Solution to Final Exercise — Model Theory (1), 80616

February 5, 2026

Question 1

Urysohn Space. Let $L = \{D_q \mid q \in \mathbb{Q}_{\geq 0}\}$ such that D_q is a binary relation symbol for any q. For any metric space (X, d) we can define L-structure $\mathcal{X}$ with $|\mathcal{X}| = X$ and $(x, y) \in D_q^{\mathcal{X}} \iff d(x, y) = q$.

Part a

Let $\mathcal{K}$ be the class of all finite metric spaces with rational distances.
We will show that $\mathcal{K}$ is a Fraïssé class.

Proof. We will follow the definition of Fraïssé class.

Closure under isomorphism If $\mathcal{X} \in \mathcal{K}$ and $\mathcal{Y} \cong \mathcal{X}$ and notate $(X, d), (Y, \rho)$ the equivalent metric spaces, then $f : \mathcal{X} \to \mathcal{Y}$ that witness the isomorphism fulfills,

$$d(x_1, x_2) = q \iff D_q^{\mathcal{X}}(x_1, x_2) \iff D_q^{\mathcal{Y}}(f(x_1), f(x_2)) \iff \rho(f(x_1), f(x_2)) = q.$$

Meaning that (Y, ρ) is a finite metric space with rational distance and therefore $\mathcal{Y} \in \mathcal{K}$ as well.

Essential countability Let $X_n = [n]$ and $d : X_n^2 \to \mathbb{Q}$ be a metric, define $\mathcal{X}_n^d$ as the equivalent of the metric space (X_n, d). We define the collection,

$$\mathcal{M} = \{\mathcal{X}_n^d \mid n < \omega, d \text{ is a metric}\}.$$

and notice that for each $n < \omega$, the collection of metric functions defined on X_n is countable as it is a subset of $\mathbb{Q}^{n^2}$. It follows that $\mathcal{M}$ is a countable collection, and notice that $\mathcal{M} \subseteq \mathcal{K}$.
Let $\mathcal{Y} \in \mathcal{K}$, and let (Y, ρ) be the equivalent metric space, notate $n = |Y|$ and let us define $f : [n] \to Y$ a bijection, then,

$$d : X_n^2 \to \mathbb{Q}, \qquad d(x_1, x_2) = \rho(f(x_1), f(x_2)).$$

is a metric on X_n and therefore $\mathcal{X}_n^d \in \mathcal{M}$ and $\mathcal{X}_n^d \cong \mathcal{Y}$.

Hereditary property Assume that $\mathcal{X} \subseteq \mathcal{Y} \in \mathcal{K}$ and show that $\mathcal{X} \in \mathcal{K}$, we omit the requirement that $\mathcal{X}$ is finitely-generated as there are no function symbols and $|Y| < \omega$. Metric property is closed under restrictions then if (Y, ρ) a metric space and $X \subseteq Y$ then $(X, \rho \restriction X^2)$ is a metric space as well.

The joint embedding property Suppose that $\mathcal{X}, \mathcal{Y} \in \mathcal{K}$ and let $(X, d), (Y, \rho)$ be their respective equivalencies. Assume that $X \cap Y \neq \emptyset$ then let $x_0 \in X \cap Y$ be some element, and let $C = A \cup B$, define,

$$f : C^2 \to \mathbb{Q}_{\geq 0}, \qquad f(a, b) = \begin{cases} d(a, b) & a, b \in X \\ \rho(a, b) & a, b \in Y \\ d(a, x_0) + \rho(b, x_0) & a \in X, b \in Y \\ d(b, x_0) + \rho(a, x_0) & b \in X, a \in Y \end{cases}.$$

Then (C, f) is a metric space and it follows that $\mathrm{id}_X, \mathrm{id}_Y$ are embeddings of X, Y into C respectively.
If $X \cap Y = \emptyset$ then denote some arbitrary elements $x_0 \in X, y_0 \in Y$, let $f : C^2 \to \mathbb{Q}_{\geq 0}$ defined by,

$$f(a, b) = \begin{cases} d(a, b) & a, b \in X \\ \rho(a, b) & a, b \in Y \\ d(x_0, a) + \rho(y_0, b) + 1 & a \in X, b \in Y \\ d(x_0, b) + \rho(y_0, a) + 1 & \text{otherwise} \end{cases}.$$

Thus f is a metric over C, as it is symmetric, non-negative and fulfills triangle inequality. If $\mathcal{C}$ is the structure of (C, f) then $\mathcal{C} \in \mathcal{K}$ and $\mathrm{id}_X, \mathrm{id}_Y$ are embedding of $\mathcal{X}, \mathcal{Y}$ into $\mathcal{C}$.

The amalgamation property Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathcal{K}$ with $(X, d_x), (Y, d_y), (Z, d_z)$ such that $g_x : \mathcal{Z} \to \mathcal{X}, g_y : \mathcal{Z} \to \mathcal{Y}$ are embeddings. Let $\mathcal{C} \in \mathcal{K}$ as in the last statement, and assume without loss of generality that $\mathcal{C} \subseteq \mathcal{X}, \mathcal{Y}$ (otherwise we can use closure under isomorphism) and therefore g_x, g_y are both identity. Let $f_x : \mathcal{X} \to \mathcal{C}, f_y : \mathcal{Y} \to \mathcal{C}$ be embeddings, it is implied that,

$$f_y \circ g_y = f_y \circ \mathrm{id}_Z = f_x \circ \mathrm{id}_Z = f_x \circ g_x.$$

and the property holds as wished. □

Let us denote $\mathbb{U}_\mathbb{Q}$ the Fraïssé limit of $\mathcal{K}$, and let $(U_\mathbb{Q}, d)$ be its equivalent metric space.

Part b

Let (X, d) be a metric space, a Katětov map is a function $f : X \to \mathbb{R}_{\geq 0}$ such that,

$$\forall x, y \in X,\ |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y).$$

A rational Katětov map is a Katětov map such that $\mathrm{Im}\, f \subseteq \mathbb{Q}$.
We will show that for every finite $A \subseteq U_\mathbb{Q}$, any rational Katětov map $f : A \to \mathbb{Q}_{\geq 0}$ is realized, namely there is some $b \in U_\mathbb{Q}$ such that $\forall x \in U_\mathbb{Q},\ f(x) = d(x, b)$.

Proof. Note that if $\mathcal{X} \cong \mathcal{Y}$ and $f : \mathcal{X} \to \mathcal{Y}$ isomorphism, then $R_q(x, y) \iff R_q(f(x), f(y))$, meaning that f is an isometry of X, Y as metric spaces.

Let f be some rational Katětov map of A. Let $B = A \uplus \{c\}$ for some element, we define a metric on B, $\rho : B^2 \to \mathbb{Q}_{\geq 0}$ by,

$$\rho(x, y) = \begin{cases} d(x, y) & x, y \in A \\ d(c, x) = f(x) & x \in A \\ d(c, c) = 0 & \end{cases}.$$

ρ is symmetrical and non-negative, and fulfills the triangle inequality for elements of A, let us verify that for c as well,

$$\rho(c, c) = 0 \leq 2f(x) = \rho(c, x) + \rho(x, c).$$

and,

$$\rho(x, y) = d(x, y) \leq f(x) + f(y) = \rho(x, c) + \rho(c, y).$$

Then (B, ρ) is indeed metric space, but B is finite and thus $\mathcal{B}$ is isomorphic to some substructure of $\mathbb{U}_\mathbb{Q}$, $\mathcal{B} \cong \mathcal{C} \in \mathcal{K}$, and let $g : \mathcal{B} \leftrightarrow \mathcal{C}$ be the isomorphism. $\mathbb{U}_\mathbb{Q}$ is ultra-homogeneous then the isomorphism of $A \to \mathcal{C} \setminus \{g(c)\}$ can be extended to $\sigma \in \mathrm{Aut}(\mathbb{U}_\mathbb{Q})$, and let $e = \sigma^{-1}(c')$. For any $x \in A$, $f(x) = \rho(x, c) = d(g(x), g(c)) = d(\sigma^{-1}(g(x)), \sigma^{-1}(g(c))) = d(x, e)$ as wished. □

We will show that $\mathbb{U}_\mathbb{Q}$ is the unique countable rational metric space such that for any $A \subseteq X$ finite and rational Katětov map on A, there is $b \in X$ realizing f.

Proof. Let (X, ρ) be a metric space with this property, we will show that $(X, \rho) \cong (U_\mathbb{Q}, d)$. We will show by induction that for every X_n^f (as defined in part 1) there is $Y \subseteq X$ such that $(X_n^f, f) \cong (Y, \rho \restriction Y^2)$. The case of $n = 1$ is trivial as the single metric on single valued space is 0.
Let us assume the claim holds for $m < n$ and let X_n^f be some finite and rational metric space. Take $(X_{n-1}^{f \restriction X_{n-1}^2}, f \restriction X_{n-1}^2)$, we get a sub-metric space of size $n-1$, meaning that there is some such $Y \subseteq X$ and let $\sigma : X_{n-1} \leftrightarrow Y$. Define $g : Y \to \mathbb{Q}_{\geq 0}$ by $g(x) = f(n-1, \sigma^{-1}(x))$, then g is a Katětov map, as a result of the last subpart. There is an element $b \in X$ that realizes g, therefore $\rho(x, b) = g(x) = f(n-1, \sigma^{-1}(x))$, meaning that $f(\sigma^{-1}(\cdot), \sigma^{-1}(\cdot)) = \rho \restriction (Y \cup \{b\})^2$.

We have shown that $\{X_n^f \mid n < \omega, f \text{ is a metric}\} \subseteq \mathrm{Age}(\mathcal{X})$ for $\mathcal{X}$ the equivalent of (X, ρ), therefore $\mathrm{Age}(\mathcal{X}) = \mathcal{K} = \mathrm{Age}(\mathbb{U}_\mathbb{Q})$, then $\mathcal{X} \cong \mathbb{U}_\mathbb{Q}$. □

Part c

Let $\mathbb{U}$ be the completion of $\mathbb{U}_{\mathbb{Q}}$, we will show that any finite metric space (X, d) is isometrically embedded into $\mathbb{U}$.

Proof. It is going to be induction over the size of the finite subspace yet again, then using the fact that for given set of points, there is a point in distance $|d - d_0| < \varepsilon$, then taking limit of that. $\square$