Exercise 5 Answer Sheet — Logic Theory (2), 80424

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Question 1

Part a

Suppose that $\varphi(z, y, t_0, \dots, t_{n-1})$ is a formula in the language of L_{PA}. We will show that,

$$\text{PA} \vdash \forall z \forall t_0 \dots \forall t_{n-1}(\forall x \le z \exists y \varphi \leftrightarrow \exists w \forall x \le z \exists y \le w \varphi)$$

Proof. Let us fix $z, t_0, \dots, t_{n-1}$, then is suffice to show that $\text{PA} \vdash \forall x \le z \exists y \varphi \leftrightarrow \exists w \forall x \le z \exists y \le w \varphi$. We assume toward a contradiction that the formula does not hold, then $\forall x \le z \exists y \varphi$ as well $\neg \exists w \forall x \le z \exists y \le w \varphi$ holds as well. The latter is equivalent to $\forall w \exists x \le z \forall y \le w(\neg \varphi)$. We fix w and it derives that $\exists x \le z \forall y \le w(\neg \varphi)$ holds. Let us fix such x as well (such that it is disjoint from previous fixations according to KE). From $\forall x \le z \exists y \varphi$ we can derive that for our fixed x the statement holds, meaning that $\exists y \varphi$, and $\forall y \le w(\neg \varphi)$. We finally fix some y and then φ holds, but w can be arbitrarily fixed, we assume that $y \le w$ and it follows that $\neg \varphi$, a contradiction. □

Part b

Suppose that $\psi(y_0, y_1, t_0, \dots, t_{n-1})$ is some formula in the language L_{PA}. We will show that,

$$\text{PA} \vdash \forall t_0 \dots \forall t_{n-1}(\exists y_0 \exists y_1 \psi \leftrightarrow \exists w \exists y_0 \le w \exists y_1 \le w \psi)$$

Proof. The proof is very similar to part a, Let $t_0, \dots, t_{n-1}$ be some values, and let us assume toward a contradiction that $\exists y_0 \exists y_1 \psi$, and that $\neg \exists w \exists y_0 \le w \exists y_1 \le w \psi \equiv \forall w \forall y_0 \le w \forall y_1 \le w(\neg \psi)$ hold. We fix y_0, y_1 such that ψ holds, and fix some $w \ge \sup\{y_0, y_1\}$, hence $\neg \psi$ is derived by the second formula, resulting in a contradiction. □

Question 2

We define $\Sigma_n(\mathrm{PA})$ as the set of formulas $\psi(x_0, \ldots, x_{n-1})$, for any $n < \omega$, such that for some Σ_n formula $\psi'(x_0, \ldots, x_{n-1})$,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}(\psi \leftrightarrow \psi')$$

Part a

We will show that the class of $\Sigma_0(\mathrm{PA})$ formulas in PA are closed under boolean operations and bounded quantifiers.

Proof. □