CS 2305: Discrete Mathematics for Computing I

Lecture 08

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Restricted Quantifiers (1)

- Unrestricted quantifiers apply to the entire domain of discourse
  - Most of what we have seen so far
- A restricted quantifier has the same semantics as an unrestricted quantifier except that the variables in the domain must satisfy a certain condition in order for the quantification to apply
  - $(\forall x)_{x < 0}\ (x^2 > 0)$
  - $(\forall y)_{y \neq 0}\ (y^3 \neq 0)$
  - $(\exists z)_{z > 0}\ (z^2 = 2)$

Assumption: Domain of discourse is **R**, the set of real numbers

"For all/some «variable» satisfying the domain constraint …"

Restricted Quantifiers (2)

- Sometimes we need to express a restricted quantifier as an unrestricted quantifier
  - Express $(\exists x)_{P(x)}\ (Q(x))$ using the unrestricted existential quantifier
    - Solution
      - The logical expression states that there exists some value in the domain for which P(x) is true such that Q(x) is true
      - It can be expressed with an unrestricted existential quantifier as follows:
        - » $\exists x(P(x) \land Q(x))$

Restricted Quantifiers (3)

- Express $(\forall x)_{P(x)}\ (Q(x))$ using the unrestricted universal quantifier
  - Solution
    - Step 1
      - The logical expression states that for all values in the domain for which $P(x)$ is true, $Q(x)$ is true as well. We start with $\forall x(P(x) \land Q(x))$. Although it satisfies our basic condition, it makes the unreasonable assumption that $P(x)$ is true for all values of x
    - Step 2
      - Being a universal quantifier, we need to account for all values in the domain: those for which our claim applies and those for which it does not
    - Step 3
      - We need to exclude those values from the domain for which our claim does not apply by observing that $\forall x(\neg P(x) \lor (P(x) \land Q(x)))$
    - Step 4
      - We simplify the above expression

$$\forall x(\neg P(x) \lor (P(x) \land Q(x))) \equiv \forall x((\neg P(x) \lor P(x)) \land (\neg P(x) \lor Q(x)))$$
$$\equiv \forall x(\mathbf{T} \land (\neg P(x) \lor Q(x))) \equiv \forall x(\neg P(x) \lor Q(x)) \equiv \forall x(P(x) \rightarrow Q(x))$$

Section 1.3, Table 6

- **Distributive Law**
- **Negation Law**
- **Identity Law**

Restricted Quantifiers (4)

- To summarize
  - $(\exists x)_{P(x)}\,(Q(x)) \equiv \exists x(P(x) \wedge Q(x))$
    - "The restriction of an existential quantification is the same as the existential quantification of a conjunction"
  - $(\forall x)_{P(x)}\,(Q(x)) \equiv \forall x(P(x) \rightarrow Q(x))$
    - "The restriction of a universal quantification is the same as the universal quantification of a conditional statement"

Translating from English to Logic (1)

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class then we can use an unrestricted universal quantifier.

Define a propositional function J(*x*) denoting "x has taken a course in Java"

The logical representation is simply $\forall x\, J(x)$.

Solution 2: But if *U* is all people, we start with a restricted universal quantifier

Define a propositional function S(x) denoting "x is a student in this class"

The logical representation using the restricted quantifier is $(\forall x)_{S(x)}(J(x))$

Next we express it using an unrestricted universal quantifier as $\forall x\, (S(x) \rightarrow J(x))$

$\forall x\, (S(x) \land J(x))$ is not correct. What does it mean?

↳ Every person is a student of this class and every person has taken a course in Java

Translating from English to Logic (2)

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain *U*.

Solution 1: If *U* is all students in this class then we can use an unrestricted existential quantifier.

Define a propositional function J(*x*) denoting "x has taken a course in Java"

The logical representation is simply $\exists x\, J(x)$.

Solution 2: But if *U* is all people, we start with a restricted existential quantifier

Define a propositional function S(x) denoting "x is a student in this class"

The logical representation using the restricted quantifier is $(\exists x)_{S(x)}(J(x))$

Next we express it using an unrestricted existential quantifier as $\exists x\, (S(x) \land J(x))$

$\exists x(S(x) \rightarrow J(x))$ is not correct. What does it mean?

↳ There exists a person such that if the person is a student of this class then the student has taken Java

↳ Vacuously we will include someone who is not a student

Convention for translating from English to Logic

- Instead of restricting the domain of discourse, we generally start by using restricted quantifiers on the universal domain and then expressing the logical statement in terms of the unrestricted quantifiers
  - $(\exists x)_{P(x)}\ (Q(x)) \equiv \exists x(P(x) \land Q(x))$
  - $(\forall x)_{P(x)}\ (Q(x)) \equiv \forall x(P(x) \rightarrow Q(x))$

Translating from English to Logic (3)

Example 3: Translate the following sentence into predicate logic: “Every student in this class has visited Canada or Mexico.” The domain of discourse U is all people

Solution:

Let the propositional function S(x) denote “x is a student in this class”

Let the propositional function C(x) denote “x has visited Canada”

Let the propositional function M(x) denote “x has visited Mexico”

Using the restricted universal quantifier we have:

$(\forall x)_{S(x)}(C(x) \lor M(x))$

Expressing this using the unrestricted universal quantifier we have:

$\forall x(S(x) \rightarrow (C(x) \lor M(x)))$

Translating from English to Logic (4)

Example 3: Given that the domain of discourse U is all birds, translate the following sentences into predicate logic:

a) All hummingbirds are richly colored.

b) No large birds live on honey.

c) Birds that do not live on honey are dull in color.

d) Hummingbirds are small.

Solution:

Let P(x) denote "x is a hummingbird"

Let Q(x) denote "x is large"

Let R(x) denote "x lives on honey"

Let S(x) denote "x is richly colored"

a) $(\forall x)_{P(x)} S(x) \equiv \forall x(P(x) \rightarrow S(x))$

b) $\neg\exists x(Q(x) \land R(x))$

c) $\forall x(\neg R(x) \rightarrow \neg S(x))$

d) $(\forall x)_{P(x)} \neg Q(x) \equiv \forall x(P(x) \rightarrow \neg Q(x))$

Nested Quantifiers

In nested quantifiers one quantifier is within the scope of another quantifier

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example 1: “Every real number has an inverse” is

$$\forall x \, \exists y(x + y = 0)$$

where the domains of x and y are the real numbers.

Example 2: “The product of a positive real number and a negative real number is always a negative real number” is

$$\forall x \, \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$$

where the domains of x and y are the real numbers.

Thinking of Nested Quantification

Nested loops in programming languages provide a good way of thinking about nested quantifiers

- To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
- To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
  - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

Order of Quantifiers (1)

The order of the nested quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers

We will not prove this formally but we will show it through examples

Order of Quantifiers (2)

Example 1:

Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers.

$\forall x\ \forall y P(x,y)$ says that "For all real numbers x, for all real numbers y, x + y = y + x"

$\forall y\ \forall x P(x,y)$ says that "For all real numbers y, for all real numbers x, x + y = y + x"

Clearly $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same meaning.

Order of Quantifiers (3)

Example 2:

Let $P(x,y)$ be the statement "$x + y = 100$" Assume that U is the real numbers.

$\exists x\ \exists y P(x,y)$ says that "There exists some real number x, such that for some real number y, x + y = 100"

$\exists y\ \exists x P(x,y)$ says that "There exists some real number y, such that for some real number x, x + y = 100"

Clearly $\exists x\ \exists y P(x,y)$ and $\exists y\ \exists x P(x,y)$ have the same meaning.