Solution to Exercise 1 — Model Theory (1), 80616

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Question 1

Definition 0.1. A formula φ is called *basic Horn* formula if,

$$\varphi = (\theta_0(\bar{x}) \wedge \cdots \wedge \theta_{m-1}(\bar{x})) \to \theta_m(\bar{x})$$

where θ_i is atomic for $i \leq m$.

Remark. In the case $m = 0$ we get $\varphi = \theta_0(\bar{x})$. In the case that $\theta_m = \perp$, $\varphi \equiv \neg\theta_0(\bar{x}) \vee \cdots \neg\theta_{m-1}(\bar{x})$.

Definition 0.2. The set of Horn formulas is the minimal set of formulas containing the basic Horn formulas and closed under conjugation and quantification.

Part a

Let $\psi(x_0, \dots, x_{n-1})$ be a Horn formula and let $F \subseteq \mathcal{P}(I)$ be a filter. Let $\langle \mathcal{M}_i \mid i \in I \rangle$ be a sequence of structures and $a_j \in \prod_{i \in I} \mathcal{M}_i$ for $j < n$ such that,

$$\{i \in I \mid \mathcal{M}_i \models \psi(a_0(i), \dots, a_{n-1}(i))\} \in F$$

We will show that $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / F \models \psi([a_0]_F, \dots, [a_{n-1}]_F)$.

Proof. Let us prove by induction over Horn set.

Assume that $\varphi(\bar{x}) = P(\bar{x})$ and $\{i \in I \mid \mathcal{M}_i \models P(\bar{x})\} \in F$, then by definition $\mathcal{N} \models \varphi$. Let us assume that $\varphi = (\theta_0 \wedge \cdots \wedge \theta_{n-1}) \to \theta_n$ for atomic θ_n, then the claim holds directly from definition as in the last part.

The case where $\theta_m = \perp$ is equivalent as $\varphi \equiv \theta_0 \vee \cdots \vee \theta_{m-1}$.

We move to assume that the claim holds for φ, ψ and prove for $\varphi \wedge \psi$. This part of the proof is identical to the proof of Łoś theorem,

$$\{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}) \wedge \psi(\bar{a}(i))\} = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \cap \{i \in I \mid \mathcal{M}_i \models \psi(\bar{a}(i))\} \in F \implies \mathcal{N} \models \varphi, \psi \implies \mathcal{N} \models \varphi \wedge \psi$$

where the second equation is derived from filter definition.

We move to the case $\varphi = \exists x\ \psi$ for ψ that fulfills the claim. If $J = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \in F$ then for $j \in J$ we define $b_j \in M_j$ as a witness to $\mathcal{M}_j \models \psi(b_j, \bar{a}(j))$, meaning that,

$$\{i \in I \mid \mathcal{M}_i \models \psi(b(i), \bar{a}(i))\} = J \in F$$

where $b(i) = b_i$ for $i \in J$ and arbitrary otherwise. Then by the induction hypothesis $\mathcal{N} \models \psi([b], [\bar{a}])$ and therefore $\mathcal{N} \models \varphi([\bar{a}])$.

Lastly we will assume that $\varphi = \forall x\ \psi$ for ψ that fulfills the claim and show that φ does as well.

If $J = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \in F$, then if $b : I \to \bigcup M_i$ some choice function then,

$$\{i \in I \mid \mathcal{M}_i \models \psi(b(i), \bar{a})\} \supseteq J \in F$$

therefore $\mathcal{N} \models \psi([b], [\bar{a}])$, then $\mathcal{N} \models \varphi([\bar{a}])$ as required. □

Part b

We will find a language and a sentence φ such that for any set I and filter F,

there is $\langle \mathcal{M}_i \mid i \in I \rangle$ such that $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / F \models \varphi$ if and only if F is an ultrafilter.

Solution. Define $L = \{=\}$ as the trivial language, and $\varphi = \exists x \exists y\ (x \neq y \wedge \forall z\ (z = x \vee z = y))$.

Let I be some indices set and $F \subseteq \mathcal{P}(F)$. Let $M = \{0, 1\}$ and $\mathcal{N} = \mathcal{M}^I / F$.

If F is an ultrafilter then by Łoś theorem $\mathcal{N} \models \varphi$. Otherwise there is a set $J \subseteq I$ such that $J \notin F, I \setminus J \notin F$. Define,

$$f(x) = \begin{cases} 0 & x \in J \\ 1 & x \in I \setminus J \end{cases}$$

then $\mathcal{N} \models [f] \neq [c_0]$ as well $\mathcal{N} \models [f] \neq [c_1]$, and thus $\mathcal{N} \models \neg\varphi$.

Part c

Let $\mathcal{M}_n$ be finite models in the language of equality, let $\mathcal{U} \subseteq \mathcal{P}(\omega)$ be a non-principal ultrafilter, and define $\mathcal{M}_\omega = \prod \mathcal{M}_n/\mathcal{U}$. We will show that M_ω is either finite or uncountable.

Proof. Let us assume for contradiction that $|M_\omega| = \omega$ and let $\langle f_n : \omega \to \bigcup M_i \mid n < \omega\rangle$ be sequence such that $\{[f_n]\} = M_\omega$. Let $B_0 = \{n < \omega \mid f_0(n) \neq f_1(n)\} \in \mathcal{U}$, and by recursion for each B_k we define,

$$B_{k+1} = B_k \cap \{n < \omega \mid \forall i \leq k,\ f_i(n) \neq f_{k+1}(n)\} \in \mathcal{U}$$

Then $\{B_k\} \subseteq \mathcal{U}$ and $B_k \supseteq B_{k+1}$ for any k, and let $B = \inf B_n$. If $B = \emptyset$ then there is minimal $k < \omega$ such that $B_k \neq \emptyset$, but $B_k, B_{k+1} \in \mathcal{U}$, therefore $\emptyset \in \mathcal{U}$, a contradiction. Then there is $b \in B$, then $S = \{f_n(b) \mid n < \omega\}$ is a set such that $|S| = \aleph_0$, but $S \subseteq M_b$ and $|M_b| < \aleph_0$. We conclude that $|M_\omega|$ cannot be countable.

We will show that $|M_\omega|$ if and only if $|M_n| < K$ for $n \in J \in \mathcal{U}$ and $K < \omega$. Let us assume that $\forall n \in J,\ |M_n| < K$, and define,

$$\varphi_K = \forall x_0 \cdots \forall x_{K-1} \left(\bigvee_{i<j<K} x_i = x_j \right)$$

then $\mathcal{M}_n \models \varphi_K$ for any $n \in J$, therefore by Łoś we get $\mathcal{M}_\omega \models \varphi_K$ as well, in particular $|M_\omega| < K$.
In the other direction the claim holds similarly by Łoś.

We will show that $|M_\omega| = 2^{\aleph_0}$ if $|M_\omega|$ is uncountable. By the last claim, $\mathcal{M}_\omega \models \neg\varphi_K$ for any $K < \omega$, meaning that $|M_\omega|$ is unbounded. Let us assume that $h : \omega \to \omega$ is a function such that $|M_{h(n)}| \geq n$. We can also define $h_n : n \to M_n$ by the last cardinality inequality (and more choice). It is known that the cardinality of functions $\omega \to \omega$ that strictly increasing is $2^{\aleph_0}$, then it suffices to show that any such function g can be mapped uniquely to $[f] \in M_\omega$. We can define $f' = \{\langle h(g(n)), h_{h(g(n))}(g(n))\rangle\}$ and,

$$f(n) = \begin{cases} f'(n) & n \in \operatorname{dom} f' \\ h_n(0) & \text{otherwise} \end{cases}$$

Then $[f] \in M_\omega$. If g, g' are two different strictly increasing functions, and f, f' are their respective constructions, then $[f] \neq [f']$ directly from definition. We deduce that $2^{\aleph_0} \leq |M_\omega| \leq |\omega^\omega| = |2^{\aleph_0}|$. $\square$

Question 2

Part a

Suppose that T has quantifier elimination and that there is a model $\mathcal{M}_0$ that embeds in every model of T.
We will show that T is complete.

Proof. Let's assume for sake of contradiction that T is not complete, meaning that there is a sentence φ such that $T_- = T \cup \{\neg\varphi\}, T_+ = T \cup \{\varphi\}$ are both consistent. Let $\mathcal{M}_- \models T_-, \mathcal{M}_+ \models T_+$ be two models to witness that, then $\mathcal{M}_0$ embeds in both $\mathcal{M}_-$ and $\mathcal{M}_+$. T has quantifier elimination then let $\psi \equiv \varphi$ be a quantifier-free sentence. It follows that $\mathcal{M}_- \models \neg\psi$ and $\mathcal{M}_+ \models \psi$, but $\mathcal{M}_0 \subseteq \mathcal{M}_-, \mathcal{M}_+$, and by embedding properties we deduce that $\mathcal{M}_0 \models \psi, \neg\psi$, a contradiction. $\square$

Part b

Let T be the theory of torsion free divisible Abelian groups, meaning that the language is $\{0, +\}$ with the axioms,

$$\forall n > 0,\ \text{tor}_n = \forall x\,(x \neq 0 \to \underline{n} \cdot x \neq 0), \quad \text{div}_n = \forall x\,(\exists y\ \underline{n} \cdot y = x)$$

We will show that $T \cup \{\exists x\ x \neq 0\}$ has quantifier elimination, and describe its completion.

Proof. By the equivalency to quantifier elimination theorem, it is suffices to show that for any $\mathcal{M}, \mathcal{N} \models T$ and $A \subseteq M \cap N$ finite and $\exists$-primitive sentence $\varphi \in \text{sent}_{L(A)}$, $\mathcal{M}_A \models \varphi \iff \mathcal{N}_A \models \varphi$. If $\mathcal{A} = \langle A \rangle$ then $\mathcal{A} \subseteq \mathcal{M}, \mathcal{N}$, and in particular $\mathcal{A}$ is also torsion-free Abelian group, and subgroup of both $\mathcal{M}$ and $\mathcal{N}$. It follows that $\mathcal{A} \trianglelefteq \mathcal{M}, \mathcal{N}$ as well, but I don't see a way its important.

assume that $a \in \mathcal{A}$, then $\underline{n} \cdot a \in \mathcal{A}$ as well, and by div_m also $\frac{\underline{n}}{\underline{m}} \cdot a \in \mathcal{A}$. We deduce that $\langle A \rangle = \text{cl}_+(\mathbb{Q} \cdot A)$, meaning that any $a \in \mathcal{A}$ can be expressed as $\sum_{i<N} \frac{p_i}{q_i} a_i$ where $A = \{a_i \mid i < N\}$. If $t(x)$ is a term in $L(\langle A \rangle)$ then as result, $t = qx + a$ for $q \in \mathbb{Q}, a \in \mathcal{A}$. It follows that if $t(x), t'(x) \in \text{term}_{L(\langle A \rangle)}$ then $t = t' \iff t - t' = 0$ where $t - t'$ is a valid term as well. If $\theta(x)$ is an atomic formula, then $\theta \equiv t = 0$ or $\theta \equiv t \neq 0$ for $t(x)$ term as described above.

φ is a $\exists$-primitive formula, then without loss of generality,

$$\varphi = \exists x \left(\bigwedge_{i<n} t_i = 0\right) \wedge \left(\bigwedge_{j<m} s_{n+j} \neq 0\right)$$

where $t_i(x) = q_i x + b_i$ is a term for $i < n + m$.

Let us assume that $\mathcal{M} \models \varphi$, then there exists $\alpha \in M$ such that $q_i\alpha + b_i = 0 \iff \alpha = -\frac{b_i}{q_i}$, meaning that α is fixed by each $i < n$. but $b_i \in N$ and by $\mathcal{N}$ properties $-\frac{b_i}{q_i}$ is definable and thus $\alpha \in \mathcal{N}$ (in its sense as definable term) and $\mathcal{N} \models \varphi$ as testified by α. note that we did not cover the case of $n = 0, m > 0$, in this case we can construct explicit witness in each model.

We have shown that $\mathcal{M} \models \varphi \implies \mathcal{N} \models \varphi$, then by symmetry of the proof we can conclude that $\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$, meaning that T has quantifier elimination. As a side-note, we have also shown that $0 \leq n \leq 1$ for φ. $\square$