Gathering the Dual

Note: For temporary simplicity I will skip the coverage constraint.

We have a program

MINIMIZE $\sum_{q \in \mathcal{Q}} C^q z^q$

st

[1] $\sum_{q \in \mathcal{Q}} u_i^q z^q = 1 \ \forall i \in \mathcal{I}$ (dual: π_i)

[2] $\sum_{q \in \mathcal{Q}} z^q \leq K$ (dual: ρ)

[3] $y_{it}^S \leq y_{i,t+1}^S \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: α_{it})

[4] $y_{it}^E \leq y_{i,t+1}^E \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: β_{it})

[5] $y_{it}^S = 0 \ \forall t \leq T_i^S \ \forall i \in \mathcal{I}$ (dual: ζ_{it}) - zeta

[6] $y_{it}^E = 1 \ \forall t \geq T_i^E \ \forall i \in \mathcal{I}$ (dual: η_{it}) - eta

[7] $y_{it}^S - y_{it}^E \leq \sum_{q \in \mathcal{Q}} z_q \delta_{it}^q \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: μ_{it})

[8] $\sum_{t \in \mathcal{T}} (y_{it}^S - y_{it}^E) \geq \tau_i^W \ \forall i \in \mathcal{I}$ (dual: θ_i)

Because our problem is MINIMIZE, we will try to write every constraint as a LHS ≥ RHS, except for the equals, which should be preserved as they are.

Also, observe that every one of our decision variables is ≥ 0.

MINIMIZE $\sum_{q \in \mathcal{Q}} C^q z^q$

st

[1] $\sum_{q \in \mathcal{Q}} u_i^q z^q = 1 \ \forall i \in \mathcal{I}$ (dual: π_i)

[2] $\sum_{q \in \mathcal{Q}} z^q \geq -K$ (dual: ρ)

[3] $-y_{it}^S + y_{i,t+1}^S \geq 0 \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: α_{it})

[4] $-y_{it}^E + y_{i,t+1}^E \geq 0 \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: β_{it})

[5] $y_{it}^S = 0 \ \forall t \leq T_i^S \ \forall i \in \mathcal{I}$ (dual: ζ_{it}) - zeta

[6] $y_{it}^E = 1 \ \forall t \geq T_i^E \ \forall i \in \mathcal{I}$ (dual: η_{it}) - eta

[7] $\sum_{q \in \mathcal{Q}} z_q \delta_{it}^q - y_{it}^S + y_{it}^E \geq 0 \ \forall i \in \mathcal{I}, t \in \mathcal{T}$ (dual: μ_{it})

[8] $\sum_{t \in \mathcal{T}} (y_{it}^S - y_{it}^E) \geq \tau_i^W \ \forall i \in \mathcal{I}$ (dual: θ_i)

[1], [5], and [6] are equalities, while [2], [3], [4], [7], [8] are ≥.

Notice that we have a total of $Q + 2IT$ decision variables, we'll list them starting with the Q variables q, then the IT variables y_{it}^S, then the IT variables y_{it}^E.

Meanwhile, we have: $I + 1 + IT + IT + IT + IT + IT + I = 5IT + 2I + 1$ constraints.

So if we write our formulation as

MINIMIZE $c^T x$

ST $Ax \geq b$

then x has dimensions $Q + 2IT$ by 1
c has dimensions $Q + 2IT$ by 1

A has dimensions $5IT + 2I + 1$ by $Q + 2IT$
b has dimensions $5IT + 2I + 1$ by 1.

Then we have a dual which is

MAXIMIZE $b^T y$

ST $A^T y \leq c$

y has dimensions $5IT + 2I + 1$ by 1
b has dimensions $5IT + 2I + 1$ by 1

A^T has dimensions $Q + 2IT$ by $5IT + 2I + 1$
c has dimensions $Q + 2IT$ by 1

Note that the direction of the inequalities in the DUAL is dependent on whether the decision variable x_i uses ≥ 0, ≤ 0, or unconstrained in the PRIMAL. Since all our PRIMAL decision variables are ≥ 0, every DUAL constraint will use $\leq$.

Meanwhile, the sign of each DUAL variable is the same as the sign of the corresponding constraint (because we are solving a minimization problem). This means $\pi_i, \zeta_{it}, \eta_{it}$ are all just reals, whereas $\rho, \alpha_{it}, \beta_{it}, \mu_{it}, \theta_i$ are all non-negative reals.

Writing the Dual
Observe that the dual equations are gathered by: taking the first column in A and dot-product-multiplying with the dual variables. Then do the same for the second column, etc. The RHS of the constraint comes from the appropriate coefficient in the PRIMAL objective. There are a total of $Q + 2IT$ constraints.

We can "block" these constraints for easier writing.

The first block: let's look at each of the first Q columns in the PRIMAL. Note that these correspond to coefficients on z^q.

Only [1], [2], and [7] actually contribute to coefficients on z^q.

For an arbitrary q:

from [1]: we have I constraints total. They look like

$\ldots + u_1^q z^q + \ldots = 1$ [dual: π_1]
$\ldots + u_2^q z^q + \ldots = 1$ [dual: π_2]
...
$\ldots + u_I^q z^q + \ldots = 1$ [dual: π_I]

from [2]: we have 1 constraint, with coefficients

$\ldots + 1z^q + \ldots \geq -K$ [dual: ρ]

from [7]: we have IT constraints, with coefficients

$\ldots + \delta_{it}^q z_q + \ldots \geq 0$ [dual: μ_{it}]

Meaning if we sum up all the contributions, we have q constraints that look like:

$\sum_{i \in \mathcal{I}} \pi_i u_i^q + \rho + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \mu_{it} \delta_{it}^q \leq C^q$ FOR EACH $q \in \mathcal{Q}$.

Note that the dual formulation written in the Overleaf document uses $-\rho$; that's because my ρ here reversed the sign to be $\geq$ instead of $\leq$. The two are equal.

(see next page for second block)

The second block: Looking at the next IT columns in the PRIMAL. Here, only [3], [5], [7], [8] have anything to do with y_{it}^S variables. We examine their columns. Notice that the RHS will always be 0, because the coefficient in the PRIMAL objective was 0 for y's.

For a certain choice of i, t:

from [3] we get:
Well, there are at most two terms of concern. For a certain i and certain t, the coefficient only shows up in at most TWO of the IT constraints: $-y_{it}^S + y_{i,t+1}^S \geq 0$ and $-y_{i,t-1}^S + y_{it}^S \geq 0$.

Meaning for most cases, when $2 \leq t \leq T-1$, the columns will contribute $-\alpha_{it} + \alpha_{i,t-1}$.

Except watch out for when $t = 1$, because there is no such thing as the second constraint. When $t = 1$, there is only a constraint contribution $-\alpha_{it}$.

And also when $t = T$, because there is no such thing as the first constraint. When $t = T$, there is only a constraint contribution $+\alpha_{i,t-1}$.

from [5] we get:
These constraints really only exist if, for a given i and t, $t \leq T_i^S$. Otherwise we're not saying anything. So, for a choice of i and then t:

If $t \leq T_i^S$, then we have a contribution of ζ_{it}. Otherwise it's 0.

from [7] we get:
At last, an easier one. Here every coefficient will be $-\mu_{it}$.

from [8] we get:
Coefficients are θ_i, regardless of what t is.

So to summarize: For these IT constraints,
For when $t = 1$ and $t \leq T_i^S$, $-\alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = 1$ and $t > T_i^S$, $-\alpha_{it} - \mu_{it} + \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t \leq T_i^S$, $\alpha_{i,t-1} - \alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t > T_i^S$, $\alpha_{i,t-1} - \alpha_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = T$ and $t \leq T_i^S$, $\alpha_{i,t-1} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$
For when $t = T$ and $t > T_i^S$, $\alpha_{i,t-1} - \mu_{it} + \theta_i \leq 0$

Cases 2 ($t = 1$ and $t > T_i^S$) and 5 ($t = T$ and $t \leq T_i^S$) will probably never appear.

(see next page for third block)

The third block: The final IT constraints formed by y^E. Here, [4], [6], [7], and [8] are the relevant equations. This is very similar to the previous part.

from [4] we get:
Depends again...we do the same analysis as we did for [3] in the previous part.

If $t = 1$ then we only show up at $-y_{it}^E + y_{i,t+1}^E \geq 0$, so there's a contribution of $-\beta_{it}$.

If $2 \leq t \leq T-1$ then we have $\beta_{i,t-1} - \beta_{it}$.

If $t = T$ then there's a contribution of $\beta_{i,t-1}$.

from [6] we get:
If $t \geq T_i^E$ then contribution is η_{it}, else nothing.

from [7] we get:
No matter what, contribution is μ_{it}.

from [8] we get:
No matter what, coefficient is $-\theta_i$.

Summary: For these IT constraints,
For when $t = 1$ and $t \geq T_i^E$, $-\beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = 1$ and $t < T_i^E$, $-\beta_{it} + \mu_{it} - \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t \geq T_i^E$, $\beta_{i,t-1} - \beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $2 \leq t \leq T-1$ and $t < T_i^E$, $\beta_{i,t-1} - \beta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = T$ and $t \geq T_i^E$, $\beta_{i,t-1} + \eta_{it} + \mu_{it} - \theta_i \leq 0$
For when $t = T$ and $t < T_i^E$, $\beta_{i,t-1} + \mu_{it} - \theta_i \leq 0$

In practice, cases 1 ($t = 1$ and $t \geq T_i^E$) and 6 ($t = T$ and $t < T_i^E$) will never happen.

Finally, we need to compute our objective. But this is formed by reading down the right hand column:

$$\sum_{i \in \mathcal{I}} \pi_i \text{ - } K\rho + \sum_{i \in \mathcal{I}} \sum_{t \in \mathcal{T}} \eta_{it} + \sum_{i \in \mathcal{I}} \tau_i^W \theta_i$$

(see next page for full formulation)

FULL FORMULATION

We have 8 sets of dual variables: π_i, ρ, α_{it}, β_{it}, ζ_{it}, η_{it}, μ_{it}, θ_i. Together they take up $5IT + 2I + 1$ rows as a column vector. For π_i, ζ_{it}, η_{it} they can be any real, for ρ, α_{it}, β_{it}, μ_{it}, θ_i we require non-negativity.

MAXIMIZE $\sum_{i \in \mathscr{I}} \pi_i$ - $K\rho$ + $\sum_{i \in \mathscr{I}} \sum_{t \in \mathscr{T}} \eta_{it}$ + $\sum_{i \in \mathscr{I}} \tau_i^W \theta_i$

SUBJECT TO

$\sum_{i \in \mathscr{I}} \pi_i u_i^q$ + ρ + $\sum_{i \in \mathscr{I}} \sum_{t \in \mathscr{T}} \mu_{it} \delta_{it}^q \leq C^q$ FOR EACH $q \in \mathscr{Q}$

For when $t = 1$ and $t \leq T_i^S$, $-\alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$

For when $t = 1$ and $t > T_i^S$, $-\alpha_{it} - \mu_{it} + \theta_i \leq 0$

For when $2 \leq t \leq T - 1$ and $t \leq T_i^S$, $\alpha_{i,t-1} - \alpha_{it} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$

For when $2 \leq t \leq T - 1$ and $t > T_i^S$, $\alpha_{i,t-1} - \alpha_{it} - \mu_{it} + \theta_i \leq 0$

For when $t = T$ and $t \leq T_i^S$, $\alpha_{i,t-1} + \zeta_{it} - \mu_{it} + \theta_i \leq 0$

For when $t = T$ and $t > T_i^S$, $\alpha_{i,t-1} - \mu_{it} + \theta_i \leq 0$

For when $t = 1$ and $t \geq T_i^E$, $-\beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$

For when $t = 1$ and $t < T_i^E$, $-\beta_{it} + \mu_{it} - \theta_i \leq 0$

For when $2 \leq t \leq T - 1$ and $t \geq T_i^E$, $\beta_{i,t-1} - \beta_{it} + \eta_{it} + \mu_{it} - \theta_i \leq 0$

For when $2 \leq t \leq T - 1$ and $t < T_i^E$, $\beta_{i,t-1} - \beta_{it} + \mu_{it} - \theta_i \leq 0$

For when $t = T$ and $t \geq T_i^E$, $\beta_{i,t-1} + \eta_{it} + \mu_{it} - \theta_i \leq 0$

For when $t = T$ and $t < T_i^E$, $\beta_{i,t-1} + \mu_{it} - \theta_i \leq 0$

$\rho, \alpha_{it}, \beta_{it}, \mu_{it}, \theta_i \geq 0$

CRITICAL WARNING

[5] $y_{it}^S = 0 \ \forall t \leq T_i^S \ \forall i \in \mathscr{I}$ (dual: ζ_{it}) - zeta

[6] $y_{it}^E = 1 \ \forall t \geq T_i^E \ \forall i \in \mathscr{I}$ (dual: η_{it}) - eta

Notice that from [5] and [6], we do not actually have IT constraints. The true number of ζ_{it} and η_{it} variables is almost certainly smaller than IT, because in the cases for which $t > T_i^S$ or $t < T_i^E$, **nowhere are these "nonexistent" ζ_{it} and η_{it} variables constrained!** In fact, they can be as big as possible and destroy our objective!

So: for ζ_{it} where $t > T_i^S$, AND for η_{it} where $t < T_i^E$, we need to force it to be zero.

Otherwise, because as you can see these variables are not actually constrained in any of the 12 equations governing them (and technically these variables don't exist, but **do** calamitously appear in the dual objective), they will just make the objective infinity.

Also, we're not done. We also notice that ζ_{it} contributes nothing to the objective, BUT in the first set of 6 equations, what if ζ_{it} were very very negative? Then θ_i could be infinity, forcing the objective to infinity. So force $\zeta_{it} \geq 0$.

Setting η_{it} but not ζ_{it} to non-negative also works...I will try to find out the reason.