Exercise 6 Answer Sheet — Axiomatic Set Theory, 80650

January 8, 2025

Question 1

Let λ be an infinite cardinal. A (λ^+, λ)-Ulam matric is a collection of sets $\langle A_{\alpha,\rho} \mid \alpha < \lambda^+, \rho < \lambda \rangle$ such that,

1. For every $\alpha < \beta < \lambda^+$ and $\rho < \lambda$, $A_{\alpha,\rho} \cap A_{\beta,\rho} = \emptyset$.
2. For every $\alpha < \lambda^+$, $|\lambda^+ \setminus (\bigcup_\rho A_{\alpha,\rho})| \leq \lambda$.

a

We will show that for every infinite λ, a (λ^+, λ)-Ulam matrix exists.

Proof. Let us define for each $0 < \xi < \lambda^+$ a surjection $f_\xi : \lambda \to \xi$ by $f_\xi(x) = x$ if $x < \xi$ and to arbitrary value otherwise. Define $A_{\alpha,\rho} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\}$, and we will show that this definition is fulfilling Ulam matrix definition.
Let $\alpha < \beta < \lambda^+$ and let us fix $\rho < \lambda$, then

$$A_{\alpha,\rho} \cap A_{\beta,\rho} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\} \cap \{\xi < \lambda^+ \mid f_\xi(\rho) = \beta\} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha = \beta\} = \emptyset$$

Let us fix $\alpha < \lambda^+$, then

$$\left|\lambda^+ \setminus \left(\bigcup_\rho A_{\alpha,\rho}\right)\right| = \left|\lambda^+ \setminus \left(\bigcup_\rho \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\}\right)\right| = \left|\lambda^+ \setminus \{\xi < \lambda^+ \mid f_\xi(\lambda) \ni \alpha\}\right|$$

But for each α, $f_{\alpha+1}(\alpha) = \alpha$ we can deduce

$$\left|\lambda^+ \setminus \{\xi < \lambda^+ \mid f_\xi(\lambda) \ni \alpha\}\right| \left|\lambda^+ \setminus \{\xi < \lambda^+ \mid \alpha + 1 < \lambda^+\}\right| \leq \lambda$$

□

b

Let κ be the least cardinal such that there is a σ-additive, non-trivial, non-atomic measure μ with $\operatorname{dom} \mu = \mathcal{P}(\kappa)$.
We will prove that κ is not a successor cardinal.

Proof. Let us assume for contradiction that κ is indeed a successor cardinal such that $\lambda^+ = \kappa$.
By the last part there is a (λ^+, λ)-Ulam matrix, A, for this specified λ.
Fixing $\alpha < \lambda^+$, we will find $\rho < \lambda$ such that $\mu(A_{\alpha,\rho}) > 0$. We know that $|\kappa \setminus \bigcup_\rho A_{\alpha,\rho}| \leq \lambda$, then the assumption that all these elements of A fulfilling $A_{\alpha,\rho} \leq \lambda$ would lead to contradiction, as their union would be $< \kappa$. Then there is an element $A_{\alpha,\rho} > \lambda$ for every $\alpha < \kappa$, for each for these $\mu(A_{\alpha,\rho}) > 0$ as μ is non-atomic. Let $\gamma = \{\rho < \lambda \mid \alpha < \kappa, \mu(A_{\alpha,\rho}) > 0\}$, then define $B_\alpha = \bigcup_{\rho \in \gamma} A_{\alpha,\rho}$. $B_\alpha \cap B_\beta = \emptyset$ for every $\alpha < \beta < \kappa$ as deducted from Ulam matrix, and then by σ-additivity of μ we get contradiction to $\mu(\bigcup B_\alpha) \leq 1$. By the contradiction it is followed that there is no such λ, meaning κ is not a successor cardinal. □

Question 2

Let κ be an uncountable regular cardinal such that there is non-principle filter $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ with the following properties,

1. For every $\langle x_\alpha \in \mathcal{F} \mid \alpha < \kappa \rangle$ also $\bigcap_{\alpha<\kappa} x_\alpha \in \mathcal{F}$.
2. For every collection $\{X_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{P}(\kappa)$ such that $\forall \alpha, \kappa \setminus X_\alpha \notin \mathcal{F}$, there are $\alpha < \beta$ such that $X_\alpha \cap X_\beta \neq \emptyset$.

Such an $\mathcal{F}$ is called non-trivial σ-saturated κ-complete filter on κ.
We will show that either there is a κ-complete ultrafilter on κ or $\kappa \leq 2^{\aleph_0}$ and κ is a limit cardinal.

Proof. Let $\mathcal{F}^+ = \{X \subseteq \kappa \mid \kappa \setminus X \notin \mathcal{F}\}$, This set represent the elements of $\mathcal{F}$ which are non zero in a sense, a positive subset of the filter.

Let us assume that for every $B \subseteq A$, $B \in \mathcal{F}^+$ or $A \setminus B \in \mathcal{F}^+$, this is in a sense the atomic case, in which there is a set that acts as an atom. We will show that in this case there is a κ-complete ultrafilter on κ. Let us define A an atom of $\mathcal{F}^+$, from the assumption we made it is clear that there is an atom, such can be constructed by intersecting decreasing series of sets that are all in $\mathcal{F}^+$. Define $\mathcal{U} = \{x \in \mathcal{F}^+ \mid A \subseteq x\}$, this is an ultrafilter which is κ-complete as required[1]

Assuming the contrary of our initial assumption, it is directly follows that for every $A \in \mathcal{F}^+$ there is $B \subseteq A$ such that $B, A \setminus B \in \mathcal{F}^+$. This is case is in a sense non-atomic, as for each positive-measure set there is a split of disjoint positive-measure subsets. Let us define a left centered standard binary tree $\langle T, f \rangle$ such that $f : 2^{\omega_1} \to \mathcal{F}^+$, defined by $f(\langle\rangle) = \kappa$ and for each $t \in \operatorname{dom} f$, if $f(t) = A$, from the assumption there is $B \subseteq A$ satisfying the noted property, then we define $f(t \frown \langle 0 \rangle) = B$, $f(t \frown \langle 1 \rangle) = A \setminus B$. Let us define $\text{level}_\alpha(\langle T, f \rangle) = \{f(t) \mid \text{len}(t) = \alpha$, and $M = \text{level}_{\omega_1}$. We know that $|\mathbb{R}| \leq \omega_1$, then by the non-atomic property of μ we conclude that $\mu(f(t)) > 0$ for all $t \in \operatorname{dom} f$, but then $\sum M = \infty$ and in particular larger than 1. This is of course a contradiction to the implicit assumption that $\kappa > \omega_1$, then we conclude directly $\kappa \leq 2^{\aleph_0}$. $\square$

[1] Jech T. Set Theory. 2003, 1, 77.