Solution to Exercise 4 — Model Theory (1), 80616

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Question 1

Definition 0.1 (algebraic formula)**.** Let $\mathcal{M}$ be a structure. A formula $\varphi(x) \in \text{form}_{L_{\mathcal{M}}}$ is called algebraic if,

$$|\{a \in M \mid \mathcal{M} \models \varphi(a)\}| < \aleph_0$$

Definition 0.2 (definable element)**.** An element $a \in M$ is called definable if there is a formula $\psi(x)$ such that $\mathcal{M} \models \psi(x) \iff x = a$ for any $x \in M$.

Definition 0.3 (acl and dcl)**.** If $\mathcal{M}$ is a structure and $A \subseteq M$ a set of elements, then let,

$$\text{acl}_{\mathcal{M}}(A) = \bigcup\{\varphi(M) \mid \varphi \in \text{form}_{L_A}, \varphi \text{ is algebraic}\}, \quad \text{dcl}\,\mathcal{M}(A) = \{a \in M \mid a \text{ is definable using an } L_A \text{ formula}\}$$

be the set of elements that has a formula with parameters from A that is algebraic and are true for them, and the set of elements uniquely definable by formulas with parameters.

Part a

Let $A \subseteq M$ for $\mathcal{M} \prec \mathcal{N}$. We will show that $\text{acl}_{\mathcal{M}}(A) = \text{acl}_{\mathcal{N}}(A)$ and $\text{dcl}_{\mathcal{M}}(A) = \text{dcl}_{\mathcal{M}}(A)$.

Proof. Let $a \in M$ be $\in \text{acl}_{\mathcal{M}}(A)$ and let $\varphi \in \text{form}_{L_A}$ be an algebraic formula to witness it. $\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(a)$ by the given elementary embedding and the fact that $A \subseteq N$.

Let $a \in \text{acl}_{\mathcal{N}}(A)$ and assume toward contradiction that $a \notin \text{acl}_{\mathcal{M}}(A)$, meaning that there is $\varphi(x)$ over L_A such that $\mathcal{N} \models \varphi(a)$ and φ is algebraic. Let $B = A \cup \varphi(M)$, denote $\varphi(M) = \{b_i \mid i < N\}$ for $N < \omega$,

$$\psi(x) = \varphi \wedge \left(\bigwedge_{i<N} x \neq b_i\right)$$

Then ψ is algebraic over both $\mathcal{M}$ and $\mathcal{N}$ and $\psi(M) = \emptyset$, while $a \in \psi(N)$. Let $\phi = \exists x\ \psi$, then $\mathcal{N} \models \phi$ as a witnesses exactly this, while $\mathcal{M} \models \neg\phi$ in contradiction to $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$.

The case for dcl is equivalent, we take $\varphi(x)$ such that $\varphi(M) = \{a\}$ and therefore $\varphi(N) = \{a\}$ as otherwise we can define $\varphi \wedge (x \neq a)$. □

We conclude that the relative model of acl and dcl can be omitted.

Part b

Let $A \subseteq M$, we will show that $\text{acl}(\text{acl}(A)) = \text{acl}(A)$ and $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$.

Proof. Let $a \in \text{dcl}(\text{dcl}(A))$ be an element and let $\varphi(x)$ be its witness. Suppose that $\{b_i \mid i < N\}$ is the set of elements that φ consists of such that they are not in A. $b_i \in \text{dcl}(A)$ for any i, and let $\theta_i(x)$ witness that. Let $\{y_i \mid i < N\}$ be a set of distinct and disjoint to φ variables, and let us define,

$$\varphi' = \varphi^{b_0,\dots,b_{N-1}}_{y_0,\dots,y_{N-1}}, \quad \phi(x) = \exists y_0 \cdots \exists y_{N-1} \left(\bigwedge_{i<N-1} \theta(y_i)\right) \wedge \varphi'$$

then $\phi \in \text{form}_{L_A}$ and $\phi(M) = \{a\}$, therefore $a \in \text{dcl}(A)$ and thus $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$.

The case of acl is equivalent. □

Part c

Let $L = \{0, 1, +, \cdot\}$ and K be an algebraically closed field and let $A \subseteq K$.

We will compute $\text{acl}(A)$ and $\text{dcl}(A)$.

Solution. K has quantifier elimination therefore we can omit the discussion about quantifiers. Each formula $\varphi(x) \in \text{form}_{L_A}$ is (without loss of generality) of the form,

$$\varphi = (\bigwedge_{i<n} p_i(x) = 0) \wedge (\bigwedge_{j<m} p_{n+j}(x) \neq 0)$$

where $p_i(x)$ is a polynomial over A. An element $a \in K$ is $\in \text{dcl}(A)$ if it is the unique solution of some such polynomial. Each polynomial in algebraically closed field can be deconstructed to product of linear polynomials, meaning that $p_i(x)$ has a single solution if $p_i(x) = (x - a)$. We conclude that $\text{dcl}(A) = \text{cl}_{+,\cdot}(A)$.

By similar proposition, $\text{acl}(A) = \langle A \rangle \leq K$, meaning the subfield generated by A.

Question 2

Part a

Let κ be an infinite cardinal and let $\lambda \leq \kappa$ be the minimal cardinal such that $\kappa^\lambda > \kappa$. Consider the order $(\kappa^{<\lambda}, <_l)$ the lexicographic order on $\kappa^{<\lambda}$.
We will show that this order has more than κ many cuts and construct dense linear order of cardinality κ with $> \kappa$ different cuts.

Proof. A cut is defined as a collection $X \subseteq \kappa^{<\lambda}$ such that if $y \in \kappa^{<\lambda}$ and there is $x \in X$ such that $y < x$, then $y \in X$. Let $\delta \in \kappa^\lambda$ be a sequence, and let,

$$\eta_\delta = \{x \in \kappa^{<\lambda} \mid \exists \alpha < \lambda,\ x < \delta \restriction \alpha\}$$

notice that if $x < \delta \restriction \alpha$ then for any $\alpha < \beta$ also $x < \delta \restriction \beta$, thus the definition of ϵ_δ holds. We also note that ϵ_δ is a cut of ϵ^δ as $y < x < \delta \restriction \alpha \implies y < \delta \restriction \alpha$.
Let $\delta \neq \gamma \in \kappa^\lambda$ and assume that β is the least ordinal such that $\delta(\beta) \neq \gamma(\beta)$ and $\delta(\beta) > \gamma(\beta)$ without loss of generality. It follows that $\delta \restriction \beta \in \eta_\delta$ but $\delta \restriction \beta \notin \eta_\gamma$, therefore $\eta_\delta \neq \eta_\gamma$. The injection $\delta \mapsto \epsilon_\delta$ witness $|\kappa| < |\kappa^\lambda| \leq |H|$ for $H \subseteq \kappa^{<\lambda}$ the collection of cuts.

Let $X = \{f \in \kappa^\lambda \mid \exists \alpha < \lambda \forall \alpha < \beta < \lambda,\ f(\beta) = 0\}$, then there is a bijection of X and $\kappa^{<\lambda}$ and therefore $|X| = \kappa$. The proof that $|H_X| > \kappa$ remains the same in this case, and we will show that $\langle X, <_l \rangle$ is dense. If $f, g \in X$ with $f <_l g$ then f is not an initial segment of g and therefore there exists least α such that $f(\alpha) < g(\alpha)$. We can define,

$$h(x) = \begin{cases} f(x) & x \neq \alpha + 1 \\ f(x) + 1 & x = \alpha + 1 \end{cases}$$

Then $f <_l h$ but $h(\alpha) < g(\alpha)$ therefore $f <_l h <_l g$, as intended. □

Part b

Let T be a theory and let us assume that there is $\mathcal{M} \models T$, a sequence $\langle \bar{a}_n \mid n < \omega \rangle \subseteq M^k$ and a formula $\varphi(\bar{x}, \bar{y})$ such that for any $n \neq m$, $\mathcal{M} \models \varphi(\bar{a}_n, \bar{a}_m) \iff n < m$.
We will show that T is not κ-stable for any $\kappa \geq |T|$.

Proof. We will show that there is $A \subseteq \mathcal{C}_T$ with $|A| \leq \kappa$, such that $\kappa < |S_1(A)|$. Note that $\mathcal{M} \prec \mathcal{C}_T$, then $\mathcal{C}_T \models \varphi(\bar{a}_n, \bar{a}_m) \iff n < m$.

If T is ω-stable then it is κ-stable for any $\kappa \geq |T|$, then it is sufficient to show that T is not ω-stable. By the equivalency to δ-stableness theorem, T is ω-stable if and only if $|S_1(\emptyset)| \leq \omega$ and $\forall n \in \mathbb{N},\ |S_n(N)| \leq \omega$ for any $\mathcal{N} \models T$. Therefore if $|S_k(M)| > \omega$ then T is not κ-stable.

We intend to define lexicographic order over $\mathcal{M}$ using φ, assume that $\langle \bar{b}_n \mid n < m \rangle$ for some $m < \omega$. Let us define the formula that extends φ's order to m-tuples against m'-tuples,

$$\psi_{m,m'}(\bar{x}_0, \ldots, \bar{x}_{m-1}, \bar{y}_0, \ldots, \bar{y}_{m'-1}) = (m < m') \vee \varphi(\bar{x}_0, \bar{y}_0) \vee \cdots \vee \varphi(\bar{x}_{m-1}, \bar{y}_{m-1})$$

Notice that $\mathcal{M} \models \varphi(\langle \bar{a}_{n_i} \mid i < m \rangle, \langle \bar{a}_{n_j} \mid j < m' \rangle) \iff \langle n_i \mid i < m \rangle <_l \langle n_j \mid j < m' \rangle$.

Let $\langle \omega^{<\omega}, < \rangle$ be a dense linear order from the last part, and let H be the collection of cuts, we saw that $|H| > \omega$. Let $F : \omega \to M^k$ be the map defined by $n \mapsto \bar{a}_n$ and $G : \omega^{<\omega} \to (M^k)^{<\omega}$ defined as $G(f)(i) = F(f(i))$.
Notate $H_\mathcal{M} = G(H)$, the collection of cuts of M^k in relation to the order induced by ψ. Given $X \in H_\mathcal{M}$, let us define $p_X \in S_1(M)$ as,

$$p(\bar{x}) = \mathrm{cl}_\vdash \{\varphi(\bar{y}, \bar{x}) \mid \bar{y} \in X\}$$

Namely, $\bar{x}$ is an k-tuple of elements such that acts as the supremum of X. It follows that there are $|H_{\mathcal{M}}| = |H| > \omega$ such types, then $|S_1(M)| > \omega$ as well, concluding our proof. $\square$

Part c

We will show that if L is a language with $|L| \geq \kappa$ and T is a theory such that $\forall n < \omega,\ |S_n(\emptyset)| \leq \kappa$, then there is $L' \subseteq L$ with $|L'| = \kappa$ such that for any $\varphi(x_0, \ldots, x_{n-1}) \in \text{form}_L$, there is an L'-formula $\varphi'(x_0, \ldots, x_{n-1})$ with $T \models \forall \bar{x}\ (\varphi \leftrightarrow \varphi')$.

Proof. Let us observe $tp(c) = \{\varphi(x) \mid T \models \varphi(c)\} \in S_1(\emptyset)$ for some constant symbol $c \in L$. If there were $\{c_\alpha \mid \alpha < \lambda\}$ for $\lambda > \kappa$ then by $|\{tp(c_\alpha) \mid \alpha < \lambda\}| \leq |S_1(\emptyset)| \leq \kappa$, we would get that up to equality in T there are at least κ constants, then let us add those constants to L'.

In similar manner, if $R \in L$ is an n-placed relation symbol, we can define $p_R(\bar{x}) = \{\varphi(\bar{x}) \mid T \models \varphi(\bar{x}) \to R(\bar{x})\}$. We get up to κ n-placed relation symbols $\{R_\alpha^n \mid \alpha < \kappa\}$ such that there is $\alpha < \kappa$ for which $T \models \forall \bar{x}\ (R \leftrightarrow R_\alpha^n)$. We enrich L' with $\{R_\alpha^n \mid n < \omega, \alpha < \kappa\}$, note that $\kappa \cdot \kappa = \kappa$ therefore $|L'| = \kappa$.

We move to function symbols. Let F be an n-placed function symbol, and let,

$$p_F(\bar{x}) = \{\exists y\ \varphi(\bar{x}, y) \mid T \models \varphi(\bar{x}, y) \to (F(\bar{x}) = y)\}$$

and define in the same sense $\{F_\alpha^n\}$, the enrichment of L' by them finishes the proof. $\square$

Question 3

We say that $\mathcal{M}$ is minimal if $\mathcal{M}$ has no proper elementary substructure.

Part a

Let T be a countable and complete theory. We will show that if T has a prime model, then any minimal model is prime.

Proof. Let $\mathcal{M} \models T$ be a prime model and let $\mathcal{N} \models T$ be a minimal model. Then there exists $\iota : \mathcal{M} \hookrightarrow \mathcal{N}$ an elementary embedding. But $\mathcal{N}$ is minimal, therefore $\mathcal{M} \supseteq \mathcal{N}$ (without loss of generality), in particular $\mathcal{M} \cong \mathcal{N}$ thus $\mathcal{M} = \mathcal{N}$, so $\mathcal{N}$ is prime as well. $\square$

Part b

We will find and example for a countable atomic model over countable language such that it is not minimal.
Solution. Let $\mathcal{M} = \langle \frac{1}{2}\mathbb{Z}, +, 0, 1 \rangle$, namely $\frac{x}{2} \in M$ for any $x \in \mathbb{Z}$. By part c (which we will cover shortly) for any $a \in M$, there is $\varphi(x) \in$ form that isolating $a + a$, then $\psi(x) = \varphi(x + x)$ isolates a, therefore $\mathcal{M}$ is atomic.
We will show that $\mathcal{N} = \langle \mathbb{Z}, +, 0, 1 \rangle$ is a proper elementary sub-model of $\mathcal{M}$ using Tarski-Vaught test. Let $\varphi(x, \bar{y})$ be a formula and $\bar{a} \in M$ parameters, and assume that $\mathcal{M} \models \exists x\ \varphi(x, \bar{a})$. By the second Gauss lemma the witness is $\in \mathbb{Z}$ as well.

Part c

We will show that $\mathcal{M} = \langle \mathbb{Z}, +, 0, 1 \rangle$ is minimal and atomic.

Proof. Each element can be isolated by $x = \overbrace{1 + \cdots + 1}^{n \text{ times}}$ or by $\exists y\ (y = \overbrace{1 + \cdots + 1}^{n \text{ times}} \wedge y + x = 0)$.
Moving to show minimality, assume that $\mathcal{N} \prec \mathcal{M}$, then $\mathcal{N} \subseteq \mathcal{M}$ and closed under addition in particular. But $0, 1 \in N$ therefore $\mathbb{Z} \subseteq N$, meaning that $\mathcal{N} = \mathcal{M}$ and therefore not a proper substructure, thus $\mathcal{M}$ is minimal. $\square$

Part d

We will show that $\mathcal{M} = \langle \mathbb{Z}, +, 0 \rangle$ is minimal and not atomic.

Proof. Let $\mathcal{N} \prec \mathcal{M}$ be a subgroup, then it is generated by a single element $a \in \mathbb{Z}$, if $a = 1$ then $\mathcal{N} = \mathcal{M}$. Otherwise let us define $\varphi(x, y) = y\ (\overbrace{y + \cdots + y}^{a \text{ times}} = x)$, then $\mathcal{M} \models \exists x\ \varphi(x, a)$ but it follows that $x = 1$ only, therefore $\mathcal{M} \not\models \exists x\ \varphi(x, a)$. Tarski-Vaught test fails, it is implied that $\mathcal{N}$ is not a substructure in this case, concluding our proof that $\mathcal{M}$ is minimal.
The theory $\mathrm{Th}(\mathcal{M})$ has quantifier-elimination, and let $p(x) = tp(1)$. Each formula in p is of the form $\varepsilon(\underline{n} \cdot x) = 0$ by closure to conjunction of p, therefore,

$$p = \{\underline{n} \cdot x \neq 0 \mid n \geq 1\}$$

but for the same reason we get $tp(2) = p$, meaning that if φ isolates p, then $\varphi(2) \wedge \varphi(1)$ holds. Then $\mathcal{M}$ has no formula isolating 1, therefore it is not atomic. $\square$