Exercise 1 Answer Sheet — Logic Theory (2), 80424

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Question 1

Let us assume that D is an ultrafilter on given set X.

a

Suppose that there is some $s \in D$ such that $|s| < \omega$, we will show that D is principal.

Proof. We will assume that s is minimal in relation to cardinality in D, otherwise let us choose the minimal such s by using the well ordering principle. There is a unique such $s \in D$, otherwise $|s| = |s'|$ and $|s \cap s'| < |s|$, a contradiction to the minimality of the cardinality of s.
For every $x \in D$, there is some $y \in D$ such that $y \subseteq x \cap s$, but by s minimality, $y = x \cap s$, hence $s \subseteq x$. In the other direction, if $s \subseteq x$, then $x \in D$ by filter properties.
Therefore $D = \{x \in \mathcal{P}(X) \mid s \subseteq x\}$, D is a principle ultrafilter defined by s. $\square$

b

We will show that if $D \subseteq \mathcal{P}(I)$ is a principal ultrafilter, $\langle \mathcal{M}_i \mid i \in I \rangle$ is a sequence of L-structures, then $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / D \cong M_i$ for some $i \in I$.

Proof. Let $s \subseteq I$ be the defining set for D, and let us fix $i \in s$. Let $f : N \to M_i$ such that $f([g]_D) = g(i)$. f is well defined, as every for every $h, h' \in [h]$, it is follows that $h \restriction s = h' \restriction s$ by D's definition as principle.

We will show that f is an isomorphism between $\mathcal{N}$ and $\mathcal{M}_i$. L-constants are being preserved directly from definition of ultra-products. Assume $F \in \text{Func}_{L,n}$, and $t_0, \dots, t_{n-1} \in \text{term}_L$, then,
$$f(F^{\mathcal{N}}(t_0^{\mathcal{N}}, \dots, t_{n-1}^{\mathcal{N}})) = f([\{F^{\mathcal{M}_j}(t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j}) \mid j \in I\}]) = F^{\mathcal{M}_i}(t_0^{\mathcal{M}_i}, \dots, t_{n-1}^{\mathcal{M}_i})$$
meaning f preserves terms. Let $R \in \text{rel}_{L,n}$, and let $t_0, \dots, t_{n-1} \in \text{term}_L$, then,
$$\mathcal{N} \models R(t_0, \dots, t_{n-1}) \iff \{j \in I \mid \langle t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j} \rangle \in R^{\mathcal{M}_j}\} \in D \implies \langle t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j} \rangle \in R^{\mathcal{M}_i}$$
We conclude that f is indeed a structures isomorphism. $\square$

Question 2

We will show that if F is a filter on a set X such that if $s \in F$ then s is infinite, then there is a non-principle ultrafilter extending F.

Proof. Initially, we claim that there is no $x \in F$ such that $|x|, |x^C| \geq \omega$, when $x^C = X \setminus x$. If there is one, then $\emptyset \subseteq x \cap x^C = \emptyset$ is in F, in contradiction to F being a filter. We define the ultrafilter $U = \{x \subseteq X \mid |x| \geq \omega\}$, this is indeed an ultrafilter as noted in lecture. it is sufficient to show that $F \subseteq U$. Indeed, if $s \in F$ then $|s| \geq \omega$, implies that $s \in U$.

We found an ultrafilter extending F, it is required to prove that U is not a principle ultrafilter as well. We will assume otherwise, then it is follows that there is $S \subseteq X$ such that $\forall x, S \subseteq x \iff x \in U$. Let $a \in S$ be an element, then $b = X \setminus \{a\}$ is a set such that $|b| \geq \omega$, implies that $b \in U$, a contradiction as $S \not\subseteq b$. $\square$

Question 3

a

Let $\mathcal{N} = (\mathbb{N}; R^{\mathcal{N}})$ be graph such that $(a, b) \in R^{\mathcal{N}} \iff |b - a| = 1$. Let D be a non-principle ultrafilter on $\mathbb{N}$ (An example of one is the construction from exercise 2). Let $\mathcal{M} = \mathcal{N}^{\mathbb{N}}/D$. We will prove that $\mathcal{M}$ is not path-connected.

Proof. TODO $\square$