Solution to Exercise 0 — Model Theory (1), 80616

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Question 1

Let $L = \{P\}$ a language where P is unary relation. Define,

$$\varphi_n = \exists x_0 \ldots \exists x_n \left(\bigwedge_{i \leq n} P(x_i) \wedge \bigwedge_{i < j \leq n} x_i \neq x_j \right), \quad \psi_n = \exists x_0 \ldots \exists x_n \left(\bigwedge_{i \leq n} \neg P(x_i) \wedge \bigwedge_{i < j \leq n} x_i \neq x_j \right)$$

and let $T = \{\varphi_n, \psi_n \mid n < \omega\}$.
We will show that $\mathrm{cl}_{\vdash} T$ is ω-categorical.

Proof. Let $\mathcal{M} \models T$ be some model. It can be proved by direct induction that $|P^{\mathcal{M}}| = \omega$ as well as $|\neg P^{\mathcal{M}}| = \omega$. Let us construct $f : \omega \to M$ such that $f(n) \in P^{\mathcal{M}}$ for any $n < \omega$. $\mathcal{M} \models \varphi_0 \iff \mathcal{M} \models \exists x\, P(x)$ then let $f(0)$ be such witness. Let us assume that $f \restriction n$ is defined, then $\mathcal{M} \models \varphi_{n+1}$, then by the pigeonhole principle there is some $a \in \mathcal{M}$ such that $a \notin f''n$, and let $f(n+1) = a$. For the sake of convenience let us redefine f as $2 \times \omega \to M$ injective function such that $f(0, n)$ is the same as $f(n)$ and $f(1, n) \notin P^{\mathcal{M}}$. By CSB we can assume that f is bijection as well, and by the selection of $\mathcal{M}$ as an arbitrary model of T we can deduce that for any $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \cong \mathcal{N}$ by composition of functions as was constructed. $\square$

Question 2

Let $L = \{c_n \mid n < \omega\}$ be language consists of constant symbols. Let us define the theory $T = \{c_i \neq c_j \mid i < j < \omega\}$. We will show that there are countably many non-isomorphic countable models of T, and that T is complete.

Proof. Let us define the model $\mathcal{M}_n$ such that $M = \omega$ and,

$$c_i^{\mathcal{M}} = i + n$$

for any $i < j < \omega$,

$$c_i^{\mathcal{M}} = i + n \neq j + n = c_j^{\mathcal{M}}$$

therefore $\mathcal{M}_n \models T$. $\mathcal{M}_n \models k \neq c_i$ for all $i < \omega$, in particular $\mathcal{M}_n \models \exists x\, x \neq c_i$. It is implied that also,

$$\mathcal{M}_n \models \exists x_0 \ldots \exists x_{k-1} \left(\bigwedge_{i<j<k} x_i \neq x_j \wedge x_i \neq c_l \right) = \varphi_l^k$$

for all $l < \omega$. Finally, $\mathcal{M}_n \not\models \varphi_l^k$ for any $k > n$, we deduce that $\mathcal{M}_n \ncong \mathcal{M}_m$ for any $n \neq m$.

We move to show that T is complete. Let us assume toward a contradiction that φ is a sentence such that $\varphi \notin T$ and $T \cup \{\varphi\}$ is consistent. By construction of Henkin models we can deduce that $\mathcal{M}_0 \models \varphi$, but $\mathcal{M}_0$ is minimal, namely if $\mathcal{N} \models T$ then $\mathcal{M}_0 \subseteq \mathcal{N}$, then by definition $T \models \varphi$, a contradiction. $\square$

Question 3

We will show that $T = \mathrm{Th}(\mathbb{N}, +, \cdot)$ has $2^{\aleph_0}$ non-isomorphic countable models.

Proof. Observe the fact that numbers are definable in T, by formula as such,

$$\varphi_n(x) = \forall y\ x \cdot y = \overbrace{y + \cdots + y}^{n \text{ times}}$$

If $\mathcal{M} \models T$ then we denote by $\underline{n}$ the single element of M that fulfills φ_n.
By the fact that $\exists x\ \varphi_n(x) \in T$ it follows that $\{\underline{n} \mid n < \omega\} \subseteq M$ for any such model.
We also denote by $x \mid y$ the formula $\exists z\ x \cdot z = y$.

Let $P \subseteq \mathbb{N}$ be the set of prime numerals, namely $\varphi(x) = \forall y,\ (y \mid x \to (y = x \vee y = \underline{1}))$. We add new constant symbol c to the language of T, and let $P' \subseteq P$ be some infinite set of primes. Let us define a new theory,

$$T' = T \cup \{\underline{p} \mid c \mid p \in P'\} \cup \{\underline{p} \nmid c \mid p \notin P'\}$$

For any $T_0' \subseteq T'$ finite, either $T_0' \subseteq T$ and satisfiable or $\{c \mid \underline{p} \mid p \in P_0'\} \in T_0'$ for some finite $P_0' \subseteq P'$, and then $\mathbb{N} \models \prod_{p \in P_0'} \underline{p}$. From the compactness theorem we conclude that T' is satisfiable and let $\mathcal{M}_{P'}' \models T'$ be a witness. We can now remove the constant symbol c and get a model $\mathcal{M}_{P'} \models T$.

By downwards Löwenheim Skolem theorem we can assume that $|M_{P'}| = \omega$ for any $P' \subseteq P$. Let us assume that $|\{\mathcal{M}_{P'} \mid P' \subseteq P, |P'| = \omega\}| < 2^\omega$, then there must be model $\mathcal{N}$ such that it has non-countable non-standard elements, in contradiction to being countable. Then $|\{\mathcal{M}_{P'} \mid P' \subseteq P, |P'| = \omega\}| \geq 2^\omega$. $\square$

Question 4

Let $\kappa \geq \omega$ be some cardinal and let L be some language. Let T be a κ-categorical L-theory such that it has no finite models. We will show that T can be incomplete.

Solution. Let $L = \{c_\alpha \mid \alpha < \delta\} \cup \{P\}$ for $\kappa < \delta$, where c_α is a constant symbol and P is unary relation.

$$T = \{c_\alpha \neq c_\beta \mid \alpha < \beta < \delta\}$$

It follows from the definition of T that if $\mathcal{M} \models T$ then $|M| \geq \delta > \kappa$, therefore there are no models of T of cardinality κ, then the theory is vacuously κ-categorical. T is not complete, as $P(c_0) \notin T$ as well $\neg P(c_0) \notin T$.

Question 5

Let $T = \mathrm{Th}(\mathbb{Q}, \leq)$ be DLO.
We will show that T is not κ-categorical for some uncountable cardinal κ.

Proof. Define $(\mathbb{R}, \leq)$ and $(\mathbb{R} + \mathbb{Q}, \leq)$, these are both models of DLO, and let us assume that $f : \mathbb{R} + \mathbb{Q} \to \mathbb{R}$ is model isomorphism and thus also an order isomorphism. Let $y = f(\langle 1, 0\rangle)$, then,

$$|\{x \geq y \mid x \in \mathbb{R}\}| = 2^{\omega}$$

but

$$|\{f^{-1}(x) \geq^{\mathbb{R}+\mathbb{Q}} \langle 1, 0\rangle \mid x \in \mathbb{R}\}| \leq |\mathbb{Q}| = \omega$$

a contradiction, then $(\mathbb{R}, \leq) \ncong (\mathbb{R} + \mathbb{Q}, \leq)$. $\square$