Exercise 1 Answer Sheet — Logic Theory (2), 80424

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Question 1

Let us assume that D is an ultrafilter on given set X.

Part a

Suppose that there is some $s \in D$ such that $|s| < \omega$, we will show that D is principal.

Proof. We will assume that s is minimal in relation to cardinality in D, otherwise let us choose the minimal such s by using the well ordering principal. There is a unique such $s \in D$, otherwise $|s| = |s'|$ and $|s \cap s'| < |s|$, a contradiction to the minimality of the cardinality of s.
For every $x \in D$, there is some $y \in D$ such that $y \subseteq x \cap s$, but by s minimality, $y = x \cap s$, hence $s \subseteq x$. In the other direction, if $s \subseteq x$, then $x \in D$ by filter properties.
Therefore $D = \{x \in \mathcal{P}(X) \mid s \subseteq x\}$, D is a principal ultrafilter defined by s. $\square$

Part b

We will show that if $D \subseteq \mathcal{P}(I)$ is a principal ultrafilter, $\langle \mathcal{M}_i \mid i \in I \rangle$ is a sequence of L-structures, then $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / D \cong M_i$ for some $i \in I$.

Proof. Let $s \subseteq I$ be the defining set for D, and let us fix $i \in s$. Let $f : N \to M_i$ such that $f([g]_D) = g(i)$. f is well defined, as every for every $h, h' \in [h]$, it is follows that $h \restriction s = h' \restriction s$ by D's definition as principal.

We will show that f is an isomorphism between $\mathcal{N}$ and $\mathcal{M}_i$. L-constants are being preserved directly from definition of ultra-products. Assume $F \in \mathrm{Func}_{L,n}$, and $t_0, \dots, t_{n-1} \in \mathrm{term}_L$, then,

$$f(F^{\mathcal{N}}(t_0^{\mathcal{N}}, \dots, t_{n-1}^{\mathcal{N}})) = f([\{F^{\mathcal{M}_j}(t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j}) \mid j \in I\}]) = F^{\mathcal{M}_i}(t_0^{\mathcal{M}_i}, \dots, t_{n-1}^{\mathcal{M}_i})$$

meaning f preserves terms. Let $R \in \mathrm{rel}_{L,n}$, and let $t_0, \dots, t_{n-1} \in \mathrm{term}_L$, then,

$$\mathcal{N} \models R(t_0, \dots, t_{n-1}) \iff \{j \in I \mid \langle t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j} \rangle \in R^{\mathcal{M}_j}\} \in D \implies \langle t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j} \rangle \in R^{\mathcal{M}_i}$$

We conclude that f is indeed a structures isomorphism. $\square$

Question 2

We will show that if F is a filter on a set X such that if $s \in F$ then s is infinite, then there is a non-principal ultrafilter extending F.

Proof. Initially, we claim that there is no $x \in F$ such that $|x|, |x^C| \geq \omega$, when $x^C = X \setminus x$. If there is one, then $\emptyset \subseteq x \cap x^C = \emptyset$ is in F, in contradiction to F being a filter. We define the ultrafilter $U = \{x \subseteq X \mid |x| \geq \omega\}$, this is indeed an ultrafilter as noted in lecture. it is sufficient to show that $F \subseteq U$. Indeed, if $s \in F$ then $|s| \geq \omega$, implies that $s \in U$.

We found an ultrafilter extending F, it is required to prove that U is not a principal ultrafilter as well. We will assume otherwise, then it is follows that there is $S \subseteq X$ such that $\forall x, S \subseteq x \iff x \in U$. Let $a \in S$ be an element, then $b = X \setminus \{a\}$ is a set such that $|b| \geq \omega$, implies that $b \in U$, a contradiction as $S \nsubseteq b$. □

Question 3

Part a

Let $\mathcal{N} = (\mathbb{N}; R^{\mathcal{N}})$ be graph such that $(a, b) \in R^{\mathcal{N}} \iff |b - a| = 1$. Let D be a non-principal ultrafilter on $\mathbb{N}$ (An example of one is the construction from exercise 2). Let $\mathcal{M} = \mathcal{N}^{\mathbb{N}}/D$. We will prove that $\mathcal{M}$ is not path-connected.

Proof. Let us define $a = [0], b = [id]$, in the sense that the constant valued function $f(n) = 0$, it derived that $f \in a$. We claim that there is no finite path between $a, b \in N$. We assume otherwise, and define $\langle [v_i] \mid i < n \rangle$ for some $n < \omega$, such that $([v_i], [v_{i+1}]) \in R^{\mathcal{N}}$ for every $i < n$, and $[v_0] = a, [v_{n-1}] = b$. D is non-principal then there is some $i < \omega$ such that $v_i(i)$ testifies to the relation, and by the last exercise i is as large as we desire, then we assume $i > n$. But $v_0(i) = 0, v_{n-1}(i) = i$, and by the relation definition $v_j(i) = j$ for every $0 \leq j \leq n$, it is implies that $i = v_{n-1}(i) < n < i$, a contradiction. We conclude that $\mathcal{M}$ is not path-connected. $\square$

Part b

Let $\mathcal{R} = (\mathbb{R}; +, \cdot, 0, 1, <)$. Let D be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{R}^* = \mathcal{R}^{\mathbb{N}}/D$.
We will show that in $\mathcal{R}^*$ there is an infinitesimal element, namely $\epsilon \in R^*$ such that $\forall n \in \mathbb{N}, 0 < \epsilon < \frac{1}{n}$.

Proof. Let $d \in D$ be an unbounded subset of $\mathbb{N}$, and let $r : \mathbb{N} \to \mathbb{R}$ such that $(r \restriction d)(n) = \frac{1}{n+1}$, We define $\epsilon = [r]_{\mathcal{R}^*}$. We observe that $1^{\mathcal{R}^*} = [\{\langle n, 1 \rangle \mid n \in \mathbb{N}\}]$ by definition, and it is follows that $(\frac{1}{n})^{\mathcal{R}^*} = [\{\langle n, \frac{1}{n} \rangle \mid n \in \mathbb{N}\}]$. Let $m \in \mathbb{N}$ be some number, we will show that $\mathcal{R}^* \models \epsilon < \frac{1}{n}$,

$$\mathcal{R}^* \models \epsilon < \frac{1}{n} \iff \{n \in \mathbb{N} \mid \mathbb{R} \models \epsilon < \frac{1}{m}\} \in D \iff \{n \in d \mid \epsilon(n) = \frac{1}{m+1} < \frac{1}{m}\} \in D$$

and by d's definition the statement indeed holds. Lastly we will note that $\mathcal{R}^* \models 0 < \epsilon$, it is derived by the same method. $\square$

Question 4

We will define a filter F on X as σ-complete if and only if for every $\{s_\alpha \mid \alpha < \omega\} \subseteq F$, $\bigcap_{\alpha<\omega} s_\alpha \in F$.

Part a

We will show that if F if a σ-complete ultrafilter and $|X| = \omega$ then F is principal.

Proof. An answer using AC without necessity,
Let $\{s_\alpha \mid \alpha < \omega\} \subseteq F$ be some collection, and let us define $t_0 = s_0$ and for every $\alpha < \omega$, $t_{\alpha+1} = s_\alpha \cap t_\alpha$. We also define $t_\omega = \bigcap_{\alpha<\omega} t_\alpha$, this is indeed a set, and $\in F$ in particular, due to F being σ-complete. The item t_ω is a supremum in relation of $\supseteq$ in the chain $\langle t_\alpha \mid \alpha < \omega\rangle$, and it follows that every chain has such supremum. By Zorn's lemma we conclude that there is $s_m \in F$ which is maximal according to the relation $\supseteq$, then $\forall s \in F, s_m \subseteq s$. If $s_m \subseteq s$ for any $s \subseteq X$, then by filters properties $s \in F$, concluding that $F = \{s \subseteq X \mid s_m \subseteq s\}$.

We can also use the well-founded order of $\subseteq$ of X and take a minimal of it, omitting the part of Choice axiom.

We can also assume that it is non-principal, therefore for every $x \in X$, $X \setminus \{x\} \in F$ (as it is an ultrafilter), then $\bigcap_{\alpha<\omega} X \setminus \{x_\alpha\} = \emptyset \in F$, a contradiction. $\square$

Part b

We will provide an example of a σ-complete ultrafilter non-principal filter on an infinite set.
Solution. Let $\mu < \kappa$ be cofinal cardinals such that $\omega_1 < \mu$. We define $U = \{\delta < \kappa \mid \mu < \delta\}$. This is indeed an ultrafilter as concluded from the cofinality of μ, and it is non-principal as derived from extension of question 2. It is clear that U is σ-complete, as ω_1 intersections are preserving cofinality for μ.

Part c

We will show that if $\mathcal{M}_i = (X_i, <_i)$ are nonempty well-ordered sets for $i \in I$ and D is a σ-complete ultrafilter on I, then $\mathcal{N} = \prod_{i\in I} \mathcal{M}_i / D$ is well-ordered.

Proof. Using Löwenheim–Skolem we can assume without loss of generality that $\mathcal{N}$ is a countable model. Let $\{[f_j] \in N \mid j < J\}$ be some collection, we will show that there if $[f] \in Y$ minimal in $<^{\mathcal{N}}$. Let $f(i) = \min_{<^{\mathcal{M}_i}}\{f_j(i) \mid j < J\}$, this definition holds as $<^{\mathcal{M}_i}$ is a well-order. By σ-completeness we deduce that $\{i < I \mid f_j(i) = f(i), j < J\} = \bigcap_{j<J}\{i < I \mid f_j(i) = f(i)\}$, implying that indeed $[f] \in N$, concluding our claim. $\square$

Part d

We will show that the statement of part c does not hold without the assumption that D is σ-complete.
Solution. Let $\mathcal{N} = (\mathbb{N}; <)^{\mathbb{N}}/D$ for the ultrafilter defined in question 2. It is clear that $\mathcal{N}$ fulfills the requirements, and that D is not σ-complete. We will construct a set and show there is no minimum in it. Let $s = \{[f] : \mathbb{N} \to \mathbb{N} \mid \forall n \in \mathbb{N} f(n) \le f(n+1)\} \subseteq N$, and let $[f] \in s$, we choose the function $[g] \in s$ such that $f(n) = g(n)$ for $n > M \in \mathbb{N}$, and $g(n) = f(M)$ otherwise. Its follows that $[g] <^{\mathcal{N}} [f]$, therefore f is not minimal, we conclude that indeed s witnessing that $<^{\mathcal{N}}$ is not a well-order.