CS 2305: Discrete Mathematics for Computing I

Lecture 07

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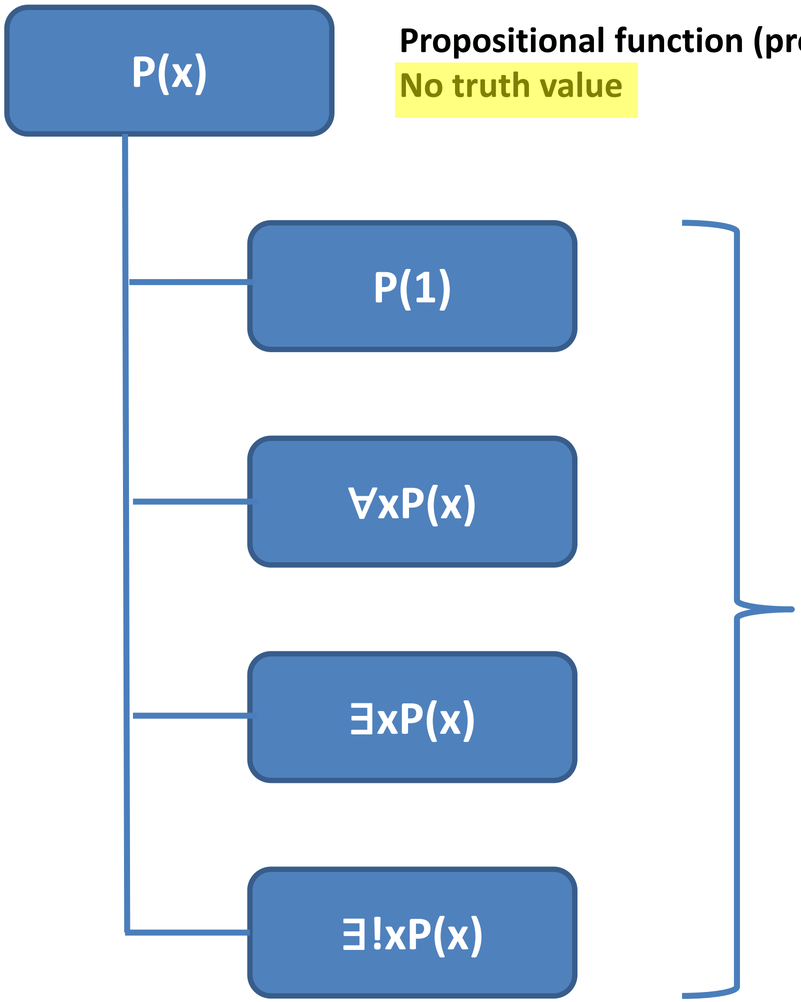
From Propositional Functions to Propositions (Recap)



Propositional function (predicate) with predicate variable x
No truth value

Propositions
Easily verifiable truth value

Quantifiers with Restricted Domains

- Sometimes it is not feasible to enumerate the domain of a quantifier. In such instances, an abbreviated notation is often used
  - a condition a variable must satisfy is included after the quantifier
  - Such quantifiers are called <u>restricted quantifiers</u>
- Examples (assumption: the domain in each case consists of the real numbers)
  - $(\forall x)_{x < 0} (x^2 > 0)$
    - For all –ve real numbers, the square of the number is positive
  - $(\forall y)_{y \neq 0} (y^3 \neq 0)$
    - For all non-zero real numbers, the cube of the number is non-zero
  - $(\exists z)_{z > 0} (z^2 = 2)$
    - There exists a non-zero number whose square is the number 2 (i.e. there is a positive square root of 2)

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
  – $\forall x\, (P(x) \vee Q(x))$ means something different.

Equivalences in Predicate Logic

- The notion of logical equivalences can extended to expressions involving predicates and quantifiers
- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- **Example**: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

$\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$

- "A universal quantifier can be distributed over a conjunction"
  - Assume $\forall x(P(x) \land Q(x))$ is true
    - For any element a in the domain, $P(a) \land Q(a)$ is true

      For any element a in the domain, P(a) is true AND Q(a) is true

      $\therefore \forall x P(x) \land \forall x Q(x)$ is true
  - Assume $\forall x P(x) \land \forall x Q(x)$ is true
    - Select any element a in the domain

      P(a) is true

      AND

      Q(a) is true

      Hence $P(a) \land Q(a)$ is true for any element in the domain

      $\therefore \forall x(P(x) \land Q(x))$ is true

$\exists x(P(x) \vee Q(x)) \equiv \exists xP(x) \vee \exists xQ(x)$

- "An existential quantifier can be distributed over a disjunction"
  - Assume $\exists x(P(x) \vee Q(x))$ is true
    - For some element a in the domain, $P(a) \vee Q(a)$ is true

      For some element a in the domain P(a) is true OR Q(a) is true [or both]

      $\therefore \exists xP(x) \vee \exists xQ(x)$ is true
  - Assume $\exists xP(x) \vee \exists xQ(x)$ is true
    - There is some element a in the domain such that

      P(a) is true

      OR

      Q(a) is true

      Hence $P(a) \vee Q(a)$ is true for some element in the domain

      $\therefore \exists x(P(x) \vee Q(x))$ is true

$\forall x(P(x) \vee Q(x)) \neg\equiv \forall x P(x) \vee \forall x Q(x)$

- "A universal quantifier cannot be distributed over a disjunction"
  - Proof by counter-example
    - Let the domain of discourse be the positive integers
    - Let P(x): x is an even number
    - Let Q(x): x is an odd number
    - The proposition $\forall x(P(x) \vee Q(x))$ is true because every positive integer is either odd or even
    - The proposition $\forall x P(x)$ is false because all numbers are not even
    - The proposition $\forall x Q(x)$ is false because all numbers are not odd
    - $\therefore$ proposition $\forall x P(x) \vee \forall x Q(x)$ is false

$\exists x(P(x) \land Q(x)) \neg\equiv \exists xP(x) \land \exists xQ(x)$

- "An existential quantifier cannot be distributed over a conjunction"
  - Proof by counter-example
    - Let the domain of discourse be the positive integers
    - Let $P(x)$: x is an even number
    - Let $Q(x)$: x is an odd number
    - The proposition $\exists x(P(x) \land Q(x))$ is false because no number can be simultaneously even and odd
    - The proposition $\exists xP(x)$ is true because there exist numbers that are even
    - The proposition $\exists xQ(x)$ is true because there exist numbers that are odd
    - $\therefore$ proposition $\exists xP(x) \land \exists xQ(x)$ is true

Negating Quantified Expressions (1)

Consider $\forall x\, J(x)$

"Every student in your class has taken a course in Java."

Here $J(x)$ is "x has taken a course in Java" and

the domain is students in your class.

Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg \forall x\, J(x)$ and $\exists x \neg J(x)$ are equivalent

Negating Quantified Expressions (2)

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	***Equivalent Statement***	***When Is Negation True?***	***When False?***
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every *x*, *P*(*x*) is false.	There is *x* for which *P*(*x*) is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an *x* for which *P* (*x*) is false.	*P*(*x*) is true for every *x*.

The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

These are important. You will use these.

Negating Quantified Expressions (3)

- Find the negation of the statement $\forall x(x^2 > x)$

  $\neg\forall x(x^2 > x) \equiv \exists x \neg(x^2 > x)$ [De Morgan's Law]

  $\equiv \exists x(x^2 \le x)$ [Algebraic knowledge]

- Find the negation of the statement $\exists x(x^2 = 2)$

  $\neg\exists x(x^2 = 2) \equiv \forall x \neg(x^2 = 2)$ [De Morgan's Law]

  $\equiv \forall x(x^2 \ne 2)$ [Conventional notation]

Negating Quantified Expressions (4)

- Show that $\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$

$\neg\forall x(P(x) \rightarrow Q(x)) \equiv \exists x(\neg(P(x) \rightarrow Q(x)))$	By De Morgan's Law for Quantifiers $\neg\forall x P(x) \equiv \exists x \neg P(x)$
$\equiv \exists x(\neg(\neg P(x) \vee Q(x)))$	Conditional statement in disjunction form $p \rightarrow q \equiv \neg p \vee q$ (Table 7, Section 1.3)
$\equiv \exists x(\neg\neg P(x) \wedge \neg Q(x))$	By De Morgan's Law $\neg(p \vee q) \equiv \neg p \wedge \neg q$
$\equiv \exists x(P(x) \wedge \neg Q(x))$	Double Negation Law (Table 6, Section 1.3)

Restricted Quantifiers (1)

- Unrestricted quantifiers apply to the entire domain of discourse
  - Most of what we have seen so far
- A restricted quantifier has the same semantics as an unrestricted quantifier except that the variables in the domain must satisfy a certain condition in order for the quantification to apply
  - $(\forall x)_{x < 0}\ (x^2 > 0)$
  - $(\forall y)_{y \neq 0}\ (y^3 \neq 0)$
  - $(\exists z)_{z > 0}\ (z^2 = 2)$

Assumption: Domain of discourse is **R**, the set of real numbers

"For all/some «variable» satisfying the domain constraint …"

Restricted Quantifiers (2)

- Sometimes we need to express a restricted quantifier as an unrestricted quantifier
  - Express $(\exists x)_{P(x)}\,(Q(x))$ using the unrestricted existential quantifier
    - Solution
      - The logical expression states that there exists some value in the domain for which P(x) is true such that Q(x) is true
      - It can be expressed with an unrestricted existential quantifier as follows:
        - » $\exists x(P(x) \land Q(x))$

Restricted Quantifiers (3)

- Express $(\forall x)_{P(x)}\,(Q(x))$ using the unrestricted universal quantifier
  - Solution
    - Step 1
      - The logical expression states that for all values in the domain for which $P(x)$ is true, $Q(x)$ is true as well. We start with $\forall x(P(x) \land Q(x))$. Although it satisfies our basic condition, it makes the unreasonable assumption that $P(x)$ is true for all values of x
    - Step 2
      - Being a universal quantifier, we need to account for all values in the domain: those for which our claim applies and those for which it does not
    - Step 3
      - We need to exclude those values from the domain for which our claim does not apply by observing that $\forall x(\neg P(x) \lor (P(x) \land Q(x)))$
    - Step 4
      - We simplify the above expression

$$\forall x(\neg P(x) \lor (P(x) \land Q(x))) \equiv \forall x((\neg P(x) \lor P(x)) \land (\neg P(x) \lor Q(x)))$$
$$\equiv \forall x(\mathbf{T} \land (\neg P(x) \lor Q(x))) \equiv \forall x(\neg P(x) \lor Q(x)) \equiv \forall x(P(x) \rightarrow Q(x))$$

Section 1.3, Table 6
- **Distributive Law**
- **Negation Law**
- **Identity Law**

Restricted Quantifiers (4)

- To summarize
  - $(\exists x)_{P(x)}\,(Q(x)) \equiv \exists x(P(x) \land Q(x))$
    - "The restriction of an existential quantification is the same as the existential quantification of a conjunction"
  - $(\forall x)_{P(x)}\,(Q(x)) \equiv \forall x(P(x) \rightarrow Q(x))$
    - "The restriction of a universal quantification is the same as the universal quantification of a conditional statement"