CS 2305: Discrete Mathematics for Computing I

Lecture 14

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Tuples

Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.

The *ordered n-tuple* $(a_1,a_2,\ldots..,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product 1

Definition: The *Cartesian Product* of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

René Descartes (1596-1650)

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

Cartesian Product 2

Example: Let $A = \{0,1, 2, 3\}$

What is $A \times \emptyset$?

Solution: $\emptyset$

Definition: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.

- Subset of ordered pairs where the first element belongs to A and the second to B
- will be covered in depth later

A relation from a set A to itself is called a *relation* on A.

Cartesian Product 3

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n =$$

$$\{(a_1, a_2 \cdots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C$ **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Cartesian Product 4

We use the notation A^2 to denote $A \times A$

Similarly $A^3 = A \times A \times A$

In general

$$A^n = \{(a_1, a_2, a_3, \ldots, a_n) \mid a_i \in A \text{ for } I = 1, 2, 3, \ldots\ldots n\}$$

Example: If $A = \{1, 2\}$, $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

Truth Sets of Quantifiers

Given a predicate *P* and a domain *D*, we define the *truth set* of *P* to be the set of elements in *D* for which *P*(*x*) is true. The truth set of *P*(x) is denoted by

$$\{x \in D \mid P(x)\}$$

Example: The truth set of *P*(*x*) where the domain is the integers and *P*(*x*) is "$|x| = 1$" is the set {-1,1}

Set Operations

Section 2.2

Boolean Algebra

Propositional calculus and set theory are both instances of an algebraic system called a *Boolean Algebra*. This is discussed in Chapter 12.

The operators in set theory are analogous to the corresponding operator in propositional calculus.

As always there must be a universal set U. All sets are assumed to be subsets of U.

Set Operations

- Set theory defines various operations that can be applied to arbitrary sets in some domain in order to generate new sets
- Most commonly used set theory operations are:
  - Union
  - Intersection
  - Difference
  - Complementation
  - Symmetric Difference

Union

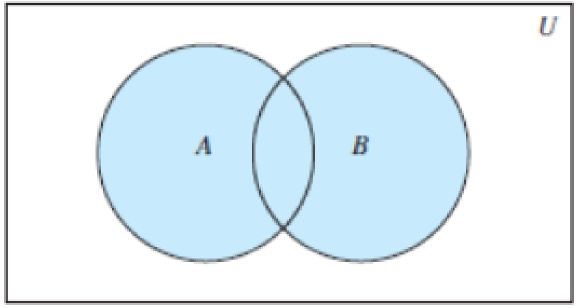
Definition: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Example: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



$A \cup B$ is shaded.

Intersection

Definition: The *intersection* of sets A and B, denoted by $A \cap B$, is

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

Note if the intersection is empty, then A and B are said to be *disjoint*.

Example: What is? $\{1,2,3\} \cap \{3,4,5\}$?

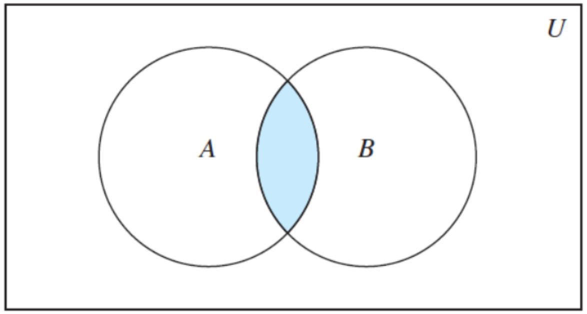
Solution: $\{3\}$

Example: What is?

$\{1,2,3\} \cap \{4,5,6\}$?

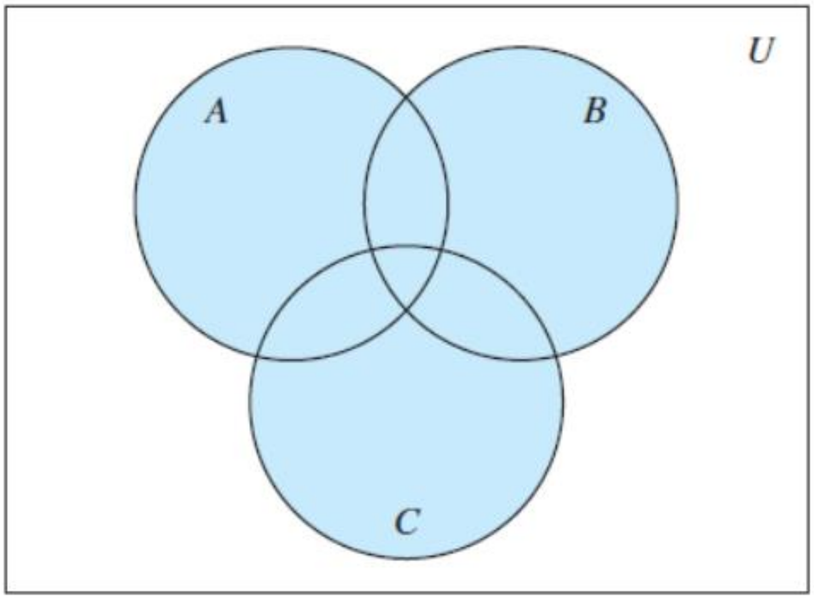
Solution: $\emptyset$

Venn Diagram for $A \cap B$

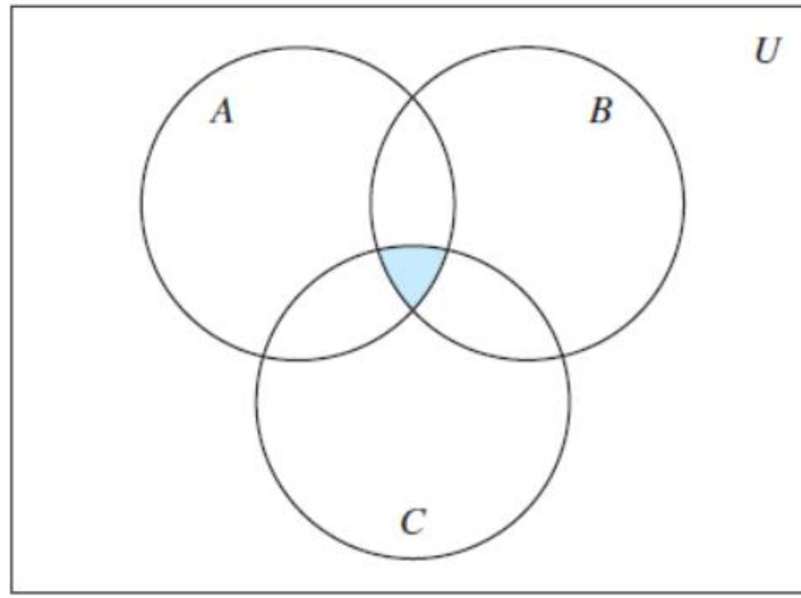


$A \cap B$ is shaded.

Union and Intersection of 3 Sets



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

Difference

Definition: Let *A* and *B* be sets. The *difference* of *A* and *B*, denoted by *A* – *B*, is the set containing the elements of *A* that are not in *B*.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

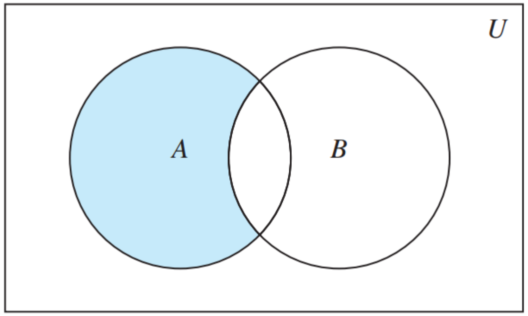
Example: Given A = {1, 3, 5} and B = {1, 2, 3}, what is A – B and B - A

Solution:

A – B = {5}

B – A = {2}

Venn Diagram for *A* – *B*



A − *B* is shaded.

Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

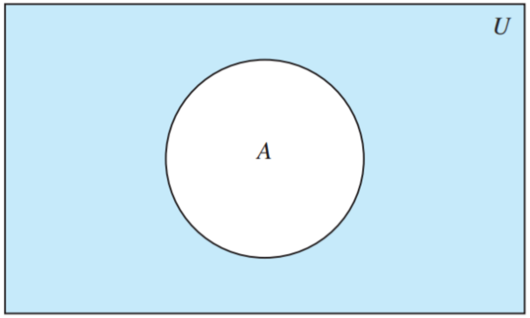
$$\bar{A} = \{x \mid x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

$$\text{Solution}: \{x \mid x \leq 70\}$$

Venn Diagram for Complement



$\bar{A}$ is shaded.

Another Interpretation of Set Difference

Since the *difference* of two sets *A* and *B* (denoted by *A – B)* is the set containing the elements of *A* that are not in *B,* it can also be interpreted as the intersection of *A* and the *complement* of B (i.e. $\overline{B}$)

The difference of *A* and *B* is also called the complement of *B* with respect to *A*.

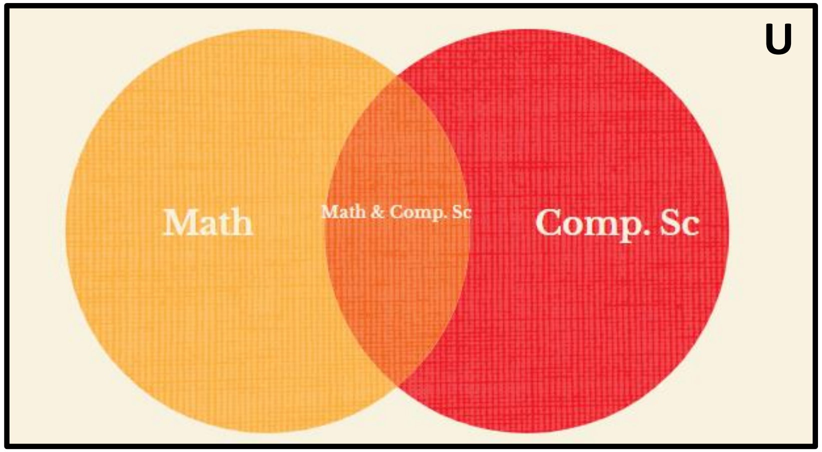
$$A - B = \left\{ x \mid x \in A \wedge x \notin B \right\} = A \cap \overline{B}$$

The Cardinality of the Union of Two Sets

Principle of Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.



Principle of Inclusion-Exclusion (1)

100 students at UTD were surveyed to find out whether they had taken courses in Discrete Math, Java, and Computer Architecture. The results of the survey were as follows:

- 21 students had taken Discrete Math
- 45 students had taken Java
- 38 students had taken Computer Architecture
- 9 students had taken Discrete Math and Java
- 4 students had taken Discrete Math and Computer Architecture
- 18 students had taken Java and Computer Architecture
- 23 students has not taken any of these three courses

How many students had taken all three courses?

Principle of Inclusion-Exclusion (2)

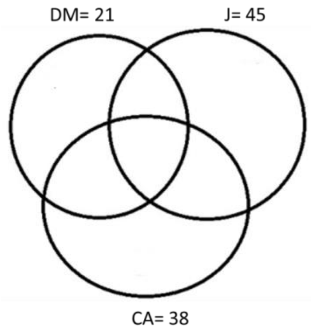
Let us assume that the number of students taking all three courses is x.

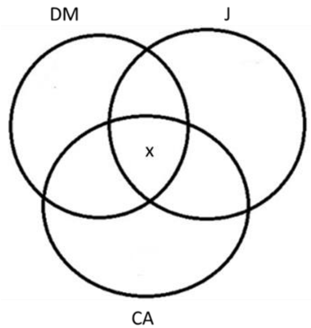
Let us fill up the rest of the overlap regions in the Venn Diagram

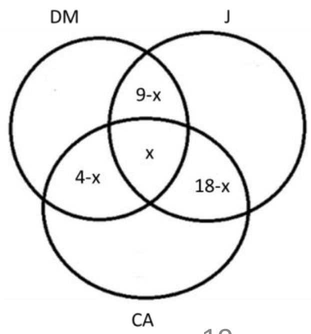
Number of students taking Discrete Math only = 21 - [9-x + 4-x +x] = 21 – [13-x] = 8 + x

Number of students taking Java only = 45 - [9-x + 18-x +x] = 45 – [27-x] = 18 + x

Number of students taking Computer Architecture only = 38 - [4-x + 18-x +x] = 38 – [22-x] = 16 + x

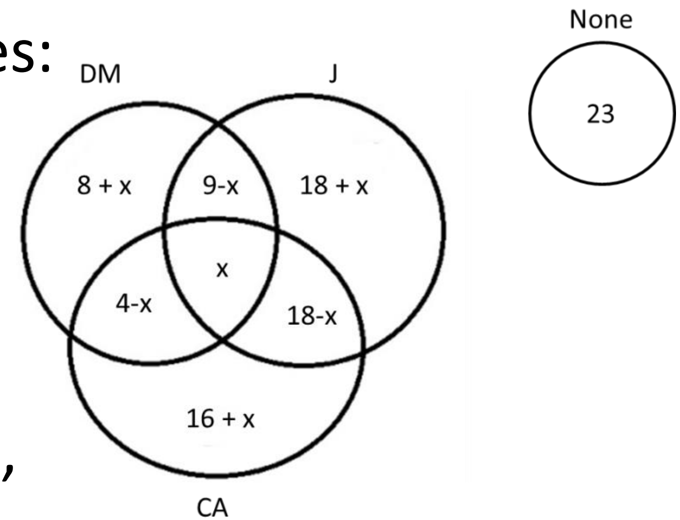






Principle of Inclusion-Exclusion (3)

Since 23 students have not taken any of these courses, the Venn Diagram becomes:



Since the total number of students is 100, we have

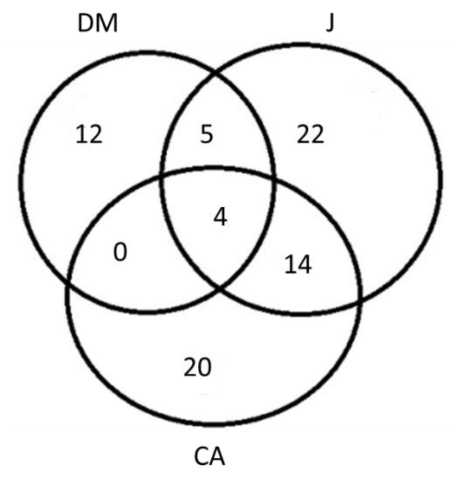
(8+x) + (18+x) + (16+x) + (9-x) + (4-x) + (18-x) + x + 23 = 100

96 + x = 100

x = 4

Principle of Inclusion-Exclusion (4)

∴ the number of students who have taken all the courses is 4.



Symmetric Difference

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

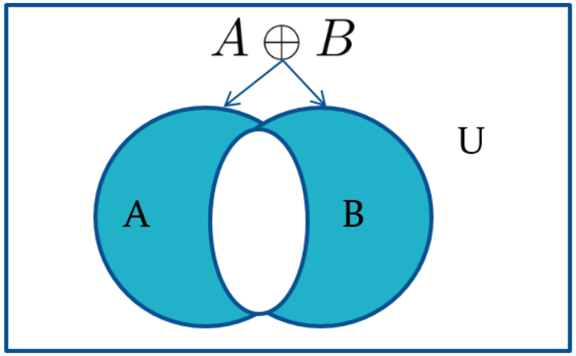
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

Solution: $\{1,2,3,6,7,8\}$

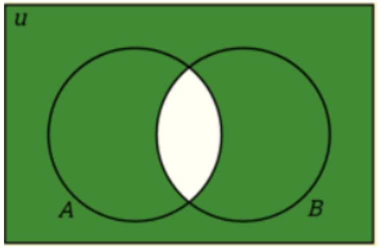


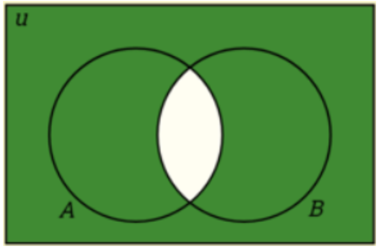
Venn Diagram

Is Symmetric Difference the Complement of Intersection?

- When dealing with only two sets, the complement of the intersection turns out to be the same as the symmetric difference. However, this is not true in general

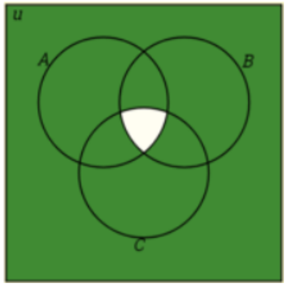
$\overline{A \cap B}$

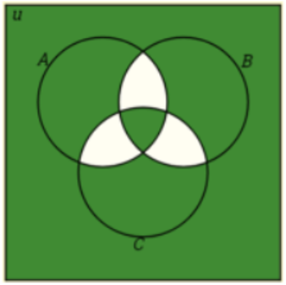




$A \oplus B$

$\overline{A \cap B \cap C}$





$A \oplus B \oplus C$

Set Identities[1]

Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Domination laws

$$A \cup U = U \qquad A \cap \varnothing = \varnothing$$

Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

Complementation law

$$\left(\overline{\overline{A}}\right) = A$$

Set Identities 2

Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities 3

De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

Complement laws

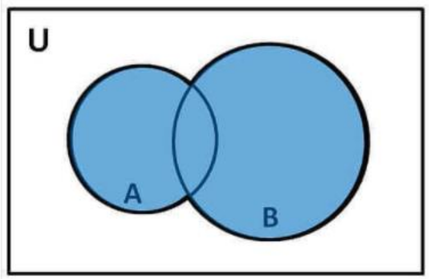
$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Proving Set Identities[1]

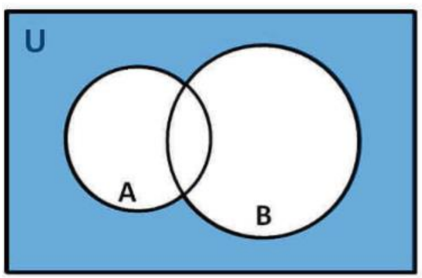
Venn Diagrams give a lot of useful insight into sets constructed using two or three atomic sets. However they provide far less insight when four or more atomic sets are involved

Venn diagrams “proofs” are considered informal

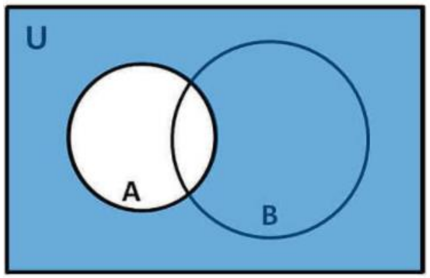
Proving Set Identities$_2$



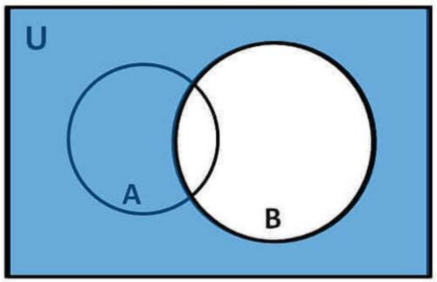
A ∪ B



(A ∪ B)′



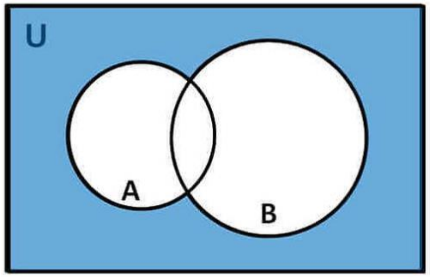
A′



B′

$$\overline{\boldsymbol{A} \cup \boldsymbol{B}} = \overline{\boldsymbol{A}} \cap \overline{\boldsymbol{B}}$$

De Morgan’s Law



A′ ∩ B′

Proving Set Identities3

Different ways to formally prove set identities:

1. Prove that each set (side of the identity) is a subset of the other
2. Use set builder notation and propositional logic to transform one side of the identity to the other
3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

Proof of Second De Morgan Law 1

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$$1)\ \overline{A \cap B} \subseteq \overline{A} \cup \overline{B} \qquad \text{and}$$

$$2)\ \overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

Proof of Second De Morgan Law 2

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg\big((x \in A) \wedge (x \in B)\big)$	by defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

Proof of Second De Morgan Law

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$\left(x \in \overline{A}\right) \vee \left(x \in \overline{B}\right)$	by defn. of union
$\left(x \notin A\right) \vee \left(x \notin B\right)$	defn. of complement
$\neg\left(x \in A\right) \vee \neg\left(x \in B\right)$	defn. of negation
$\neg\left(\left(x \in A\right) \wedge \quad \left(x \in B\right)\right)$	1st De Morgan law for Prop Logic
$\neg\left(x \in A \cap B\right)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

Proof of Second De Morgan Law 4

Because we have shown that:

$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$$

and

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

the two sets are equal, and the identity is proved