CS 2305: Discrete Mathematics for Computing I

Lecture 02

- KP Bhat

Logical Operators

- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- XOR $\oplus$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

<u>Connectives</u>

Logical Operator: Disjunction

- The *disjunction* of propositions *p* and *q* is denoted by $p \vee q$ and has this truth table:

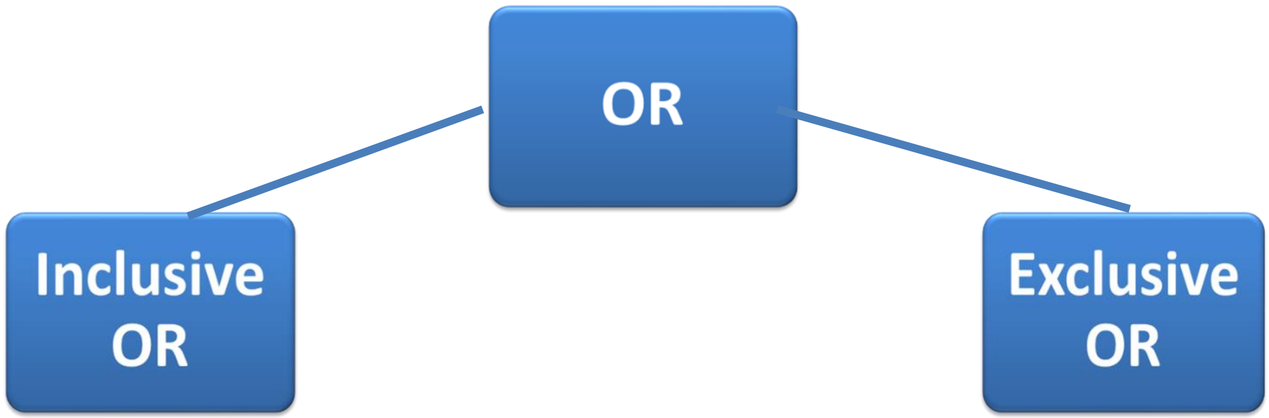
p	*q*	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.
- **Example**: If *p* denotes "I am at home." and *q* denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

Classroom Exercise

- Create a disjunction from the following simple propositions:
  - Rebecca's PC has more than 16 GB free hard disk space
  - The processor in Rebecca's PC runs faster than 1 GHz
    - Rebecca's PC has at least 16 GB free hard disk space, or the processor in Rebecca's PC runs faster than 1 GHz

The Connective “Or” in the English Language



Inclusive OR

- disjunction (∨)
- For $p \vee q$ to be true, either one or both of p and q must be true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive OR

- xor (⊕)
- For $p \oplus q$ to be true, one of p and q must be true, but not both.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Exclusive OR

- Sample proposition
  - p: I will use all my savings to travel to Europe
  - q: I will use all my savings to buy an electric car
  - p $\oplus$ q: I will use all my savings to travel to Europe or to buy an electric car
- Sample application
  - A program accepts as its input a file or a directory. If the input is a file it is processed. If the input is a directory, all files in the directory are processed

Input is a file	Input is a directory	Output
T	T	Error condition
T	F	Operate on the file
F	T	Operate on all files in the directory
F	F	Error condition

Truth Table of a Compound Proposition (p ∧ ¬q)

p	q	¬q	p ∧ ¬q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Equivalent Propositions

- Two propositions are *equivalent* (≡) if they always have the same truth value
  - The output columns of the truth tables are the same

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\neg(p \land q)$ is logically equivalent (≡) to $\neg p \lor \neg q$

Conditional Statement: IF p THEN q ($p \rightarrow q$)

- Consider the propositions
  - p: It is raining
  - q: The ground is wet
- What conclusions can you draw from IF p THEN q?
- There are two conclusions that you can draw
  1. If it rains, the ground is wet
  2. If the ground is not wet it did not rain
- You cannot draw the conclusion that if the ground is wet then it rained
  - Somebody could have turned on the hose or sprinkler

$p \rightarrow q$

$$\textbf{IF } \overset{p}{\underbrace{\text{it is raining}}_{\substack{\text{hypothesis /} \\ \text{antecedent /} \\ \text{premise}}}} \textbf{ THEN } \overset{q}{\underbrace{\text{the ground is wet}}_{\substack{\text{conclusion /} \\ \text{consequence}}}}$$

Vacuous Truth

- A conditional statement that is true by virtue of the fact that its hypothesis is false is often called vacuously true or true by default

Truth Table for IF p THEN q (1)

- Things to remember when filling out the truth table for "**IF** it is raining **THEN** the ground is wet"
  - If the hypothesis is false, we are not making any claim
  - If there is no contradiction, the truth value is **T**
  - If there is a contradiction, the truth value is **F**
  - If there is no claim (i.e. hypothesis is false), the truth value is **T** (vacuously)

Truth Table for IF p THEN q (2)

p	q	IF p THEN q	
T	T	T	No contradiction
T	F	F	Contradiction
F	T	T	No claim: vacuously true
F	F	T	No claim: vacuously true

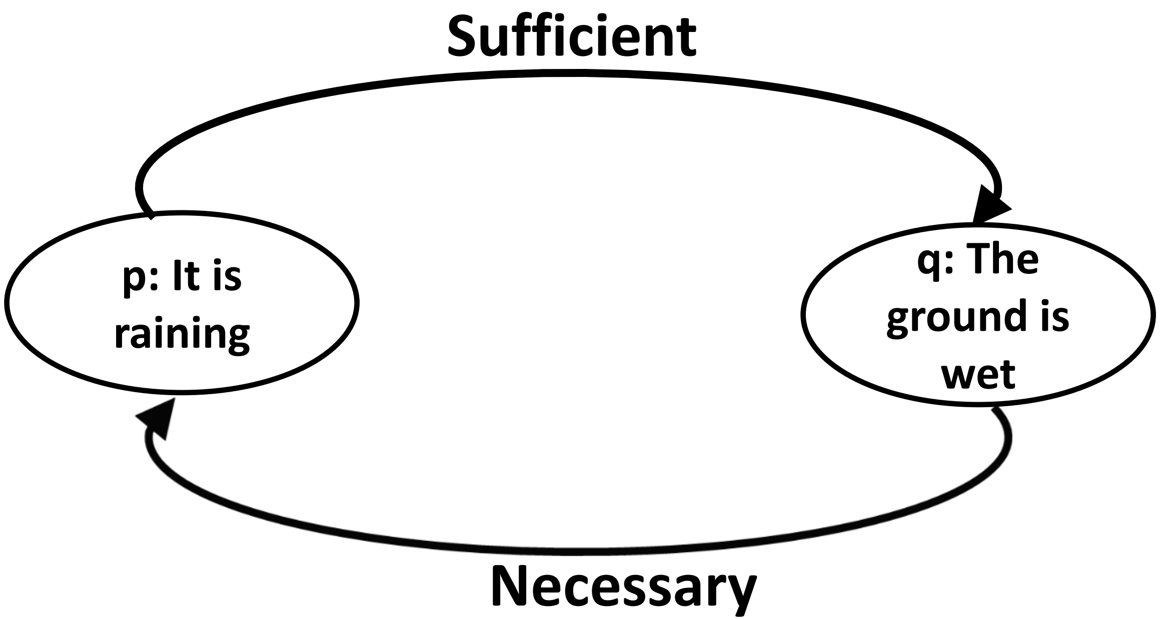
Implication/Conditional Statement

- If p and q are propositions, then $p \to q$ is a *conditional statement* or *implication* which is read as "p implies q" or "if p, then q" and has this truth table:

p	q	$P \to q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \to q$ denotes "If I am at home then it is raining."

p → q: Necessary & Sufficient Conditions



- p is sufficient for q
- q is necessary for p

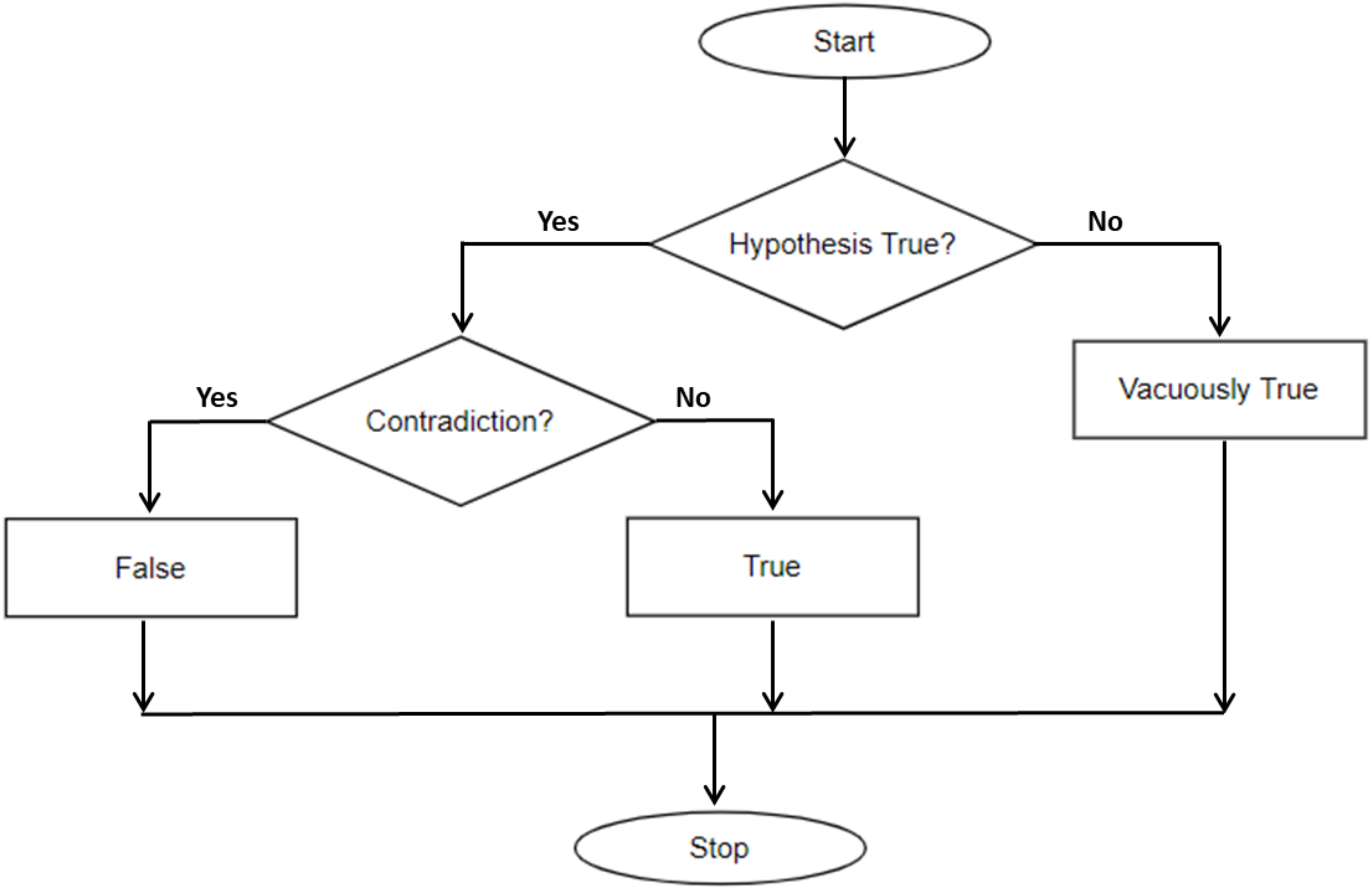
- **Sufficient**: “It is raining” is sufficient for “The ground is wet”
  - There could be many ways for the ground to get wet and rain is one of them
- **Necessary**: “The ground is wet” is necessary for “It is raining”
  - It is ground is not wet, we know for sure that it did not rain

Different Ways of Expressing $p \rightarrow q$

- **if** p**, then** q
- **if** p, q
- q **unless** $\neg p$
- q **if** p
- q **whenever** p
- q **follows from** p
- p **implies** q
- p **only if** q
- q **when** p
- p **is sufficient for** q
- q **is necessary for** p

- **a necessary condition for** p **is** q
- **a sufficient condition for** q **is** p

Evaluating the Conditional Statement



Simplified Version of the Implication Statement

- $(p \rightarrow q) \equiv (\neg p \vee q)$

p	q	¬p	p → q	¬p ∨ q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Exercise

- Given
    - p: It is a weekday
    - q: Parking lot is full
  - express "Parking lot is full **only if** it is a weekday" in symbolic form
- $q \rightarrow p$

Remember:
if p then q ≡ p only if q

Truth Values of Implication

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- If Austin is the capital of Texas then 100 is an even number
  - True
- If Austin is the capital of Texas then 100 is an odd number
  - False
- If Austin is the capital of Oklahoma then 100 is an even number
  - True
- If Austin is the capital of Oklahoma then 100 is an odd number
  - True

$p \to q$ versus IF-THEN-ELSE Programming Language Constructs

- $p \to q$ describes the relationship between **<u>two</u>** propositions
  - IF it is raining THEN the ground is wet
  - when the "if" part (hypothesis) of an if-then statement is false, the statement as a whole is true, regardless of whether the "then" part (conclusion) is true or false
- IF p THEN {…} ELSE {…} programming construct specifies what actions are performed, depending upon the truth value of the proposition p
  - Only **<u>one</u>** proposition
  - Perhaps IF p PERFORM {…} OTHERWISE {…} could have been less confusing

Converse, Contrapositive, and Inverse (1)

- From $p \rightarrow q$ we can form new conditional statements .
  - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
  - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$

Converse, Contrapositive, and Inverse (2)

- Find the converse, inverse, and contrapositive of “It is raining is a sufficient condition for my not going to town.”
- **Solution:**
  - Step 1: Find the propositions
    - p: It is raining
    - q: I will not go to town
  - Step 2a: Converse:
    - **If** I will not go to town **then** it is raining
  - Step 2b: Inverse:
    - **If** it is not raining **then** I will go to town
  - Step 2c: Contrapositive:
    - **If** I will go to town **then** it is not raining

Comparing Implication and its Contapositive

p	q	¬p	¬q	p → q	¬q → ¬p
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Implication is logically equivalent (≡) to its contrapositive

Biconditional/Bi-implication

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

P	Q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- p $\leftrightarrow$ q is true if p and q are either both true or both false
- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways of expressing biconditional propositions in English:
  - p **if and only if** q
  - p **iff** q
  - p **is necessary and sufficient for** q
  - **if** p **then** q, **and conversely**
  - p **exactly when** q

Dissecting the Biconditional (1)

p	q	p → q	q → p	(p → q) ∧ (q → p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

p ↔ q is logically equivalent (≡) to (p → q) ∧ (q → p)

Dissecting the Biconditional (2)

- An important observation:
  - $p \leftrightarrow q$ is true whenever p and q have the same truth values
  - Any relationship with XOR?

Truth table for the biconditional is the negation of the truth table for XOR

- Alternate forms of the biconditional statement
  - $(p \rightarrow q) \wedge (q \rightarrow p)$
  - $(\neg p \vee q) \wedge (\neg q \vee p)$

Truth Value of Biconditionals

- 1+1 = 2 iff 2+2 = 4
  - True

T	T	T

- 1+1 = 3 iff 2 + 2 = 4
  - False

F	T	F

- 1+1 = 3 iff 2+2 = 5
  - True

F	F	T

- 1+1 = 2 iff 2+2 = 5
  - False

T	F	F

- It is summer in the Northern Hemisphere iff it is winter in the Southern Hemisphere
  - True

T	T	T