CS 2305: Discrete Mathematics for Computing I

Lecture 03

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Precedence of Logical Operators

- ¬p ∧ q
  - Is it (¬p) ∧ q or ¬(p ∧ q)?
- To unambiguously interpret compound propositions, the following precedence of logical operators has been agreed upon:

Operator	Precedence
¬	1
∧	2
∨	3
→	4
↔	5

- By default the proposition is (¬p) ∧ q
- Use parentheses to override the default precedence
- Use parentheses when in doubt

Translating English Sentences (1)

- Translate the following English sentence into a logical expression
  - "Sylvia is very smart but she is careless"
- Steps:

i. Extract simple propositions from the sentence, assigning them to propositional variables
   - p: Sylvia is very smart
   - q: She is careless

ii. Extract the logical operators used in the sentence
   - ???

iii. Rewrite the original sentence to use more commonly used logical operators
   - Sylvia is very smart and she is careless

iv. Extract the logical operators used in the sentence
   - and: $\wedge$

v. Convert to logical expression, using propositional variables and logical operators
   - $p \wedge q$

Translating English Sentences (2)

- Translate the following English sentence into a logical expression
  - "Neither is he interested in higher studies nor is he interested in a job"
- Steps:

  i. Extract simple propositions from the sentence, assigning them to propositional variables
     - p: He interested in higher studies
     - q: He interested in a job

  ii. Extract the logical operators used in the sentence
     - ???

  iii. Rewrite the original sentence
     - He is not interested in higher studies and he is he not interested in a job

  iv. Extract the logical operators used in the sentence
     - not : $\neg$
     - and: $\land$

  v. Convert to logical expression, using propositional variables and logical operators
     - $\neg p \land \neg q$

Translating English Sentences (3)

- Translate the following English sentence into a logical expression
  - "You can access the Internet from campus only if you are a computer science major or you are not a freshman"
- Steps:

  i. Extract simple propositions from the sentence, assigning them to propositional variables
     - p: You can access the Internet from campus
     - q: You are a computer science major
     - r: You are a freshman

  ii. Extract the logical operators used in the sentence
     - only if: $\rightarrow$
     - or: $\vee$
     - not: $\neg$

  iii. Convert to logical expression, using propositional variables and logical operators
     - $p \rightarrow q \vee \neg r$
     - May add parentheses for clarity: $p \rightarrow (q \vee \neg r)$

A Word About "Unless"

- Unless has the same semantics as if … not
  - Unless I work hard, I will fail the exam
    - If I do not work hard then I will fail the exam
  - I will not go to Best Buy unless I need a TV
    - I will not go to Best Buy if I do not need a TV
    - If I do not need a TV then I will not go to Best Buy

Remember:
q if p ≡ if p then q

Translating English Sentences (4.1)

- Translate the following English sentence into a logical expression
  - "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old"
- Steps:

i. Extract simple propositions from the sentence, assigning them to propositional variables
  - p: You can ride the roller coaster
  - q: You are under 4 feet tall
  - r: You are older than 16 years old

ii. Extract the logical operators used in the sentence
  - not: ¬
  - if …:

iii. Rewrite the original sentence to use more commonly used logical operators
  - p unless q ≡ p if not q ≡ if ¬q then p

q if p ≡ if p then q

  - If you are under 4 feet tall then you cannot ride the roller coaster unless you are older than 16 years old
  - If you are not older than 16 years old then if you are under 4 feet tall then you cannot ride the roller coaster

Translating English Sentences (4.2)

– "If you are not older than 16 years old then if you are under 4 feet tall then you cannot ride the roller coaster"

- Steps (Cont'd):
  - iv. Extract the logical operators used in the modified sentence
    - not: $\neg$
    - if then: $\rightarrow$
  - v. Convert to logical expression, using propositional variables and logical operators
    - $\neg r \rightarrow (q \rightarrow \neg p)$

Note: This solution is different from that in the text.
Verify that the two logical expressions are equivalent.

Tautologies, Contradictions, and Contingencies

- A ***tautology*** (represented by **T** or **1**) is a proposition which is always true.
  - Example: $p \vee \neg p$
- A ***contradiction*** (represented by **F** or **0**) is a proposition which is always false.
  - Example: $p \wedge \neg p$
- A ***contingency*** is a proposition which is neither a tautology nor a contradiction

An Example of a Tautology

- $p \vee \neg(p \wedge q)$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Logically Equivalent (1)

- We already know that two compound propositions *p* and *q* are equivalent if and only if the columns in a truth table giving their truth values agree.
- Formally two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- We write this as $p \Leftrightarrow q$ or as $p \equiv q$ where *p* and *q* are compound propositions

Logically Equivalent (2)

- Formally, two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
  - Example: ¬(r ∧ s) ≡ ¬r ∨ ¬s

r	s	p ¬(r ∧ s)	q ¬r ∨ ¬s
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

p	q	p↔q
F	F	T
T	T	T
T	T	T
T	T	T

Tautology

De Morgan's Laws (1)

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan
1806-1871

- Amongst the most important results in logic
  - Lots of applications in computer programming, VLSI chip design, set theory, formal logic etc.
- Truth table for De Morgan's Second Law

p	q	¬p	¬q	(p ∨ q)	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

De Morgan's Laws (2)

- De Morgan's laws can be extend to multiple variables
  - $\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$
  - $\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$

- Symbolically

$$\neg\left(\bigwedge_{j=1}^{n} p_j\right) \equiv \bigvee_{j=1}^{n} \neg p_j$$

$$\neg\left(\bigvee_{j=1}^{n} p_j\right) \equiv \bigwedge_{j=1}^{n} \neg p_j$$

Key Logical Equivalences [1]

Identity Laws: $p \wedge T \equiv p, \qquad p \vee F \equiv p$

Domination Laws: $p \vee T \equiv T, \qquad p \wedge F \equiv F$

Idempotent laws: $p \vee p \equiv p, \qquad p \wedge p \equiv p$

Double Negation Law: $\neg(\neg p) \equiv p$

Negation Laws: $p \vee \neg p \equiv T, \qquad p \wedge \neg p \equiv F$

Key Logical Equivalences 2

Commutative Laws: $p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$

Associative Laws: $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws: $\left(p \vee (q \wedge r)\right) \equiv (p \vee q) \wedge (p \vee r)$

$\left(p \wedge (q \vee r)\right) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption Laws: $p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$