An Automated Market Maker for Perpetual Futures Contracts[*]

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Abstract

This paper presents an Automated Market Maker (AMM) for decentralized perpetual futures contracts. Protocol deployers provide initial liquidity that is locked into the protocol. Additional liquidity providers can also add and remove liquidity to the protocol. Liquidity providers do not face impermanent loss, but instead participate in the AMM's profit and loss, paid for in the collateral currency. Liquidity providers have no immediate impact on prices. Perpetuals with the same collateral currency can share a liquidity pool, adding to a capital efficient setup. The AMM allows for linear, inverse, and quanto perpetuals. The AMM pricing approach is based on risk-neutral valuation, which sets this AMM apart from other DeFi perpetuals.

Key Words: Perpetual Futures Contract, AMM, DeFi

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1 Introduction

A Perpetual Future, or in short "perpetual", is a derivative product similar to a futures but without expiry date. Market participants trade futures contracts to profit from price movements or to hedge against losses. Traders can have a leveraged long or short position and sell their contract at any point in time. The profit & loss of a future is determined by the price differential at the time when the future is sold compared to when the future is bought. See [CME Group, 2022] for a detailed explanation of a futures profit & loss calculation.

To pull the price of the perpetual future towards the spot price, perpetuals feature funding payments. Funding payments add to the profit & loss. Funding payments have to be made from the long traders to the short traders if the perpetual contract trades at a higher price than the spot. This payment will make the long position more expensive and the short position more attractive. This causes the perpetual price to move towards the spot. Equivalently, if instead, the perpetual price trades below the spot, the funding payment is made from the short to the long.

The spot price is represented by an index price that normally consists of an aggregation of different spot price sources. Funding payments are based on a funding rate that corresponds to the percentage deviation of the perpetual price from the mark-price, often represented as a time and volume averaged perpetual price. The funding rate is applied to the notional position size to arrive at the payment amount. The convention is that the funding rate is an 8-hour rate (as opposed to e.g., a 30/360 rate) that is updated on a frequent basis (e.g., milliseconds for some centralized exchanges). Often the accrued funding rate is paid every eight hours. Funding payments are subtracted or added to the margin collateral.

We distinguish between *linear*, *inverse*, and *quanto* perpetuals. The distinction lies between the type of collateral currency compared to the instrument base currency and quote currency. For a BTCUSD instrument the base currency is BTC, the quote currency is USD. With linear perpetuals, the collateral is held in the quote currency (or a stablecoin with that currency such as USDC). With inverse perpetuals, the collateral is held in the base currency (e.g., ETHUSD collateralized in ETH). With quanto perpetuals, the collateral is held in a third currency (e.g., ETHUSD collateralized in BTC).

Some DeFi products marketed as perpetuals deviate from what is described above in a way that renders them unfit for hedging purposes and arguably confines them as a retail speculative product. One example is that the funding rate is replaced by a borrowing rate that is charged to both the long and the short. Another example is that this rate is then interpreted as a one-hour rate (instead of the standard 8-hour rate), increasing the payments 8-fold for the same rate. In turn, leverage of up the area of 100x is offered.

We deviate from the standard perpetual contracts in that the funding rate is paid with every trader interaction, as opposed to the discrete 8-hour intervals. For inverse perpetuals, our AMM quotes the notional amount of the contract in units of base currency (e.g., in ETH for a ETHUSD perpetual collateralized in ETH), as opposed to quoting the size in USD as in standard centralized contract specifications such as the one from [Bitmex, 2022] or [Deribit, 2022]. We discuss the difference of the conventions on trade notional in Appendix A. Similar to centralized exchanges, we also have a maximal position size (per address). In our case, the maximal position size is dynamic and depends on the total capital in the system. Finally, centralized exchanges have limit order books. Instead, the counterparty to each trade in our system is the Automated Market Maker (AMM). The protocol, however, allows for limit and stop-limit orders that are cryptographically signed by the trader and executed by a third party once the AMM prices are in line with the conditions of the order.

The paper is structured as follows. We finish this introductory chapter with an overview of the challenges arising when the order-book is replaced with an automated market maker. In Section 6 we explain the AMM's capital reserves that are used to protect the traders and we detail how the capital reserves vary over time. We then describe the default waterfall, i.e., how the capital reserves are used in case the AMM runs out of margin collateral. Section 4 describes our pricing approach. Section 5 explains the concept of Mark Prices and how funding rates are defined. Section 6 determines the target sizes for the AMM fund and the default fund. Section 7 determines the maximal trade size that traders face. Finally, in Section 8, we touch on the simulation framework that we use to calibrate and stress test the system. The last section concludes.

1.1 AMMs for Perpetuals

In DeFi, there are exchanges that follow the order-book based approach. In this case, the market price (resulting from a market order) is determined by the market makers' limit orders. The price is determined by market makers and need not to be defined on a product level. Other decentralized exchanges follow the AMM (Automated Market Maker) approach, in which the price is determined algorithmically. In a centralized exchange, each trade is closed with a limit order posted by another market participant, and thus the exchange steps in only in case one of the counterparties has to be liquidated. In the AMM approach, each trade has the AMM as its counterparty, and is thus subject to price risk. However, if the AMM exposure nets to zero, the AMM is hedged against price moves. We illustrate this in Table 1. The AMM, much like a traditional market maker, needs to manage the additional price exposure risk. That is, if the long positions are not equal to the short position amount, the AMM makes a loss or profit when the underlying price moves.

Table 1: **Example: Market Maker Exposure.** In this example Alice enters a position of -1 at $3000. The price moves down to 2,900$ and so the AMM has a loss of $100 which is Alice's profit. Bob enters at this point with a position of +1, which nets the AMM position to zero. From now on, any price move will not change the AMM profit: the AMM is hedged against price moves. At time 2, Alice exits the position. Alice's losses incured from time 1 to time 2 fully pay for Bob's gains, because the AMM net position was zero. From time 2 onwards, the AMM has a net position not equal to zero and is exposed to price moves. At time 3, Bob exits with a gain of $1200. The AMM was again exposed to price moves and hence accounts for a loss of $100 for the period 2-3, in total $200.

Time	Action	Perpetual Price, $	Change Trader Pos.	AMM Net Position	Alice P&L, $	Bob P&L, $	AMM P&L, $
0	Alice enters	3000	-1	1	0	0	0
1	Bob enters	2900	1	0	100	0	-100
2	Alice exits	4000	1	-1	-1000	1100	-100
3	Bob exits	4100	-1	0	-	1200	-200

The classical and well known example for an AMM approach used for spot trading is Uniswap v1 that uses the Bancor constant product pricing (which we don't detail here). For perpetual futures, the current AMM approaches can broadly be divided into two categories:

1. Borrowed pricing: the pricing is based on a pricing formula borrowed from another protocol, such as Uniswap. Examples: FutureSwap, Perpetual Protocol v2

2. Risk based AMM: the pricing is determined so that traders are incentivized to reduce the AMM risk. Example: MCDEX v3

We follow the second category, building upon the design presented in [Serchev and Cadis, 2022], pricing-in the risk exposure taken by the AMM, calculated in a risk-neutral pricing setup.

2 Capital Structure

Each liquidity pool consists of one or several perpetuals that share the same collateral currency. Collateral currency is the currency in which the margin is held and payments are made (e.g., a BTCUSD and ETHBTC perpetual can both be collateralized in BTC and share a liquidity pool). The capital structure of a liquidity pool is illustrated in Figure 1. Each perpetual has a margin account like a regular trader, which we term *AMM margin*. The AMM margin is the first capital tranche used in the system. Each perpetual has a specific *AMM fund*, termed fund 1 to n in the figure, and there is a *participation fund*. If the AMM margin is above the initial margin, capital is sent to the AMM fund of the perpetual and the participation fund. If the AMM margin is below the initial margin (a loss), capital is taken from the AMM fund and participation fund and added to the AMM margin of the perpetual that incurs the loss. For a regular trader, the initial margin determines the amount they have to deposit to initiate a trade. For the AMM, the initial margin acts as a "target margin". This process of balancing towards the initial margin is called rebalancing. Rebalancing occurs with every action that a participant can perform (such as trade, withdrawal, deposit).

There is a *default fund* that is used to replenish AMM funds if they fall below their target size. Similarly, the default fund is paid if an AMM fund size is above its target.

Anyone can deposit into the *participation fund* in the collateral currency of its liquidity pool. Depositors are termed PnL participants, because they participate in profit and loss of the perpetuals and earn trading fees. PnL participants can add and withdraw funds. The default fund has a target size and if it fills up beyond the target size, proceeds can be withdrawn by protocol governance. All other funds than the participation fund and the amount of default fund beyond its target are protocol owned and cannot be withdrawn. Protocol owned funds other than a share of the AMM margin are never transferred to the PnL participation pool.

At initiation, the deployer of the smart contracts funds the AMM and the default fund with their own funds. AMM and participation funds are kept separate from each other for the following main reason. First, this setup allows us to maintain protocol owned liquidity that is not at risk to be withdrawn. This leads to better prices for the traders and mitigates a "bank run" situation, where liquidity providers flee, followed by a flight of traders due to higher slippage.

Second, this approach helps with the price stability as we detail now. The amount of capital has an impact on the prices that traders face. The general approach is that more capital leads to prices that are less sensitive to AMM exposure and trade size. With this setup, the full AMM fund size can be counted towards capital for the pricing of perpetuals in our setup, because it is not influenced by any liquidity provider action. The participation fund, however, is only partially used for pricing to avoid that traders that are also liquidity providers can manipulate prices. In Figure 1 this is emphasized by showing that part of the participation fund is considered "virtual" and not counted towards pricing. We detail the amount of capital considered for pricing

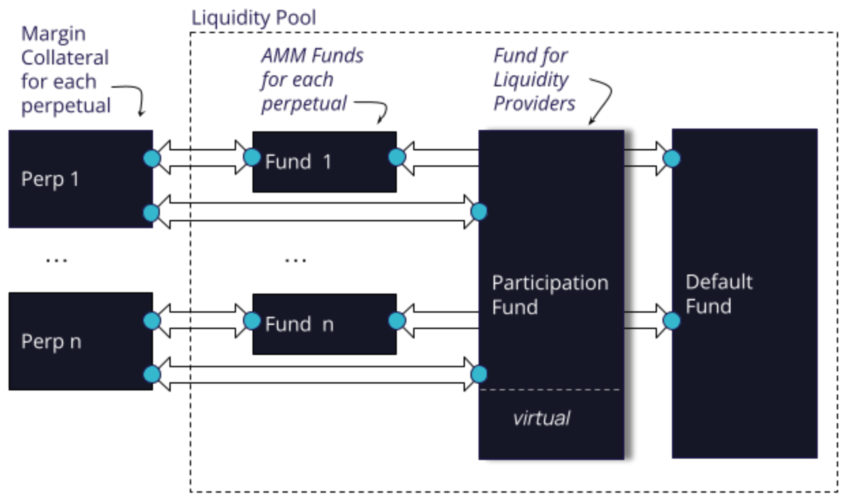

Figure 1: Capital Structure. Perpetuals with the same collateral currency can share one 'liquidity pool'. A perpetual is connected to one liquidity pool. This figure illustrates the capital structure of such a pool. Each perpetual has its own margin account and its own AMM fund. There is a PnL participation pool, and a default fund. Each perpetual has a margin account like a trader. The Profit & Loss ('PnL') in the perpetual margin is regularly exchanged with the perpetual's AMM fund and with the participation fund. The default fund is built up by excess AMM funds and is used if AMM fund size is below its target level. The liquidity counted towards the pricing of the perpetual consists of its AMM fund and a fraction of the participation fund.

in the next paragraph.[1]

2.1 Avoiding Price Manipulation from Liquidity Provision

At the initiation of a perpetual, the corresponding AMM fund and the AMM margin belonging to that perpetual are stocked with an initial amount. No funds are required in the participation fund for the system to be functional.

To ensure that a potential malicious trader is not able to manipulate prices by transferring liquidity to or from the participation fund, we introduce the concept of 'virtual' funds: only participation funds that are not virtual are counted towards the AMM liquidity used for pricing calculations, and virtual funds are realized in a tightly controlled fashion on three fronts:

[1] A trader and PnL participant can enter a trade if the price deviates from spot, subsequently deposit liquidity and thereby change the price, reaping the gain as a trader and socializing the PnL participant loss between AMM fund and the other PnL participants.

1. All participant funds are considered virtual when they are first added to the liquidity pool,

2. Over time, virtual funds are gradually turned into real funds:
   - The rate of growth is parameterized in terms of the total time it should take to transform all virtual funds into real ones,
   - The absolute growth of real funds is capped, overriding the parameterized rate, and thus eliminating the impact of sudden large deposits,
   - Real funds are distributed equally across all perpetuals in a liquidity pool,
   - The total amount of real funds is capped to a pre-determined fraction of the default fund

3. When some or all participant funds are removed, the default fund is used to transfer an equivalent amount into the AMM funds:
   - This eliminates any instantaneous price impact on the perpetuals in the corresponding liquidity pool,
   - The cap on realized participation funds ensures such a transfer is always possible and does not increase the risk to the protocol on its own,
   - If the resulting funds in any of the individual AMM funds exceed their target after participation funds are replaced by protocol owned funds, then they will gradually be transferred into the default fund with every transaction

In all, these restrictions ensure that, if there is any price impact to be expected from a change in externally provided liquidity, this should always occur gradually over time. This introduces significant risk for price manipulators by the inherent stochastic nature of the spot-price and the ongoing trading activity in the perpetual being targeted, thus making such a strategy not viable for potential attackers.

2.2 Protocol Earnings

The AMM prices contain a bid-ask spread and slippage that contribute to the protocol earnings. The protocol can also be parameterized to charge a trading fee relative to the trade notional.

The AMM is exposed to market risk when the positions do not net to zero. This can lead to either profits or losses. The pricing approach is so that arbitrage traders are incentivized to minimize the AMM net exposure.

Conditional orders such as limit orders or stop limit orders, are executed by a third party, termed *referrer*, once the AMM prices meet the order conditions. The referrer earns a fee for their service, paid for by the trader.

If the position is liquidated because the margin is below the maintenance margin, the *liquidator* earns a fee that is subtracted from the trader margin. The liquidation fee is shared between the protocol and the liquidator.

When profit/loss is re-balanced between the AMM margin account and its connected liquidity pool, the profit/loss is shared between AMM funds and participation fund proportionally to the respective sizes. E.g., 0.01 ETH of the inverse perpetual ETHUSD are sent to the liquidity pool, then the corresponding AMM fund receives/pays a share equal to $\max[a/(a+p), 25\%]$, where a is the sum of all AMM fund sizes in its liquidity pool, and p the PnL participation fund size. The participation fund receives/pays $\min[p/(a+p), 75\%]$.

Each AMM fund has a target size, and so does the default fund. Each AMM fund pays to the default fund as soon as its target size is met. If the default fund target size is met, protocol governance can withdraw the amount above the target from the default fund. We detail how we arrive at the target sizes in Section 6.

Because the default fund needs to be replenished from AMM gains, we impose the cap of 75% for the maximal relative amount that the PnL participants receive and pay. If an AMM fund is below its target size, the missing funds are drawn from the default fund and participation fund every time the profit is exchanged. This exchange amount is also capped, at 75% of available funds. Once the default fund also reached its target size, the profits can be withdrawn to the protocol treasury wallet.

3 Default Waterfall

Trader losses The trader places an *initial margin* before they can trade. When the trader's loss leads to a margin balance below *maintenance margin*, market participants can earn a fee by liquidating the position so that the initial margin is re-instantiated (or the position is closed).

Anyone can be a liquidator and request the AMM to liquidate a given trader. The AMM then either rejects the request if the trader margin is above the initial margin, or proceeds. The liquidation amount is determined so that the trader is brought back to the initial margin, if possible. Hence, this can result in a partial or a full liquidation of the position.

Unlike in centralized exchanges, every trade is with the AMM and hence the exchange does not need to replace the defaulter's position to guarantee the counterparty profit & loss. However, the net exposure of the AMM might grow or shrink due to the trader default and this can influence prices like liquidations do in order-book based markets. The mark price that is the relevant price for liquidations, however, is not changed within the same block.

AMM losses Similar to a trader, each perpetual has a margin account. The profit and loss for each perpetual are exchanged between the margin account and the liquidity pool. The AMM margin of each perpetual in a given liquidity pool is re-balanced with AMM/trader interactions (deposit, trade, withdrawal), so that the AMM margin resets to the initial margin level. For example, if the initial margin rate is 20%, and the position value decreased so that the margin balance stands at 15%, the funds are taken from the liquidity pool to re-establish a margin of 20%.

Certain events, as defined below, call for the liquidation of one or all perpetuals in the liquidity pool, which we term perpetual settlement. In case of perpetual settlement, all traders are closed out in the given perpetual. Traders with no outstanding position keep their posted collateral and are not affected. The liquidation price corresponds to the 'mark price' which is an exponentially weighted moving average of the perpetual mid-price. After the pool liquidation, the liquidity pool is functional with the non-liquidated perpetuals in the pool. If there are not enough funds to settle all traders, the funds are distributed proportionally, that is, each trader receives their margin balance (at the mark price) scaled by the ratio of total capital over total trader margin. Where we define total capital as the sum of: (1) the AMM fund and (2) margin of the perpetual being liquidated, (3) the participation fund, (4) the default fund.

Perpetuals settlements are subject to the following rules illustrated in Figure 2. Trader gains are paid by the margin account of the perpetual. If the margin is not enough, the AMM fund and participation fund are used to pay the trader. If the AMM fund for the given perpetual is empty, the perpetual is settled. If the default fund runs out of funds, all perpetuals of the liquidity pool are settled If a price oracle used for the given perpetual is terminated, the perpetual is settled

4 Pricing

Mid-prices for perpetual futures often deviate from the corresponding spot in order-book based exchanges. In practice, we observe price deviations in the direction of

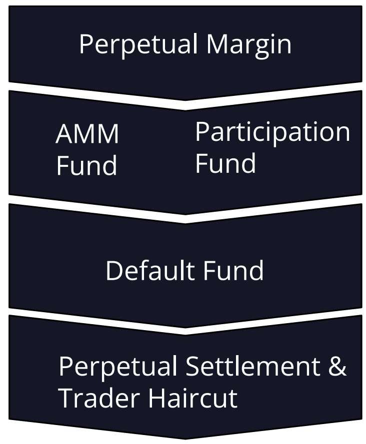

Figure 2: Default Waterfall. Profit and losses of the AMM above/below the perpetual margin account are shared between the perpetual's AMM fund and the participation fund. In case the participation fund and AMM fund are used up, the default fund is used as a last resort. If the default fund is not sufficient to pay the traders, the perpetual enters into settlement mode where all traders are closed out at mark price and traders share the loss proportionally. Settlement can enter earlier under different conditions, as outlined in the section.

higher demand of the traders that are not market makers. We can explain a short-term deviation of this type at a market micro-structure level: a higher demand in one direction removes the limit orders on that side of the order-book and hence the mid-price defined as the average between best bid and ask moves in this direction. In an efficient market and neglecting default risk, the spot mid-price and the corresponding futures mid-price should only differ by the cost of carry: the spot trade requires the full amount of funds, and the futures trade requires only a fractional amount of funds (for the required margin collateral) to get the otherwise same PnL exposure.[2] This can be shown by an arbitrage argument (basis trade). This explanation made for futures should also be valid for perpetuals in the presence of a funding rate. Another friction to the efficient market setup that could explain price differences between spot and perpetuals might be the difference of liquidity in the spot and perpetual markets. Finally, and most important for our purpose, the exposure of the involved parties is only partially funded in the futures and perpetual market, and hence the risk of default comes into play. Even if the centralized exchange takes over the defaulted trader positions, default risk is non-negligible if the vast majority of the traders that the market-makers face trade in one direction.

[2]The cost of carry in crypto-currency markets should correspond to the funding cost in the fiat currency market.

In AMM-based markets, the AMM takes the opposite side of each trade, so the market makers are replaced by the AMM. Ideally, the AMM follows qualitatively the same pricing patterns to remain operational. When the AMM has a net-zero exposure (e.g., taking the opposite sides for both a 1 BTCUSD long trade and a 1 BTCUSD short trade), the AMM profit/loss is hedged against price moves, as long as the traders do not exit or default and no new traders enter. On the other hand, if there is a net exposure for the AMM (e.g., taking the opposite side of only 1 BTCUSD trade), the AMM could exceed its margin and would no longer be able to pay the trader.

Because the AMM guarantees the trade, we could view a price deviation from spot as a default risk insurance premium for the AMM. If a trader enters a position, the AMM is at risk and the AMM needs to have additional measures in place to guarantee the contract. The insurance premium corresponds to the costs of these additional measures.

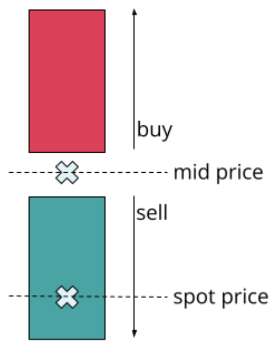

Figure 3: Pricing Concept. This figure illustrates the pricing concept for the case in which the majority of the traders are long. The perpetual mid-price deviation from spot corresponds to the value of the insurance that the AMM provides. Long traders further increase the risk and pay a price above the mid-price. Short traders decrease the risk and receive a premium compared to the spot-price. If, however, the short trade size is large enough to flip the AMM exposure, the perpetual price ends up below the spot price. A trade that nets the AMM to minimal risk exposure brings the mid-price back to the spot price.

Let $Q(0)$ be the unit-cost of insurance for an infinitesimally small trade, given the current state of the AMM. If a trader goes long with a notional of κ, the trader will bring the AMM to a different state so that the unit-cost of insurance is at $Q(\kappa)$. The trader causes the AMM insurance costs of $Q(\kappa)$ per contract unit.

We define the mid-price as the price that corresponds to the spot price plus the unit-value of the insurance that the AMM currently covers. Figure 3 explains the concept with an example of an AMM in which the aggregated net position of the traders is long. Traders increases the AMM risk with a long trade and therefore long traders get charged a price above the mid-price (and above the spot); see 'buy' in Figure 3. If a trade decreases the AMM default risk, the trade happens at a favorable price; see 'sell' in Figure 3. This price is favorable for the short because the price is expected to move towards spot and hence the short trader makes an additional profit by the difference of their price and the spot. The AMM is thus charging the risk-increasing side a premium relative to the spot, that is later being credited to the short as spot and perpetual mid-price converge. If the AMM has minimal price exposure, the perpetual price equals the spot price. The higher the trading amount, the higher the AMM default risk and thus the premium, i.e., the trader observes slippage. The larger the short trader's position, the lower the price they get, i.e., they too observe slippage. There is an optimal trade size, $\kappa^{\star}$, that brings the AMM to its minimal risk exposure and is associated with a default risk premium of zero.

The same logic applies if the majority of the traders are short. Hence, qualitatively this pricing strategy is analogous to the price patterns we observe in order-book-based markets. We derive the insurance premium in the sequel.

4.1 Illustrative Setting: Single Trader

We assume a one-period model and calculate the price of the insurance using risk-neutral valuation, in the spirit of the [Merton, 1974] bond pricing model. We have a single period and hence two points in time that we denote by '0' (now) and 't' (end of period). Market prices at time 0 are known, whereas the end-of-period prices are unknown at the beginning of the period. Let $M_2 > 0$ be the collateral available for the AMM to cover trader gains, deposited in the collateral currency which we choose to be the base currency BTC. We define the AMM default as the event that the AMM capital is not sufficient to cover the traders' profits.

For illustration, we consider the case where the exchange only serves one trader. In the absence of funding rates, the profit/loss of the individual trader at the end of the single period is given by

$$\kappa(s_{2,t} - s_{2,0}), \tag{1}$$

where $s_{2,0}$ is the trader's entry price for BTCUSD (in USD), $s_{2,t}$ the exit price, and κ is the position size. The sign indicates the direction; e.g., $\kappa = -1$ for one short contract of BTCUSD.

The amount owed to the trader in excess of what the AMM capital can cover is

$$\max\{\kappa(s_{2,t} - s_{2,0}) - M_2 s_{2,t}, 0\} \tag{2}$$

The problem of determining the premium that the AMM would charge a trader to open a position κ thus becomes equivalent to finding the price of the insurance, $i(\kappa)$, needed in case of a default.

According to risk-neutral pricing theory, the price of the insurance is given by the expected value of (2) under the unique risk-neutral measure:

$$i(\kappa) = \mathbf{E}\left[\max\{\kappa(\tilde{s}_{2,t} - s_{2,0}) - M_2\tilde{s}_{2,t}, 0\}\right] \tag{3}$$

The price $\tilde{s}_{2,t}$ is not yet known at time 0, so this is a random variable, which the notation emphasizes using a tilde ($\sim$). To calculate the expectation, we express the price at the end of the period in terms of returns, $\tilde{s}_{2,t} = s_{2,0}\exp(\tilde{r}_2)$. With a normal distribution of $\tilde{r}_2 \sim N(\mu - \sigma^2/2, \sigma^2)$, we can rewrite Equation (3) as

$$\begin{aligned} i(\kappa) &= \mathbf{E}\left[\max\{(\kappa - M_2)\exp(\tilde{r}_2) - \kappa, 0\}\right] s_{2,0} \\ &= \frac{1}{\sqrt{2\pi\sigma}} \int \max\{(\kappa - M_2)\exp(x) - \kappa, 0\} \exp\left[-\frac{1}{2\sigma^2}(x - \mu + \sigma^2/2)^2\right] dx \end{aligned} \tag{4}$$

This integral can be found analytically, and we provide the details of the calculation in Appendix B.1. The result, in the case $\mu = 0$, is

$$i(\kappa) = \begin{cases} 0 & \text{if } 0 \leq \kappa \leq M_2 \\ (\kappa - M_2)\Phi(-d_1) - \kappa s_{2,0}\Phi(-d_2) & \text{if } \kappa > M_2 \\ (\kappa - M_2)\Phi(d_1) - \kappa s_{2,0}\Phi(d_2) & \text{if } \kappa < 0 \end{cases} \tag{5}$$

where $d_1 = \dfrac{\Theta - \sigma^2/2}{\sigma}$, $d_2 = \dfrac{\Theta + \sigma^2/2}{\sigma}$, and $\Theta = \ln\left(\dfrac{\kappa}{\kappa - M_2}\right)$, $\Phi(\cdot)$ is the standard normal cumulative distribution function, and $\ln(\cdot)$ the logarithm with basis e.

At this point, we modify Equation (4) to replace the insurance value by the value of a binary option times the position value, $|\kappa| s_{2,0}$. The binary option equals one if the AMM defaults, zero otherwise. In Appendix B.1, we show that this is a conservative approximation, and at the same time the approximation significantly simplifies the formulas that we need to implement on the blockchain.

The probability at time 0 that the AMM defaults at the end of the single period is given by

$$Q(\kappa) = \mathbf{P}\{M_2\tilde{s}_{2,t} \leq \kappa(\tilde{s}_{2,t} - s_{2,0})\}. \tag{6}$$

To calculate the probability we again express the price at the end of the period in terms of returns, $\tilde{s}_{2,t} = s_{2,0}\exp(\tilde{r}_2)$. We can then rewrite Equation (6) as

$$Q(\kappa) = \mathbf{P}\left\{\exp(\tilde{r}_2)\left(s_{2,0}(M_2 - \kappa)\right) \leq -\kappa s_{2,0}\right\} \tag{7}$$

$$= \begin{cases} \Phi\left(\text{sgn}(\kappa)\frac{\ln((\kappa - M_2)/\kappa) + (\mu - \sigma^2/2)}{\sigma}\right), & \text{if } M_2 < \kappa \text{ or } \kappa < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Similar to (5), if the collateral held is larger than the traded amount ($M_2 > \kappa$, $\kappa > 0$), the AMM cannot default because the collateral is in the same currency and direction as the contract, and thus the collateral can not be exceeded.

According to the risk-neutral pricing theory, the price of a digital option that pays one unit in case of a default is given by the discounted expected value of the future payoff under the unique risk-neutral measure, see e.g., [Björk, 2009]. If $N(\mu, \sigma)$ is the risk-neutral distribution, the insurance price per unit corresponds to the probability $Q(\kappa)$ presented in Equation (7).

We embed the insurance premium in the price that the AMM offers to the trader, i.e., we could determine the price as

$$p = s_{2,0}(1 + \text{sgn}(\kappa)Q(\kappa)), \tag{9}$$

where $\text{sgn}(\cdot)$ is the sign function. The short trader would get a lower price than the spot index $s_{2,0}$. The long trader would get a higher price than the spot. The larger the trade, the more the price moves against the trader. This concludes our simplified example and we proceed to the realistic setup in the next section. In what follows we focus on the digital option approach to pricing, and leave the details and comparison to the exact insurance approach to Appendix B.1.

4.2 AMM Price Derivation

Building on the simplified example from the previous section, we now assume that we have an arbitrary amount of traders, and we also allow for the liquidity pool to have any pre-determined currency. We stay in the simple one-period world which differs from the underlying instrument that can continuously default and has no defined maturity.[3]

We denote the first currency of a pair (e.g., BTC/USD) as *base currency* and the second currency as *quote currency*, as is market convention. The pricing framework also allows for collateral denoted in a third currency termed *quanto currency*. For example, if we had an S&P-500 perpetual collateralized in BTC, then the quote currency is USD, the base "currency" is S&P-500 index points, and the quanto currency is BTC.

[3]However, we note that in the traditional financial markets, credit default swaps that could also be triggered continuously, are often priced with a one-period model too. Furthermore, we are looking for a relatively simple closed-form solution for the price to be able to reasonably implement the solution on-chain. We suggest to calibrate the model based on a typical holding period for leveraged traders (e.g., 1 trading day). With this we expect that our simplified pricing method captures the risks and dynamics of the underlying instrument reasonably well, so that traders are incentivized to close the AMM exposure. We think that this is an improvement on the virtual AMM approach for which the price is largely independent of the AMM risk.

In the sequel we derive the insurance premium. The final price consists of the following two components:

1. To charge the premium $Q(\kappa)$, we factor it into the perpetual price. If the traders in the perpetual have a net long exposure after the trade, $\kappa - \kappa^\star > 0$ (where $\kappa^\star$ is the trade-size that brings the AMM to its minimal risk), the premium is added to the spot price. That is, the long traders face an unfavorable price, the short a favorable price. Vice versa for $\kappa - \kappa^\star < 0$, where the short get an unfavorable price.
2. We allow for a minimal half bid-ask spread δ.

With these components, the trader's fill price (measured in quote currency) is given by the following equation:

$$p(\kappa) = s_{2,0}\Big(1 + \mathsf{sgn}(\kappa - \kappa^\star)Q(\kappa) + \delta\mathsf{sgn}(\kappa) + \delta_i G(\kappa)\Big), \tag{10}$$

where $Q(\kappa)$ is the risk-neutral price of a digital option when trading a position of (signed) size κ, $s_{2,0}$ is the spot index price of the underlying instrument that we observe in the spot market, $\kappa^\star$ is the trade size that minimizes the AMM risk, δ is the minimal half bid-ask spread, G is an additional slippage function detailed in Appendix B.5, and δ_i is the maximal additional slippage introduced thereby. Figure 4 illustrates how the perpetual price deviates from spot. The premium, $p(\kappa) - s_{2,0}$, is directly charged or rebated to the trader's margin account after the trade, so this part of the P&L is realized instantly. We now derive $Q(\kappa)$ and derive $\kappa^\star$ next in Section 4.3.

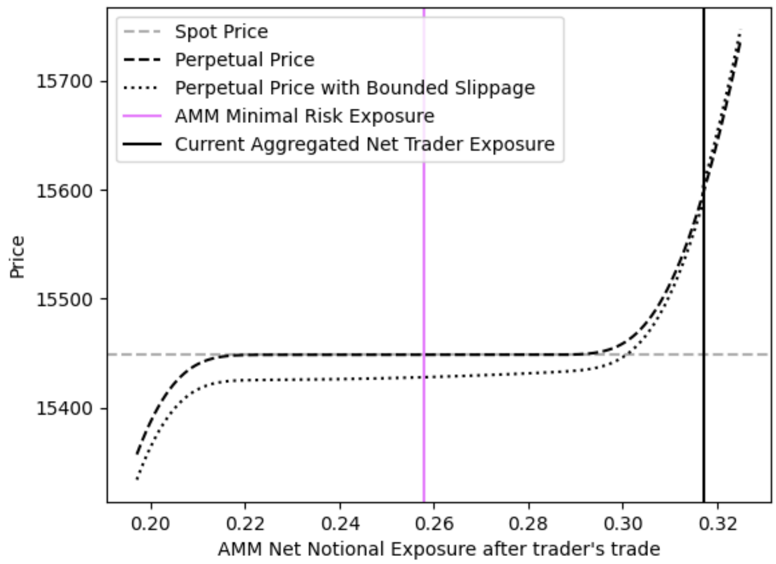

Figure 4: Pricing Curve. This figure plots the perpetual price (y-axis), as a function of the resulting aggregate AMM position $K_2 + \kappa$ (x-axis). The vertical line at K_2 (≈ 0.32) represents the current net aggregated trader positions. In this case, most traders are long ($K_2 > 0$) and the price deviates from spot with a positive spread. The dotted curve shows the pricing curve when including bounded slippage, an extra slippage term.

AMM fund. The AMM can maintain collateral in quote currency, base currency, or quanto currency. We define the following variables:

$$M_1 : \text{Size of AMM capital held in quote currency} \tag{11}$$

$$M_2 : \text{Size of AMM capital held in base currency} \tag{12}$$

$$M_3 : \text{Size of AMM capital held in quanto currency} \tag{13}$$

In practice, we only have one collateral currency per liquidity pool, but we keep the pricing framework general so that the same formulas can be used across different liquidity pools by setting the non-existent collateral amounts to zero (e.g., $M_1 = 0, M_2 = 0, M_3 \neq 0$ for a perpetual collateralized in quanto-currency).

The pricing approach conservatively ignores the AMM margin in the calculation of the default probabilities, and so the only protocol-owned funds that count toward pricing are those in the AMM fund. In addition, as detailed in section 2.1, the externally provided funds available in the PnL participation fund, if any, count toward the AMM capital in a restricted fashion: we let R be the capital in the participation fund that is not virtual at the time of pricing, and distribute it uniformly across perpetuals, so that the available capital, $M_\cdot$, is given by R divided by the number of perpetuals plus the capital in the corresponding AMM fund.

AMM profit/loss. We define K_2 as the net exposure that the AMM owes to the traders (in terms of base currency, e.g., BTC for BTCUSD and S&P-500 contracts for S&P-500/USD). We update K_2 with each new trade of size κ as $K_2 \leftarrow K_2 + \kappa$. The traded amount κ is negative if the trader went short.

We define L_1 as the "locked-in value" denoted in quote currency, that is initialized to zero and updated with each trade as $L_1 \leftarrow L_1 + p \cdot \kappa$, where assuming p is the fill price the trader received.

The single trader's profit/loss is given by $\kappa(p_{2,t} - p_{2,0})$, not accounting for funding rates. Now that L_1 contains $+\kappa p$ for each trader (we do not index κ and p with a trader or time index for brevity), we see that we obtain the sum of all trader profits/losses by $-L_1 + K_2 s_{2,t}$, if we value the profit/loss at the index spot price $s_{2,t}$.[4]

Return distributions. To assess the magnitudes of future profit/loss, we define the current spot index prices as $s_{2,0}$ and $s_{3,0}$ for the base and quanto currency respectively. The prices of the base and quanto currency for the end of the period are unknown at the beginning of the period and represented by $s_{2,0}\exp(\tilde{r}_2)$ and $s_{3,0}\exp(\tilde{r}_3)$

[4] The contract is always sized in base currency (e.g., BTC for BTCUSD), hence the index 2 for K_2. The profit/loss is measured in quote currency (e.g., USD for BTCUSD), hence the index 1 for L_1.

respectively (the notation is to use tilde ($\sim$) for random variables). We specify the following log-return distributions (in terms of the quote currency):

$$\tilde{r}_2 \sim N\left(r - \frac{1}{2}\sigma_2^2, \sigma_2^2\right) : \text{log-return distribution of base currency} \tag{14}$$

$$\tilde{r}_3 \sim N\left(r - \frac{1}{2}\sigma_3^2, \sigma_3^2\right) : \text{log-return distribution of quanto currency,} \tag{15}$$

and we assume that the returns are coupled by a correlation coefficient ρ. The constant r corresponds to the risk-free rate of return in risk-neutral valuation theory.

AMM default probability. If the trader profit (in quote currency) exceeds the AMM collateral, the AMM defaults. Assuming a trader aims to enter with a trade of κ contracts, we can write the AMM default probability after the trade as follows

$$Q(\kappa) = \mathbf{P}\{L_1 + \kappa s_{2,0} - \kappa s_{2,0}e^{\tilde{r}_2} - K_2 s_{2,0}e^{\tilde{r}_2} + M_1 + M_2 s_{2,0}e^{\tilde{r}_2} + M_3 s_{3,0}e^{\tilde{r}_3} \leq 0\} \tag{16}$$

$$= \mathbf{P}\left\{e^{\tilde{r}_2}\left(s_{2,0}(M_2 - \kappa - K_2)\right) + e^{\tilde{r}_3} M_3 s_{3,0} \leq -L_1 - \kappa s_{2,0} - M_1\right\}, \tag{17}$$

where the index 0 denotes the beginning of the period which in practice corresponds to the current time.

There is no closed-form solution for the distribution of the sum of two log-normally distributed random variables as we have in Equation (17). We therefore distinguish two cases

1. No pool M_3. We have a log-normal distribution and an exact solution.
2. The liquidity pool is held in quanto-currency for the given perpetual. We approximate the distribution with a normal distribution.

In general, the solution to $Q(\kappa)$ is as follows

Without quanto-pool.

$$Q(\kappa) = \begin{cases} Q^+(\kappa) & \text{if } M_2 - \kappa - K_2 > 0 \text{ and } -L_1 - \kappa s_{2,0} - M_1 > 0 \\ 0 & \text{if } M_2 - \kappa - K_2 \geq 0 \text{ and } -L_1 - \kappa s_{2,0} - M_1 \leq 0 \\ 1 - Q^+(\kappa) & \text{if } M_2 - \kappa - K_2 < 0 \text{ and } -L_1 - \kappa s_{2,0} - M_1 < 0 \\ 1 & \text{if } M_2 - \kappa - K_2 \leq 0 \text{ and } -L_1 - \kappa s_{2,0} - M_1 > 0 \text{ and } M_3 = 0 \end{cases} \tag{18}$$

where $Q^+(\kappa)$ is defined as follows.

$$Q^+(\kappa) = \Phi\left(\frac{1}{\sigma_2}\left[\ln\left(\frac{-L_1 - \kappa s_{2,0} - M_1}{s_{2,0}(M_2 - \kappa - K_2)}\right) - \mu_Y\right]\right) \tag{19}$$
$$\mu_Y = r - \frac{1}{2}\sigma_Y^2$$

With quanto-pool. Appendix B.2 details the approximation. The solution is

$$Q(\kappa) = 1 - \Phi\left(\frac{L_1 + \kappa s_{2,0} + M_1 + e^r \mu_Z}{e^r \sigma_Z}\right) \tag{20}$$
$$\mu_Z = s_{2,0}(M_2 - \kappa - K_2) + s_{3,0}M_3 \tag{21}$$
$$\sigma_Z^2 = s_{3,0}^2 M_3^2 (e^{\sigma_3^2} - 1) + s_{2,0}^2 (M_2 - \kappa - K_2)^2 (e^{\sigma_2^2} - 1) + 2 s_{2,0} s_{3,0} (M_2 - \kappa - K_2) M_3 (e^{\rho\sigma_2\sigma_3} - 1)$$

To understand the cases in which the probability goes to one or zero, it helps to think in terms of Equation (16).

4.3 Optimal Trade Size $\kappa^\star$

We define the optimal trade size $\kappa^\star$ as the trade that brings the exposure of the AMM to its minimum, given the current state of the AMM. For $\kappa^\star$ we have that, approximately, $Q(\kappa^\star + \delta\kappa) = Q(\kappa^\star - \delta\kappa)$, for any $\delta\kappa$. Figure 5 illustrates this. Note that the 'distance to default' graph in Figure 5 plots an artificial value of -100 for the area where the default probability is zero. The current aggregated trader position K_2 is plotted in the top graph (the vertical line close to 0.32). The K_2 that results from an optimal trade of size $\kappa^\star$ is represented by the vertical line in the figure.

For the case of $M_3 = 0$, making use of equation (18), the AMM risk is seen to be identically zero for all κ in the interval

$$\left[-s_{2,0}^{-1}(L_1 + M_1),\ M_2 - K_2\right]$$

This interval is non-empty: otherwise the AMM would have already defaulted, c.f. the event in equation (17) with $\kappa = \tilde{r}_2 = \tilde{r}_3 = 0$. Therefore, in this case, κ^* can take any value in the given range.

In the case $M_3 \neq 0$, and assuming further that $r \equiv 0$, equation (20) can be directly differentiated with respect to κ to get that $Q'(\kappa) = 0$ if and only if $\kappa = \kappa^*$, where

$$\kappa^* = M_2 + \frac{s_{3,0}(e^{\rho\sigma_2\sigma_3} - 1)}{s_{2,0}(e^{\sigma_2^2} - 1)} M_3 - K_2$$

Noting that the above reduces to the right end-point of the interval for the no-quanto case, we choose to define κ^* by this formula in the general case, thus removing the need for two separate definitions of an optimal trade size.

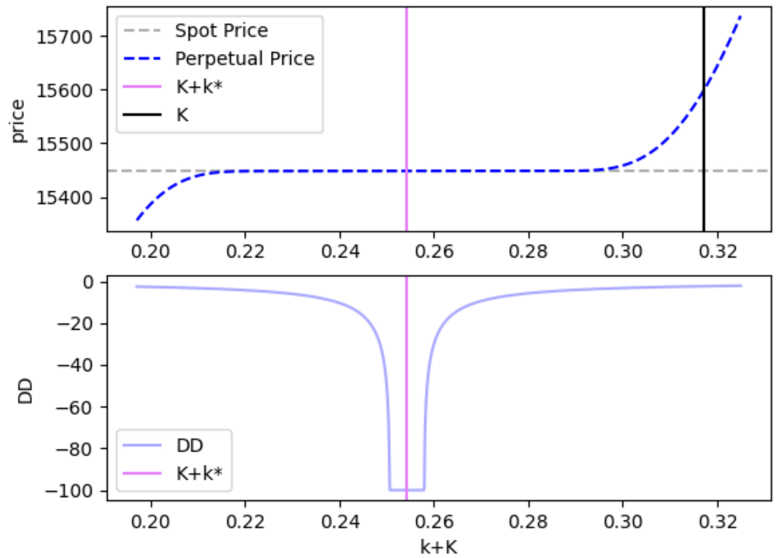

Figure 5: Optimal Trade Size. The lower plot shows the argument to the normal cumulative distribution function of $Q(\kappa)$, in the default risk literature often termed distance to default (DD). The optimal trade size lies where the distance to default is minimized. The vertical line denotes the center of the area where there is a minimum. The dashed curve in the top graph shows the corresponding price (without minimal spreads and bounded slippage for simplicity).

5 Mark Price and Funding Rate

The funding rate is a payment made from one side of the trade to the other (from long or short or vice versa). When the funding rate is negative, shorts pay longs, when funding is positive longs pay shorts. The purpose of the funding rate is to pull the perpetual price back to the spot price, and with that to decrease the AMM net exposure.

We follow other exchanges like BitMEX and Deribit and define a *Mark Price*. Typically the Mark Price is defined as a an exponentially weighted moving average of the difference between the perpetual "mid-price" and index price and that average is added to the index price. Our simulations have shown that with a blockchain implementation where we face delays, it is preferable to define the moving average on a relative difference between mid-price and index, rather than an absolute difference, hence we define a *rate*.

We define the *Mark Premium Rate* as the exponentially weighted moving average of the relative deviation of the perpetual mid-price p_t from the spot index s_t:

$$\bar{r}_t = \lambda \bar{r}_{t-1} + (1-\lambda)(p_t/s_t - 1), \tag{22}$$

where λ is the ewma parameter.

We calculate the funding rate from the premium rate as follows:

$$f_t = \max\left[\bar{r}_t, \Delta\right] + \min\left[\bar{r}_t, -\Delta\right] + \text{sgn}(K_2) b, \tag{23}$$

where b is a rate that is exchanged even if the premium rate is between the clamps (e.g., 0.05%). Figure 6 plots this function. The rate b incentivizes traders to net the AMM position K_2 to zero, even if the premium rate is insignificant. If f_t is positive, the long pays the short, if f_t is negative, the short pays the long. There is no fee on funding rates and the rate is paid directly peer to peer.

For each block, we use the last observed $\bar{r}_t$ of the previous block as the relevant mark price. This avoids within-block price manipulation.

Like BitMEX, we impose a cap on the Funding Rate to ensure the maximum leverage can still be utilized. The absolute Funding Rate is capped at 90% of Initial Margin - Maintenance Margin. For example, if the difference between Initial Margin and the Maintenance Margin is 2%, the maximum Funding Rate will be 1.8%.

Finally, we obtain the *Mark Price*, $\bar{s}_2$, by imputing the Mark Premium Rate to the index price, omitting time-indices:

$$\bar{s}_2 = s_2(1+\bar{r}). \tag{24}$$

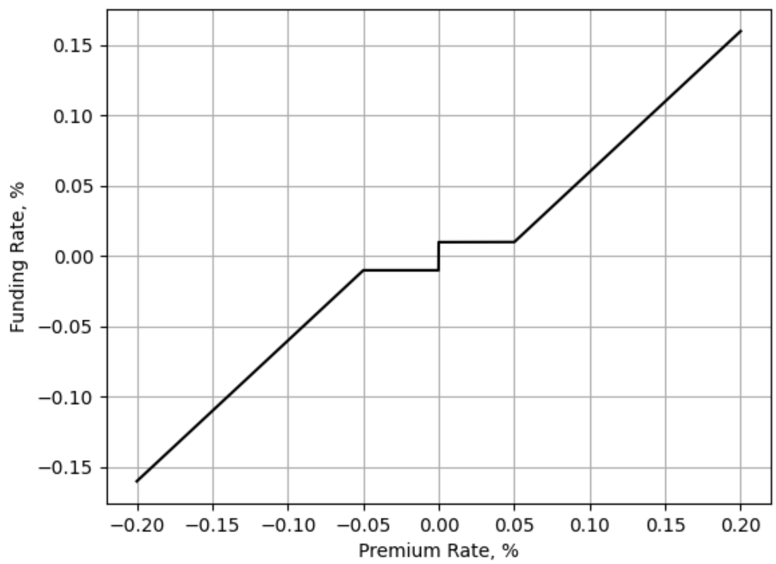

Figure 6: Funding Rate. The funding rate is equal to a signed constant when the mark premium rate is between bounds, here between -0.05% and 0.05%. The payment of the (absolute) rate flows from long to short traders if the net trader exposure is greater than zero. A negative rate corresponds to a payment from the short to the long that occurs if the traders' net exposure is negative. The sign switches where the aggregate trader exposure K equals zero. The (absolute) funding rate increases linearly with the mark premium rate outside the bounds. The mark premium rate is defined as an exponentially weighted moving average of the relative deviation of the perpetual price to the spot index price.

where s_2 is the index price and $\bar{r}$ the Mark Premium Rate. The Mark Price is used as the relevant price for liquidations and unrealized P&L displayed in the front-end. Any market participant can call the smart contract with a trader address to liquidate the trader. The smart contract calculates the liquidation amount, and, if the trader is liquidated, the liquidator earns a fee. Appendix C, 'Lemmas', provides formulas for leverage and liquidation, which is more challenging in the multi currency setup presented here, as opposed to the single currency case (such as BTCUSDC collateralized in USDC).

6 Capital Limits

In this section, we define a target size for the AMM fund size, and the default fund size. If the target sizes are exceeded, proceeds from trading are no longer used to increase these two funds.

Within a liquidity pool, we sum up each AMM fund to arrive at the total AMM fund size. As long as the total pool size does not reach the sum of the individual target sizes, profit and fees are used to increase the corresponding AMM fund. If the perpetual specific AMM fund is filled (but not all pools), the profits are split equally over all funds to help capitalize. Once the target size is reached, the funds are sent to the default fund.

6.1 Default Fund Target Size

In line with the AMM funds, default funds are held in the liquidity pool specific collateral currency which can be any currency:

$$I_1 : \text{default fund size held in quote currency} \tag{25}$$

$$I_2 : \text{default fund size held in base currency} \tag{26}$$

$$I_3 : \text{default fund size held in quanto currency.} \tag{27}$$

The default fund covers the liquidation fees paid to the liquidator if the trade could not be closed before the trader margin is used up. Its secondary and more important use is to serve as a last resort to cover AMM losses after all other funds are used up (AMM margin, AMM fund, PnL participation fund). Traditional clearing houses have default funds that are setup to cover trader losses. Default funds are usually sized according to the "cover 2" standard in which the fund should cover a simultaneous default of the largest two members' during the most extreme but plausible market stress. See e.g., [LCH, 2021]. We follow a similar approach.

Cover two. We replace the "cover 2" method with a more computationally efficient version of a "cover n" approach. That is, we use a summary-statistics of trader position sizes to arrive at a

$$\text{"representative" position size} : \bar{\Pi}. \tag{28}$$

Instead of fixing n we set a parameter to a relative amount n_r e.g., 5% of all active accounts that we aim to cover in case of default, but at least 5 traders:

$$n = \max[n_R A, 5], \tag{29}$$

where n_R is a parameter, and A the current number of active traders. Since every trade is closed with one or several opposite trades of the same aggregated size, we do not register long and short seperately for $\bar{\Pi}$. We then set the target size of the default fund so that the fund alone is able to cover a simultaneous default of the AMM and n representative traders with position size $\bar{\Pi}$ in a severe but plausible stress scenario. We use a similar statistic to maintain a representative AMM exposure for K_2. Here we distinguish typical long and short sizes. We term the representative AMM exposure sizes $\bar{K}_2^+$ and $\bar{K}_2^-$ for long and short respectively.

We choose a version of an exponentially weighted moving average (EWMA). We define an EWMA with a decay factor, λ, that differs if the position value after the trade exceeds the average:

$$\lambda = 1_{\{x>\bar{x}\}}\lambda_1 + (1 - 1_{\{x>\bar{x}\}})\lambda_2 \tag{30}$$

$$\bar{x} \leftarrow \lambda\bar{x} + (1 - \lambda)x. \tag{31}$$

Figure 7 simulates the behavior of this equation for fictious trade sizes (we chose $\lambda_1 = 0.5$, $\lambda_2 = 0.99$). Once there is a large trade, the representative position size jumps up and only slowly recovers.

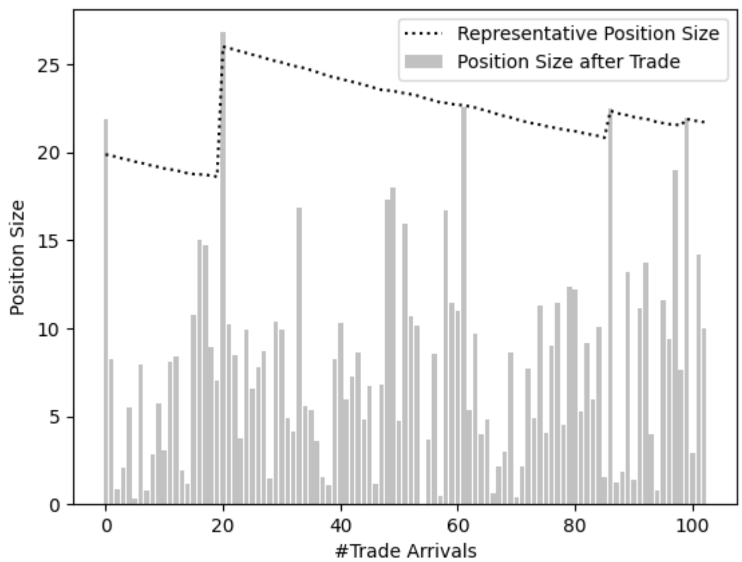

Figure 7: Representative Trader Position Size. For this figure we simulate trade arrivals and show the resulting position size (for the position corresponding to the trade). We then calculate the "representative" position size to explain our variant of the "cover 2"-rule applied at clearing houses. The representative position size is chosen via an exponentially weighted moving average (EWMA) of the positions resulting from new trades. The EWMA weights depend on whether the trader’s post-trade position size exceeds the current EWMA or not. Once there is a large trade, the representative position size (dotted line) jumps up and only slowly recovers. Only opening trades enter this calculation.

6.1.1 Stress Scenarios

The exchange could default either if the market has a sharp upwards move, or if there is a market crash. We have the following two situations:

- $K_2 < 0$: the traders gain if the price drops and the AMM is exposed to their gain with K_2 contracts. If at the same time, n long traders ($\kappa > 0$) default, the AMM has to cover the gains that these long traders would have covered with their margin losses
- $K_2 > 0$: the traders gain if the price increases and the AMM is exposed to their gain with K_2 contracts. If at the same time, n short traders ($\kappa < 0$) default, the AMM has to cover the gains that these short traders would have covered with their margin losses

To calculate a loss for the stress event we define extreme but plausible returns for the relevant currencies (the base currency and the quanto currency). We only need to define *negative* extreme returns for the quanto currency, because the AMM has no short exposure to the quanto currency. We define two stress scenarios:

$$(r_2^-, r_3^-) : \begin{cases} r_2^- \text{ is a negative extreme return for the base currency} \\ r_3^- \text{ a simultaneous negative extreme return for quanto currency} \end{cases} \tag{32}$$

$$(r_2^+, r_3^\mp) : \begin{cases} r_2^+ \text{ is a positive extreme return for the base currency} \\ r_3^\mp \text{ a simultaneous negative extreme return for the quanto currency} \end{cases} \tag{33}$$

If there is positive correlation between the quanto currency and the base currency, we would choose the negative quanto return in the upward scenario $r_3^\mp$ smaller than the negative quanto return in the downward scenario r_3^-.

The goal is to evaluate the losses in the two stress events (r_2^-, r_3^-) and $(r_2^+, r_3^\mp)$ to size the default fund so that we can cover both losses. We express the relative loss in the two scenarios as ℓ^- and ℓ^+ respectively.

$$\ell^- = (\bar{K}_2^- + n\bar{\Pi})(1 - e^{r_2^-}), \tag{34}$$

$$\ell^+ = (\bar{K}_2^+ + n\bar{\Pi})(e^{r_2^+} - 1), \tag{35}$$

where the representative trade sizes $\bar{K}_2^-$ (short aggregated trader exposure), $\bar{K}_2^+$ (long aggregated trader exposure), and $\bar{\Pi}$ are all represented with absolute values. The dollar-loss (or the loss expressed in quote currency) is $s_{2,0}\ell^+$ or $s_{2,0}\ell^-$ respectively if the corresponding stress event realizes, measured at the spot index price.

Now, the target fund size has to be set so that the collateral is larger than the larger of the two stress losses, that is,

$$I_1 + I_2 s_{2,0} e^{r_2^+} + I_3 s_{3,0} e^{r_3^{\mp}} \overset{!}{\geq} s_{2,0}\ell^+ \qquad \text{if } \ell^+ > \ell^- \tag{36}$$

$$I_1 + I_2 s_{2,0} e^{r_2^-} + I_3 s_{3,0} e^{r_3^-} \overset{!}{\geq} s_{2,0}\ell^- \qquad \text{if } \ell^+ \leq \ell^- \tag{37}$$

Since typically the collateral is held in one currency only, Equations (36) and (37) can be solved for the corresponding fund size. We denote the amount that satisfies equality with a star:

$$I_1^\star = s_{2,0}\max[\ell^+, \ell^-], \quad I_2^\star = \max\left[\frac{\ell^+}{e^{r_2^+}}, \frac{\ell^-}{e^{r_2^-}}\right] \quad I_3^\star = \frac{s_{2,0}}{s_{3,0}}\max\left[\frac{\ell^+}{e^{r_3^{\mp}}}, \frac{\ell^-}{e^{r_3^-}}\right] \tag{38}$$

To summarize, the parameters required are the stress returns $r_2^+, r_2^-, r_3^-, r_3^{\pm}$, the relative number of defaulters n_R to consider for the cover-n rule, and finally the 2 parameters (λ_1, λ_2) for each of $\bar{\Pi}$ and $\bar{K}_2^{-/+}$ used for the EWMA.

6.2 AMM Fund Target Size

The AMM capital available to each perpetual, plus a fraction of the LP funds, are used in the pricing approach. If the AMM is not well stocked compared to the current AMM exposure, the trader who further increase the exposure will face adverse prices. On the other hand, if the AMM is filled with collateral, traders can expose the AMM to a large exposure K_2 and observe little slippage but profit from the "cheap capital" supplied by the AMM. Therefore, we aim to stock the AMM fund so that we reach a target insurance premium, q^*, for non-zero exposure, while incorporating LP funds.

We are looking for the target AMM fund sizes M_1°, M_2°, or M_1°, having the collateral in either quote, base, or quanto currency respectively. To derive $M_\cdot^\circ$ (where the dot stands for any of the 3 currencies), we proceed as follows. First, we define the AMM target size as a function of a target insurance premium in the next paragraph. Then we use this function in Section 6.2.2 to define a target that dynamically adjusts based on how well the default fund is funded. Finally we arrive at $M_\cdot^\circ$ in Section 6.2.3, where we incorporate the liquidity provider funds.

6.2.1 Target Insurance Premium

We define a target insurance premium q^* and solve the insurance premium-equation $Q^+(0)$ for the relevant capital M_1, M_2, or M_3. We denote the resulting formula by the function $m_k(q^*)$ for $k \in \{1, 2, 3\}$ indexing the collateral type. Again, we assume that the AMM capital is held in only one of the three currencies involved.

To keep a minimal amount of capital even if the AMM has zero net exposure, we define the constants C_{M1}, C_{M2}, and C_{M3} below which the AMM capital target size should not fall. These definitions lead to the following equations.

1. Collateral in quote currency: $M_1 \neq 0, M_2 = 0, M_3 = 0$.

$$m_1(q^*) = \begin{cases} K_2 s_{2,0} \exp\left(\mu_2 + \sigma_2 \Phi^{-1}(q^*)\right) - L_1 & \text{if } K_2 < 0 \\ K_2 s_{2,0} \exp\left(\mu_2 - \sigma_2 \Phi^{-1}(q^*)\right) - L_1 & \text{if } K_2 > 0 \\ \emptyset & \text{if } K_2 = 0 \end{cases} \tag{39}$$

$$m_1^*(q^*) = \max[m_1(q^*), C_{M1}] \tag{40}$$

2. Collateral in base currency: $M_1 = 0, M_2 \neq 0, M_3 = 0$.

$$m_2(q^*) = \begin{cases} K_2 - \frac{L_1}{s_{2,0} \exp(\mu_2 + \sigma_2 \Phi^{-1}(q^*))} & \text{if } L_1 < 0 \text{ and } K_2 \neq 0 \\ K_2 - \frac{L_1}{s_{2,0} \exp(\mu_2 - \sigma_2 \Phi^{-1}(q^*))} & \text{if } L_1 > 0 \text{ and } K_2 \neq 0 \\ \emptyset & \text{if } K_2 = 0 \text{ or } L_1 = 0 \end{cases} \tag{41}$$

$$m_2^*(q^*) = \max[m_2(q^*), C_{M2}] \tag{42}$$

3. Collateral in quanto currency: $M_1 = 0, M_2 = 0, M_3 \neq 0$.

$$m_3(q^*) = \text{...see Appendix B.3} \tag{43}$$

$$m_3^*(q^*) = \max[m_3(q^*), C_{M3}] \tag{44}$$

$m_3(q^*)$ is a solution to a quadratic equation.

The function $\Phi^{-1}(\cdot)$ is the inverse of the normal cumulative distribution function, so that $\Phi^{-1}(\Phi(x)) = x$. The empty set, $\emptyset$, corresponds to situations in which the target probability cannot be achieved. For example, if collateral is held in base currency and $L_1 = 0$ we see from Equation (16) that the default probability is either 1 if $K_2 > M_2$ or 0 if $K_2 \leq M_2$, but there is no state between. The amounts $m_\cdot(q^*)$ that lead to the target insurance premium q^* can be negative, meaning the target default probability can only be reached with a hypothetical negative AMM collateral so with zero collateral, the default probability is smaller than the target probability.

6.2.2 Risk-dependent Target Sizes

We now use the functions $m^*_\cdot(q^*)$ derived above to arrive at risk-dependent AMM target sizes.

We maintain two different target default probabilities, one is our baseline, the other is used when the system is stressed. If the AMM fund reached its baseline target size, we send profit and fee earnings to the default fund, otherwise the proceeds are kept in the AMM fund. If the AMM fund is below its stress target size, we send funds from the default fund to the AMM fund. This method avoids that we drain the default fund by offering overly small spreads when markets are volatile, as we learned from simulations.

In light of the above, it is clear that the baseline target size plays a two-fold role: on one hand it determines how competitive the slippage offered by the AMM will be, in average, provided the protocol has sufficient capital; on the other hand, it also determines how quickly profits from the AMM margin flow into the default fund. Should the default fund fall below its target size, this logic remains in place, but the baseline target size adapts accordingly: it linearly decreases, proportionally to the size of the gap between the current default fund size and its target. This allows the default fund to acquire funds faster at the expense of increased slippage, thus reflecting more accurately the risk to the protocol. In symbols, the target size for the available capital then becomes

$$M^* \leftarrow m^*_\cdot(q^*_{\text{stress}}) + r\left[m^*_\cdot(q^*_{\text{baseline}}) - m^*_\cdot(q^*_{\text{stress}})\right], \tag{45}$$

omitting the index 1,2,3, for brevity, where r is the ratio between the default fund size and its target size, capped at 1.

We adjust K_2 and L_1 that enter the formula for $M^*_\cdot$ above in an adverse direction:

$$K_2 \leftarrow K_2 - \text{sgn}(\kappa^\star)\bar{\Pi} \tag{46}$$

$$L_1 \leftarrow L_1 - \text{sgn}(\kappa^\star)\bar{\Pi}s_{2,0} \tag{47}$$

where $\bar{\Pi}$ is the trader position size EWMA used previously. This prevents that the AMM is stuck in a narrow trading band with low mid-price premium yet large slippage as we observed in simulations.

6.2.3 Accounting for external Liquidity Provision

Finally, we incorporate the PnL participation fund to arrive at the AMM target size $M^\circ_\cdot$.

The available capital M_1, M_2, or M_3 used to compute the insurance premium may contain funds that are not protocol-owned; namely, any portion of the PnL participation fund that is not virtual at the time of pricing. Consequently, in practice, the AMM fund target size, $M^\circ_\cdot$, can be set lower than M^* above, as it needs to account for the liquidity provider funds, if any:

$$M^\circ_\cdot = \max\{M^* - R/n_{\text{perp}}, 0\} \tag{48}$$

where R is the amount of non-virtual PnL participation funds, and n_{perp} is the number of perpetuals in the corresponding liquidity pool.

Thus, while $M^\circ_\cdot$ above accounts for both the protocol-owned and externally provided funds, M^* only accounts for the AMM fund. The AMM target size, $M^\circ\cdot$., could in theory vanish, provided there is sufficient external liquidity in the system.

7 Maximal Trade Sizes

We limit the maximal trade size to have another risk mitigating measure. We use the following principles to guide the maximal trade size.

1. First, each trader is always allowed to close their position.
2. Second, each trader should always be allowed to trade towards the exposure that minimizes the risk of the AMM, that is, a trade of size k^* is always allowed, where k^* is the position delta that minimizes the AMM risk. We can further relax this assumption and always allow a trade size of $2k^*$, because this will bring us to the same default probability, due to symmetry.
3. Third, the maximal position size should not substantially exceed the representative position size $\bar{\Pi}$, defined in Section 6.1, that is used to size the default fund.

We implement these guiding principles as follows. The maximal position size is given by

$$\bar{\Pi}\varsigma, \tag{49}$$

where ς is a scaling factor. We set ς to a fixed value for a given perpetual (e.g., 1.5) if the default fund is fully funded (fund balance > target size). Otherwise we scale the fixed value by the ratio of the default fund balance to its target size.

Now the maximal trade size is given by the difference between the maximal position size and the trader's current position Π. To implement principle (2), we allow for a trade of size $2k^*$, hence

$$\delta k^{(m)} = \begin{cases} \max\left[\bar{\Pi}\varsigma - \Pi, 2k^*\right] & \text{trader opens long} \\ \min\left[-\bar{\Pi}\varsigma - \Pi, 2k^*\right] & \text{trader opens short} \end{cases} \tag{50}$$

where Π is the trader's current position.

8 Agent based Simulations

We implemented agent-based simulations to assess the profit/loss over time and investigate inverse scenarios. Simulations are a great tool to investigate the perpetual parameters.

The simulation mimics the AMM and we also benchmark smart contract implementation against the simulation results in a small setting. Perpetual index prices are taken from historical data for the corresponding perpetual. The agents involved are liquidity providers and traders. Liquidity providers randomly add and remove funds to the system. Traders have randomized individual trading preferences with respect to their cash holdings, leverage choices, long/short choices, trading frequency and they have different strategies. The strategies are momentum trading, noise trading (essentially not a strategy), and arbitrage traders that compare the perpetual prices to the index prices when deciding whether to trade.

We parameterize the simulation so that there is a growing number of traders over time and simulate up to 1,000 traders by the end of one quarter.

The simulation results are out of scope for this paper.

9 Conclusion

Our approach for the AMM to price perpetual contracts is based on the risk-neutral valuation approach. The long pays the short an insurance premium if the AMM has a net short exposure, and vice versa. This incentivizes traders to hedge the AMM risk.

Shared liquidity pools allow for an efficient capital setup. We have measures in place to avoid sudden impacts on prices by liquidity providers. A liquidity pool consists of AMM funds, a participation fund for external liquidity providers, and a default fund. AMM funds and the default fund each have target sizes. The AMM fund size varies with profit and loss. Balances exceeding the target size are sent to the default fund. The target size of the default fund is set similar to that of default funds in clearing houses that aim to cover the default of the n largest counterparties for a conservative risk exposure under severe but realistic stress conditions. The target size of the AMM fund is set so that with current AMM exposure, the AMM default probability is below a threshold, or (equivalently) traders face a price which is not deviating too much from the spot price.

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A BitMEX Notation

BitMEX uses USD contract sizes and has an inverse notation to denote the profit and loss. We explain this difference between BitMEX and our setting using a particular example, specifically a BTCUSD perpetual collateralized in BTC.

In our setting, contract sizes k are specified in BTC. Profit and loss in USD and BTC, respectively, are calculated as

$$\pi^{(USD)} = k(P_t - P_0), \tag{51}$$

$$\pi^{(BTC)} = k(P_t - P_0)/P_t, \tag{52}$$

where k is the position size in BTC, P_t and P_0 are exit and entry prices respectively. The second equation is just the USD-profit (51) expressed in BTC at the current (time t) price.

BitMEX specify contracts in USD as follows

$$\text{BTC position} : c \text{ contracts} * 1 \text{ USD} * 1/P_0 = k \text{ BTC} \tag{53}$$

and calculate profit and loss as

$$\pi^{(BTC)} = c(1/P_0 - 1/P_t), \tag{54}$$

In this notation the position sizes are denoted in number of dollar contracts c.

We now derive equation (54) from our P&L equation (52) which shows that the two P&L are equivalent. From (53) we see that the contract size c (at contract size 1$) corresponds to the dollar value of position size k, hence $k = c/P_0$. We replace k in (52) to get

$$\pi^{(BTC)} = c/P_0(P_t - P_0)/P_t, \tag{55}$$

$$= c(1/P_0 - 1/P_t) \tag{56}$$

which corresponds to the P&L of equation (54).

To summarize, in the BitMEX setting, we denote the contract size in dollars. A trade with contract size c, defined as above, is closed with a trade of contract size $-c$, regardless of the current BTCUSD price.

However, in order to establish equivalence to BitMEX, we would have to close/initiate a trade with an exact dollar-value c – not more or less. But our pricing function is a function of k (BTC), not c (current USD value of k). Without re-engineering the pricing function to have it dependent on c, one needs to iterate over different k to find a value that matches a given c. The analogue in a non-AMM world is that in our setting quantities are quoted in BTC but in the BitMEX setting quantities are quoted in USD (see their order book).

Our model is not unique with this approach, e.g., MCDEX uses BTC (base currency) quoting.

B Pricing

In this section we detail derivations for pricing.

B.1 Insurance

The AMM is said to default if the amount owed to a trader opening a position κ exceeds the AMM margin account balance. The risk-neutral price of the corresponding insurance premium is thus given by the expected value of this excess amount under the risk-neutral measure, that is,

$$i(\kappa) = \mathbf{E}[\max\{\kappa(s_{2,t} - s_{2,0}) - (M - K_2 s_{2,t} + L_1), 0\}]$$

where we used the fact that the AMM position is $-K_2$, its locked-in value is $-L_1$, and its total cash is given by

$$M = M_1 + M_2 s_{2,t} + M_3 s_{3,t}$$

Rearranging terms, and defining $K_2' = K_2 + \kappa$ and $L_1' = L_1 + \kappa s_{2,0}$, the insurance takes the more familiar form

$$i(\kappa) = \mathbf{E}[\max\{(K_2' - M_2)s_{2,0}e^{\tilde{r}_2} - M_3 s_{3,0} e^{\tilde{r}_3} - M_1 - L_1', 0\}]$$

We define θ as the event that the integrand is positive. That is,

$$\theta := \{\kappa(s_{2,t} - s_{2,0}) - (M - K_2 s_{2,t} + L_1) > 0\} \tag{57}$$

Then we can remove the maximum from the integrand and write the insurance as

$$i(\kappa) = \mathbf{E}[\big((K_2 + \kappa - M_2)s_{2,0}e^{\tilde{r}_2} - M_3 s_{3,0} e^{\tilde{r}_3} - M_1 - L_1 - \kappa s_{2,0}\big)\, \mathbf{1}_\theta]$$

from where it is clear that, in addition to the default probability $\mathbf{P}(\theta)$, it suffices to compute the integrals

$$\mathbf{E}[e^{\tilde{r}_2}\mathbf{1}_\theta] \quad \text{and} \quad \mathbf{E}[e^{\tilde{r}_3}\mathbf{1}_\theta]$$

For this, we recall that the joint density of $x = (\tilde{r}_2, \tilde{r}_3)$ is given by

$$p(x) = \text{Const.} \exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right) \tag{58}$$

where $\mu = (r - \sigma_2^2/2, r - \sigma_3^2/2)$ and Σ is the variance-covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_2^2 & \rho\sigma_2\sigma_3 \\ \rho\sigma_2\sigma_3 & \sigma_3^2 \end{pmatrix}.$$

Thus, for an arbitrary two-dimensional vector $u = (u_2, u_3)'$, we can write

$$\begin{aligned}\mathbf{E}[e^{u_2\tilde{r}_2+u_3\tilde{r}_3}\mathbf{1}_\theta] &= \int e^{x'u}p(x)\mathbf{1}_\theta(x)dx \\ &= \text{Const.} \int \exp\left(x'u - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right)\mathbf{1}_\theta(x)dx \end{aligned} \tag{59}$$

Next we make use of the identity

$$x'u - \frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu) = \mu'u + \frac{1}{2}u'\Sigma u - \frac{1}{2}(x-\mu-\Sigma u)'\Sigma^{-1}(x-\mu-\Sigma u)$$

from where it follows that

$$\mathbf{E}[e^{\tilde{r}'u}\mathbf{1}_\theta] = e^{\mu'u+\frac{1}{2}u'\Sigma u}\int p(x-\Sigma u)\mathbf{1}_\theta(x)dx.$$

Regarding the indicator function, we observe that

$$\begin{aligned}\mathbf{1}_\theta(r) &= \mathbf{1}_{\{L_1'+M_1+(M_2-K_2')s_{2,0}e^{r_2}+M_3s_{3,0}e^{r_3}\leq 0\}} \\ &= \mathbf{1}_{\{L_1'+M_1+(M_2-K_2')s_{2,0}e^{v_2}e^{r_2-v_2}+M_3s_{3,0}e^{v_3}e^{r_3-v_3}\leq 0\}} \\ &= \mathbf{1}_{\theta_v}(r-v)\end{aligned}$$

where θ_v is the same event as θ, with the terms $M_2 - K_2'$ and M_3 are replaced by $(M_2 - K_2')e^{v_2}$ and $M_3e^{v_3}$, respectively.

Combining all of the above we see that

$$\mathbf{E}[e^{\tilde{r}'u}\mathbf{1}_\theta] = e^{\mu'u+\frac{1}{2}u'\Sigma u}\int p(x-\Sigma u)\mathbf{1}_{\theta_{\Sigma u}}(x-\Sigma u)dx = e^{\mu'u+\frac{1}{2}u'\Sigma u}\mathbf{P}(\theta_{\Sigma u}).$$

Applying this to $u = (1, 0)'$ and $u = (0, 1)'$ we obtain the final results

$$\mathbf{E}\left[e^{\tilde{r}_2}\mathbf{1}_\theta\right] = e^r\mathbf{P}\left(\theta_{(\sigma_2^2,\rho\sigma_2\sigma_3)}\right),$$

and

$$\mathbf{E}\left[e^{\tilde{r}_3}\mathbf{1}_\theta\right] = e^r\mathbf{P}\left(\theta_{(\rho\sigma_2\sigma_3,\sigma_3^2)}\right).$$

From where we conclude that

$$i(\kappa) = (K_2+\kappa-M_2)s_{2,0}e^r\mathbf{P}(\theta_{(\sigma_2^2,\rho\sigma_2\sigma_3)}) - M_3s_{3,0}e^r\mathbf{P}(\theta_{(\rho\sigma_2\sigma_3,\sigma_3^2)}) - (M_1+L_1+\kappa s_{2,0})\mathbf{P}(\theta) \tag{60}$$

B.1.1 Without Quanto

In the no-quanto case, $M_3 = 0$, we know how to compute the probability of θ analytically, as it corresponds in fact to the probability of default. We recall also that this probability is computed piece-wise, depending the sign of $M_2 - K_2 - \kappa$ and $M_1 + L_1 + \kappa s_{2,0}$, see Equation (18); it is then worth noting that the change of scale used above to arrive at (60) does not modify the signs of these quantities, meaning that the final formulae can be expressed in terms of the same piece-wise division.

The final result in this case can be written as

$$i(\kappa) = \begin{cases} (K_2+\kappa-M_2)s_{2,0}e^r\Phi(d_2) - (M_1+L_1+\kappa s_{2,0})\Phi(d_1) & \text{if } \begin{array}{c} M_2-\kappa-K_2>0, \\ -L_1-\kappa s_{2,0}-M_1>0 \end{array} \\ 0 & \text{if } \begin{array}{c} M_2-\kappa-K_2\geq 0, \\ -L_1-\kappa s_{2,0}-M_1\leq 0 \end{array} \\ (K_2+\kappa-M_2)s_{2,0}e^r(1-\Phi(d_2)) - (M_1+L_1+\kappa s_{2,0})(1-\Phi(d_1)) & \text{if } \begin{array}{c} M_2-\kappa-K_2<0, \\ -L_1-\kappa s_{2,0}-M_1<0 \end{array} \\ (K_2+\kappa-M_2)s_{2,0}e^r - (M_1+L_1+\kappa s_{2,0}) & \text{if } \begin{array}{c} M_2-\kappa-K_2\leq 0, \\ -L_1-\kappa s_{2,0}-M_1\geq 0 \end{array} \end{cases} \tag{61}$$

where

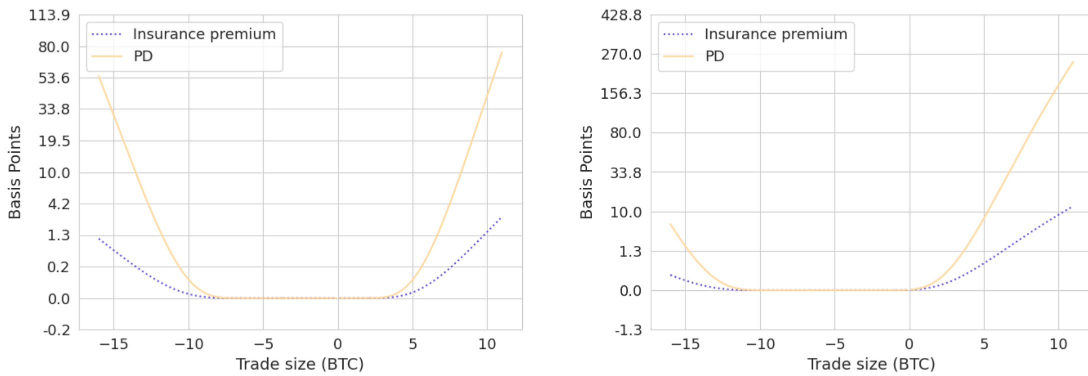
$$d_1 = \frac{1}{\sigma_2}\left(\ln\left(\frac{M_1+L_1+\kappa s_{2,0}}{(K_2+\kappa-M_2)s_{2,0}}\right) - r + \frac{\sigma_2^2}{2}\right) \tag{62}$$

$$d_2 = \frac{1}{\sigma_2}\left(\ln\left(\frac{M_1+L_1+\kappa s_{2,0}}{(K_2+\kappa-M_2)s_{2,0}}\right) - r - \frac{\sigma_2^2}{2}\right) \tag{63}$$

In Figure 8 we compare the premia derived from the insurance approach with the probability of default. The latter is seen to be more conservative for the range of parameters that we are interested in. This, in combination with its much simpler formulation and clear interpretation as the price of a digital option, justifies using this formulae for the pricing of perpetual futures.

B.1.2 With Quanto

There is no explicit formula for the probability of default, and consequently neither for (60), when the collateral is held in quanto currency, $M_3 \neq 0$. However, we observe in figurethat the approximation detailed in B.2 is in fact more conservative than the insurance approach, which we compute directly from the definition by Monte-Carlo.



(a) Base Collateral **(b)** Quote Collateral

Figure 8: Insurance per position value and Probability of Default: Quote collateral. The solid line is the probability of default of the AMM, accounting for a newly opened position of size κ (on the x-axis). The dashed line is the insurance cost per position value, $i(\kappa)/|\kappa s_{2,0}|$. We see that the PD is always more conservative than the insurance.

B.2 Moment Matching for Quanto Currency

In this section, we approximate the probability given by

$$\mathbf{P}\left\{s_{2,0}(M_2 - \kappa - K_2)e^{\tilde{r}_2} + s_{3,0}M_3 e^{\tilde{r}_3} \le -L_1 - \kappa s_{2,0} - M_1\right\}, \tag{64}$$

where

$$\tilde{r}_2 \sim N(r - \frac{1}{2}\sigma_2^2, \sigma_2^2), \qquad \tilde{r}_3 \sim N(r - \frac{1}{2}\sigma_3^2, \sigma_3^2) \tag{65}$$

for the general case where M_3 is not necessarily zero, i.e. collateral is held in a third currency on top of the base currency and quote currency.

The left hand side of the probability is a sum of two log-normal random variables of the form $\tilde{Y} = a\tilde{X}_2 + b\tilde{X}_3$ with $\tilde{X}_2 = e^{\tilde{r}_2}$, $\tilde{X}_3 = e^{\tilde{r}_3}$, where

$$a = s_{2,0}(M_2 - \kappa - K_2), \quad b = s_{3,0}M_3. \tag{66}$$

Often, a log-normal approximation is a good choice when approximating the sum of two log-normals, see e.g., [Henriksen, 2008]. However, in our case the variable a can be negative, leading to non-zero probability for negative values of Y. This makes the log-normal distribution, having a strictly positive support, a weak choice. [Borovkova et al., 2007] face a similar challenge in which the random variable is a weighted sum of log-normals with possibly negative weights. [Borovkova et al., 2007] suggest to use a shifted log-normal distribution, or a negative shifted log-normal distribution approximated using moment-matching. We prefer to use a normal

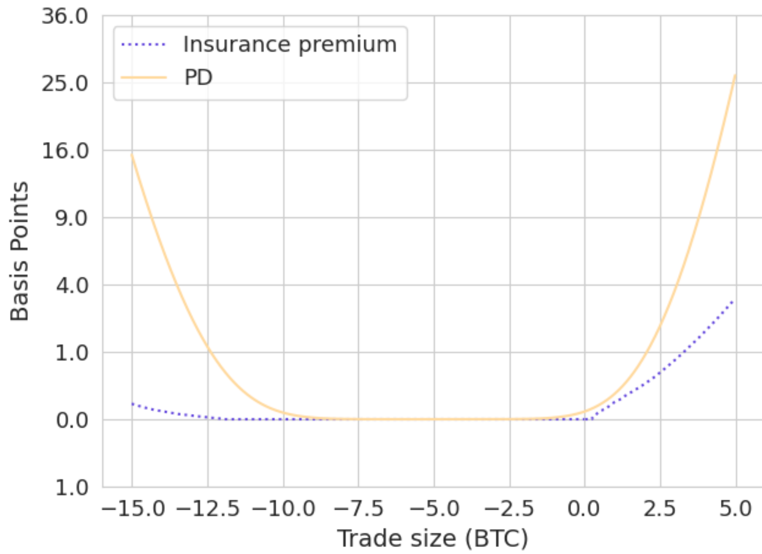


Figure 9: Insurance per position value and Probability of Default: Quanto collateral. The solid line is the approximation to the probability of default of the AMM using moment matching, and accounting for a newly opened position of size κ (on the x-axis). The dashed line is the insurance cost per position value, $i(\kappa)/|\kappa s_{2,0}|$, obtained by Monte-Carlo. We see that the approximate PD is always more conservative than the insurance premium.

approximation for the following reasons. First, the moment matching procedure is available in closed form using the normal distribution. Second, the (negative) shifted log-normals has a minimal (or maximal) value for its support which is equal to the shift parameter. Using moment matching, a considerable amount of the exact distribution can be cut off because of this limit, resulting in sub-optimal probability estimates.

We approximate $\tilde{Y}$ with a normal random variable $\tilde{Z}$ of the form

$$\tilde{Z} \sim N(\mu_Z, \sigma_Z) \tag{67}$$

by matching the first and second moment of $\tilde{Z}$ with those of $\tilde{Y}$.

For a log-normal random variable with $\ln(\tilde{X}) \sim N(\mu, \sigma^2)$, the mean and variances are given by

$$\mathbb{E}(\tilde{X}) = e^{\mu + \frac{1}{2}\sigma^2} \qquad \text{Var}(\tilde{X}) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

We use this to derive mean and variance of $\tilde{Y}$. Further, we recall that $\text{Var}(\tilde{X}_1 + \tilde{X}_2) = \text{Var}(\tilde{X}_1) + \text{Var}(\tilde{X}_2) + 2\text{Cov}(\tilde{X}_1, \tilde{X}_2)$ and $\text{Cov}(\tilde{X}_1, \tilde{X}_2) = \mathbb{E}(\tilde{X}_1\tilde{X}_2) - \mathbb{E}(\tilde{X}_1)\mathbb{E}(\tilde{X}_2)$ to arrive at the variance:

$$\mu_Y = e^r(a+b) \tag{68}$$
$$\sigma_Y^2 = e^{2r}\left((e^{\sigma_2^2}-1)a^2 + (e^{\sigma_3^2}-1)b^2 + 2(e^{\rho\sigma_2\sigma_3}-1)ab\right) \tag{69}$$

By the method of moments, we set $\mu_Z = e^{-r}\mu_Y$ and $\sigma_Z = e^{-r}\sigma_Y$ and we can solve the approximate default probability

$$\mathbf{P}\left\{s_{2,0}(M_2-\kappa-K_2)e^{\tilde{r}_2} + s_{3,0}M_3e^{\tilde{r}_3} \leq \theta\right\} \approx \mathbf{P}\left(\tilde{Z} \leq \theta\right), \tag{70}$$

where θ is the relevant threshold. Normalizing leads to the final formula. Figure 10 shows an example of the approximation.

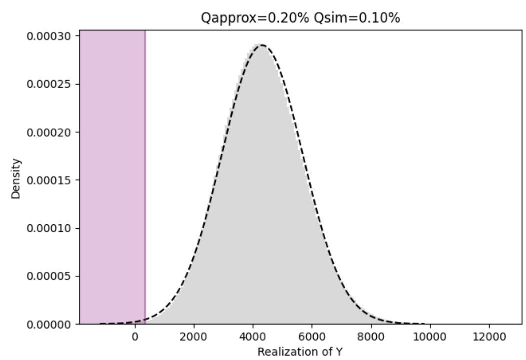


Figure 10: Normal approximation. If capital is held in quanto currency, the distribution needs negative support. This figure compares 1e7 simulations of the random variable Y (histogram-area) that we approximate via moment-matching of a normal pdf (dashed line). We also calculate the approximated insurance premium (Qapprox) and the simulated one (Qsim). The vertical, shaded area to the left denotes the area of default.

Approximated Default Probability for $M_3 \neq 0$

$$\mathbf{Q} \approx \Phi\left(\frac{1}{\sigma_Y}\left[L_1 - \kappa s_{2,0} - M_1 - \mu_Y\right]\right) \tag{71}$$
$$= 1 - \Phi\left(\frac{L_1 + \kappa s_{2,0} + M_1 + e^r\mu_Z}{e^r\sigma_Z}\right) \tag{72}$$

where

$$\mu_Z = s_{2,0}(M_2-\kappa-K_2) + s_{3,0}M_3 \tag{73}$$
$$\sigma_Z^2 = s_{2,0}^2(e^{\sigma_2^2}-1)(M_2-\kappa-K_2)^2 + (e^{\sigma_3^2}-1)s_{3,0}^2M_3^2 + 2(e^{\rho\sigma_2\sigma_3}-1)(M_2-\kappa-K_2)s_{2,0}s_{3,0}M_3 \tag{74}$$

B.3 Optimal AMM Capital in Quanto Currency

We determine the amount of capital M_3 that leads to a specified default probability q^*.

To find the optimal M_3, we solve Equation (72) with $\kappa = 0$ for M_3 setting the left hand side equal to q^*. We find that the equation that satisfies a probability of q^*,

$$\sigma_Z \Phi^{-1}(1-q^*) = e^{-r}(L_1 + M_1) + \mu_Z \tag{75}$$

leads to a quadratic equation in M_3

$$aM_3^2 + bM_3 + c = 0 \tag{76}$$

where

$$a = s_3^2 \left[1 - (e^{\sigma_3^2} - 1)\Phi^{-1}(1-q^*)^2\right] \tag{77}$$

$$b = 2s_3 \left[e^{-r}(L_1 + M_1) + s_2(M_2 - K_2)\left(1 - \Phi^{-1}(1-q^*)^2(e^{\rho\sigma_2\sigma_3} - 1)\right)\right] \tag{78}$$

$$c = (e^{-r}(L_1 + M_1) + s_2(M_2 - K_2))^2 - s_2^2(M_2 - K_2)^2(e^{\sigma_2^2} - 1)\Phi^{-1}(1-q^*)^2 \tag{79}$$

(80)

One of the two solutions of the quadratic equation is the optimal amount $m_3(q^*)$. Since a larger amount of capital gives us a smaller probability of default, we choose the correct solution by choosing the larger of the two solutions $M_{3,1}$ and $M_{3,2}$:

$$m_3(q^*) = \max[M_{3,1}, M_{3,2}]. \tag{81}$$

B.4 Derivation of Optimal Trade Size $k^\star$

Assuming the moment-matching approximation holds, we have the identity

$$Q'(\kappa) = -\Phi'(z(\kappa))z'(\kappa)$$

where

$$z(\kappa) = \frac{L_1 + \kappa s_{2,0} + M_1 + e^r \mu_Z}{e^r \sigma_Z}$$

Given that Φ' is always strictly positive, the problem of minimizing Q is reduced to that of finding zeros of z'.

The numerator above is linear in κ, whereas the denominator is the root of a quadratic function. By further assuming that $r \equiv 0$, the numerator can be seen to be in fact constant as a function of κ:

$$L_1 + \kappa s_{2,0} + M_1 + \mu_Z = L_1 + M_1 + s_{2,0}(M_2 - K_2) + s_{3,0}M_3$$

It follows that, when $r = 0$, $z' = 0$ if and only if $(\sigma_Z^2)' = 0$, which is a linear equation of the form

$$-s_{2,0}(e^{\sigma_2^2} - 1)(M_2 - \kappa - K_2) - (e^{\rho\sigma_2\sigma_3} - 1)s_{3,0}M_3 = 0$$

from where the value of κ^* can be readily found, namely,

$$\kappa^* = M_2 - K_2 + \frac{s_{3,0}}{s_{2,0}} \frac{e^{\rho\sigma_2\sigma_3} - 1}{e^{\sigma_2^2} - 1} M_3$$

B.5 Bounded Slippage

The probability of default can vanish identically or become negligible, as we see for instance in Equation (18). This leaves room for an additional, risk-independent premium to be applied on trades within a certain range, so as to not interfere with the risk-based premium, while at the same time increasing the protocol earnings in absence of risk.

More precisely, the price quoted at time t for a trade of size k is given by

$$p(k) = s_{2,t}(1 + \mathsf{sign}(k)\delta + \mathsf{sign}(k - k^*)Q(k))$$

and so, differentiating with respect to k, we see that

$$p'(k) = \mathsf{sign}(k - k^*)s_{2,t}Q'(k) = s_{2,t}|Q'(k)|.$$

Clearly, $p' \geq 0$, with equality if and only if $Q' = 0$, which holds in a neighborhood of k^*. When k^* is small enough, this causes the risk-based premium to vanish for small trades. In order to remedy this, we add to the above an additional slippage function g:

$$p'(k) = s_2(t)(|Q'(k)| + g(k)).$$

We choose g to be a bounded non-negative function of the absolute trade size, bounded away from zero for small trades, and vanishing when the probability of default becomes significant. By design, the risk-based premium becomes significant near the EWMA computed from the representative trader exposure of Equation (28), $\bar{\Pi}$, and thus we choose this value as a threshold for g:

$$g(k) = \max\left\{1 - |k|/\bar{\Pi}, 0\right\},$$

which vanishes for $|k| > \bar{\Pi}$ and attains its maximum value at $k = 0$.

We denote by δ_i the maximum spread to be added to the pricing function, so that it can be written as

$$p(k) = s_{2,t}[1 + \delta\mathsf{sign}(k) + \delta_i G(k) + \mathsf{sign}(k - k^*)Q(k)],$$

where we obtain $G(k)$ through integration of $g(k)$, ignoring constants:

$$G(k) = \begin{cases} -1 \quad \text{if } k \leq -\bar{\Pi}, \\ (1 - |k|/\bar{\Pi})^2 - 1 \quad \text{if } -\bar{\Pi} < k \leq 0, \\ 1 - (1 - |k|/\bar{\Pi})^2 \quad \text{if } 0 < k \leq \bar{\Pi}, \\ 1 \quad \text{if } k > \bar{\Pi}. \end{cases}$$

C Lemmas

C.1 Partial Liquidation

$$\tau : \text{target margin rate} \tag{82}$$
$$\bar{s}_2 : \text{current mark price} \tag{83}$$
$$s_3 : \text{index price collateral currency, coll. to quote conversion} \tag{84}$$
$$s_2 : \text{index price base currency, base to quote conversion} \tag{85}$$
$$\ell : \text{trader's locked-in value} \tag{86}$$
$$\Pi : \text{trader's current position} \tag{87}$$
$$\delta : \text{required position amount to be liquidated} \tag{88}$$
$$m_c : \text{trader collateral (in collateral currency)} \tag{89}$$
$$f : \text{fee rate applied to notional trade amount} \tag{90}$$

The margin balance of the trader at mark price is given by

$$b_0 = (\Pi\bar{s}_2 - \ell)/s_3 + m_c, \tag{91}$$

where $(\Pi\bar{s}_2 - \ell)$ is the unrealized PnL. When the amount δ of the trader position is sold at mark price, the trader observes the following PnL, Δm_c, and change in locked-in value, $\Delta\ell$:

$$\Delta m_c = (\delta\bar{s}_2 - \frac{\ell}{\Pi}\delta)/s_3 \tag{92}$$
$$\Delta\ell = -\delta\frac{\ell}{\Pi} \tag{93}$$

where $\frac{\ell}{\Pi}$ is the average price the trader got when opening the position of size Π. The collateral m_c is in 'collateral currency', hence the division by s_3. We define δ as positive if the trader has a long position, and vice versa, hence sgn$(\delta) =$ sgn(Π).

Let f be the fee rate that is applied to the traded notional δ. The margin balance corresponds to the unrealized PnL plus collateral. We now express the margin balance after the liquidation trade at mark price using Eq. (92) and (93):

$$\begin{aligned} b &= \left((\Pi - \delta)\bar{s}_2 - (\ell + \Delta\ell)\right)/s_3 + m_c + \Delta m_c - f|\delta|s_2/s_3 && (94)\\ &= \left((\Pi - \delta)\bar{s}_2 - \ell + \delta\frac{\ell}{\Pi}\right)/s_3 + m_c + \left(\delta\bar{s}_2 - \frac{\ell}{\Pi}\delta\right)/s_3 - f|\delta|s_2/s_3 && (95)\\ &= b_0 - f|\delta|s_2/s_3 && (96)\end{aligned}$$

where $(\Pi - \delta)$ is the new position after selling δ, and the margin balance b is in collateral currency.

After liquidating the amount ℓ at mark price, we want the margin balance b to be equal to the margin requirement at target margin rate τ, after fees:

$$b \stackrel{!}{=} |\Pi - \delta|\tau\bar{s}_2/s_3, \tag{97}$$

where τ is the target margin rate. As for the margin balance, we value the margin requirement at mark-price $\bar{s}_2$. Now, our goal is to set the balance from Eq.(96) equal to the rhs of Eq.(97) and solve for δ. We replace $|\delta|$ by sgn$(\Pi)\delta$ and $|\Pi - \delta|$ by sgn$(\Pi)(\Pi - \delta)$, and therefore impose $|\delta| < |\Pi|$, to get

$$b_0 - f\text{sgn}(\Pi)\delta s_2/s_3 = \text{sgn}(\Pi)(\Pi - \delta)\tau\bar{s}_2/s_3 \tag{98}$$

$$\delta = \frac{|\Pi|\tau\bar{s}_2 - b_0 s_3}{\text{sgn}(\Pi)(\bar{s}_2\tau - s_2 f)} \tag{99}$$

For this equation to hold, we require

$$b_0 s_3 < |\Pi|\tau\bar{s}_2 \qquad \text{we start below target} \tag{100}$$

$$b_0 - |\Pi| f s_2/s_3 > 0 \qquad \text{liquidating the whole position pays the fees} \tag{101}$$

C.2 Leverage

The leverage ratio is defined as the inverse of the margin rate:

$$\lambda = \frac{1}{\tau}. \tag{102}$$

Using this definition, and Equation (97) (with $\delta = 0$), we get

$$\lambda = \frac{|\Pi|\bar{s}_2/s_3}{b}, \tag{103}$$

where b is the margin balance, $|\Pi|$ the absolute notional of the position (in base currency), $\bar{s}_2$ the conversion from base to quote currency at the mark-price, s_3 the conversion from collateral to quote currency.

We now determine the required collateral for a desired position size $|\Pi|$ and leverage λ, including fees f. As established earlier, the margin balance in collateral currency is given by

$$b = (\Pi s - \ell)/s_3 + m_c \tag{104}$$

where s is the price at which we value the balance (typically the mark-price), ℓ is the locked-in value (position times purchase price), m_c is the collateral.

For the margin requirement, the margin balance is evaluated at the mark-price $\bar{s}_2$, so at the time of the trade, the margin balance at mark price b' is given by

$$b' = \Pi(\bar{s}_2 - p(\Pi))/s_3 + m_c, \tag{105}$$

where $p(\Pi)$ is the purchase price. Inserting into Equation (103), we can solve for the collateral required for a given position and leverage:

$$m_c = \frac{|\Pi|\bar{s}_2/s_3}{\lambda} - \Pi\left(\bar{s}_2 - p(\Pi)\right)/s_3, \tag{106}$$

Fees are charged on the position notional, that is, the trader pays $|\Pi| f s_2/s_3$. Therefore, including fees, the trader needs to deposit the following amount of collateral to initiate a new position of size Π and leverage λ:

$$m_c = \left(\frac{|\Pi|\bar{s}_2}{\lambda} - \Pi\left(\bar{s}_2 - p(\Pi)\right) + |\Pi| f s_2\right)/s_3 + f^{(R)}, \tag{107}$$

where $f^{(R)}$ corresponds to the relayer fees, a fee charged to reimburse the entity that sends the conditional order to the smart contract. Note that the initial margin rate restricts the maximal leverage ratio through Equation (102).

C.3 Leverage with existing Position

Now we consider the case in which the trader already has an existing position, and wishes to achieve a given leverage λ_0.

The position at the time when a conditional order is executed is not known at the time when the conditional order is posted. E.g., the user can reduce their position size after posting the conditional order and before the conditional order is executed. When the trader reduces his position size by a trade, the leverage will always decrease even if no margin is added.

This uncertainty about the position size at the time of execution raises the question how the trader should be choosing a leverage when posting a conditional order (limit or stop orders). We take the following approach to deal with leverage choices.

- For all order types (market orders and conditional orders) that increase the position size, the amount of margin is added that corresponds to the trade leverage.
- For conditional orders that decrease the position size, the leverage of the new position (after the order is executed) is set so that it equals the leverage of the position before the execution. If the initial position does not meet initial margin requirements, then the leverage is set so that the new position just meets these requirements.
- Leverage choice is disabled for market orders if the trade size decreases the position size.

- Instead, we allow the trader the option to keep position leverage.
- If the trade flips the position sign (from short to long, or from long to short), the margin added/withdrawn is chosen so that the resulting position leverage equals the trade leverage.[5]

Therefore we need to calculate the amount of margin collateral to remove from the trader's margin account so that the leverage remains constant, λ_0, when decreasing the position size. By the definition of leverage, Equation (103), we have

$$\lambda_0 \stackrel{!}{=} \frac{|\Pi+\delta|\bar{s}_2/s_3}{\left[(\Pi+\delta)\bar{s}_2-\ell-\delta p\right]/s_3+m_c-m_c'}, \tag{108}$$

$$= \frac{|\Pi+\delta|\bar{s}_2/s_3}{b_0+\left[\delta\bar{s}_2-\delta p\right]/s_3-m_c'}, \tag{109}$$

where δ is the trade size, p the price for this trade, m_c' the margin collateral to be removed so that we achieve a leverage of λ_0. By b_0 we refer to the margin balance prior to the trade. See the previous section for the remaining parameters. Solving for m_c':

$$m_c' = b_0+\delta\left[\bar{s}_2-p\right]/s_3-\frac{|\Pi+\delta|\bar{s}_2/s_3}{\lambda_0}. \tag{110}$$

The variable m_c' is the amount of margin that is removed from the trader's margin account, so that the new position of size $\Pi+\delta$ has the leverage λ_0. To incorporate fees, we reduce the margin collateral by

$$m_c' - f|\delta|s_2/s_3 - f^{(R)}, \tag{111}$$

to obtain the target leverage λ_0 after fees, where f is the total fee rate, $f^{(R)}$ the relayer fees (for conditional orders)

[5] It would not be possible in all cases to a-priori calculate a correct 'approval amount'. Imagine the current position is small 0.01 BTC and has a low leverage (e.g. leverage 1), then there is a larger negative trade (-0.10 BTC) with very high leverage set (e.g., 10x so a margin of about 0.01BTC is reserved for that trade). Now the trade is executed and we should have a margin of 0.10BTC to keep a leverage of 1x (but we only have 0.01 BTC). Hence we should reserve more but how much depends on the position before the trade is executed. In contrast, if we only shrink the trade (no change of position sign), we know the leverage will decrease without adding margin - so we can deal with that case.