

## Solution to Exercise 4 — Model Theory (1), 80616

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## Question 1

**Definition 0.1** (algebraic formula). Let  $\mathcal{M}$  be a structure. A formula  $\varphi(x) \in \text{form}_{L_{\mathcal{M}}}$  is called algebraic if,

$$|\{a \in M \mid \mathcal{M} \models \varphi(a)\}| < \aleph_0$$

**Definition 0.2** (definable element). An element  $a \in M$  is called definable if there is a formula  $\psi(x)$  such that  $\mathcal{M} \models \psi(x) \iff x = a$  for any  $x \in M$ .

**Definition 0.3** (acl and dcl). If  $\mathcal{M}$  is a structure and  $A \subseteq M$  a set of elements, then let,

$$\text{acl}_{\mathcal{M}}(A) = \bigcup \{\varphi(M) \mid \varphi \in \text{form}_{L_A}, \varphi \text{ is algebraic}\}, \quad \text{dcl}_{\mathcal{M}}(A) = \{a \in M \mid a \text{ is definable using an } L_A \text{ formula}\}$$

be the set of elements that has a formula with parameters from  $A$  that is algebraic and are true for them, and the set of elements uniquely definable by formulas with parameters.

### Part a

Let  $A \subseteq M$  for  $\mathcal{M} \prec \mathcal{N}$ . We will show that  $\text{acl}_{\mathcal{M}}(A) = \text{acl}_{\mathcal{N}}(A)$  and  $\text{dcl}_{\mathcal{M}}(A) = \text{dcl}_{\mathcal{N}}(A)$ .

*Proof.* Let  $a \in M$  be  $\in \text{acl}_{\mathcal{M}}(A)$  and let  $\varphi \in \text{form}_{L_A}$  be an algebraic formula to witness it.  $\mathcal{M} \models \varphi(a) \iff \mathcal{N} \models \varphi(a)$  by the given elementary embedding and the fact that  $A \subseteq N$ .

Let  $a \in \text{acl}_{\mathcal{N}}(A)$  and assume toward contradiction that  $a \notin \text{acl}_{\mathcal{M}}(A)$ , meaning that there is  $\varphi(x)$  over  $L_A$  such that  $\mathcal{N} \models \varphi(a)$  and  $\varphi$  is algebraic. Let  $B = A \cup \varphi(M)$ , denote  $\varphi(M) = \{b_i \mid i < N\}$  for  $N < \omega$ ,

$$\psi(x) = \varphi \wedge \left( \bigwedge_{i < N} x \neq b_i \right)$$

Then  $\psi$  is algebraic over both  $\mathcal{M}$  and  $\mathcal{N}$  and  $\psi(M) = \emptyset$ , while  $a \in \psi(N)$ . Let  $\phi = \exists x \psi$ , then  $\mathcal{N} \models \phi$  as  $a$  witnesses exactly this, while  $\mathcal{M} \models \neg \phi$  in contradiction to  $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$ .

The case for dcl is equivalent, we take  $\varphi(x)$  such that  $\varphi(M) = \{a\}$  and therefore  $\varphi(N) = \{a\}$  as otherwise we can define  $\varphi \wedge (x \neq a)$ . □

We conclude that the relative model of acl and dcl can be omitted.

### Part b

Let  $A \subseteq M$ , we will show that  $\text{acl}(\text{acl}(A)) = \text{acl}(A)$  and  $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$ .

*Proof.* Let  $a \in \text{dcl}(\text{dcl}(A))$  be an element and let  $\varphi(x)$  be its witness. Suppose that  $\{b_i \mid i < N\}$  is the set of elements that  $\varphi$  consists of such that they are not in  $A$ .  $b_i \in \text{dcl}(A)$  for any  $i$ , and let  $\theta_i(x)$  witness that. Let  $\{y_i \mid i < N\}$  be a set of distinct and disjoint to  $\varphi$  variables, and let us define,

$$\varphi' = \varphi_{y_0, \dots, y_{N-1}}^{b_0, \dots, b_{N-1}}, \quad \phi(x) = \exists y_0 \cdots \exists y_{N-1} \left( \bigwedge_{i < N-1} \theta(y_i) \right) \wedge \varphi'$$

then  $\phi \in \text{form}_{L_A}$  and  $\phi(M) = \{a\}$ , therefore  $a \in \text{dcl}(A)$  and thus  $\text{dcl}(\text{dcl}(A)) = \text{dcl}(A)$ .

The case of acl is equivalent. □

### Part c

Let  $L = \{0, 1, +, \cdot\}$  and  $K$  be an algebraically closed field and let  $A \subseteq K$ .

We will compute  $\text{acl}(A)$  and  $\text{dcl}(A)$ .

*Solution.*  $K$  has quantifier elimination therefore we can omit the discussion about quantifiers. Each formula  $\varphi(x) \in \text{form}_{L_A}$  is (without loss of generality) of the form,

$$\varphi = \left( \bigwedge_{i < n} p_i(x) = 0 \right) \wedge \left( \bigwedge_{j < m} p_{n+j}(x) \neq 0 \right)$$

where  $p_i(x)$  is a polynomial over  $A$ . An element  $a \in K$  is  $\in \text{dcl}(A)$  if it is the unique solution of some such polynomial. Each polynomial in algebraically closed field can be deconstructed to product of linear polynomials, meaning that  $p_i(x)$  has a single solution if  $p_i(x) = (x - a)$ . We conclude that  $\text{dcl}(A) = \text{cl}_{+, \cdot}(A)$ .

By similar proposition,  $\text{acl}(A) = \langle A \rangle \leq K$ , meaning the subfield generated by  $A$ .

## Question 2

### Part a

Let  $\kappa$  be an infinite cardinal and let  $\lambda \leq \kappa$  be the minimal cardinal such that  $\kappa^\lambda > \kappa$ . Consider the order  $(\kappa^{<\lambda}, <_l)$  the lexicographic order on  $\kappa^{<\lambda}$ .

We will show that this order has more than  $\kappa$  many cuts and construct dense linear order of cardinality  $\kappa$  with  $> \kappa$  different cuts.

*Proof.* A cut is defined as a collection  $X \subseteq \kappa^{<\lambda}$  such that if  $y \in \kappa^{<\lambda}$  and there is  $x \in X$  such that  $y < x$ , then  $y \in X$ . Let  $\delta \in \kappa^\lambda$  be a sequence, and let,

$$\eta_\delta = \{x \in \kappa^{<\lambda} \mid \exists \alpha < \lambda, x < \delta \restriction \alpha\}$$

notice that if  $x < \delta \restriction \alpha$  then for any  $\alpha < \beta$  also  $x < \delta \restriction \beta$ , thus the definition of  $\epsilon_\delta$  holds. We also note that  $\epsilon_\delta$  is a cut of  $\epsilon^\delta$  as  $y < x < \delta \restriction \alpha \implies y < \delta \restriction \alpha$ .

Let  $\delta \neq \gamma \in \kappa^\lambda$  and assume that  $\beta$  is the least ordinal such that  $\delta(\beta) \neq \gamma(\beta)$  and  $\delta(\beta) > \gamma(\beta)$  without loss of generality. It follows that  $\delta \restriction \beta \in \eta_\delta$  but  $\delta \restriction \beta \notin \eta_\gamma$ , therefore  $\eta_\delta \neq \eta_\gamma$ . The injection  $\delta \mapsto \epsilon_\delta$  witness  $|\kappa| < |\kappa^\lambda| \leq |H|$  for  $H \subseteq \kappa^{<\lambda}$  the collection of cuts.

Let  $X = \{f \in \kappa^\lambda \mid \exists \alpha < \lambda \forall \alpha < \beta < \lambda, f(\beta) = 0\}$ , then there is a bijection of  $X$  and  $\kappa^{<\lambda}$  and therefore  $|X| = \kappa$ . The proof that  $|H_X| > \kappa$  remains the same in this case, and we will show that  $\langle X, <_l \rangle$  is dense. If  $f, g \in X$  with  $f <_l g$  then  $f$  is not an initial segment of  $g$  and therefore there exists least  $\alpha$  such that  $f(\alpha) < g(\alpha)$ . We can define,

$$h(x) = \begin{cases} f(x) & x \neq \alpha + 1 \\ f(x) + 1 & x = \alpha + 1 \end{cases}$$

Then  $f <_l h$  but  $h(\alpha) < g(\alpha)$  therefore  $f <_l h <_l g$ , as intended.  $\square$

### Part b

Let  $T$  be a theory and let us assume that there is  $\mathcal{M} \models T$ , a sequence  $\langle \bar{a}_n \mid n < \omega \rangle \subseteq M^k$  and a formula  $\varphi(\bar{x}, \bar{y})$  such that for any  $n \neq m$ ,  $\mathcal{M} \models \varphi(\bar{a}_n, \bar{a}_m) \iff n < m$ .

We will show that  $T$  is not  $\kappa$ -stable for any  $\kappa \geq |T|$ .

*Proof.* We will show that there is  $A \subseteq \mathcal{C}_T$  with  $|A| \leq \kappa$ , such that  $\kappa < |S_1(A)|$ . Note that  $\mathcal{M} \prec \mathcal{C}_T$ , then  $\mathcal{C}_T \models \varphi(\bar{a}_n, \bar{a}_m) \iff n < m$ .

If  $T$  is  $\omega$ -stable then it is  $\kappa$ -stable for any  $\kappa \geq |T|$ , then it is sufficient to show that  $T$  is not  $\omega$ -stable. By the equivalency to  $\delta$ -stability theorem,  $T$  is  $\omega$ -stable if and only if  $|S_1(\emptyset)| \leq \omega$  and  $\forall n \in \mathbb{N}, |S_n(N)| \leq \omega$  for any  $\mathcal{N} \models T$ . Therefore if  $|S_k(M)| > \omega$  then  $T$  is not  $\kappa$ -stable.

We intend to define lexicographic order over  $\mathcal{M}$  using  $\varphi$ , assume that  $\langle \bar{b}_n \mid n < m \rangle$  for some  $m < \omega$ . Let us define the formula that extends  $\varphi$ 's order to  $m$ -tuples against  $m'$ -tuples,

$$\psi_{m,m'}(\bar{x}_0, \dots, \bar{x}_{m-1}, \bar{y}_0, \dots, \bar{y}_{m'-1}) = (m < m') \vee \varphi(\bar{x}_0, \bar{y}_0) \vee \dots \vee \varphi(\bar{x}_{m-1}, \bar{y}_{m-1})$$

Notice that  $\mathcal{M} \models \varphi(\langle \bar{a}_{n_i} \mid i < m \rangle, \langle \bar{a}_{n_j} \mid j < m' \rangle) \iff \langle n_i \mid i < m \rangle <_l \langle n_j \mid j < m' \rangle$ .

Let  $\langle \omega^{<\omega}, < \rangle$  be a dense linear order from the last part, and let  $H$  be the collection of cuts, we saw that  $|H| > \omega$ . Let  $F : \omega \rightarrow M^k$  be the map defined by  $n \mapsto \bar{a}_n$  and  $G : \omega^{<\omega} \rightarrow (M^k)^{<\omega}$  defined as  $G(f)(i) = F(f(i))$ .

Notate  $H_{\mathcal{M}} = G(H)$ , the collection of cuts of  $M^k$  in relation to the order induced by  $\psi$ . Given  $X \in H_{\mathcal{M}}$ , let us define  $p_X \in S_1(M)$  as,

$$p_X(\bar{x}) = \text{cl}_{\vdash} \{ \varphi(\bar{y}, \bar{x}) \mid \bar{y} \in X \}$$

Namely,  $\bar{x}$  is an  $k$ -tuple of elements such that acts as the supremum of  $X$ . It follows that there are  $|H_{\mathcal{M}}| = |H| > \omega$  such types, then  $|S_1(M)| > \omega$  as well, concluding our proof.  $\square$

### Part c

We will show that if  $L$  is a language with  $|L| \geq \kappa$  and  $T$  is a theory such that  $\forall n < \omega$ ,  $|S_n(\emptyset)| \leq \kappa$ , then there is  $L' \subseteq L$  with  $|L'| = \kappa$  such that for any  $\varphi(x_0, \dots, x_{n-1}) \in \text{form}_L$ , there is an  $L'$ -formula  $\varphi'(x_0, \dots, x_{n-1})$  with  $T \models \forall \bar{x} (\varphi \leftrightarrow \varphi')$ .

*Proof.* Let us observe  $tp(c) = \{\varphi(x) \mid T \models \varphi(c)\} \in S_1(\emptyset)$  for some constant symbol  $c \in L$ . If there were  $\{c_\alpha \mid \alpha < \lambda\}$  for  $\lambda > \kappa$  then by  $|\{tp(c_\alpha) \mid \alpha < \lambda\}| \leq |S_1(\emptyset)| \leq \kappa$ , we would get that up to equality in  $T$  there are at least  $\kappa$  constants, then let us add those constants to  $L'$ .

In similar manner, if  $R \in L$  is an  $n$ -placed relation symbol, we can define  $p_R(\bar{x}) = \{\varphi(\bar{x}) \mid T \models \varphi(\bar{x}) \rightarrow R(\bar{x})\}$ . We get up to  $\kappa$   $n$ -placed relation symbols  $\{R_\alpha^n \mid \alpha < \kappa\}$  such that there is  $\alpha < \kappa$  for which  $T \models \forall \bar{x} (R \leftrightarrow R_\alpha^n)$ . We enrich  $L'$  with  $\{R_\alpha^n \mid n < \omega, \alpha < \kappa\}$ , note that  $\kappa \cdot \kappa = \kappa$  therefore  $|L'| = \kappa$ .

We move to function symbols. Let  $F$  be an  $n$ -placed function symbol, and let,

$$p_F(\bar{x}) = \{\exists y \varphi(\bar{x}, y) \mid T \models \varphi(\bar{x}, y) \rightarrow (F(\bar{x}) = y)\}$$

and define in the same sense  $\{F_\alpha^n\}$ , the enrichment of  $L'$  by them finishes the proof.  $\square$

### Question 3

We say that  $\mathcal{M}$  is minimal if  $\mathcal{M}$  has no proper elementary substructure.

#### Part a

Let  $T$  be a countable and complete theory. We will show that if  $T$  has a prime model, then any minimal model is prime.

*Proof.* TODO

□