Gathering the Dual

Note: For temporary simplicity I will skip the coverage constraint.

We have a program

MINIMIZE $\sum_{q \in \mathcal{Q}} C^q z^q$

st

[1] $\sum_{q \in \mathcal{Q}} u_i^q z^q = 1 \ \forall i \in \mathcal{I}$ (dual: π_i)

[2] $\sum_{q \in \mathcal{Q}} z^q \leq K$ (dual: ρ)

[3] $y_{it}^S \leq y_{i,t+1}^S \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: α_{it})

[4] $y_{it}^E \leq y_{i,t+1}^E \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: β_{it})

[5] $y_{it}^S = 0 \ \forall t \leq T_i^S \ \forall i \in \mathcal{I}$ (dual: ζ_{it}) - zeta

[6] $y_{it}^E = 1 \ \forall t \geq T_i^E \ \forall i \in \mathcal{I}$ (dual: η_{it}) - eta

[7] $y_{it}^S - y_{it}^E \leq \sum_{q \in \mathcal{Q}} z_q \delta_{it}^q \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: μ_{it})

[8] $\sum_{t \in \mathcal{T}} (y_{it}^S - y_{it}^E) \geq \tau_i^W \ \forall i \in \mathcal{I}$ (dual: θ_i)

Because our problem is MINIMIZE, we will try to write every constraint as a LHS ≥ RHS, except for the equals, which should be preserved as they are.

Also, observe that every one of our decision variables is ≥ 0.

MINIMIZE $\sum_{q \in \mathcal{Q}} C^q z^q$

st

[1] $\sum_{q \in \mathcal{Q}} u_i^q z^q = 1 \ \forall i \in \mathcal{I}$ (dual: π_i)

[2] $\sum_{q \in \mathcal{Q}} z^q \geq -K$ (dual: ρ)

[3] $-y_{it}^S + y_{i,t+1}^S \geq 0 \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: α_{it})

[4] $-y_{it}^E + y_{i,t+1}^E \geq 0 \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: β_{it})

[5] $y_{it}^S = 0 \ \forall t \leq T_i^S \ \forall i \in \mathcal{I}$ (dual: ζ_{it}) - zeta

[6] $y_{it}^E = 1 \ \forall t \geq T_i^E \ \forall i \in \mathcal{I}$ (dual: η_{it}) - eta

[7] $\sum_{q \in \mathcal{Q}} z_q \delta_{it}^q - y_{it}^S + y_{it}^E \geq 0 \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: μ_{it})

[8] $\sum_{t \in \mathcal{T}} (y_{it}^S - y_{it}^E) \geq \tau_i^W \ \forall i \in \mathcal{I}$ (dual: θ_i)

[1], [5], and [6] are equalities, while [2], [3], [4], [7], [8] are ≥.

Notice that we have a total of $Q + 2IT$ decision variables, we'll list them starting with the Q variables q, then the IT variables y_{it}^S, then the IT variables y_{it}^E.

Meanwhile, we have: $I + 1 + IT + IT + IT + IT + IT + I = 5IT + 2I + 1$ constraints.

So if we write our formulation as

MINIMIZE $c^T x$

ST $Ax \geq b$

then x has dimensions $Q + 2IT$ by 1
c has dimensions $Q + 2IT$ by 1

A has dimensions $5IT + 2I + 1$ by $Q + 2IT$
b has dimensions $5IT + 2I + 1$ by 1.

Then we have a dual which is

MAXIMIZE $b^T y$

ST $A^T y \leq c$

y has dimensions $5IT + 2I + 1$ by 1
b has dimensions $5IT + 2I + 1$ by 1

A^T has dimensions $Q + 2IT$ by $5IT + 2I + 1$
c has dimensions $Q + 2IT$ by 1

Note that the direction of the inequalities in the DUAL is dependent on whether the decision variable x_i uses ≥ 0, ≤ 0, or unconstrained in the PRIMAL. Since all our PRIMAL decision variables are ≥ 0, every DUAL constraint will use $\leq$.

Meanwhile, the sign of each DUAL variable is the same as the sign of the corresponding constraint (because we are solving a minimization problem). This means $\pi_i, \zeta_{it}, \eta_{it}$ are all just reals, whereas $\rho, \alpha_{it}, \beta_{it}, \mu_{it}, \theta_i$ are all non-negative reals.

Writing the Dual
Observe that the dual equations are gathered by: taking the first column in A and dot-product-multiplying with the dual variables. Then do the same for the second column, etc. The RHS of the constraint comes from the appropriate coefficient in the PRIMAL objective. There are a total of $Q + 2IT$ constraints.

We can "block" these constraints for easier writing.

The first block: let's look at each of the first Q columns in the PRIMAL. Note that these correspond to coefficients on z^q.

Only [1], [2], and [7] actually contribute to coefficients on z^q.

For an arbitrary q:

from [1]: we have I constraints total. They look like

$\ldots + u_1^q z^q + \ldots = 1$ [dual: π_1]
$\ldots + u_2^q z^q + \ldots = 1$ [dual: π_2]
...
$\ldots + u_I^q z^q + \ldots = 1$ [dual: π_I]

from [2]: we have 1 constraint, with coefficients

$\ldots + 1z^q + \ldots \geq -K$ [dual: ρ]

from [7]: we have IT constraints, with coefficients

$\ldots + \delta_{it}^q z_q + \ldots \geq 0$ [dual: μ_{it}]

Meaning if we sum up all the contributions, we have q constraints that look like:

$\sum_{i \in \mathcal{I}} \pi_i u_i^q + \rho + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \mu_{it} \delta_{it}^q \leq C^q$ FOR EACH $q \in \mathcal{Q}$.

Note that the dual formulation written in the Overleaf document uses $-\rho$; that's because my ρ here reversed the sign to be $\geq$ instead of $\leq$. The two are equal.

(see next page for second block)

The second block: Looking at the next IT columns in the PRIMAL. Here, only [3], [5], [7], [8] have anything to do with y_{it}^S variables. We examine their columns. Notice that the RHS will always be 0, because the coefficient in the PRIMAL objective was 0 for y's.

For a certain choice of i, t:

from [3] we get:
Well, there are at most two terms of concern. For a certain i and certain t, the coefficient only shows up in at most TWO of the IT constraints: $-y_{it}^S + y_{i,t+1}^S \geq 0$ and $-y_{i,t-1}^S + y_{it}^S \geq 0$.

Meaning for most cases, when $2 \leq t \leq T-1$, the columns will contribute $-\alpha_{it} + \alpha_{i,t-1}$.

Except watch out for when $t = 1$, because there is no such thing as the second constraint. When $t = 1$, there is only a constraint contribution $-\alpha_{it}$.

And also when $t = T$, because there is no such thing as the first constraint. When $t = T$, there is only a constraint contribution $+\alpha_{i,t-1}$.

from [5] we get:
These constraints really only exist if, for a given i and t, $t \leq T_i^S$. Otherwise we're not saying anything. So, for a choice of i and then t:

If $t \leq T_i^S$, then we have a contribution of ζ_{it}. Otherwise it's 0.

from [7] we get:
At last, an easier one. Here every coefficient will be $-\mu_{it}$.

from [8] we get:
Coefficients are θ_i, regardless of what t is.

So to summarize: For these IT constraints,
For when $t = 1$ and $t \leq T_i^S$, $-\alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = 1$ and $t > T_i^S$, $-\alpha_{it} - \mu_{it} + \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t \leq T_i^S$, $\alpha_{i,t-1} - \alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t > T_i^S$, $\alpha_{i,t-1} - \alpha_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = T$ and $t \leq T_i^S$, $\alpha_{i,t-1} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = T$ and $t > T_i^S$, $\alpha_{i,t-1} - \mu_{it} + \theta_i \leq 0$

Cases 2 ($t = 1$ and $t > T_i^S$) and 5 ($t = T$ and $t \leq T_i^S$) will probably never appear.

(see next page for third block)

The third block: The final IT constraints formed by y^E. Here, [4], [6], [7], and [8] are the relevant equations. This is very similar to the previous part.

from [4] we get:
Depends again...we do the same analysis as we did for [3] in the previous part.

If $t = 1$ then we only show up at $-y_{it}^E + y_{i,t+1}^E \geq 0$, so there's a contribution of $-\beta_{it}$.

If $2 \leq t \leq T-1$ then we have $\beta_{i,t-1} - \beta_{it}$.

If $t = T$ then there's a contribution of $\beta_{i,t-1}$.

from [6] we get:
If $t \geq T_i^E$ then contribution is η_{it}, else nothing.

from [7] we get:
No matter what, contribution is μ_{it}.

from [8] we get:
No matter what, coefficient is $-\theta_i$.

Summary: For these IT constraints,
For when $t = 1$ and $t \geq T_i^E$, $-\beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = 1$ and $t < T_i^E$, $-\beta_{it} + \mu_{it} - \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t \geq T_i^E$, $\beta_{i,t-1} - \beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t < T_i^E$, $\beta_{i,t-1} - \beta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = T$ and $t \geq T_i^E$, $\beta_{i,t-1} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = T$ and $t < T_i^E$, $\beta_{i,t-1} + \mu_{it} - \theta_i \leq 0$

In practice, cases 1 ($t = 1$ and $t \geq T_i^E$) and 6 ($t = T$ and $t < T_i^E$) will never happen.

(see next page for objective and full formulation)

Finally, we need to compute our objective. But this is formed by reading down the right hand column:

$$\sum_{i \in \mathcal{I}} \pi_i - K\rho + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \eta_{it} + \sum_{i \in \mathcal{I}} \tau_i^W \theta_i$$

FULL FORMULATION

We have 8 sets of dual variables: $\pi_i, \rho, \alpha_{it}, \beta_{it}, \zeta_{it}, \eta_{it}, \mu_{it}, \theta_i$. Together they take up $5IT + 2I + 1$ rows as a column vector. For $\pi_i, \zeta_{it}, \eta_{it}$ they can be any real, for $\rho, \alpha_{it}, \beta_{it}, \mu_{it}, \theta_i$ we require non-negativity.

MAXIMIZE $\sum_{i \in \mathcal{I}} \pi_i - K\rho + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \eta_{it} + \sum_{i \in \mathcal{I}} \tau_i^W \theta_i$

SUBJECT TO

$\sum_{i \in \mathcal{I}} \pi_i u_i^q + \rho + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \mu_{it} \delta_{it}^q \leq C^q$ FOR EACH $q \in \mathcal{Q}$

For when $t = 1$ and $t \leq T_i^S$, $-\alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = 1$ and $t > T_i^S$, $-\alpha_{it} - \mu_{it} + \theta_i \leq 0$
For when $2 \leq t \leq T - 1$ and $t \leq T_i^S$, $\alpha_{i,t-1} - \alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $2 \leq t \leq T - 1$ and $t > T_i^S$, $\alpha_{i,t-1} - \alpha_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = T$ and $t \leq T_i^S$, $\alpha_{i,t-1} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = T$ and $t > T_i^S$, $\alpha_{i,t-1} - \mu_{it} + \theta_i \leq 0$

For when $t = 1$ and $t \geq T_i^E$, $-\beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = 1$ and $t < T_i^E$, $-\beta_{it} + \mu_{it} - \theta_i \leq 0$
For when $2 \leq t \leq T - 1$ and $t \geq T_i^E$, $\beta_{i,t-1} - \beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $2 \leq t \leq T - 1$ and $t < T_i^E$, $\beta_{i,t-1} - \beta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = T$ and $t \geq T_i^E$, $\beta_{i,t-1} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = T$ and $t < T_i^E$, $\beta_{i,t-1} + \mu_{it} - \theta_i \leq 0$

$\rho, \alpha_{it}, \beta_{it}, \mu_{it}, \theta_i \geq 0$