Exercise 8 Answer Sheet — Logic Theory (2), 80424

May 31, 2025

Question 2

We will show that the universal recursive function $U : \mathbb{N}^2 \rightharpoonup \mathbb{N}$ is not total, and that there is no total recursive function U' containing it.

Proof. We defined the universal function over the domain of coded Σ_1-formulas, and directly by past constructions the number 1 does not codify such a formula. Let us assume towards a contradiction that there is $U' \supseteq U$ total universal function.

The idea is to take some recursive non-total function $f : \mathbb{N} \rightharpoonup \mathbb{N}$ that cannot be extended into total recursive function. Let us define,

$$\varphi(x, y) = x \leq \underline{6} \land x = y$$

It corresponds to the function $f(x) = x$ for $x \leq 6$. To ensure that it is indeed recursive, note that $\varphi \equiv \exists y, x = y \land x \leq \underline{6}$, which is Σ_1 and indeed recursive. Let e be the code for f in $\operatorname{dom} U'$ (note that it does not has to codify f in $\operatorname{dom} U$). Then $U'(e, x) = f(x) = x$ for all $x \leq 6$, but U' is total and defined for every x. In particular $U'(e, 7) = f(7) = y'$, but there is no y' such that $\varphi(7, y')$ holds.

We can also see that,

$$\{0, \ldots, 6\} = \operatorname{dom} f \neq \{x \in \mathbb{N} \mid \langle e, x \rangle \in \operatorname{dom} U'\} = \mathbb{N}$$

In contradiction to the definition of universal recursive functions. $\square$

Question 3

For every $l \in \mathbb{N}$ we define $b_l : \mathbb{N} \to \mathbb{N}$ by $b_0(n) = 2n$, $b_{l+1}(n) = b_l^{n+1}(1)$ in the sense of composition. Let $B(l, n) = b_l(n)$, such that $B(0, n) = 2n$, $B(l + 1, 0) = B(l, 1)$ and $B(l + 1, n + 1) = B(l, B(l + 1, n))$.

Part a

We will show that for any $l \in \mathbb{N}$, b_l is primitive-recursive.

Proof. □