Solution to Exercise 3 — Model Theory (1), 80616

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Question 1

Part a

We will show that there is a collection $X \subseteq \mathcal{P}(\omega)$ such that $|X| = 2^{\aleph_0}$ and if $x \neq y \in X$ then $x \cap y$ is bounded.

Proof. For any $A \subseteq \omega$ infinite let us define $Y(A) = \{\sum_{n \in A \cap m} 2^n \mid m < \omega\}$, then $X(A) \in \mathcal{P}(\omega)$. It follows that $Y : \omega^\omega \to \mathcal{P}(\omega)$ is a function, note that it is injective, as if $A \neq B \in \operatorname{dom} Y$ then there is $m \in A \setminus B$, witnessing $Y(A) \neq Y(B)$. The injectivity of Y implies that $|\operatorname{Im} Y| = |\mathcal{P}(\omega)|$, and let $X = \operatorname{Im} Y$.

We move to show that if $A' \neq B' \in X$ then $A' \cap B' < \alpha$ for some $\alpha < \omega$. Let $A, B \subseteq \omega$ be elements such that $Y(A) = A', Y(B) = B'$. $A \neq B \implies \exists m < \omega,\ k \in A, k \notin B$ without loss of generality. Let $k < m < \omega$, then it suffices to show that $\sum_{n<m} 2^n(\mathbb{1}_A(n) - \mathbb{1}_B(n)) \neq 0$, but as a result from number theory and by the fact that $A \cap m \neq B \cap m$ the expression indeed does not nullify. We deduce that $|A' \cap B'| \leq k < \omega$ as wished. □

Part b

Let $\mathcal{L} = \{0, S, \leq\} \cup \{R_X \mid X \subseteq \omega\}$ be an uncountable language such that 0 is constant symbol, S is unary function symbol, $\leq$ is binary relation symbol and R_X is an unary relation symbol for all $X \subseteq \omega$.
Let $\mathcal{A}$ be a model over $\mathcal{L}$ with $A = \omega, 0^\mathcal{A} = 0, S^\mathcal{A}(n) = n+1, x \leq^\mathcal{A} y \iff x \in y$ and $R_X^\mathcal{A}(n) \iff n \in X$, and let $T = \operatorname{Th}(\mathcal{A})$.
We will show that T is ω-categorical, that it has infinitely many nonequivalent formulas and that there are infinitely many non-isolate types in $S_1(T)$.

Proof. Let $\mathcal{M} \models T$ be some countable model, we will show that $\mathcal{M} \cong \mathcal{A}$. We define recursively $f : A \to M$ by $f(0) = 0^\mathcal{M}$ and $f(n+1) = S^\mathcal{M}(f(n))$. It follows that f preserves $0, S$, it remains to show it also preserves $\leq, R_X$. The claim that f preserves $\leq$ can be shown using double induction by fixing each n and proving $n \leq m \implies f(n) \leq f(m)$. Definability of $n \in \mathcal{A}$ as $S^n(0)$ and the fact for each X, $R_X^\mathcal{A}(n) \iff S^n(0) \in X$ implies that R_X is preserved under f as well.
$\mathcal{A}$ fulfills the axiom scheme of induction, meaning that T does as well. Each $X \subseteq A$ is definable using R_X, meaning that the sentence $(0 \in \mathbb{N} \land (x \in \mathbb{N} \to S(x) \in \mathbb{N})) \to \forall x \in \mathbb{N}$ exists in T (to be precise it is symbolically appears and not the actual form written) where $\mathbb{N}$ is some X.

Let $\varphi_n(x) = (x = S^{\underline{n}}(0))$, as an informal way to define that $\varphi_0(x) = (x = 0), \varphi_1 = (x = S(0))$ and so on. $\mathcal{A} \models \forall x\ x \neq S(x)$, then we can deduce that $\varphi_n \not\equiv \varphi_m$ for all $n < m < \omega$. Then $\{\varphi_n \mid n < \omega\} \subseteq T$ is a set of non-equivalent formulas.

We have shown that f is model embedding, and now we will show that it is also surjective. Let $\gamma \in M$ be a non-standard element, namely $\mathcal{M} \models \neg\varphi_n(\gamma)$ for all $n < \omega$. By f's construction, $\gamma \notin \operatorname{Im} f$, and therefore also $\underline{n}\gamma \in M \setminus \operatorname{Im} f$ as well. Let $X_0 \in X$ where X is from the last part, then $X_0' = \{\underline{n}\gamma \mid n \in X_0\}$ is subset of M. Let $X_0 \neq X_1 \in X$ be some other set, and let us define X_1' accordingly. Let $g : X \to M^{<\omega}$ by $X_1 \mapsto X_0' \cap X_1'$, but $|M^{<\omega}| = |M| = \omega$ but g is injective and $|X| = 2^\omega$, a contradiction. We deduce that f is a bijection.

Let $g : \omega \to \{0,1\}, |g^{-1}(1)| = |g^{-1}(0)| = \omega$ be a function and let $Q = \{q_n\}_{n=1}^\infty$ be the set of the primes. Let X_q be the set such that $x \in X_q \iff q|x$, note that divisibility is definable, and let $R^i = R_{X_{q_i}}$ for any $i < \omega$.

$$p_g'(x) = \{x \neq \underline{n} \mid n < \omega\} \cup \{R^i(x) \mid g(i) = 1\} \cup \{\neg R^i(x) \mid g(i) = 0\}$$

p_g' is consistent from the compactness theorem, and let p_g be its closure, such that p_g is complete and consistent. p_g cannot be isolated as otherwise p_g' can be isolated as well, a contradiction to g's definition by taking the minimal $i < \omega$ such that R^i does not show up in isolating formula. There are infinitely many such functions g, implying that there are also infinitely many such types p_g. To be precise there are $2^{\aleph_0}$ such functions, then $2^{\aleph_0}$ such non-isolated types. □

Question 2

Let $\omega \leq \kappa$ be some cardinal. We say that a model $\mathcal{M}$ is κ-saturated if for every $A \subseteq M$ with $|A| < \kappa$, any type in $S_1(A)$ is realized.

Part a

Let T be a consistent complete theory over a language of size $\leq \kappa$ with an infinite model.
We will show that there is a κ^+ saturated model of T of cardinality 2^κ.

Proof. By Löwenheim-Skolem theorem we can assume that there is a model $\mathcal{M} \models T$ of cardinality κ. We will construct recursively a κ-saturated model, by recursion on $|A|$ for $A \subseteq M$. For $|A| = \emptyset$ we get $A = \emptyset$, $|\,\text{form}\,| \leq \kappa$ implies that $S_1(\emptyset) = S_1(T)$ is of cardinality $\leq 2^\kappa$ as well. Let $\mathcal{M}_0' \models T \cup \{p(c_p) \mid p \in S_1(\emptyset)\}$ be a model of the language $\mathcal{L} \cup \{c_p \mid p \in S_1(\emptyset)\}$ and let $\mathcal{M}_0$ be its reduction to $\mathcal{L}$. $|M_0| \leq 2^\kappa$ and it is 1-saturated.

Let us assume that $\alpha < \kappa$ is some cardinal and that $\mathcal{M}_\alpha \models T$ is a α^+-saturated model with $|M_\alpha| \leq 2^\kappa$. $|\mathcal{P}_{=\alpha}(M_\alpha)| < \kappa$ and $|\,\text{form}_{\mathcal{L}(A)}\,| < 2^\kappa$ then $|S_1(A)| < 2^\kappa$ for any such A as well, meaning that,

$$\Sigma_\alpha = \bigcup_{\substack{A \subseteq M_\alpha \\ |A| = \alpha}} S_1(A)$$

is of cardinality $\leq 2^\kappa$ as well. We enrich $\mathcal{L}$ by $\{c_p \mid p \in \Sigma_\alpha\}$ and define $\mathcal{M}'_{\alpha^+} \models T \cup \{p(c_p) \mid p \in \Sigma_\alpha\}$ be a models, $\mathcal{M}_{\alpha^+}$ be its reduction to $\mathcal{L}$. Then $\mathcal{M}_{\alpha^+}$ is α^{++}-saturated and $|M_{\alpha^+}| \leq 2^\kappa$.

κ^+ is regular, thus $\mathcal{M}_{\kappa^+} = \bigcup_{\alpha < \kappa^+} \mathcal{M}_\alpha$ is of cardinality 2^κ and κ^+-saturated as wished. $\square$

Part b

We will show that if $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent κ-saturated models of cardinality κ then $\mathcal{M} \cong \mathcal{N}$, and that any partial elementary map $f : A \to B$ with $A \subseteq M, B \subseteq N, |A| < \kappa$ can be extended to an isomorphism.

Proof. The proof is similar to the case of ω-saturation, the second statement implies the first one and it suffices to show it. The proof is by induction on ordinals $< \kappa$, the proof for the case of $\kappa = \omega$ can be used as a base for the induction.

Without loss of generality we can assume $A, B = \alpha$ for $\alpha < \kappa$ by the well order principle. The case of successor ordinal is trivial from the case ω. Let us assume that the statement is true for any $\alpha < \beta < \kappa$. Then there exists f_α and if $\alpha < \gamma < \beta$, $f_\gamma \restriction \alpha = f_\alpha$, then let us define $f_\beta = \bigcup_{\alpha < \beta} f_\alpha$. The induction step is completed therefore the statement holds for any $\alpha < \kappa$, and by an identical step there is also a function $f = f_\kappa$ fulfilling our requirements. $\square$

Part c

We will show that if $\mathcal{M} \models T$ is κ-saturated model of a complete theory T, then every model $\mathcal{N} \models T$ of size κ can be embedded into $\mathcal{M}$.

Proof. We will construct an embedding by recursion. Let $f_0 = \{\langle d^{\mathcal{N}}, d^{\mathcal{M}} \rangle \mid d \in L, d \text{ is constant symbol}\}$, this is an embedding by T's completeness.

Let $f_\delta : A \to B$ be a partial embedding, $A \subseteq N, B \subseteq M$, $A = \{a_i\}_{i<\delta}$. Let $a \in N \setminus A$ and let $p = tp(a/N_A)$ be the complete type such that $\mathcal{N} \models p(a)$. Then $q = p^{a_0,\dots}_{f(a_0),\dots}$ is a type $\in S_1(B)$, it is consistent by compactness and complete as closure under inference. By κ-saturation of $\mathcal{M}$ there exists $b \in M$ such that $\mathcal{M} \models q(b)$, let $f_{\delta+1} = f_\delta \cup \{\langle a, b \rangle\}$.

Let δ be a limit ordinal and let us assume that f_α is defined for $\alpha < \delta$, such that $f_\alpha \subseteq f_\beta$ for all $\alpha < \beta < \delta$. Then $f_\delta = \bigcup_{\alpha<\delta} f_\alpha$ is defined and acts as embedding as any term or formula are of finite length and thus embedded correctly in some $\alpha < \delta$.

We get that f_α exists for each $\alpha \leq \kappa$, note that by the definition of κ-saturation this process cannot be extended after reaching κ. Note that by using the well order principle we can assume that $N = \kappa$ and therefore f_κ is a total function, and therefore an embedding $\mathcal{N} \hookrightarrow \mathcal{M}$. $\square$