Solution to Exercise 0 — Model Theory (1), 80616

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Question 1

Let $L = \{P\}$ a language where P is unary relation. Define,

$$\varphi_n = \exists x_0 \dots \exists x_n \left(\bigwedge_{i \leq n} P(x_i) \wedge \bigwedge_{i<j \leq n} x_i \neq x_j \right), \quad \psi_n = \exists x_0 \dots \exists x_n \left(\bigwedge_{i \leq n} \neg P(x_i) \wedge \bigwedge_{i<j \leq n} x_i \neq x_j \right)$$

and let $T = \{\varphi_n, \psi_n \mid n < \omega\}$.
We will show that $\mathrm{cl}_{\vdash} T$ is ω-categorical.

Proof. Let $\mathcal{M} \models T$ be some model. It can be proved by direct induction that $|P^{\mathcal{M}}| = \omega$ as well as $|\neg P^{\mathcal{M}}| = \omega$. Let us construct $f : \omega \to M$ such that $f(n) \in P^{\mathcal{M}}$ for any $n < \omega$. $\mathcal{M} \models \varphi_0 \iff \mathcal{M} \models \exists x\, P(x)$ then let $f(0)$ be such witness. Let us assume that $f \restriction n$ is defined, then $\mathcal{M} \models \varphi_{n+1}$, then by the pigeonhole principle there is some $a \in \mathcal{M}$ such that $a \notin f''n$, and let $f(n+1) = a$. For the sake of convenience let us redefine f as $2 \times \omega \to M$ injective function such that $f(0, n)$ is the same as $f(n)$ and $f(1, n) \notin P^{\mathcal{M}}$. By CSB we can assume that f is bijection as well, and by the selection of $\mathcal{M}$ as an arbitrary model of T we can deduce that for any $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \cong \mathcal{N}$ by composition of functions as was constructed. □

Question 2

Let $L = \{c_n \mid n < \omega\}$ be language consists of constant symbols. Let us define the theory $T = \{c_i \neq c_j \mid i < j < \omega\}$. We will show that there are countably many non-isomorphic countable models of T, and that T is complete.

Proof. Let us define the model $\mathcal{M}_n$ such that $M = \omega$ and,

$$c_i^{\mathcal{M}} = i + n$$

for any $i < j < \omega$,

$$c_i^{\mathcal{M}} = i + n \neq j + n = c_j^{\mathcal{M}}$$

therefore $\mathcal{M}_n \models T$. $\mathcal{M}_n \models k \neq c_i$ for all $i < \omega$, in particular $\mathcal{M}_n \models \exists x\, x \neq c_i$. It is implied that also,

$$\mathcal{M}_n \models \exists x_0 \ldots \exists x_{k-1} \left(\bigwedge_{i<j<k} x_i \neq x_j \wedge x_i \neq c_l \right) = \varphi_l^k$$

for all $l < \omega$. Finally, $\mathcal{M}_n \not\models \varphi_l^k$ for any $k > n$, we deduce that $\mathcal{M}_n \ncong \mathcal{M}_m$ for any $n \neq m$.

We move to show that T is complete. Let us assume toward a contradiction that φ is a sentence such that $\varphi \notin T$ and $T \cup \{\varphi\}$ is consistent. By construction of Henkin models we can deduce that $\mathcal{M}_0 \models \varphi$, but $\mathcal{M}_0$ is minimal, namely if $\mathcal{N} \models T$ then $\mathcal{M}_0 \subseteq \mathcal{N}$, then by definition $T \models \varphi$, a contradiction. $\square$

Question 3

We will show that $\mathrm{Th}(\mathbb{N}, +, \cdot)$ has $2^{\aleph_0}$ non-isomorphic countable models.

Proof. Let $f : \omega \to P$ be the map between number and its respective prime in the order induced from $\mathbb{N}$, namely $f(0) = 2, f(1) = 3, \ldots$. assuming that $A \subseteq \omega$ is some set, we define $\mathcal{M}_A = (\mathrm{cl}_{+,\cdot}(f(A)), +, \cdot)$.

$$\mathcal{M}_A \vDash \exists p \forall x \; p \cdot x = \underline{p} \cdot x$$

when $\underline{p}$ acts as numeral, and $\underline{p} \cdot x$ is abbreviation to $x + \cdots + x$ p times. Let us denote this sentence as φ_p, then,

$$\mathcal{M}_A \vDash \{\varphi_{f(a)} \mid a \in A\}$$

as well,

$$\mathcal{M}_A \nvDash \{\varphi_q \mid q \notin \mathrm{cl}_+ A\}$$

Therefore if $A, B \subseteq \omega$ and $A \neq B$ then $\mathcal{M}_A \ncong \mathcal{M}_B$, thus there are $|\mathcal{P}(\omega)|$ non-isomorphic countable models. $\square$

Question 4

Let $\kappa \geq \omega$ be some cardinal and let L be some language. Let T be a κ-categorical L-theory such that it has no finite models. We will show that T can be incomplete.

Solution. Let $L = \{c_\alpha \mid \alpha < \delta\} \cup \{P\}$ for $\kappa < \delta$, where c_α is a constant symbol and P is unary relation.

$$T = \{c_\alpha \neq c_\beta \mid \alpha < \beta < \delta\} \cup \{P(c_\alpha) \mid \alpha < \delta\}$$

It follows from the definition of T that if $\mathcal{M} \models T$ then $|M| \geq \delta > \kappa$, therefore there are no models of T of cardinality κ, then the theory is vacuously complete.

Just take $\kappa = \omega$ and L and T from question 1, we saw that T is ω-categorical and has no finite models. We also know that if $\mathcal{M} \models T$ and $|M| > \kappa$ then $|P^{\mathcal{M}}|, |\neg P^{\mathcal{M}}| \geq \omega$, and therefore it is possible that $|P^{\mathcal{M}}| = |M|$ or $|P^{\mathcal{M}}| < |M|$, then T cannot be complete.

Question 5

Let $T = \text{Th}(\mathbb{Q}, \leq)$ be DLO.
We will show that T is not κ-categorical for some uncountable cardinal κ.

Proof. The key here is similar to question 4, we can define $P(x) \iff x \leq c$ for some arbitrary value. This is equivalent to the theory of 4 and 1, and we just talked about why this is not necessarily κ-categorical. $\square$