CS 2305: Discrete Mathematics for Computing I

Lecture 27+

Bonus Material ---- Not on the finals

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Generalized Permutations and Combinations

Section 6.5

Background Information

- In our discussion on permutations and combinations so far each element could be used at most once
  - While discussing the product rule we did look at some repetitions
- In some counting problems elements may be used repeatedly
  - e.g. letter or a digit may be used more than once on a license plate
- Some counting problems involve indistinguishable elements
  - e.g. in how many ways can the letters of the word SUCCESS be rearranged

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition.

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Comparison: Repetition vs Nonrepetition

Non-repetition

- First element can be chosen in n ways
- Second element can be chosen in n-1 ways
- …
- r^{th} element can be chosen in n-(r-1) ways
- By product rule total number of ways to choose the elements = n * (n-1) * … * (n-r+1) = $\frac{n!}{(n-r)!}$

Repetition

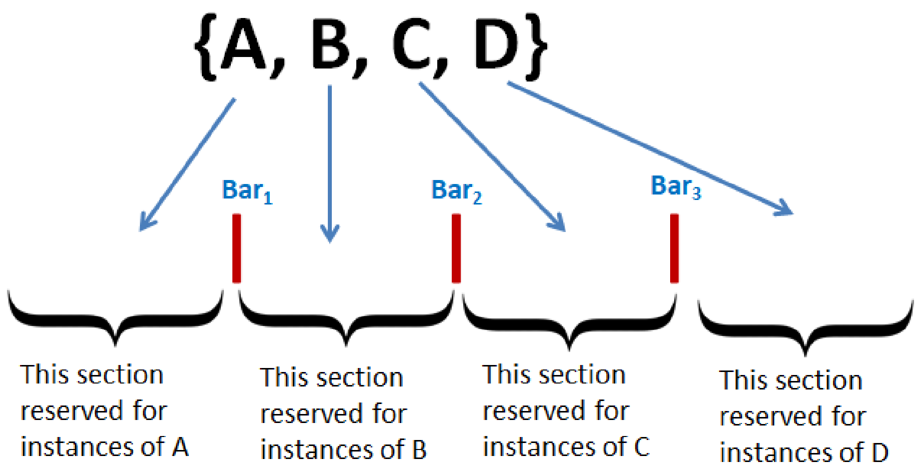
- First element can be chosen in n ways
- Second element can be chosen in n ways
- …
- r^{th} element can be chosen in n ways
- By product rule total number of ways to choose the elements = n * n * … * n = n^r

Background: Combinations with Repetition [1]

Let us devise a way to represent an *r*-combination of a set with *n* elements with repetition.

First we will create a list of n-1 bars, which can be used to represent the n elements of the set.

For example if our set is {A, B, C, D} we introduce the bars as follows:



Background: Combinations with Repetition 2

For each instance of an element in the unordered arrangement, we will put an asterisk symbol in the corresponding section. If there is no instance of an element in the unordered arrangement, we will leave it blank.

For example, the arrangement AACDDD will be represented as

** | | * | ***

There is a unique pattern of bars and asterisks that represents each pattern in the *r*-combination. Each pattern has a total of *n*-1+*r* elements, comprising *n*-1 bars and *r* asterisks.

The problem of counting the number of *r*-combinations from a set with *n* elements when repetition of elements is allowed now reduces to the problem of counting the number of ways we can arrange *n*-1 bars within a set with *n*-1+*r* elements

Combinations with Repetition 1

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1)$$

Recall $C(n, r) = C(n, n - r)$

Proof: Let us represent each r-combination of a set with n elements with repetition allowed by a list of $n - 1$ bars and r stars

Each r-combination of a set with n elements with repetition allowed can be represented by a list of $n - 1$ bars and r stars. The bars mark the n cells containing a star for each time the i^{th} element of the set occurs in the combination.

The number of such lists is $C(n + r - 1, r)$, because each list is a choice of the r positions to place the stars, from the total of $n + r - 1$ positions to place the stars and the bars. This is also equal to $C(n + r - 1, n - 1)$, which is the number of ways to place the $n - 1$ bars.

Combinations with Repetition 2

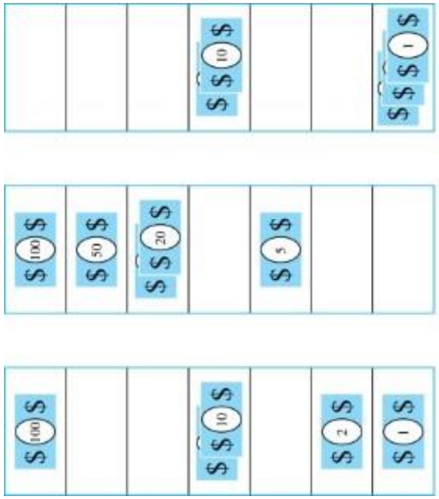
Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100?

Solution: Place the selected bills in the appropriate position of a cash box illustrated below:

Combinations with Repetition 3

Some possible ways of placing the five bills:

The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.



| | | * * | | | * * *

* | * | * * | | * | |

* | | | * * | | * | *

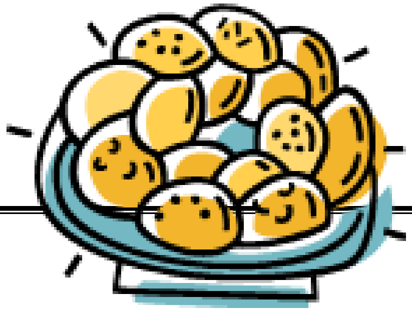
This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

$C(n + r - 1, r)$ [$= C(n + r - 1, n - 1)$]
$n = 7, r = 5$

ways to choose five bills with seven types of bills.

Combinations with Repetition 4



Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(4+6-1,\ 6) \ = \ C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

$C(n + r - 1, r)$ [$= C(n + r - 1, n - 1)$]
$n = 4, r = 6$

is the number of ways to choose six cookies from the four kinds.

Combinations with Repetition

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1 , x_2 and x_3 are nonnegative integers?

Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.
By Theorem 2 it follows that there are

C(3 + 11 -1, 11) = C(13, 11) = C(13, 2) = $\frac{13 * 12}{2 * 1}$ = 78 solutions.

$C(n + r - 1, r)\ [= C(n + r - 1, n - 1)]$
$n = 3, r = 11$

Combinations with Repetition

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$?

Solution: In this case we have at least one item of type x_1, two items of type x_2, and three items of type x_3. So the problem boils down to selecting 5 items [11 – (1 + 2 + 3)] from a set with three elements, with repetition.

By Theorem 2 it follows that there are

C(3 + 5 -1, 5) = C(7, 5) = C(7, 2) = $\frac{7 * 6}{2 * 1}$ = 21 solutions.

$C(n + r - 1, r)$ [$= C(n + r - 1, n - 1)$]
$n = 3, r = 5$

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

Permutations with Indistinguishable Objects [1]

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof:

The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.

Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.

Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways.

By the product rule the total number of permutations is:

$C(n, n_1)\, C(n - n_1, n_2)\, C(n - n_1 - n_2, n_3) \cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$

Permutations with Indistinguishable Objects 2

Proof (Cont’d):
The product $C(n, n_1)\, C(n - n_1, n_2)\, C(n - n_1 - n_2, n_3) \cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!\,\cancel{(n-n_1)}!}\;\frac{\cancel{(n-n_1)}!}{n_2!\,\cancel{(n-n_1-n_2)}!}\;\frac{\cancel{(n-n_1-n_2)}!}{n_3!\,\cancel{(n-n_1-n_2-n_3)}!}\cdots\frac{\cancel{(n-n_1-\cdots-nk_{-1})}!}{n_k!\,0!}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\ldots nk!}$$

“telescoping cancellation”

Permutations with Indistinguishable Objects 3

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!}\cdot\frac{4!}{2!2!}\cdot\frac{2!}{1!1!}\cdot\frac{1!}{3!2!1!1!} = 420.$$

Distributing Objects into Boxes [1]

Many counting problems can be solved by counting the ways objects can be placed in boxes.

- The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
- The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Distinguishable Objects into Distinguishable Boxes [1]

Theorem 4: The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, i = 1, 2,…, k, equals

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof:
(Very similar to the proof for permutations with indistinguishable objects)
The first box holds n_1 objects, and there are $C(n,n_1)$ ways to choose those objects from among the n objects in the collection.
Once these objects are chosen, we can choose the objects to be placed in the second box in $C(n - n_1,n_2)$ ways, since there are $n - n_1$ objects not yet placed, and we need to put n_2 of them into the second box.
Similarly, there are then $C(n - n_1 - n_2, n_3)$ ways to choose objects for the third box.
We continue in this way, until finally there are $C(n - n_1 - n_2, \ldots - n_{k-1}, n_k)$ ways to choose the objects to put in the last (k^{th}) box.
By the product rule the total number of permutations is:
$C(n, n_1)\, C(n - n_1, n_2)\, C(n - n_1 - n_2, n_3) \cdots C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$, which has been shown to be equal to

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots nk!}$$

Distributing Distinguishable Objects into Distinguishable Boxes 2

Example: How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution:

ab initio approach

The first player can be dealt 5 cards in $C(52, 5)$ ways. The second player can be dealt 5 cards in $C(47, 5)$ ways, because only 47 cards are left. The third player can be dealt 5 cards in $C(42, 5)$ ways. Finally, the fourth player can be dealt 5 cards in $C(37, 5)$ ways.

Hence, the total number of ways to deal four players 5 cards each is

$$C(52, 5)C(47, 5)C(42, 5)C(37, 5) = \frac{52!}{47!5!} * \frac{47!}{42!5!} * \frac{42!}{37!5!} * \frac{37!}{32!5!}$$

$$= \frac{52!}{5!5!5!5!32!}$$

Distributing Distinguishable Objects into Distinguishable Boxes

Solution (Cont'd):

Distinguishable objects, distinguishable boxes theorem approach

The distinguishable objects are the 52 cards, and the five distinguishable boxes are the hands of the four players (size 4 each) and the rest of the deck (size 32).

Hence from the distinguishable objects, distinguishable boxes theorem the total number of ways to deal is $\dfrac{52!}{5!5!5!5!32!}$

Distributing Indistinguishable Objects into Distinguishable Boxes[1]

Theorem: The number of ways to distribute n indistinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals $C(n + k - 1, k - 1) = C(n + k - 1, n)$

Proof: (Very similar to proof for number of r-combinations from a set with n elements when repetition of elements is allowed).

Let us consider a slotted box with adjustable dividers.

k-1 dividers are needed to convert the slotted box into k bins

Each arrangement of objects in the bins is represented by a unique combination of dividers and objects.

There are total of k-1+n dividers and objects. The number of combinations of k-1 dividers among k-1+n dividers and objects is $C(k\text{-}1+n, k\text{-}1)$

Example: There are C(8 + 10 − 1, 8-1) = C(17,7) = 19,448 ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Indistinguishable Boxes

- There are no closed formulae to count the ways to distribute objects, distinguishable or indistinguishable, into indistinguishable boxes
  - A closed formula is an expression that can be evaluated using a finite number of operations
    - arithmetic operations, rational powers, exponential and logarithmic functions, trigonometric functions, and the factorial function
    - infinite series are excluded