Solution to Exercise 0 — Model Theory (1), 80616

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Question 1

Let $L = \{P\}$ a language where P is unary relation. Define,

$$\varphi_n = \exists x_0 \ldots \exists x_n \left(\bigwedge_{i \leq n} P(x_i) \wedge \bigwedge_{i<j \leq n} x_i \neq x_j \right), \quad \psi_n = \exists x_0 \ldots \exists x_n \left(\bigwedge_{i \leq n} \neg P(x_i) \wedge \bigwedge_{i<j \leq n} x_i \neq x_j \right)$$

and let $T = \{\varphi_n, \psi_n \mid n < \omega\}$.
We will show that $\mathrm{cl}_{\vdash} T$ is ω-categorical.

Proof. Let $\mathcal{M} \models T$ be some model. It can be proved by direct induction that $|P^{\mathcal{M}}| = \omega$ as well as $|\neg P^{\mathcal{M}}| = \omega$. Let us construct $f : \omega \to M$ such that $f(n) \in P^{\mathcal{M}}$ for any $n < \omega$. $\mathcal{M} \models \varphi_0 \iff \mathcal{M} \models \exists x\, P(x)$ then let $f(0)$ be such witness. Let us assume that $f \upharpoonright n$ is defined, then $\mathcal{M} \models \varphi_{n+1}$, then by the pigeonhole principle there is some $a \in \mathcal{M}$ such that $a \notin f''n$, and let $f(n+1) = a$. For the sake of convenience let us redefine f as $2 \times \omega \to M$ injective function such that $f(0, n)$ is the same as $f(n)$ and $f(1, n) \notin P^{\mathcal{M}}$. By CSB we can assume that f is bijection as well, and by the selection of $\mathcal{M}$ as an arbitrary model of T we can deduce that for any $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \cong \mathcal{N}$ by composition of functions as was constructed. $\square$

Question 2

Let $L = \{c_n \mid n < \omega\}$ be language consists of constant symbols. Let us define the theory $T = \{c_i \neq c_j \mid i < j < \omega\}$. We will show that there are countably many non-isomorphic countable models of T, and that T is complete.

Proof. Let us define the model $\mathcal{M}_n$ such that $M = \omega$ and,

$$c_i^{\mathcal{M}} = i + n$$

for any $i < j < \omega$,

$$c_i^{\mathcal{M}} = i + n \neq j + n = c_j^{\mathcal{M}}$$

therefore $\mathcal{M}_n \models T$. $\mathcal{M}_n \models k \neq c_i$ for all $i < \omega$, in particular $\mathcal{M}_n \models \exists x\, x \neq c_i$. It is implied that also,

$$\mathcal{M}_n \models \exists x_0 \dots \exists x_{k-1} \left(\bigwedge_{i<j<k} x_i \neq x_j \wedge x_i \neq c_l \right) = \varphi_l^k$$

for all $l < \omega$. Finally, $\mathcal{M}_n \not\models \varphi_l^k$ for any $k > n$, we deduce that $\mathcal{M}_n \not\cong \mathcal{M}_m$ for any $n \neq m$.

We move to show that T is complete. Let us assume toward a contradiction that φ is a sentence such that $\varphi \notin T$ and $T \cup \{\varphi\}$ is consistent. By construction of Henkin models we can deduce that $\mathcal{M}_0 \models \varphi$, but $\mathcal{M}_0$ is minimal, namely if $\mathcal{N} \models T$ then $\mathcal{M}_0 \subseteq \mathcal{N}$, then by definition $T \models \varphi$, a contradiction. $\square$

Question 3

We will show that $T = \text{Th}(\mathbb{N}, +, \cdot)$ has $2^{\aleph_0}$ non-isomorphic countable models.

Proof. Observe the fact that numbers are definable in T, by formula as such,

$$\varphi_n(x) = \forall y\; x \cdot y = \overbrace{y + \cdots + y}^{n \text{ times}}$$

If $\mathcal{M} \models T$ then we denote by $\underline{n}$ the single element of M that fulfills φ_n.
By the fact that $\exists x\; \varphi_n(x) \in T$ it follows that $\{\underline{n} \mid n < \omega\} \subseteq M$ for any such model.

Let us define an explicit model of T, $M = \mathbb{N}(\mu_n)$ where $n \in \mathbb{N}$ and μ_n is primitive root of unity of order n, namely,

$$\mathcal{M} \models (\forall m < n,\; \mu_n^m \neq \underline{1}) \wedge \mu_n^n = \underline{1}$$

when power is notation for repeated use of $\cdot$ symbol. This is a first-order sentence with parameters.
This cannot work as this sentence is false in T.

Maybe using vector spaces over $\mathbb{Q}$? We first need to prove that $\mathbb{Q} \models T$. But $\mathbb{N} \models \neg\exists x, x \cdot x = \underline{2}$.

Let $\epsilon_0 = \omega$ and $\epsilon_{n+1} = \mathcal{P}(\epsilon_n)$, this generates a recursive sequence of elements of different cardinality such that they are all infinite. This won't work as well.

Last try, let us define $M = 2 \times \omega$ and let $\{n\} \times \omega$ act as the given model $(\mathbb{N}, +, \cdot)$ within its domain, for $n \in 2$. $T \models \forall x, y,\; (x + y = y + x \wedge x \cdot y = y \cdot x)$ then it suffices to define the operations $\langle 0, n\rangle + \langle 1, m\rangle$ and $\langle 0, n\rangle \cdot \langle 1, m\rangle$. Let $f : \omega^2 \to \omega$ be some bilinear form, and define the function symbols such that,

$$\langle 0, n\rangle + \langle 1, m\rangle = \langle 1, f(n, m)\rangle$$

The dot product symbol is defined by decomposition to plus symbols chain in $\{1\} \times \mathbb{N}$. Consider the model countable $\mathcal{M}_f = (M, +, \cdot)$ as defined, we claim that $\mathcal{M} \models T$. $\square$

Question 4

Let $\kappa \geq \omega$ be some cardinal and let L be some language. Let T be a κ-categorical L-theory such that it has no finite models. We will show that T can be incomplete.

Solution. Let $L = \{c_\alpha \mid \alpha < \delta\} \cup \{P\}$ for $\kappa < \delta$, where c_α is a constant symbol and P is unary relation.

$$T = \{c_\alpha \neq c_\beta \mid \alpha < \beta < \delta\} \cup \{P(c_\alpha) \mid \alpha < \delta\}$$

It follows from the definition of T that if $\mathcal{M} \models T$ then $|M| \geq \delta > \kappa$, therefore there are no models of T of cardinality κ, then the theory is vacuously complete.

Just take $\kappa = \omega$ and L and T from question 1, we saw that T is ω-categorical and has no finite models. We also know that if $\mathcal{M} \models T$ and $|M| > \kappa$ then $|P^{\mathcal{M}}|, |\neg P^{\mathcal{M}}| \geq \omega$, and therefore it is possible that $|P^{\mathcal{M}}| = |M|$ or $|P^{\mathcal{M}}| < |M|$, then T cannot be complete.

Question 5

Let $T = \mathrm{Th}(\mathbb{Q}, \leq)$ be DLO.
We will show that T is not κ-categorical for some uncountable cardinal κ.

Proof. The key here is similar to question 4, we can define $P(x) \iff x \leq c$ for some arbitrary value. This is equivalent to the theory of 4 and 1, and we just talked about why this is not necessarily κ-categorical. $\square$