Exercise 7 Answer Sheet — Logic Theory (2), 80424

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Question 1

For any $\Gamma \subseteq \text{form}_{L_{\text{PA}}}$ let $I\Gamma$ be the induction scheme over Γ.

For $\varphi(z, y, t_0, \ldots, t_{n-1}) \in \text{form}_{L_{\text{PA}}}$ we define

$$\text{Coll}(\varphi) = \forall z \forall t_0 \ldots \forall t_{n-1} (\forall x \leq z \exists y \varphi \leftrightarrow \exists w \forall x \leq z \exists y \leq w \varphi)$$

By the last exercise, $\text{PA} \vdash \text{Coll}(\varphi)$ for any such φ.

Let $\text{Coll} = \{\text{Coll}(\varphi) \mid \varphi \in \text{form}_{L_{\text{PA}}}\}$, Q be the first 9 axioms of PA, and $\text{PA}_0 = Q \cup I\Sigma_0 \cup \text{Coll}$.

We will show that $\text{PA}_0 \vdash \text{PA}$.

Proof. For $Q \subseteq \text{PA}$ we assumed that $\text{PA}_0 \vdash Q$, then it is sufficient to show that $\text{PA}_0 \vdash \text{Ind}(\varphi)$ for any $\varphi \in \text{form}_{L_{\text{PA}}}$.

If $\varphi \in \Sigma_0$ then $\text{PA}_0 \vdash \text{Ind}(\varphi) \ni I\Sigma_0$.

In the previous exercises we have shown that for any φ there is n such that $\varphi \in \Sigma_n^0$, and that there is $\psi \in \Sigma_n$, for which, $\text{PA} \vdash (\varphi \leftrightarrow \psi)$. We will prove in induction that $\text{PA}_0 \vdash (\varphi \leftrightarrow \psi)$ as well, and that $\text{PA}_0 \vdash \text{Ind}(\varphi)$ in such case. The case of $n = 0$ was proven in the last exercise.

Let us assume that $\varphi \in \Sigma_n(\text{PA})$ and $\psi \in \Sigma_n$ such that $\text{PA} \vdash (\varphi \leftrightarrow \psi)$ and $\psi = \exists v \phi$ for $\phi \in \Pi_{n-1}$. By the induction hypothesis ψ fulfills the claim, meaning that $\text{PA}_0 \vdash \text{Ind}(\psi)$. We will show that $\text{PA}_0 \vdash (\varphi \leftrightarrow \psi)$ and that $\text{PA}_0 \vdash \text{Ind}(\varphi)$.

As for the first part, by $\text{Coll}(\phi)$ and negation we infer that $\text{PA}_0 \vdash (\varphi \leftrightarrow \psi)$ as intended, and it is left to show that $\text{PA}_0 \vdash \text{Ind}(\psi)$. The proof is the same as of question 1 of exercise 6.

Let $\mathcal{M} \models \text{PA}_0$ be some model and we assume that $\exists x \phi(x, y, z)$ satisfies the induction condition, meaning that $\mathcal{M} \models \phi(x, 0, c)$ for some $c \in M$; as well,

$$\mathcal{M} \models \forall y (\exists x \varphi(x, y, c)) \rightarrow (\exists x \varphi(x, S(y), c))$$

We want to show that $\mathcal{M} \models \forall y \exists x \varphi(x, y, c)$.

Let $b \in M$ be some arbitrary element, we want to show that,

$$\mathcal{M} \models \exists x \varphi(x, b, c) \tag{1}$$

thus proving the last statement. By using $\text{Coll} \subseteq \text{PA}_0$ we infer that

$$\mathcal{M} \models \forall b' \leq b \exists x \varphi(x, b', c) \tag{2}$$

if and only if exists $d \in M$ such that,

$$\mathcal{M} \models \forall b' \leq b \; \exists x \leq d \; \varphi(x, b', c) \tag{3}$$

In the set-theoretical sense, let $X = \{x \mid \mathcal{M} \models \exists b' \leq b, \; \varphi(x, b', c)\}$, this is a finite set as the order induced on M by $\leq^{\mathcal{M}}$ is a well-order. Let us select $d = \sup_{\leq^{\mathcal{M}}} X$, this d fulfills (3), thus also proving (2), but as $b \in M$ is arbitrarily chosen, (1) holds as well, completing our proof. $\qquad\square$

Question 3

We will show that Sat_0 is primitive-recursive.

Proof. In the lectures we saw that Sat_0 is total recursive.

It follows that Sat_0 is a Σ_1^0 formula, meaning that it is equivalent to some Σ_1 formula, φ. By the presented proof, we also know that $\varphi(x,y) = \exists z \psi(x,y,z)$ for x code for a Σ_0-formula, y a finite-support assignment and z an inference sequence to represent the formula and the assignment. We will bound z, showing that φ is indeed Σ_0.

The sequence z fulfills, $z = \#\langle (f_i, s_i, \epsilon_i) \mid i < n \rangle$ for some $n \in \mathbb{N}$, and it is derived by definition that $z \leq \#(f, s, \epsilon)$ for some such values. Let us choose $s = \#\sigma$, $f = \#x$ and fixing some arbitrary large enough $\epsilon \geq \mathbb{T}, \mathbb{F}$. As both x and σ are variables and the fact that they bind each element of the sequence (by 7.3 and 7.4) we infer that they are indeed a bound to ψ.

The previous statement is not true at all, as the process of evaluation of the formula and the assignment produce larger finite numbers. We will show that there is a bound such that it is dependant on only x and y, using induction over the structure of the formula. For atomic formulas it is sufficient to choose $\#\sigma(v_i)$ for the maximal such $t \in x$, there are up to two such values and it follows that it is Σ_0 to take their maximum. For unary and binary relations we choose maximal value in similar way, the proof that it is maximum is trivial by the choice of the bound. It is remaining to show that there is a bound for bounded universal quantifier, but for this we choose the bound of the quantifier (up to choosing its value in σ as a variable). $\qquad\square$

Question 4

Let t be a term and σ some finite-supported assignment.

We will prove that $t^{\hat{\mathbb{N}}}(\sigma) \leq K^{\lceil t \rceil + 1}$ for $K = \max\{\sigma(v_i) \mid i \in \mathbb{N}\} + 2$.

Proof. We will prove the claim by induction over the structure of the term. Note that the proof is equivalent the case of trees and sequences, then we shall use trees without diverge from past assumptions.

For $t = 0$ (the only constant), $\lceil t \rceil = 1$ and $K \geq 1$, then $0^{\hat{\mathbb{N}}}(\sigma) = 0^{\hat{\mathbb{N}}} = 0 \leq 1 \leq K^1$.

For $t = v_i$ for some variable v_i ($i \in \mathbb{N}$), then $t^{\hat{\mathbb{N}}}(\sigma) = \sigma(v_i)$. From the other side, $K \geq \sigma(v_i)$ as a maximum value, and $\lceil t \rceil + 1 \geq 2$, then $\sigma(v_i) \leq \sigma(v_i)^2 \leq K^2$.

Let us move to the induction step, we assume that the claim holds for $s_0, s_1 \in \text{term}_{\text{PA}}$ and for some assignment σ. Either $t = S(s_0), t = s_0 + s_1$ or $t = s_0 \cdot s_1$.

For $t = S(s_0)$ we get $t^{\hat{\mathbb{N}}}(\sigma) = S(s_0^{\hat{\mathbb{N}}}(\sigma)) \leq K^{\lceil s_0 \rceil + 1} + 1$ by the induction hypothesis. By our definition of numbering also $\lceil s_0 \rceil \cdot 13 \leq \lceil t \rceil$, in particular $K^{\lceil s_0 \rceil + 1} + 1 \leq K^{\lceil t \rceil + 1}$.

The remaining cases are similar, we will prove it for $t = s_0 + s_1$,

$$t^{\hat{\mathbb{N}}}(\sigma) = s_0^{\hat{\mathbb{N}}}(\sigma) + s_0^{\hat{\mathbb{N}}}(\sigma) \leq K^{\lceil s_0 \rceil + 1} + K^{\lceil s_1 \rceil + 1}$$

We assume without loss of generality that $\lceil s_0 \rceil \leq \lceil s_1 \rceil$, then,

$$K^{\lceil s_0 \rceil + 1} + K^{\lceil s_1 \rceil + 1} \leq 2K^{\lceil s_1 \rceil + 1}$$

By our definition of Gödel numbering $\lceil t \rceil \geq 13^{\lceil s_0 \rceil} \lceil s_1 \rceil$, then the claim follows almost immediately. $\square$