CS 2305: Discrete Mathematics for Computing I

Lecture 15

- KP Bhat

Set-Builder Notation: First De Morgan Law

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

- Very similar to truth table
- 1 indicates that an element belongs to the set
- 0 indicates that an element does not belong to the set

A	*B*	*C*	*B*∩*C*	*A*∪ (*B*∩*C*)	*A*∪*B*	*A*∪*C*	(*A*∪*B*) ∩ (*A*∪*C*)
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

For $i = 1,2,\ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\} = A_1$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n$$

Multisets

- A multiset (short for multiple-membership set) is an unordered collection of elements where an element can occur as a member more than once
- Multisets use a similar notation as sets but for each element we also list its multiplicity i.e. the number of times it occurs
  - e.g. {4 . a, 1 . b, 3 . c}
- The cardinality of a multiset is defined to be the sum of the multiplicities of its elements

Multiset Operations (1)

Let P and Q be multisets.

The **union** of the multisets P and Q ($P \cup Q$) is the multiset in which the multiplicity of an element is the maximum of its multiplicities in P and Q

The **intersection** of P and Q ($P \cap Q$) is the multiset in which the multiplicity of an element is the minimum of its multiplicities in P and Q

The **difference** of P and Q ($P - Q$) is the multiset in which the multiplicity of an element is the multiplicity of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is 0

The **sum** of P and Q ($P + Q$) is the multiset in which the multiplicity of an element is the sum of multiplicities in P and Q

Multiset Operations (2)

Example: Suppose that P and Q are the multisets $\{4 \cdot a, 1 \cdot b, 3 \cdot c\}$ and $\{3 \cdot a, 4 \cdot b, 2 \cdot d\}$, respectively. Find $P \cup Q$, $P \cap Q$, $P - Q$, and $P + Q$.

Solution:

$P \cup Q = \{\max(4, 3) \cdot a, \max(1, 4) \cdot b, \max(3, 0) \cdot c, \max(0, 2) \cdot d\}$
$= \{4 \cdot a, 4 \cdot b, 3 \cdot c, 2 \cdot d\}$

$P \cap Q = \{\min(4, 3) \cdot a, \min(1, 4) \cdot b, \min(3, 0) \cdot c, \min(0, 2) \cdot d\}$
$= \{3 \cdot a, 1 \cdot b, 0 \cdot c, 0 \cdot d\} = \{3 \cdot a, 1 \cdot b\}$

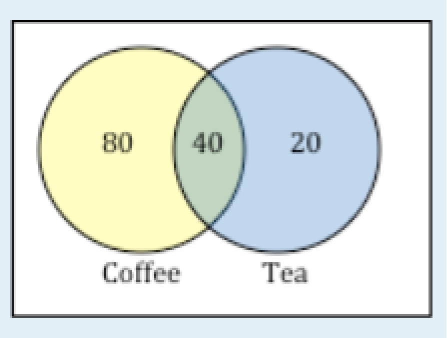
$P - Q = \{\max(4 - 3, 0) \cdot a, \max(1 - 4, 0) \cdot b, \max(3 - 0, 0) \cdot c, \max(0 - 2, 0) \cdot d\}$
$= \{1 \cdot a, 0 \cdot b, 3 \cdot c, 0 \cdot d\} = \{1 \cdot a, 3 \cdot c\}$

$P + Q = \{(4 + 3) \cdot a, (1 + 4) \cdot b, (3 + 0) \cdot c, (0 + 2) \cdot d\} = \{7 \cdot a, 5 \cdot b, 3 \cdot c, 2 \cdot d\}$

Revisiting Set Cardinality

Question: A survey asks 200 people "What beverage do you drink in the morning"

60 report tea, 120 report coffee, 40 report both. How many people drink neither tea nor coffee?



Solution:

200 – (80 + 40 + 20) = 60

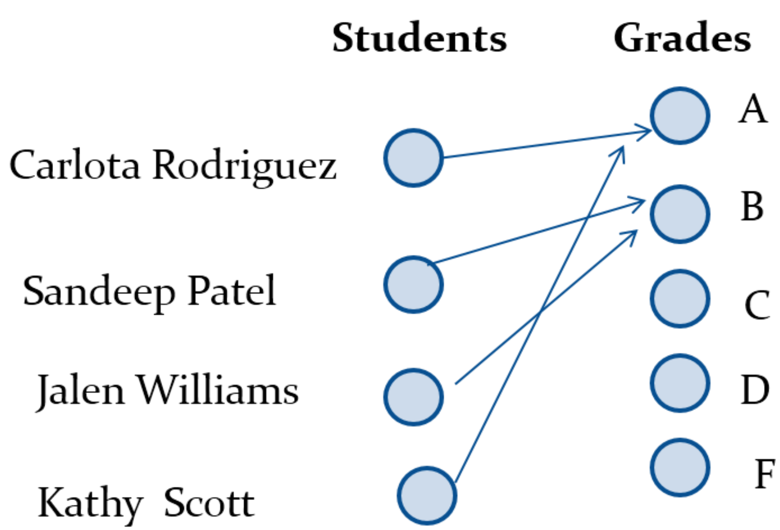
Functions

Section 2.3

Functions [1]

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.

- Functions are sometimes called *mappings* or *transformations*.



Functions $_2$

A function f: $A \rightarrow B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.

Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a, b*) for every element $a \in A$.

$$\forall x\left[x \in A \rightarrow \exists y\left[y \in B \wedge (x, y) \in f\right]\right]$$

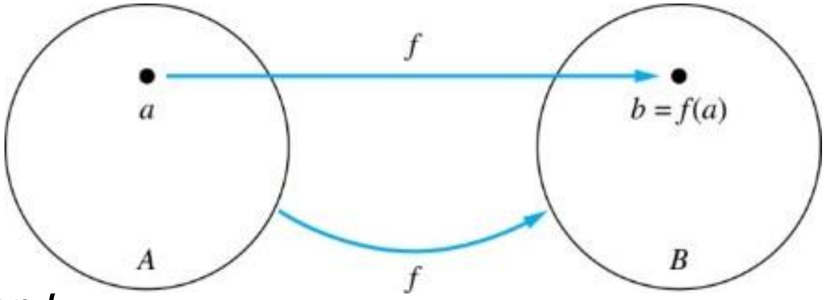
and

$$\forall x, y_1, y_2\left[\left[(x, y_1) \in f \wedge (x, y_2) \in f\right] \rightarrow y_1 = y_2\right]$$

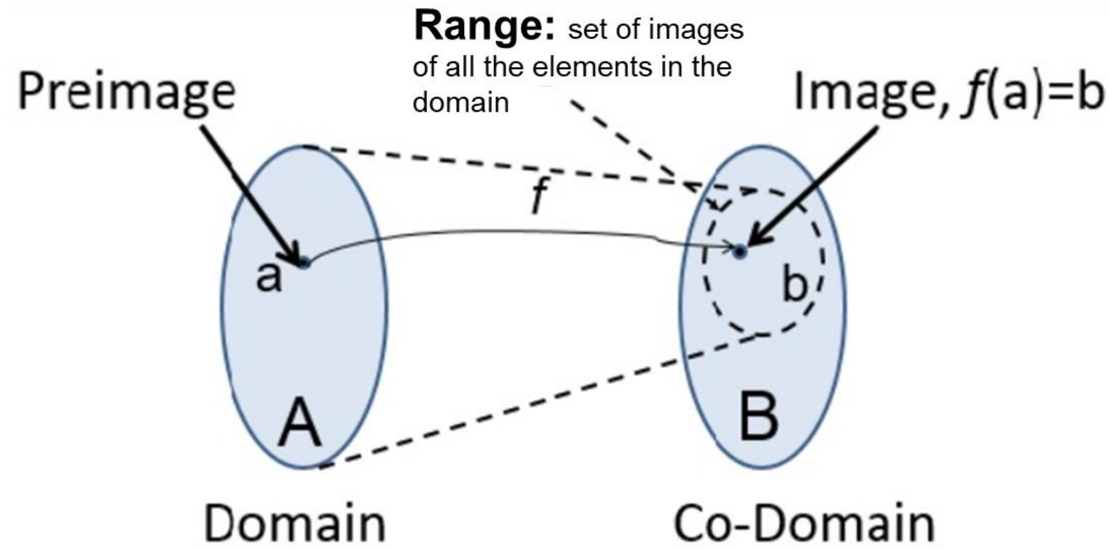
Functions 3

Given a function f: $A \rightarrow B$**:**

- We say f *maps* A to B *or* f is a *mapping* from A to B.
- A is called the *domain* of f.
- B is called the *codomain* of f.
- If $f(a) = b$,
  - then b is called the *image* of a under f.
  - a is called the *preimage* of b.
- The range of f is the set of all images of points in **A** under f. We denote it by $f(\boldsymbol{A})$.



Function: Visualization[1]



A function, f: A $\rightarrow$ B

Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Domain, Codomain and Range

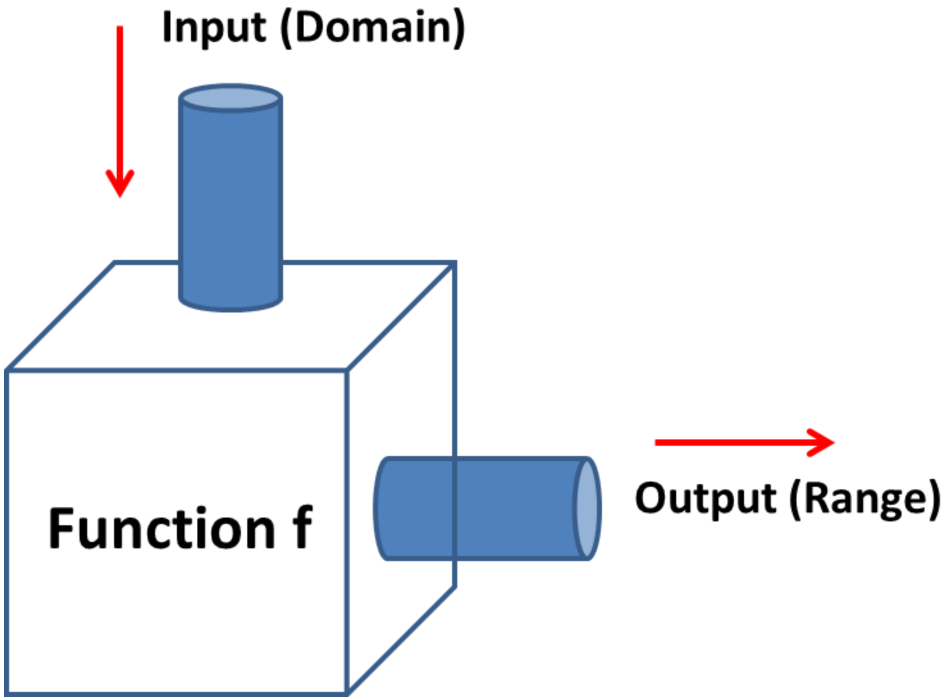
Domain, Codomain and Range

There are special names for **what can go into**, and **what can come out** of a function:

- What can go **into** a function is called the **Domain**
- What **may possibly come out** of a function is called the **Codomain**
- What **actually comes out** of a function is called the **Range**

Note: The range is always a subset (and sometimes a proper subset) of the domain.

Function: Visualization$_2$



Examples of Functions (1)

Example: Suppose that each student in a discrete mathematics class is assigned a letter grade from the set {A, B, C, D, F}. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. Find the domain, codomain, and range of the function that assigns a grade to the students.

Solution:

Domain:

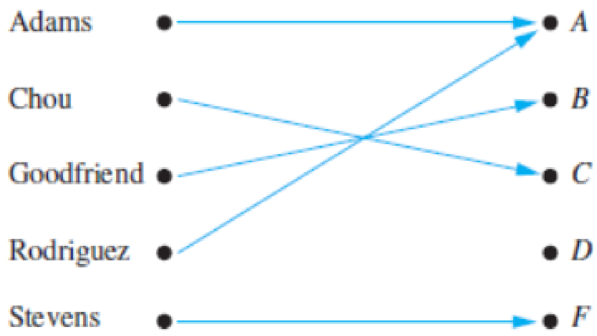
{Adams, Chou, Goodfriend, Rodriguez, Stevens}

Codomain:

{*A, B, C, D, F*}

Range:

{A, B , C , F}



Examples of Functions (2)

Example: Let f : **Z** → **Z** assign the square of an integer to this integer [i.e. $f(x) = x^2$]. Find the domain, codomain, and range of the function

Solution:

Domain:

Z

Codomain:

Z

Range:

{x ∈ **Z** | x is a perfect square}

Representing Functions

Functions may be specified in different ways:

- An explicit statement of the assignment. Students and grades example.
- A formula.

$$f(x) = x + 1$$

- A computer program.
  - A Java program that when given an integer *n*, produces the *n*th Fibonacci Number.

Questions

f(a) = ? z

The image of d is ? z

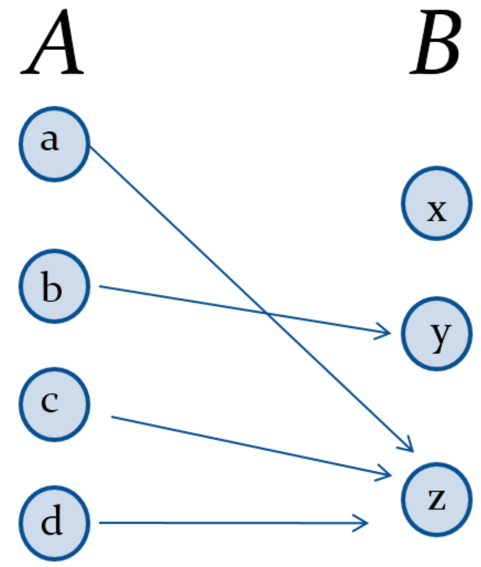
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

$f(A)$ = ? {y,z}

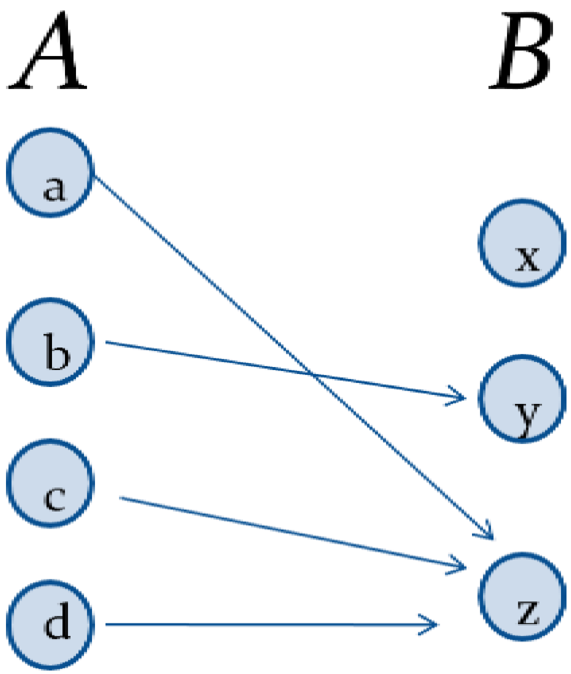
The preimage(s) of z is (are) ? {a,c,d}



Question on Functions and Sets

If $f : A \rightarrow B$ and S is a subset of A, then the range of S is the set of all images of the elements of S

$$f(S) = \{ f(s) \mid s \in S \}$$



f {a,b,c,} is ? {y,z}

f {c,d} is ? {z}