From external to swap regret 2.0

An Efficient Reduction for Large Action Spaces

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Tel-Aviv University

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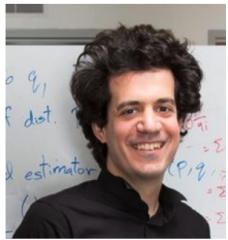
Authors

Yuval Dagan



Costis Daskalakis

Maxwell Fishelson

Noah Golowich

Overview

Example - The Prisoner's Dilemma

Example (Prisoner's Dilemma)

- Two bank robbers are arrested and being interrogated.
- Police only has evidence of minor offenses.
- The robbers are interrogated separately and can't communicate with each other.
- Each robber can either cooperate with the police 🐀 or remain silent 🤐.

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$A_1 \backslash A_2$	🐀2	🤐2
🐀1	$(-5,-5)$	$(0,-20)$
🤐1	$(-20,0)$	$(-1,-1)$

Strategic game setting

Definition (Strategic game)

A strategic game is a triplet $\langle K, A, f \rangle$, where $A = \bigtimes_{i=1}^{K} A_i$ is the set of all player's actions, and $f = (f_1, \ldots, f_K) : A \to \mathbb{R}^K$ is the payoff (or reward) function.

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A normal form game

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$A_i = \{🪨_i, 📄_i, ✂️_i\}$

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✂️$_1$	$(-1,1)$	$(1,-1)$	$(0,0)$

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$A_i = \{$🪨$_i$, 📄$_i$, ✂️$_i\}$

$A = \{$(🪨,🪨),(📄,📄), (✂️,✂️), (🪨,✂️), (✂️,🪨), $\ldots\}$

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(✂️,✂️), (🪨,✂️), (✂️,🪨), . . . }

f(🪨,🪨)=f(✂️,✂️)=f(📄,📄)= $(0,0), \ldots$

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Definition (Strategy profile)

$s = (s_i; s_{-i}) = (s_1, \ldots, s_n) \in A$ is called a strategy profile, where $s_i \in A_i$ is the strategy (or action) of player i

Equilibrium

Game Equilibrium

Strategy profile where no player wants to deviate



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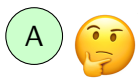

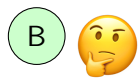

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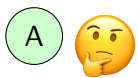
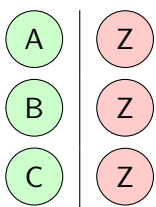
Strategy profile where no player wants to deviate



Coarse Deviations



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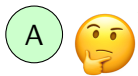

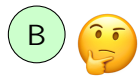

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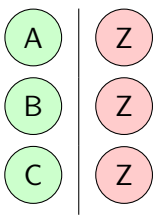
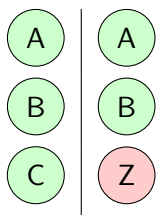


Swap Deviations



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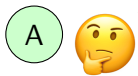

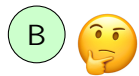

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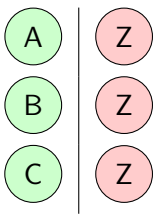

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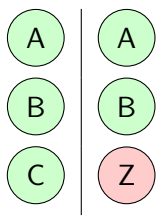

CCE: No player wants to make a coarse deviation
CE: No player wants to make a swap deviation
NE: No player wants to make a swap deviation
AND profile must be independent

Nash Equilibrium

Definition (Nash equilibrium)

A strategy profile s^* in a game $\langle K, A, f \rangle$ is called a **Nash equilibrium** if for all $i \in [K]$:

$$\forall d \in A_i : \; f_i(s_i^*; s_{-i}^*) \geq f_i(d; s_{-i}^*)$$

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Let Q be a **distribution** over A. Q is called a Correlated equilibrium, if for all $i \in [K]$, and for every function $\phi : A_i \to A_i$:

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Definition (ε-approximate-CE)

Q is called an ε-Correlated equilibrium if for all $i \in [K]$, and for every function $\phi : A_i \to A_i$:

$$\varepsilon + \mathop{\mathbb{E}}_{s\sim Q}[f_i(s)] \geq \mathop{\mathbb{E}}_{s\sim Q}[f_i(\phi(s_i); s_{-i})]$$

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Correlated Equilibrium:
$p(S, C) = \frac{1}{2}, p(C, S) = \frac{1}{2},$
$p(C, C) = p(S, S) = 0$

Coarse Correlated Equilibrium (CCE)

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CE VS CCE

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CE

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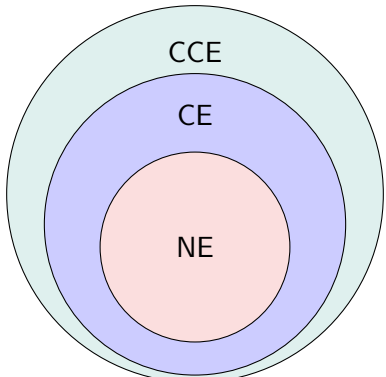
Hierarchy Of Equilibrium



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- The player achieves $x^{(t)} \cdot f^{(t)}$ reward at round t.

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The difference between the obtained reward and a benchmark

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Definition (Regret against Φ)

The regret of an online learner is defined as:

$$\text{Regret}_\Phi(x^{(1:T)}; f^{(1:T)}) = \left(\max_{\phi \in \Phi} \sum_{t=1}^{T} \phi(x^{(t)}) \cdot f^{(t)} \right) - \sum_{t=1}^{T} x^{(t)} \cdot f^{(t)}$$

Swap Regret

Definition (Swap Regret)

The swap regret of an online learner is defined as the regret against the set of **all maps**. namely:

$$\mathrm{SR}(x^{(1:T)}; f^{(1:T)}) = \left(\max_{\phi:\Delta_N \to \Delta_N} \sum_{t=1}^{T} \phi(x^{(t)}) \cdot f^{(t)} \right) - \sum_{t=1}^{T} x^{(t)} \cdot f^{(t)}$$

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Theorem (ε-SR induces an ε-CE)

If each player achieves swap regret bounded above by ε, then the joint distribution of all player actions is an ε-Correlated equilibrium.

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The swap regret of an online learner is defined as the regret against the set of **all maps**. namely:

$$\mathrm{SR}(x^{(1:T)}; f^{(1:T)}) = \left(\max_{\phi:\Delta_N \to \Delta_N} \sum_{t=1}^{T} \phi(x^{(t)}) \cdot f^{(t)} \right) - \sum_{t=1}^{T} x^{(t)} \cdot f^{(t)}$$

Theorem (ε-SR induces an ε-CE)

If each player achieves swap regret bounded above by ε, then the joint distribution of all player actions is an ε-Correlated equilibrium.

• **Proof sketch**: for each player, the maximum over all maps is at least as big as over a swap function. If we now define the same swap on the distribution, we get the definition of ε-CE.

ε-SR induces an ε-CE

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ε-SR induces an ε-CE

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We have:

$$\mathop{\mathbb{E}}_{s\sim X}[f_i(\phi(s_i); s_{-i})] - \mathop{\mathbb{E}}_{s\sim X}[f_i(s)] = \mathop{\mathbb{E}}_{s\sim\hat{\phi}(X)}[f_i(s)] - \mathop{\mathbb{E}}_{s\sim X}[f_i(s)] \leq \varepsilon$$

External regret

Definition (External regret)

The external regret of an online learner is defined as the regret against the set of all **constant maps**. (Map $x^{(t)}$ to a distribution where $\exists i \in [N] : P(i) = 1$)
Namely:

$$\mathrm{ER}(x^{(1:T)}; f^{(1:T)}) = \left(\max_{i \in [N]} \sum_{t=1}^{T} f_i^{(t)} \right) - \sum_{t=1}^{T} x^{(t)} \cdot f^{(t)}$$

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- External regret minimization is a much(❗❗) easier task then swap regret minimization
- This is enough to guarantee swap regret minimization! (*foreshadowing* 👨‍🍳)

No-external-regret learning

Multiplicative Weight Updates

Look at the past, do the best thing (with some normalization)

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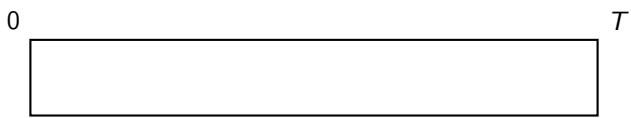

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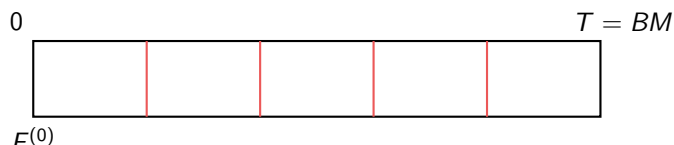

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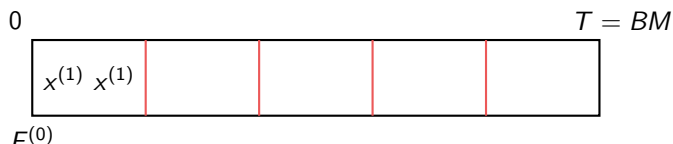

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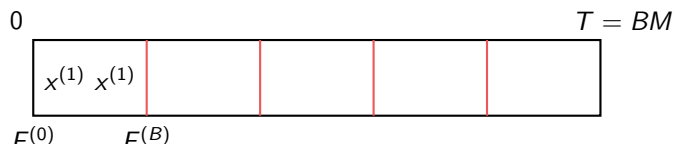

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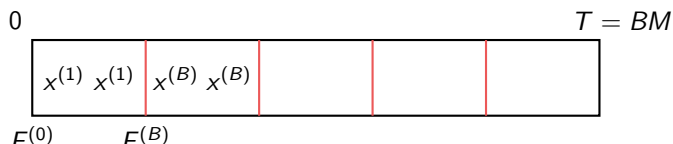

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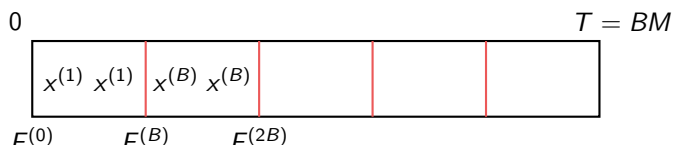

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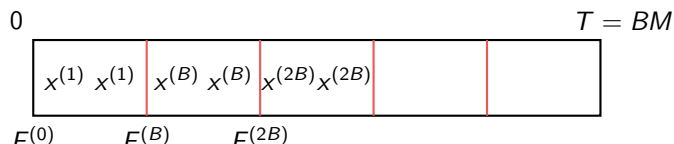

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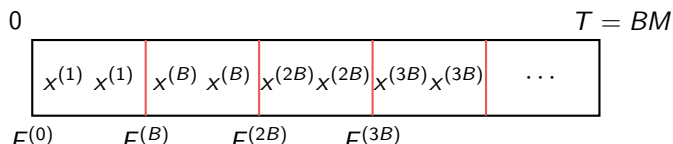

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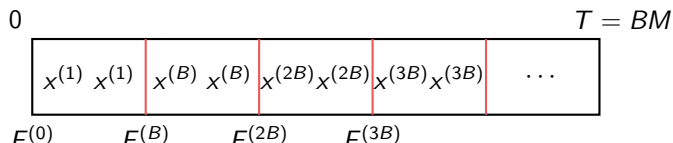

MWU: ε-ER for $T = \Omega\left(\frac{\log N}{\varepsilon^2}\right)$

No-external-regret learning

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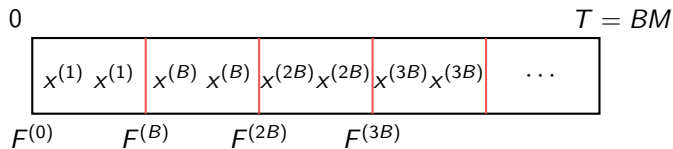
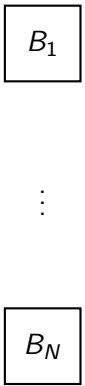


MWU: ε-ER for $T = \Omega\left(\frac{\log N}{\varepsilon^2}\right)$

Lazy MWU: ε-ER for $M = \Omega\left(\frac{\log N}{\varepsilon^2}\right)$

Blum-Mansour algorithm



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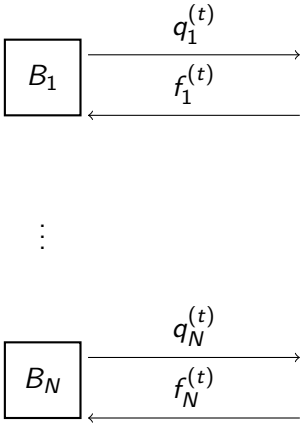

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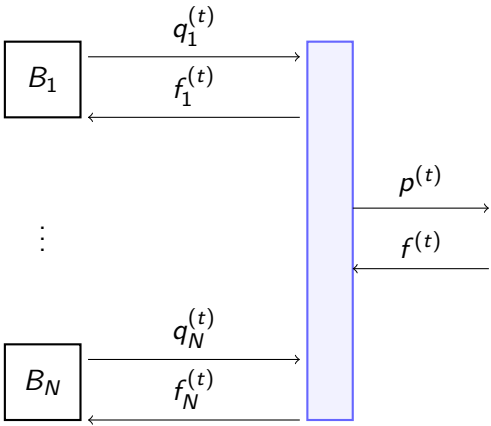

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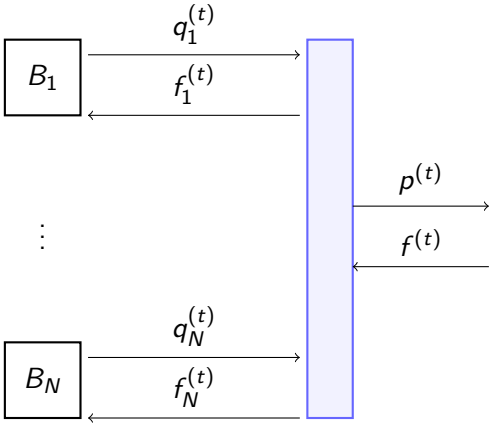
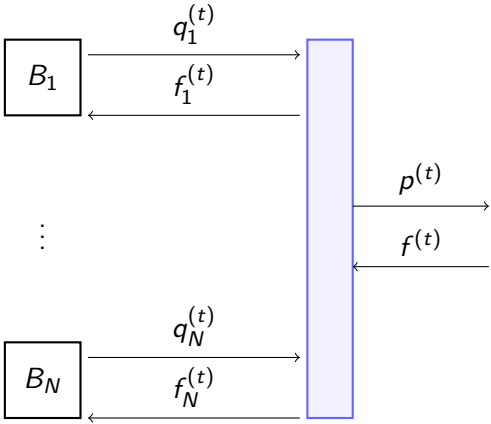


i) Calculating $p^{(t)}$ from $\vec{q}_1^{(t)}, \ldots, \vec{q}_N^{(t)}$.

Blum-Mansour algorithm



i) Calculating $p^{(t)}$ from $\vec{q}_1^{(t)}, \ldots, \vec{q}_N^{(t)}$.

ii) Distributing the reward across the algorithms $B_1, \ldots, B_N$.

Blum-Mansour algorithm

Distribute: $B_i \leftarrow f_i^{(t)} = f^{(t)} \cdot p_i^{(t)}$

Blum-Mansour algorithm

Distribute: $B_i \leftarrow f_i^{(t)} = f^{(t)} \cdot p_i^{(t)}$
Calculate p: LATER!

Blum-Mansour algorithm

Distribute: $B_i \leftarrow f_i^{(t)} = f^{(t)} \cdot p_i^{(t)}$

Calculate p: LATER!

Denote:

$$Q = \begin{pmatrix} q_{11} & \cdots & q_{1N} \\ \vdots & \ddots & \vdots \\ q_{N1} & \cdots & q_{NN} \end{pmatrix} = \begin{pmatrix} \vec{q}_1^{(t)} \\ \vdots \\ \vec{q}_N^{(t)} \end{pmatrix}$$

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$F_{B_i}^T := \sum_{i=1}^T F_{B_i}^{(t)}$

Proof.

B_i's Reward is:

$$F_{B_i}^{(t)} = (p_i^{(t)} f^{(t)}) \cdot \vec{q}_i^{(t)} = p_i^{(t)} (f^{(t)} \cdot \vec{q}_i^{(t)})$$

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Hence our total reward is:

$$\varepsilon + F^T = \varepsilon + \sum_{i=1}^{N} F_{B_i}^T \geq \sum_{i=1}^{N} \sum_{t=1}^{T} p_i^{(t)} \cdot f_{\phi(i)}^{(t)}$$

Tree-Swap

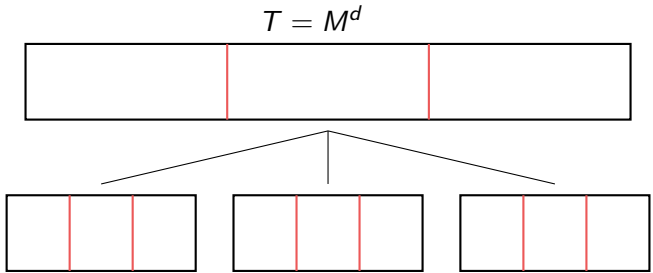
Theorem (From external to swap regret)

Let $M, d \in \mathbb{N}$, and let $\mathcal{F}$ be a function class.
If there exists a learner for $\mathcal{F}$ that achieves ε external-regret after M rounds, then there is a learner for $\mathcal{F}$ that achieves swap-regret of at most $\varepsilon + \frac{1}{d}$ after $T = M^d$ rounds.

Tree-Swap

$T = M^d$

Tree-Swap



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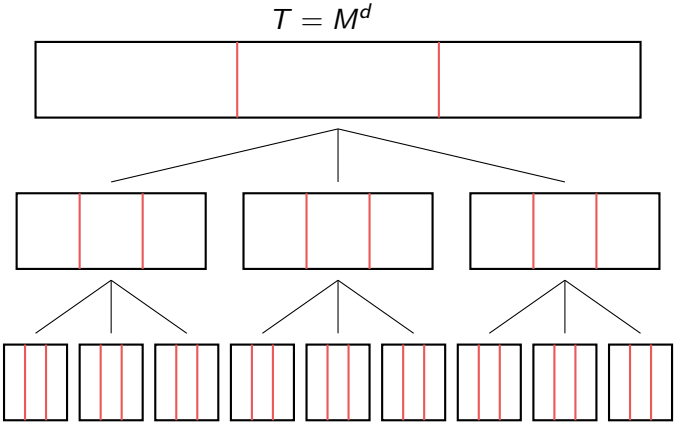

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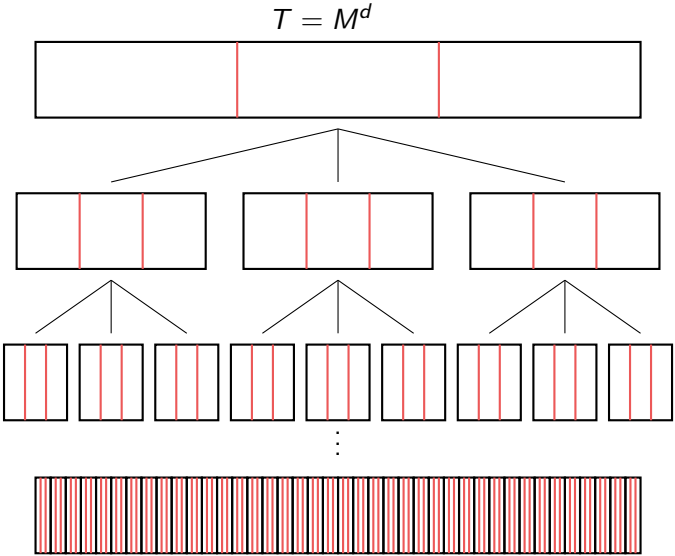

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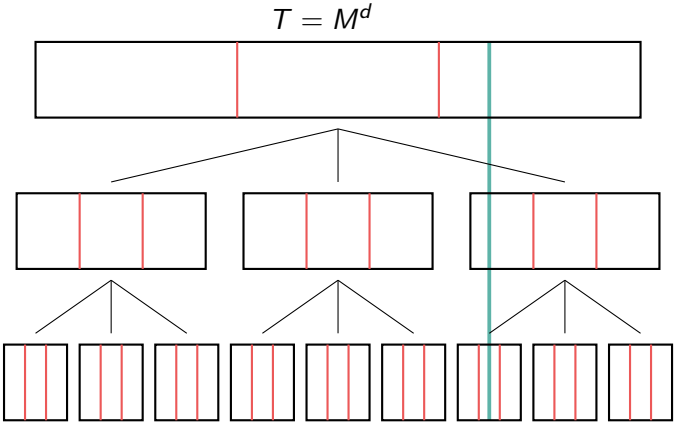

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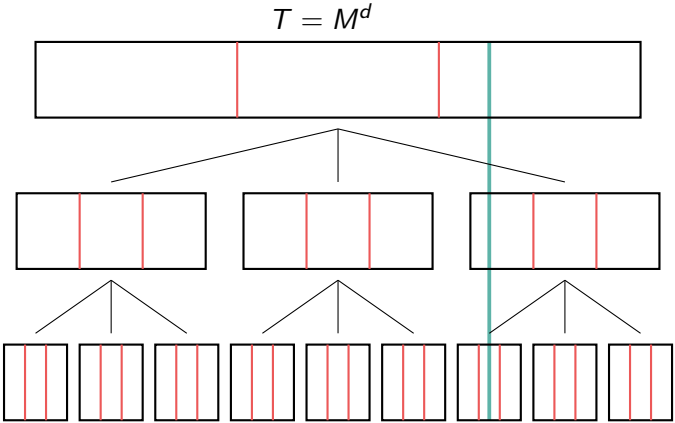

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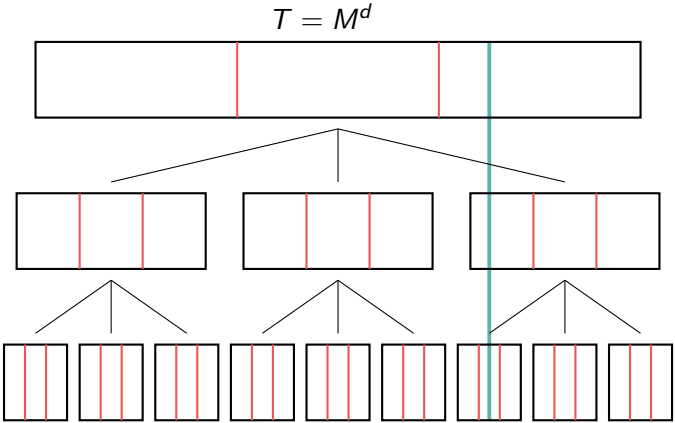

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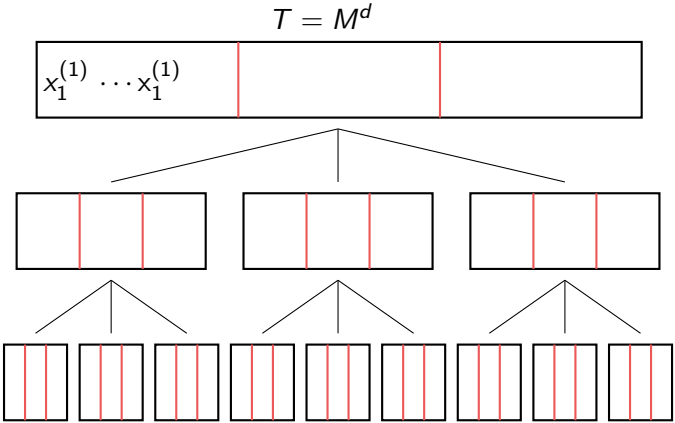

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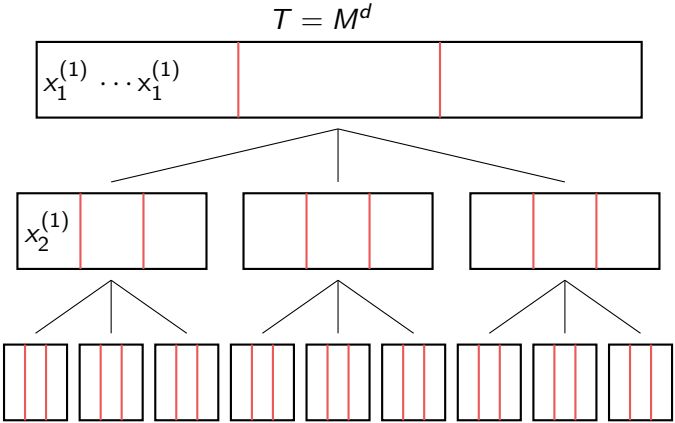

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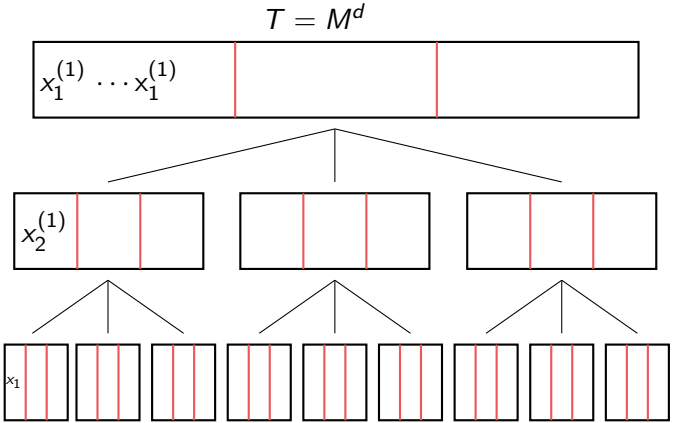

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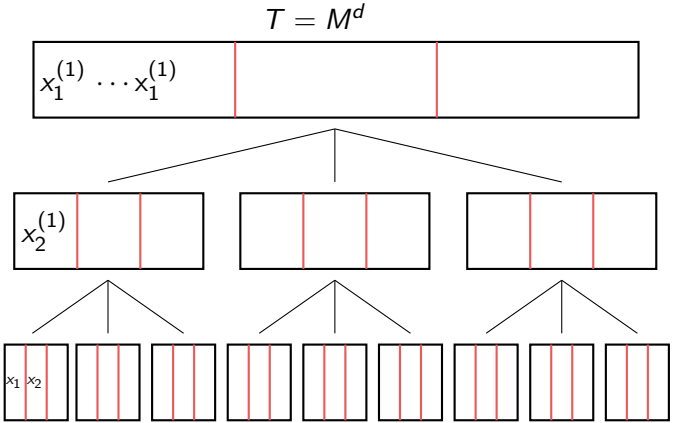

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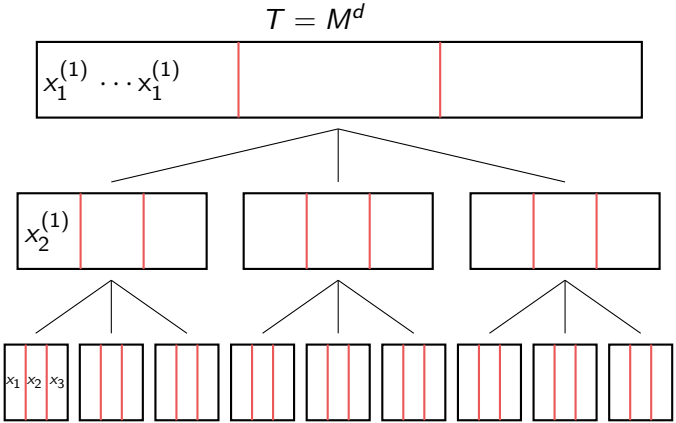

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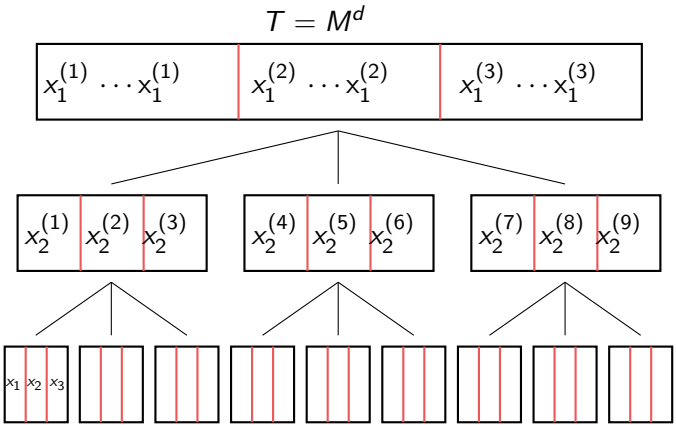

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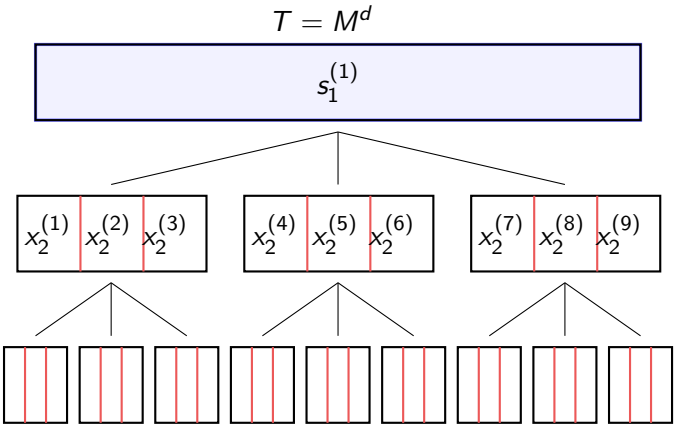

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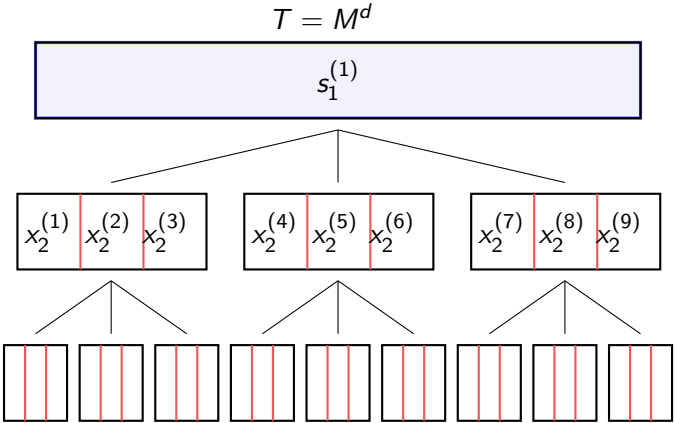

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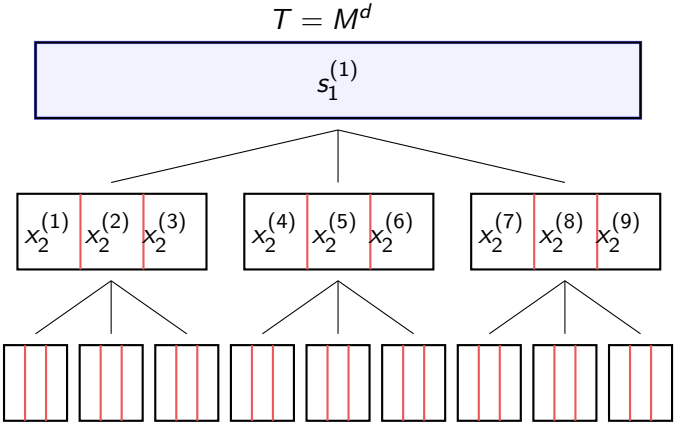

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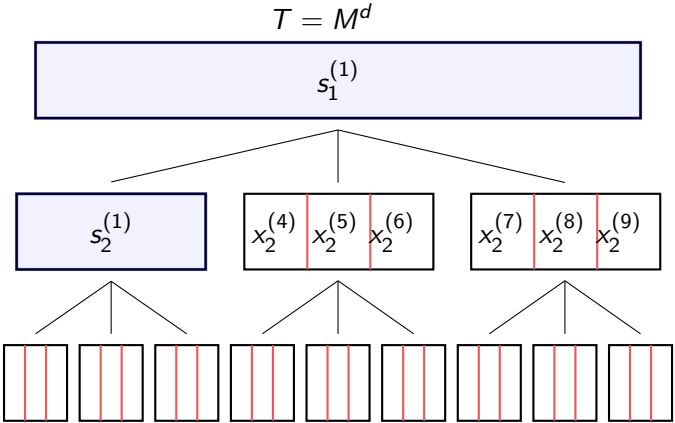

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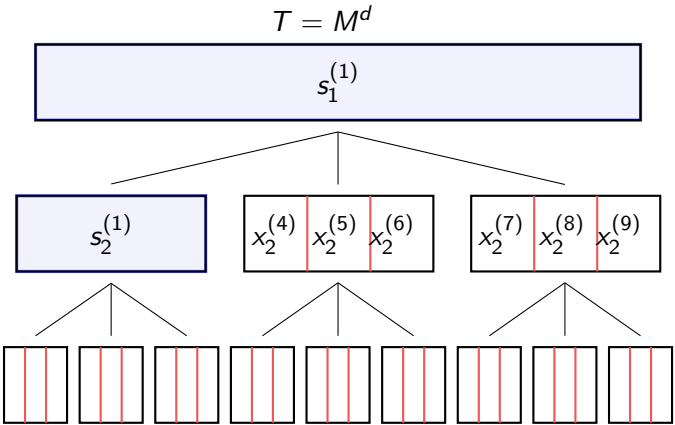

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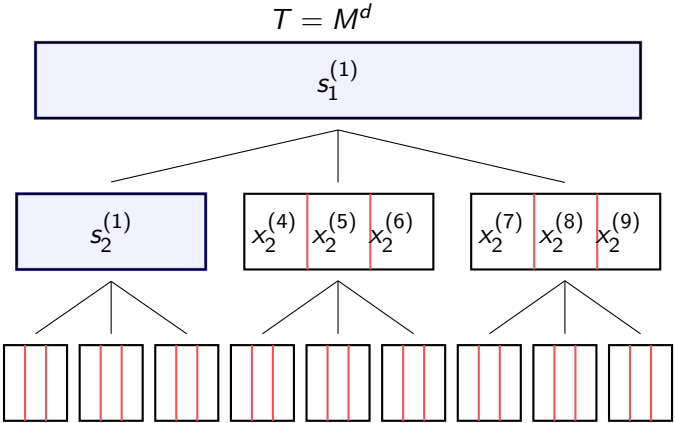

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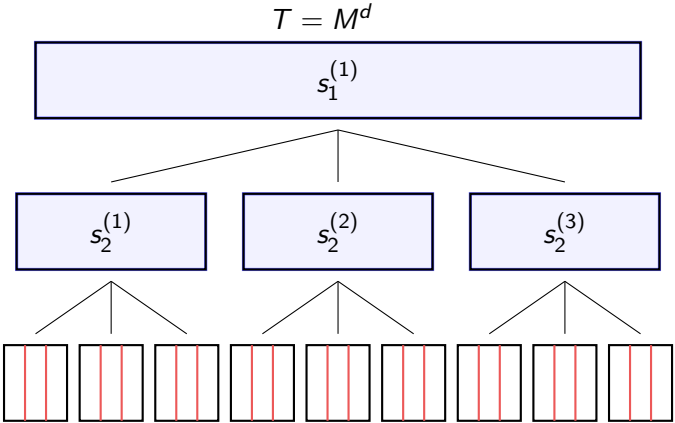

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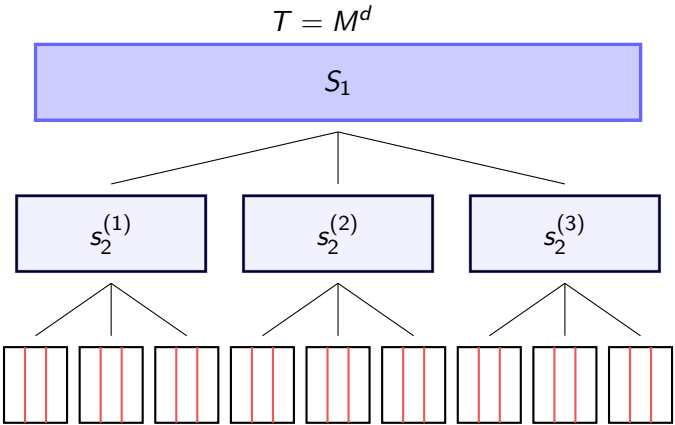

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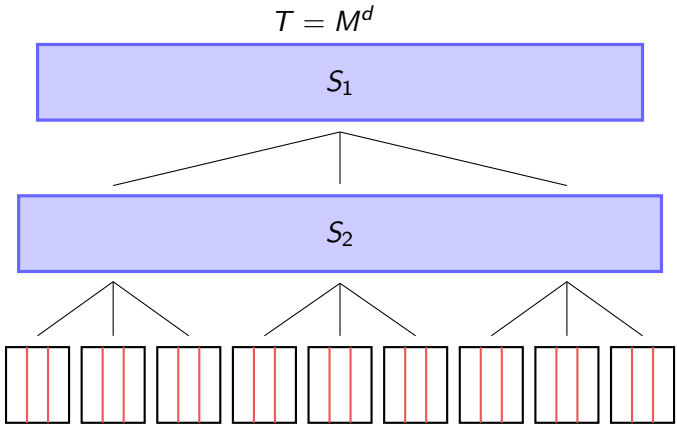

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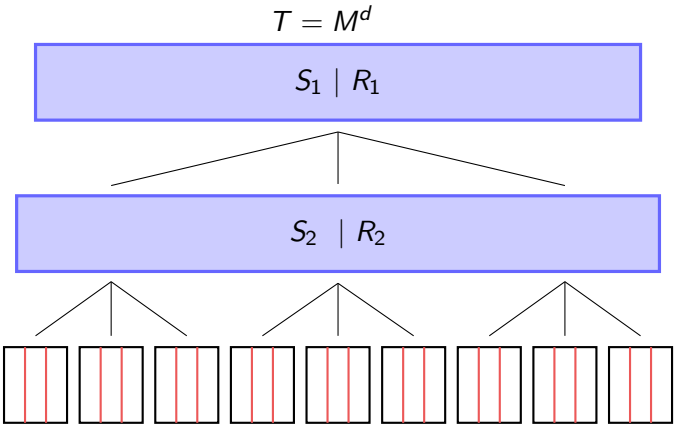

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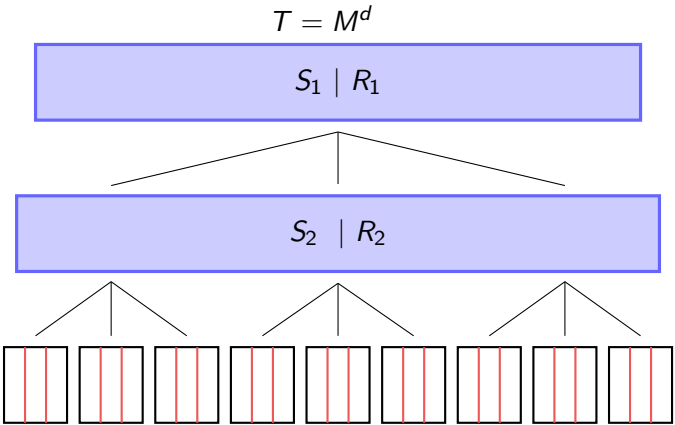

- By the *ER* assumption: $S_i - R_i \leq \varepsilon$

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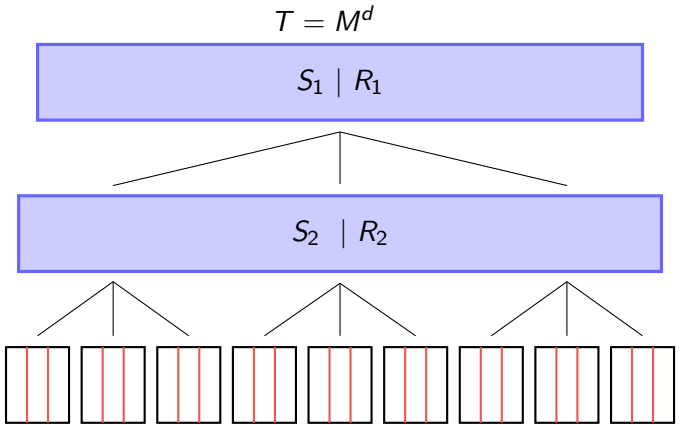

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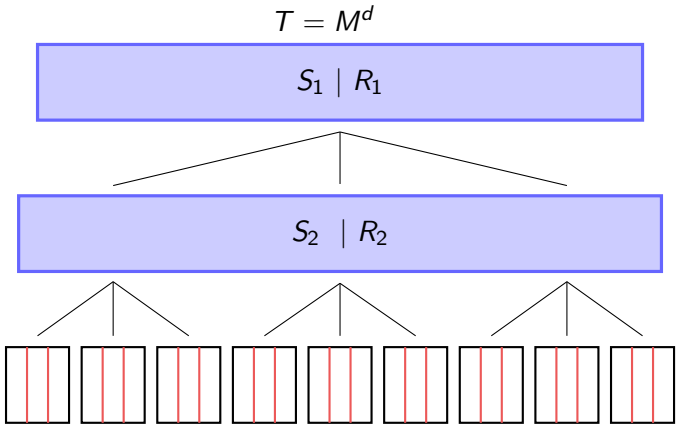

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Tree-swap (Pseudo code)

Require: Action set $\mathcal{X}$, utility class $\mathcal{F}$, no-external regret algorithm Alg, time horizon T, parameters M,d with $T \leq M^d$.

For each sequence $\sigma \in \bigcup_{h=1}^{d} \{0, 1, \ldots, M-1\}^{h-1}$, initialize an instance of Alg with time horizon M, denoted Alg_σ.
for $1 \leq t \leq T$ **do**
  Let $\sigma = (\sigma_1, \ldots, \sigma_d)$ denote the base-M representation of $t-1$.
  for $1 \leq h \leq d$ **do**
    if $\sigma_{h+1} = \cdots = \sigma_d = 0$ or $h = d$ **then**
      If $\sigma_h > 0$, call $Alg_{\sigma_{1:h-1}}$.update $\left(\frac{1}{M^{d-h}} \cdot \sum_{s=t-M^{d-h}}^{t-1} f^{(s)}\right)$.
      $Alg_{\sigma_{1:h-1}}$.curAction $\leftarrow$ $Alg_{\sigma_{1:h-1}}$.act().
    end if
  end for
  Output the uniform mixture $x^{(t)} := \frac{1}{d} \sum_{h=1}^{d} Alg_{\sigma_{1:h-1}}$.curAction, and observe $f^{(t)}$.
end for

Main results

	Best previously known	Tree-swap's result
Upper bound on T for ε SR		

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- Note that even the existence of approximate correlated equilibria in games of finite Littlestone dimension was previously unknown.

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Upper bound on T for ε SR	$\tilde{\Omega}(N/\varepsilon^2)$	$(\log(N)/\varepsilon^2)^{\Omega(1/\varepsilon)}$
Per-iteration runtime	$\Omega(N^3)$	$\tilde{O}(N)$
Finite Littlestone dim. upper bound	😳	$(L_{dim}(\mathcal{X})/\varepsilon^2)^{\Omega(1/\varepsilon)}$

- Note that even the existence of approximate correlated equilibria in games of finite Littlestone dimension was previously unknown.
- Additionally, the paper proved a lower bound against an oblivious adversary:

$$\text{Let: } T \leq O(1) \cdot \min\left\{\exp(O(\varepsilon^{-1/6})), \frac{N}{\log^{12}(N) \cdot \varepsilon^2}\right\}$$

Then, there exists an oblivious adversary, such that any learning algorithm run over T steps will incur swap regret at least ε.

Critical Points

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- No known practical use for correlated equilibria in games 😔

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- No known practical use for correlated equilibria in games 😔
- Upper bound is pretty bad... exponential dependence on $\frac{1}{\varepsilon}$ ‼️

Future Research

References

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The End