Exercise 3 Answer Sheet — Logic Theory (2), 80424

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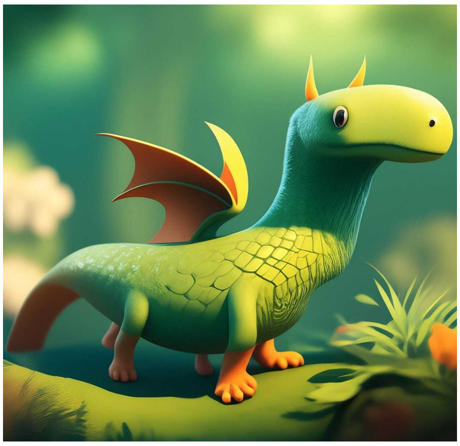

Question 1

Let $T = \mathrm{Th}(\hat{\mathbb{N}})$ We will show that T has 2^ω non-isomorphic countable models.

Proof. Let us define $P \subseteq \hat{\mathbb{N}}$ the set of prime numbers. We assume that $S \subseteq P$ is some infinite set. We enrich the language by a new constant symbol, c. Let $T_S = T \cup \{\exists x, (x \cdot p = c) \wedge \neg(\exists y, y \cdot p = x) \mid p \in S\}$, namely extension of T such that $c = \prod_{p \in S} p$. T_S is finitely satisfiable by $\hat{\mathbb{N}}$ as for every finite S, $\prod_{p \in S} p \in \mathbb{N}$, then by compactness T_S is also satisfiable. It follows that there is a model $\mathcal{M}_S \models T_S$ over $L \cup \{c\}$, then it is also a model of T by reduction of the language.

Let $S \neq S'$ be subsets of P and let $\mathcal{M}, \mathcal{N}$ be some of their respective models. We will show that $\mathcal{M} \not\simeq \mathcal{N}$. It is clear for $L \cup \{c\}$-isomorphisms, as there is $p \in S \setminus S'$ (without loss of generality) such that $\mathcal{M} \models p \mid c$ but $\mathcal{N} \not\models p \mid c$. Note that p is definable, let us define $c' \in M$ as the element such that $c' = c^{\mathcal{M}}$. If there was an L-isomorphism f between $\mathcal{M}$ and $\mathcal{N}$, then $f(c')$ would not satisfy the formula $\varphi(x) = p \mid x$, in contradiction to $\mathcal{M} \models \varphi(c')$ and the fact that f is a models isomorphism.

We know that $P \subseteq \mathbb{N}$, and that this set is not finite, then $|P| = \omega$, it derives that there are $|2^P| = 2^\omega$ non-isomorphic models $\mathcal{M}_S \models T$. □

Question 2

Let F be an infinite field, and let $L_{F\,\text{VS}}$ be the language of vector spaces over F. Let $V \subseteq U$ be two non-trivial vector spaces over F, we will show that $V \prec U$.

Proof. Let P be a $L_{F\,\text{VS}}$-disjoint unary relation symbol to $L_{F\,\text{VS}}$, and let us define $L' = L_{F\,\text{VS}} \cup \{P\}$. We extend U to L' by $P^U(x) \iff x \in V$, namely U'. For every $\psi(x_0, \ldots, x_{n-1}) \in \text{form}_{L\,\text{VS}}$, let $\varphi_\psi(x_0, \ldots, x_{n-1}) = \psi(x_0, \ldots, x_{n-1}) \wedge P(x_0) \wedge \ldots \wedge P(x_{n-1})$, a formula such that $U' \models \psi \iff V \models \psi$. It follows that there is a theory T such that $U' \models T \iff V \prec U$, meaning that T testifies to elementary substructure of V in U.

Assuming that $|F| = \kappa$ we use upward Löwenheim–Skolem theorem to extend U' to a model U'' such that $|U''| > \kappa$ and $U' \prec U''$. We define $V'' = P^{U''}$, meaning that V'' is the extension of V. We note that $|F| < |U''|, |V''|$, hence both spaces are infinite dimensional. From question 1 of exercise 2 we infer that indeed $V'' \prec U''$, implying that $U'' \models T$, then $U' \models T$ as well, which lead us to the conclusion that $V \prec U$, as intended. $\square$

Question 3

Let $\mathcal{M}$ be the model $\langle \mathbb{Q}(\pi); + \rangle$, we will show that the function $f_\pi : \mathbb{Q}(\pi) \to \mathbb{Q}(\pi)$ such that $f_\pi(u) = \pi u$ is not definable in $\mathcal{M}$.

Proof. We will show that there is no formula $\varphi(x, y) \in \mathrm{form}_L$ such that f_π is definable as the separation of φ on M, meaning that $\varphi(x, y) \iff y = \pi \cdot x$. We assume toward a contradiction that such a formula indeed exists. Let $\mathcal{N} \succ \mathcal{M}$ be an elementary extension such that $|M| < |N|$, namely a model of larger cardinality, which derived from Löwenheim–Skolem theorem. We fix some $e \in N \setminus M$, there is indeed one as $|N \setminus M| = |N|$. Let $B = M \cup \{e\}$ and $\mathcal{N}' = \operatorname{Sp} B$, the vector space over the field $\mathbb{Q}$. It directly follows that $M \subseteq N'$, as well the language enrichment $\mathcal{M}'$ of $\mathcal{M}$ to vector spaces over $\mathbb{Q}$ fulfills $\mathcal{M}' \prec \mathcal{N}'$ by the last question. We reduce the language to infer that $\mathcal{M} \prec \mathcal{N}'$ as well. We assumed that $\mathcal{M} \models \forall x \exists y, \varphi(x, y)$ (the function condition over the formula φ), it follows that $\mathcal{N}' \models \forall x \exists y, \varphi(x, y)$ as well. In particular, $\mathcal{N}' \models \exists y, \varphi(e, y)$, meaning that $y = \pi \cdot e$, but $e \notin M$ and is not closed to multiplication by π, which implies that $\pi \cdot e \notin \mathcal{N}'$, a contradiction to the assumption that φ exists. $\square$

Question 4

Part a

We will show that PA proves that the addition is associative and commutative.

Proof. we will prove that

$$\text{PA} \models \forall x, y(S(x+y) = S(x) + y) \tag{C1}$$

by induction over y. For $y = 0$ it derives that $S(x+0) = S(x) = S(x)+0$ from axiom N3. If PA $\models \forall x, (S(x+n) = S(x)+n)$ then,

$$\begin{aligned}\forall x, (S(x+n) = S(x) + n) &\iff \forall x, (S(S(x+n)) = S(x) + S(n)) \\ &\iff \forall x, (S(x + S(n)) = S(x) + S(n))\end{aligned}$$

Directly by axiom N4.

Let $\varphi(y) = \forall x, z(x + (y + z) = (x + y) + z)$, and $\psi = \forall y \psi(y)$ be the sentence that testifies to associativity, we will show that PA $\models \psi$, by showing that $\text{Ind}(\varphi)$ holds. PA $\models \varphi(0)$ directly by the axioms N3 and N4. Let us assume that $\varphi(n)$ holds, we will show that $\varphi(S(n))$ holds as well,

$$\begin{aligned}\text{PA} \models \varphi(n) &\iff \forall x, z(x + (n + z) = (x + n) + z) \\ &\overset{\text{N1}}{\iff} \forall x, z(S(x + (n + z)) = S((x + n) + z)) \\ &\overset{\text{N4, C1}}{\iff} \forall x, z(x + S(n + z) = S(x + n) + z) \\ &\iff \forall x, z(x + (S(n) + z) = (x + S(n)) + z) \\ &\iff \varphi(S(n))\end{aligned}$$

We conclude by induction axiom scheme that indeed PA $\models \psi$.

Let $\varphi(x) = \forall y(x + y = y + x)$ and $\psi = \forall x, \varphi(x)$, sentence such that PA $\models \psi$ if and only if PA is commutative. We will show that PA $\models \text{Ind}(\varphi)$. For $x = 0$, $0 + y = S(0 + y') = \cdots = S(S(\cdots S(0 + 0))) = y = y + 0$ by axioms N3 and N4. Let us assume that $\varphi(n)$ holds, then,

$$\begin{aligned}\text{PA} \models \varphi(n) &\iff \forall y(n + y = y + n) \\ &\overset{\text{N1}}{\iff} \forall y(S(n + y) = S(y + n)) \\ &\overset{\text{N4, C1}}{\iff} \forall y((S(n) + y) = y + S(n)) \\ &\iff \text{PA} \models \varphi(S(n))\end{aligned}$$

It follows that $\text{Ind}(\varphi)$ holds, meaning that PA $\models \psi$. $\square$

Part b

We will show that PA proves that $(+, \cdot, \leq)$ is a semi-ring, namely $+, \cdot$ are associative and commutative, $0, 1$ are neutral and $\leq$ is linear order.

Proof. associativity and commutativity of $+$ was shown in the first part, the proof for $\cdot$ is similar. By N3 and N5 we infer that $0, 1$ are indeed neutral elements.

We move to show $\leq$ is a linear order. From N9, $\leq$ is reflexive. To show transitivity we use induction, for the base N7 and for the step iterative usage of N8. For linearity we use N9. We also need to show that both $+$ and $\cdot$ are order-preserving for $\leq$. This statement is proven in induction over the added value to both sides of every inequality. $\square$