Solution to Exercise 0 — Model Theory (1), 80616

November 15, 2025

Question 1

Definition 0.1. A formula φ is called *basic Horn* formula if,

$$\varphi = (\theta_0(\bar{x}) \wedge \cdots \wedge \theta_{m-1}(\bar{x})) \to \theta_m(\bar{x})$$

where θ_i is atomic for $i \leq m$.

Remark. In the case $m = 0$ we get $\varphi = \theta_0(\bar{x})$. In the case that $\theta_m = \perp$, $\varphi \equiv \neg\theta_0(\bar{x}) \vee \cdots \neg\theta_{m-1}(\bar{x})$.

Definition 0.2. The set of Horn formulas is the minimal set of formulas containing the basic Horn formulas and closed under conjugation and quantification.

Part a

Let $\psi(x_0, \ldots, x_{n-1})$ be a Horn formula and let $F \subseteq \mathcal{P}(I)$ be a filter. Let $\langle \mathcal{M}_i \mid i \in I \rangle$ be a sequence of structures and $a_j \in \prod_{i \in I} \mathcal{M}_i$ for $j < n$ such that,

$$\{i \in I \mid \mathcal{M}_i \models \psi(a_0(i), \ldots, a_{n-1}(i))\} \in F$$

We will show that $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / F \models \psi([a_0]_F, \ldots, [a_{n-1}]_F)$.

Proof. Let us prove by induction over Horn set.

Assume that $\varphi(\bar{x}) = P(\bar{x})$ and $\{i \in I \mid \mathcal{M}_i \models P(\bar{x})\} \in F$, then by definition $\mathcal{N} \models \varphi$. Let us assume that $\varphi = (\theta_0 \wedge \cdots \wedge \theta_{n-1}) \to \theta_n$ for atomic θ_n, then the claim holds directly from definition as in the last part.

We will now check the case when $\theta_m = \perp$. □

Part b

We will find a language and a sentence φ such that for any set I and filter F, there is $\langle \mathcal{M}_i \mid i \in I \rangle$ such that $\prod_{i \in I} \mathcal{M}_i / F \models \varphi$ if and only if F is an ultrafilter.

Solution.