Exercise 5 Answer Sheet — Logic Theory (2), 80424

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Question 1

Part a

Suppose that $\varphi(z, y, t_0, \ldots, t_{n-1})$ is a formula in the language of L_{PA}. We will show that,

$$\mathrm{PA} \vdash \forall z \forall t_0 \ldots \forall t_{n-1} (\forall x \leq z \exists y \varphi \leftrightarrow \exists w \forall x \leq z \exists y \leq w \varphi)$$

Proof. Let us fix $z, t_0, \ldots, t_{n-1}$, then is suffice to show that $\mathrm{PA} \vdash \forall x \leq z \exists y \varphi \leftrightarrow \exists w \forall x \leq z \exists y \leq w \varphi$. We assume toward a contradiction that the formula does not hold, then $\forall x \leq z \exists y \varphi$ as well $\neg \exists w \forall x \leq z \exists y \leq w \varphi$ holds as well. The latter is equivalent to $\forall w \exists x \leq z \forall y \leq w (\neg \varphi)$.

We want to show that there is an upper bound to y in the formula $\forall x \leq z \exists y \varphi$. This will be proven using induction over z. For the case $z = 0$ there is only the case $x = 0$ and the y that fulfills $\exists y \varphi$ for $x = 0$ is an upper bound. We assume that m is an upper bound, meaning that $\forall x \leq z \exists y \leq m \varphi$ and check the case for $\forall x \leq z + 1 \exists y \varphi$. There is y witnessing the case $x = z + 1$, and either $m \leq y$ or $y \leq m$, we define $m' = \max\{y, m\}$ and it follows that it is an upper bound for the case $z + 1$.

Back to the initial claim, there is m by the induction above such that it is a bound for y in the formula $\forall x \leq z \exists y \varphi$. We fix $w = m$ and it derives that $\exists x \leq z \forall y \leq w (\neg \varphi)$ holds, and we fix such $x \leq z$, then $\forall \leq w (\neg \varphi)$. But by the first formula $\exists y \leq w \varphi$ in contradiction to this claim. □

Part b

Suppose that $\psi(y_0, y_1, t_0, \ldots, t_{n-1})$ is some formula in the language L_{PA}. We will show that,

$$\mathrm{PA} \vdash \forall t_0 \ldots \forall t_{n-1} (\exists y_0 \exists y_1 \psi \leftrightarrow \exists w \exists y_0 \leq w \exists y_1 \leq w \psi)$$

Proof. The proof is very similar to part a, Let $t_0, \ldots, t_{n-1}$ be some values, and let us assume toward a contradiction that $\exists y_0 \exists y_1 \psi$, and that $\neg \exists w \exists y_0 \leq w \exists y_1 \leq w \psi \equiv \forall w \forall y_0 \leq w \forall y_1 \leq w (\neg \psi)$ hold. We fix y_0, y_1 such that ψ holds, and fix some $w \geq \sup\{y_0, y_1\}$, hence $\neg \psi$ is derived by the second formula, resulting in a contradiction. □

Question 2

We define $\Sigma_n(\mathrm{PA})$ as the set of formulas $\psi(x_0, \ldots, x_{n-1})$, such that for some Σ_n formula $\psi'(x_0, \ldots, x_{n-1})$,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}(\psi \leftrightarrow \psi')$$

Part a

We will show that the class of $\Sigma_0(\mathrm{PA})$ formulas in PA is closed under boolean operations and bounded quantifiers.

Proof. Let $\psi, \varphi \in \Sigma_0(\mathrm{PA})$ be some formulas. There exist φ', ψ' such that,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}(\psi \leftrightarrow \psi'), \forall x_0, \ldots, \forall x_{n-1}(\varphi \leftrightarrow \varphi')$$

Then it follows that,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}((\psi \leftrightarrow \psi') \wedge (\varphi \leftrightarrow \varphi'))$$

Therefore by an identity from the previous course,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}((\psi \wedge \varphi) \leftrightarrow (\psi' \wedge \varphi'))$$

Namely, $\varphi \wedge \psi \in \Sigma_n(\mathrm{PA})$.

By another identity it also derives that,

$$\varphi \leftrightarrow \varphi' \equiv (\neg\varphi) \leftrightarrow (\neg\varphi')$$

We deduce that $\Sigma_n(\mathrm{PA})$ is closed under boolean operations.

We use the fact that Σ_0 is closed under bounded quantifiers and deduce immediately that $\Sigma_0(\mathrm{PA})$ is closed under bounded quantifiers as well. □

Part b

We will show that for some $n > 0$, the class of $\Sigma_n(\mathrm{PA})$ formulas are closed under bounded quantifiers, existential quantifiers, disjunctions and conjunctions.

Proof. Let $\varphi \in \Sigma_n(\mathrm{PA})$, and suppose $\varphi' \in \Sigma_n$ testifies to that. $\forall v \leq w\varphi' \in \Sigma_n^0$, then by the last exercise we deduce that there is $\psi \in \Sigma_n$ such that $\forall v \leq w\varphi \equiv \forall v \leq w\varphi' \equiv \psi$. φ testifies to $\forall v \leq w\varphi \in \Sigma_n(\mathrm{PA})$. The proof is identical for bounded existential quantifier.

We will show that the class in closed under existential quantifiers. Suppose that $\varphi \in \Sigma_n(\mathrm{PA})$, and let this be testified by $\varphi' \in \Sigma_n$. There is $\phi \in \Pi_{n-1}$ such that $\varphi' = \exists v\phi$. We define $\psi = \exists u\varphi$, then by the second part of the last question,

$$\mathrm{PA} \vdash \forall t_0 \ldots \forall t_{n-1}(\exists u \exists v\varphi \leftrightarrow \exists w \exists v \leq w \exists u \leq w\varphi)$$

Then $\psi \in \Sigma_n(\mathrm{PA})$.

We claim that $\Sigma_n(\mathrm{PA})$ is also closed under disjunction and conjunction, as the proof is the same as the case of $n = 0$. □

Part c

We will show that for $n > 0$, the class $\Pi_n(\mathrm{PA})$ is closed under bounded quantifiers, universal quantifiers, disjunctions and conjunctions.

Proof. The proof for closeness under bounded quantifiers, as well disjunctions and conjunctions is the same as of the above, we move to show closeness under universal quantifiers. Let $\varphi \in \Pi_n(\mathrm{PA})$ and $\varphi' \in \Pi_n$ be the formula to testify that. We define $\psi = \forall v\varphi$, note that $\psi \equiv \forall v\varphi'$. $\varphi' = \forall u\phi$ for some $\phi \in \Sigma_{n-1}$. Using part a of question 1 we deduce that $\forall u\psi \equiv \forall w\forall v \leq w\forall u \leq w\phi$, it is implied that indeed $\psi \in \Pi_n(\text{PA})$. $\square$

Part d

We will show that the negation of a $\Sigma_n(\text{PA})$ formula is $\Pi_n(\text{PA})$.

Proof. In the previous exercise we had shown that the negation of Σ_n^0 formula is a Π_n^0 formula and vice versa. We also know that every Σ_n formula is in particular a Σ_n^0 formula. Let $\varphi \in \Sigma_n(\text{PA})$ and $\varphi' \in \Sigma_n$ be formulas such that,

$$\text{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}(\varphi \leftrightarrow \varphi')$$

It follows that,

$$\text{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}((\neg\varphi) \leftrightarrow (\neg\varphi'))$$

But by the statement above, $\neg\varphi' \in \Pi_n^0$ and there is $\varphi'' \in \Pi_n$ such that $\varphi' \equiv \neg\varphi''$, which implies that,

$$\text{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}((\neg\varphi) \leftrightarrow \varphi'')$$

We deduce that indeed $\varphi \in \Pi_n(\text{PA})$. $\square$

Part e

We will show that every formula is in $\Sigma_n(\text{PA})$ or in $\Pi_n(\text{PA})$ for some $n \in \mathbb{N}$.

Proof. Similarly to the last exercise, we will prove the claim by induction over the structure of the formula. For any atomic formula the claim holds for $n = 0$ from part a. For negation we have shown in the last part that there is closeness, and in parts b and c we had shown closeness for conjunctions.

Lastly, for universal quantifiers we either use definition under $\Sigma_n(\text{PA})$, and question 1 part a for formulas from $\Pi_n(\text{PA})$. We handle existential quantifiers similarly. $\square$

Question 3

Part a

We will show that PA proves that $\forall x \exists y > 0(\forall 0 < z \leq x(z \mid y))$.

Proof. Let us notate $\varphi(x) = \exists y > 0(\forall 0 < z \leq x(z \mid y))$, We want to prove that $\forall x \varphi$ by induction over x. For $x = 0$ we will arbitrarily set $y = 1$, and the formula holds as there is not $0 < z \leq 0$.

Let us assume that PA $\models \varphi(x)$ for some x, we will show that $\varphi(x+1)$ as well. There is y such that it satisfies the formula $\varphi(x)$, then we define $y' = y \cdot (x+1)$. Let $0 < z \leq x+1$, then if $z < x+1$ by the induction hypothesis $z \mid y$ and in particular $z \mid y \cdot (x+1) = y'$. If $z = x+1$ then $z \mid x+1$ and in particular $z \mid y'$ as well. Then y' satisfies $\varphi(x+1)$, and the induction step is concluded. □

Part b

We will show that PA proves that for all x, if y satisfies φ of the last part, and $x' < x'' \leq x$ then $\alpha = (x'+1) \cdot y + 1$ and $\beta = (x''+1) \cdot y + 1$ are co-primes ($\gcd(\alpha, \beta) = 1$).

Proof. α, β are co-primes in PA if and only if PA $\vdash \forall n((n \mid \alpha \wedge n \mid \beta) \to n = 1)$. By KE we assume otherwise, and let $n > 1$ be a number that divides both α and β. Then $n \mid \beta - \alpha = (x'' - x') \cdot y$. $x'' - x' < x$ and by part a it is implied that $n \mid y$. We infer that $n \mid y \cdot (x'+1)$ as well, and by the assumption in contradiction also $n \mid (x'+1) \cdot y + 1$, then $n \mid \alpha - (x'+1) \cdot y = 1$. We conclude that $n = 1$, but $n \neq 1$, a contradiction. □

Part c

We will prove that for every formula $\psi(x, y_0, \ldots, y_{n-1})$, PA proves that for all $y_0, \ldots, y_{n-1}$ and all w, k, m such that m satisfies $\varphi(k)$, meaning that $\forall 0 < z \leq k(z \mid m)$, and $w \leq k$, then there is n such that,

1. If $x < w$ then $\psi(x, y_0, \ldots, y_{n-1})$ if and only if $(x+1) \cdot m + 1 \mid n$
2. If p is a prime and $p \mid n$ then there is $x < w$ such that $\psi(x, y_0, \ldots, y_{n-1})$ and $p \mid (x+1) \cdot m + 1$.

Proof. We will prove the claim by induction over w.

For the basis we set $w = 0$. In this case there is not $x < 0$, then the first statement is vacuously true. We fix $n = 1$ and thus the second statement is also vacuously true.

Let us assume that the statement holds for w, such that n_0 satisfies both of the claims, and let us examine $w+1$. If we would choose n_0 for $w+1$, then the first statement holds for any $x < w$, and it remains to examine $x = w$. $(w+1) \cdot m + 1 \nmid n_0$ by the induction hypothesis, then if $\neg\psi(w, y_0, \ldots, y_{n-1})$ we can choose $n = n_0$. If $\psi(w, y_0, \ldots, y_{n-1})$ then we choose $n = n_0 \cdot (w+1) \cdot m + 1$.

For the second statement, let $p \mid n$ for the n we found above. By the second part of the question, there is some unique $x < w+1$ such that $p \mid (x+1) \cdot m + 1 \mid n$. By the induction hypothesis $(x+1) \cdot m + 1 \mid n_0 \mid n$ then $\psi(x, y_0, \ldots, y_{n-1})$ is true, concluding the statement. □

Question 4

Part a

We will show that there is a coded empty set, namely exist n, m, k such that $\neg FS(x, n, m, k)$ for all x.

Proof. By the last question it is suffice to fix $n = 0$, as the formula to represent the empty set if $\psi(x) = \perp$ (or some other sentence like $0 = 1$). We also choose $k = 1$. ☐

Part b

We will show that there are coded singletons, for all y there are n, m, k such that for all x, $FS(x, m, n, k) \iff x = y$.

Proof. We define $\psi(x, y) = x = y$, as well $k > y$ some fixed value, then by question 3 part c, when y is playing the role of y_0 of 3c, there is n such that $\psi(x, y)$ is and only if $x < k \land x = y$, or more simply, $x = y$. ☐

Part c

We will show that PA proves that there are coded unions for coded finite sets. For all n_1, m_1, k_1 and n_2, m_2, k_2 there are n, m, k such that for all x, $FS(x, m, n, k) \iff FS(x, m_1, n_1, k_1) \lor FS(x, m_2, n_2, k_2)$.

Proof. We define $k = \max\{k_1, k_2\}$, as well $m = k!$ and $n = n_1 \cdot n_2$. It follows that $FS(x, m, n, k) \iff (x + 1) \cdot m + 1 \mid n = n_1 \cdot n_2 \iff (x + 1) \cdot m + 1 \mid n_1 \lor (x + 1) \cdot m + 1 \mid n_2 \iff FS(x, m_1, n_1, k_1) \lor FS(x, m_2, n_2, k_2)$.

We could also define $\psi(x) = FS(x, m_1, n_1, k_1) \lor FS(x, m_2, n_2, k_2)$ directly and use question 3 part c. ☐

Part d

We will show that PA proves that there are coded intersections for coded finite sets. For all n_1, m_1, k_1 and n_2, m_2, k_2 there are n, m, k such that for all x, $FS(x, m, n, k) \iff FS(x, m_1, n_1, k_1) \land FS(x, m_2, n_2, k_2)$.

Proof. We define $k = \min\{k_1, k_2\}$ as well $m = k!$ and $n = \gcd(n_1, n_2)$. The proof is almost identical to that of the last part.

We could also define $\psi(x) = FS(x, m_1, n_1, k_1) \land FS(x, m_2, n_2, k_2)$ directly and use question 3 part c. ☐