# Exercise 1 Answer Sheet — Logic Theory (2), 80424

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# Question 1

Let us assume that $D$ is an ultrafilter on given set $X$.

## Part a

Suppose that there is some $s \in D$ such that $|s| < \omega$, we will show that $D$ is principal.

*Proof.* We will assume that $s$ is minimal in relation to cardinality in $D$, otherwise let us choose the minimal such $s$ by using the well ordering principal. There is a unique such $s \in D$, otherwise $|s| = |s'|$ and $|s \cap s'| < |s|$, a contradiction to the minimality of the cardinality of $s$.
For every $x \in D$, there is some $y \in D$ such that $y \subseteq x \cap s$, but by $s$ minimality, $y = x \cap s$, hence $s \subseteq x$. In the other direction, if $s \subseteq x$, then $x \in D$ by filter properties.
Therefore $D = \{x \in \mathcal{P}(X) \mid s \subseteq x\}$, $D$ is a principal ultrafilter defined by $s$. □

## Part b

We will show that if $D \subseteq \mathcal{P}(I)$ is a principal ultrafilter, $\langle \mathcal{M}_i \mid i \in I \rangle$ is a sequence of $L$-structures, then $\mathcal{N} = \prod_{i \in I} \mathcal{M}_i / D \cong M_i$ for some $i \in I$.

*Proof.* Let $s \subseteq I$ be the defining set for $D$, and let us fix $i \in s$. Let $f : N \to M_i$ such that $f([g]_D) = g(i)$. $f$ is well defined, as every for every $h, h' \in [h]$, it is follows that $h \restriction s = h' \restriction s$ by $D$'s definition as principal.

We will show that $f$ is an isomorphism between $\mathcal{N}$ and $\mathcal{M}_i$. $L$-constants are being preserved directly from definition of ultra-products. Assume $F \in \mathrm{Func}_{L,n}$, and $t_0, \dots, t_{n-1} \in \mathrm{term}_L$, then,

$$f(F^{\mathcal{N}}(t_0^{\mathcal{N}}, \dots, t_{n-1}^{\mathcal{N}})) = f([\{F^{\mathcal{M}_j}(t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j}) \mid j \in I\}]) = F^{\mathcal{M}_i}(t_0^{\mathcal{M}_i}, \dots, t_{n-1}^{\mathcal{M}_i})$$

meaning $f$ preserves terms. Let $R \in \mathrm{rel}_{L,n}$, and let $t_0, \dots, t_{n-1} \in \mathrm{term}_L$, then,

$$\mathcal{N} \models R(t_0, \dots, t_{n-1}) \iff \{j \in I \mid \langle t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j} \rangle \in R^{\mathcal{M}_j}\} \in D \implies \langle t_0^{\mathcal{M}_j}, \dots, t_{n-1}^{\mathcal{M}_j} \rangle \in R^{\mathcal{M}_i}$$

We conclude that $f$ is indeed a structures isomorphism. □

## Question 2

We will show that if $F$ is a filter on a set $X$ such that if $s \in F$ then $s$ is infinite, then there is a non-principal ultrafilter extending $F$.

*Proof.* The statement that $F$ has not finite elements is of first-order then can be using to separate on the family of all extensions of $F$, by Zorn's-lemma we get some ultrafilter $D$ such that $F \subseteq D$ and $\forall s \in D \implies |s| \geq \omega$.

We will prove that is $D$ is not principal by assuming it is in order to get contradiction. We denote $s_0 \subseteq F$ as the generating element of $D$, meaning $s \in D \iff s_0 \subseteq s$. We assume that $\emptyset \subsetneq s_1 \subsetneq s_0$, then $s_1 \not\supseteq s_0 \implies s_1 \notin D \iff X \setminus s_1 \in D \implies (X \setminus s_1) \cap s_0 \in D$. But $s_0 \setminus s_1 \subsetneq s_0$, it follows that $s_0 \setminus s_1 \notin D$, then there is not such $s_1$, meaning that $s_0$ is a singleton. Lastly, it derives that $|s_0| = 1 < \omega$, contradicting $D$'s definition. □

# Question 3

## Part a

Let $\mathcal{N} = (\mathbb{N}; R^{\mathcal{N}})$ be graph such that $(a, b) \in R^{\mathcal{N}} \iff |b-a| = 1$. Let $D$ be a non-principal ultrafilter on $\mathbb{N}$ (An example of one is the construction from exercise 2). Let $\mathcal{M} = \mathcal{N}^{\mathbb{N}}/D$. We will prove that $\mathcal{M}$ is not path-connected.

*Proof.* Let us define $a = [0], b = [id]$, in the sense that the constant valued function $f(n) = 0$, it derived that $f \in a$. We claim that there is no finite path between $a, b \in N$. We assume otherwise, and define $\langle [v_i] \mid i < n\rangle$ for some $n < \omega$, such that $([v_i], [v_{i+1}]) \in R^{\mathcal{N}}$ for every $i < n$, and $[v_0] = a, [v_{n-1}] = b$. $D$ is non-principal then there is some $i < \omega$ such that $v_i(i)$ testifies to the relation, and by the last exercise $i$ is as large as we desire, then we assume $i > n$. But $v_0(i) = 0, v_{n-1}(i) = i$, and by the relation definition $v_j(i) = j$ for every $0 \le j \le n$, it is implies that $i = v_{n-1}(i) < n < i$, a contradiction. We conclude that $\mathcal{M}$ is not path-connected. $\square$

## Part b

Let $\mathcal{R} = (\mathbb{R}; +, \cdot, 0, 1, <)$. Let $D$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{R}^* = \mathcal{R}^{\mathbb{N}}/D$.
We will show that in $\mathcal{R}^*$ there is an infinitesimal element, namely $\epsilon \in R^*$ such that $\forall n \in \mathbb{N}, 0 < \epsilon < \frac{1}{n}$.

*Proof.* Let $d \in D$ be an unbounded subset of $\mathbb{N}$, and let $r : \mathbb{N} \to \mathbb{R}$ such that $(r \restriction d)(n) = \frac{1}{n+1}$, We define $\epsilon = [r]_{\mathcal{R}^*}$. We observe that $1^{\mathcal{R}^*} = [\{\langle n, 1\rangle \mid n \in \mathbb{N}\}]$ by definition, and it is follows that $(\frac{1}{n})^{\mathcal{R}^*} = [\{\langle n, \frac{1}{n}\rangle \mid n \in \mathbb{N}\}]$. Let $m \in \mathbb{N}$ be some number, we will show that $\mathcal{R}^* \models \epsilon < \frac{1}{n}$,

$$\mathcal{R}^* \models \epsilon < \frac{1}{n} \iff \{n \in \mathbb{N} \mid \mathbb{R} \models \epsilon < \frac{1}{m}\} \in D \impliedby \{n \in d \mid \epsilon(n) = \frac{1}{m+1} < \frac{1}{m}\} \in D$$

and by $d$'s definition the statement indeed holds. Lastly we will note that $\mathcal{R}^* \models 0 < \epsilon$, it is derived by the same method. $\square$

# Question 4

We will define a filter $F$ on $X$ as $\sigma$-complete if and only if for every $\{s_\alpha \mid \alpha < \omega\} \subseteq F$, $\bigcap_{\alpha<\omega} s_\alpha \in F$.

## Part a

We will show that if $F$ if a $\sigma$-complete ultrafilter and $|X| = \omega$ then $F$ is principal.

*Proof.* An answer using AC without necessity,
Let $\{s_\alpha \mid \alpha < \omega\} \subseteq F$ be some collection, and let us define $t_0 = s_0$ and for every $\alpha < \omega$, $t_{\alpha+1} = s_\alpha \cap t_\alpha$. We also define $t_\omega = \bigcap_{\alpha<\omega} t_\alpha$, this is indeed a set, and $\in F$ in particular, due to $F$ being $\sigma$-complete. The item $t_\omega$ is a supremum in relation of $\supseteq$ in the chain $\langle t_\alpha \mid \alpha < \omega\rangle$, and it follows that every chain has such supremum. By Zorn's lemma we conclude that there is $s_m \in F$ which is maximal according to the relation $\supseteq$, then $\forall s \in F, s_m \subseteq s$. If $s_m \subseteq s$ for any $s \subseteq X$, then by filters properties $s \in F$, concluding that $F = \{s \subseteq X \mid s_m \subseteq s\}$.

We can also use the well-founded order of $\subseteq$ of $X$ and take a minimal of it, omitting the part of Choice axiom.

We can also assume that it is non-principal, therefore for every $x \in X$, $X \setminus \{x\} \in F$ (as it is an ultrafilter), then $\bigcap_{\alpha<\omega} X \setminus \{x_\alpha\} = \emptyset \in F$, a contradiction. ☐

## Part b

We will provide an example of a $\sigma$-complete ultrafilter non-principal filter on an infinite set.
*Solution.* Let $\mu < \kappa$ be cofinal cardinals such that $\omega_1 < \mu$. We define $U = \{\delta < \kappa \mid \mu < \delta\}$. This is indeed an ultrafilter as concluded from the cofinality of $\mu$, and it is non-principal as derived from extension of question 2. It is clear that $U$ is $\sigma$-complete, as $\omega_1$ intersections are preserving cofinality for $\mu$.

## Part c

We will show that if $\mathcal{M}_i = (X_i, <_i)$ are nonempty well-ordered sets for $i \in I$ and $D$ is a $\sigma$-complete ultrafilter on $I$, then $\mathcal{N} = \prod_{i\in I} \mathcal{M}_i/D$ is well-ordered.

*Proof.* Using Löwenheim–Skolem we can assume without loss of generality that $\mathcal{N}$ is a countable model. Let $\{[f_j] \in N \mid j < J\}$ be some collection, we will show that there if $[f] \in Y$ minimal in $<^{\mathcal{N}}$. Let $f(i) = \min_{<^{\mathcal{M}_i}}\{f_j(i) \mid j < J\}$, this definition holds as $<^{\mathcal{M}_i}$ is a well-order. By $\sigma$-completeness we deduce that $\{i < I \mid f_j(i) = f(i), j < J\} = \bigcap_{j<J}\{i < I \mid f_j(i) = f(i)\}$, implying that indeed $[f] \in N$, concluding our claim. ☐

## Part d

We will show that the statement of part c does not hold without the assumption that $D$ is $\sigma$-complete.
*Solution.* Let $\mathcal{N} = (\mathbb{N}; <)^{\mathbb{N}}/D$ for the ultrafilter defined in question 2. It is clear that $\mathcal{N}$ fulfills the requirements, and that $D$ is not $\sigma$-complete. We will construct a set and show there is no minimum in it. Let $s = \{[f] : \mathbb{N} \to \mathbb{N} \mid \forall n \in \mathbb{N} f(n) \le f(n+1)\} \subseteq N$, and let $[f] \in s$, we choose the function $[g] \in s$ such that $f(n) = g(n)$ for $n > M \in \mathbb{N}$, and $g(n) = f(M)$ otherwise. Its follows that $[g] <^{\mathcal{N}} [f]$, therefore $f$ is not minimal, we conclude that indeed $s$ witnessing that $<^{\mathcal{N}}$ is not a well-order.