Exercise 9 Answer Sheet — Logic Theory (2), 80424

June 5, 2025

Question 1

Part a

We will show that the set,

$$H = \{e \in \mathbb{N} \mid (e, 0) \in \operatorname{dom} U\}$$

is recursively-enumerable but not recursive.

Proof. H is Σ_1^0 by the definition of U, implies that it is recursively-enumerable, then it is sufficient to show that it is not recursive. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by,

$$f(x) = x.$$

This is indeed a recursive function (even primitive-recursive), and for every $e \in D$, when $D = \{e \in \mathbb{N} \mid (e, e) \in \operatorname{dom} U\}$, we get $f(e) = e \in H$, as e witnessing bound larger than 0. It follows that f is a recursive reduction of D to H and implies that H is not recursive. $\square$

Part b

We will conclude that $H^C = \mathbb{N} \setminus H$ is not recursively-enumerable.

Proof. H is a relation, meaning that if it were to be Π_1^0 it would be Δ_1^0 and recursive. If H^C is recursivel y-enumerable then $H^C = \{e \in \mathbb{N} \mid (e, 0) \in \operatorname{dom}(\neg U)\}$ is Σ_1^0, and implying by negation that $H = \neg H^C$ (in the sense of relations) is Π_1^0. But H is recursively-enumerable and Σ_1^0, meaning that it is indeed Δ_1^0, in contradiction to its being not recursive. $\square$

Part c

Let us define the set,

$$T = \{e \in \mathbb{N} \mid \{x \mid (e, x) \in \operatorname{dom} U\} = \mathbb{N}\}$$

We will show that T is not recursively-enumerable.

Proof. We assume in contradiction that T is recursively-enumerable.
By the equivalence to recursively-enumerable functions proposition we can assume that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that f is primitive-recursive and $T = \{f(x) \mid x \in \mathbb{N}\}$. Let us define $g : \mathbb{N}^2 \to \mathbb{N}$ by,

$$g(n, x) = U(f(n), x) + 1$$

g is indeed total recursive function as U is recursive and f is total such that $U(f(n), \cdot)$ is total recursive function. Then by Kleene's recursion theorem there is $e \in \mathbb{N}$ such that,

$$U(e, x) = g(e, x) = U(f(e), x) + 1.$$

$\square$