Solution to Exercise 3 — Model Theory (1), 80616

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Question 1

Part a

We will show that there is a collection $X \subseteq \mathcal{P}(\omega)$ such that $|X| = \aleph_1$ and if $x \neq y \in X$ then $x \cap y$ is bounded.

Proof. For any $A \subseteq \omega$ infinite let us define $Y(A) = \{\sum_{n \in A \cap m} 2^n \mid m < \omega\}$, then $X(A) \in \mathcal{P}(\omega)$. It follows that $Y : \omega^\omega \to \mathcal{P}(\omega)$ is a function, note that it is injective, as if $A \neq B \in \operatorname{dom} Y$ then there is $m \in A \setminus B$, witnessing $Y(A) \neq Y(B)$. The injectivity of Y implies that $|\operatorname{Im} Y| = |\mathcal{P}(\omega)|$, and let $X = \operatorname{Im} Y$.

We move to show that if $A' \neq B' \in X$ then $A' \cap B' < \alpha$ for some $\alpha < \omega$. Let $A, B \subseteq \omega$ be elements such that $Y(A) = A', Y(B) = B'$. $A \neq B \implies \exists m < \omega,\ k \in A, k \notin B$ without loss of generality. Let $k < m < \omega$, then it suffices to show that $\sum_{n<m} 2^n(\mathbb{1}_A(n) - \mathbb{1}_B(n)) \neq 0$, but as a result from number theory and by the fact that $A \cap m \neq B \cap m$ the expression indeed does not nullify. We deduce that $|A' \cap B'| \leq k < \omega$ as wished. □

Part b

Let $\mathcal{L} = \{0, S, \leq\} \cup \{R_X \mid X \subseteq \omega\}$ be an uncountable language such that 0 is constant symbol, S is unary function symbol, $\leq$ is binary relation symbol and R_X is an unary relation symbol for all $X \subseteq \omega$.
Let $\mathcal{A}$ be a model over $\mathcal{L}$ with $A = \omega, 0^\mathcal{A} = 0, S^\mathcal{A}(n) = n+1, x \leq^\mathcal{A} y \iff x \in y$ and $R_X^\mathcal{A}(n) \iff n \in X$, and let $T = \operatorname{Th}(\mathcal{A})$.
We will show that T is ω-categorical, that it has infinitely many nonequivalent formulas and that there are infinitely many non-isolate types in $S_1(T)$.

Proof. Let $\mathcal{M} \models T$ be some countable model, we will show that $\mathcal{M} \cong \mathcal{A}$. We define recursively $f : A \to M$ by $f(0) = 0^\mathcal{M}$ and $f(n+1) = S^\mathcal{M}(f(n))$. It follows that f preserves $0, S$, it remains to show it also preserves $\leq, R_X$. The claim that f preserves $\leq$ can be shown using double induction by fixing each n and proving $n \leq m \implies f(n) \leq f(m)$. Definability of $n \in \mathcal{A}$ as $S^n(0)$ and the fact for each X, $R_X^\mathcal{A}(n) \iff S^n(0) \in X$ implies that R_X is preserved under f as well.
$\mathcal{A}$ fulfills the axiom scheme of induction, meaning that T does as well. Each $X \subseteq A$ is definable using R_X, meaning that the sentence $(0 \in \mathbb{N} \wedge (x \in \mathbb{N} \to S(x) \in \mathbb{N})) \to \forall x \in \mathbb{N}$ exists in T (to be precise it is symbolically appears and not the actual form written) where $\mathbb{N}$ is some X. f is defined as injection, let $m \in M$ be some element. If $m \in \mathbb{N}$ then by the recursive definition of f, $m \in \operatorname{Im} f$. Otherwise, $\mathcal{M} \models m \notin \mathbb{N}$, a contradiction to the sentence $\in T$. f is bijection and thus a model isomorphism, thus T is ω-categorical.

Let $\varphi_n(x) = (x = S^{\underline{n}}(0))$, as an informal way to define that $\varphi_0(x) = (x = 0), \varphi_1 = (x = S(0))$ and so on. $\mathcal{A} \models \forall x\ x \neq S(x)$, then we can deduce that $\varphi_n \not\equiv \varphi_m$ for all $n < m < \omega$. Then $\{\varphi_n \mid n < \omega\} \subseteq T$ is a set of non-equivalent formulas.

Let $p(x) = \{\underline{n} \leq x \mid n < \omega\}$ be a partial type, and let $A \in X$ for X of the previous part. $T \models \forall x\ (x = 0 \vee \exists y\ S(y) = x)$ therefore if $S = T \cup p(\gamma)$ for new constant symbol γ then $\exists x\ x = S^z(\gamma)$ for any $z \in \mathbb{Z}$. Let $p_A(x) = \operatorname{cl}_\vdash p(x) \cup \{R_A(S^a(x)) \mid a \in A\} \cup \{\neg R_Y(S^z(x)) \mid A \neq Y \subseteq \omega, z \in \mathbb{Z}\}$. $S_A = T \cup p_A(\gamma)$ is complete as a closure under consequences. It can be shown using induction over the structure of the formula that p_A cannot be omitted for any $A \in X$.

We want to show that there are infinitely many different types p_A. Let $A \neq B \in X$ and let us consider S_A, S_B. If $\mathcal{M} \models S_A \cap S_B$ then $\mathcal{M}$ thinks that there is a maximal element in $A \cap B$, meaning that the class is bounded, in oppose to the fact that S_A, S_B think that A, B are unbounded. It follows that $S_A \neq S_B$ and that there is no common theory such that it contain T. We conclude that there are $2^{\aleph_0}$ such types. □

Question 2

Let $\omega \leq \kappa$ be some cardinal. We say that a model $\mathcal{M}$ is κ-saturated if for every $A \subseteq M$ with $|A| < \kappa$, any type in $S_1(A)$ is realized.

Part a

Let T be a consistent complete theory over a language of size $\leq \kappa$ with an infinite model.
We will show that there is a κ^+ saturated model of T of cardinality 2^κ.

Proof. By Löwenheim-Skolem theorem we can assume that there is a model $\mathcal{M} \models T$ of cardinality κ. Let,

$$K = \{p \subseteq \text{form}_{L(A)} \mid A \subseteq M, \forall \varphi \in p,\ \text{FV}(\varphi) = \{x\}, T \cup p(c) \text{ is complete}\}$$

Note that $\text{form}_{L(M)}$ is of size 2^κ therefore K is bounded by 2^κ as well. Let $\mathcal{N}' \models T \cup \bigcup K$ be enrichment of $\mathcal{M}$ by up to 2^κ new constants such that each type is realized and let $\mathcal{N}$ be its reduction to $\mathcal{M}$'s language. Then $\mathcal{N} \models T$, is saturated and κ^+ saturated directly by its construction. Lastly, if $|N| < 2^\kappa$, we can use upwards Löwenheim-Skolem again on $\mathcal{N}'$. ☐

Part b

We will show that if $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent κ-saturated models of cardinality κ then $\mathcal{M} \cong \mathcal{N}$, and that any partial elementary map $f : A \to B$ with $A \subseteq M, B \subseteq N, |A| < \kappa$ can be extended to an isomorphism.

Proof. TODO ☐