Exercise 8 Answer Sheet — Logic Theory (2), 80424

June 2, 2025

Question 2

We will show that the universal recursive function $U : \mathbb{N}^2 \rightharpoonup \mathbb{N}$ is not total, and that there is no total recursive function U' containing it.

Proof. We defined the universal function over the domain of coded Σ_1-formulas, and directly by past constructions the number 1 does not codify such a formula. Let us assume towards a contradiction that there is $U' \supseteq U$ total recursive function.

$U' : \mathbb{N}^2 \to \mathbb{N}$ is recursive, then using projections we get that $g : \mathbb{N} \to \mathbb{N}$ which defined by $g(x) = U'(\text{Le}(x), \text{Re}(x))$ is recursive as well. There is a code for g such that $U(e, cd(x, y)) = g(cd(x, y)) = U'(x, y)$ for all values. In particular for $x = e, U(e, cd(e, y)) = U'(e, y)$

$$f(x) = \begin{cases} 0 & \exists z\, T(x, x, z) \\ 1 & \text{Otherwise} \end{cases}$$

f is total recursive by the closure to existential quantifiers of Σ^0_1, and □

Question 3

For every $l \in \mathbb{N}$ we define $b_l : \mathbb{N} \to \mathbb{N}$ by $b_0(n) = 2n$, $b_{l+1}(n) = b_l^{n+1}(1)$ in the sense of composition. Let $B(l, n) = b_l(n)$, such that $B(0, n) = 2n$, $B(l + 1, 0) = B(l, 1)$ and $B(l + 1, n + 1) = B(l, B(l + 1, n))$.

Part a

We will show that for any $l \in \mathbb{N}$, b_l is primitive-recursive.

Proof. □