CS 2305: Discrete Mathematics for Computing I

Lecture 13

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The Role of Open Problems

Unsolved problems have motivated much work in mathematics. Fermat's Last Theorem was conjectured more than 300 years ago. It has only recently been finally solved.

Fermat's Last Theorem: The equation $x^n + y^n = z^n$

has no solutions in integers x, y, and z, with $xyz \neq 0$ whenever n is an integer with $n > 2$.

A proof was found by Andrew Wiles in the 1990s.

An Open Problem

The $3x + 1$ Conjecture: Let T be the transformation that sends an even integer x to $x/2$ and an odd integer x to $3x + 1$. For all positive integers x, when we repeatedly apply the transformation T, we will eventually reach the integer 1.

For example, starting with $x = 13$:

$T(13) = 3 \cdot 13 + 1 = 40$, $T(40) = 40/2 = 20$, $T(20) = 20/2 = 10$,

$T(10) = 10/2 = 5$, $T(5) = 3 \cdot 5 + 1 = 16$,$T(16) = 16/2 = 8$,

$T(8) = 8/2 = 4$, $T(4) = 4/2 = 2$, $T(2) = 2/2 = 1$

The conjecture has been verified using computers up to $5.48 * 10^{18}$.

https://www.youtube.com/watch?v=5mFpVDpKX70
https://www.youtube.com/watch?v=O2_h3z1YgEU

Additional Proof Methods

Time permitting, we will see many other proof methods:

- Mathematical induction, which is a useful method for proving statements of the form $\forall n\, P(n)$, where the domain consists of all positive integers.
- Structural induction, which can be used to prove such results about recursively defined sets.
- Cantor diagonalization is used to prove results about the size of infinite sets.
- Combinatorial proofs use counting arguments.

Basic Structures: Sets, Functions~~,~~ ~~Sequences, Sums, and Matrices~~

Chapter 2

Sets

Section 2.1

Sets

A *set* is an unordered collection of objects.

- the students in this class
- the chairs in this room

The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.

The notation $a \in A$ denotes that a is an element of the set A.

If a is not a member of A, write $a \notin A$

By convention sets are denoted using uppercase letters while lowercase letters are used to denote elements of sets.

Describing a Set: Roster Method

In this method all members of the set are explicitly listed between open and closed braces

$S = \{a,b,c,d\}$

Order not important

$$S = \{a,b,c,d\} = \{b,c,a,d\}$$

Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a,b,c,d, \ldots\ldots,z\}$$

Roster Method

Set of all vowels in the English alphabet:

$V = \{a,e,i,o,u\}$

Set of all odd positive integers less than 10:

$O = \{1,3,5,7,9\}$

Set of all positive integers less than 100:

$S = \{1,2,3,\ldots\ldots..,99\}$

Set of all integers less than 0:

$S = \{\ldots., -3,-2,-1\}$

Some Important Sets

N = *natural numbers* = {0,1,2,3….}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

Z⁺ = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

R⁺ = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

"Beware that mathematicians disagree whether 0 is a natural number. We consider it quite natural."
Rosen, 8e, Pg 122

Set-Builder Notation

Specify the property or properties that all members must satisfy

The general form of this notation is {x | x has property P} and is read "the set of all x such that x has property P."

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

A predicate may be used:

$S = \{x \mid \mathrm{P}(x)\}$

Example: $S = \{x \mid \mathrm{Prime}(x)\}$

Positive rational numbers*:*

$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

The interval notation is used to denote the set of real numbers, given two endpoint numbers *a* and *b*

$$[a,b] = \{x \mid a \le x \le b\}$$

Includes both a and b

$$[a,b) = \{x \mid a \le x < b\}$$

Includes a but excludes b

$$(a,b] = \{x \mid a < x \le b\}$$

Excludes a but includes b

$$(a,b) = \{x \mid a < x < b\}$$

Excludes both a and b

closed interval [a,b]

open interval (a,b)

Universal Set, Empty Set and Singleton Set

The *universal set U* is the set containing everything currently under consideration.

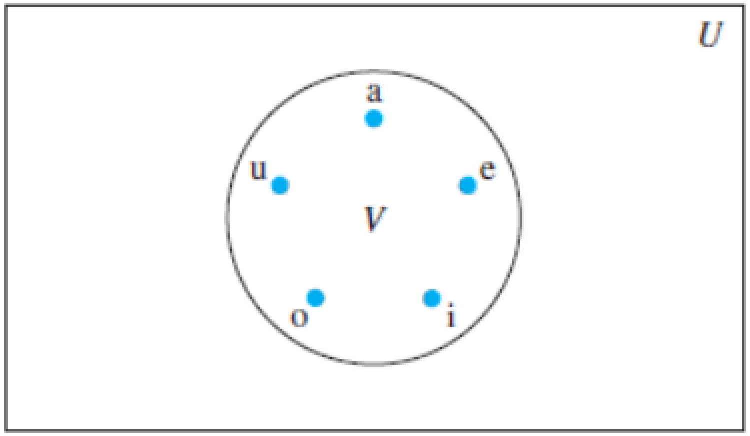
- Sometimes implicit
- Sometimes explicitly stated.
- Contents depend on the context.

The *empty set* (aka *null set*) is the set with no elements. Symbolized $\emptyset$, but {} also used.

A set with one element is called a *singleton set*.

Venn Diagrams

- Sets are often represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881
- In Venn diagrams the universal set *U* is represented by a rectangle
- Inside the Venn Diagram for *U*, circles or other geometrical figures are used to represent sets
- Sometimes points are used to represent the particular elements of the set



The Venn diagram for *V*, the set of vowels

John Venn (1834-1923)
Cambridge, UK

Some things to remember

Sets can be elements of sets.

$\{\{1,2,3\}, a, \{b,c\}\}$, $\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

Let $A = \{ \{a\}, \{b\}, \{a, b\}\}$

In this case $\{a\} \in A$, but $a \notin A$

The empty set is different from a set containing the empty set.

Analogy: An empty folder is not the same thing as a folder containing an empty folder.

$$\emptyset \neq \{ \emptyset \}$$

Singleton Set

Naive vs Axiomatic Set Theory

- Set theory was developed by the German mathematician Georg Cantor in the late 19^{th} century
- Cantor defined a set as collection of objects, without clearly specifying what an object is
- This simplistic version of set theory is now called the naive set theory and it gives rise to some interesting paradoxes (e.g. next slide)
- These paradoxes are avoided in axiomatic set theory, which is extremely abstract and beyond the scope of this course

Russell's Paradox

Henry is a barber who shaves all people who do not shave themselves. A paradox results from trying to answer the question "Does Henry shave himself?"

More generically:

Let *S* be the set of all sets which are not members of themselves. A paradox results from trying to answer the question "Is *S* a member of itself?"

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x\left(x \in A \leftrightarrow x \in B\right)$
- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Remember:

- order is immaterial
- multiplicity is ignored

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x\left(x \in A \rightarrow x \in B\right)$ is true.

1. Because $x \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.

   Vacuous Truth

2. Because $x \in S \rightarrow x \in S$, $S \subseteq S$, for every set S.

"Every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself"

Supersets

Definition: If set A is a *subset* of set B, then set B is a *superset* of set A.

- The notation $B \supseteq A$ is used to indicate that B is a superset of the set A.

$A \subseteq B$ and $B \supseteq A$ are equivalent statements.

Showing a Set is or is not a Subset of Another Set

Showing that A is a Subset of B: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a Subset of B: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.
   - {-9, -8, -7, …, 0, 1, 2, …9} $\nsubseteq$ {0, 1, 2, 3, ……}

Another look at Equality of Sets

Recall that two sets A and B are *equal*, denoted by $A = B$, iff

$$\forall x\left(x \in A \leftrightarrow x \in B\right)$$

Using logical equivalences we have that $A = B$ iff

$$\forall x\left[\left(x \in A \to x \in B\right) \wedge \left(x \in B \to x \in A\right)\right]$$

This is equivalent to

$$A \subseteq B \text{ and } B \subseteq A$$

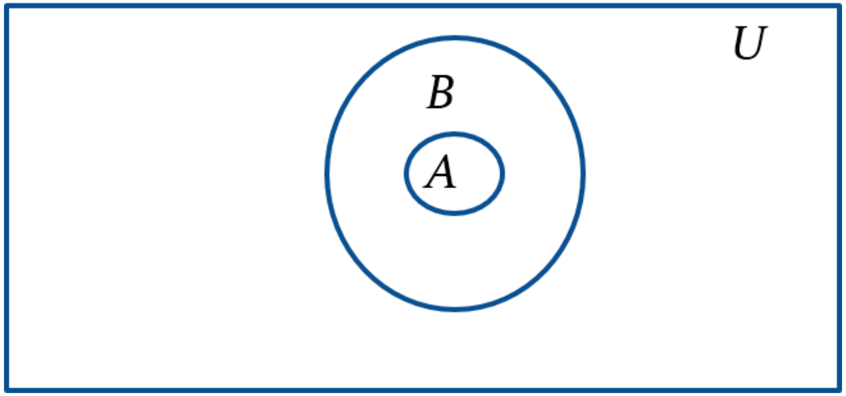
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets 1

Definition: The set of all subsets of a set *A*, denoted *P*(*A*), is called the *power set* of *A*.

Example: If *A* = {a,b} then

$$P(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

1

If a set has *n* elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)

Note: The empty set and the set itself are members of the *power set*

Power Sets [2]

Q) What is the *power set* of ø?

A) P(ø) = {ø}

Q) What is the *power set* of *{ø}*?

A) P({ø}) = {ø, {ø}}

Tuples

Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.

The *ordered n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.

Two n-tuples are equal if and only if their corresponding elements are equal.

2-tuples are called *ordered pairs*.

The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.