CS 2305: Discrete Mathematics for Computing I

Lecture 11

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Proof By Contraposition

- Sometimes it is easier to prove theorems using proof by contraposition
  - the conditional statement $p \to q$ is proved by showing that its contrapositive, $\neg q \to \neg p$ is true

Proof By Contraposition: $\neg q \rightarrow \neg p$ (1)

Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution:

We will show that if n is even then 3n+2 is even

Let us take an arbitrary even number n.

$n = 2k$ for some integer k.

Thus $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) = 2j$ for $j = 3k + 1$

Therefore $3n + 2$ is even.

Since we have shown $\neg q \rightarrow \neg p$, $p \rightarrow q$ must hold as well.

We have proved by contraposition that if $3n + 2$ is odd, then n is odd

QED

Proof By Contraposition: $\neg q \rightarrow \neg p$ (2)

Example: Prove that for an integer *n*, if n^2 is odd, then *n* is odd.

Solution*:*

We will show that if n is even then n^2 is even

Let us take an arbitrary even number n.

$n = 2k$ for some integer *k*.

Thus $n^2 = 4k^2 = 2\,(2k^2)$

Therefore n^2 is even.

Since we have shown $\neg q \rightarrow \neg p$, $p \rightarrow q$ must hold as well.

We have proved by contraposition that if n^2 is odd, then *n* is odd.

QED

Proof By Contraposition: $\neg q \rightarrow \neg p$ (3)

Example: Prove that if $n = a*b$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Solution:

We will show that if $(a \leq \sqrt{n} \vee b \leq \sqrt{n})$ is false then $n \neq a*b$

Since we are assuming that $(a \leq \sqrt{n} \vee b \leq \sqrt{n})$ is false, $\neg(a \leq \sqrt{n} \vee b \leq \sqrt{n})$ must be true.

By De Morgan's Law, $\neg(a \leq \sqrt{n}) \wedge \neg(b \leq \sqrt{n})$ is true

$\therefore \neg(a \leq \sqrt{n})$ is true and $\neg(b \leq \sqrt{n})$ is true

$\therefore a > \sqrt{n})$ and $b > \sqrt{n}$

$\therefore$ a*b > $(\sqrt{n} * \sqrt{n})$

$\therefore$ a*b > n i.e. a*b ≠ n

Since we have shown $\neg q \rightarrow \neg p$, $p \rightarrow q$ must hold as well.

We have proved by contraposition that if $n = a*b$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

QED

Vacuous & Trivial Proofs

- $p \rightarrow q \equiv \neg p \vee q$
- A proof that makes use of the fact that $p \rightarrow q$ must be true when p is false is called a **vacuous** proof
- A proof that makes use of the fact that $p \rightarrow q$ must be true when q is true is called a **trivial** proof
- These proofs are never treated as complete proofs but they are used in conjunction with other proof techniques (like proof by cases and mathematical induction) to establish that special cases of a theorem are not in violation of the generalized theorem
  - Show that the proposition $P(0)$ is true, where $P(n)$ is "If $n > 1$, then $n^2 > n$" and the domain consists of all integers
  - Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$," where the domain consists of all nonnegative integers. Show that $P(0)$ is true

Proof by Contradiction

- Also known as *reductio ad absurdum*
- To prove $p \rightarrow q$ we make the opposite assumption i.e. we assume that if p is true then q is false. From this assumption we try to reach an absurd or a demonstrably false statement. Clearly, therefore, the assumption that if p is true then q is false is wrong. In other words if p is true then q must be true.

Proof by Contradiction (1)

Example: Prove that if you pick 22 days from the calendar, at least 4 must fall on the same day of the week.

Solution: Assume that no more than 3 of the 22 days fall on the same day of the week. Because there are 7 days of the week, we could only have picked 21 days. This contradicts the assumption that we have picked 22 days.

QED

Proof by Contradiction (2)

Example: Use a proof by contradiction to give a proof that √2 is irrational.

Solution: Suppose √2 is rational. Then there exists integers *a* and *b* with √2 = *a/b*, where $b \neq 0$ and *a* and *b* have no common factors (see Chapter 4). Then

$$2 = \frac{a^2}{b^2} \qquad\qquad 2b^2 = a^2$$

Therefore a^2 must be even. If a^2 is even then *a* must be even.

(The proof is very similar to "if n^2 is odd, then *n* is odd", shown earlier)

Since *a* is even, $a = 2c$ for some integer *c*. Thus,

$$2b^2 = 4c^2 \qquad\qquad b^2 = 2c^2$$

Therefore b^2 is even. Again then *b* must be even as well.

But then 2 must divide both *a* and *b*. This contradicts our assumption that *a* and *b* have no common factors. We have proved by contradiction that our initial assumption must be false and therefore √2 is irrational.

QED

Background Information

- Fundamental theorem of arithmetic (also called the unique factorization theorem)
  - Every number is a prime or a unique product of primes

Proof by Contradiction (3)

Example: Prove that there is no largest prime number.

Solution:

Assume there are a finite number of primes. Let p_n be the largest prime number. Consider the number:

$$r = p_1 \times p_2 \times \ldots \times p_n + 1$$

This number is the product of all the prime numbers there are, plus 1.

If we try to divide r by any prime number, we get a remainder of 1. In other words, none of the prime numbers divides r.

By the Fundamental Theorem of Arithmetic, either r is prime or it is a unique product of primes that is not in the set $\{p_1, p_2, \ldots\ldots, p_n\}$.

This contradicts the assumption that there is a largest prime. Therefore we have proved by contradiction that there is no largest prime.

QED

Proof by Contradiction (4)

Example: Give a proof by contradiction of the theorem "If 3*n* + 2 is odd, then *n* is odd."

Solution:

Let us assume that 3n + 2 is odd but n is even.

Since n is even, there exists some k such that n = 2k.

$\therefore$ 3n + 2 = 3(2k) +2 = 6k + 2 = 2(3k+1), which is even

This contradicts the assumption that 3n + 2 is odd. Therefore we have proved by contradiction that if 3*n* + 2 is odd, then *n* is odd.

QED

Theorems that are Biconditional Statements

To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true.

Example: Prove the theorem: "If n is an integer, then n is odd if and only if n^2 is odd."

Solution: We have already shown (previous slides) that both $p \rightarrow q$ and $q \rightarrow p$. Therefore we can conclude $p \leftrightarrow q$.

Sometimes *iff* is used as an abbreviation for "if and only if," as in

"If n is an integer, then n is odd iff n^2 is odd."

Proof By Counterexample

- Statements of the form $\forall x P(x)$ can be proved to be false by providing a counterexample $\exists x(\neg P(x))$
  - **Example**: Show that the statement “Every positive integer is the sum of the squares of two integers” is false.

The number 3 can be represented as the sum of two numbers in one of two ways (order being immaterial)

$3 = 0 + 3 = 0^2 + 3$

$3 = 1 + 2 = 1^2 + 2$

In neither case can be number 3 be represented as a sum of two squares.

Therefore we have proved by counterexample that “Every positive integer is the sum of the squares of two integers” is false.

QED