Solution to Exercise 0 — Model Theory (1), 80616

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Question 1

Definition 0.1. A formula φ is called *basic Horn* formula if,

$$\varphi = (\theta_0(\bar{x}) \wedge \cdots \wedge \theta_{m-1}(\bar{x})) \to \theta_m(\bar{x})$$

where θ_i is atomic for $i \leq m$.

Remark. In the case $m = 0$ we get $\varphi = \theta_0(\bar{x})$. In the case that $\theta_m = \bot$, $\varphi \equiv \neg\theta_0(\bar{x}) \vee \cdots \neg\theta_{m-1}(\bar{x})$.

Definition 0.2. The set of Horn formulas is the minimal set of formulas containing the basic Horn formulas and closed under conjugation and quantification.

Part a

Let $\psi(x_0, \ldots, x_{n-1})$ be a Horn formula and let $F \subseteq \mathcal{P}(I)$ be a filter. Let $\langle \mathcal{M}_i \mid i \in I \rangle$ be a sequence of structures and $a_j \in \prod_{i \in I} \mathcal{M}_i$ for $j < n$ such that,

$$\{i \in I \mid \mathcal{M}_i \models \psi(a_0(i), \ldots, a_{n-1}(i))\} \in F$$

We will show that $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / F \models \psi([a_0]_F, \ldots, [a_{n-1}]_F)$.

Proof. Let us prove by induction over Horn set.

Assume that $\varphi(\bar{x}) = P(\bar{x})$ and $\{i \in I \mid \mathcal{M}_i \models P(\bar{x})\} \in F$, then by definition $\mathcal{N} \models \varphi$. Let us assume that $\varphi = (\theta_0 \wedge \cdots \wedge \theta_{n-1}) \to \theta_n$ for atomic θ_n, then the claim holds directly from definition as in the last part.

The case where $\theta_m = \bot$ is equivalent as $\varphi \equiv \theta_0 \vee \cdots \vee \theta_{m-1}$.

We move to assume that the claim holds for φ, ψ and prove for $\varphi \wedge \psi$. This part of the proof is identical to the proof of Łoś theorem,

$$\{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}) \wedge \psi(\bar{a}(i))\} = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \cap \{i \in I \mid \mathcal{M}_i \models \psi(\bar{a}(i))\} \in F \implies \mathcal{N} \models \varphi, \psi \implies \mathcal{N} \models \varphi \wedge \psi$$

where the second equation is derived from filter definition.

We move to the case $\varphi = \exists x \, \psi$ for ψ that fulfills the claim. If $J = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \in F$ then for $j \in J$ we define $b_j \in M_j$ as a witness to $\mathcal{M}_j \models \psi(b_j, \bar{a}(j))$, meaning that,

$$\{i \in I \mid \mathcal{M}_i \models \psi(b(i), \bar{a}(i))\} = J \in F$$

where $b(i) = b_i$ for $i \in J$ and arbitrary otherwise. Then by the induction hypothesis $\mathcal{N} \models \psi([b], [\bar{a}])$ and therefore $\mathcal{N} \models \varphi([\bar{a}])$.

Lastly we will assume that $\varphi = \forall x \, \psi$ for ψ that fulfills the claim and show that φ does as well.

If $J = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \in F$, then if $b : I \to \bigcup M_i$ some choice function then,

$$\{i \in I \mid \mathcal{M}_i \models \psi(b(i), \bar{a})\} \supseteq J \in F$$

therefore $\mathcal{N} \models \psi([b], [\bar{a}])$, then $\mathcal{N} \models \varphi([\bar{a}])$ as required. $\square$

Part b

We will find a language and a sentence φ such that for any set I and filter F,
there is $\langle \mathcal{M}_i \mid i \in I \rangle$ such that $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / F \models \varphi$ if and only if F is an ultrafilter.

Solution. Let L be the language of equality, and. Maybe some theory that is ω-categorical.