Exercise 7 Answer Sheet — Axiomatic Set Theory, 80650

January 16, 2025

Question 1

We will assume ZFC unless stated otherwise.

Let $j : V \to M$ be a non-trivial elementary embedding such that $\operatorname{crit} j = \kappa$, where M is transitive. Let $\mathcal{U} = \mathcal{U}_j = \{X \subseteq \kappa \mid \kappa \in j(X)\}$, we know that $\mathcal{U}$ is a κ-complete ultrafilter on κ.

a

We will show that for every $A \in \mathcal{U}$ and regressive function $f : A \to \kappa$ there is $B \subseteq A, B \in \mathcal{U}$, such that $\operatorname{rng}(f \restriction B) = \{\gamma\}$ for some $\gamma < \kappa$.

Proof. Assumption that $A = \kappa$ will result in contradiction to f being regressive, then $|\operatorname{dom} f| < \kappa$. $\forall x \in A, f(x) < x$ then $\sup x \in Af(x) < \kappa$. Then we will notate $\mu = |\operatorname{Im} f|$, we can assume $\mu < \kappa$.
We will imitate Fodor's lemmas proof.
If there is $\gamma < \kappa$ such that the conditions are met then the proof is done, then we will assume there is no such set B.
For every $\gamma < \kappa$ we define $B_\gamma = f^{-1}(\{\gamma\})$, $B_\gamma \notin \mathcal{U}$ then $U_\gamma = \kappa \setminus B_\gamma \in \mathcal{U}$ and disjoint from B_γ. We then use $\mathcal{U}$'s κ-completeness to deduce that for $U = \bigcap_{\gamma<\mu} U_\gamma$, $U \in \mathcal{U}$. Then $A \cap U \in \mathcal{U}$ as well, and in particular $A \cap U \neq \emptyset$, let $\alpha \in A \cap U$. $f(\alpha) = \gamma < \alpha$ for certain γ, but then $\alpha \in B_\alpha$, meaning $\alpha \notin U$, a contradiction. $\square$

b

We notate $\operatorname{Ma}(x) \iff x$ is Mahlo cardinal.
We will show that $\{\alpha \mid \operatorname{Ma}(x)\} \in \mathcal{U}$.

Proof. Let $A = \{\alpha \leq \kappa \mid \operatorname{Ma}(\alpha)\}$, from a theorem $\kappa \in A$.
It follows that $j(A) = \{\alpha \leq j(\kappa) \mid M \models \operatorname{Ma}(\alpha)\}$, but $\kappa < j(\kappa)$ and κ is definable and therefore $\kappa \in M$ and $M \models \operatorname{Ma}(\kappa)$.
We can assume then $\kappa \in j(A)$, this is of course implies that $A \in \mathcal{U}$ as we wanted to show. $\square$

c

Let $S \subseteq \kappa$ be stationary set. We define $\varphi_s(x, y) \iff x$ is stationary in y.
We will prove that $A = \{\alpha \subseteq \kappa \mid \varphi_s(S \cap \alpha, \alpha)\} \in \mathcal{U}$.

Proof. $V \models \varphi_s(S, \kappa) \iff M \models \varphi_s(j(S), j(\kappa))$ as j is elementary embedding and φ_s is first order (I carefully hope). Also $j(A) = \{\alpha \leq j(\kappa) \mid M \models \varphi(j(S \cap \alpha), j(\alpha))\}$, but $\kappa \leq j(\kappa)$, and $j(S \cap \kappa) = j(S)$ then by the last statement $\kappa \in j(A)$ as intended. $\square$

Question 2

Let κ be a measurable cardinal and let $\mathcal{U}$ be a non-principle κ-complete normal ultrafilter on κ.
Let j be the elementary embedding derived from the ultrapower by $\mathcal{U}$ and let $\mathcal{V} = \{X \subseteq \kappa \mid \kappa \in j(X)\}$.
We will show that $\mathcal{V} = \mathcal{U}$.

Proof. We define $M = V^\kappa/\mathcal{U}$ and $j : V \to M$. We proved during a lecture that $\kappa^M \leq [id] < j(\kappa)$ by

$$\forall \alpha < \kappa, \{i \mid c_\alpha(i) \leq id(i)\} = \kappa \setminus \alpha \in \mathcal{U}$$

and

$$\{i < \kappa \mid id(i) < j(\kappa)\} = \kappa \in \mathcal{U}$$

We assume that $\kappa^M < [id]$ then there is $g : \kappa \to \kappa$ such that $[g] = \kappa^M$, it follows that $[g] < [id]$, by Łoś $\{i \mid g(i) < id(i) = i\} \in \mathcal{U}$. $\mathcal{U}$ is normal, therefore by the last statement g is regressive, then there is $\gamma < \kappa$ such that $\{i \mid g(i) = \gamma = c_\gamma(i)\} \in \mathcal{U} \iff [g] = \gamma^M$. This is a contradiction to $[g] = \kappa^M$, we can deduce that $\kappa^M = [id]$.

Note: this statement is already true due to the lecture. We will show that for every $X \subseteq \kappa$, $j(X) = c_X$. For every $\alpha \in X$, $\alpha \in Ord$ and therefore definable, then $j(\alpha) = [c_\alpha]$. $\forall \alpha \in X, \{i \mid c_\alpha(i) \in j(X)(i)\} \in \mathcal{U} \implies \forall i < \kappa, X \subseteq j(X)(i)$, then by the definition of $[f]$ as the least ranked functions, we can assume $j(X) = c_X$.

As a result,

$$X \in \mathcal{V} \iff \kappa^M \in j(X) \iff [id] \in c_X \iff \{i \mid id(i) \in c_X(i)\} \in \mathcal{U} \iff \{i \mid i \in X\} \in \mathcal{U} \iff X \in \mathcal{U}$$

Then $\mathcal{U} = \mathcal{V}$ as intended. $\square$