finite element methods

consider a set of N points:

$$\{P_1, P_2, \dots, P_N\} =: \Lambda$$

and a set of M rods:

$$\{R_1, R_2, \dots, R_M\} =: \Gamma$$

each of these rods is the connection of two points from Λ

the <u>master stiffnes matrix</u> of this finite element system is $F \in \mathrm{Mat}_{2N\times 2N}(\mathbb{R})$ and is of the following form:

$$F = \begin{pmatrix} P_1P_1 & P_1P_2 & P_1P_3 & \cdots & P_1P_N \\ P_2P_1 & P_2P_2 & \cdots & & \\ & & \ddots & & \\ & & & & P_NP_N \end{pmatrix}$$

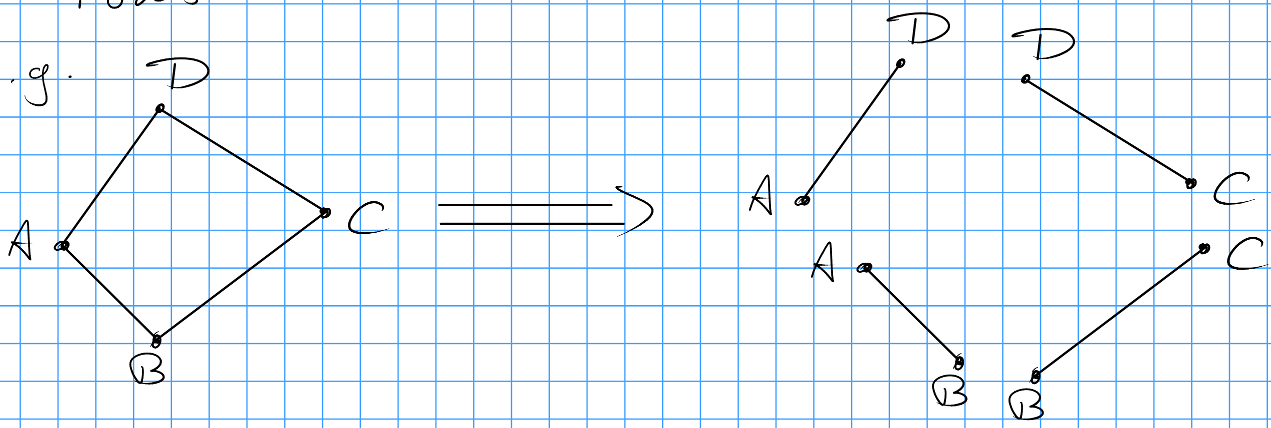
where P_iP_j is a 2×2 matrix

intermission:

as we are only considering small perturbations, we can do the following to calculate the master stiffness equation:

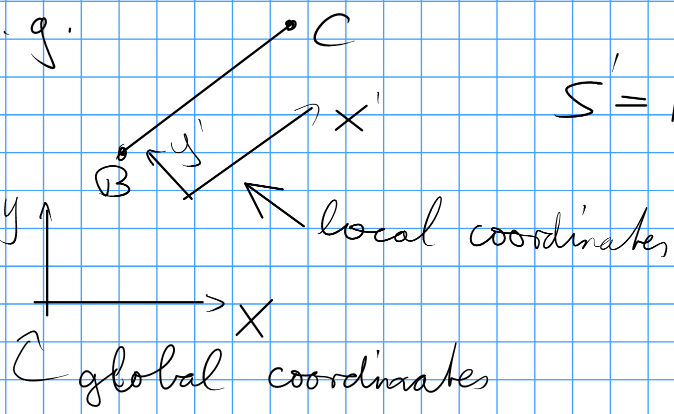
1. break down the system into single rods

e.g.



2. calculate the member stiffness matrix of each rod in its local coordinate system

e.g.



$$S' = K \cdot \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

stiffness coefficient

this matrix solves the following equation in local coordinates:

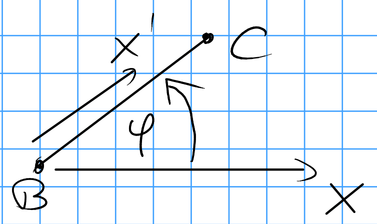
$$\begin{pmatrix} f_x^{B'} \\ f_y^{B'} \\ f_x^{C'} \\ f_y^{C'} \end{pmatrix} = S' \cdot \begin{pmatrix} \delta x_B' \\ \delta y_B' \\ \delta x_C' \\ \delta y_C' \end{pmatrix}$$

3. transform the member stiffness equation to global coordinates

$$S = \left(\begin{array}{c|c} K_{BC} & -K_{BC} \\ \hline -K_{BC} & K_{BC} \end{array}\right)$$

where $K_{BC} = k \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$

angle φ:



$\varphi = \measuredangle(x, x')$

φ = angle between x & x' in positive CCW direction

side-note: the order of the points does not matter as your angle can take up to 2 values:

$$\varphi \quad \text{and} \quad \varphi + \pi$$

fortunatelly: $\sin(\varphi) = -\sin(\varphi + \pi)$
& $\cos(\varphi) = -\cos(\varphi + \pi)$

$$\hookrightarrow K_{BC}(\varphi) = K_{BC}(\varphi + \pi)$$

this makes the later implementation much easier, as the order of the points does not matter

4. include the other points in the displacement - force equation of one rod

e.g. for BC

$$\begin{pmatrix} \vec{f}_A \\ \vec{f}_B \\ \vec{f}_C \\ \vec{f}_D \end{pmatrix} = \underbrace{\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & K_{CB} & -K_{CB} & 0 \\ \hline 0 & -K_{CB} & K_{CB} & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)}_{=: F_{CB}} \begin{pmatrix} \delta \vec{x}_A \\ \delta \vec{x}_B \\ \delta \vec{x}_C \\ \delta \vec{x}_D \end{pmatrix}$$

if there would be a rod between B&D you'll get F_{BD}:

$$\begin{pmatrix} \vec{f}_A \\ \vec{f}_B \\ \vec{f}_C \\ \vec{f}_D \end{pmatrix} = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & K_{BD} & 0 & -K_{BD} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & -K_{BD} & 0 & K_{BD} \end{array}\right) \cdot \begin{pmatrix} \delta \vec{x}_A \\ \delta \vec{x}_B \\ \delta \vec{x}_C \\ \delta \vec{x}_D \end{pmatrix}$$

5. get the master stiffness equation by adding the F_{xx} matrises (see step 4) together

why can we do this?

we are only considering <u>small perturbations</u> that do <u>not change the geometry</u>

$\hookrightarrow$ we are in a linear regime and can simply add displacements and forces

<u>intermission end</u>

From the derivation in the intermission we can find the following algorith to get the master stiffness equation:

$$F = \begin{pmatrix} P_1P_1 & P_1P_2 & P_1P_3 & \cdots & P_1P_N \\ P_2P_1 & P_2P_2 & \cdots & & \vdots \\ P_3P_1 & P_3P_2 & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ P_NP_1 & \cdots & & \cdots & P_NP_N \end{pmatrix}$$

the block matrises P_iP_j are given by:

I if $i \neq j$: 1) P_i and P_j are connected by a rod $\rightarrow$ $P_iP_j = K_{P_iP_j}$ (see step 3 for the form of $K_{P_iP_j}$)

or

2) P_i and P_j are not connected by a rod $\rightarrow$ $P_iP_j = \underline{0}$

zero matrix

II $i = j$: $P_iP_i = \sum\limits_{\substack{\text{all points } l \text{ connected} \\ \text{to } i \text{ via a rod}}} K_{P_iP_l}$