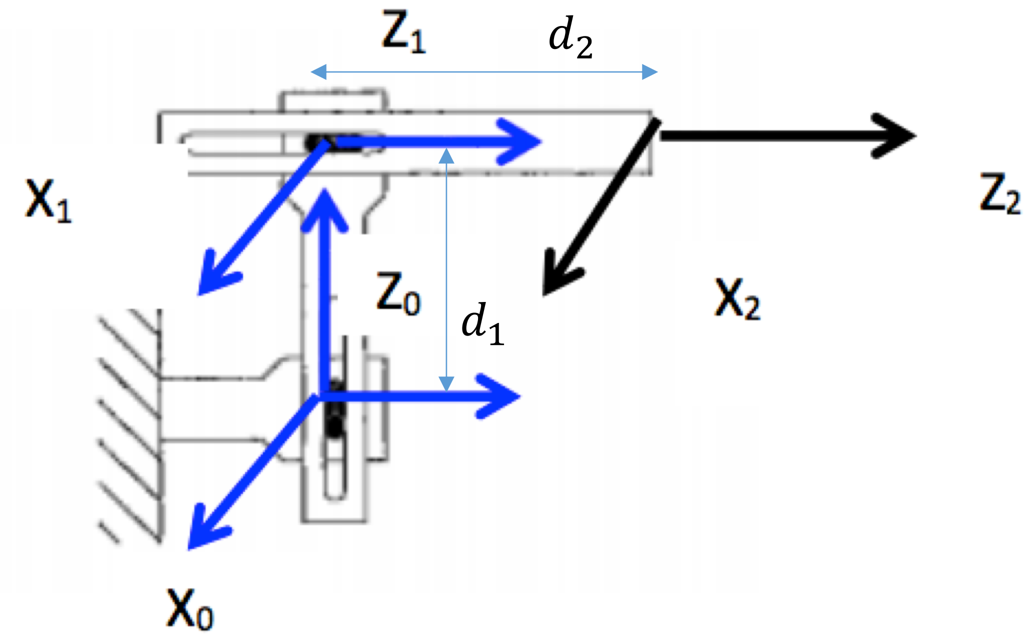
DH1: The axis x_{i+1} is perpendicular to the axis z_i

DH2: The axis x_{i+1} intersects the axis z_i



Denavit Hartenberg parameters

Link	θ_i	d_i	a_i	α_i
1				
2				



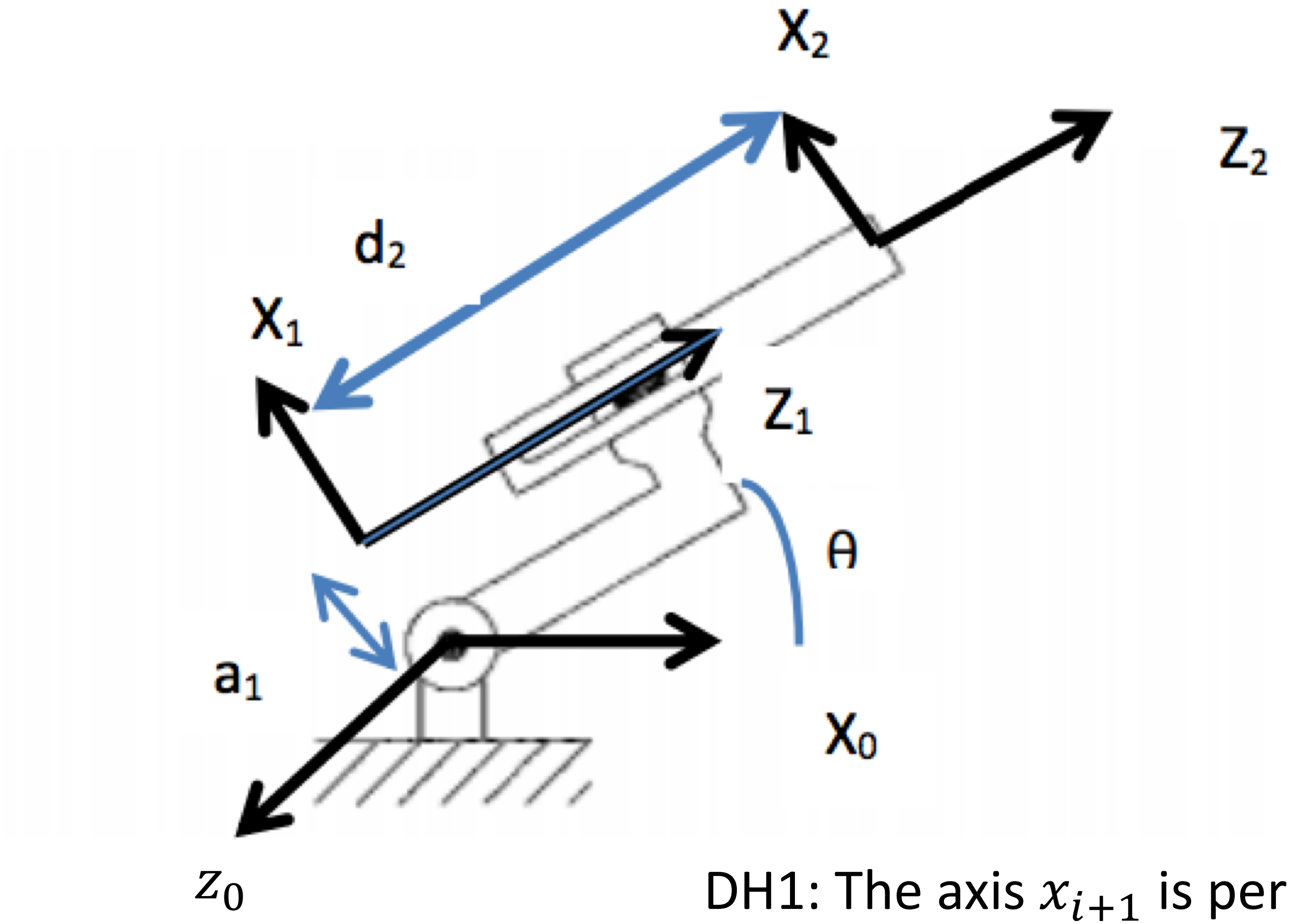
Denavit Hartenberg parameters

Link	θ_i	d_i	a_i	α_i
1	0	d_1	0	$-\frac{\pi}{2}$
2	0	d_2	0	0

$$
{}^{j-1}A_j = \begin{pmatrix} \cos\theta_j & -\sin\theta_j\cos\alpha_j & \sin\theta_j\sin\alpha_j & a_j\cos\theta_j \\ \sin\theta_j & \cos\theta_j\cos\alpha_j & -\cos\theta_j\sin\alpha_j & a_j\sin\theta_i \\ 0 & \sin\alpha_j & \cos\alpha_j & d_j \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

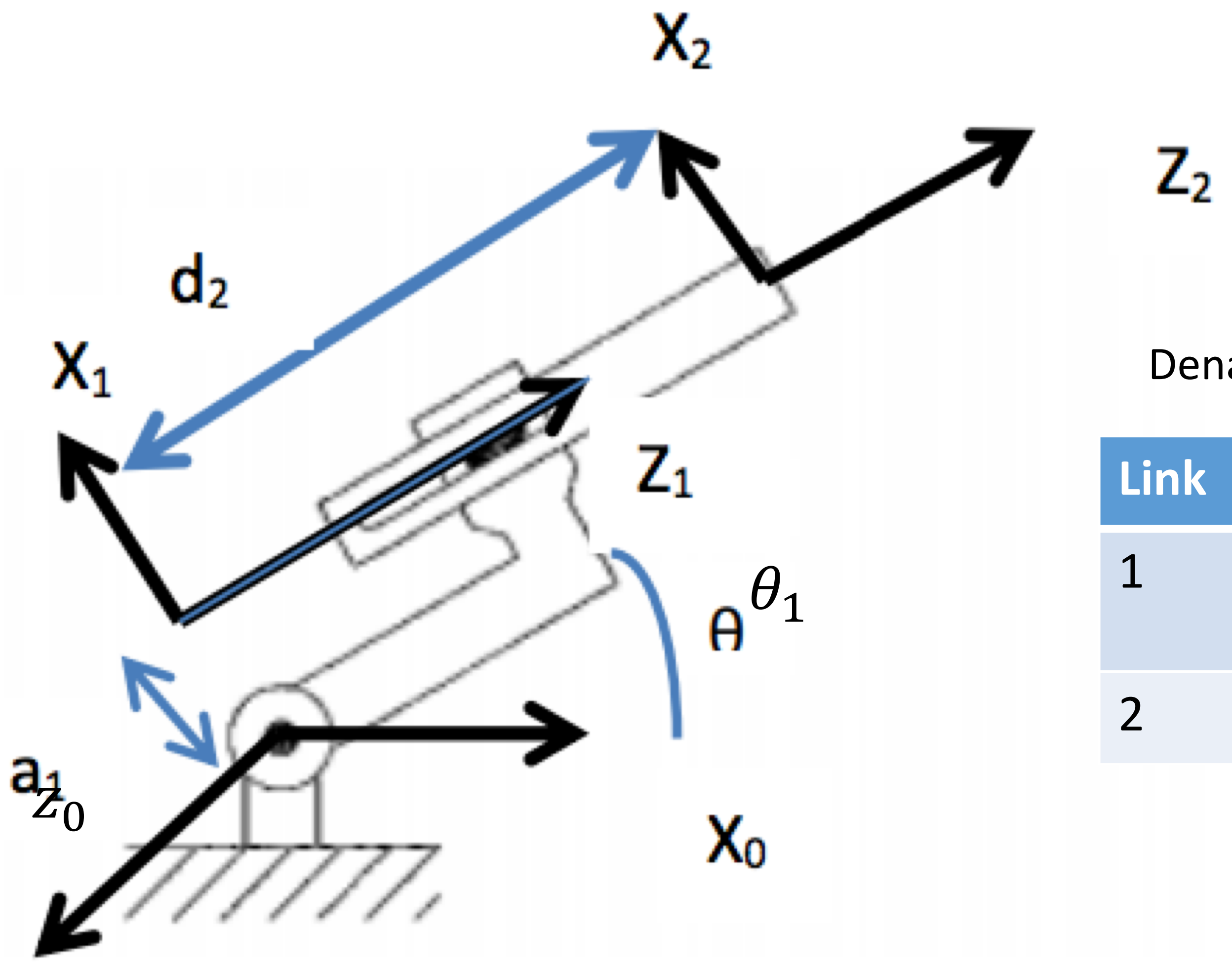
$$
A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



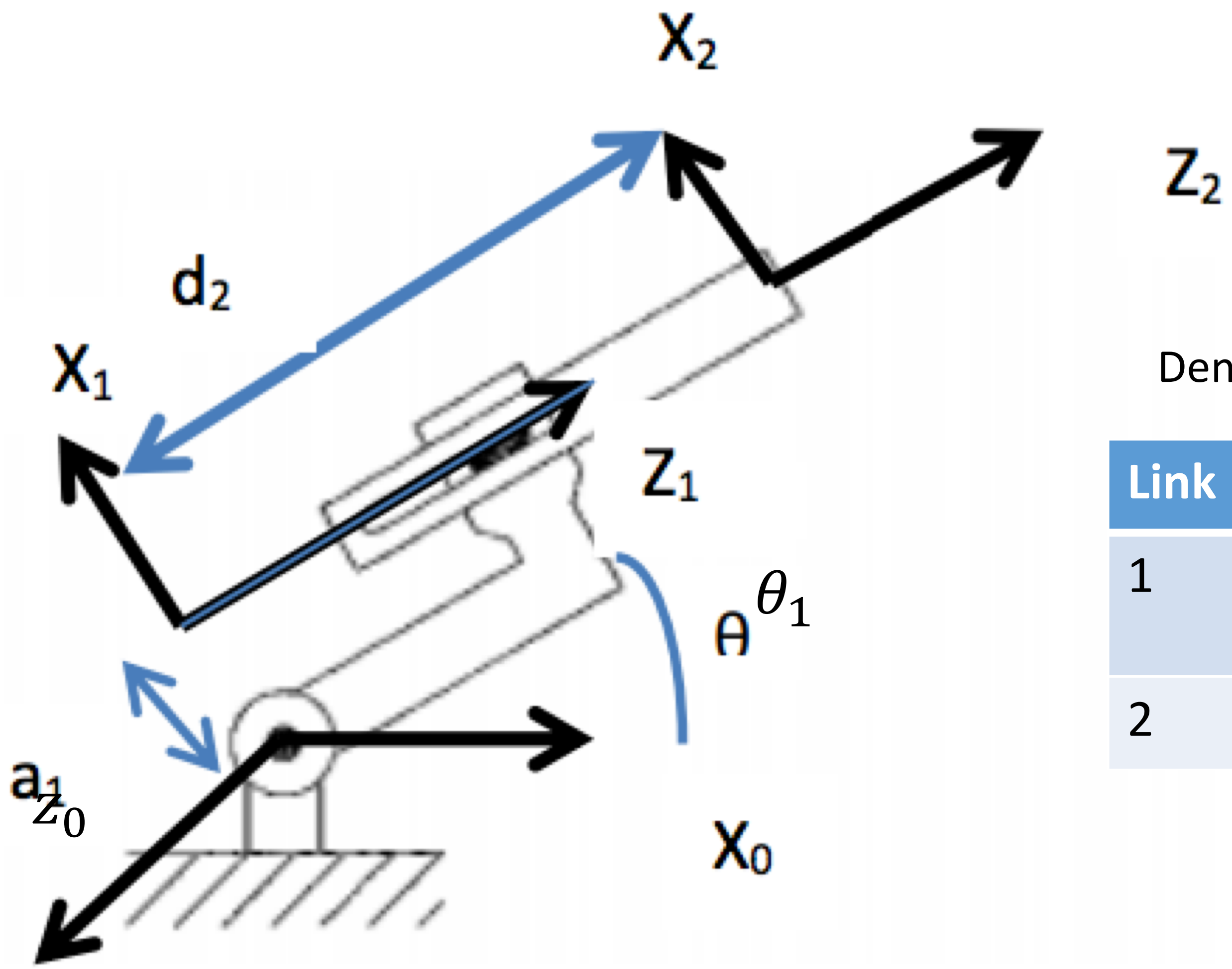
DH1: The axis x_{i+1} is perpendicular to the axis z_i

DH2: The axis x_{i+1} intersects the axis z_i



Denavit Hartenberg parameters

Link	θ_i	d_i	a_i	α_i
1				
2				



Denavit Hartenberg parameters

Link	θ_i	d_i	a_i	α_i
1	$\theta_1 + \frac{\pi}{2}$	0	a_1	$\frac{\pi}{2}$
2	0	d_2	0	0

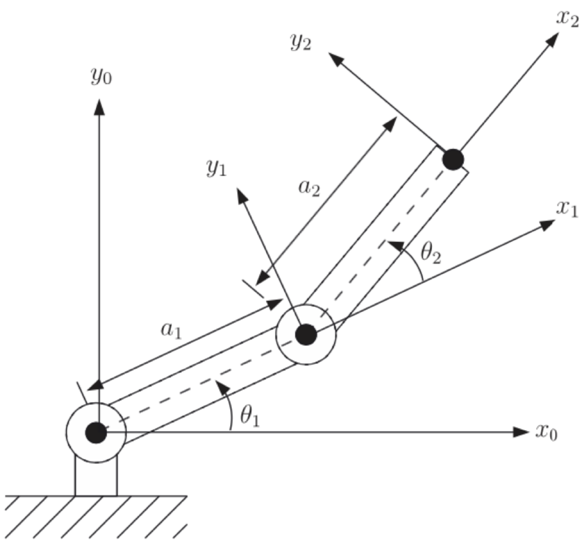
Link	θ_i	d_i	a_i	α_i
1	$\theta_1 + \frac{\pi}{2}$	0	a_1	$\frac{\pi}{2}$
2	0	d_2	0	0

$$A_1 = \begin{bmatrix} c_{\theta+90} & 0 & s_{\theta+90} & a_1 c_{\theta+90} \\ s_{\theta+90} & 0 & -c_{\theta+90} & a_1 s_{\theta+90} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -s_\theta & 0 & c_\theta & -a_1 s_\theta \\ c_\theta & 0 & s_\theta & a_1 c_\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = A_1 A_2 = \begin{bmatrix} -s_\theta & 0 & c_\theta & d_2 c_\theta - a_1 s_\theta \\ c_\theta & 0 & s_\theta & d_2 s_\theta + a_1 c_\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link	θ_i	d_i	a_i	α_i	σ_i
1	q_1	0	1	0	0
2	q_2	0	1	0	0




```
>> L(1) = Link([0 0 1 0]);
>> L(2) = Link([0 0 1 0]);
>> L
L =
 theta=q1, d=0, a=1, alpha=0 (R,stdDH)
 theta=q2, d=0, a=1, alpha=0 (R,stdDH)
```

which are passed to the constructor `SerialLink`

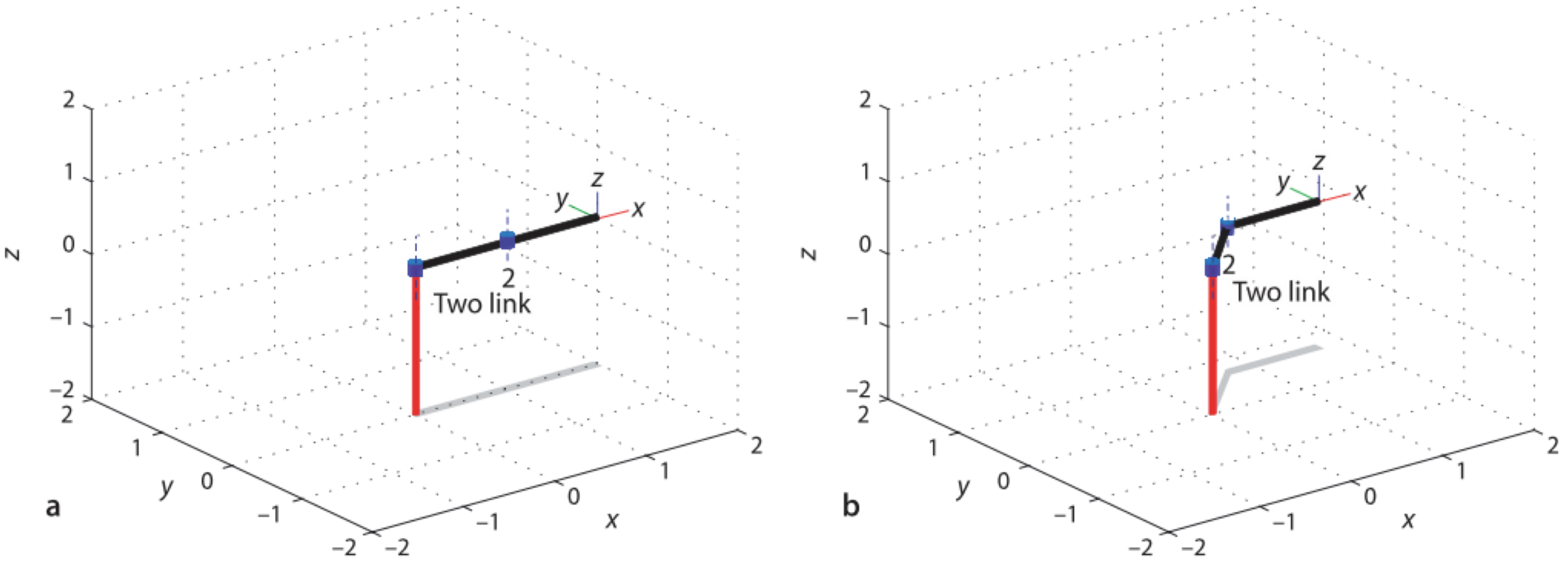
```
>> two_link = SerialLink(L, 'name', 'two link');
```

```
>> mdl_twolink
```

```
>> twolink.fkine([0 0])
ans =
     1     0     0     2
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

the method returns the homogenous transform that represents the pose of the second link coordinate frame of the robot, T_2. For a different configuration the tool pose is

```
>> twolink.fkine([pi/4 -pi/4])
ans =
    1.0000         0         0    1.7071
         0    1.0000         0    0.7071
         0         0    1.0000         0
         0         0         0    1.0000
```




```
>> twolink.plot([0 0])
>> twolink.plot([pi/4 -pi/4])
```

```
>> mdl_puma560
```

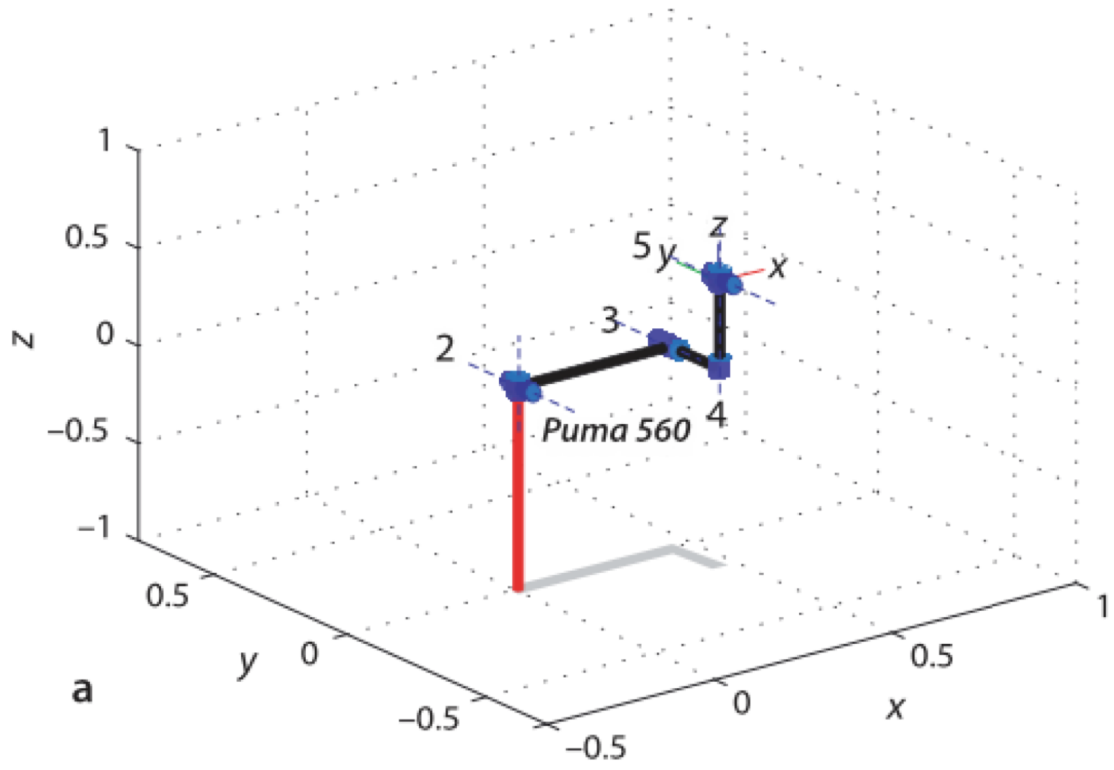
which creates a `SerialLink` object, `p560`, in the workspace

```
>> p560
p560 =
Puma 560 (6 axis, RRRRRR, stdDH)
 Unimation; viscous friction; params of 8/95;
+---+-----------+-----------+-----------+-----------+
| j |     theta |         d |         a |     alpha |
+---+-----------+-----------+-----------+-----------+
|  1|         q1|          0|          0|      1.571|
|  2|         q2|          0|     0.4318|          0|
|  3|         q3|       0.15|     0.0203|     -1.571|
|  4|         q4|     0.4318|          0|      1.571|
|  5|         q5|          0|          0|     -1.571|
|  6|         q6|          0|          0|          0|
+---+-----------+-----------+-----------+-----------+

grav =     0  base = 1  0  0  0   tool =  1  0  0  0
           0         0  1  0  0           0  1  0  0
        9.81         0  0  1  0           0  0  1  0
                     0  0  0  1           0  0  0  1
```

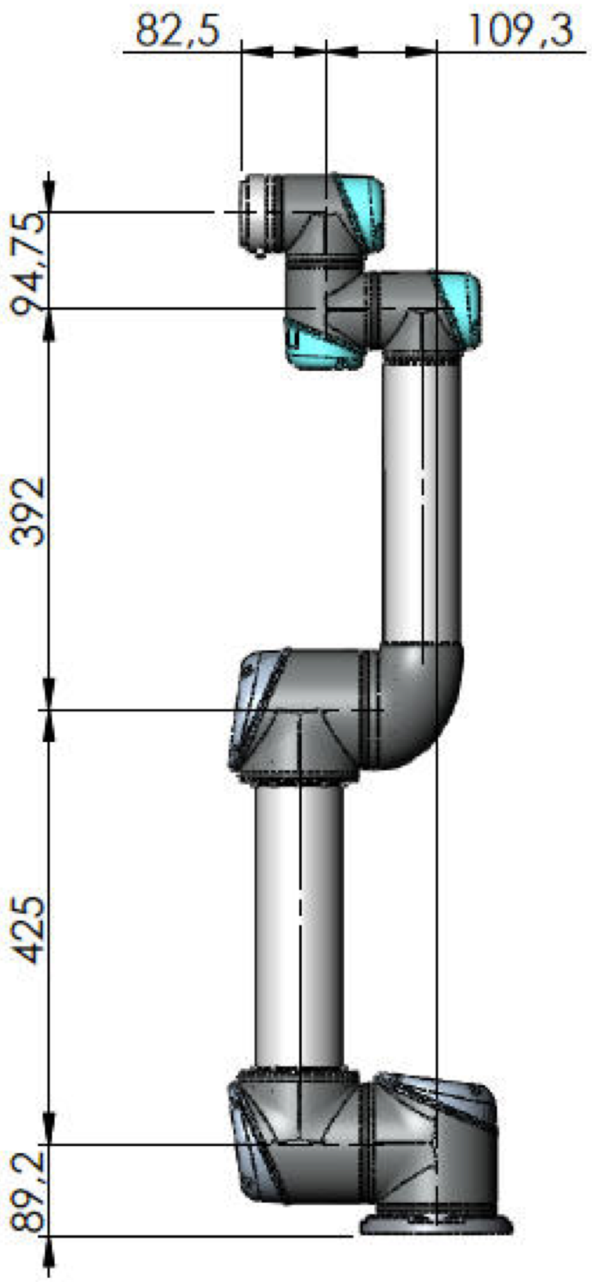
`qz`	$(0, 0, 0, 0, 0, 0)$	*zero angle*
`qr`	$(0, \frac{\pi}{2}, -\frac{\pi}{2}, 0, 0, 0)$	*ready*, the arm is straight and vertical
`qs`	$(0, 0, -\frac{\pi}{2}, 0, 0, 0)$	*stretch*, the arm is straight and horizontal
`qn`	$(0, \frac{\pi}{4}, -\pi, 0, \frac{\pi}{4}, 0)$	*nominal*, the arm is in a dextrous working pose

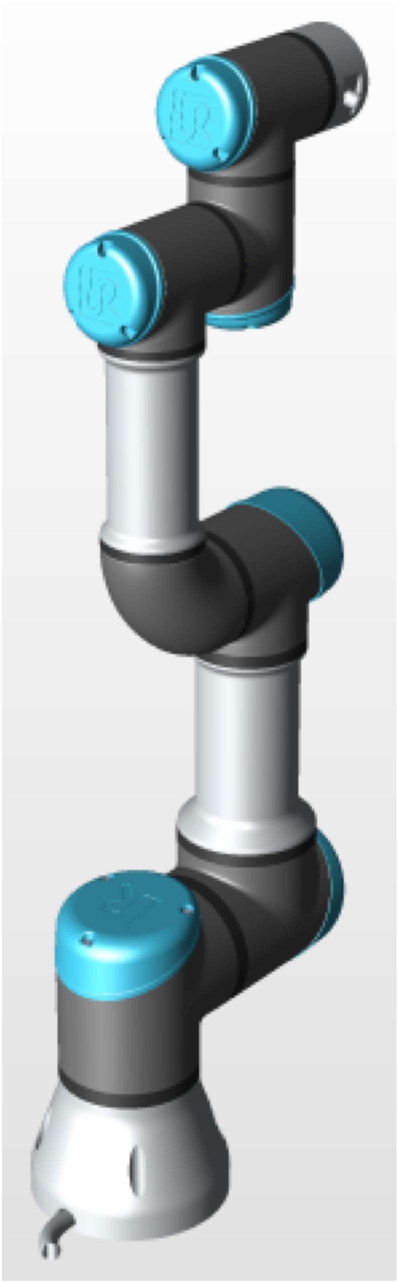
```
>> p560.plot(qz)
```



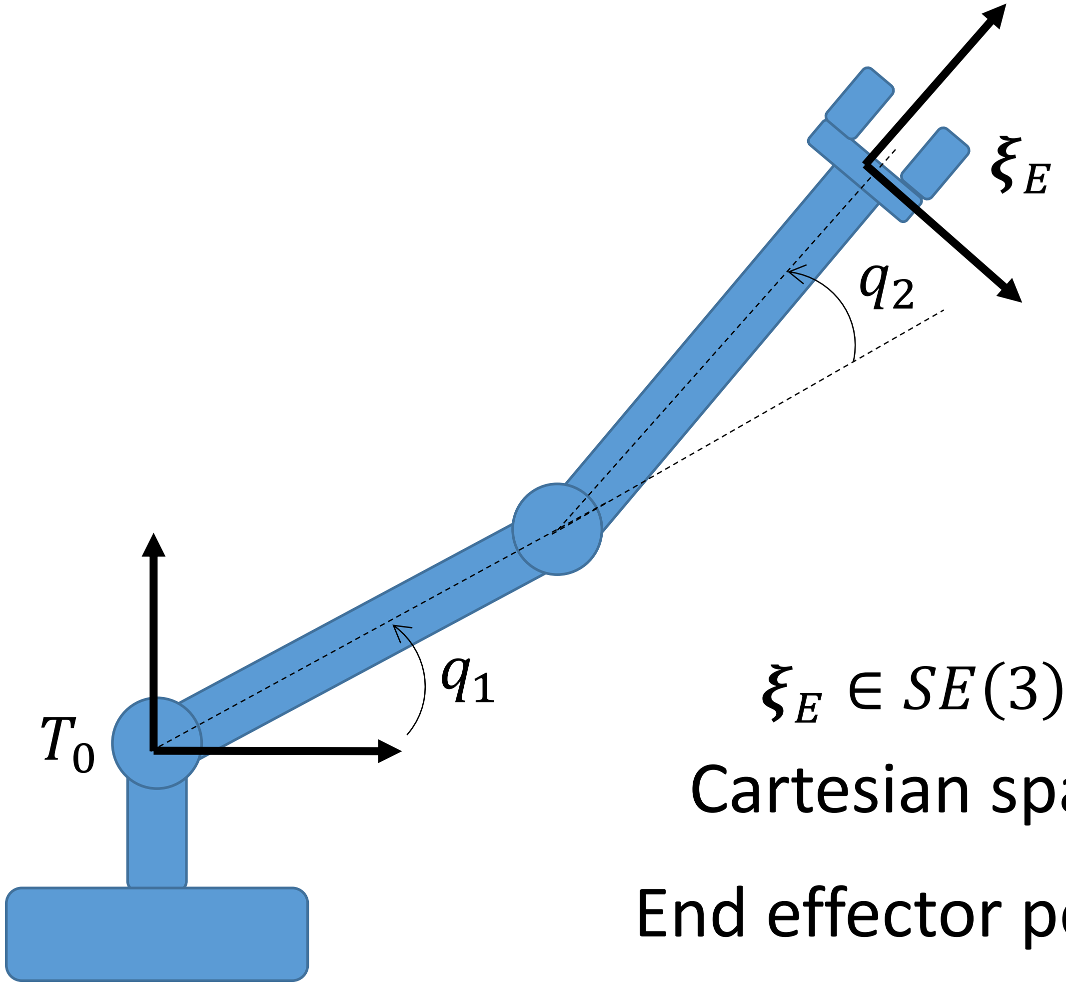
a

```
>> p560.fkine(qz)
ans =
    1.0000         0         0    0.4521
         0    1.0000         0   -0.1500
         0         0    1.0000    0.4318
         0         0         0    1.0000
```


82,5
109,3
94,75
392
425
89,2



Inverse kinematics



$$q = \mathcal{K}^{-1}(\boldsymbol{\xi}_E)$$

Inverse kinematics

$\boldsymbol{\xi}_E \in SE(3)$

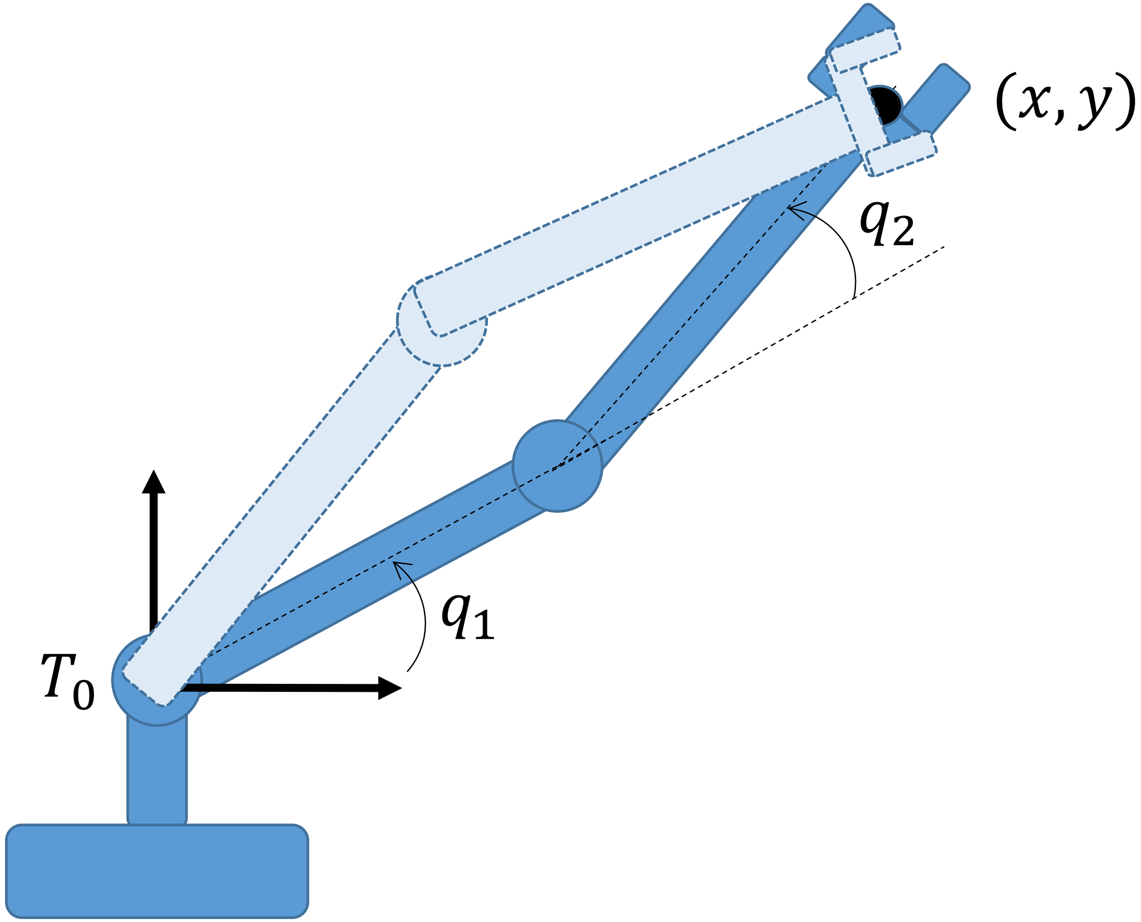
Cartesian space

End effector pose

$\boldsymbol{\theta} = (\theta_1, \theta_2)$

Joint space

Inverse kinematics



$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

$$x^2 = l_1^2 \cos^2 \ (\theta_1) + l_2^2 \cos^2(\theta_1 + \theta_2) + l_1 l_2 \cos(\theta_1) \cos(\theta_1 + \theta_2)$$

$$y^2 = l_1^2 \sin^2 \ (\theta_1) + l_2^2 \sin^2(\theta_1 + \theta_2) + l_1 l_2 \sin(\theta_1) \sin(\theta_1 + \theta_2)$$

$$x^2 + y^2 = l_1^2(\cos^2 \ (\theta_1) + \sin^2 \ (\theta_1)) + l_2^2(\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2))$$

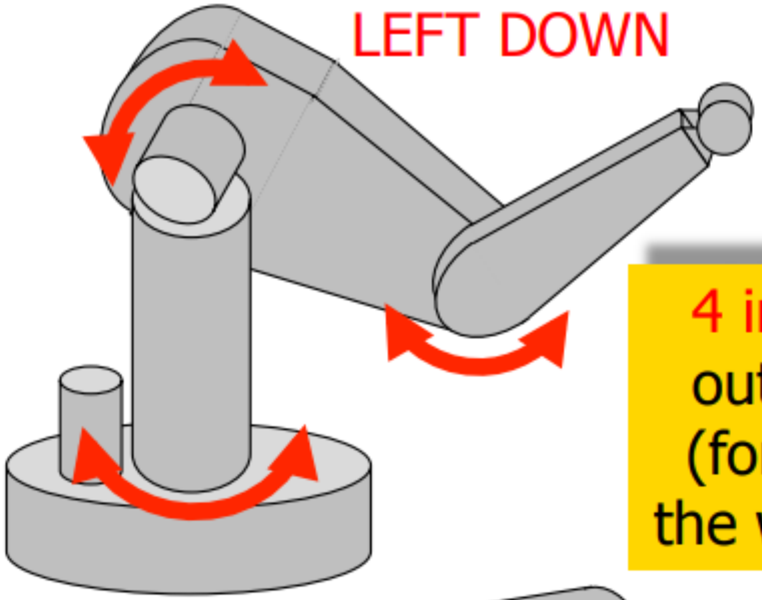
$$+ l_1 l_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) + l_1 l_2 \sin(\theta_1) \sin(\theta_1 + \theta_2)$$

$$x^2 + y^2 = l_1^2 + l_2^2 + +2 l_1 l_2 \cos(\theta_2)$$

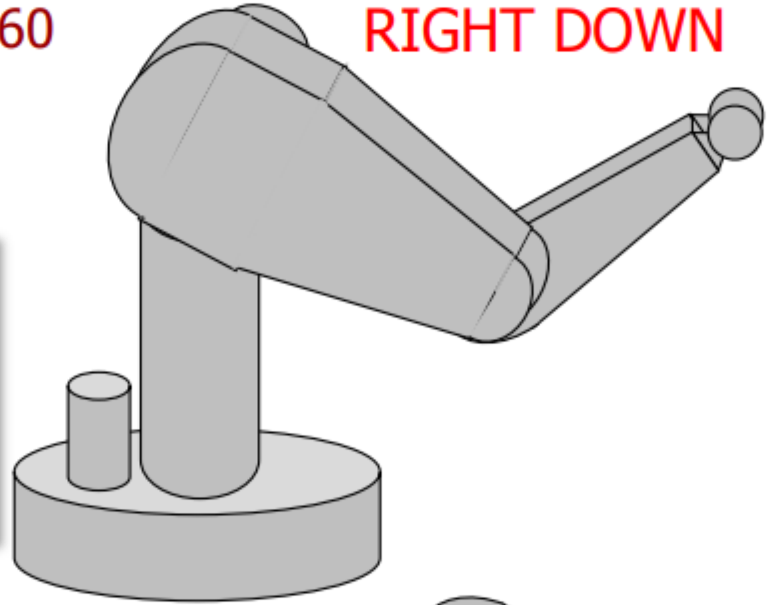
$$\cos(\theta_2) = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\sin(\theta_2) = \pm\sqrt{1 - \cos(\theta_2)^2}$$

Unimation PUMA 560

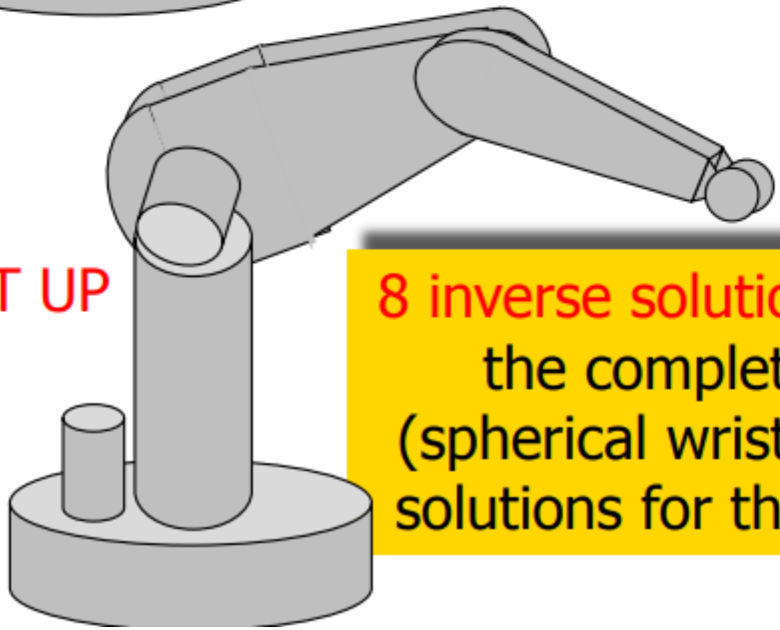


LEFT DOWN

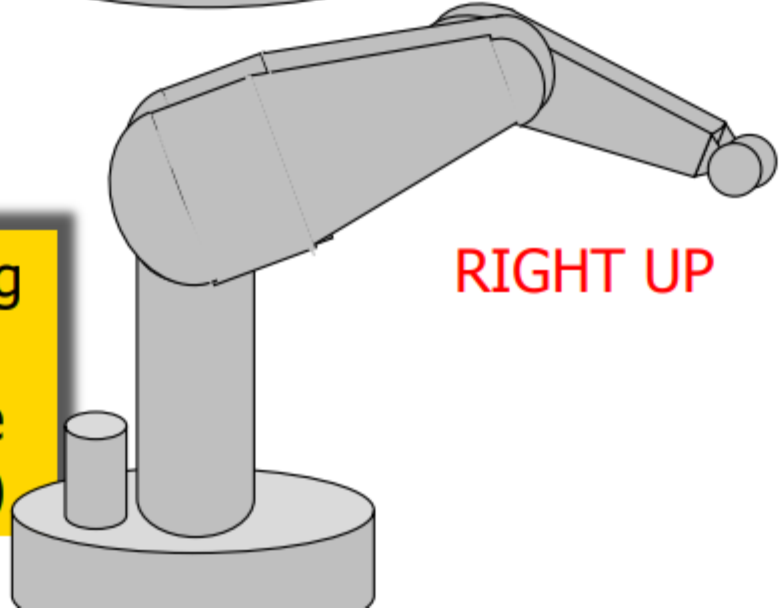


RIGHT DOWN

4 inverse solutions
out of singularities
(for the position of
the wrist center only)



LEFT UP



RIGHT UP

8 inverse solutions considering
the complete E-E pose
(spherical wrist: 2 alternative
solutions for the last 3 joints)

Inverse kinematics

- Closed form solutions / numerical optimization
- Uniqueness of solutions
- Constraints, collisions, etc

Newtons method Inverse Kinematics

Desired cartesian position

$$\xi^*$$

$$\xi^* = \mathcal{K}(\boldsymbol{q}^*)$$

$$\boldsymbol{q}^*??$$

Initial estimate

$$\boldsymbol{q}_k$$

$$\xi_k = \mathcal{K}(\boldsymbol{q}_k)$$

Iteration

$$\boldsymbol{q}_{k+1}$$

$$\xi_{k+1} = \mathcal{K}(\boldsymbol{q}_{k+1})$$

$$\xi^* = \mathcal{K}(\boldsymbol{q}_0 + \Delta\boldsymbol{q}) \approx \mathcal{K}(\boldsymbol{q}_0) + J(\boldsymbol{q}_0)\Delta\boldsymbol{q}$$

Jacobian

$$q_{k+1} = q_k + J^{-1}(q_k)(\xi^* - \xi_k)$$

$$\xi^* = \mathcal{K}(\boldsymbol{q}_k + \Delta\boldsymbol{q}) \approx \mathcal{K}(\boldsymbol{q}_k) + J(\boldsymbol{q}_k)\Delta\boldsymbol{q}$$

$$\Delta\boldsymbol{q} = \boldsymbol{q}_{k+1} - \boldsymbol{q}_k$$

$$\boldsymbol{q}_{k+1} = \boldsymbol{q}_k + \Delta\boldsymbol{q}$$

$$\xi^* = \xi_k + J(\boldsymbol{q}_k)\Delta\boldsymbol{q}$$

$$\xi_k = \mathcal{K}(\boldsymbol{q}_k)$$

$$J(\boldsymbol{q}_k)\Delta\boldsymbol{q} = \xi^* - \xi_k$$

$$\Delta\boldsymbol{q} = J(\boldsymbol{q}_k)^+\big(\xi^* - \mathcal{K}(\boldsymbol{q}_k)\big) = J(\boldsymbol{q}_k)^+(\xi^* - \xi_k)$$

$$\boldsymbol{q}_{k+1} = \boldsymbol{q}_k + \Delta\boldsymbol{q} = \boldsymbol{q}_k + J(\boldsymbol{q}_k)^+(\xi^* - \xi_k)$$

Jacobian

$$x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial \theta_1} & \cdots & \frac{\partial x_1}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial \theta_1} & \cdots & \frac{\partial x_m}{\partial \theta_n} \end{pmatrix}$$

$$J = \begin{pmatrix} -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

Trajectories

- Closed form solutions / numerical optimization
- Uniqueness of solutions
- Constraints, collisions, etc