Assumption 1: Ideal weights are bounded by known positive values:

$$\|V\|_F \le V_M, \quad \|W\|_F \le W_M, \quad \|Z\|_F \le Z_M$$

$$\text{where } Z = \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix}$$

Assumption 2: desired trajectory is bounded ~~by~~:

$$\| [q_d^T \ \dot{q}_d^T \ \ddot{q}_d^T]^T \| \le Q_d$$

Fact: $x(t) = (e_m^T \ \dot{e}_m^T \ q_d^T \ \dot{q}_d^T \ \ddot{q}_d^T)^T$

is bounded by: $\|x\| \le c_1 Q_d + c_2 \|r\|$

$c_1, c_2 \rightarrow$ computable constants.

- Closed loop dynamics using control algorithm we get the following error dynamic equation:

$$\bar{M}\dot{r} = -L^T K_v L r - \bar{V}_1 r + L^T\left[W^T\sigma(V^T x) - \hat{W}^T\sigma(\hat{V}^T x)\right] + L^T\left[\tau_d + \varepsilon + v\right] \quad (1)$$

- By expanding $\sigma(V^T x)$ in Taylor series about $\sigma(\hat{V}^T x)$ and other manipulations:

$$\bar{M}\dot{r} = -L^T K_v L r - \bar{V}_1 r + L^T\left[\tilde{W}^T(\hat{\sigma} - \hat{\sigma}'\hat{V}^T x) + \hat{W}^T\hat{\sigma}'\tilde{V}^T x\right] + L^T(w + v)$$

where disturbance:

$$w(t) = \tilde{W}^T\hat{\sigma}'V^T x + W^T O(\tilde{V}^T x)^2 + \tau_d + \varepsilon$$

Theorem: Let NN weight tuning be provided by:

$$\dot{\hat{W}} = F\sigma(\hat{V}^T x)(Lr)^T - F\hat{\sigma}'\hat{V}^T x(Lr)^T - K F \|Lr\| \hat{W}$$

$$\dot{\hat{V}} = Gx(\hat{\sigma}'^T \hat{W}(Lr))^T - KG\|(Lr)\|\hat{V}$$

where F, G are positive definite matrices and $K > 0$, then filtered tracking error $r(t)$, force errors and NN weight estimates are globally uniformly bounded.

Proof: Consider the Lyapunov candidate:

$$L = \frac{1}{2} r^T \bar{M} r + \frac{1}{2} tr(\tilde{W}^T F^{-1} \tilde{W}) + \frac{1}{2} tr(\tilde{V}^T G^{-1} \tilde{V})$$

$$\Rightarrow \frac{\partial L}{\partial t} = \frac{\partial L}{\partial r} \dot{r} + \frac{\partial L}{\partial \tilde{W}} \dot{\tilde{W}} + \frac{\partial L}{\partial \tilde{V}} \dot{\tilde{V}}$$

$$\Rightarrow \dot{L} = r^T \bar{M} \dot{r} + tr(\tilde{W} F^{-} \dot{\tilde{W}}) + tr(\tilde{V} G^{-} \dot{\tilde{V}})$$

On using ①: ~~[illegible]~~

$$\dot{L} = -r^T L^T K_v L r + r^T L^T (w + v)$$
$$+ tr[\tilde{W}^T F^{-} \dot{\tilde{W}} - \tilde{W}^T (\hat{\sigma} - \hat{\sigma}' \hat{V}^T x)(Lr)^T]$$
$$+ tr[\tilde{V}^T G^{-} \dot{\tilde{V}} - \tilde{V}^T x (\hat{\sigma}'^T \hat{W} (Lr))^T]$$

On using the weight tuners ~~weights~~ and also assuming $\dot{V} = \dot{W} = 0$, we get

$$\dot{L} = -(Lr)^T K_v (Lr) + (Lr)^T (w+v) + K\|Lr\| \left[tr[\tilde{W}^T(W-\tilde{W})] + tr[\tilde{V}^T(V-\tilde{V})] \right]$$

Using inequality:

$$tr(\tilde{Z}^T(Z-\tilde{Z})) = \langle \tilde{Z}, Z \rangle_F - \|Z\|_F^2$$
$$\leq \|\tilde{Z}\|_F \|Z\|_F - \|Z\|_F^2$$

$$\Rightarrow \dot{V} \leq (-Lr)^T K_v (Lr) + (Lr)^T (w+v) + K\|Lr\| \left[\|\tilde{Z}\|_F Z_M - \|\tilde{Z}\|_F^2 \right]$$

- The disturbance term is bounded according to:

$$\|w(t)\| \le C_0 + C_1\|\tilde{Z}\|_F + C_2\|\tilde{Z}\|_F\|M\|$$

$C_i \to$ known constants

- Using the above fact, we get:

$$\dot{L} \le -\|L_M\|\Big(K_v\|L_M\| - K\big(\|\tilde{Z}\|_F Z_M - \|\tilde{Z}_F\|^2\big) - C_0 - C_1\|\tilde{Z}\|_F$$

• If $\|L_M\| > \left(\dfrac{\dfrac{K(2M + C_1/K)^2}{4} + C_0}{K_V} \right)$ or

$$\|\tilde{Z}_{R}\|_F > \frac{(2M + C_1/K)}{2} + \sqrt{\left(2M + \frac{C_1}{K}\right)^2 + \frac{C_0}{K}}$$

then : $\dot{V} \leq 0$

• Let $\delta_1 := \left(\dfrac{\dfrac{K(2M + C_1/K)^2}{4} + C_0}{K_V} \right)$

$$\delta_2 := \frac{(2M + C_1/K)}{2} + \sqrt{\left(2M + \frac{C_1}{K}\right)^2 + \frac{C_0}{K}}$$

- $\zeta_1 := (Lr)$, $\zeta_2 = \tilde{z}$

- Let $D : \{\zeta \mid \zeta_1 < \delta_1 , \zeta_2 < \delta_2\}$ then $\exists$

  $\Omega : \{\zeta \mid \zeta_1 < \bar{\delta}_1 , \zeta_2 < \bar{\delta}_2\}$ where $\bar{\delta}_i > \delta_i$

  $\Rightarrow D \subset \Omega$

- whenever, $\zeta_i > \delta_i \Rightarrow V(\zeta)$ won't increase

  $\Rightarrow$ ζ will stay in Ω

  $\Rightarrow$ Ω is an invariant set.

  $\Rightarrow$ (Lr) and $\tilde{z}$ will always stay inside Ω after a finite period of time.