Solution to Final Exercise — Model Theory (1), 80616

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Question 1

Urysohn Space. Let $L = \{D_q \mid q \in \mathbb{Q}_{\geq 0}\}$ such that D_q is a binary relation symbol for any q. For any metric space (X, d) we can define L-structure $\mathcal{X}$ with $|\mathcal{X}| = X$ and $(x, y) \in D_q^{\mathcal{X}} \iff d(x, y) = q$.

Part a

Let $\mathcal{K}$ be the class of all finite metric spaces with rational distances.
We will show that $\mathcal{K}$ is a Fraïssé class.

Proof. We will follow the definition of Fraïssé class.

Closure under isomorphism If $\mathcal{X} \in \mathcal{K}$ and $\mathcal{Y} \cong \mathcal{X}$ and notate $(X, d), (Y, \rho)$ the equivalent metric spaces, then $f : \mathcal{X} \to \mathcal{Y}$ that witness the isomorphism fulfills,

$$d(x_1, x_2) = q \iff D_q^{\mathcal{X}}(x_1, x_2) \iff D_q^{\mathcal{Y}}(f(x_1), f(x_2)) \iff \rho(f(x_1), f(x_2)) = q.$$

Meaning that (Y, ρ) is a finite metric space with rational distance and therefore $\mathcal{Y} \in \mathcal{K}$ as well.

Essential countability Let $X_n = [n]$ and $d : X_n^2 \to \mathbb{Q}$ be a metric, define $\mathcal{X}_n^d$ as the equivalent of the metric space (X_n, d). We define the collection,

$$\mathcal{M} = \{\mathcal{X}_n^d \mid n < \omega, d \text{ is a metric}\}.$$

and notice that for each $n < \omega$, the collection of metric functions defined on X_n is countable as it is a subset of $\mathbb{Q}^{n^2}$. It follows that $\mathcal{M}$ is a countable collection, and notice that $\mathcal{M} \subseteq \mathcal{K}$.
Let $\mathcal{Y} \in \mathcal{K}$, and let (Y, ρ) be the equivalent metric space, notate $n = |Y|$ and let us define $f : [n] \to Y$ a bijection, then,

$$d : X_n^2 \to \mathbb{Q}, \qquad d(x_1, x_2) = \rho(f(x_1), f(x_2)).$$

is a metric on X_n and therefore $\mathcal{X}_n^d \in \mathcal{M}$ and $\mathcal{X}_n^d \cong \mathcal{Y}$.

Hereditary property Assume that $\mathcal{X} \subseteq \mathcal{Y} \in \mathcal{K}$ and show that $\mathcal{X} \in \mathcal{K}$, we omit the requirement that $\mathcal{X}$ is finitely-generated as there are no function symbols and $|Y| < \omega$. Metric property is closed under restrictions then if (Y, ρ) a metric space and $X \subseteq Y$ then $(X, \rho \restriction X^2)$ is a metric space as well.

The joint embedding property Suppose that $\mathcal{X}, \mathcal{Y} \in \mathcal{K}$ and let $(X, d), (Y, \rho)$ be their respective equivalencies. Assume that $X \cap Y \neq \emptyset$ then let $x_0 \in X \cap Y$ be some element, and let $C = A \cup B$, define,

$$f : C^2 \to \mathbb{Q}_{\geq 0}, \qquad f(a, b) = \begin{cases} d(a, b) & a, b \in X \\ \rho(a, b) & a, b \in Y \\ d(a, x_0) + \rho(b, x_0) & a \in X, b \in Y \\ d(b, x_0) + \rho(a, x_0) & b \in X, a \in Y \end{cases}.$$

Then (C, f) is a metric space and it follows that $\mathrm{id}_X, \mathrm{id}_Y$ are embeddings of X, Y into C respectively.
If $X \cap Y = \emptyset$ then denote some arbitrary elements $x_0 \in X, y_0 \in Y$, let $f : C^2 \to \mathbb{Q}_{\geq 0}$ defined by,

$$f(a, b) = \begin{cases} d(a, b) & a, b \in X \\ \rho(a, b) & a, b \in Y \\ d(x_0, a) + \rho(y_0, b) + 1 & a \in X, b \in Y \\ d(x_0, b) + \rho(y_0, a) + 1 & \text{otherwise} \end{cases}.$$

Thus f is a metric over C, as it is symmetric, non-negative and fulfills triangle inequality. If $\mathcal{C}$ is the structure of (C, f) then $\mathcal{C} \in \mathcal{K}$ and $\mathrm{id}_X, \mathrm{id}_Y$ are embedding of $\mathcal{X}, \mathcal{Y}$ into $\mathcal{C}$.

The amalgamation property Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{K}$ with $(X, d_x), (Y, d_y), (Z, d_z)$ such that $g_x : \mathcal{Z} \to \mathcal{X}, g_y : \mathcal{Z} \to \mathcal{Y}$ are embeddings. Let $\mathcal{C} \in \mathcal{K}$ as in the last statement, and assume without loss of generality that $\mathcal{C} \subseteq \mathcal{X}, \mathcal{Y}$ (otherwise we can use closure under isomorphism) and therefore g_x, g_y are both identity. Let $f_x : \mathcal{X} \to \mathcal{C}, f_y : \mathcal{Y} \to \mathcal{C}$ be embeddings, it is implied that,

$$f_y \circ g_y = f_y \circ \mathrm{id}_Z = f_x \circ \mathrm{id}_Z = f_x \circ g_x.$$

and the property holds as wished. □

Let us denote $\mathbb{U}_\mathbb{Q}$ the Fraïssé limit of $\mathcal{K}$, and let $(U_\mathbb{Q}, d)$ be its equivalent metric space.

Part b

Let (X, d) be a metric space, a Katětov map is a function $f : X \to \mathbb{R}_{\geq 0}$ such that,

$$\forall x, y \in X,\ |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y).$$

A rational Katětov map is a Katětov map such that $\operatorname{Im} f \subseteq \mathbb{Q}$.
We will show that for every finite $A \subseteq U_\mathbb{Q}$, any rational Katětov map $f : A \to \mathbb{Q}_{\geq 0}$ is realized, namely there is some $b \in U_\mathbb{Q}$ such that $\forall x \in U_\mathbb{Q},\ f(x) = d(x, b)$.

Proof. Note that if $\mathcal{X} \cong \mathcal{Y}$ and $f : \mathcal{X} \to \mathcal{Y}$ isomorphism, then $R_q(x, y) \iff R_q(f(x), f(y))$, meaning that f is an isometry of X, Y as metric spaces.

Let f be some rational Katětov map of A. Let $B = A \uplus \{c\}$ for some element, we define a metric on B, $\rho : B^2 \to \mathbb{Q}_{\geq 0}$ by,

$$\rho(x, y) = \begin{cases} d(x, y) & x, y \in A \\ d(c, x) = f(x) & x \in A \\ d(c, c) = 0 \end{cases}.$$

ρ is symmetrical and non-negative, and fulfills the triangle inequality for elements of A, let us verify that for c as well,

$$\rho(c, c) = 0 \leq 2f(x) = \rho(c, x) + \rho(x, c).$$

and,

$$\rho(x, y) = d(x, y) \leq f(x) + f(y) = \rho(x, c) + \rho(c, y).$$

Then (B, ρ) is indeed metric space, but B is finite and thus $\mathcal{B}$ is isomorphic to some substructure of $\mathbb{U}_\mathbb{Q}$, $\mathcal{B} \cong \mathcal{C} \in \mathcal{K}$, and let $g : \mathcal{B} \leftrightarrow \mathcal{C}$ be the isomorphism. $\mathbb{U}_\mathbb{Q}$ is ultra-homogeneous then the isomorphism of $A \to \mathcal{C} \setminus \{g(c)\}$ can be extended to $\sigma \in \operatorname{Aut}(\mathbb{U}_\mathbb{Q})$, and let $e = \sigma^{-1}(c')$. For any $x \in A$, $f(x) = \rho(x, c) = d(g(x), g(c)) = d(\sigma^{-1}(g(x)), \sigma^{-1}(g(c))) = d(x, e)$ as wished. □

We will show that $\mathbb{U}_\mathbb{Q}$ is the unique countable rational metric space such that for any $A \subseteq X$ finite and rational Katětov map on A, there is $b \in X$ realizing f.

Proof. Let (X, ρ) be a metric space with this property, we will show that $(X, \rho) \cong (U_\mathbb{Q}, d)$. We will show by induction that for every X_n^f (as defined in part 1) there is $Y \subseteq X$ such that $(X_n^f, f) \cong (Y, \rho \restriction Y^2)$. The case of $n = 1$ is trivial as the single metric on single valued space is 0.
Let us assume the claim holds for $m < n$ and let X_n^f be some finite and rational metric space. Take $(X_{n-1}^{f \restriction X_{n-1}^2}, f \restriction X_{n-1}^2)$, we get a sub-metric space of size $n-1$, meaning that there is some such $Y \subseteq X$ and let $\sigma : X_{n-1} \leftrightarrow Y$. Define $g : Y \to \mathbb{Q}_{\geq 0}$ by $g(x) = f(n-1, \sigma^{-1}(x))$, then g is a Katětov map, as a result of the last subpart. There is an element $b \in X$ that realizes g, therefore $\rho(x, b) = g(x) = f(n-1, \sigma^{-1}(x))$, meaning that $f(\sigma^{-1}(\cdot), \sigma^{-1}(\cdot)) = \rho \restriction (Y \cup \{b\})^2$.
We have shown that $\{X_n^f \mid n < \omega, f \text{ is a metric}\} \subseteq \operatorname{Age}(\mathcal{X})$ for $\mathcal{X}$ the equivalent of (X, ρ), therefore $\operatorname{Age}(\mathcal{X}) = \mathcal{K} = \operatorname{Age}(\mathbb{U}_\mathbb{Q})$, then $\mathcal{X} \cong \mathbb{U}_\mathbb{Q}$. □

Part c

Let $\mathbb{U}$ be the completion of $\mathbb{U}_{\mathbb{Q}}$, we will show that any finite metric space (X, d) is isometrically embedded into $\mathbb{U}$.

Proof. The proof is by induction over $n = |X|$. For $n = 1$ the proposition is trivial. Assume that the proposition holds for $m < n$ and prove for n. Let $x_0 \in X$ be some element and let $Y = X \setminus \{x_0\}$ and therefore $(Y, d \upharpoonright Y^2)$ is a finite metric space of size $n - 1$. The induction hypothesis holds for Y, and let us denote $\varphi : (Y, d \upharpoonright Y^2) \to \mathbb{U}$ isometry embedding of Y in $\mathbb{U}$.

Denote $X = \{x_m \mid m < n\}$, and let $\varepsilon > 0$. For any $0 < m$ there is $|u_m^\varepsilon - \varphi(x_m)| < \varepsilon$ such that $u_m^\varepsilon \in U_{\mathbb{Q}}$. By the last part we also deduce that there is $u_0^\varepsilon \in U_{\mathbb{Q}}$ such that if (U, ρ) is the equivalent metric space of $\mathbb{U}$, then $|d(x_i, x_j) - \rho(u_i^\varepsilon, u_j^\varepsilon)| < \varepsilon$ (Oh yes, this is a diagonal proposition). Define the sequences $\langle x_m^{\frac{1}{k}} \mid 0 < k < \omega\rangle$, then $x_m^{\frac{1}{k}} \to x_m$ and let us define $u_0^{\frac{1}{k}}$ as well. By our definition of $\langle u_0^{\frac{1}{k}} \mid 0 < k < \omega\rangle$, it is a Cauchy sequence and therefore it converges, let us denote its limit with u_0. It follows that $|d(x_i, x_j) - \rho(u_i, u_j)| = 0$, meaning that the map $x_i \mapsto u_i$ is an isometry into $\mathbb{U}$. $\square$

Part d

We will show that for any $A \subseteq \mathbb{U}$ finite and Katětov map f on A is realized by point $b \in \mathbb{U}$.

Proof. The proof is identical to the proof of the last two parts, using induction over the size of A and constructing space using rational Katětov functions and their realization. $\square$

Part e

We will show that every complete separable metric space is isometrically embedded into $\mathbb{U}$.

Proof. By combining part 3 and 4 we get that if (X, d) is a finite metric space and $(X \cup \{y\}, f)$ is an extension of (X, d), namely f is a metric and $f \upharpoonright X^2 = d$, then the isometry embedding of (X, d) into $\mathbb{U}$ is a sub-extension of $(X \cup \{y\}, f)$ into $\mathbb{U}$.

Let (X, d) be a countable metric space and suppose that $X = \{x_i \mid i < \omega\}$, then $(\{x_i \mid i < n\}, d \upharpoonright (\{x_i \mid i < n\})^2)$ is isometrically embedded by σ_n into $\mathbb{U}$, and by our assumption $\sigma_i \subseteq \sigma_j$ for all $i < j < \omega$. Let us denote $\sigma = \bigcup_{i<\omega} \sigma_n$, then $\sigma : X \to \mathbb{U}$ is an isometry embedding as for any choice of points x, y, there is n such that $x, y \in \{x_i \mid i < n\}$.

We can now move to the case of (X, d) complete separable metric space. Take $Y \subseteq X$ dense subset, and let $\sigma : Y \to \mathbb{U}$ be isometry embedding. For each $x \in X$ there is a sequence $(y_n)_{n=1}^\infty \subseteq Y$ such that $y_n \to x$, and therefore $\sigma(y_n) \to b$ for some $b \in \mathbb{U}$, we define $x \mapsto b$ for all $x \in X$. The result is an isometry embedding $\sigma' : X \to \mathbb{U}$. $\square$

Question 2

Groups.

Part a

Let $\mathcal{G}$ be a group in the language of group theory.

i

Assume that $\mathrm{Th}(\mathcal{G})$ is ω-categorical, we will show that for every finite $A \subseteq G$, it holds that $\mathcal{A} = \langle A \rangle$ is finite, where $\langle A \rangle$ is the subgroup generated by A.

Proof. Assume that $|G| = \omega$, then $\mathcal{G}$ is saturated. We assume for the sake of contradiction that the proposition does not hold, meaning that there is $A \subseteq G$ finite with $\langle A \rangle = \mathcal{G}$. We define $T : A^{<\omega} \to G$ by,

$$\forall f : [n] \to A, \quad T(f) = \prod_{i=0}^{n-1} f(i).$$

and let $p \in S_1(\mathcal{G})$ defined as,

$$p(x) = \{T(f) \neq x \mid f \in A^{<\omega}\}.$$

p is realized in $\mathcal{G}$, but by p's definition it holds that if $c \in G$ witnessing p then $c \notin \langle A \rangle$, a contradiction. □

ii

Let us assume that for every finite $A \subseteq G$, $|\langle A \rangle|$ is finite as well. We will show that $\mathrm{Th}(\mathcal{G})$ is not ω-categorical.

Proof. Let us construct a counter example. Let μ_p denote Prüfer p-group over $\mathbb{C}$, namely,

$$\mu_p = \{z \in \mathbb{C} \mid z^p = 1\}.$$

and $\mu_\infty = \bigcup_{0<n} \mu_p$. Each finite $A \subseteq \mu_\infty$ equals to $\mu_{\gcd A}$, in particular $\langle A \rangle$ is finite. Let $\nu_\infty = \langle \bigcup_{2<p\in P} \mu_n \rangle$ be the subset of μ_∞ such that it has no elements of order 2. As a subgroup of μ_∞, it holds that ν_∞ also follows the required property, and it is clear that it is countable, therefore we found two such groups that are not isomorphic. □

Part b

Let $\mathcal{G}$ be a group and let us assume that $\mathrm{Th}(\mathcal{G})$ is totally transcendental. We will show that there is no infinite strictly decreasing sequence of definable subgroups $\langle H_n \mid n < \omega \rangle$.

Proof. We will show that if $\mathcal{G}$ has definable strictly decreasing subgroups then $\mathrm{Th}(\mathcal{G})$ has the binary tree property. Let us assume that $\langle H_n \mid n < \omega \rangle$ is such a sequence, meaning that $\mathcal{G} = \mathcal{H}_0 > \mathcal{H}_1 > \cdots$. H_n is definable and let $\varphi_n(x)$ witness it, meaning that $\mathcal{G} \models \varphi_i(x) \to \varphi_j(x)$ for any $i < j < \omega$. Let $\psi_n(x) = \varphi_n(x) \wedge (\neg \varphi_{n+1}(x))$, namely $\mathcal{M} \models \psi_n(x)$ if $x \in H_n \setminus H_{n+1}$, the sequence is strictly decreasing therefore $\mathcal{G} \models \exists x\, \psi_n(x)$ for any n.

For any n, we denote h_n as a witness of $\varphi_n \wedge (\neg \varphi_{n+1})$, meaning that $h_n \in H_n \setminus H_{n+1}$, meaning that $h_n H_{n+1}$ is a proper coset of H_{n+1} in H_n (and in G). We deduce that $H_{n+1} \cap h_n H_{n+1} = \emptyset$, and define $\varphi_{n+1}^{h_n}(x) = \varphi_{n+1}(h_n^{-1} x)$, hence $\mathcal{G} \models \varphi_{n+1}^{g}(x) \iff x \in h_n H_{n+1}$. Given a formula φ_n^g for some $g \in H$, we can define the following formulas,

$$\varphi_{n+1}^{g}(x) = \varphi_{n+1}(g^{-1} x), \qquad \varphi_{n+1}^{h_n g}(x) = \varphi(g^{-1} h_n^{-1} x).$$

By $H_{n+1} \subsetneq H_n$ we get that $gH_{n+1} \subsetneq gH_n$, therefore $\{\varphi_n^g, \varphi_{n+1}^g\}$ is consistent, We also get that $\{\varphi_n^g, \varphi_{n+1}^{h_n g}\}$ is consistent as a result of the previous statement, but $\mathcal{G} \models \varphi_{n+1}^g(x) \wedge \varphi_{n+1}^{h_n g}(x) \iff x \in gH_n, h_n gH_n$, but $gH_n \cap h_n gH_n \neq \emptyset \iff h_n H_n \cap H_n \neq \emptyset$, implying that $\varphi_{n+1}^g \wedge \varphi_{n+1}^{h_n g}$ is inconsistent.

We can now get to the construction of the binary tree. Let $T : 2^{<\omega} \to G$ be defined as,

$$T(\eta) = \prod_{n \in \eta^{-1}(\{1\})} h_n.$$

and let $\Gamma : 2^{<\omega} \to$ form defined by,

$$\forall \eta : 2^n \to G, \quad \Gamma(\eta) = \varphi_{n+1}^{T(\eta)}.$$

Γ fulfills the binary tree property, and therefore witnessing that $\mathcal{G}$ is not totally transcendental.

The proposition holds by contra positive principle. $\square$

Question 3

Part a

Let (X, cl) be a pre-geometry, we will show that there is a basis for X and that every two bases have the same cardinality.

Proof. We say that a set $B \subseteq X$ is a basis if $\mathrm{cl}(B) = X$ and $\forall b \in B,\ b \notin \mathrm{cl}(B \setminus \{b\})$ (independent). We will show that B is a basis if and only if B is maximally independent.
Assume that B is a basis, and let $x \in X \setminus B$, then $x \in \mathrm{cl}(B)$ and therefore B is maximally independent.
In the other direction, let us assume that B is maximally independent, we will show that $\mathrm{cl}(B) = X$. We assume towards contradiction that there is $x \in X \setminus \mathrm{cl}(B)$, and let $B' = B \cup \{x\}$, thus $x \in \mathrm{cl}(B')$ but $x \notin \mathrm{cl}(B' \setminus \{x\}) = \mathrm{cl}(B)$. For any $b \in B$ we already assumed that $b \notin \mathrm{cl}(B \setminus \{b\})$, therefore B' is independent and $B' \supsetneq B$, a contradiction.

Let $\mathcal{B} = \{B \subseteq X \mid B \text{ is independent}\}$, notice that $\emptyset$ is independent then $\mathcal{B}$ is not empty. In intention of using Zorn's lemma, let $C \subseteq \mathcal{B}$ be a chain under $\subseteq$, and let $B^* = \bigcup C$. If $b \in B^*$, then,

$$\mathrm{cl}(B^* \backslash \{b\}) = \bigcup \{\mathrm{cl}(B_0) \mid B_0 \subseteq B^* \backslash \{b\}, |B_0| < \omega\} \overset{(1)}{=} \bigcup \{\mathrm{cl}(B_0) \mid \exists B \in C\, B_0 \subseteq B \backslash \{b\}, |B_0| < \omega\} = \bigcup_{B \in C} \mathrm{cl}(B \backslash \{b\}).$$

where (1) is following from the fact that if B_0 is finite and $B_0 \subseteq \bigcup C$ then there is an element in the chain such that $B_0 \subseteq B \in C$. All $B \in C$ are independent then $b \notin \mathrm{cl}(B \setminus \{b\})$, it follows that $b \notin \mathrm{cl}(B^* \setminus \{b\})$ as well, B^* is independent and maximal. The conditions for Zorn's lemma are fulfilled, as a result $\mathcal{B}$ has a maximal element, in other words there is $B \in \mathcal{B}$ that is maximally independent, and thus a basis.

Let $B_1, B_2 \subseteq X$ be bases for (X, cl), we will show that $|B_1| = |B_2|$. Let $B_1 = \{b_\alpha \mid \alpha < \kappa\}$, for any α, $\alpha \in \mathrm{cl}(B_2)$, so there is finite set $B_0^\alpha \subseteq B_2$ such that $b_\alpha \in \mathrm{cl}(B_0^\alpha)$. Take $C = \bigcup_{\alpha < \kappa} B_0^\alpha$, then $B \subseteq \mathrm{cl}(C) \implies X = \mathrm{cl}(B) \subseteq \mathrm{cl}(\mathrm{cl}(C)) = \mathrm{cl}(C) \implies \mathrm{cl}(C) = X$ and $|C| = \kappa$ as a union of finite sets. Notice that $C \subseteq B_2$, then $|B_1| = |C| \leq |B_2|$, by symmetry it also follows that $|B_2| \leq |B_1|$ and by CSB $|B_1| = |B_2|$. $\square$

Part b

Let T be a countable complete $\aleph_1$-categorical theory. We will show that T has at most $\aleph_0$ different isomorphism classes for countable models.

Proof. $\square$