Exercise 6 Answer Sheet — Axiomatic Set Theory, 80650

January 9, 2025

Question 1

Let λ be an infinite cardinal. A (λ^+, λ)-Ulam matric is a collection of sets $\langle A_{\alpha,\rho} \mid \alpha < \lambda^+, \rho < \lambda \rangle$ such that,

1. For every $\alpha < \beta < \lambda^+$ and $\rho < \lambda$, $A_{\alpha,\rho} \cap A_{\beta,\rho} = \emptyset$.
2. For every $\alpha < \lambda^+$, $|\lambda^+ \setminus (\bigcup_\rho A_{\alpha,\rho})| \leq \lambda$.

a

We will show that for every infinite λ, a (λ^+, λ)-Ulam matrix exists.

Proof. Let us define for each $0 < \xi < \lambda^+$ a surjection $f_\xi : \lambda \to \xi$ by mapping $f_\xi(x) = x$ if $x < \xi$ and into an arbitrary unique (as per ξ) value $< \xi$ otherwise. We define a matrix $A_{\alpha,\rho} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\}$ and want to show that the definition of A fulfills Ulam matrix definition as depicted earlier. Let $\alpha < \beta < \lambda^+$ as well let us fix $\rho < \lambda$, therefore

$$A_{\alpha,\rho} \cap A_{\beta,\rho} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\} \cap \{\xi < \lambda^+ \mid f_\xi(\rho) = \beta\} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha, f_\xi(\rho) = \beta\} = \emptyset$$

Pay attention that the last step follows directly from the fact we fixed ρ and that f_ξ is a function.
Let us fix $\alpha < \lambda^+$, then

$$\left|\lambda^+ \setminus \left(\bigcup_{\rho<\lambda} A_{\alpha,\rho}\right)\right| = \left|\lambda^+ \setminus \left(\bigcup_{\rho<\lambda} \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\}\right)\right| = \left|\lambda^+ \setminus \{\xi < \lambda^+ \mid f_\xi(\lambda) \ni \alpha\}\right|$$

when $f_\xi(\lambda) = \operatorname{Im} f_\xi = \xi$ as f_ξ is surjective. Then

$$\left|\lambda^+ \setminus \{\xi < \lambda^+ \mid \alpha < \xi\}\right| |\lambda^+ \setminus \{\xi \mid \alpha < \xi < \lambda^+\}| \leq \lambda$$

As for every choice of $\alpha < \lambda^+$ the cardinality of α is at most λ. $\square$

b

Let κ be the least cardinal such that there is a σ-additive, non-trivial, non-atomic measure μ with $\operatorname{dom} \mu = \mathcal{P}(\kappa)$.
We will prove that κ is not a successor cardinal.

Proof. Let us assume in order to prove by contradiction that κ is indeed a successor cardinal such that $\lambda^+ = \kappa$. By the last proposition, there is a (λ^+, λ)-Ulam matrix, namely A, for the specified λ. Fixing $\alpha < \lambda^+$, we will show that there is $\rho < \lambda$ such that $\mu(A_{\alpha,\rho}) > 0$.
It is known that $|\kappa \setminus \bigcup_\rho A_{\alpha,\rho}| \leq \lambda$, then the assumption that all these elements of A fulfilling $A_{\alpha,\rho} \leq \lambda$ would lead to contradiction as their union would be $< \kappa$. Then there is an element $A_{\alpha,\rho} > \lambda$ for every $\alpha < \kappa$. For each of these, $\mu(A_{\alpha,\rho}) > 0$, as μ is non-atomic (and the elements are *big* in relation to κ). Let $\gamma = \{\rho < \lambda \mid \alpha < \kappa, \mu(A_{\alpha,\rho}) > 0\}$ in order to define $B_\alpha = \bigcup_{\rho \in \gamma} A_{\alpha,\rho}$. From its definition, $B_\alpha \cap B_\beta = \emptyset$ for every $\alpha < \beta < \kappa$, as deducted from Ulam matrix properties, and then by σ-additivity of μ we get contradiction to $\mu(\bigcup B_\alpha) \leq 1$. From the contradiction it is followed that there is no such λ, meaning κ is not a successor cardinal. $\square$

Question 2

Let κ be an uncountable regular cardinal such that there is non-principle filter $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ with the following properties,

1. For every $\langle x_\alpha \in \mathcal{F} \mid \alpha < \kappa \rangle$ also $\bigcap_{\alpha<\kappa} x_\alpha \in \mathcal{F}$.
2. For every collection $\{X_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{P}(\kappa)$ such that $\forall \alpha, \kappa \setminus X_\alpha \notin \mathcal{F}$, there are $\alpha < \beta$ such that $X_\alpha \cap X_\beta \neq \emptyset$.

Such an $\mathcal{F}$ is called non-trivial σ-saturated κ-complete filter on κ.
We will show that either there is a κ-complete ultrafilter on κ or $\kappa \leq 2^{\aleph_0}$ and κ is a limit cardinal.

Proof. Let $\mathcal{F}^+ = \{X \subseteq \kappa \mid \kappa \setminus X \notin \mathcal{F}\}$, This set represents the elements of $\mathcal{F}$ which are *non zero* in a sense, a positive subset of the filter.

Let us assume that for every $B \subseteq A$, $B \in \mathcal{F}^+$ or $A \setminus B \in \mathcal{F}^+$, this is in a sense the atomic case, in which there is a set that acts as an atom. We will show that in this case there is a κ-complete ultrafilter on κ. We will construct a sequence $\langle x_\alpha \in \mathcal{F}^+ \mid \alpha < \kappa \rangle \subseteq \mathcal{P}(\kappa)$ by taking such list of decreasing ordinals and using the defined property about inclusion in $\mathcal{F}^+$. By κ-completeness of $\mathcal{F}$ the intersection, A, of the stated sequence is in $\mathcal{F}$ and in particular from the assumption we made $A \in \mathcal{F}^+$. We define $\mathcal{U} = \{x \in \mathcal{F}^+ \mid A \subseteq x\}$, this is an principal ultrafilter which is κ-complete as required[1] In details, we found a set which fulfills the definition of an atom, and by defining a measure using this atom, we get κ-complete ultrafilter, this property carries from $\mathcal{F}$ κ-completeness.

Assuming the contrary of our initial assumption, it directly follows that for every $A \in \mathcal{F}^+$ there is $B \subseteq A$ such that $B, A \setminus B \in \mathcal{F}^+$ (There is also the case that $B, A \setminus B \notin \mathcal{F}^+$, which lead to contradiction to the definition of $\mathcal{F}$ as a filter). This is case is in a sense non-atomic, as for each positive-measure set there is a split of disjoint positive-measure subsets. Let us define a left centered standard binary tree $\langle T, f \rangle$ such that $f : 2^{\omega_1} \to \mathcal{F}^+$, defined by $f(\langle\rangle) = \kappa$ and for each $t \in \operatorname{dom} f$, if $f(t) = A$, from the assumption there is $B \subseteq A$ satisfying our assumption, then we define $f(t \frown \langle 0 \rangle) = B$, $f(t \frown \langle 1 \rangle) = A \setminus B$. From our definition, for every $t \in 2^{\omega_1}$, $f(t \frown \langle 0 \rangle) \cap f(t \frown \langle 1 \rangle) = \emptyset$ and $f(t \frown \langle 0 \rangle) \cup f(t \frown \langle 1 \rangle) = f(t)$. Let $M = \{f(t) \mid t \in 2^\omega\}$, then $|M| = 2^{\aleph_0}$ and by σ-saturation there are $m, m' \in M$ such that $m \cap m' \neq \emptyset$, which is a contradiction. Therefore we assume $M \cap \mathcal{F}^+ = \emptyset$ (the process to get a contradiction can be done for each branch to obtain this claim). But $\bigcup M = \kappa$, then we can conclude that $\kappa \leq 2^{\aleph_0}$ as well.
We want to show that κ is a limit cardinal, to do that we will use the conclusion of question 1. To fulfill the propositions requirements it is needed to define a measure which is σ-additive, non-trivial and non-atomic. We can use the principal measure of $\mathcal{F}$ (Which requires extending it to ultrafilter, which in turn requires AC) to obtain such a measure. $\square$

[1] Jech T. Set Theory. 2003, 1, 77.