

Solution to Exercise 2 – Model Theory (1), 80616

December 1, 2025



Question 1

Part a

Let \mathbb{F} be a field and let T_{vec} be the theory of infinite vector space over the field \mathbb{F} with the language $L = \{0, +\} \cup \{\lambda_a \mid a \in \mathbb{F}\}$, such that λ_a represents multiplication by scalar $a \in \mathbb{F}$. We will show that T_{vec} satisfies quantifier elimination.

Proof. Let \mathcal{V}, \mathcal{W} are vector spaces over \mathbb{F} such that there is some finite $A \subseteq V \cap W$ and $\langle A \rangle = V \cap W$, and let \mathcal{A} be the model over $\langle A \rangle$.

Let $t(x)$ be a term over $L(A)$ (as every $a \in \langle A \rangle$ is definable in $L(A)$). Any definable $v(x) : \mathcal{A} \rightarrow \mathcal{A}$ is of the form $v = u + ax$, then $t(x) = u + \lambda_a x$ for some $u \in A$ and $a \in \mathbb{F}$.

Any atomic formula with single free variable is of the form $t(x) = s(x)$ or $t(x) \neq s(x)$, or equivalently $t(x) = 0$ or $t(x) \neq 0$ for some term. In turn, any \exists -primitive formula is of one of the forms,

$$\varphi(x) = \exists x \ t(x) = 0, \quad \varphi(x) = \exists x \ \bigwedge_{i < n} t_i(x) \neq 0$$

We know that $u \in V, W$ for each $u \in \langle A \rangle$, as a linear combination it follows that $\mathcal{V} \models u + av = 0$ for $v \in V$ if and only if $\mathcal{V} \models v = \frac{u}{-a}$. But $\lambda_{-1/a} \in \mathbb{F}$ as it is a field, and it follows that v is definable in $L(A)$ if $\mathcal{V} \models u + av = 0$, we deduce that $v \in \mathcal{W}$ as well. Then if φ is of the first form, then $\mathcal{V} \models \varphi \iff \mathcal{W} \models \varphi$, and it remains to check the second form.

If φ is of the second form, then for any $i < n$, $t_i(x) \neq 0$ is equivalent to $x \neq \frac{u_i}{-a_i} = c_i$ for $u_i \in A, a_i \in \mathbb{F}$, and by the assumption that T is of infinite we infer that $T \models \exists x (\bigwedge_{i < n} x \neq c_i)$, meaning that if $\mathcal{V}, \mathcal{W} \models \varphi$.

By quantifier elimination equivalently theorem, T is eliminating quantifiers.

If \mathcal{V} was not infinite (and we would change T as well) then the last step won't hold. In turn, we would have to divide into cases by the character of \mathbb{F} , if it would be 0 then the proof will hold. If on the other hand $\text{char } \mathbb{F} < \infty$, then the claim would not be true anymore. \square

Part b

Let $L = \{\leq\} \cup \{c_n \mid n < \omega\}$ be a language, and let $T = \text{DLO} \cup \{c_n < c_{n+1} \mid n < \omega\}$.

We will show that T has quantifier elimination, and find all non-isolated types in $S_1(T)$.

Proof. The proof is identical to the case of DLO, using back and forth method on two models $\mathcal{M}, \mathcal{N} \models T$. The key difference is that if $A \subseteq M \cap N$ with $|A| < \omega$ then the back and forth isomorphism construction has to start with,

$$(\{c_i^{\mathcal{M}} \mid i < \omega\} \cup \{c_i^{\mathcal{N}} \mid i < \omega\}) \cap A$$

This way we get an isomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ over $L(A)$, and in particular $\mathcal{M}_A \models \psi \iff \mathcal{N}_A \models \psi$ as ψ is existential primitive and being preserved by φ . \square

We will find all non-isolated types in $S_1(T)$.

Solution. Let $p \in S_1(T)$ be some type, and let c be new constant such that $\varphi(c)$ is true. Then $\varphi(x) = c = c_i$ for some $i < \omega$ is a type, but it is (by definition) isolated, and thus we can assume p does not consist of such formulas. T has only c_i as constant symbols, then any type such that $c_i \leq x \leq c_{i+1} \in p$ is isolated, but if all of the formulas are of the form $c_i \leq x$ we get partial type. We can deduce that there are no non-isolated types in $S_1(T)$.

Question 2

Let T_{EQ} be the theory of equivalence relation over $L = \{E\}$. We will show that T_{EQ}^* , the model companion of T_{EQ} , is the theory of equivalence relation with infinitely many infinite equivalence classes. Moreover, we will show that T_{EQ}^* has quantifier elimination.

Proof. Notice that,

$$T_{EQ} = \{\forall x E(x, x), \forall x \forall y (E(x, y) \leftrightarrow E(y, x)), \forall x \forall y \forall z (E(x, y) \wedge E(y, z)) \rightarrow E(x, z)\}$$

meaning that T_{EQ} consists of global sentences only, as a corollary from class T_{EQ}^* exists uniquely.

By proposition from class we know that T_{EQ}^* is closed existentially closed over $T_V = T$. Let us observe,

$$\varphi_n = \exists x_0 \cdots \exists x_{n-1} (\bigwedge_{i < j < n} x_i \neq x_j \wedge E(x_i, x_j))$$

then $\varphi_n \in T_{EQ}^*$ for any $n < \omega$, meaning that any equivalence class is infinite. We also define,

$$\psi_n = \exists x_0 \cdots \exists x_{n-1} (\bigwedge_{i < j < n} \neg E(x_i, x_j))$$

then $T_{EQ}^* \models \psi_n$ as well for any $n < \omega$, meaning that there are infinite equivalence classes.

We move to show that T_{EQ}^* eliminating quantifiers. The proof is by using the equivalence theorem and by defining $A \subseteq M, N$ as in other proofs. In this case the primitive existential formula over $L(A)$ will be of the form $\exists x E(x, a_0) \wedge \cdots \wedge E(x, a_{n-1}) \wedge \neg E(x, a_n) \wedge \cdots \wedge \neg E(x, a_{n+m-1})$, and formulas over equation symbols. In turn we can use the properties of T_{EQ}^* to show $\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi$ in the exact way of question 1. \square

Question 3

We will show that there is a complete theory T over a countable language, and a collection of 2^{\aleph_0} non-isolated types in $S_1(T)$ such that every model $\mathcal{M} \models T$ satisfies at least one of these types.

Proof. Let $L = \{P_i \mid i < \omega\}$ be a language consists of countable many unary relation symbols. Let,

$$T = \left\{ \varphi_{A,B} = \exists x \left(\bigwedge_{i \in A} P_i(x) \right) \wedge \left(\bigwedge_{j \in B} \neg P_j(x) \right) \mid A, B \subseteq \omega, |A|, |B| < \omega, A \cap B = \emptyset \right\}$$

be the theory such that for any finite selection of predicates, there is an element that is true for them for any choice.

We will show that T is complete. In intention of showing quantifier elimination, let $\mathcal{M}, \mathcal{N} \models T$ be some models and let $A \subseteq M, N$ be some finite set such that $A \subseteq \mathcal{M}, \mathcal{N}$. Let $\varphi(x)$ be some primitive existential formula over $L(A)$, then,

$$\varphi = \exists x \left(\bigwedge_{i < n_0} P_i(x) \right) \wedge \left(\bigwedge_{j < n_1} \neg P_j(x) \right) \wedge \left(\bigwedge_{k < n_2} x = a_k \right) \wedge \left(\bigwedge_{l < n_3} x \neq a_l \right)$$

which means that $\varphi \equiv \varphi_{C,B}$ or $\varphi \equiv \exists x x = a$ or $\varphi \equiv \exists x, x \neq a_0 \wedge \dots \wedge x \neq a_{n-1}$. $\mathcal{M}, \mathcal{N} \models \varphi_{C,B}, \exists x x = a$ for any C, B, a from T and A definition. The last case is achieved by choosing a witness in each model of $\varphi_{C,B}$, such that $C \supseteq \bigcup C_i, B \supseteq \bigcup B_i$, when a_i are witnesses of φ_{C_i, B_i} for $i < n$. By equivalently to quantifier elimination T is such theory, and has no constant symbols, then it is complete.

Let $f : \omega \rightarrow 2$ be a function, and let us consider the type,

$$p_f(x) = \{\varphi_{A,B} \mid A = f^{-1}(1) \cap [n], B = f^{-1}(0) \cap [n], n < \omega\}$$

type such that $T_{\{c\}} \models \varphi(c)$ is true if and only if $P_n(c) \iff f(n) = 1$. T is complete and c has a sentence defining its value for any P_n , therefore $p_f \in S_1(T)$.

We will now show that p is not isolated. Let $M = N = \{f \in \{0, 1\}^{[m]} \mid m < \omega\}$ and,

$$P_i^M(f) \iff i \in \text{dom } f \wedge f(i) = 1, \quad P_i^N(f) \iff i \in \text{dom } f \vee f(i) = 1$$

We now define the types p_{c_0} and p_{c_1} , then $f = \{(0, 0)\} \in M, N$, and,

$$\mathcal{M} \models p_{f \cup \{\langle n, 0 \rangle \mid 0 < n < \omega\}}(f), \mathcal{M} \not\models p_{f \cup \{\langle n, 1 \rangle \mid 0 < n < \omega\}}(f), \mathcal{N} \not\models p_{f \cup \{\langle n, 0 \rangle \mid 0 < n < \omega\}}(f), \mathcal{N} \models p_{f \cup \{\langle n, 1 \rangle \mid 0 < n < \omega\}}(f)$$

Then p_f cannot be isolated, as otherwise there would be undecidable sentence in T . The above claim can be extended to any arbitrary function using appropriate construction of models, we infer that p_f is non-isolated for all $f \in \{0, 1\}^\omega$.

Let $\mathcal{M} \models T$ be some model. $\mathcal{M} \models \exists x \varphi_{\{0\}, \emptyset}$, and let $d \in M$ be a witness. Let $f : \omega \rightarrow \{0, 1\}$ be a function such that $f(n) = 1 \iff P_n^M(d)$ ($f(0) = 1$ by definition). $\mathcal{M} \models p_f(d)$ directly from the definition of p_f . \square

Question 4

Let L and T be as in the last question. We will show that T is complete and that there is no isolated type in $S_1(T)$.

Proof. Was proved in the last question. □

Question 5

We will show that there is a complete theory T over the language L such that $|L| = \aleph_1$ and a non-isolated type $p(c)$ that cannot be omitted.

Proof. Let $L = \{P\} \cup \{E_f \mid f : \omega_1 \leftrightarrow \omega_1\} \cup \{c_i \mid i < \omega\} \cup \{d_i \mid i < \omega_1\}$, where P is an unary predicate, E_f is binary predicate for any $f : \omega_1 \leftrightarrow \omega_1$, c_i, d_i are constant symbols for any i . Let us define the theory,

$$\begin{aligned} T = & \{\forall x (P(x) \rightarrow \exists!y(\neg P(y) \wedge E_f(x, y))), \forall y (\neg P(y) \rightarrow \exists!x(P(x) \wedge E_f(x, y))) \mid f : \omega \leftrightarrow \omega_1\} \\ & \cup \{c_i \neq c_j \mid i < j < \omega\} \cup \{d_i \neq d_j \mid i < j < \omega_1\} \cup \{P(c_i) \mid i < \omega\} \cup \{\neg P(d_i) \mid i < \omega_1\} \\ & \cup \{E_f(c_n, d_{f(n)}) \mid f : \omega_1 \leftrightarrow \omega_1, n < \omega\} \\ & \cup \{\exists x E_f(x, d_\alpha) \wedge E_g(x, d_\beta) \mid f, g : \omega_1 \leftrightarrow \omega_1, \alpha, \beta < \omega_1, f^{-1}(\alpha) = g^{-1}(\beta)\} \end{aligned}$$

the theory such that if $\mathcal{M} \models T$ then $E_f^{\mathcal{M}} : A_M \rightarrow B_M$, for $A_M = \{x \in M \mid P^{\mathcal{M}}(x)\}, B_M = \{y \in M \mid \neg P^{\mathcal{M}}(y)\}$, is a bijection. It also follows from definition that $\{c_i^{\mathcal{M}} \mid i < \omega\} \subseteq A_M$ and $\{d_i^{\mathcal{M}} \mid i < \omega_1\} \subseteq B_M$, therefore $|B_M| \geq \aleph_1$. $E_f^{\mathcal{M}}$ is a bijection, implying that $|A_M| = |B_M|$.

We will show that T is complete as a theory that eliminates quantifiers and has a common embeddable model. Let $\mathcal{M}, \mathcal{N} \models T$ be some models, $A \subseteq M \cap N$ finite, and φ be some primitive existential formula over $L(A)$. If φ consists of the formula $x = c_i$ or $x = d_i$ for some i then $\mathcal{M}_A \models \varphi \iff \mathcal{N}_A \models \varphi$ and we can assume otherwise. In similar manner, if $E_f(c_i, x), E_f(x, d_i)$ are found in φ then both models fulfill φ together, thus we assume otherwise as well. Any other case is that φ consists of conjunction of negated atomic formulas and due to size consideration fulfill the requirement of quantifier elimination equivalence theorem. It follows that T indeed has quantifier elimination.

Let $\mathcal{M}_0 \models T$ be the model such that $M = 2 \times \omega_1$, $\mathcal{M}_0 \models P(\langle a, b \rangle) \iff a = 0, E_f(x, y) \iff f(x) = y$, and $c_n^{\mathcal{M}_0} = \langle 0, n \rangle, d_i^{\mathcal{M}_0} = \langle 1, i \rangle$. Each $e \in M_0$ is definable using some sentence, as either $e = d_i^{\mathcal{M}_0}$ for some $i < \omega_1$, or $E_f^{\mathcal{M}_0}(e, d_i)$, therefore $\mathcal{M}_0 \subseteq \mathcal{M}$ for any $\mathcal{M} \models T$. Therefore T is complete.

We move to define a type that cannot be isolated nor being omitted. Let,

$$p(x) = \{P(x)\} \cup \{x \neq c_i \mid i < \omega\}$$

p is complete, $T \cup p$ is equivalent to the theory where there are c_n constant symbols for any $n < \omega + 1$, by the last claim $T \cup p$ is complete. T eliminates quantifiers implying that if p can be isolated then there is a quantifier-free formula $\varphi(x)$ such that $\forall x (\varphi \rightarrow p)$. We can deduce from the definition of p that φ does not consist of $x = c_i$ or $x = d_i$, leaving only the option that $E_f(x, d_i)$ appears in φ . In this case we can show that there is a contradiction directly by taking \mathcal{M}_0 and observe that both \mathcal{M}_0 thinks both $p(\langle 0, \omega \rangle)$ and $p(\langle 0, \omega + 1 \rangle)$.

This fact also shows that p cannot be omitted, as $\mathcal{M}_0 \overset{\iota}{\hookrightarrow} \mathcal{M}$ and therefore $\mathcal{M} \models p(\iota(\langle 0, \omega \rangle))$. \square