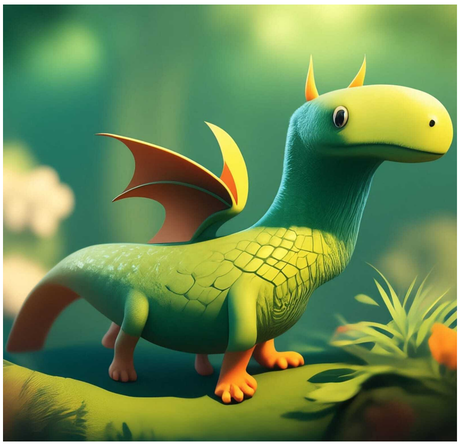
Exercise 3 Answer Sheet — Logic Theory (2), 80424

April 14, 2025



Question 1

Let $T = \mathrm{Th}(\hat{\mathbb{N}})$ We will show that T has 2^{ω} non-isomorphic countable models.

Proof. Let us define $P \subseteq \hat{\mathbb{N}}$ the set of prime numbers. We assume that $S \subseteq P$ is some set. We enrich the language by a new constant symbol, c. Let $T_S = T \cup \{\exists x, (x \cdot p = c) \wedge \neg(\exists y, y \cdot p = x) \mid p \in S\}$, namely extension of T such that $c = \prod_{p \in S} p$. T_S is finitely satisfiable by $\hat{\mathbb{N}}$ as for every finite S, $\prod_{p \in S} p \in \mathbb{N}$, then by compactness T_S is also satisfiable. It follows that there is a model $\mathcal{M}_S \models T_S$ over $L \cup \{c\}$, then it is also a model of T by reduction of the language.

Let $S \neq S'$ be subsets of P and let $\mathcal{M}, \mathcal{N}$ be some of their respective models. We will show that $\mathcal{M} \not\cong \mathcal{N}$. It is clear for $L \cup \{c\}$-isomorphisms, as there is $p \in S \setminus S'$ (without loss of generality) such that $\mathcal{M} \models p \mid c$ but $\mathcal{N} \not\models p \mid c$. By the last claim we can deduce that if there was such L-isomorphism, then for $c^{\mathcal{M}}$ there would be a contradiction, meaning there is no such isomorphism.

We know that $P \subseteq \mathbb{N}$, and that this set is not finite, then $|P| = \omega$, it derives that there are $|2^P| = 2^{\omega}$ non-isomorphic models $\mathcal{M}_S \models T$. $\square$

Question 2

Let F be an infinite field, and let $L_{F\,\text{VS}}$ be the language of vector spaces over F. Let $V \subseteq U$ be two non-trivial vector spaces over F, we will show that $V \prec U$.

Proof. Let P be a disjoint unary relation symbol to $L_{F\,\text{VS}}$, and let us define $L' = L_{F\,\text{VS}} \cup \{P\}$. We extend U to L' by $P^U(x) \iff x \in V$. For every $\psi(x_0, \ldots, x_{n-1}) \in \text{Th}(V)$, let $\varphi_\psi = \exists_{i<n} x_i (\bigwedge_{i<n} P(x_i) \to \exists x_n \psi(x_0, \ldots, x_{n-1}))$ $\square$