Exercise 2 Answer Sheet — Logic Theory (2), 80424

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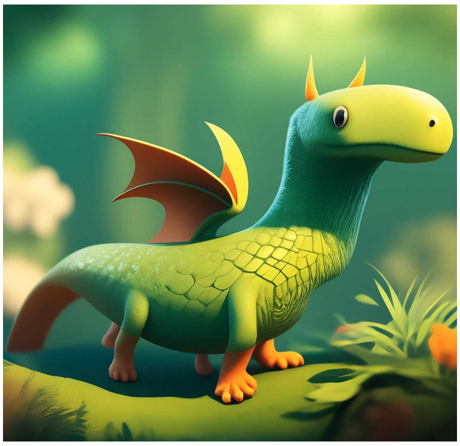

Question 1

Let F be a field and let $L_{F\,\mathrm{VS}}$ be the language of vector spaces over F, $L_{F\,\mathrm{VS}} = \{0, +\} \cup \{\lambda_a \mid a \in F\}$, such that λ_a resolved to scalar multiplication. We assume that $\mathcal{V} \subseteq \mathcal{U}$ are both infinite dimensional vector spaces over F, and will prove that $\mathcal{V} \prec \mathcal{U}$.

Proof. Let us assume that $\psi(x_0, \ldots, x_{n-1})$ is a wff, as well $\varphi(x_0, \ldots, x_n) = \exists x_n \psi(x_0, \ldots, x_{n-1})$. Let $a_0, \ldots, a_{n-1} \in V$, we will show that if $\mathcal{U} \models \varphi(a_0, \ldots, a_{n-1})$ then there is $a_n \in V$ such that $\mathcal{U} \models \psi(a_0, \ldots, a_n)$. By the assumption that $\mathcal{U} \models \varphi(a_0, \ldots, a_{n-1})$ we assume that there is $b \in U$ such that $\mathcal{U} \models \psi(a_0, \ldots, a_{n-1}, b)$. If $b \in V$, then the criteria is fulfilled, then let us assume that $b \notin V$. Let B_V be a basis for $\mathcal{V}$, it is clear that b is linear-independent from B_V, otherwise it would follow that $b \in V$. We choose some $c \in V$ such that $c \in V \setminus \mathrm{Sp}\{a_0, \ldots, a_{n-1}\}$, there must be one by $\mathcal{V}$'s infinite dimension. We construct two bases for $\mathcal{U}$, $B_U^1 \supset B_V \cup \{b\}$, and the other one would be $B_U^2 \supset B_V \cup \{c\}$. Let $M : \mathcal{U} \to \mathcal{U}$ be an automorphism such that $M(a_i) = a_i$ for all $i < n$, and let $M(b) = c$.

We claim that $\mathcal{U} \models \psi(M(a_0), \ldots, M(a_{n-1}), M(b))$, as an automorphism is preserving any relation and function defined over its respective vector-space. then $c \in M$, thus being a witness to out initial claim that there is $c \in \mathcal{V}$ such that $\mathcal{U} \models \psi(a_0, \ldots, a_{n-1}, c)$. Tarski-Vaught tests requirements are all met, it follows that $\mathcal{V} \prec \mathcal{U}$. □

Question 2

Part a

We will show that if $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{K}$ and $\mathcal{M} \prec \mathcal{K}, \mathcal{N} \prec \mathcal{K}$, then $\mathcal{M} \prec \mathcal{N}$.

Proof. Let $\varphi(x_0, \ldots, x_{n-1})$ be some wff, and $a_0, \ldots, a_{n-1} \in M$ then,

$$\mathcal{M} \models \varphi(a_0, \ldots, a_{n-1}) \iff \mathcal{K} \models \varphi(a_0, \ldots, a_{n-1}) \iff \varphi(a_0, \ldots, a_{n-1})$$

as $M \subseteq N$, hence $a_0, \ldots, a_{n-1} \in N$. By the definition of sub-elementary embedding, $\mathcal{M} \prec \mathcal{N}$. □

Part b

We will show that $\mathcal{M} = (\mathbb{N}, <) \prec (\mathbb{N}, <) + (\mathbb{Z}, <) = \mathcal{N}$, where addition of order models is defined as the disjoint union of the universes and the order is the lexicographic order.

Proof. By the EF-games we showed that $\mathcal{M} \equiv \mathcal{N}$. We intend to use Tarski-Vaught test, it followed that we assume that $\psi(x_0, \ldots, x_n)$ is a wff over L, and let $\varphi(x_0, \ldots, x_{n-1}) = \exists x_n \psi(x_0, \ldots, x_n)$. We assume that $a_0, \ldots, a_{n-1} \in \mathbb{N}$ such that $\mathcal{N} \models \varphi(a_0, \ldots, a_{n-1})$. We will prove that there is $a \in \mathcal{M}$ such that $\mathcal{N} \models \psi(a_0, \ldots, a_{n-1}, a)$. From $\mathcal{M} \equiv \mathcal{N}$ it derives that if $\phi = \exists x_0 \ldots \exists x_{n-1} \varphi(x_0, \ldots, x_{n-1})$ then $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$. But we assumed that $\mathcal{N} \models \varphi(a_0, \ldots, a_{n-1})$, it follows that $\mathcal{N} \models \phi$, then $\mathcal{M} \models \phi$ as well. Let $b_0, \ldots, b_n \in \mathbb{N}$ such that $\mathcal{M} \models \psi(b_0, \ldots, b_n)$, the witnesses to $\mathcal{M} \models \phi$. It is sufficient to show that $b_i \mapsto a_i$ is an embedding of $\mathcal{M}$ into $\mathcal{M}$. We can assume that there is such mapping, as otherwise, it would follow that $\mathcal{M} \not\models \phi$. $\mathcal{M} \models \psi(a_0, \ldots, a_{n-1}, b)$ for b such that the embeddings value at b_n, therefore $b \in N$ and $\mathcal{N} \models \psi(a_0, \ldots, a_{n-1}, b)$. Then by Tarski-Vaught test we deduce $\mathcal{M} \prec \mathcal{N}$. □

Part c

We will find an example for three models $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{K}$ such that $\mathcal{M} \prec \mathcal{N}, \mathcal{M} \prec \mathcal{K}$ but $\mathcal{N} \not\prec \mathcal{K}$.
Solution. We define, $\mathcal{M} = (\mathbb{N}, <), \mathcal{K} = (\mathbb{N}, <) + (\mathbb{Z}, <)$ and $\mathcal{N} = (\mathbb{N}, <) + (2\mathbb{Z}, <)$, TODO