Solution to Exercise 2 — Model Theory (1), 80616

November 22, 2025

Question 1

Part a

Let $\mathbb{F}$ be a field and let T_{vec} be the theory of infinite vector space over the field $\mathbb{F}$ with the language $L = \{0, +\} \cup \{\lambda_a \mid a \in \mathbb{F}\}$, such that λ_a represents multiplication by scalar $a \in \mathbb{F}$. We will show that T_{vec} satisfies quantifier elimination.

Proof. Let $\mathcal{V}, \mathcal{W}$ are vector spaces over $\mathbb{F}$ such that there exists $A \subseteq V \cap W$ and $\langle A \rangle = V \cap W$, and let $\mathcal{A}$ be the model over $\langle A \rangle$.

Let $t(x)$ be a term over $L(A)$ (as every $a \in \langle A \rangle$ is definable in $L(A)$). Any definable $v(x) : \mathcal{A} \to \mathcal{A}$ is of the form $v = u + ax$, then $t(x) = u + \lambda_a x$ for some $u \in A$ and $a \in \mathbb{F}$.

Any atomic formula with single free variable is of the form $t(x) = s(x)$ or $t(x) \neq s(x)$, or equivalently $t(x) = 0$ or $t(x) \neq 0$ for some term. In turn, any $\exists$-primitive formula is of one of the forms,

$$\varphi(x) = \exists x \; t(x) = 0, \quad \varphi(x) = \exists x \bigwedge_{i<n} t_i(x) \neq 0$$

We know that $u \in V, W$ for each $u \in \langle A \rangle$, as a linear combination it follows that $\mathcal{V} \models u + av = 0$ for $v \in V$ if and only if $\mathcal{V} \models v = \frac{u}{-a}$. But $\lambda_{-1/a} \in \mathbb{F}$ as it is a field, and it follows that v is definable in $L(A)$ if $\mathcal{V} \models u + av = 0$, we deduce that $v \in \mathcal{W}$ as well. Then if φ is of the first form, then $\mathcal{V} \models \varphi \iff \mathcal{W} \models \varphi$, and it remains to check the second form.

If φ is of the second form, then for any $i < n$, $t_i(x) \neq 0$ is equivalent to $x \neq \frac{u_i}{-a_i} = c_i$ for $u_i \in A, a_i \in \mathbb{F}$, and by the assumption that T is of infinite we infer that $T \models \exists x \left(\bigwedge_{i<n} x \neq c_i \right)$, meaning that if $\mathcal{V}, \mathcal{W} \models \varphi$.

By quantifier elimination equivalently theorem, T is eliminating quantifiers.

If $\mathcal{V}$ was not infinite (and we would change T as well) then the last step won't hold. In turn, we would have to divide into cases by the character of $\mathbb{F}$, if it would be 0 then the proof will hold. If on the other hand $\operatorname{char} \mathbb{F} < \infty$, then the claim would not be true anymore. $\square$

Part b

Let $L = \{\leq\} \cup \{c_n \mid n < \omega\}$ be a language, and let $T = \text{DLO} \cup \{c_n < c_{n+1} \mid n < \omega\}$.
We will show that T has quantifier elimination, and find all non-isolated types in $S_1(T)$.

Proof. The proof is identical to the case of DLO, using back and forth method on two models $\mathcal{M}, \mathcal{N} \models T$. The key difference is that if $A \subseteq M \cap N$ with $|A| < \omega$ then the back and forth isomorphism construction has to start with,

$$(\{c_i^{\mathcal{M}} \mid i < \omega\} \cup \{c_i^{\mathcal{N}} \mid i < \omega\}) \cap A$$

This way we get an isomorphism $\varphi : \mathcal{M} \to \mathcal{N}$ over $L(A)$, and in particular $\mathcal{M}_A \models \psi \iff \mathcal{N}_A \models \psi$ as ψ is existential primitive and being preserved by φ. $\square$

We will find all non-isolated types in $S_1(T)$.
Solution. Let $p \in S_1(T)$ be some type, and let c be new constant such that $\varphi(c)$ is true. Then $\varphi(x) = c = c_i$ for some $i < \omega$ is a type, but it is (by definition) isolated, and thus we can assume p does not consist of such formulas. T has only c_i as constant symbols, then any type such that $c_i \leq x \leq c_{i+1} \in p$ is isolated, but if all of the formulas are of the form $c_i \leq x$ we get partial type. We can deduce that there are no non-isolated types in $S_1(T)$.

Question 2

Let T_{EQ} be the theory of equivalence relation. We will show that T_{EQ}^*, the model companion of T_{EQ}, is the theory of equivalence relation with infinitely many infinite equivalence classes. Moreover, we will show that T_{EQ}^* has quantifier elimination.

Proof. TODO $\square$

Question 3

We will show that there is a complete theory T over a countable language, and a collection of $2^{\aleph_0}$ non-isolated types in $S_1(T)$ such that every model $\mathcal{M} \models T$ satisfies at least one of these types.

Proof. Let $L = \{P_i \mid i < \omega\}$ be a language consists of countable many unary relation symbols. Let,

$$T = \left\{ \varphi_{A,B} = \exists x \left(\bigwedge_{i \in A} P_i(x) \right) \wedge \left(\bigwedge_{j \in B} \neg P_j(x) \right) \mid A, B \subseteq \omega, |A|, |B| < \omega, A \cap B = \emptyset \right\}$$

be the theory such that for any finite selection of predicates, there is an element that is true for them for any choice.

We will show that T is complete. Assuming otherwise, there is $\varphi \in \mathrm{sent}_L$ such that both $T \cup \{\neg\varphi\}$ and $T \cup \{\varphi\}$ are consistent. Let $L' \subseteq L$ be the maximal language such that $\varphi \in \mathrm{sent}_{L'}$, and let $T' \subseteq T$ be the maximal theory over L'. Without loss of generality $L' = \{P_i \mid i < N\}$ for $N < \omega$, and $T' = \{\varphi_{A,B} \mid A, B \subseteq N, A \cap B = \emptyset\}$.

Every sentence $\varphi \in \mathrm{sent}_L$ is equivalent to sentence over $\varphi_{A,B}$ for some amount of them. Prove that. $\square$