Solution to Exercise 4 — Model Theory (1), 80616

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Question 1

Definition 0.1 (algebraic formula)**.** Let $\mathcal{M}$ be a structure. A formula $\varphi(x) \in \text{form}_{L_\mathcal{M}}$ is called algebraic if,

$$|\{a \in M \mid \mathcal{M} \models \varphi(a)\}| < \aleph_0.$$

Definition 0.2 (definable element)**.** An element $a \in M$ is called definable if there is a formula $\psi(x)$ such that $\mathcal{M} \models \psi(x) \iff x = a$ for any $x \in M$.

Definition 0.3 (acl and dcl)**.** If $\mathcal{M}$ is a structure and $A \subseteq M$ a set of elements, then let,

$$\text{acl}_\mathcal{M}(A) = \bigcup\{\varphi(M) \mid \varphi \in \text{form}_{L_A}, \varphi \text{ is algebraic}\}, \quad \text{dcl}\,\mathcal{M}(A) = \{a \in M \mid a \text{ is definable using an } L_A \text{ formula}\}.$$

be the set of elements that has a formula with parameters from A that is algebraic and are true for them, and the set of elements uniquely definable by formulas with parameters.

Part a

Let $A \subseteq M$ for $\mathcal{M} \prec \mathcal{N}$. We will show that $\text{acl}_\mathcal{M}(A) = \text{acl}_\mathcal{N}(A)$ and $\text{dcl}_\mathcal{M}(A) = \text{dcl}_\mathcal{M}(A)$.

Proof. Let $a \in M$ be $\in \text{acl}_\mathcal{M}(A)$ and let $\varphi \in \text{form}_{L_A}$ be an algebraic formula to witness it. $\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(a)$ by the given elementary embedding and the fact that $A \subseteq N$.

Let $a \in \text{acl}_\mathcal{N}(A)$ and assume toward contradiction that $a \notin \text{acl}_\mathcal{M}(A)$, meaning that there is $\varphi(x)$ over L_A such that $\mathcal{N} \models \varphi(a)$ and φ is algebraic. Let $B = A \cup \varphi(M)$, denote $\varphi(M) = \{b_i \mid i < N\}$ for $N < \omega$,

$$\psi(x) = \varphi \land \left(\bigwedge_{i<N} x \neq b_i\right).$$

Then ψ is algebraic over both $\mathcal{M}$ and $\mathcal{N}$ and $\psi(M) = \emptyset$, while $a \in \psi(N)$. Let $\phi = \exists x\ \psi$, then $\mathcal{N} \models \phi$ as a witnesses exactly this, while $\mathcal{M} \models \neg\phi$ in contradiction to $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$.

The case for dcl is equivalent, we take $\varphi(x)$ such that $\varphi(M) = \{a\}$ and therefore $\varphi(N) = \{a\}$ as otherwise we can define $\varphi \land (x \neq a)$. □

We conclude that the relative model of acl and dcl can be omitted.

Part b

Let $A \subseteq M$, we will show that $\text{acl}(\text{acl}(A)) = \text{acl}(A)$ and $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$.

Proof. Let $a \in \text{dcl}(\text{dcl}(A))$ be an element and let $\varphi(x)$ be its witness. Suppose that $\{b_i \mid i < N\}$ is the set of elements that φ consists of such that they are not in A. $b_i \in \text{dcl}(A)$ for any i, and let $\theta_i(x)$ witness that. Let $\{y_i \mid i < N\}$ be a set of distinct and disjoint to φ variables, and let us define,

$$\varphi' = \varphi^{b_0,\dots,b_{N-1}}_{y_0,\dots,y_{N-1}}, \quad \phi(x) = \exists y_0 \cdots \exists y_{N-1} \left(\bigwedge_{i<N-1} \theta(y_i)\right) \land \varphi'.$$

then $\phi \in \text{form}_{L_A}$ and $\phi(M) = \{a\}$, therefore $a \in \text{dcl}(A)$ and thus $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$.

The case of acl is equivalent. □

Part c

Let $L = \{0, 1, +, \cdot\}$ and K be an algebraically closed field and let $A \subseteq K$.

We will compute $\text{acl}(A)$ and $\text{dcl}(A)$.

Solution. K has quantifier elimination therefore we can omit the discussion about quantifiers. Each formula $\varphi(x) \in \text{form}_{L_A}$ is (without loss of generality) of the form,

$$\varphi = (\bigwedge_{i<n} p_i(x) = 0) \wedge (\bigwedge_{j<m} p_{n+j}(x) \neq 0).$$

where $p_i(x)$ is a polynomial over A. An element $a \in K$ is $\in \text{dcl}(A)$ if it is the unique solution of some such polynomial. Each polynomial in algebraically closed field can be deconstructed to product of linear polynomials, meaning that $p_i(x)$ has a single solution if $p_i(x) = (x - a)$. We conclude that $\text{dcl}(A) = \text{cl}_{+,\cdot}(A)$.

By similar proposition, $\text{acl}(A) = \langle A \rangle \leq K$, meaning the subfield generated by A.

Question 2

Part a

Let κ be an infinite cardinal and let $\lambda \leq \kappa$ be the minimal cardinal such that $\kappa^\lambda > \kappa$. Consider the order $(\kappa^{<\lambda}, <_l)$ the lexicographic order on $\kappa^{<\lambda}$.