CS 2305: Discrete Mathematics for Computing I

Lecture 27

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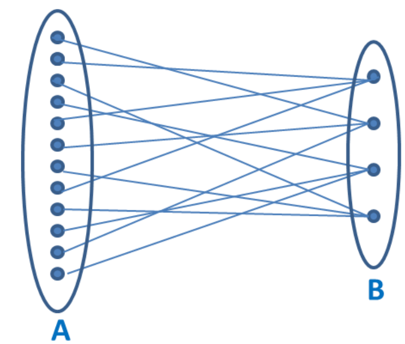
Basic Counting Principles: Division Rule[1]

Division Rule: There are *n*/*d* ways to do a task if it can be done using a procedure that can be carried out in *n* ways, and for every way *w*, there are *d* equivalent ways of doing *w*.

Restated <u>in terms of sets</u>: If the finite set *A* is the union of *n* pairwise disjoint subsets each with *d* elements, then $n = |A|/d$.

<u>In terms of functions</u>: If *f* is a function from *A* to B, where both are finite sets, and for every value $y \in B$ there are exactly *d* values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

The Division Rule is commonly used for counting arrangements when some of the objects are indistinguishable or when order doesn't matter.



Division Rule: An Illustrative Example

- Assume that there are 12 ways to carry out a procedure:
  - A, B, C, D, E, F, G, H, I, J, K, L

A
E
I

Indistinguishable alternatives --- let us represent these by α

B
F
J

Indistinguishable alternatives ---let us represent these by β

C
G
K

Indistinguishable alternatives ---let us represent these by γ

D
H
L

Indistinguishable alternatives ---let us represent these by δ

- Because of indistinguishable alternatives there are effectively only $\frac{12}{3} = 4$ ways to carry out the procedure
  - α, β, γ, δ

Basic Counting Principles: Division Rule[2]

Example: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

Solution: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are 4! = 24 ways to order the four people. Since two seatings are the same when each person has the same left and right neighbor, corresponding to each of the 24 ways of seating around the table, there are 4 equivalent ways of seating the people, which differ only in the choice of the person in seat 1. This corresponds to rotating the table, which does not change a person's neighbors.

Therefore, by the division rule, there are 24/4 = 6 different seating arrangements.

Alternate Solution (using the product rule): The choice of seat 1 is immaterial. There are 3 ways to select seat 2, 2 ways to select seat 3 and 1 way to select seat 4.

Therefore, by the product rule, there are 3 * 2 * 1 = 6 different seating arrangements.

Basic Counting Principles: Division Rule

Example: How many ways are there to seat six people around a circular table where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?

Solution: There are 6! = 720 ways to order the six people. For reasons explained on the previous slide, this overcounts the seating arrangements by a multiple of 6, corresponding to rotating the table. Furthermore, we are overcounting by a multiple of 2 because seating arrangements that reverse the right and the left neighbor are equivalent.

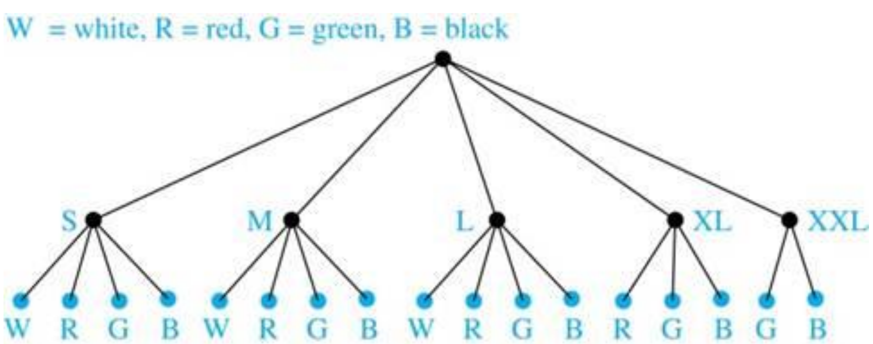
Therefore, by the division rule, there are 720/6/2 = 60 different seating arrangements.

Tree Diagrams[1]

Tree Diagrams: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Example: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S,M,L,XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?

Solution: Draw the tree diagram.



The store must stock at least 17 T-shirts.

Alternate Solution:

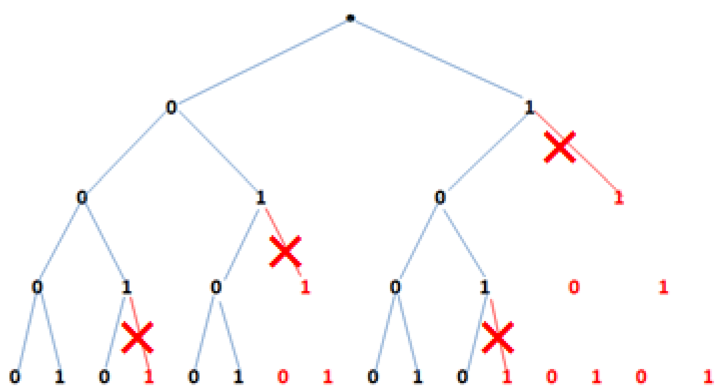
By sum rule 4S + 4M + 4L + 3XL + 2XXL = 17 types of shirts

Tree Diagrams₂

Example: How many bit strings of length four do not have two consecutive 1s?

Solution: The tree diagram shows that there are 8 bit strings of length four that do not have two consecutive 1s

- 0000
- 0001
- 0010
- 0100
- 0101
- 1000
- 1001
- 1010



Note: Tree diagrams are most feasible when the number of leaves in the tree is small.

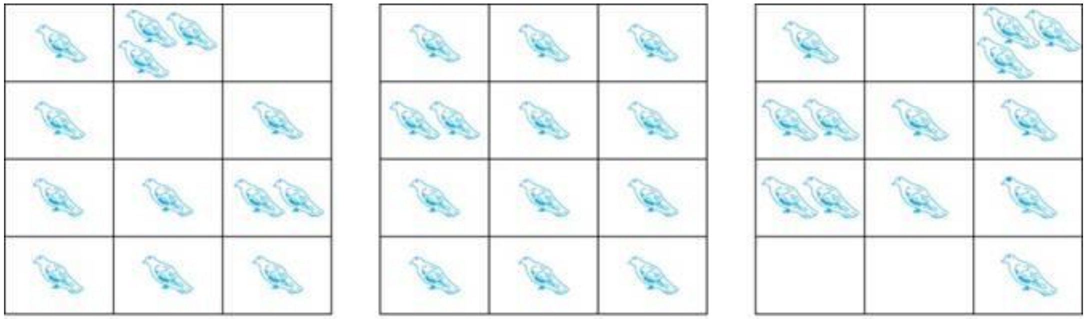
The Pigeonhole Principle

Section 6.2

The Pigeonhole Principle[1]

If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.

- Also known as the Dirichlet Drawer Principle



(a) (b) (c)

The Pigeonhole Principle[2]

Pigeonhole Principle: If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition.

Let p: k + 1 or more objects are placed into k boxes

Let q: at least one box contains two or more objects

The contrapositive states that $\neg q \rightarrow \neg p$

Our hypothesis is that each box contains at most 1 object.

Since there are k boxes, the total number of objects is at most k.

This is a contradiction since there are at least k+1 boxes.

Contradiction of the contrapositive

Jump to long description

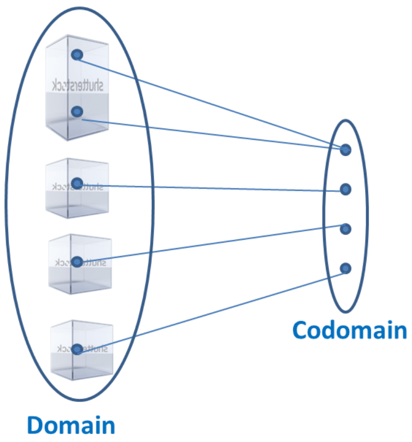
The Pigeonhole Principle₃

Corollary 1: A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box corresponding to each element y in the codomain of f .
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are at least $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one.



The Pigeonhole Principle[4]

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example Show that for every integer *n* there is a multiple of *n* that has only 0s and 1s in its decimal expansion.
Solution: Let *n* be a positive integer. Consider the *n* + 1 integers 1, 11, 111, …., 11…1 (where the last has *n* + 1 1s). There are *n* possible remainders when an integer is divided by *n*. By the pigeonhole principle, when each of the *n* + 1 integers is divided by *n*, at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of *n* that has only 0s and 1s in its decimal expansion.

Smallest multiple of the first 34 numbers that is composed entirely of 0s and 1s in the decimal expansion:
http://oeis.org/A004290/list

Let $a \equiv b \pmod{c}$
$a = b + km$
$a - b = km$
$m \mid (a-b)$

The Generalized Pigeonhole Principle 1

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contradiction of contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < \lceil N/k \rceil + 1$ has been used. This is a contradiction because there are a total of N objects.

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

The Generalized Pigeonhole Principle 2

Typically with the Generalized Pigeonhole Principle we look for the smallest number that will guarantee an outcome.

Problem: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is N = [(6 - 1) * 5] + 1 = 26.

The Generalized Pigeonhole Principle 3

Example: a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

b) How many must be selected to guarantee that at least three hearts are selected?

Solution: a) We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards. At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is

$N = [(3 - 1) * 4] + 1 = 9$.

b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts.

The Generalized Pigeonhole Principle

Example: What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.

Solution: Total number of phone numbers, without the area code = $8*10^6$

Since we have to provide 25 million phone numbers, the state will need a minimum of $\lceil 25*10^6/8*10^6 \rceil = 4$ area codes

Permutations and Combinations

Section 6.3

Permutations and Combinations[1]

- Many counting problems deal with finding the number of ways to arrange a specified number of elements of a set, without actually listing them
  - either all *n* elements of a set or a subset of *r* elements of the set
- There are two distinct methods that can be used to select r objects from a set of n elements: ordered and unordered
  - In an ordered selection, it is not only what elements are chosen but also the order in which they are chosen that matters (e.g. number of ways to seat people around a table)
    - An ordered selection of r elements from a set of n elements is called an r-permutation of the set.
  - In an unordered selection it is only the identity of the chosen elements that matters while order is immaterial (e.g. number of ways to select members to a committee)
    - An unordered selection of r elements from a set of n elements is called the r-combination of the set.

Permutations and Combinations[2]

- The product rule is a generic rule that allows us to count the number of ways to complete <u>any procedure</u> that is composed of multiple tasks
- Permutations and combinations make heavy use of the product rule to count the number of ordered and unordered selections, of a desired size, that can be made from a given set

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

The number of r-permutations of a set with n elements is denoted by $P(n,r)$.

- The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

r-permutations of a set with n distinct elements.
Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Proof:

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1) = n(n - 1)(n - 2) \cdots (n - r + 1)\frac{(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Permutation Formula, Special Cases

$P(n, r) = \frac{n!}{(n-r)!}$

- $P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$
  - There are $n!$ ordered arrangements involving n elements
- $P(n, 0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
  - There is only 1 ordered arrangement involving 0 elements
- $P(n, 1) = \frac{n!}{(n-1)!} = n$
  - There are n ordered arrangements involving 1 element

Solving Counting Problems by Counting Permutations [1]

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Solving Counting Problems by Counting Permutations [2]

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations 3

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: Because the letters ABC must occur as a block, we can consider them to be a single object. The problem now reduces to finding the number of permutations of six objects viz. *ABC*, *D*, *E*, *F*, *G*, and *H*.

So the answer is $P(6, 6) = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Combinations[1]

Definition: An *r-combination* of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*.

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

$C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations [2]

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!\,r!}.$$

Proof:

The number of r-Combinations of the set is *C*(*n, r*) (by definition).

If we order each of these combinations, we get *P*(*n, r*) the *r*-permutations of the set.

By the product rule *P*(*n, r*) = *C*(*n*,*r*) · *P*(*r*,*r*). Therefore

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!\,(n-r)!}$$

Combinations $_3$

Corollary 2: Let n and r be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)!\left[n-(n-r)\right]!} = \frac{n!}{(n-r)!r!}.$$

Hence, $C(n, r) = C(n, n - r)$.

Combination Formula, Special Cases

$C(n, r) = \frac{n!}{r!(n-r)!}$

- $C(n, n) = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1$
  - There is only 1 unordered selection involving n elements
- $C(n, 0) = \frac{n!}{0!(n-0)!} = \frac{n!}{1.n!} = 1$
  - There is only 1 unordered arrangement involving 0 elements
- $C(n, 1) = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = n$
  - There are n unordered arrangements involving 1 element

Combinations[4]

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$

$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960$$

Combinations [5]

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775$$

Combinations [6]

Problem: How many bit strings of length 10 contain

a) exactly four 1s?

b) at most four 1s?

c) at least four 1s?

d) an equal number of 0s and 1s?

Solution:

a) C(10, 4) = $\frac{10!}{(4!)(6!)}$ = 210

b) C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) + C(10, 4) = 1 + 10 + 45 + 120 + 210 = 386

c) C(10, 4) + C(10, 5) + C(10, 6) + C(10, 7) + C(10, 8) + C(10, 9) + C(10, 10) = 210 + 252 + 210 + 120 + 45 + 10 + 1 = 848

d) C(10, 5) = 252

Combinations [7]

Problem: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: Number of ways to select 3 faculty members from the mathematics department = C(9, 3)

Number of ways to select 4 faculty members from the computer science department = C(11, 4)

By product rule the total ways to select the committee is C(9, 3) * C(11, 4) = 84 * 330 = 27, 720

Combinations: A Shortcut

$$C(n, r) = \frac{n!}{r! * (n-r)!} = \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)!}{r! * (n-r)!} =$$

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \frac{\overbrace{n(n-1)(n-2)\ldots(n-r+1)}^{\text{r terms}}}{\underbrace{r(r-1)(r-2)\ldots(2)(1)}_{\text{r terms}}}$$

$$\text{e.g. } C(10, 3) = \frac{\overbrace{10 * 9 * 8}^{\text{3 terms}}}{\underbrace{3 * 2 * 1}_{\text{3 terms}}} = \frac{720}{6} = 120$$

Since C(n, r) = C(n, n-r), for using the shortcut the trick is to use the formula where the second argument of the n-Combinations function is the smaller of the two numbers (viz. r and n-r).

For example, to compute C(10, 7) we first convert C(10, 7) to C(10, 10-7) = C(10, 3) and then apply the shortcut

Permutation vs Combinations: Special Cases

$P(n, r) = \frac{n!}{(n-r)!}$

- $P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$
  - There are $n!$ ordered arrangements involving n elements
- $P(n, 0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
  - There is only 1 ordered arrangement involving 0 elements
- $P(n, 1) = \frac{n!}{(n-1)!} = n$
  - There are n ordered arrangements involving 1 element

$C(n, r)$ = C(n, n-r) = $\frac{n!}{r!(n-r)!}$

- $C(n, n) = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1$
  - There is only 1 unordered selection involving n elements
- $C(n, 0) = \frac{n!}{0!(n-0)!} = \frac{n!}{1.n!} = 1$
  - There is only 1 unordered arrangement involving 0 elements
- $C(n, 1) = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!}$ = n
  - There are n unordered arrangements involving 1 element