Final Exercise Answer Sheet — Logic Theory (2), 80424

August 15, 2025

Question 1

Part a

Let $U \subseteq \mathcal{P}(\mathbb{N})$ be a non-principal ultrafilter, let $\langle \mathcal{M}_n \mid n < \omega \rangle$ be a sequence of L-structures, and $\mathcal{M} = \prod_{n<\omega} \mathcal{M}_n / U$. We will show that for every countable consistent set of formulas $\Gamma(x)$ with parameters from M is realized in $\mathcal{M}$, namely that $\mathcal{M}$ is countably saturated.

Proof. Take a coverage $\langle \Sigma_n(x) \mid n < \omega \rangle \subseteq \Gamma(x)$ such that $|\Sigma_n| < \omega$ for all n. Then Σ_n is realized, and let $[f_n] \in M$ be such that $\mathcal{M} \models \Sigma_n([f_n])$. Then $a_n = \{j < \omega \mid \mathcal{M}_n \models \Sigma_n(f_n(j))\} \in U$. Filters are closed to intersection, then let us assume that $a_{n+1} \subseteq a_n$, otherwise we could define,

$$g_{n+1}(i) = \begin{cases} f_{n+1}(i) & i \in a_n \\ c_n & \text{otherwise} \end{cases}$$

where $c_n \in M_n$ is some arbitrary value.

We now take $a = \bigcap_{n<\omega} a_n$ and $[f] \in M$ such that for $n \in a$, $f(n) \in \{f_i(n) \mid i < \omega\}$. If $a \in U$ then $\mathcal{M} \models \Gamma([f])$, then let us assume $a \notin U$, conversely $a^C = \mathbb{N} \setminus a \in U$. $a^C \cap a_n \in U$ for all n and therefore $a^C \cap a_n \neq \emptyset$. It immediately follows that $\emptyset \in U$, a contradiction. □

Part b

We define σ-complete ultrafilter U as an ultrafilter such that it is closed to countable intersections.
Let U be some σ-complete ultrafilter, $L = \{=\}$, $\mathcal{M} = (\mathbb{N}, =)^I / U$ for some index set I.
We will show that $|M| = \omega$ and deduce that $\mathcal{M}$ is not countably saturated.

Proof. Directly by Łoś theorem and sentence of the form $\varphi_N = \bigwedge_{n<N} \exists x (x \neq c_n)$ we deduce that $|M| \geq \omega$. Define $C_x = \{x\}^I$ the constant function, we will show that for every $[f] \in M$ there is $n < \omega$ such that $[f] = [C_n]$. Note that this is equivalent to the claim that $\{j < \omega \mid f(j) = n\} \in U$. We will assume otherwise in contradiction, then $a_n = \{j < \omega \mid f(j) \neq n\}$ is in U, and $a_n \cap a_m$ is non-empty for all $n \neq m$. We take $a = \bigcap_{n<\omega} a_n$, U is σ-complete therefore $a \in U$. It follows that $f(j) \neq n$ for all $j \in I, n < \omega$, a contradiction to f being $I \to \mathbb{N}$ function. □

Part c

We will show that if U is an ultrafilter on some indices set I such that U is not σ-complete and $\langle \mathcal{M}_i \mid i \in I \rangle$,
then $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / U$ is countably saturated.

Proof. By σ-incompleteness we can assume that there is decreasing chain $\langle u_n \mid j < \omega \rangle \subseteq U$ such that $\bigcap_{n<\omega} u_n \notin U$. We can assume without loss of generality that $\bigcap u_n = \emptyset$ as $I \setminus \bigcap u_n \in U$ and therefore we can take $u_n \setminus \bigcap u_m$.

Let $\Gamma(x)$ be countably realized set of formulas. By countability let us denote $\Gamma(x) = \{\gamma_n \mid n < \omega\}$. For every $N < \omega$ we also define $\Gamma_N = \{\gamma_n \mid n < N\}$. Then Γ_N is finite set of formulas, and realized by $[f_N] \in M$. We can take f_N such that,

$$a_N = \{j \in I \mid \mathcal{M}_j \models \Gamma_N(f_N(j))\} \subseteq \{j \in I \mid \mathcal{M}_j \models \Gamma_N(f_M(j))\}$$

for $M < N$ by finite intersections. The sequence $\langle a_n \mid n < \omega \rangle \subseteq U$ is decreasing. Let us take $U_n = u_n \cap a_n$ for every n, clearly $U_n \in U$. Note that for every $i \in I$ there is maximal $n < \omega$ such that $i \in U_n$, and let us define $f(i) = f_n(i)$. Indeed $f \in M$ and,

$$\mathcal{M} \models \gamma_n([f]) \iff \{j \in I \mid \mathcal{M}_j \models \gamma_n(f(j))\} \in U \impliedby \{j \in I \mid \mathcal{M}_j \models \gamma_n(f_n(j))\} \in U$$

but the latter holds directly by the definition of f. □