Exercise 8 Answer Sheet — Axiomatic Set Theory, 80650

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Question 1

Definition 0.1. Let X be a set. A tree T is set such that,

1. For every $\eta \in T$, η is a function from an ordinal α to X.
2. If $\eta \in T$ and $\operatorname{dom} \eta = \alpha > \beta$ then $\eta \restriction \beta \in T$.

If $X = 2$ then we say that T is binary tree.

The height of T is the least ordinal α such that $\forall \eta \in T, \operatorname{dom} \eta < \alpha$. We define $\operatorname{Lev}(\eta) = \operatorname{dom} \eta$ (the level of η), and we denote $T_\alpha = \{\eta \in T \mid \operatorname{Lev}(\eta) = \alpha\}$. For $\eta, \eta' \in T$ we define $\eta \leq_T \eta'$ if $\eta = \eta' \restriction \operatorname{dom} \eta$.

Definition 0.2. Let κ be a regular cardinal, we say that a tree T is a κ-tree if the height of T is κ and for every $\alpha < \kappa$, $|T_\alpha| < \kappa$.

Definition 0.3. Let T be a tree of height α. A function $b : \alpha \to X$ is a cofinal branch in T if for every $\beta < \alpha$, $b \restriction \beta \in T$. We would also use the term cofinal branch for the set $\{b \restriction \beta \mid \beta < \alpha\}$.

Let κ be an infinite regular cardinal. Let T be a binary κ-tree.

We will prove that there is $T' \subseteq T$ of height κ such that for every $\alpha < \beta < \kappa$ and $x \in T'$ with $\operatorname{Lev}(x) = \alpha$, there is $y \in T'$ with $\operatorname{Lev}(y) = \beta$ and $x \leq_T y$.

Proof. Let us define $T_0 = \{x \in T \mid \forall \operatorname{Lev}(x) < \alpha < \kappa, \exists y \in T, y \in T_\alpha, x \leq_T y\}$. If T_0 is a κ-tree then it satisfies the required property, then we will show it is indeed a tree of height κ.

For every $x \in T_0$, $x \in T$, hence is a map from an ordinal to X. Assume $y \in T_0$, and let $x = y \restriction \beta$ for $\beta < \kappa$. For every $\beta < \gamma < \kappa$ there is $x \leq_T y \leq_T z$ such that the given property is satisfied. We assume $\alpha < \gamma < \beta$, then $y \restriction \gamma \in T_\gamma$ as $y \in T$. We can conclude $x \in T_0$, meaning T_0 satisfies definition, namely T_0 is a tree.

We move to proving T_0 is of height κ. For certain $x \in T_0$ for every $\operatorname{dom} x < \alpha$ there is $y_\alpha \in T_0$ such that $\operatorname{dom} y_\alpha = \alpha$ for evert such ordinal, then $\sup_{\alpha<\kappa} y_\alpha = \kappa$ as intended. The claim is not about T' being κ-tree (I hope), but we know that each level of T_0 must be bounded by the equivalent level in T, meaning it is bounded by κ.

Lastly, we will check if T_0 in not empty, fulfilling our claims stated above. By the definition of T, if we select $\alpha = 0$, by the height of T the statement is indeed true, indicating $\emptyset \in T_0$.

We showed that there is such $T' = T_0$. $\square$

Question 2

We will show that every binary ω-tree has a cofinal branch.

Proof. From the last question, we can assume $T' \subseteq T$ fulfills the property of arbitrary elements, then we will define recursively the function $b : \omega \to X$ by the following,

1. $b(0) = \eta(0)$, when η is any branch $\in T$ (the root of ordered tree is unique).
2. If $b \restriction n$ is already set, then $b(n) \in T_n$ such that $b(n-1) \leq_T b(n)$, there exists such in T'.

The result is indeed $b : \omega \to X$ such that $b \restriction n \in T' \subseteq T$ for all $n < \omega$, meaning b is cofinal branch of T as desired. $\square$

Question 3

We will prove that if there is some cardinal μ such that $\mu^+ < \kappa$ and $|T_\alpha| \le \mu$ for all α, then T has a cofinal branch.

Proof. We assume such μ exists, as well without loss of generality the arbitrary height of branches is fulfilled. For each $x \in T$ such that $\alpha = \mathrm{Lev}(x)$, we let β_x be the largest ordinal such that there is no other $y \in T_\alpha$ such that $y \restriction \beta_x = x \restriction \beta_x$. In other words, we get the highest level in which x is the only continuation (as of branch) of some branch of that level. For each level α we define $f(\alpha) = \sup_{x \in T_\alpha} \beta_x$, f mapping each level to the least level below it such that there is uniquely-extendable branch between the levels. Let $\operatorname{dom} f = S = \{\alpha < \kappa \mid \mu^+ < \alpha\}$, then S is stationary in κ, and $f : S \to \kappa$. By the definition, $f(\alpha) \le \alpha$. For every $x \in T_\alpha$ for $\alpha \in S$, we know that $|T_\alpha| \le \mu$, then there cannot be more than μ levels such that there are more continuations to the restricted branch of x, but $\operatorname{cf} \alpha \ge \mu^+$, meaning the set of such levels is bounded by $\beta < \mu^+$, in particular $\beta_x < \alpha$, then $f(\alpha) < \alpha$, namely f is regressive. By Fodors lemma there is $T \subseteq S$ stationary in κ such that $\forall x \in T, f(x) = \gamma$ for $\gamma < \kappa$. For some arbitrary $\alpha \in T$, let $x \in T_\alpha$ be a branch for which $\beta_x = \gamma$. For each $\alpha \in T$, we can conclude $\beta_x = \gamma$. T is stationary therefore unbounded in κ, then for every $\delta < \kappa$, there is $\delta < \delta' \in T$. For this δ' there is a branch $x \le_T y \in T_{\delta'}$ by the arbitrary height claim, and $\beta_y = \gamma$ as well. We can define then $b : \kappa \to X$ by setting $b \restriction \delta = y$ for each $\delta < \kappa$, therefore b is a cofinal branch of T. $\square$

Question 4

Let us assume that there is a cardinal $\mu < \kappa$ and a function $f : T \to \mu$ such that for all $x, y \in T$, if $x <_T y$ then $f(x) \neq f(y)$. We will prove that there is no cofinal branch in T.

Proof. We assume by contradiction that $b : \kappa \to X$ is a cofinal branch of T. By transitivity of $<_T$ it follows that $f(b \restriction \alpha) \neq f(b \restriction \beta)$ for all $\alpha < \beta < \kappa$. We can deduce that for $X = f''\{b \restriction \alpha \mid \alpha < \kappa\}$, $|X| = \kappa$, in contradiction to $X \subseteq \operatorname{rng} f = \mu < \kappa$. $\square$

Question 5

Let κ be a measurable cardinal.
We will prove that every κ-tree has a cofinal branch.

Proof. If $\kappa \leq 2^{\aleph_0}$ then we already know that every κ-tree has cofinal branch. We assume that $2^{\aleph_0} < \kappa$, then the measure is κ-complete and we can assume that κ is inaccessible (strong?).

By the inaccessibility and κ being regular we can deduce there is no mapping from $\alpha < \kappa$ to $|T_\alpha|$ such that $|T_\alpha| = \kappa$. Then there is an ordinal μ, by the inaccessibility of κ, such that $|T_\alpha| \leq \mu$ for all $\alpha < \kappa$. Then we can use the previous question to deduce that there is a cofinal branch in T. $\square$