lunes, 15 de abril de 2024 9:41 p. m.

Ejercicio 1 (2 ptos). Demuestre que $\sum_{k=1}^{n}(k/3^k) \leq 1$. Puede suponer que es conocido que $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ cuando $|x| < 1$.

$S = \sum_{k=1}^{n} a_k$, Si : $\frac{a_{k+1}}{a_k} \leq r$, $0 < r < 1$ $\forall k \geq k_0$

$a_{k_0} \leq a_{k_0}$

$a_{k_0+1} \leq a_{k_0} \cdot r^1$

$a_{k_0+2} \leq a_{k_0+1} \cdot r \leq a_{k_0} \cdot r^2$ (+)

$\vdots$

$a_n \leq a_{n-1} \cdot r \leq a_{k_0} \cdot r^{n-k_0}$

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots a_{k_0-1} + \sum_{k=k_0}^{n} a_k$$

$$\leq \sum_{i=0}^{n} a_{k_0} \cdot r^i$$

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{k_0-1} a_k + a_{k_0} \cdot \underbrace{\sum_{i=0}^{n} r^i}_{\leq \sum_{i=0}^{\infty} r^i} \rightarrow \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{k_0-1} a$$

$$\leq \sum_{k=1}^{k_0-1} a_k + a_{k_0} \cdot \sum_{i=1}^{\infty} r^i = \sum_{k=1}^{k_0-1} a_k + \frac{a_{k_0}}{1-r}$$

$$\sum_{k=1}^{n} \frac{k}{3^k} \qquad a_k = \frac{k}{3^k} \rightarrow \frac{a_{k+1}}{a_k} = \left[\frac{\frac{k+1}{3^{k+1}}}{\frac{k}{3^k}}\right] = \frac{(k+1) \cdot 3^k}{k \cdot 3^{k+1}}$$

$$\rightarrow \frac{a_{k+1}}{a_k} = \left(\frac{k+1}{k}\right)\cdot\frac{1}{3} = \left(1+\frac{1}{k}\right)\frac{1}{3} \leq \frac{2}{3}$$

$$k \geq 1 \rightarrow \frac{1}{k} \leq 1 \rightarrow 1+\frac{1}{k} \leq 2 \rightarrow \frac{1}{3}\left(1+\frac{1}{k}\right) \leq \frac{2}{3}$$

$$k_0 = 1$$

$$r = \frac{2}{3}$$

$$\sum_{k=1}^{n} \frac{k}{3^k} \leq a_1 \cdot \frac{1}{1-\frac{2}{3}} \leq \frac{\frac{1}{3}}{\frac{1}{3}} \leq \underline{1}$$

$$a_1 = \frac{1}{3}$$

Ejercicio 2 (6 ptos). Pruebe las siguientes afirmaciones.

(a) $\frac{2}{3}n^2 - 7n - 300 = \Omega(n^2)$.

(b) $n^2 = O(n^2/3 - 5n \lg n)$. Puede suponer que es conocido que $n/\lg n$ es creciente. Sugerencia: tome como n_0 a una potencia de 2.

$$\frac{2}{3}n^2 - 7n - 300 = \Omega(n^2)$$

$$n^2 \geq 35 \qquad \forall n \geq 35$$

$$\frac{n^2}{5} \geq 7n$$

$$-\frac{n^2}{5} \leq -7n$$

$$\forall n \geq 3($$

$$n^2 \geq 900$$

$$-n^2 \leq -900$$

$$-\frac{n^2}{3} \leq -300$$

$\underline{\forall n \geq 35}$:

$$-\frac{n^2}{5} \leq -7n$$

$$-\frac{n^2}{3} \leq -300 \qquad (+)$$

$$\frac{2}{3}n^2 \leq \frac{2}{3}n^2$$

$$\frac{2}{15}n^2 \leq \frac{2}{3}n^2 - 7n - 300$$

$\therefore \ n_0 = 35 \quad c = \frac{2}{15}$

(b) $n^2 = O(n^2/3 - 5n\lg n)$. Puede suponer que es conocido que $n/\lg n$ es una función creciente. Sugerencia: tome como n_0 a una potencia de 2.

$$0 \leq n^2 \leq c\left(\frac{n^2}{3} - 5n\lg n\right)$$

$$\frac{1}{c}n^2 \leq \frac{n^2}{3} - 5n\lg n$$

$\underline{\forall n \geq 2^k} \rightarrow \quad \frac{n}{\lg n} \geq \frac{2^k}{\lg 2^k} = \frac{2^k}{k} \rightarrow \frac{k}{2^k} \geq$

$$-\frac{5kn^2}{2^k} \leq -5n\lg n$$

$$n^2 - 5k\,n^2 \leq -5n\lg n + n^2$$

$$\overline{3} \qquad \overline{2^k} \qquad - \qquad \overline{3}$$

$$n^2\left(\frac{1}{3} - \frac{5k}{2^k}\right) \leq \frac{n^2}{3} - 5n\lg n$$

Si ! $k=7 \Rightarrow \forall n \geq 128$

$$n^2\left(\frac{128-105}{128\cdot 3}\right) \leq \frac{n^2}{3} - 5n\lg n$$

$$n^2\left(\frac{23}{384}\right) \leq \frac{n^2}{3} - 5n\lg n$$

$\forall n \geq 128$ $\qquad n^2 \leq \left(\frac{384}{23}\right)\left[\frac{n^2}{3} - 5n\lg n\right]$

$\underline{n_0 = 2^7}$ $\qquad c = \frac{384}{23}$ ✓

(h) $\frac{1}{3}(n+1)(n-2) - 5 = \Theta(n^2)$.

(i) $n \lg n - \lceil 2n/3 \rceil - \lg n + 4$ es $\Omega(2n \lg n)$

(j) $\lg n!$ es $\Omega(n \lg n)$

$\frac{1}{3}(n+1).(n-2) - \underset{\sim}{5}$

$\leq 2n \quad \leq 2n$

$n+1 \leq 2n$

$1 \leq n$

$\forall n \geq 1$

$\forall n \geq 4 \qquad n^2 \geq 16 \geq 5$

$\rightarrow n^2 > 5$

$-n^2 < -5$

$\forall n \geq 3$

$n+1 \leq 2n$

$n-2 \leq 2n$ $\downarrow (\times)$

$\frac{1}{3}(n+1)(n-2) \leq \frac{4n^2}{3}$

$\frac{1}{3}(n+1)(n-2) - 5 \leq \frac{4}{3}n^2 - 5 \leq \frac{4}{3}n^2$

$m_0 = 3 \qquad C_1 = \frac{4}{3}$ ✓

$\forall m \geq 4 \qquad \frac{m}{2} < m < m+1$

$\frac{m}{2} < m+1$

$\frac{m}{2} < m-2 \qquad (*)$

$\frac{1}{3} \frac{m^2}{4} < \frac{1}{3}(m+1)(m-2)$

$\frac{1}{12} m^2 \leq \frac{1}{3}(m+1)(m-2) \quad ---- (\alpha)$

$\forall m \geq 5k \qquad \forall m^2 \geq 25k^2$

$-m^2 \leq -25k^2$

$-\frac{m^2}{5k^2} \leq -5 \quad ---- (\beta)$

$m^2\left[\frac{1}{12} - \frac{1}{5k^2}\right] \leq \frac{1}{3}(m+1)(m-2) - 5$

$k = 2$

$\forall m \geq 10 \quad \rightarrow \quad m^2\left[\frac{1}{12} - \frac{1}{20}\right] \leq \frac{1}{3}(m+1)(m-2) - 5$

$\frac{8}{\cancel{20} \times \cancel{12}}$

$\frac{m^2}{30} \leq \frac{1}{3}(m+1)(m-2) - 5$

$C_2 = \frac{1}{30}$ $\quad n_0 = 10$ ✓

(i) $n \lg n - \lceil 2n/3 \rceil - \lg n + 4$ es $\Omega(2n \lg n)$

$$x \leq \underbrace{\lceil x \rceil < x+1}$$

$$-\lceil x \rceil > -(x+1)$$

$$n \lg n + -\left\lceil \frac{2n}{3} \right\rceil > -\left(\frac{2n}{3}+1\right) + n\lg n - \lg n + 4$$
$$-\lg n \quad + 4$$

$$\underbrace{n \lg n - \left\lceil \frac{2n}{3} \right\rceil - \lg n + 4}_{f(n)} > 3 + n \lg n - \lg n - \frac{2n}{3}$$

$$> 3 + (n-1) \lg n - \frac{2n}{3}$$

$\frac{2}{3}$

$$\forall n \geq 4 \rightarrow n-1 \geq \frac{7}{10} n .$$

$$(n-1) \cdot \lg n \geq \frac{7}{10} n \lg n \text{ - - - } (\alpha)$$

$$n \leq n \lg n$$

$$-\frac{2}{3} n \geq -\frac{2}{3} n \lg n \text{ - - - - - } (\beta)$$

$$(n-1) \lg n - \frac{2}{3} n \geq \left(\frac{7}{10} - \frac{2}{3}\right) n \lg n$$

$>$

$$3 + (n-1) \cdot \lg n - \frac{2}{3} n > (n-1) \cdot \lg n - \frac{2}{3} n \geq \boxed{\frac{1}{30 \cdot 2}} \cdot 2 n \lg n$$

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$u_1 = \frac{7}{60}$ $n_0 = 4$