Exercise 5 Answer Sheet — Logic Theory (2), 80424

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Question 1

Part a

Suppose that $\varphi(z, y, t_0, \ldots, t_{n-1})$ is a formula in the language of L_{PA}. We will show that,

$$\mathrm{PA} \vdash \forall z \forall t_0 \ldots \forall t_{n-1}(\forall x \leq z \exists y \varphi \leftrightarrow \exists w \forall x \leq z \exists y \leq w \varphi)$$

Proof. Let us fix $z, t_0, \ldots, t_{n-1}$, then is suffice to show that $\mathrm{PA} \vdash \forall x \leq z \exists y \varphi \leftrightarrow \exists w \forall x \leq z \exists y \leq w \varphi$. We assume toward a contradiction that the formula does not hold, then $\forall x \leq z \exists y \varphi$ as well $\neg \exists w \forall x \leq z \exists y \leq w \varphi$ holds as well. The latter is equivalent to $\forall w \exists x \leq z \forall y \leq w(\neg \varphi)$. We fix w and it derives that $\exists x \leq z \forall y \leq w(\neg \varphi)$ holds. Let us fix such x as well (such that it is disjoint from previous fixations according to KE). From $\forall x \leq z \exists y \varphi$ we can derive that for our fixed x the statement holds, meaning that $\exists y \varphi$, and $\forall y \leq w(\neg \varphi)$. We finally fix some y and then φ holds, but w can be arbitrarily fixed, we assume that $y \leq w$ and it follows that $\neg \varphi$, a contradiction. $\square$

Part b

Suppose that $\psi(y_0, y_1, t_0, \ldots, t_{n-1})$ is some formula in the language L_{PA}. We will show that,

$$\mathrm{PA} \vdash \forall t_0 \ldots \forall t_{n-1}(\exists y_0 \exists y_1 \psi \leftrightarrow \exists w \exists y_0 \leq w \exists y_1 \leq w \psi)$$

Proof. The proof is very similar to part a, Let $t_0, \ldots, t_{n-1}$ be some values, and let us assume toward a contradiction that $\exists y_0 \exists y_1 \psi$, and that $\neg \exists w \exists y_0 \leq w \exists y_1 \leq w \psi \equiv \forall w \forall y_0 \leq w \forall y_1 \leq w(\neg \psi)$ hold. We fix y_0, y_1 such that ψ holds, and fix some $w \geq \sup\{y_0, y_1\}$, hence $\neg \psi$ is derived by the second formula, resulting in a contradiction. $\square$

Question 2

We define $\Sigma_n(\text{PA})$ as the set of formulas $\psi(x_0, \dots, x_{n-1})$, such that for some Σ_n formula $\psi'(x_0, \dots, x_{n-1})$,

$$\text{PA} \vdash \forall x_0, \dots, \forall x_{n-1}(\psi \leftrightarrow \psi')$$

Part a

We will show that the class of $\Sigma_0(\text{PA})$ formulas in PA is closed under boolean operations and bounded quantifiers.

Proof. Let $\psi, \varphi \in \Sigma_0(\text{PA})$ be some formulas. There exist φ', ψ' such that,

$$\text{PA} \vdash \forall x_0, \dots, \forall x_{n-1}(\psi \leftrightarrow \psi'), \forall x_0, \dots, \forall x_{n-1}(\varphi \leftrightarrow \varphi')$$

Then it follows that,

$$\text{PA} \vdash \forall x_0, \dots, \forall x_{n-1}((\psi \leftrightarrow \psi') \land (\varphi \leftrightarrow \varphi'))$$

Therefore by an identity from the previous course,

$$\text{PA} \vdash \forall x_0, \dots, \forall x_{n-1}((\psi \land \varphi) \leftrightarrow (\psi' \land \varphi'))$$

Namely, $\varphi \land \psi \in \Sigma_n(\text{PA})$.

By another identity it also derives that,

$$\varphi \leftrightarrow \varphi' \equiv (\neg\varphi) \leftrightarrow (\neg\varphi')$$

We deduce that $\Sigma_n(\text{PA})$ is closed under boolean operations.

We use the fact that Σ_0 is closed under bounded quantifiers and deduce immediately that $\Sigma_0(\text{PA})$ is closed under bounded quantifiers as well. $\square$

Part b

We will show that for some $n > 0$, the class of $\Sigma_n(\text{PA})$ formulas are closed under bounded quantifiers, existential quantifiers, disjunctions and conjunctions.

Proof. Let $\varphi \in \Sigma_n(\text{PA})$, and suppose $\varphi' \in \Sigma_n$ testifies to that. $\forall v \le w\varphi' \in \Sigma_n^0$, then by the last exercise we deduce that there is $\psi \in \Sigma_n$ such that $\forall v \le w\varphi \equiv \forall v \le w\varphi' \equiv \psi$. φ testifies to $\forall v \le w\varphi \in \Sigma_n(\text{PA})$. The proof is identical for bounded existential quantifier.

We will show that the class in closed under existential quantifiers. Suppose that $\varphi \in \Sigma_n(\text{PA})$, and let this be testified by $\varphi' \in \Sigma_n$. There is $\phi \in \Pi_{n-1}$ such that $\varphi' = \exists v\phi$. We define $\psi = \exists u\varphi$, then by the second part of the last question,

$$\text{PA} \vdash \forall t_0 \dots \forall t_{n-1}(\exists u\exists v\varphi \leftrightarrow \exists w\exists v \le w\exists u \le w\varphi)$$

Then $\psi \in \Sigma_n(\text{PA})$.

We claim that $\Sigma_n(\text{PA})$ is also closed under disjunction and conjunction, as the proof is the same as the case of $n = 0$. $\square$

Part c

We will show that for $n > 0$, the class $\Pi_n(\text{PA})$ is closed under bounded quantifiers, universal quantifiers, disjunctions and conjunctions.

Proof. The proof for closeness under bounded quantifiers, as well disjunctions and conjunctions is the same as of the above, we move to show closeness under universal quantifiers. Let $\varphi \in \Pi_n(\text{PA})$ and $\varphi' \in \Pi_n$ be the formula to testify that. We define $\psi = \forall v \varphi$, note that $\psi \equiv \forall v \varphi'$. $\varphi' = \forall u \phi$ for some $\phi \in \Sigma_{n-1}$. Using part a of question 1 we deduce that $\forall u \psi \equiv \forall w \forall v \leq w \forall u \leq w \phi$, it is implied that indeed $\psi \in \Pi_n(\mathrm{PA})$. $\square$

Part d

We will show that the negation of a $\Sigma_n(\mathrm{PA})$ formula is $\Pi_n(\mathrm{PA})$.

Proof. In the previous exercise we had shown that the negation of Σ_n^0 formula is a Π_n^0 formula and vice versa. We also know that every Σ_n formula is in particular a Σ_n^0 formula. Let $\varphi \in \Sigma_n(\mathrm{PA})$ and $\varphi' \in \Sigma_n$ be formulas such that,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}(\varphi \leftrightarrow \varphi')$$

It follows that,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}((\neg\varphi) \leftrightarrow (\neg\varphi'))$$

But by the statement above, $\neg\varphi' \in \Pi_n^0$ and there is $\varphi'' \in \Pi_n$ such that $\varphi' \equiv \neg\varphi''$, which implies that,

$$\mathrm{PA} \vdash \forall x_0, \ldots, \forall x_{n-1}((\neg\varphi) \leftrightarrow \varphi'')$$

We deduce that indeed $\varphi \in \Pi_n(\mathrm{PA})$. $\square$

Part e

We will show that every formula is in $\Sigma_n(\mathrm{PA})$ or in $\Pi_n(\mathrm{PA})$ for some $n \in \mathbb{N}$.

Proof. Similarly to the last exercise, we will prove the claim by induction over the structure of the formula. For any atomic formula the claim holds for $n = 0$ from part a. For negation we have shown in the last part that there is closeness, and in parts b and c we had shown closeness for conjunctions.

Lastly, for universal quantifiers we either use definition under $\Sigma_n(\mathrm{PA})$, and question 1 part a for formulas from $\Pi_n(\mathrm{PA})$. We handle existential quantifiers similarly. $\square$

Question 3

TODO