Exercise 8 Answer Sheet — Axiomatic Set Theory, 80650

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Question 1

Definition 0.1. Let X be a set. A tree T is set such that,

1. For every $\eta \in T$, η is a function from an ordinal α to X.
2. If $\eta \in T$ and $\operatorname{dom} \eta = \alpha > \beta$ then $\eta \restriction \beta \in T$.

If $X = 2$ then we say that T is binary tree.
The height of T is the least ordinal α such that $\forall \eta \in T, \operatorname{dom} \eta < \alpha$. We define $\operatorname{Lev}(\eta) = \operatorname{dom} \eta$ (the level of η), and we denote $T_\alpha = \{\eta \in T \mid \operatorname{Lev}(\eta) = \alpha\}$. For $\eta, \eta' \in T$ we define $\eta \leq_T \eta'$ if $\eta = \eta' \restriction \operatorname{dom} \eta$.
Definition 0.2. Let κ be a regular cardinal, we say that a tree T is a κ-tree if the height of T is κ and for every $\alpha < \kappa$, $|T_\alpha| < \kappa$.
Definition 0.3. Let T be a tree of height α. A function $b : \alpha \to X$ is a cofinal branch in T if for every $\beta < \alpha$, $b \restriction \beta \in T$. We would also use the term cofinal branch for the set $\{b \restriction \beta \mid \beta < \alpha\}$.

Let κ be an infinite regular cardinal. Let T be a binary κ-tree.

We will prove that there is $T' \subseteq T$ of height κ such that for every $\alpha < \beta < \kappa$ and $x \in T'$ with $\operatorname{Lev}(x) = \alpha$, there is $y \in T'$ with $\operatorname{Lev}(y) = \beta$ and $x \leq_T y$.

Proof. The assumption that this condition isn't met at all will lead to T being of height less than κ, then there are such elements in T. Why can't we just take an arbitrary κ branch, this would create some linear order of order κ and would satisfy the proposition. $\square$

Question 2

We will show that every binary ω-tree has a cofinal branch.

Proof. It is direct by the definition of height of a tree. From the last question, we can assume $T' \subseteq T$ fulfills the property of arbitrary elements, then we will define recursively the function $b : \omega \to X$ by the following,

1. $b(0) = \eta(0)$, when η is any branch $\in T$ (the root of ordered tree is unique).
2. For every $0 \leq n < \omega$, let $b \restriction n + 1 = x$ when $x \in T', \mathrm{Lev}(x) = n + 1$ such that $b \restriction n \leq_T x$, given by the last question.

This recursive definition ensures us that for every $\alpha < \omega$, indeed $b \restriction \alpha \in T' \subseteq T$, as desired. $\square$

Question 3

We will prove that if there is some cardinal μ such that $\mu^+ < \kappa$ and $|T_\alpha| \leq \mu$ for all α, then T has a cofinal branch.