Exercise 9 Answer Sheet — Logic Theory (2), 80424

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Question 1

Part a

We will show that the set,

$$H = \{e \in \mathbb{N} \mid (e, 0) \in \operatorname{dom} U\}$$

is recursively-enumerable but not recursive.

Proof. H is Σ_1^0 by the definition of U, implies that it is recursively-enumerable, then it is sufficient to show that it is not recursive. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by,

$$f(x) = x.$$

This is indeed a recursive function (even primitive-recursive), and for every $e \in D$, when $D = \{e \in \mathbb{N} \mid (e, e) \in \operatorname{dom} U\}$, we get $f(e) = e \in H$, as e witnessing bound larger than 0. It follows that f is a recursive reduction of D to H and implies that H is not recursive. □

Part b

We will conclude that $H^C = \mathbb{N} \setminus H$ is not recursively-enumerable.

Proof. H is a relation, meaning that if it were to be Π_1^0 it would be Δ_1^0 and recursive. If H^C is recursively-enumerable then $H^C = \{e \in \mathbb{N} \mid (e, 0) \in \operatorname{dom}(\neg U)\}$ is Σ_1^0, and implying by negation that $H = \neg H^C$ (in the sense of relations) is Π_1^0. But H is recursively-enumerable and Σ_1^0, meaning that it is indeed Δ_1^0, in contradiction to its being not recursive. □

Part c

Let us define the set,

$$T = \{e \in \mathbb{N} \mid \{x \mid (e, x) \in \operatorname{dom} U\} = \mathbb{N}\}$$

We will show that T is not recursively-enumerable.

Proof. We assume in contradiction that T is recursively-enumerable.
By the equivalence to recursively-enumerable functions proposition we can assume that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that f is primitive-recursive and $T = \{f(x) \mid x \in \mathbb{N}\}$. Let us define $g : \mathbb{N}^2 \to \mathbb{N}$ by,

$$g(n, x) = U(f(n), x) + 1$$

g is indeed total recursive function as U is recursive and f is total such that $U(f(n), \cdot)$ is total recursive function. $h(x) = g(x, x) + 1 = U(f(x), x) + 1$ is a total recursive function. This means that $\#h \in T$, and there is $m \in \mathbb{N}$ such that $f(m) = \#h$.

$$h(m) = U(f(m), m) + 1 = U(\#h, m) + 1 = h(m) + 1$$

This is contradiction by diagonalization. □

Part d

We will show that $T^C = \mathbb{N} \setminus T$ is not recursively-enumerable.

Proof. We will find a function that maps the code of a recursive function to the code of a constant function to return the value at zero of the initial function. In other words, we want a function g such that,

$$U(g(e), x) = U(e, 0)$$

Let us define,

$$F(e,x) = U(e,0)$$

By the fixed point theorem there exists g recursive such that,

$$U(g(e),x) = F(e,x) = U(e,0)$$

Then g is exactly the function we wanted.

For any $e \in H$ we get some total recursive function code, and in particular $g(e) \in T$. For $e \notin H \iff e \in H^C$ it follows that $g(e)$ is not defined for any $x \in \mathbb{N}$, in particular $g(e) \in T^C$. We conclude that $g^{-1}(T^C) = H^C$, if we assume in contradiction that T^C is recursively-enumerable then by the pre-image theorem it follows that H^C is recursively-enumerable, a contradiction to the previous parts conclusion. $\square$

Question 2

We will show that any infinite recursively-enumerable set contains an infinite recursive subset.

Proof. Let X be some recursively-enumerable set. Let $f : \mathbb{N} \to \mathbb{N}$ be a total primitive-recursive function such that $X = f(\mathbb{N})$. We saw that max is primitive-recursive, then,

$$g(x) = \max\{f(n) \mid n < x\}$$

is primitive-recursive as well. Note that f is unbounded, therefore g is unbounded as well. We define $h : \mathbb{N} \to \{0, 1\}$ by,

$$h(x) = \begin{cases} 1 & g(x) = f(x) \\ 0 & \text{otherwise} \end{cases}$$

This is primitive-recursive function as the cases operation is primitive, and $h = \chi_Y$ for $Y \subseteq X$ recursive as we wanted. $\square$

Question 3

We define the sets $A, B \subseteq \mathbb{N}$ as recursively-separated if there is some recursive set $C \subseteq \mathbb{N}$ such that $A \subseteq C$ and $B \cap C = \emptyset$. We will find an example of two disjoint recursively-enumerable subsets of $\mathbb{N}$ which cannot be recursively-separated.

Solution. Let $f : \mathbb{N} \to \mathbb{N}$ be the function such that $f(\mathbb{N}) = H$ from the first question. We define $A = \{f(2n) \mid n \in \mathbb{N}\}$ and $B = \{f(2n+1) \mid n \in \mathbb{N}\}$, these are clearly recursively-enumerable but not recursive. We also see that $A \cap B = \emptyset$ directly by definition.

Let us assume in contradiction that there is a recursive $C \subseteq \mathbb{N}$ such that $A \subseteq C$ and $B \cap C = \emptyset$. But the set of recursive functions R is recursive and the reduction of recursive function is recursive, then $R \cap C = A$ is recursive, in contradiction to question 1.

Note that f might not be injective, meaning that $A \cap B$ might not be null. To solve this, we define new recursive function, $g : \mathbb{N} \to \mathbb{N}$ by,

$$g(n) = \min\{f(k) \mid i > n, f(k) > g(n-1)\}$$

This function is recursive directly as the minimization of recursive function f. We also derive from $|\operatorname{Im} f| = \omega$ that g is total and that $\operatorname{Im} g \subseteq \operatorname{Im} f$. In particular, using g instead of f we get $A \cap B = \emptyset$ as wished.