Solution to Exercise 0 — Model Theory (1), 80616

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Question 1

Let $L = \{P\}$ a language where P is unary relation. Define,

$$\varphi_n = \exists x_0 \dots \exists x_n \left(\bigwedge_{i \le n} P(x_i) \wedge \bigwedge_{i<j\le n} x_i \neq x_j \right), \quad \psi_n = \exists x_0 \dots \exists x_n \left(\bigwedge_{i \le n} \neg P(x_i) \wedge \bigwedge_{i<j\le n} x_i \neq x_j \right)$$

and let $T = \{\varphi_n, \psi_n \mid n < \omega\}$.
We will show that $\mathrm{cl}_{\vDash} T$ is ω-categoric.

Proof. Let us define the model $\mathcal{M} \vDash T$ such that $M = 2 \times \mathbb{N}$ and $\langle 0, n \rangle \in P^{\mathcal{M}}$ for $n < \omega$, as well $\langle 0, n \rangle \notin P^{\mathcal{M}}$. It is clear that for any n, we can choose $\langle \langle 0, i \rangle \mid i < n \rangle$ as witnesses for φ_n, and similarly choose $\langle 1, i \rangle$ for ψ_n.
Let $\mathcal{N} \vDash T$ be some countable model, we will show that $\mathcal{N} \cong \mathcal{M}$. For $n = 0$, $\mathcal{N} \vDash \varphi_n \implies \mathcal{N} \vDash \exists x\ P(x)$, and let $f(0,0) = x$, and in similar manner $f(1,0) = y$ for the witness of $\psi_0 = \exists \neg P(x)$. We use this as a basis for recursive definition of a function $f : M \to N$, it will be required to show in induction over n that we can choose explicitly x_i in φ_n for $i < n$.
Let us assume the induction and recursion step, meaning that $f \upharpoonright 2 \times n$ is defined. $\mathcal{N} \vDash \varphi_{n+1}$, meaning that there are at lest $n+1$ elements of $\mathcal{N}$ such that they are in $P^{\mathcal{N}}$, by the pigeonhole principle there is at least one element $a \in N$ such that $f(0,k) \neq a$ for $k < n+1$, then we can define $f(0, n+1) = a$. This conclude our step, hence there is such function f, and by our construction it also holds that f is embedding of $\mathcal{M}$ into $\mathcal{N}$, but

This is all irrelevant, I can just use the set definition of $\mathcal{N}$. $\square$

Question 2

Let $L = \{c_n \mid n < \omega\}$ be language consists of constant symbols. Let us define the theory $T = \{c_i \neq c_j \mid i < j < \omega\}$. We will show that there are countably many non-isomorphic countable models of T, and that T is complete.

Proof. Let us define the model $\mathcal{M}_n$ such that $M = \omega$ and,

$$c_i^{\mathcal{M}} = i + n$$

for any $i < j < \omega$,

$$c_i^{\mathcal{M}} = i + n \neq j + n = c_j^{\mathcal{M}}$$

therefore $\mathcal{M}_n \models T$. $\mathcal{M}_n \models k \neq c_i$ for all $i < \omega$, in particular $\mathcal{M}_n \models \exists x\, x \neq c_i$. It is implied that also,

$$\mathcal{M}_n \models \exists x_0 \ldots \exists x_{k-1} \left(\bigwedge_{i<j<k} x_i \neq x_j \land x_i \neq c_l \right) = \varphi_l^k$$

for all $l < \omega$. Finally, $\mathcal{M}_n \not\models \varphi_l^k$ for any $k > n$, we deduce that $\mathcal{M}_n \ncong \mathcal{M}_m$ for any $n \neq m$.

We move to show that T is complete. Let us assume toward a contradiction that φ is a sentence such that $\varphi \notin T$ and $T \cup \{\varphi\}$ is consistent. By construction of Henkin models we can deduce that $\mathcal{M}_0 \models \varphi$, but $\mathcal{M}_0$ is minimal, namely if $\mathcal{N} \models T$ then $\mathcal{M}_0 \subseteq \mathcal{N}$, then by definition $T \models \varphi$, a contradiction. $\square$

Question 3

We will show that $\mathrm{Th}(\mathbb{N}, +, \cdot)$ has $2^{\aleph_0}$ non-isomorphic countable models.

Proof. Let $f : \omega \to P$ be the map between number and its respective prime in the order induced from $\mathbb{N}$, namely $f(0) = 2, f(1) = 3, \ldots$. assuming that $A \subseteq \omega$ is some set, we define $\mathcal{M}_A = (\mathrm{cl}_{+,\cdot}(f(A)), +, \cdot)$.

$$\mathcal{M}_A \models \exists p \forall x\, p \cdot x = \underline{p} \cdot x$$

when $\underline{p}$ acts as numerator, and $\underline{p} \cdot x$ is abbreviation to $x + \cdots + x$ p times. Let us denote this sentence as φ_p, then,

$$\mathcal{M}_A \models \{\varphi_{f(a)} \mid a \in A\}$$

as well,

$$\mathcal{M}_A \not\models \{\varphi_q \mid q \notin \mathrm{cl}_+ A\}$$

Therefore if $A, B \subseteq \omega$ and $A \neq B$ then $\mathcal{M}_A \ncong \mathcal{M}_B$, thus there are $|\mathcal{P}(\omega)|$ non-isomorphic countable models. $\square$

Question 4

Let $\kappa \geq \omega$ be some cardinal and let L be some language. Let T be a κ-categorical L-theory such that it has no finite models. We will show that T is complete.

Proof. Let us assume for the sake of contradiction that T is incomplete, and let $\varphi \in \mathrm{sent}_L$ be a sentence such that $T_+ = T \cup \{\varphi\}$ is consistent. Let $T_- = T \cup \{\neg\varphi\}$ and let $\mathcal{M}_+ \vDash T_+, \mathcal{M}_- \vDash T_-$ be models witnessing the theories consistency. $|M_+|, |M_-| = |L|$ without loss of generality.

Why not choose $L = \{c_\alpha \mid \alpha < \delta\}$ for $\kappa < \delta$ and,

$$T = \{c_\alpha \neq c_\beta \mid \alpha < \beta < \kappa\}$$

If $\mathcal{M} \vDash T$ then $|M| \geq \kappa$ and then let $\kappa < \epsilon < \delta$ be some ordinal, the sentence $\varphi = c_0 = c_\epsilon$ is consistent with T and $\neg\varphi$ as well. $\square$

Question 5

Let $T = \mathrm{Th}(\mathbb{Q}, \leq)$ be DLO.

We will show that T is not κ-categorical for some uncountable cardinal κ.

Proof. □