Solution to Exercise 2 — Model Theory (1), 80616

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Question 1

Part a

Let $\mathbb{F}$ be a field and let T_{vec} be the theory of infinite vector space over the field $\mathbb{F}$ with the language $L = \{0, +\} \cup \{\lambda_a \mid a \in \mathbb{F}\}$, such that λ_a represents multiplication by scalar $a \in \mathbb{F}$. We will show that T_{vec} satisfies quantifier elimination.

Proof. Let $\mathcal{V}, \mathcal{W}$ are vector spaces over $\mathbb{F}$ such that there exists $A \subseteq V \cap W$ and $\langle A \rangle = V \cap W$, and let $\mathcal{A}$ be the model over $\langle A \rangle$.

Let $t(x)$ be a term over $L(A)$ (as every $a \in \langle A \rangle$ is definable in $L(A)$). Any definable $v(x) : \mathcal{A} \to \mathcal{A}$ is of the form $v = u + ax$, then $t(x) = u + \lambda_a x$ for some $u \in A$ and $a \in \mathbb{F}$.

Any atomic formula with single free variable is of the form $t(x) = s(x)$ or $t(x) \neq s(x)$, or equivalently $t(x) = 0$ or $t(x) \neq 0$ for some term. In turn, any $\exists$-primitive formula is of one of the forms,

$$\varphi(x) = \exists x \; t(x) = 0, \quad \varphi(x) = \exists x \bigwedge_{i<n} t_i(x) \neq 0$$

We know that $u \in V, W$ for each $u \in \langle A \rangle$, as a linear combination it follows that $\mathcal{V} \models u + av = 0$ for $v \in V$ if and only if $\mathcal{V} \models v = \frac{u}{-a}$. But $\lambda_{-1/a} \in \mathbb{F}$ as it is a field, and it follows that v is definable in $L(A)$ if $\mathcal{V} \models u + av = 0$, we deduce that $v \in \mathcal{W}$ as well. Then if φ is of the first form, then $\mathcal{V} \models \varphi \iff \mathcal{W} \models \varphi$, and it remains to check the second form.

If φ is of the second form, then for any $i < n$, $t_i(x) \neq 0$ is equivalent to $x \neq \frac{u_i}{-a_i} = c_i$ for $u_i \in A, a_i \in \mathbb{F}$, and by the assumption that T is of infinite we infer that $T \models \exists x \; (\bigwedge_{i<n} x \neq c_i)$, meaning that if $\mathcal{V}, \mathcal{W} \models \varphi$.

By quantifier elimination equivalently theorem, T is eliminating quantifiers.

If $\mathcal{V}$ was not infinite (and we would change T as well) then the last step won't hold. In turn, we would have to divide into cases by the character of $\mathbb{F}$, if it would be 0 then the proof will hold. If on the other hand $\operatorname{char} \mathbb{F} < \infty$, then the claim would not be true anymore. $\square$

Part b

Let $L = \{\leq\} \cup \{c_n \mid n < \omega\}$ be a language, and let $T = \text{DLO} \cup \{c_n < c_{n+1} \mid n < \omega\}$.
We will show that T has quantifier elimination, and find all non-isolated types in $S_1(T)$.

Proof. The proof is identical to the case of DLO, using back and forth method on two models $\mathcal{M}, \mathcal{N} \models T$. The key difference is that if $A \subseteq M \cap N$ with $|A| < \omega$ then the back and forth isomorphism construction has to start with,

$$(\{c_i^{\mathcal{M}} \mid i < \omega\} \cup \{c_i^{\mathcal{N}} \mid i < \omega\}) \cap A$$

This way we get an isomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ over $L(A)$, and in particular $\mathcal{M}_A \models \psi \iff \mathcal{N}_A \models \psi$ as ψ is existential primitive and being preserved by φ. $\square$

We will find all non-isolated types in $S_1(T)$.
Solution. Let $p \in S_1(T)$ be some type, and let c be new constant such that $\varphi(c)$ is true. Then $\varphi(x) = c = c_i$ for some $i < \omega$ is a type, but it is (by definition) isolated, and thus we can assume p does not consist of such formulas. T has only c_i as constant symbols, then any type such that $c_i \leq x \leq c_{i+1} \in p$ is isolated, but if all of the formulas are of the form $c_i \leq x$ we get partial type. We can deduce that there are no non-isolated types in $S_1(T)$.

Question 2

Let T_{EQ} be the theory of equivalence relation over $L = \{E\}$. We will show that T^*_{EQ}, the model companion of T_{EQ}, is the theory of equivalence relation with infinitely many infinite equivalence classes. Moreover, we will show that T^*_{EQ} has quantifier elimination.

Proof. Notice that,

$$T_{EQ} = \{\forall x\, E(x,x), \forall x \forall y\, (E(x,y) \leftrightarrow E(y,x)), \forall x \forall y \forall z\, (E(x,y) \wedge E(y,z)) \to E(x,z)\}$$

meaning that T_{EQ} consists of global sentences only, as a corollary from class T^*_{EQ} exists uniquely.

By proposition from class we know that T^*_{EQ} is closed existentially closed over $T_\forall = T$. Let us observe,

$$\varphi_n = \exists x_0 \cdots \exists x_{n-1}\, (\bigwedge_{i<j<n} x_i \neq x_j \wedge E(x_i, x_j))$$

then $\varphi_n \in T^*_{EQ}$ for any $n < \omega$, meaning that any equivalence class is infinite. We also define,

$$\psi_n = \exists x_0 \cdots \exists x_{n-1}\, (\bigwedge_{i<j<n} \neg E(x_i, x_j))$$

then $T^*_{EQ} \models \psi_n$ as well for any $n < \omega$, meaning that there are infinite equivalence classes.

We move to show that T^*_{EQ} eliminating quantifiers. The proof is by using the equivalence theorem and by defining $A \subseteq M, N$ as in other proofs. In this case the primitive existential formula over $L(A)$ will be of the form $\exists x\, E(x, a_0) \wedge \cdots \wedge E(x, a_{n-1}) \wedge \neg E(x, a_n) \wedge \cdots \wedge \neg E(x, a_{n+m-1})$, and formulas over equation symbols. In turn we can use the properties of T^*_{EQ} to show $\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$ in the exact way of question 1. $\square$

Question 3

We will show that there is a complete theory T over a countable language, and a collection of $2^{\aleph_0}$ non-isolated types in $S_1(T)$ such that every model $\mathcal{M} \models T$ satisfies at least one of these types.

Proof. Let $L = \{P_i \mid i < \omega\}$ be a language consists of countable many unary relation symbols. Let,

$$T = \left\{ \varphi_{A,B} = \exists x \left(\bigwedge_{i \in A} P_i(x) \right) \wedge \left(\bigwedge_{j \in B} \neg P_j(x) \right) \mid A, B \subseteq \omega, |A|, |B| < \omega, A \cap B = \emptyset \right\}$$

be the theory such that for any finite selection of predicates, there is an element that is true for them for any choice.

We will show that T is complete. In intention of showing quantifier elimination, let $\mathcal{M}, \mathcal{N} \models T$ be some models and let $A \subseteq M, N$ be some finite set such that $\mathcal{A} \subseteq \mathcal{M}, \mathcal{N}$. Let $\varphi(x)$ be some primitive existential formula over $L(A)$, then,

$$\varphi = \exists x \ (\bigwedge_{i<n_0} P_i(x)) \wedge (\bigwedge_{j<n_1} \neg P_j(x)) \wedge (\bigwedge_{k<n_2} x = a_k) \wedge (\bigwedge_{l<n_3} x \neq a_l)$$

which means that $\varphi \equiv \varphi_{C,B}$ or $\varphi \equiv \exists x \ x = a$ or $\varphi \equiv \exists x, x \neq a_0 \wedge \cdots \wedge x \neq a_{n-1}$. $\mathcal{M}, \mathcal{N} \models \varphi_{C,B}, \exists x \ x = a$ for any C, B, a from T and $\mathcal{A}$ definition. The last case is achieved by choosing a witness in each model of $\varphi_{C,B}$, such that $C \supsetneq \bigcup C_i, B \supsetneq \bigcup B_i$, when a_i are witnesses of φ_{C_i,B_i} for $i < n$. By equivalently to quantifier elimination T is such theory, and has no constant symbols, then it is complete.

Let $f : \omega \to 2$ be a function, and let us consider the type,

$$p_f(x) = \{\varphi_{A,B} \mid A = f^{-1}(1) \cap [n], B = f^{-1}(0) \cap [n], n < \omega\}$$

type such that $T_{\{c\}} \models \varphi(c)$ is true if and only if $P_n(c) \iff f(n) = 1$. T is complete and c has a sentence defining its value for any P_n, therefore $p_f \in S_1(T)$.

We will now show that p is not isolated. Let $M = N = \{f \in \{0,1\}^{[m]} \mid m < \omega\}$ and,

$$P_i^{\mathcal{M}}(f) \iff i \in \operatorname{dom} f \wedge f(i) = 1, \quad P_i^{\mathcal{N}}(f) \iff i \in \operatorname{dom} f \vee f(i) = 1$$

We now define the types p_{c_0} and p_{c_1}, then $f = \{\langle 0, 0 \rangle\}$ is $\in M, N$, and,

$$\mathcal{M} \models p_{f \cup \{\langle n,0 \rangle | 0<n<\omega\}}(f), \mathcal{M} \not\models p_{f \cup \{\langle n,1 \rangle | 0<n<\omega\}}(f), \mathcal{N} \not\models p_{f \cup \{\langle n,0 \rangle | 0<n<\omega\}}(f), \mathcal{N} \models p_{f \cup \{\langle n,1 \rangle | 0<n<\omega\}}(f)$$

Then p_f cannot be isolated, as otherwise there would be undecidable sentence in T. The above claim can be extended to any arbitrary function using appropriate construction of models, we infer that p_f is non-isolated for all $f \in \{0,1\}^\omega$.

Let $\mathcal{M} \models T$ be some model. $\mathcal{M} \models \exists x \ \varphi_{\{0\},\emptyset}$, and let $d \in M$ be a witness. Let $f : \omega \to \{0,1\}$ be a function such that $f(n) = 1 \iff P_n^{\mathcal{M}}(d)$ ($f(0) = 1$ by definition). $\mathcal{M} \models p_f(d)$ directly from the definition of p_f. $\square$

Question 4

Let L and T be as in the last question. We will show that T is complete and that there is no isolated type in $S_1(T)$.

Proof. Was proved in the last question. $\square$

Question 5

We will show that there is a complete theory T over the language L such that $|L| = \aleph_1$ and a non-isolated type $p(c)$ that cannot be omitted.

Proof. Let $L = \{P, E\} \cup \{c_i \mid i < \omega\} \cup \{d_i \mid i < \omega_1\}$, where P is an unary predicate, E is binary predicate, c_i, d_i are constant symbols for any i. Let us define the theory,

$$\begin{aligned} T =& \{\forall x\ (P(x) \to \exists! y(\neg P(y) \land E(x, y))), \forall y\ (\neg P(y) \to \exists! x(P(x) \land E(x, y)))\} \\ &\cup \{c_i \neq c_j \mid i < j < \omega\} \cup \{d_i \neq d_j \mid i < j < \omega_1\} \\ &\cup \{\neg E(c_i, d_j) \mid i < \omega, j < \omega_1\} \cup \{P(c_i) \mid i < \omega\} \cup \{\neg P(d_i) \mid i < \omega_1\} \end{aligned}$$

the theory such that if $\mathcal{M} \models T$ then $E^{\mathcal{M}} : A_M \to B_M$, for $A_M = \{x \in M \mid P^{\mathcal{M}}(x)\}, B_M = \{y \in M \mid \neg P^{\mathcal{M}}(y)\}$, is a bijection. It also follows from definition that $\{c_i^{\mathcal{M}} \mid i < \omega\} \subseteq A_M$ and $\{d_i^{\mathcal{M}} \mid i < \omega_1\} \subseteq B_M$, therefore $|B_M| \geq \aleph_1$. $E^{\mathcal{M}}$ is a bijection, implying that $|A_M| = |B_M|$.

We will show that T is complete as an $\aleph_1$-categorical theory with no models of size $< \aleph_1$. As was shown in the last part $|M| \geq |B_M| \geq \aleph_1$.
Let $\mathcal{M}, \mathcal{N} \models T$ be some models such that $|M| = |N| = \aleph_1$, and let $f = E^{\mathcal{M}}$ and $g = E^{\mathcal{N}}$. $A_M \cup B_M = M$ then $|A_M| = |B_M| = \aleph_1$. $|A_M| \geq \aleph_1, |\{c_i^{\mathcal{M}}\}| = \aleph_0$ therefore $|A_M \setminus \{c_i^{\mathcal{M}}\}| \geq \aleph_1$ as well, the claim holds for $\mathcal{N}$ as well. It follows that $|A_M \setminus \{c_i^{\mathcal{M}}\}| = |A_N \setminus \{c_i^{\mathcal{N}}\}|$ and let $h : A_M \setminus \{c_i\} \to A_N \setminus \{c_i\}$ be a bijection witnessing that. We can define now $F : M \to N$ by,

$$F(x) = \begin{cases} c_i^{\mathcal{N}} & x = c_i^{\mathcal{M}} \\ d_i^{\mathcal{N}} & x = d_i^{\mathcal{M}} \\ h(x) & x \in A_M \setminus \{c_i^{\mathcal{M}}\} \\ (g \circ h \circ f^{-1})(x) & x \in B_M \setminus \{d_i^{\mathcal{M}}\} \end{cases}$$

F is a bijection by its definition as composition of bijections, F is also constant preserving between $\mathcal{M}$ and $\mathcal{N}$. By A_M, A_N, B_M, B_N definitions F also preserves P, and by f, g it preserves E as well, implying that $F : \mathcal{M} \to \mathcal{N}$ is a model isomorphism.
We found that T has single model of size $\aleph_1$ up to isomorphism, therefore T is complete.

We move to define a type that cannot be isolated nor being omitted. Let,

$$p(x) = \{P(x)\} \cup \{x \neq c_i \mid i < \omega\}$$

p cannot be omitted, as to be so it must be omitted by sentence of the form $E(x, d_i)$ for some i.
If $\mathcal{M} \models T$ then we take $e \in M$ such that $\mathcal{M} \models E(e, d_0)$. □