Final Exercise Answer Sheet — Logic Theory (2), 80424

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Question 1

Part a

Let $U \subseteq \mathcal{P}(\mathbb{N})$ be a non-principal ultrafilter, let $\langle \mathcal{M}_n \mid n < \omega \rangle$ be a sequence of L-structures, and $\mathcal{M} = \prod_{n<\omega} \mathcal{M}_n / U$. We will show that for every countable consistent set of formulas $\Gamma(x)$ with parameters from M is realized in $\mathcal{M}$, namely that $\mathcal{M}$ is countably saturated.

Proof. Take a coverage $\langle \Sigma_n(x) \mid n < \omega \rangle \subseteq \Gamma(x)$ such that $|\Sigma_n| < \omega$ for all n. Then Σ_n is realized, and let $[f_n] \in M$ be such that $\mathcal{M} \models \Sigma_n([f_n])$. Then $a_n = \{j < \omega \mid \mathcal{M}_n \models \Sigma_n(f_n(j))\} \in U$. Filters are closed to intersection, then let us assume that $a_{n+1} \subseteq a_n$, otherwise we could define,

$$g_{n+1}(i) = \begin{cases} f_{n+1}(i) & i \in a_n \\ c_n & \text{otherwise} \end{cases}$$

where $c_n \in M_n$ is some arbitrary value.

We now take $a = \bigcap_{n<\omega} a_n$ and $[f] \in M$ such that for $n \in a$, $f(n) \in \{f_i(n) \mid i < \omega\}$. If $a \in U$ then $\mathcal{M} \models \Gamma([f])$, then let us assume $a \notin U$, conversely $a^C = \mathbb{N} \setminus a \in U$. $a^C \cap a_n \in U$ for all n and therefore $a^C \cap a_n \neq \emptyset$. It immediately follows that $\emptyset \in U$, a contradiction. □

Part b

We define σ-complete ultrafilter U as an ultrafilter such that it is closed to countable intersections.
Let U be some σ-complete ultrafilter, $L = \{=\}$, $\mathcal{M} = (\mathbb{N}, =)^I / U$ for some index set I.
We will show that $|M| = \omega$ and deduce that $\mathcal{M}$ is not countably saturated.

Proof. Directly by Łoś theorem and sentence of the form $\varphi_N = \bigwedge_{n<N} \exists x (x \neq c_n)$ we deduce that $|M| \geq \omega$. Define $C_x = \{x\}^I$ the constant function, we will show that for every $[f] \in M$ there is $n < \omega$ such that $[f] = [C_n]$. Note that this is equivalent to the claim that $\{j < \omega \mid f(j) = n\} \in U$. We will assume otherwise in contradiction, then $a_n = \{j < \omega \mid f(j) \neq n\}$ is in U, and $a_n \cap a_m$ is non-empty for all $n \neq m$. We take $a = \bigcap_{n<\omega} a_n$, U is σ-complete therefore $a \in U$. It follows that $f(j) \neq n$ for all $j \in I, n < \omega$, a contradiction to f being $I \to \mathbb{N}$ function. □

Part c

We will show that if U is an ultrafilter on some indices set I such that U is not σ-complete and $\langle \mathcal{M}_i \mid i \in I \rangle$,
then $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / U$ is countably saturated.

Proof. By σ-incompleteness we can assume that there is decreasing chain $\langle u_n \mid j < \omega \rangle \subseteq U$ such that $\bigcap_{n<\omega} u_n \notin U$. We can assume without loss of generality that $\bigcap u_n = \emptyset$ as $I \setminus \bigcap u_n \in U$ and therefore we can take $u_n \setminus \bigcap u_m$.

Let $\Gamma(x)$ be countably realized set of formulas. By countability let us denote $\Gamma(x) = \{\gamma_n \mid n < \omega\}$. For every $N < \omega$ we also define $\Gamma_N = \{\gamma_n \mid n < N\}$. Then Γ_N is finite set of formulas, and realized by $[f_N] \in M$. We can take f_N such that,

$$a_N = \{j \in I \mid \mathcal{M}_j \models \Gamma_N(f_N(j))\} \subseteq \{j \in I \mid \mathcal{M}_j \models \Gamma_N(f_M(j))\}$$

for $M < N$ by finite intersections. The sequence $\langle a_n \mid n < \omega \rangle \subseteq U$ is decreasing. Let us take $U_n = u_n \cap a_n$ for every n, clearly $U_n \in U$. Note that for every $i \in I$ there is maximal $n < \omega$ such that $i \in U_n$, and let us define $f(i) = f_n(i)$. Indeed $f \in M$ and,

$$\mathcal{M} \models \gamma_n([f]) \iff \{j \in I \mid \mathcal{M}_j \models \gamma_n(f(j))\} \in U \impliedby \{j \in I \mid \mathcal{M}_j \models \gamma_n(f_n(j))\} \in U$$

but the latter holds directly by the definition of f. □

Question 2

We will show that the following are equivalent,

1. The axiom of choice
2. (a) The ultrafilter lemma: Every filter extends to an ultrafilter
   (b) Alternative version of Łoś theorem: If $\mathcal{M}$ is an L-structure and U is an ultrafilter on indices set I, then for any sentence φ, $\mathcal{M}^I/U \models \varphi \iff \mathcal{M} \models \varphi$

Proof. $1 \implies 2$. We saw in class that the ultrafilter lemma is directly deduced from Zorn's lemma, which is equivalent to the axiom of choice; Let D be a filter, we define the order $X = \langle \{D \subseteq F \subseteq \mathcal{P}(I) \mid F \text{ is a filter}\}, \subseteq \rangle$. X is clearly not empty as $D \in X$. Let $C \subseteq X$ be a chain, and let $D_C = \bigcup C$. Then D_C is a filter and in particular $D \subseteq D_C$, then $D_C \in X$ and $F \subseteq D_C$ for all $F \in C$. By Zorn's lemma there is maximal element $U_D \in X$, namely $F \subseteq U_D$ for all $F \in X$.
It reminds to show that maximal filter is an ultrafilter, then let us assume in contradiction that it is not, meaning that there is $A \subseteq I$ such that $A \notin U_D$ as well $I \setminus A \notin U_D$. We can define $U = \langle U_D \cup \{A\} \rangle$, this is a filter such that $U_D \subsetneq U$, a contradiction to U_D being maximal.

As for the alternative version of Łoś theorem, it is derived from the general version for ultra-powers. While we will not recite the whole proof, we will note where it uses choice. The axiom of choice is being used in the proof at induction step for quantifiers, specifically to get a witness to global quantifiers.

$2 \implies 1$. Let A be some set, and let us assume toward a contradiction that there is no choice function $A \to \bigcup A$. We define $S \subseteq \mathcal{P}(A)$ the set of all subsets of A that do have choice function. In particular $\emptyset \in S$ as $\emptyset \to \bigcup \emptyset = \emptyset$ is witnessed by $\emptyset$.

Let $B \subseteq S$ and $f : B \to \bigcup B$ choice function, namely $\forall b \in B,\ f(b) \in b$. For every $B' \subseteq B$ consider $f' = f \restriction B'$. $f'(b) \in b$ directly by f's definition, then $\operatorname{rng} f' \subseteq \bigcup B'$. We conclude that $B' \in S$, meaning S is closed under subsets.

Let $B, C \in S$, we will show that $B \cup C \in S$ as well. We will assume that they are disjoint, as otherwise we can take $B' = B \setminus C \subseteq B$ by closeness to subsets. if $f : B \to \bigcup B, g : C \to \bigcup C$ such that $\forall b \in B, c \in C,\ f(b) \in b, g(c) \in c$, then $h = f \cup g$ is well-defined function, as for each $x \in \operatorname{dom} h$, either $x \in B$ or $x \in C$. Also $f(x) \in B \vee f(x) \in C$ then $f(x) \in B \cup C$. We deduce that S is closed to finite unions by induction.

We define $D = \mathcal{P}(A) \setminus S$, the collection of subsets such that there is no choice function over them. In particular by our assumption $A \in D$. By inversion we conclude that D is closed under super-sets and finite intersections, as well $\emptyset \notin D$, then D is a filter. Using the ultrafilter lemma we define $D \subseteq U$ an ultrafilter extending D.

Let us define $\mathcal{M} = \langle \bigcup A, R \rangle$ such that $\langle x, y \rangle \in R^{\mathcal{M}} \iff \exists a \in A,\ x, y \in a$. We assume that $a \cap b = \emptyset$ for all $a, b \in A$ as well. Then $R^{\mathcal{M}}$ is equivalency relation, and therefore $R^{\mathcal{M}^A/U}$ is as well. Let $B \in S$ be some non-trivial set and let $g : B \to M$ be choice function. then $[g] \in \mathcal{M}^A/U$. We assumed that B is not trivial then there are $x, y \in A$, $\langle g(x), g(y) \rangle \notin R^{\mathcal{M}}$. But then it immediately follows that $\langle [g], [c_{g(x)}] \rangle, \langle [g], [c_{g(y)}] \rangle \in R^{\mathcal{M}^A/U}$ in contradiction to $\langle [c_{g(x)}], [c_{g(y)}] \rangle \notin R^{\mathcal{M}^A/U}$. $\square$

Question 3

We will show that if t is a term that does not contain the variable v, and $\varphi \in \Sigma^0_0$, then $\forall v \leq t\ \varphi$ is Σ^0_0-formula as well. We will conclude that Σ^0_0 is closed under bounded quantification with general terms.

Proof. □

Question 4

Let T be a theory. We will show that the following are equivalent.

1. T is axiomatizable
2. T is recursively-enumerable
3. There is a recursively-enumerable set Σ such that $\operatorname{cl}\Sigma = T$

Proof. $1 \implies 2$. T is a theory, meaning that $T = \operatorname{cl} T$, and by the assumption that it is axiomatizable there is Σ recursive set of sentences such that $T = \operatorname{cl}\Sigma$. $\square$