Exercise 6 Answer Sheet — Axiomatic Set Theory, 80650

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Question 1

Let λ be an infinite cardinal. A (λ^+, λ)-Ulam matric is a collection of sets $\langle A_{\alpha,\rho} \mid \alpha < \lambda^+, \rho < \lambda \rangle$ such that,

1. For every $\alpha < \beta < \lambda^+$ and $\rho < \lambda$, $A_{\alpha,\rho} \cap A_{\beta,\rho} \neq \emptyset$.
2. For every $\alpha < \lambda^+$, $|\lambda^+ \setminus (\bigcup_\rho A_{\alpha,\rho})| \leq \lambda$.

a

We will show that for every infinite λ, a (λ^+, λ)-Ulam matrix exists.

Proof. Let us define for each $0 < \xi < \lambda^+$ a surjection $f_\xi : \lambda \to \xi$ by $f_\xi(x) = x$ if $x < \xi$ and to arbitrary value otherwise.
Define $A_{\alpha,\rho} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\}$, and we will show that this definition is fulfilling Ulam matrix definition.
Let $\alpha < \beta < \lambda^+$ and let us fix $\rho < \lambda$, then

$$A_{\alpha,\rho} \cap A_{\beta,\rho} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\} \cap \{\xi < \lambda^+ \mid f_\xi(\rho) = \beta\} = \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha = \beta\} = \emptyset$$

Let us fix $\alpha < \lambda^+$, then

$$\left|\lambda^+ \setminus \left(\bigcup_\rho A_{\alpha,\rho}\right)\right| = \left|\lambda^+ \setminus \left(\bigcup_\rho \{\xi < \lambda^+ \mid f_\xi(\rho) = \alpha\}\right)\right| = \left|\lambda^+ \setminus \{\xi < \lambda^+ \mid f_\xi(\lambda) \ni \alpha\}\right|$$

But for each α, $f_{\alpha+1}(\alpha) = \alpha$ we can deduce

$$\left|\lambda^+ \setminus \{\xi < \lambda^+ \mid f_\xi(\lambda) \ni \alpha\}\right| \left|\lambda^+ \setminus \{\xi < \lambda^+ \mid \alpha + 1 < \lambda^+\}\right| \leq \lambda$$

$\square$

b

Let κ be the least cardinal such that there is a σ-additive, non-trivial, non-atomic measure μ with $\operatorname{dom} \mu = \mathcal{P}(\kappa)$.
We will prove that κ is not a successor cardinal.

Proof. Let us assume for contradiction that κ is indeed a successor cardinal such that $\lambda^+ = \kappa$.
By the last part there is a (λ^+, λ)-Ulam matrix for this specified λ.
Fixing $\alpha < \lambda^+$, we will find $\rho < \lambda$ such that $\mu(A_{\alpha,\rho}) > 0$. $\square$

Question 2

Let κ be an uncountable regular cardinal such that there is non-principle filter $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ with the following properties,

1. For every $\langle x_\alpha \in \mathcal{F} \mid \alpha < \kappa \rangle$ also $\bigcap_{\alpha<\kappa} x_\alpha \in \mathcal{F}$.
2. For every collection $\{X_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{P}(\kappa)$ such that $\forall \alpha, \kappa \setminus X_\alpha \notin \mathcal{F}$, there are $\alpha < \beta$ such that $X_\alpha \cap X_\beta \neq \emptyset$.

Such an $\mathcal{F}$ is called non-trivial σ-saturated κ-complete filter on κ.
We will show that either there is a κ-complete ultrafilter on κ or $\kappa \leq 2^{\aleph_0}$ and κ is a limit cardinal.

Proof. TODO $\square$