

# Solution to Exercise 0 – Model Theory (1), 80616

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## Question 1

**Definition 0.1.** A formula  $\varphi$  is called *basic Horn* formula if,

$$\varphi = (\theta_0(\bar{x}) \wedge \cdots \wedge \theta_{m-1}(\bar{x})) \rightarrow \theta_m(\bar{x})$$

where  $\theta_i$  is atomic for  $i \leq m$ .

*Remark.* In the case  $m = 0$  we get  $\varphi = \theta_0(\bar{x})$ . In the case that  $\theta_m = \perp$ ,  $\varphi \equiv \neg\theta_0(\bar{x}) \vee \cdots \vee \neg\theta_{m-1}(\bar{x})$ .

**Definition 0.2.** The set of Horn formulas is the minimal set of formulas containing the basic Horn formulas and closed under conjugation and quantification.

### Part a

Let  $\psi(x_0, \dots, x_{n-1})$  be a Horn formula and let  $F \subseteq \mathcal{P}(I)$  be a filter. Let  $\langle \mathcal{M}_i \mid i \in I \rangle$  be a sequence of structures and  $a_j \in \prod_{i \in I} \mathcal{M}_i$  for  $j < n$  such that,

$$\{i \in I \mid \mathcal{M}_i \models \psi(a_0(i), \dots, a_{n-1}(i))\} \in F$$

We will show that  $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / F \models \psi([a_0]_F, \dots, [a_{n-1}]_F)$ .

*Proof.* Let us prove by induction over Horn set.

Assume that  $\varphi(\bar{x}) = P(\bar{x})$  and  $\{i \in I \mid \mathcal{M}_i \models P(\bar{x})\} \in F$ , then by definition  $\mathcal{N} \models \varphi$ . Let us assume that  $\varphi = (\theta_0 \wedge \cdots \wedge \theta_{n-1}) \rightarrow \theta_n$  for atomic  $\theta_n$ , then the claim holds directly from definition as in the last part.

The case where  $\theta_m = \perp$  is equivalent as  $\varphi \equiv \theta_0 \vee \cdots \vee \theta_{m-1}$ .

We move to assume that the claim holds for  $\varphi, \psi$  and prove for  $\varphi \wedge \psi$ . This part of the proof is identical to the proof of Łoś theorem,

$$\{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}) \wedge \psi(\bar{a}(i))\} = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \cap \{i \in I \mid \mathcal{M}_i \models \psi(\bar{a}(i))\} \in F \implies \mathcal{N} \models \varphi, \psi \implies \mathcal{N} \models \varphi \wedge \psi$$

where the second equation is derived from filter definition.

We move to the case  $\varphi = \exists x \psi$  for  $\psi$  that fulfills the claim. If  $J = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \in F$  then for  $j \in J$  we define  $b_j \in M_j$  as a witness to  $\mathcal{M}_j \models \psi(b_j, \bar{a}(j))$ , meaning that,

$$\{i \in I \mid \mathcal{M}_i \models \psi(b(i), \bar{a}(i))\} = J \in F$$

where  $b(i) = b_i$  for  $i \in J$  and arbitrary otherwise. Then by the induction hypothesis  $\mathcal{N} \models \psi([b], [\bar{a}])$  and therefore  $\mathcal{N} \models \varphi([\bar{a}])$ .

Lastly we will assume that  $\varphi = \forall x \psi$  for  $\psi$  that fulfills the claim and show that  $\varphi$  does as well.

If  $J = \{i \in I \mid \mathcal{M}_i \models \varphi(\bar{a}(i))\} \in F$ , then if  $b : I \rightarrow \bigcup M_i$  some choice function then,

$$\{i \in I \mid \mathcal{M}_i \models \psi(b(i), \bar{a})\} \supseteq J \in F$$

therefore  $\mathcal{N} \models \psi([b], [\bar{a}])$ , then  $\mathcal{N} \models \varphi([\bar{a}])$  as required. □

### Part b

We will find a language and a sentence  $\varphi$  such that for any set  $I$  and filter  $F$ ,

there is  $\langle \mathcal{M}_i \mid i \in I \rangle$  such that  $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / F \models \varphi$  if and only if  $F$  is an ultrafilter.

*Solution.* Define  $L = \{=\}$  as the trivial language, and  $\varphi = \exists x \exists y (x \neq y \wedge \forall z (z = x \vee z = y))$ .

Let  $I$  be some indices set and  $F \subseteq \mathcal{P}(I)$ . Let  $M = \{0, 1\}$  and  $\mathcal{N} = \mathcal{M}^I / F$ .

If  $F$  is an ultrafilter then by Łoś theorem  $\mathcal{N} \models \varphi$ . Otherwise there is a set  $J \subseteq I$  such that  $J \notin F, I \setminus J \notin F$ . Define,

$$f(x) = \begin{cases} 0 & x \in J \\ 1 & x \in I \setminus J \end{cases}$$

then  $\mathcal{N} \models [f] \neq [c_0]$  as well  $\mathcal{N} \models [f] \neq [c_1]$ , and thus  $\mathcal{N} \models \neg\varphi$ .

## Part c

Let  $\mathcal{M}_n$  be finite models in the language of equality, let  $\mathcal{U} \subseteq \mathcal{P}(\omega)$  be a non-principal ultrafilter, and define  $\mathcal{M}_\omega = \prod \mathcal{M}_n / \mathcal{U}$ . We will show that  $M_\omega$  is either finite or uncountable.

*Proof.* Let us assume for contradiction that  $|M_\omega| = \omega$  and let  $\langle f_n : \omega \rightarrow \bigcup M_i \mid n < \omega \rangle$  be sequence such that  $\{[f_n]\} = M_\omega$ . Let  $B_0 = \{n < \omega \mid f_0(n) \neq f_1(n)\} \in \mathcal{U}$ , and by recursion for each  $B_k$  we define,

$$B_{k+1} = B_k \cap \{n < \omega \mid \forall i \leq k, f_i(n) \neq f_{k+1}(n)\} \in \mathcal{U}$$

Then  $\{B_k\} \subseteq \mathcal{U}$  and  $B_k \supseteq B_{k+1}$  for any  $k$ , and let  $B = \inf B_n$ . If  $B = \emptyset$  then there is minimal  $k < \omega$  such that  $B_k \neq \emptyset$ , but  $B_k, B_{k+1} \in \mathcal{U}$ , therefore  $\emptyset \in \mathcal{U}$ , a contradiction. Then there is  $b \in B$ , then  $S = \{f_n(b) \mid n < \omega\}$  is a set such that  $|S| = \aleph_0$ , but  $S \subseteq M_b$  and  $|M_b| < \aleph_0$ . We conclude that  $|M_\omega|$  cannot be countable.

We will show that  $|M_\omega|$  if and only if  $|M_n| < K$  for  $n \in J \in \mathcal{U}$  and  $K < \omega$ . Let us assume that  $\forall n \in J, |M_n| < K$ , and define,

$$\varphi_K = \forall x_0 \dots \forall x_{K-1} \left( \bigvee_{i < j < K} x_i = x_j \right)$$

then  $\mathcal{M}_n \models \varphi_K$  for any  $n \in J$ , therefore by Łoś we get  $\mathcal{M}_\omega \models \varphi_K$  as well, in particular  $|M_\omega| < K$ .

In the other direction the claim holds similarly by Łoś.

We will show that  $|M_\omega| = 2^{\aleph_0}$  if  $|M_\omega|$  is uncountable. By the last claim,  $\mathcal{M}_\omega \models \neg\varphi_K$  for any  $K < \omega$ , meaning that  $|M_\omega|$  is unbounded. Let us assume that  $h : \omega \rightarrow \omega$  is a function such that  $|M_{h(n)}| \geq n$ . We can also define  $h_n : n \rightarrow M_n$  by the last cardinality inequality (and more choice). It is known that the cardinality of functions  $\omega \rightarrow \omega$  that strictly increasing is  $2^{\aleph_0}$ , then it suffices to show that any such function  $g$  can be mapped uniquely to  $[f] \in M_\omega$ . We can define  $f' = \{\langle h(g(n)), h_{h(g(n))}(g(n)) \rangle\}$  and,

$$f(n) = \begin{cases} f'(n) & n \in \text{dom } f' \\ h_n(0) & \text{otherwise} \end{cases}$$

Then  $[f] \in M_\omega$ . If  $g, g'$  are two different strictly increasing functions, and  $f, f'$  are their respective constructions, then  $[f] \neq [f']$  directly from definition. We deduce that  $2^{\aleph_0} \leq |M_\omega| \leq |\omega^\omega| = |2^{\aleph_0}|$ .  $\square$

## **Question 2**

### **Part a**

Suppose that  $T$  has quantifier elimination and that there is a model  $\mathcal{M}_0$  that embeds in every model of  $T$ . We will show that  $T$  is complete.

*Proof.*

□