# Exercise 6 Answer Sheet — Logic Theory (2), 80424

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# Question 1

*Notation* 0.1 (Closure over induction scheme). Suppose that $\Gamma$ is some collection of formulas over $L_{\text{PA}}$, then we define $I\Gamma = \{\text{Ind}(\varphi) \mid \varphi \in \Gamma\}$.
*Notation* 0.2 (Collection formula). Suppose that $\varphi(z, y, t_0, \ldots, t_{n-1})$ is some formula in $L_{\text{PA}}$, then we define $\text{Coll}(\varphi) = \forall z \forall t_0 \ldots \forall t_{n-1}(\forall x \leq z \exists y\ \varphi \leftrightarrow \exists w \forall x \leq z \exists y \leq w\ \varphi)$.

In the previous exercise we proved that $\text{PA} \vdash \text{Coll}(\varphi)$ for any $\varphi$. Let $\text{Coll} = \{\text{Coll}(\varphi) \mid \varphi \in \text{form}_{L_{\text{PA}}}\}$. Let $Q \subseteq \text{PA}$ be the first 9 axioms without the induction scheme, and let $\text{PA}_0 = Q \cup I\Sigma_0 \cup \text{Coll}$. We will show that $\text{PA}_0 \vdash I\Sigma_1$.

*Proof.* Let $\psi \in I\Sigma_1$, meaning that $\psi = \text{Ind}(\exists x\ \varphi(x, y, z))$ for some $\varphi \in \Sigma_0$, for localized version of $\Sigma_0$. Let us assume that $\mathcal{M} \models \text{PA}_0$ is some model such that $c = z^{\mathcal{M}}$ for some $z$ such that $\mathcal{M} \models \exists x\ \varphi(x, 0, c)$ and $\mathcal{M} \models (\exists x\ \varphi(x, y, c)) \to (\exists x\ \varphi(x, y+1, c))$. There exists such a model as this sentence is satisfiable.
We intend to show that $\mathcal{M} \models \forall y \exists x\ \varphi(x, y, c)$. We fix some $b \in M$, and want to show that $\mathcal{M} \models \exists x \varphi(x, b, c)$. We use $\text{Coll}(\varphi(x, y \leq b, c))$ to fix some $d$ such that for all $b' \leq b$, if $\mathcal{M} \models \exists x\ \varphi(x, b', c)$ then $\mathcal{M} \models \exists x \leq d\ \varphi(x, b', c)$. Note that this is an $I\Sigma_0$ formula, thus by our assumptions about $\mathcal{M}$ it holds, and it follows that $\exists x \varphi(x, b, c)$ holds as well. $\square$

# Question 3

## Part a

We will show that full recursion functions are definable.

*Proof.* Let us assume that $G$ is some primitive-recursion function, and we define $F(n) = G(\#\langle F(i) \mid i < n\rangle)$ (the sequence is coded). We will show that $F$ is primitive recursion definable by showing it is $\Sigma_0^1$.
Note that when given a finite sequence, a function to append value to the sequence is primitive-recursive. This is true as the length function is primitive-recursive, as the singleton and union, enabling us to code the pair $\langle i, x\rangle$ for some given $x$ numeral. Let $\eta : \mathbb{N}^2 \to \mathbb{N}$ be such function, meaning that $\eta(\#\langle a_i \mid i < n\rangle, a_{i+1}) = \#\langle a_i \mid i < n+1\rangle$.
Lastly we define $H(x) = \eta(x, G(x))$ and use the induction scheme over it to define $F$. □

## Part b

We will show that the relation $P(x)$ such that $P(x)$ if and only if $x$ is a prime, is primitive-recursive.

*Proof.* $P(x) \iff \forall y,\ y \mid x \to y = 1 \lor y = x$. This is clearly a $\Sigma_0$ formula, thus implies that $P$ is $\Delta_1^0$ and a primitive-recursion relation. □

## Part c

We will show that the function $n \mapsto p_n$ where $p_n$ it the $n$-th prime, is primitive-recursion function.

*Proof.* By primes definition and the order of the primes as sub-order of the elements,

$$F(n+1) = y \iff \exists p_n < y < p_n^2,\ P(y) \land \forall p_n < x < y, \neg P(y)$$

But this is full recursion using a primitive-recursion relation, then we deduce that $F$ is indeed primitive-recursive. □

# Question 4

We define the base $b$-concatenation as the function such that for every $x, y, b \in \mathbb{N}$, $x *_b y$ is the representation of the digits of $x$ followed by these of $y$ in base $b$. We will show that the function $(x, y, b) \mapsto x *_b y$ is primitive-recursive.

*Proof.* We define the operation / as the no-residue division, meaning that if $y = x \cdot b + c$ for $c < b$, then $y/b = x$. This is a primitive-recursive as it is a $\Sigma_1^0$ function.

We define the least nullifier function as the function such that $\lambda_b(x)$ is finite product of $b$, that its the minimal value such that $x/\lambda_b(x) = 0$. For example, $\lambda_2(7) = 2^3 = 8$. We also define $0 \mapsto 1$. The recursive definition is

$$\lambda_b(x) = \begin{cases} 1 & x = 0 \\ b \cdot \lambda(x/b) & x > 0 \end{cases}$$

This is a primitive-recursive function by question 3.

Directly from definition of base-$b$ representation of a number we deduce that $x \cdot \lambda_b(y) + y$ is the concatenation. $\square$