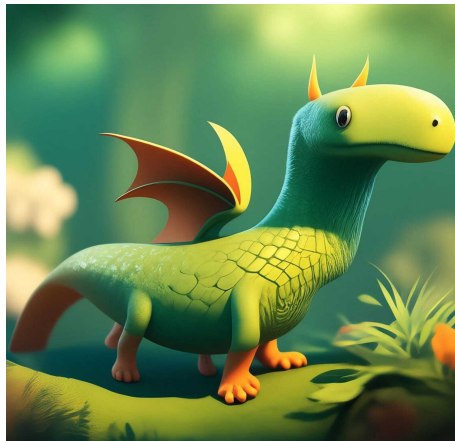


Exercise 3 Answer Sheet — Logic Theory (2), 80424

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Question 1

Let $T = \text{Th}(\hat{\mathbb{N}})$ We will show that T has 2^ω non-isomorphic countable models.

Proof. Let us define $P \subseteq \hat{\mathbb{N}}$ the set of prime numbers. We assume that $S \subseteq P$ is some set. We enrich the language by a new constant symbol, c . Let $T_S = T \cup \{\exists x, (x \cdot p = c) \wedge \neg(\exists y, y \cdot p = x) \mid p \in S\}$, namely extension of T such that $c = \prod_{p \in S} p$. T_S is finitely satisfiable by $\hat{\mathbb{N}}$ as for every finite S , $\prod_{p \in S} p \in \mathbb{N}$, then by compactness T_S is also satisfiable. It follows that there is a model $\mathcal{M}_S \models T_S$ over $L \cup \{c\}$, then it is also a model of T by reduction of the language.

Let $S \neq S'$ be subsets of P and let \mathcal{M}, \mathcal{N} be some of their respective models. We will show that $\mathcal{M} \not\cong \mathcal{N}$. It is clear for $L \cup \{c\}$ -isomorphisms, as there is $p \in S \setminus S'$ (without loss of generality) such that $\mathcal{M} \models p \mid c$ but $\mathcal{N} \not\models p \mid c$. By the last claim we can deduce that if there was such L -isomorphism, then for $c^{\mathcal{M}}$ there would be a contradiction, meaning there is no such isomorphism.

We know that $P \subseteq \mathbb{N}$, and that this set is not finite, then $|P| = \omega$, it derives that there are $|2^P| = 2^\omega$ non-isomorphic models $\mathcal{M}_S \models T$. \square

Question 2

Let F be an infinite field, and let $L_{F\text{VS}}$ be the language of vector spaces over F . Let $V \subseteq U$ be two non-trivial vector spaces over F , we will show that $V \prec U$.

Proof. Let P be a $L_{F\text{VS}}$ -disjoint unary relation symbol to $L_{F\text{VS}}$, and let us define $L' = L_{F\text{VS}} \cup \{P\}$. We extend U to L' by $P^U(x) \iff x \in V$, namely U' . For every $\psi(x_0, \dots, x_{n-1}) \in \text{form}_{L\text{VS}}$, let $\varphi_\psi(x_0, \dots, x_{n-1}) = \psi(x_0, \dots, x_{n-1}) \wedge P(x_0) \wedge \dots \wedge P(x_{n-1})$, a formula such that $U' \models \psi \iff V \models \psi$. It follows that there is a theory T such that $U' \models T \iff V \prec U$, meaning that T testifies to elementary substructure of V in U .

Assuming that $|F| = \kappa$ we use upward Löwenheim–Skolem theorem to extend U' to a model U'' such that $|U''| > \kappa$ and $U' \prec U''$. We define $V'' = P^{U''}$, meaning that V'' is the extension of V . We note that $|F| < |U''|, |V''|$, hence both spaces are infinite dimensional. From question 1 of exercise 2 we infer that indeed $V'' \prec U''$, implying that $U'' \models T$, then $U' \models T$ as well, which lead us to the conclusion that $V \prec U$, as intended. \square

Question 3

Let \mathcal{M} be the model $\langle \mathbb{Q}(\pi); + \rangle$, we will show that the function $f_\pi : \mathbb{Q}(\pi) \rightarrow \mathbb{Q}(\pi)$ such that $f_\pi(u) = \pi u$ is not definable in \mathcal{M} .

Proof. We will show that there is no formula $\varphi(x, y) \in \text{form}_L$ such that f_π is definable as the separation of φ on M , meaning that $\varphi(x, y) \iff y = \pi \cdot x$. We assume toward a contradiction that such a formula indeed exists. Let $\mathcal{N} \succ \mathcal{M}$ be an elementary extension such that $|M| < |N|$, namely a model of larger cardinality, which derived from Löwenheim–Skolem theorem. We fix some $e \in N \setminus M$, there is indeed one as $|N \setminus M| = |N|$. Let $B = M \cup \{e\}$ and $\mathcal{N}' = \text{Sp } B$, the vector space over the field \mathbb{Q} . It directly follows that $M \subseteq N'$, as well the language enrichment \mathcal{M}' of \mathcal{M} to vector spaces over \mathbb{Q} fulfills $\mathcal{M}' \prec \mathcal{N}'$ by the last question. We reduce the language to infer that $\mathcal{M} \prec \mathcal{N}'$ as well. We assumed that $\mathcal{M} \models \forall x \exists y, \varphi(x, y)$ (the function condition over the formula φ), it follows that $\mathcal{N}' \models \forall x \exists y, \varphi(x, y)$ as well. In particular, $\mathcal{N}' \models \exists y, \varphi(e, y)$, meaning that $y = \pi \cdot e$, but $e \notin M$ and is not closed to multiplication by π , which implies that $\pi \cdot e \notin \mathcal{N}'$, a contradiction to the assumption that φ exists. \square

Question 4

Part a

We will show that PA proves that the addition is associative and commutative.

Proof. we will prove that

$$\text{PA} \models \forall x, y (S(x + y) = S(x) + y) \quad (\text{C1})$$

by induction over y . For $y = 0$ it derives that $S(x+0) = S(x) = S(x)+0$ from axiom N3. If $\text{PA} \models \forall x, (S(x+n) = S(x)+n)$ then,

$$\begin{aligned} \forall x, (S(x + n) = S(x) + n) &\iff \forall x, (S(S(x + n)) = S(x) + S(n)) \\ &\iff \forall x, (S(x + S(n)) = S(x) + S(n)) \end{aligned}$$

Directly by axiom N4.

Let $\varphi(y) = \forall x, z(x + (y + z) = (x + y) + z)$, and $\psi = \forall y \varphi(y)$ be the sentence that testifies to associativity, we will show that $\text{PA} \models \psi$, by showing that $\text{Ind}(\varphi)$ holds. $\text{PA} \models \varphi(0)$ directly by the axioms N3 and N4. Let us assume that $\varphi(n)$ holds, we will show that $\varphi(S(n))$ holds as well,

$$\begin{aligned} \text{PA} \models \varphi(n) &\iff \forall x, z(x + (n + z) = (x + n) + z) \\ &\xrightarrow{\text{N1}} \forall x, z(S(x + (n + z)) = S((x + n) + z)) \\ &\xrightarrow{\text{N4, C1}} \forall x, z(x + S(n + z) = S(x + n) + z) \\ &\iff \forall x, z(x + (S(n) + z) = (x + S(n)) + z) \\ &\iff \varphi(S(n)) \end{aligned}$$

We conclude by induction axiom scheme that indeed $\text{PA} \models \psi$.

Let $\varphi(x) = \forall y(x + y = y + x)$ and $\psi = \forall x, \varphi(x)$, sentence such that $\text{PA} \models \psi$ if and only if PA is commutative. We will show that $\text{PA} \models \text{Ind}(\varphi)$. For $x = 0, 0 + y = S(0 + y') = \dots = S(S(\dots S(0 + 0))) = y = y + 0$ by axioms N3 and N4. Let us assume that $\varphi(n)$ holds, then,

$$\begin{aligned} \text{PA} \models \varphi(n) &\iff \forall y(n + y = y + n) \\ &\xrightarrow{\text{N1}} \forall y(S(n + y) = S(y + n)) \\ &\xrightarrow{\text{N4, C1}} \forall y((S(n) + y) = y + S(n)) \\ &\iff \text{PA} \models \varphi(S(n)) \end{aligned}$$

It follows that $\text{Ind}(\varphi)$ holds, meaning that $\text{PA} \models \psi$. □

Part b

We will show that PA proves that $(+, \cdot, \leq)$ is a semi-ring, namely $+, \cdot$ are associative and commutative, $0, 1$ are neutral and \leq is linear order.

Proof. associativity and commutativity of $+$ was shown in the first part, the proof for \cdot is similar. By N3 and N5 we infer that $0, 1$ are indeed neutral elements.

We move to show \leq is a linear order. From N9, \leq is reflexive. To show transitivity we use induction, for the base N7 and for the step iterative usage of N8. For linearity we use N9. We also need to show that both $+$ and \cdot are order-preserving for \leq . This statement is proven in induction over the added value to both sides of every inequality. □