Week 07 Assignment

1. exercise 6

Use a proof by case to show that $min(a, min(b, c)) = min(min(a, b), c)$, whereever a, b, c are real number

Case 1, $a = b = c$

$min(a, min(b, c)) = a = b = c$

$min(min(a, b), c)) = a = b = c$

Case 2. a is the smallest

$min(a, min(b, c)) = min(a, b \text{ or } c) = a$

$min(min(a, b), c) = min(a, c) = a$

Case 3, b is the smallest

$min(a, min(b, c)) = min(a, b) = b$

$min(min(a, b), c) = min(b, c) = b$

Case 4. c is smallest

$min(a, min(b, c)) = min(a, c) = c$

$min(min(a, b), c) = min(a \text{ or } b, c) = c$

In all cases, the $min(a, min(b, c))$ and $min(min(a, b), c)$ has the same output, therefore they are equa

Q.E.D

2 exercise 8

Prove using the notion of without loss of generality that $5x+5y$ is an odd number when x and y are integers of opposite parity

$5x+5y = 5(x+y)$

Without loss of generality, we assume $x = 2k+1$, which is odd, $y = 2h$, which is even

$x+y = 2k+1+2h = 2(k+h)+1$ which is odd

$5(2(k+h)+1) = 10(k+h)+5$ is also odd

Therefore, $5x+5y$ is always odd when x and y are integers of opposite parity.

Q.E.D

3 exercise 24

Use forward reasoning to show that if x is a non-zero real number, then $x^2+\frac{1}{x^2} \geq 2$

$x^2+\frac{1}{x^2} \geq 2 = x^2+\frac{1}{x^2}-2 \geq 0 = \frac{x^4-2x^2+1}{x^2} \geq 0 = \frac{(x^2)^2-2(x^2)\times 1+1^2}{x^2} \geq 0$

$\frac{(x^2-1)^2}{x^2} \geq 0 = \frac{(x+1)^2(x-1)^2}{x^2} \geq 0$

$(x+1)^2$ is non-negative, $(x+1)^2$ is non-negative, so $\frac{(x+1)^2(x-1)^2}{x^2}$ is always non-negative except at $x=0$ which is undefined.

Q.E.D

24. exercise 36

Prove that $\sqrt[3]{2}$ is irrational

Assume $\sqrt[3]{2}$ is rational, then $\sqrt[3]{2} = \frac{m}{n}$

$\sqrt[3]{2} = 2^{\frac{1}{3}} = \frac{m}{n}$, then $(2^{\frac{1}{3}})^3 = (\frac{m}{n})^3$

$2 = \frac{m^3}{n^3}$, $2n^3 = m^3$, so m^3 is even, hencely, m is even. Assume $m = 2k$

$n^3 = \frac{m^3}{2} = \frac{(2k)^3}{2} = \frac{8k^3}{2} = 4k^3$

$n^3 = 4k^3$, so n^3 is even, hencely, n is even

Because m, n are both even, the $\frac{m^3}{n^3}$ are not coprime because they have common factor 2, which is complicting with Assumption

Therefore, $\sqrt[3]{2}$ is irrational

Q.E.D

5. Using proof by case show that for any integer n, n^2+n is even. Since 0 is either even or odd.
Assume n is non-zero

$n^2+n = n(n+1)$

Case 1: n is odd, Assume $n=2k+1$

$n(n+1) = n^2+n$
$= (2k+1)^2+2k+1$
$= 4k^2+4k+1+2k+1$
$= 4k^2+6k+2$
$= 2(2k^2+3k+1)$ which is even

Case 2: n is even, Assume $n=2k$

$n(n+1) = n^2+n$
$= (2k)^2+2k$
$= 4k^2+2k$
$= 2(2k^2+k)$ which is even

Therefore, for any non-zero integer n, n^2+n is even

Q.E.D

6. using proof by cases show that for any integer n that is not divisible by 5, n^2 is eith of the form $5k+1$ or $5k+4$

Case 1: $n=5k+1$ (remainder $=1$)

$$n^2=(5k+1)^2=25k^2+10k+1=5(5k^2+2k)+1=5m+1 \text{ when } m=5k^2+2k$$

Case 2: $n=5k+2$

$$n^2=(5k+2)^2=25k^2+20k+4=5(5k^2+4k)+4=5m+1 \text{ when } m=5k^2+4k$$

Case 3: $n=5k+3$

$$n^2=(5k+3)^2=25k^2+30k+9=25k^2+30k+5+4=5(5k^2+6k+1)+4=5m+1, \text{ when } m=5k^2+6k+1$$

Case 4: $n=5k+4$

$$n^2=(5k+4)^2=25k^2+40k+16=25k^2+40k+15+1=5(5k^2+8k+3)+1=5m+1 \text{ When } m=5k^2+8k+3$$

Therefore, When n is not divisible by 5, n^2 is either in form of $5k+1$ or $5k+4$

Q.E.D

7. prove that the product of any three consecutive integer is even

Assume the first one is n, then the second is n+1 and third is n+2

CASE 1: n is odd, Assume $n = 2k+1$

$(2k+1)(2k+2)(2k+3) = 8k^3+24k^2+22k+6 = 2(4k^3+12k^2+11k+3)$ which is even

CASE 2: n is even, Assume $n = 2k$

$(2k)(2k+1)(2k+2) = 8k^3+12k^2+4k = 2(4k^3+6k^2+2k)$ which is even

CASE 3: $n = 0$

$0 \times 1 \times 2 = 0$ ← this depenses on how you trade it, in CS it is even

Therefore, the product of three consecutive integer is even

Q.E.D

8. Find the power set of set $A = \{a, b, c, d\}$

$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c,d\}\}$