Exercise 2 Answer Sheet — Logic Theory (2), 80424

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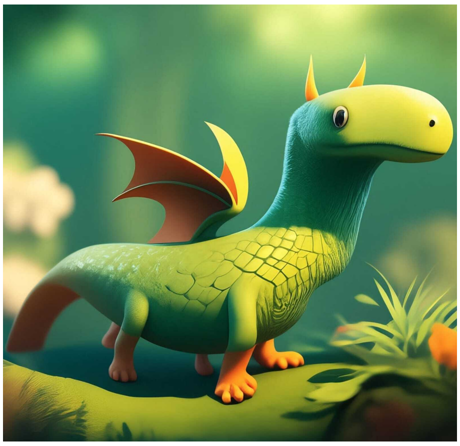

Question 1

Let F be a field and let $L_{F\,\text{VS}}$ be the language of vector spaces over F, $L_{F\,\text{VS}} = \{0, +\} \cup \{\lambda_a \mid a \in F\}$, such that λ_a resolved to scalar multiplication. We assume that $\mathcal{V} \subseteq \mathcal{U}$ are both infinite dimensional vector spaces over F, and will prove that $\mathcal{V} \prec \mathcal{U}$.

Proof. Let us assume that $\psi(x_0, \ldots, x_{n-1})$ is a wff, as well $\varphi(x_0, \ldots, x_n) = \exists x_n \psi(x_0, \ldots, x_{n-1})$. Let $a_0, \ldots, a_{n-1} \in V$, we will show that if $\mathcal{U} \models \varphi(a_0, \ldots, a_{n-1})$ then there is $a_n \in V$ such that $\mathcal{U} \models \psi(a_0, \ldots, a_n)$. By the assumption that $\mathcal{U} \models \varphi(a_0, \ldots, a_{n-1})$ we assume that there is $b \in U$ such that $\mathcal{U} \models \psi(a_0, \ldots, a_{n-1}, b)$. If $b \in V$, then the criteria is fulfilled, then let us assume that $b \notin V$. Let B_V be a basis for $\mathcal{V}$, it is clear that b is linear-independent from B_V, otherwise it would follow that $b \in V$. We choose some $c \in V$ such that $c \in V \setminus \mathrm{Sp}\{a_0, \ldots, a_{n-1}\}$, there must be one by $\mathcal{V}$'s infinite dimension. We construct two bases for $\mathcal{U}$, $B_U^1 \supset B_V \cup \{b\}$, and the other one would be $B_U^2 \supset B_V \cup \{c\}$. Let $M : \mathcal{U} \to \mathcal{U}$ be an automorphism such that $M(a_i) = a_i$ for all $i < n$, and let $M(b) = c$.

We claim that $\mathcal{U} \models \psi(M(a_0), \ldots, M(a_{n-1}), M(b))$, as an automorphism is preserving any relation and function defined over its respective vector-space. then $c \in M$, thus being a witness to out initial claim that there is $c \in \mathcal{V}$ such that $\mathcal{U} \models \psi(a_0, \ldots, a_{n-1}, c)$. Tarski-Vaught tests requirements are all met, it follows that $\mathcal{V} \prec \mathcal{U}$. $\square$

Question 2

Part a

We will show that if $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{K}$ and $\mathcal{M} \prec \mathcal{K}, \mathcal{N} \prec \mathcal{K}$, then $\mathcal{M} \prec \mathcal{N}$.

Proof. Let $\varphi(x_0, \dots, x_{n-1})$ be some wff, and $a_0, \dots, a_{n-1} \in M$ then,

$$\mathcal{M} \models \varphi(a_0, \dots, a_{n-1}) \iff \mathcal{K} \models \varphi(a_0, \dots, a_{n-1}) \iff \varphi(a_0, \dots, a_{n-1})$$

as $M \subseteq N$, hence $a_0, \dots, a_{n-1} \in N$. By the definition of sub-elementary embedding, $\mathcal{M} \prec \mathcal{N}$. □

Part b

We will show that $\mathcal{M} = (\mathbb{N}, <) \prec (\mathbb{N}, <) + (\mathbb{Z}, <) = \mathcal{N}$, where addition of order models is defined as the disjoint union of the universes and the order is the lexicographic order.

Proof. By the EF-games we showed that $\mathcal{M} \equiv \mathcal{N}$. We intend to use Tarski-Vaught test, it followed that we assume that $\psi(x_0, \dots, x_n)$ is a wff over L, and let $\varphi(x_0, \dots, x_{n-1}) = \exists x_n \psi(x_0, \dots, x_n)$. We assume that $a_0, \dots, a_{n-1} \in \mathbb{N}$ such that $\mathcal{N} \models \varphi(a_0, \dots, a_{n-1})$. We will prove that there is $a \in \mathcal{M}$ such that $\mathcal{N} \models \psi(a_0, \dots, a_{n-1}, a)$. From $\mathcal{M} \equiv \mathcal{N}$ it derives that if $\phi = \exists x_0 \dots \exists x_{n-1} \varphi(x_0, \dots, x_{n-1})$ then $\mathcal{M} \models \phi \iff \mathcal{N} \models \phi$. But we assumed that $\mathcal{N} \models \varphi(a_0, \dots, a_{n-1})$, it follows that $\mathcal{N} \models \phi$, then $\mathcal{M} \models \phi$ as well. Let $b_0, \dots, b_n \in \mathbb{N}$ such that $\mathcal{M} \models \psi(b_0, \dots, b_n)$, the witnesses to $\mathcal{M} \models \phi$. It is sufficient to show that $b_i \mapsto a_i$ is an embedding of $\mathcal{M}$ into $\mathcal{M}$. We can assume that there is such mapping, as otherwise, it would follow that $\mathcal{M} \not\models \phi$. $\mathcal{M} \models \psi(a_0, \dots, a_{n-1}, b)$ for b such that the embeddings value at b_n, therefore $b \in N$ and $\mathcal{N} \models \psi(a_0, \dots, a_{n-1}, b)$. Then by Tarski-Vaught test we deduce $\mathcal{M} \prec \mathcal{N}$. □

Part c

We will find an example for three models $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{K}$ such that $\mathcal{M} \prec \mathcal{N}, \mathcal{M} \prec \mathcal{K}$ but $\mathcal{N} \not\prec \mathcal{K}$.
Solution. We define, $\mathcal{M} = (\mathbb{N}, <), \mathcal{K} = (\mathbb{N}, <) + (\mathbb{Z}, <)$ and $\mathcal{N} = (\mathbb{N}, <) + (2\mathbb{Z}, <)$. It is inferred directly by definition that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{K}$ and by the last part, $\mathcal{M} \prec \mathcal{K}$. We also note that $\exists x(\langle 2, 1\rangle < x < \langle 4, 1\rangle)$ is a wff that holds in $\mathcal{N}$ but not in $\mathcal{K}$, implies that $\mathcal{N} \not\prec \mathcal{K}$. Lastly, by the same proof of part b, it derives that $\mathcal{M} \prec \mathcal{N}$.

Question 3

Part a

We assume that $\langle f_r \mid r \in \mathbb{R} \rangle$ are functions $f_r : \mathbb{N} \to \mathbb{N}$ such that if $r \neq q$, then there is a $k \in \mathbb{N}$, such $f_r(n) \neq f_q(n)$ for all $n \geq k$. Let $\mathcal{M} = (\mathbb{N}, <, \langle f_r \mid r \in \mathbb{R} \rangle)$ be structure in the language $\{f_r \mid r \in \mathbb{R}\}$. We will show that if $\mathcal{N} \succ \mathcal{M}$ and $\mathcal{N} \neq \mathcal{M}$ then $|N| \geq |\mathbb{R}|$.

Proof. Let $\varphi_{r,q}(k) = \forall n \geq k, f_r(n) \neq f_q(n)$. For each $r, q \in \mathbb{R}$, $\mathcal{M} \models \varphi_{r,q}(k)$ for some $k \in \mathbb{N}$. From $\mathcal{M} \prec \mathcal{N}$ it follows that $\mathcal{N} \models \varphi_{r,q}(k)$ for $k \in \mathbb{N}$ as well, for every $r, q \in \mathbb{R}$. By the elementary embedding and the formula $\psi(x, y) = \forall z \neq x, y,\ \neg(x < z < y)$ we can deduce that elements of $\mathcal{N}$ are not bounded by elements of $\mathcal{M}$. $\mathcal{N} \neq \mathcal{M}$, then there is $\alpha \in N \setminus M$, and by the last statement it derives that $n < \alpha$ for every $n \in \mathbb{N}$. For every $r \in \mathbb{R}$, $f_r(\alpha) \in N$, then by α's relation to the naturals we deduce that for every $r, q \in \mathbb{R}, \mathcal{N} \models f_r(\alpha) \neq f_q(\alpha)$. Finally, $\{f_r(\alpha) \mid r \in \mathbb{R}\} \subseteq N$, implying that $|N| \geq |\mathbb{R}|$. $\square$

Part b

We will show that such a sequence of function exists.

Proof. We define the ultrafilter $D \subseteq \mathcal{P}(\mathbb{N})$ by $X \in D \iff \exists n \in \mathbb{N}, X = [n]$, that is D is the collection of all finite beginnings of the natural numbers. We then define $\mathcal{M} = \mathbb{N}^{\mathbb{N}}/D$, in this model every elements representative is a function $\mathbb{N} \to \mathbb{N}$ such that it fulfills the requirement of the last part. Lastly, we use choice to map each such function to unique real number. $\square$

Part c

We will conclude that downwards Löwenheim-Skolem theorem does not hold without the restriction of $|L|$ in the cardinality inequality, $|N| \leq |L| + |A| + \aleph_0$.

Proof. If we were to take $\mathcal{N} = (\omega_1, <, \langle f_r \mid r \in \mathbb{R} \rangle)$, a model like in the first part, and the set $A = \omega + 1$, then the model $\mathcal{M}$ derived from the theorem satisfies,

$$\omega_1 = |N| \leq |A| + \aleph_0 = \omega$$

in contradiction to $\omega_1 > \omega$. $\square$

Question 4

Let us assume that G is a simple group. We will show that if $H \prec G$ then H is simple.

Proof. H is a group, as the formulas to represent existence of neutral element, as well the closure to operator and existence of inverted element are all first-order. If a group is simple by the first isomorphism theorem for groups every $f : G \to G$ which is not trivial is bijective, meaning it is an automorphism, but we know that we can represent automorphisms by conjugation. It follows that for every $g, g' \in G$, exists $l \in G$ such that $lgl^{-1} = g'$, or equivalently, $\psi = \forall g, g', \exists l,\ lgl^{-1} = g'$. By the elementary embedding $H \models \psi$ as well, but then by ψ there are not normal subgroups to H, as if there would be one, it would consist of H and be trivial. We conclude that H is simple. $\square$