Exercise 7 Answer Sheet — Logic Theory (2), 80424

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Question 1

For any $\Gamma \subseteq \text{form}_{L_{\text{PA}}}$ let $I\Gamma$ be the induction scheme over Γ.
For $\varphi(z, y, t_0, \dots, t_{n-1}) \in \text{form}_{L_{\text{PA}}}$ we define

$$\text{Coll}(\varphi) = \forall z \forall t_0 \dots \forall t_{n-1} (\forall x \leq z \exists y \varphi \leftrightarrow \exists w \forall x \leq z \exists y \leq w \varphi)$$

By the last exercise, $\text{PA} \vdash \text{Coll}(\varphi)$ for any such φ.
Let $\text{Coll} = \{\text{Coll}(\varphi) \mid \varphi \in \text{form}_{L_{\text{PA}}}\}$, Q be the first 9 axioms of PA, and $\text{PA}_0 = Q \cup I\Sigma_0 \cup \text{Coll}$.
We will show that $\text{PA}_0 \vdash \text{PA}$.

Proof. For $Q \subseteq \text{PA}$ we assumed that $\text{PA}_0 \vdash Q$, then it is sufficient to show that $\text{PA}_0 \vdash \text{Ind}(\varphi)$ for any $\varphi \in \text{form}_{L_{\text{PA}}}$.
If $\varphi \in \Sigma_0$ then $\text{PA}_0 \vdash \text{Ind}(\varphi) \ni I\Sigma_0$.
In the previous exercises we have shown that for any φ there is n such that $\varphi \in \Sigma_n^0$, and that there is $\psi \in \Sigma_n$, for which, $\text{PA} \vdash (\varphi \leftrightarrow \psi)$. We will prove in induction that $\text{PA}_0 \vdash (\varphi \leftrightarrow \psi)$ as well, and that $\text{PA}_0 \vdash \text{Ind}(\varphi)$ in such case. The case of $n = 0$ was proven in the last exercise.
Let us assume that $\varphi \in \Sigma_n(\text{PA})$ and $\psi \in \Sigma_n$ such that $\text{PA} \vdash (\varphi \leftrightarrow \psi)$ and $\psi = \exists v \phi$ for $\phi \in \Pi_{n-1}$. By the induction hypothesis ψ fulfills the claim, meaning that $\text{PA}_0 \vdash \text{Ind}(\psi)$. We will show that $\text{PA}_0 \vdash (\varphi \leftrightarrow \psi)$ and that $\text{PA}_0 \vdash \text{Ind}(\varphi)$.
As for the first part, by $\text{Coll}(\phi)$ and negation we infer that $\text{PA}_0 \vdash (\varphi \leftrightarrow \psi)$ as intended, and it is left to show that $\text{PA}_0 \vdash \text{Ind}(\psi)$. The proof is the same as of question 1 of exercise 6.
Let $\mathcal{M} \models \text{PA}_0$ be some model and we assume that $\exists x \phi(x, y, z)$ satisfies the induction condition, meaning that $\mathcal{M} \models \phi(x, 0, c)$ for some $c \in M$; as well,

$$\mathcal{M} \models \forall y (\exists x \varphi(x, y, c)) \to (\exists x \varphi(x, S(y), c))$$

We want to show that $\mathcal{M} \models \forall y \exists x \varphi(x, y, c)$.
Let $b \in M$ be some arbitrary element, we want to show that,

$$\mathcal{M} \models \exists x \varphi(x, b, c) \tag{1}$$

thus proving the last statement. By using $\text{Coll} \subseteq \text{PA}_0$ we infer that

$$\mathcal{M} \models \forall b' \leq b \exists x \varphi(x, b', c) \tag{2}$$

if and only if exists $d \in M$ such that,

$$\mathcal{M} \models \forall b' \leq b \, \exists x \leq d \, \varphi(x, b', c) \tag{3}$$

In the set-theoretical sense, let $X = \{x \mid \mathcal{M} \models \exists b' \leq b, \ \varphi(x, b', c)\}$, this is a finite set as the order induced on M by $\leq^{\mathcal{M}}$ is a well-order. Let us select $d = \sup_{\leq^{\mathcal{M}}} X$, this d fulfills (3), thus also proving (2), but as $b \in M$ is arbitrarily chosen, (1) holds as well, completing our proof. $\square$

Question 3

We will show that Sat_0 is primitive-recursive.

Proof. In the lectures we saw that Sat_0 is total recursive. $\square$