Module 4. Bivariate analysis: qualitative variables

Data Science

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2024–2025

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Contents

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Learning goals

- Dependent/independent variable
- Apply suitable analysis techniques for each combination of measurement levels
- Contingency tables and Cramér's *V*
- Visualization

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Bivariate analysis: overview

Independent	Dependent	Test/Metric
Qualitative	Qualitative	χ^2-test Cramér's V
Qualitative	Quantitative	two-sample t-test Cohen's d
Quantitative	Quantitative	— Regression, correlation

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Bivariate analysis

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Bivariate analysis

- ...is determining whether there is an association between two stochastic variables (X and Y).
- **Association** = you can predict (to some extent) the value of Y from the value of X
  - X — Independent variable
  - Y — Dependent variable
- **Important!** Finding an association does **NOT** imply a causal relation!

Example research questions

- Is there a difference in taste preference between two beverage brands?
- Is there a difference in spending at the campus restaurant between students and staff?
- Do smokers die more often of lung cancer than non-smokers?
- Do men and women have a different opinion on a survey question?
- …

We will use `data/rlanders.csv` from the Github repo for lab assignments to explore the last question.

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Contingency tables

Contingency tables

(also: crosstab)

See Python example code in `demo-4.01-chi-squared.ipynb`

Gender Survey	Female	Male
Strongly disagree	0	4
Disagree	17	45
Neutral	23	91
Agree	12	53
Strongly agree	0	5

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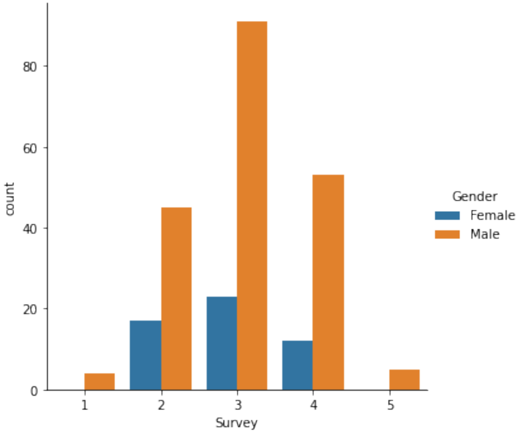
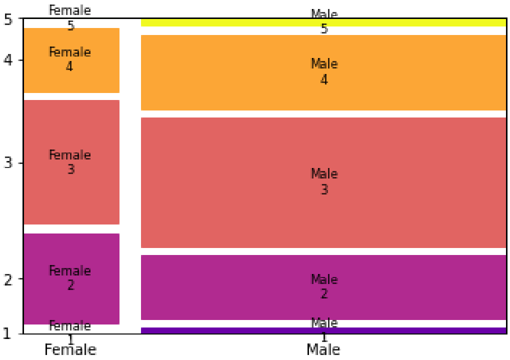
Visualization



A clustered bar chart



A mosaic plot

Marginal totals

Gender Survey	Female	Male	Total
Strongly disagree	0	4	4
Disagree	17	45	62
Neutral	23	91	114
Agree	12	53	65
Strongly agree	0	5	5
Total	52	198	250

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Expected values

If there is no difference (association), we expect the same ratios in each column of the table!

Gender Survey	Female	Male	Total
Strongly disagree	0.832	3.168	4
Disagree	12.896	49.104	62
Neutral	23.712	90.288	114
Agree	13.520	51.480	65
Strongly agree	1.040	3.960	5
Total	52	198	250

In each cell: (row total × column total) / *n*

Measuring dispersion

How far is the observed value o from the expected e?

$$\frac{(o - e)^2}{e}$$

Gender Survey	Female	Male
Strongly disagree	0.832	0.219
Disagree	1.306	0.343
Neutral	0.021	0.006
Agree	0.171	0.045
Strongly agree	1.040	0.273

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The chi-squared statistic

The sum of all these values is notated:

$$\chi^2 = \sum_i \frac{(o_i - e_i)^2}{e_i} \approx 4.255$$

- χ is the Greek letter *chi*
- o_i = number of observations in the i'th cell of the contingency table
- e_i = expected frequency
- Small value ⇒ no association
- Large value ⇒ association

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When is χ^2 large enough?

- 2×2-table with $\chi^2 = 10$
  - Relatively large difference
  - Indicates association
- 5×5-table with $\chi^2 = 10$
  - Relatively small difference
  - Does NOT indicate association

We need a metric independent of table size!

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Cramér's V

$$V = \sqrt{\frac{\chi^2}{n(k-1)}} = \sqrt{\frac{4.255}{250(2-1)}} \approx 0.130$$

with n the number of observations, $k = \min(numRows, numCols)$

Cramér's V	Interpretation
≈ 0	no association
≈ 0.1	weak association
≈ 0.25	moderate association
≈ 0.5	strong association
≈ 0.75	very strong association
≈ 1	complete association

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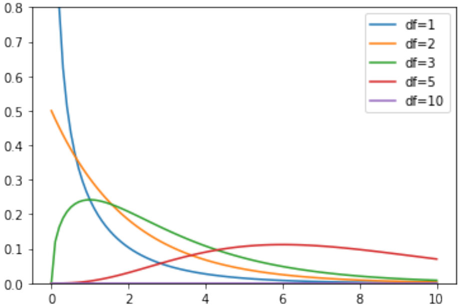
Chi-squared test for independence

χ^2 test for independence

- = Alternative to Cramér's V to investigate association between qualitative variables.
- Value of χ^2 distributed according to the χ^2 distribution



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χ^2-distribution in Python

Import `scipy.stats`
For a χ^2-distribution with `df` degrees of freedom:

Function	**Purpose**
`chi2.pdf(x, df=d)`	Probability density for `x`
`chi2.cdf(x, df=d)`	Left-tail probability $P(X <$ `x`$)$
`chi2.sf(x, df=d)`	Right-tail probability $P(X >$ `x`$)$
`chi2.isf(1-p, df=d)`	`p`% of observations is expected to be lower than this value

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Test procedure

- **Step 1.** Formulate hypotheses:
  - H_0: there is no association (χ^2 is "small")
  - H_1: there is an association (χ^2 is "large")
- **Step 2.** Choose significance level, e.g. $\alpha = 0.05$
- **Step 3.** Calculate the test statistic, $\chi^2 = 4.255$

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Test procedure (cont.)

- **Step 4.** Use $df = (numRow - 1) \times (numCol - 1)$ and either:
    - Determine critical value g so $P(\chi^2 > g) = \alpha$
    - Calculate the p-value
- **Step 5.** Draw conclusion:
    - $\chi^2 < g$: do not reject H_0; $\chi^2 > g$: reject H_0
    - $p > \alpha$: do not reject H_0; $p < \alpha$: reject H_0

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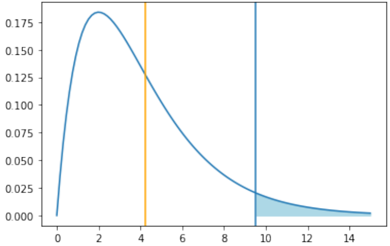
Example (Gender vs Survey)

- `g = stats.chi2.isf(0.05, df=4)` (result: 9.4877)
- `p = stats.chi2.sf(4.2555, df=4)` (result: 0.3725)



Conclusion: we do not reject the null hypothesis. Differences between expected and observed values are not significantly large.
We found no association between variables *Gender* and *Survey*

Test for independence in Python

SciPy has a function to calculate χ^2 and *p*-value from a contingency table:

```
observed = pd.crosstab(rlanders.Survey, rlanders.Gender)
chi2, p, df, expected = stats.chi2_contingency(observed)

print(”Chi-squared       : %.4f” % chi2)
print(”Degrees of freedom: %d” % df)
print(”P-value           : %.4f” % p)
```

Goodness-of-fit test

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Goodness-of-fit test

The χ^2 test can also be used to determine whether a sample is **representative** for the population.

Goodness-of-fit test

This test indicates to what degree a sample corresponds to a null hypothesis regarding the distribution of a qualitative variable over mutually exclusive classes.

Example

Type	# sample	# population
Mutant	127	35%
Human	75	17%
Alien	98	23%
God	27	8%
Demon	73	17%
Total	400	100%

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Example

Is the distribution of the sample (n = 400) representative for the full population (all superheroes)?

- What numbers would you *expect* if the sample is representative?
- How large are the differences from the *observed* numbers?
  - small ⇒ distribution is representative
  - large ⇒ distribution is **not** representative

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Example

Is the distribution of the sample (n = 400) representative for the full population (all superheroes)?

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Can you see the link with contingency tables and Cramer's V?

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Goodness of fit test

- Exactly representative ⇒ 35% of superheroes in the sample is a mutant
- The expected number therefore is $e = 0.35 \times 400 = 140$.

Therefore:

$$e = n \times \pi$$

If the differences $o - e$ are relatively small they can be attributed to random sampling errors.

Goodness of fit test

Consider χ^2:

$$\chi^2 = \sum_{i=1}^{n} \frac{(o_i - e_i)^2}{e_i}$$

Draw a conclusion based on the value of χ^2:

- small ⇒ distribution is representative
- large ⇒ distribution is **not** representative

χ^2 measures the degree of conflict with the null hypothesis

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Goodness of fit test

Superhero type	o	π	e	$\frac{(o-e)^2}{e}$
Mutant	127	35%	140	1.21
Human	75	17%	68	0.72
Alien	98	23%	92	0.39
God	27	8%	32	0.78
Demon	73	17%	68	0.37
			χ^2 =	3.47

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Goodness of fit test

- The test statistic χ^2 follows the χ^2 distribution.
- Critical value g from the χ^2 distribution: this is dependent on the number of degrees of freedom (df). In general:

  $$df = k - 1$$

  with k the number of categories.
- The critical value g for a given significance level α and number of degrees of freedom df can be calculated in Python using the function `isf()`.

$$P(\chi^2 < g) = 1 - \alpha$$

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Goodness of fit test

Testing Procedure

1. **Formulate hypotheses**
   - H_0: sample is representative for the population
   - H_1: sample is not representative for the population
2. **Choose significance level**: $\alpha = 0.05$

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Goodness of fit test

Testing Procedure

1. **Calculate test statistic**:

$$\chi^2 = \sum_{i=1}^{n} \frac{(o_i - e_i)^2}{e_i}$$

   1.1 **Critical area**: Calculate g so that $P(\chi^2 < g) = 1 - \alpha$
   1.2 **Probability value**: Calculate $p = 1 - P(X < \chi^2)$

2. Conclusion (the test is always right-tailed):
   2.1 $\chi^2 < g \Rightarrow$ do not reject H_0, $\chi^2 > g \Rightarrow$ reject H_0
   2.2 $p > \alpha \Rightarrow$ do not reject H_0, $p < \alpha \Rightarrow$ reject H_0

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Example

- `g = stats.chi2.isf(0.05, df=4)` (result: 9.4877)
- `p = stats.chi2.sf(3.4679, df=4)` (result: 0.4828)

Conclusion. $\chi^2 \approx 3.47 < g \approx 9.4877$, so we don't reject the null hypothesis. This sample is representative for the population.

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Goodness-of-fit test in Python

```python
observed   = np.array([127,  75,  98,  27, 73])
expected_p = np.array([.35, .17, .23, .08, .17])
expected   = expected_p * sum(observed)
chi2, p = stats.chisquare(f_obs=observed, f_exp=expected)

print("χ² = %.4f" % chi2)
print("p  = %.4f" % p)
```

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Standardized residuals

Example: families

Consider all families with exactly 5 children in a given community.

Example: families

Consider all families with exactly 5 children in a given community. When we look at the number of boys/girls, there are 6 possible combinations:

1. 5 boys
2. 4 boys, 1 girl
3. 3 boys, 2 girls
4. 2 boys, 3 girls
5. 1 boy, 4 girls
6. 5 girls

A survey was conducted regarding 1022 families with exactly 5 children

Are the observed numbers in the 6 classes representative for a population in which the probability of having a boy is equal to the probability of having a girl, or more concrete 0.5?

Example

i	0	1	2	3	4	5
o_i	58	149	305	303	162	45

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Example

i	0	1	2	3	4	5
o_i	58	149	305	303	162	45

If the assumption is true, the probability π_i to have i boys is determined by a binomial distribution with parameters $n = 5$ and $p = 0.5$. For example, the probability to have 2 boys out of 5 children is equal to:

$$(0.5)^2 \times (1 - 0.5)^{5-2} \times \binom{5}{2}$$

In general (you don't have to know why):

$$\pi_i = \binom{5}{i} \times 0.5^i \times 0.5^{5-i} = \frac{5!}{i!(5-i)!} \times 0.5^i$$

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Example

i	0	1	2	3	4	5	Total
o_i	58	149	305	303	162	45	1022
π_i	0,031	0,156	0,313	0,313	0,156	0,031	1
e_i	31,9	159,7	319,4	319,4	159,7	31,9	1022
$\frac{(o-e)^2}{e}$	21,268	0,715	0,647	0,840	0,033	5,343	28,846
r_i	4,686	-0,921	-0,970	-1,105	0,199	2,348	

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Example

1. **Formulate both hypotheses**
   - H_0: the sample is representative for the population
   - H_1: the sample is not representative for the population
2. **Determine α and n**: $\alpha = 0.01$ and $n = 1022$.
3. **Value of the test statistic in the sample**:

$$\chi^2 = \sum_{i=1}^{n} \frac{(o_i - e_i)^2}{e_i} \approx 28.846$$

4. **Calculate and plot critical area**: The critical value is 15.0863. Our test statistic is inside the critical area, so we can reject H_0.

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Standardized Residuals

Standardized Residuals

The standardized residuals indicate which classes make the greatest contribution to the value of χ^2.

$$r_i = \frac{o_i - n\pi_i}{\sqrt{n\pi_i(1 - \pi_i)}}$$

- $r_i \in [-2, 2] \Rightarrow$ “acceptable” values
- $r_i < -2 \Rightarrow$ underrepresented
- $r_i > 2 \Rightarrow$ overrepresented

Conclusion: families in which all children are of the same gender are overrepresented.

Cochran’s rules

Conditions

In order to apply the χ^2-test, the following conditions must be met (Rule of Cochran)

1. For all categories, the expected frequency e must be greater than 1.
2. In a maximum of 20 % of the categories, the expected frequency e may be less than 5.

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