Exercise 4 Answer Sheet — Logic Theory (2), 80424

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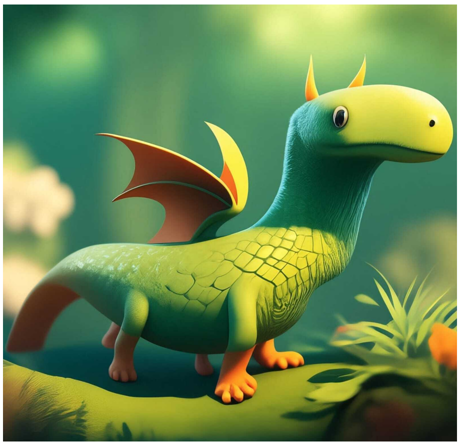

Question 1

Part a

Was solved in the last exercise.

Part b

We will show that Euclid's algorithm works in PA, meaning that we will show that PA proves that for all $0 < x, y$ there is some $z = \gcd(x, y)$, namely that for all w, $w \mid z \iff w \mid x, w \mid y$ and that $z + ax = by$ or $z + by = ax$ for some a, b.

Proof. We define $\varphi(n) = n \geq x, y > 0 \to \exists z, z \mid x, z \mid y, \forall w((w \mid x \wedge w \mid y) \to z \mid w)$, meaning that there is gcd for every two non-zero numbers there is gcd. We intend to prove the claim using induction over n. For $y = 1$ we choose $z = 1$, indeed $z \mid x$ and $z \mid y$, and by axiom N7 and N4 we deduce that if $w \mid x, w \mid y$ then $w = 1$ and $z \mid w$. We assume that φ holds for $y' < y$ and will show that it holds for y as well. It is important to note that full induction is directly proven from induction axiom scheme by using formula iterating over φ. $<$ is linear order, and by the definition of ϕ, $\gcd(x, y) = \gcd(y, x)$, then without loss of generality either $x < y$ or $x = y$. if $y = x < n$ then we assumed the claim holds, otherwise $x = y = n$. In such case we choose $z = n$ as well, and prove similarly to the case $n = 1$ that it is indeed greatest common divisor. It is remains to test $x < y = n$. If $w \mid x, w \mid y$ then $w \mid y - x$, when $y - x$ is indeed defined by the last part. Note that $y - x < n$ as $1 \leq x$, then there is such z for $x, y - x$, and therefore for x, y as well. $\square$

Part c

We will conclude that if p is a prime and $p \mid a \cdot b$ then either $p \mid a$ or $p \mid b$.

Proof. If $p \mid a$ the claim holds, then we assume otherwise that $p \nmid a$ and will show $p \mid b$. $\gcd(a, p) = 1$, as only $1, p \mid p$. Then there is k such that $1 + ka = lp$, it follows that $b + k \cdot ab = lpb$. We denote by $-$ the subtraction in the sense that if $x < y$ there is α such that $x + \alpha = y$, and then $b = lb \cdot p - k \cdot ab$. From the last part, this number is divided by p, implying that $p \mid b$. $\square$

Question 2

Part a

We will show that Σ^0_0 is closed under boolean operations and bounded quantifiers.

Proof. We redefined the boolean operations as the implication relation, and thus we will show that if $\varphi, \psi \in \Sigma^0_0$, then $\varphi \to \psi \in \Sigma^0_0$ as well. By $\varphi, \psi \in \Sigma^0_0$ we infer that (without loss of generality) all quantifiers in these formulas are bounded, and $\varphi \to \psi$ does not introduce new quantifiers to the formula. We can deduce that $\varphi \to \psi \in \Sigma^0_0$.

Σ^0_0 is closed under bounded quantifiers directly by definition.

It is important to note that the fully formalised proof would include induction over the structure of the formula. $\square$

Part b

We will show that for every $n > 0$, the class of Σ^0_n formulas is closed under bounded quantifiers, existential quantifiers and disjunctions, conjunctions.

Proof. For every formula φ such that $v_0, v_1 \notin \mathrm{FV}(\varphi)$, $\forall v_0 \forall v_1 \varphi \iff \forall v_1 \forall v_0$, it can be shown using induction over the structure of the formula. This statement holds even in the case one (or more) of the quantifiers is bounded (but not bounded by v_0 or v_1). It holds for all quantifiers as well.

We will prove closeness under bounded quantifiers of Σ^0_n, Π^0_n by induction over n. For $n = 0$ it follows from part 1. Let us assume the claim is true for n and we will show it also true for $n+1$. Assuming $\exists x_0 \varphi \in \Sigma^0_n$ be a Σ_n formula. The formula $\forall v \leq c_v \exists x_0 \varphi$ is equivalent by the last statement to $\exists x_0 \forall v \leq c_v \varphi$ (when $c_v \neq x_0$) and by the induction hypothesis is Σ^0_n.

We move to show that Σ^0_n is closed under existential quantifiers. Let $\exists x \varphi \in \Sigma^0_n$. Then $\exists x \varphi \equiv_{\text{tau}} \exists y \varphi^x_y$. It follows that $\exists x \exists y \varphi \equiv \exists x \exists y \leq x (\varphi \lor \varphi^{x,y}_{y,x})$, as for every constants fulfilling this formula, their size comparison would not affect the formula.

Lastly we will show that if $\varphi, \psi \in \Sigma^0_n$ then $\varphi \land \psi, \varphi \lor \psi \in \Sigma^0_n$ as well. Let us assume that $\varphi, \psi \in \Sigma_n$ without loss of generality. By definition $\varphi = \exists v_0, \forall v_1, \ldots, \exists v_{n-1} \varphi_0$ and $\psi = \exists v_0, \forall v_1, \ldots, \exists v_{n-1} \psi_0$ for some $\varphi_0, \psi_0 \in \Sigma_0$. By conjunction identities we can derive that $\varphi \land \psi \equiv \exists v_0, \forall v_1, \ldots, \exists v_{n-1} \varphi_0 \land \psi_0$, it follows that $\varphi \land \psi \in \Sigma^0_n$. Disjunction is similar. $\square$

Part c

We will show that for every $n > 0$, Π^0_n is closed under bounded quantifiers, universal quantifiers, disjunctions and conjunctions.

Proof. We showed closeness under bounded quantifiers, and the proof for closeness under disjunctions and conjunctions is the same, it is yet to be shown closeness under universal quantifiers.

For every x, y, there is a relation $x \leq y$ or $l \leq x$, then $\forall x \forall y \varphi$ holds if and only if $(\forall x \forall y \leq x \varphi) \land (\forall y \forall x \leq y \varphi)$. In other words, there is finite set of conjuncted formulas such that they are equivalent to the quantifying over finite number of linearly-ordered variables. By this claim we can deduce that Π^0_n is closed under global quantifiers, as it is closed under conjunction. $\square$

Part d

We will show that the negation of a Σ^0_n-formula is Π^0_n-formula.

Proof. Let $\varphi = \exists v_{n-1} \forall v_{n-2} \ldots \exists v_0 \varphi_0$ be a Σ_0-formula to testify that $\varphi' \in \Sigma_n^0$ for some such formula. By De-Morgan rule for quantifiers it follows that,

$$\neg\varphi' \equiv \neg\varphi \equiv \forall v_{n-1} \neg \ldots \varphi_0 \equiv \ldots \equiv \forall v_{n-1} \ldots \forall v_0 (\neg\varphi_0) \in \Pi_n$$

It is implied that $\neg\varphi' \in \Pi_n^0$ as intended. $\square$

Part e

We will prove that every arithmetic relation is in Σ_n^0 or Π_n^0 for some n.

Proof. We will prove by induction over the structure of the formula.

For atomic formulas, every such formula is quantifier-less and thus Σ_0 and therefore Σ_0^0 as well. For negation of formulas, if $\varphi \in \Sigma_n^0$ or $\varphi \in \Pi_n^0$ then by the last part $\neg\varphi \in \Pi_n^0$ or $\neg\varphi \in \Sigma_n^0$. By parts b and c the sets are both closed under conjunction, with the addition of completeness of $\{\wedge, \neg\}$ and the closeness to negation, it follows that the induction step for binary relations holds. Let us assume that $\varphi \in \Sigma_n^0$, then $\exists v \varphi \in \Sigma_n^0$ by part b, and $\forall v \varphi \in \Pi_{n+1}^0$ by definition of Π_{n+1}. For similar reasons if $\varphi \in \Pi_n^0$ then the formula is closed to quantifiers as well.

It derives that indeed for every $\varphi \in L_{\text{PA}}$, either $\varphi \in \Sigma_n^0$ or $\varphi \in \Pi_n^0$ for some $n \in \mathbb{N}$. $\square$

Question 3

Let PA' be a theory equivalent to PA except that the axiom scheme for induction only holds for parameter-less formulas (of the form $\varphi(v_0)$). We will show that $\mathrm{PA} \equiv \mathrm{PA}'$.

Proof. $\mathrm{PA}' \subseteqq \mathrm{PA}$ is trivial, it is sufficient to show that $\mathrm{PA}' \models \mathrm{Ind}(\varphi(v_0, \ldots, v_{n-1}))$ for every $n < \omega$ and such φ.

Let $\varphi(v_0, \ldots, v_{n-1})$ be some formula and $\mathcal{N} \models \mathrm{PA}'$ a model such that if $\mathcal{N} \models \mathrm{PA}$ then $\mathcal{N} \models \mathrm{Ind}(\varphi)$. Let $c_1, \ldots, c_{n-1} \in N$ be new constants we introduce to the language, and define $c_i^{\mathcal{N}} = a_i$ for some $a_i \in N$ for $0 < i < n$. Let $\psi(v_0) = \phi(v_0, c_1, \ldots, c_{n-1})$, then $\mathcal{N} \models \mathrm{Ind}(\psi(v_0))$ by the axiom scheme of induction in PA'. From logic 1 we infer that indeed $\mathcal{N} \models \mathrm{Ind}(\varphi)$, for every such model and formula, meaning that $\mathrm{PA}' = \mathrm{PA}$. $\square$

Question 4

Let us assume that $\mathcal{M} \models \text{PA}$. An element $e \in M$ is called *standard* if $e = \underline{n}^{\mathcal{M}}$ for some $n \in \mathbb{N}$. A model is called *standard* if all its elements are standards. Otherwise $\mathcal{M}$ is called non-standard.

Part a

We assume that $\mathcal{M} \models \text{PA}$ is non-standard, and that $\varphi(v)$ is a formula over L_{PA} with parameters from $\mathcal{M}$, such that $\mathcal{M} \models \varphi(\underline{k})$ for all $k \in \mathbb{N}$. We will show that there is some non-standard element $e \in M$ such that $\mathcal{M} \models \varphi(e)$.

Proof. It follows directly from the definition of $\mathcal{M}$ that both $\mathcal{M} \models \text{Ind}(\varphi)$ and $\mathcal{M} \models \varphi(\underline{k})$ for all naturals. It follows that $\mathcal{M} \vdash \forall v_0 \varphi(v_0)$, using specific language such that φ is indeed a formula of the model, and then restriction of the language. $\mathcal{M}$ is non-standard, then there is $e \in M$ that testifies to this fact, and by global quantifiers identities it is implied that $\mathcal{M} \models \varphi(e)$. □

Part b

We assume that $\mathcal{M}$ is a non-standard model. We will show that the order on $\mathcal{M}$ is of the form $\langle \mathbb{N}, < \rangle + C$ where C is either empty or of the form of a dense sum, with no first and last elements, of $\mathbb{Z}$, meaning that $\langle \mathcal{M}; < \rangle \cong \langle \mathbb{N} + Q \times \mathbb{Z}; < \rangle$ for some $Q \in \text{DLO}$.

Proof. From Ehrenfeucht–Fraïssé game we deduce that $\langle \mathbb{N}, < \rangle \equiv \langle \mathbb{N} + Z \times \mathbb{Z} \rangle$ for Z finite order or well-order. If $\langle M, <^{\mathcal{M}} \rangle$ is a well-order, then we can show using the previous statement that indeed it is equivalent to $\langle \mathbb{N}, < \rangle$. Let us assume that $\mathcal{M}$ is not a well-order, then the class of non-standard elements is not well-ordered and does not contain a minimum. Let $e \in M$ be some non-standard element, $e \neq 0$ implies that $0 < e$, and from axiom N8, $\underline{n}^{\mathcal{M}} < e$ for all $n \in \mathbb{N}$. Addition preserves the order, therefore $e + \underline{k} < e + e = 2 \cdot e$ for every $k \in \mathbb{N}$. It follows that the non-standard class does not contain either a maximum or a minimum. Let A be the equivalency class of non-standard elements such that $e \sim e' \iff \exists k \in \mathbb{N}, e = e' + \underline{k} \vee e' = e + \underline{k}$. We can extend the order of $\mathcal{M}$ over it. We claim that A is dense. It does not has minimum or maximum, and if $[e] < [e']$ then if $[b] < [e]$ then $[e] < [e + b] < [e']$ as a direct result from the last part and the size of $[b]$. □