Exercise 4 Answer Sheet — Axiomatic Set Theory, 80650

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Question 1

Assuming ZF.

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Assuming DC, We'll prove that a relation R of a set X is well founded if and only if there is no infinite sequence $\langle x_n \mid n < \omega\rangle \in X^\omega$ such that for all n, $x_{n+1} R x_n$.

Proof. We assume R is well founded. Let us assume for contradiction that $\langle x_n \mid n < \omega\rangle$ is a sequence such that $\forall n \in \omega, x_{n+1} R x_n$. We define a subset $Y \subseteq X$ from the list by defining $x_n \in Y$ for all n. R is well founded on X therefore exists $y \in Y$ such that $\forall n, y R x_n \in Y$. $y \in Y$ then exists $m \in \omega$, $y = x_m$, then in particular $x_{m+1} R x_m$ in contradiction to R being well founded.

In the opposite direction, let us assume there is no such sequence, we'll prove that R is well founded as a relation over X. Let $Y \subseteq X$. If there is an item $y \in Y$ such that there is no $z \in Y$ such that zRy then R is well founded, so we assume otherwise. For every element $y \in Y$ there is $z \in Y$ such that zRy from out assumption, then by DC there is an infinite sequence $\langle y_n \mid n < \omega\rangle \in Y^\omega$ such that $y_{n+1} R y_n$ for all n. Pay attention that DC was used on the reverse relation of R instead of R directly. From out initial assumption we get contradiction, then R is indeed well founded. □

b

We'll show that DC is equivalent to the following version of the Löwenheim-Skolem theorem:
For every infinite structure M over an countable language, there is a countable elementary substructure $M' \prec M$.

Proof. Watch the recording to see what Yair says □

Question 2

Assuming ZF, Let A be a class. We'll use Levý Reflection to conclude the following version of Levý Reflection for A:
For every formula φ with n free variables, there is a closed unbounded class of ordinals $C_{A,\varphi}$ such that for every $\delta \in C_{A,\varphi}$ and $p_0, \dots, p_{n-1} \in V_\delta \cap A$,

$$\langle A, \in \rangle \models \varphi(p_0, \dots, p_{n-1}) \iff \langle V_\delta \cap A, \in \rangle \models \varphi(p_0, \dots, p_{n-1})$$

Proof. From Levý Reflection there is $\alpha \in Ord$ such that,

$$\forall a_0, \dots, a_{n-1} \in V_\alpha, \varphi(a_0, \dots, a_{n-1}) \iff V_\alpha \models \varphi(a_0, \dots, a_{n-1})$$

and there is a club C_φ of such ordinals α. It is clear that $C_{A,\varphi} \overset{def}{=} C_\varphi \cap A$ is a club as well, and let $\delta \in C_{A,\varphi}$.
Assume $p_0, \dots, p_{n-1} \in V_\delta \cap A$ and that $A \models \varphi(p_0, \dots, p_{n-1})$, then $p_0, \dots, p_{n-1} \in V_\delta$ and what

We can prove by induction on A by defining $A = \{\langle x_0, \dots, x_{m-1} \rangle \in V \mid \psi(x_0, \dots, x_{m-1})\}$ and prove by induction on ψ.
For atomic formulas, we can deduce that $\psi \in \{(x = x), (x = y), (x \in y), (x \in x)\}$ and therefore $A = V, \emptyset$ and by Levý reflection the statement is true.
For ψ_0, ψ_1 we assume the statement is true, then $C_{A,\psi_0}, C_{A,\psi_1}$ exist fulfilling the statement, then we can define $C_{A,\psi_0} \cap C_{A,\psi_1}$ for $\psi_0 \wedge \psi_1$, the statement holds for similar reasons for its holding in the proof of 7.4 (in lecture notes). The same statement can be used for $\psi = \neg\psi_0$.
We assume the statement is true for ψ_0 and define $\psi = \exists v, \psi_0$ in order to finish the induction process. We can use the same argument from the proof of the theorem in its general form, as it requires only the existence of the clubs and their closeness. $\square$

Question 3

Definition (Almost Universal class)**.** Let A be a transitive class, we say that A is almost universal if for every ordinal γ, $A \cap V_\gamma \in A$.

Definition (Δ_0-Separation)**.** A class A will be called a model of Δ_0-Separation if for every $a, p_0, \ldots, p_{n-1} \in A$ and a Δ_0 formula $\varphi(x, y_0, \ldots, y_{n-1})$,

$$b = \{w \in a \mid \varphi(w, p_0, \ldots, p_{n-1})\} \in A$$

Let A be a transitive class which is almost universal, and let us assume that A is a model of Δ_0-Separation.

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We will prove that $Ord \subseteq A$.

Proof. We will prove the statement using induction over the ordinals.
As an induction basis, notice that $A \cap V_\emptyset = \emptyset \in A$.
Let us assume the statement is true for α and show that it is also true for $\alpha + 1$. We know $\alpha + 1 \in V_\alpha$, and using Δ_0-Separation on $V_{\alpha+1}$ we can deduce from the induction hypothesis that $\{\alpha\} \in A$, and from transitivity $\alpha + 1 \in A$ as well.
Assume the statement is true for all $\beta < \alpha$ and prove it also true for α. $\beta \in V_\alpha \cap A$ for every $\beta < \alpha$, therefore we can use Δ_0-Separation using localized version of formula representing an ordinal, and conclude that $\alpha \in V_\alpha \cap A$, therefore $\alpha \in A$, completing the induction step. □

Remark. A is a model of Extensionality, Foundation, Empty Set and Infinity.

b

We will show that A is a model of Union and Power Set.

Proof. Let $x \in A$ be a set, remember we defined $\bigcup x = \{z \mid \exists y \in x, z \in y\}$, this formula is Δ_0, then we can separate $\operatorname{trcl}(x)$ using it by the assumption of Δ_0-Separation. Therefore A is a model of Union.

In a similar way, we defined $\mathcal{P}(x) = \{y \mid y \subseteq x\}$, this is of course a Δ_0 formula that can be used on the least cardinal $\operatorname{trcl}(x) < \kappa$. Therefore A is also a model of Power Set. □

c

We'll prove that A satisfies the Scheme of Replacement, and in particular Pairing and Separation.

Proof. It seems to be the same exact thing with $\sup_{y \in x} \operatorname{trcl} y$ and another Δ_0-Separation. □