

Solution to Exercise 3 — Model Theory (1), 80616

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Question 1

Part a

We will show that there is a collection $X \subseteq \mathcal{P}(\omega)$ such that $|X| = 2^{\aleph_0}$ and if $x \neq y \in X$ then $x \cap y$ is bounded.

Proof. For any $A \subseteq \omega$ infinite let us define $Y(A) = \{\sum_{n \in A \cap m} 2^n \mid m < \omega\}$, then $X(A) \in \mathcal{P}(\omega)$. It follows that $Y : \omega^\omega \rightarrow \mathcal{P}(\omega)$ is a function, note that it is injective, as if $A \neq B \in \text{dom } Y$ then there is $m \in A \setminus B$, witnessing $Y(A) \neq Y(B)$. The injectivity of Y implies that $|\text{Im } Y| = |\mathcal{P}(\omega)|$, and let $X = \text{Im } Y$.

We move to show that if $A' \neq B' \in X$ then $A' \cap B' < \alpha$ for some $\alpha < \omega$. Let $A, B \subseteq \omega$ be elements such that $Y(A) = A', Y(B) = B'$. $A \neq B \implies \exists m < \omega, k \in A, k \notin B$ without loss of generality. Let $k < m < \omega$, then it suffices to show that $\sum_{n < m} 2^n (\mathbb{1}_A(n) - \mathbb{1}_B(n)) \neq 0$, but as a result from number theory and by the fact that $A \cap m \neq B \cap m$ the expression indeed does not nullify. We deduce that $|A' \cap B'| \leq k < \omega$ as wished. \square

Part b

Let $\mathcal{L} = \{0, S, \leq\} \cup \{R_X \mid X \subseteq \omega\}$ be an uncountable language such that 0 is constant symbol, S is unary function symbol, \leq is binary relation symbol and R_X is an unary relation symbol for all $X \subseteq \omega$.

Let \mathcal{A} be a model over \mathcal{L} with $A = \omega, 0^{\mathcal{A}} = 0, S^{\mathcal{A}}(n) = n + 1, x \leq^{\mathcal{A}} y \iff x \in y$ and $R_X^{\mathcal{A}}(n) \iff n \in X$, and let $T = \text{Th}(\mathcal{A})$.

We will show that T is ω -categorical, that it has infinitely many nonequivalent formulas and that there are infinitely many non-isolate types in $S_1(T)$.

Proof. Let $\mathcal{M} \models T$ be some countable model, we will show that $\mathcal{M} \cong \mathcal{A}$. We define recursively $f : A \rightarrow M$ by $f(0) = 0^{\mathcal{M}}$ and $f(n+1) = S^{\mathcal{M}}(f(n))$. It follows that f preserves 0, S , it remains to show it also preserves \leq, R_X . The claim that f preserves \leq can be shown using double induction by fixing each n and proving $n \leq m \implies f(n) \leq f(m)$. Definability of $n \in \mathcal{A}$ as $S^n(0)$ and the fact for each $X, R_X^{\mathcal{A}}(n) \iff S^n(0) \in X$ implies that R_X is preserved under f as well.

\mathcal{A} fulfills the axiom scheme of induction, meaning that T does as well. Each $X \subseteq A$ is definable using R_X , meaning that the sentence $(0 \in \mathbb{N} \wedge (x \in \mathbb{N} \rightarrow S(x) \in \mathbb{N})) \rightarrow \forall x \in \mathbb{N}$ exists in T (to be precise it is symbolically appears and not the actual form written) where \mathbb{N} is some X .

Let $\varphi_n(x) = (x = S^n(0))$, as an informal way to define that $\varphi_0(x) = (x = 0), \varphi_1(x) = (x = S(0))$ and so on. $\mathcal{A} \models \forall x x \neq S(x)$, then we can deduce that $\varphi_n \not\equiv \varphi_m$ for all $n < m < \omega$. Then $\{\varphi_n \mid n < \omega\} \subseteq T$ is a set of non-equivalent formulas.

We have shown that f is model embedding, and now we will show that it is also surjective. Let $\gamma \in M$ be a non-standard element, namely $\mathcal{M} \models \neg \varphi_n(\gamma)$ for all $n < \omega$. By f 's construction, $\gamma \notin \text{Im } f$, and therefore also $\underline{n}\gamma \in M \setminus \text{Im } f$ as well. Let $X_0 \in X$ where X is from the last part, then $X'_0 = \{\underline{n}\gamma \mid n \in X_0\}$ is subset of M . Let $X_0 \neq X_1 \in X$ be some other set, and let us define X'_1 accordingly. Let $g : X \rightarrow M^{<\omega}$ by $X_1 \mapsto X'_0 \cap X'_1$, but $|M^{<\omega}| = |M| = \omega$ but g is injective and $|X| = 2^\omega$, a contradiction. We deduce that f is a bijection.

Let $g : \omega \rightarrow \{0, 1\}, |g^{-1}(1)| = |g^{-1}(0)| = \omega$ be a function and let $Q = \{q_n\}_{n=1}^\infty$ be the set of the primes. Let X_q be the set such that $x \in X_q \iff q|x$, note that divisibility is definable, and let $R^i = R_{X_{q_i}}$ for any $i < \omega$.

$$p'_g(x) = \{x \neq \underline{n} \mid n < \omega\} \cup \{R^i(x) \mid g(i) = 1\} \cup \{\neg R^i(x) \mid g(i) = 0\}$$

p'_g is consistent from the compactness theorem, and let p_g be its closure, such that p_g is complete and consistent. p_g cannot be isolated as otherwise p'_g can be isolated as well, a contradiction to g 's definition by taking the minimal $i < \omega$ such that R^i does not show up in isolating formula. There are infinitely many such functions g , implying that there are also infinitely many such types p_g . To be precise there are 2^{\aleph_0} such functions, then 2^{\aleph_0} such non-isolated types. \square

Question 2

Let $\omega \leq \kappa$ be some cardinal. We say that a model \mathcal{M} is κ -saturated if for every $A \subseteq M$ with $|A| < \kappa$, any type in $S_1(A)$ is realized.

Part a

Let T be a consistent complete theory over a language of size $\leq \kappa$ with an infinite model.

We will show that there is a κ^+ saturated model of T of cardinality 2^κ .

Proof. By Löwenheim-Skolem theorem we can assume that there is a model $\mathcal{M} \models T$ of cardinality κ . We will construct recursively a κ -saturated model, by recursion on $|A|$ for $A \subseteq M$. For $|A| = \emptyset$ we get $A = \emptyset$, $|\text{form}| \leq \kappa$ implies that $S_1(\emptyset) = S_1(T)$ is of cardinality $\leq 2^\kappa$ as well. Let $\mathcal{M}'_0 \models T \cup \{p(c_p) \mid p \in S_1(\emptyset)\}$ be a model of the language $\mathcal{L} \cup \{c_p \mid p \in S_1(\emptyset)\}$ and let \mathcal{M}_0 be its reduction to \mathcal{L} . $|\mathcal{M}_0| \leq 2^\kappa$ and it is 1-saturated.

Let us assume that $\alpha < \kappa$ is some cardinal and that $\mathcal{M}_\alpha \models T$ is a α^+ -saturated model with $|\mathcal{M}_\alpha| \leq 2^\kappa$. $|\mathcal{P}_{=\alpha}(\mathcal{M}_\alpha)| < \kappa$ and $|\text{form}_{\mathcal{L}(A)}| < 2^\kappa$ then $|S_1(A)| < 2^\kappa$ for any such A as well, meaning that,

$$\Sigma_\alpha = \bigcup_{\substack{A \subseteq \mathcal{M}_\alpha \\ |A| = \alpha}} S_1(A)$$

is of cardinality $\leq 2^\kappa$ as well. We enrich \mathcal{L} by $\{c_p \mid p \in \Sigma_\alpha\}$ and define $\mathcal{M}'_{\alpha+} \models T \cup \{p(c_p) \mid p \in \Sigma_\alpha\}$ be a models, $\mathcal{M}_{\alpha+}$ be its reduction to \mathcal{L} . Then $\mathcal{M}_{\alpha+}$ is α^{++} -saturated and $|\mathcal{M}_{\alpha+}| \leq 2^\kappa$.

κ^+ is regular, thus $\mathcal{M}_{\kappa^+} = \bigcup_{\alpha < \kappa^+} \mathcal{M}_\alpha$ is of cardinality 2^κ and κ^+ -saturated as wished. \square

Part b

We will show that if \mathcal{M} and \mathcal{N} are elementary equivalent κ -saturated models of cardinality κ then $\mathcal{M} \cong \mathcal{N}$, and that any partial elementary map $f : A \rightarrow B$ with $A \subseteq M, B \subseteq N, |A| < \kappa$ can be extended to an isomorphism.

Proof. The proof is similar to the case of ω -saturation, the second statement implies the first one and it suffices to show it. The proof is by induction on ordinals $< \kappa$, the proof for the case of $\kappa = \omega$ can be used as a base for the induction.

Without loss of generality we can assume $A, B = \alpha$ for $\alpha < \kappa$ by the well order principle. The case of successor ordinal is trivial from the case ω . Let us assume that the statement is true for any $\alpha < \beta < \kappa$. Then there exists f_α and if $\alpha < \gamma < \beta$, $f_\gamma \upharpoonright \alpha = f_\alpha$, then let us define $f_\beta = \bigcup_{\alpha < \beta} f_\alpha$. The induction step is completed therefore the statement holds for any $\alpha < \kappa$, and by an identical step there is also a function $f = f_\kappa$ fulfilling our requirements. \square

Part c

We will show that if $\mathcal{M} \models T$ is κ -saturated model of a complete theory T , then every model $\mathcal{N} \models T$ of size κ can be embedded into \mathcal{M} .

Proof. TODO \square