Solution to Exercise 4 — Model Theory (1), 80616

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Question 1

Definition 0.1 (algebraic formula)**.** Let $\mathcal{M}$ be a structure. A formula $\varphi(x) \in \mathrm{form}_{L_\mathcal{M}}$ is called algebraic if,

$$|\{a \in M \mid \mathcal{M} \models \varphi(a)\}| < \aleph_0.$$

Definition 0.2 (definable element)**.** An element $a \in M$ is called definable if there is a formula $\psi(x)$ such that $\mathcal{M} \models \psi(x) \iff x = a$ for any $x \in M$.

Definition 0.3 (acl and dcl)**.** If $\mathcal{M}$ is a structure and $A \subseteq M$ a set of elements, then let,

$$\mathrm{acl}_\mathcal{M}(A) = \bigcup\{\varphi(M) \mid \varphi \in \mathrm{form}_{L_A}, \varphi \text{ is algebraic}\}, \quad \mathrm{dcl}\,\mathcal{M}(A) = \{a \in M \mid a \text{ is definable using an } L_A \text{ formula}\}.$$

be the set of elements that has a formula with parameters from A that is algebraic and are true for them, and the set of elements uniquely definable by formulas with parameters.

Part a

Let $A \subseteq M$ for $\mathcal{M} \prec \mathcal{N}$. We will show that $\mathrm{acl}_\mathcal{M}(A) = \mathrm{acl}_\mathcal{N}(A)$ and $\mathrm{dcl}_\mathcal{M}(A) = \mathrm{dcl}_\mathcal{M}(A)$.

Proof. Let $a \in M$ be $\in \mathrm{acl}_\mathcal{M}(A)$ and let $\varphi \in \mathrm{form}_{L_A}$ be an algebraic formula to witness it. $\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(a)$ by the given elementary embedding and the fact that $A \subseteq N$.

Let $a \in \mathrm{acl}_\mathcal{N}(A)$ and assume toward contradiction that $a \notin \mathrm{acl}_\mathcal{M}(A)$, meaning that there is $\varphi(x)$ over L_A such that $\mathcal{N} \models \varphi(a)$ and φ is algebraic. Let $B = A \cup \varphi(M)$, denote $\varphi(M) = \{b_i \mid i < N\}$ for $N < \omega$,

$$\psi(x) = \varphi \wedge \left(\bigwedge_{i<N} x \neq b_i\right).$$

Then ψ is algebraic over both $\mathcal{M}$ and $\mathcal{N}$ and $\psi(M) = \emptyset$, while $a \in \psi(N)$. Let $\phi = \exists x\ \psi$, then $\mathcal{N} \models \phi$ as a witnesses exactly this, while $\mathcal{M} \models \neg\phi$ in contradiction to $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$.

Let us move to dcl. TODO $\square$