CS 2305: Discrete Mathematics for Computing I

Lecture 16

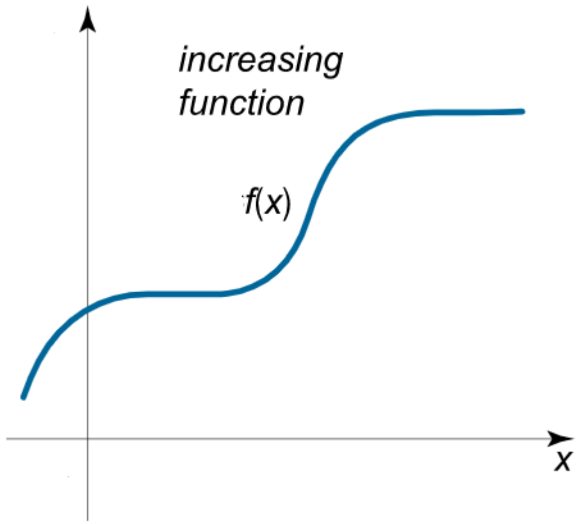
- KP Bhat

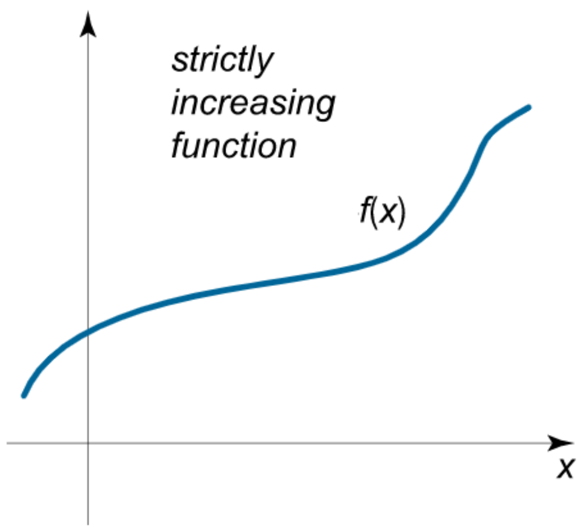
Increasing and Decreasing Functions

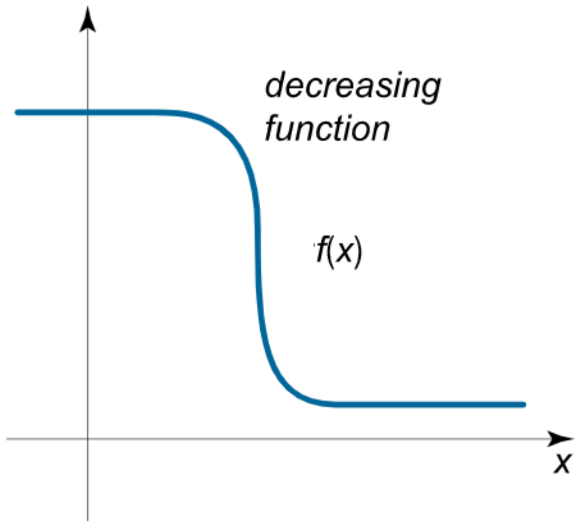
Definitions:

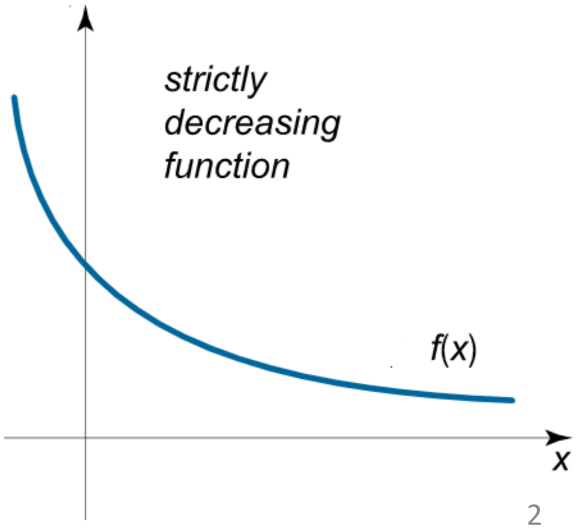
A function f is:

- *increasing* if $\forall x_1 \forall x_2 (x_1 < x_2 \rightarrow f(x_1) \leq f(x_2))$
- *strictly increasing* if $\forall x_1 \forall x_2 (x_1 < x_2 \rightarrow f(x_1) < f(x_2))$
- *decreasing* if $\forall x_1 \forall x_2 (x_1 < x_2 \rightarrow f(x_1) \geq f(x_2))$
- *strictly decreasing* if $\forall x_1 \forall x_2 (x_1 < x_2 \rightarrow f(x_1) > f(x_2))$





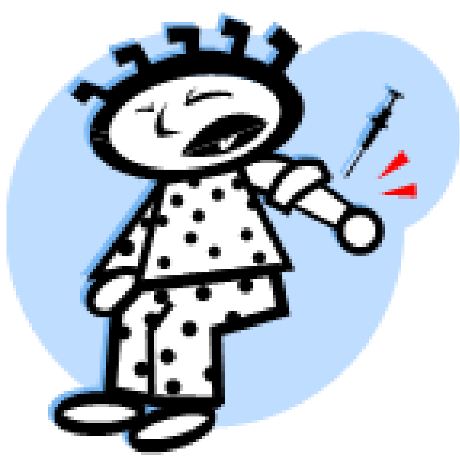


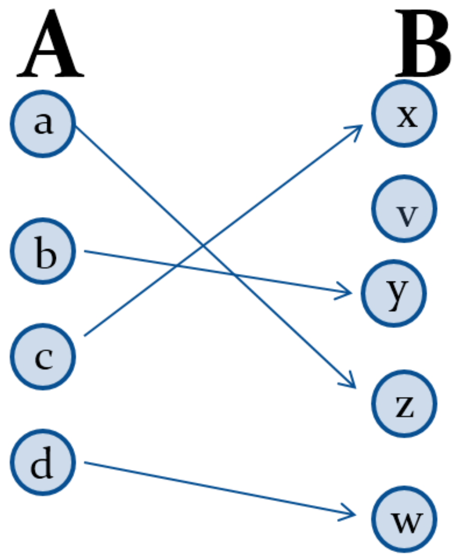


Injections

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an *injection* if it is one-to-one.

- Every image has a unique pre-image





Examples of Injections

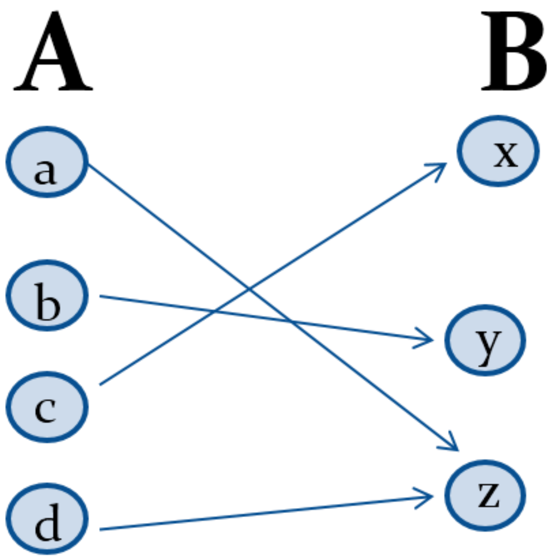
- $f(x) = x + 1$, from **R** to **R**
- $f(x) = x^2$, from $\mathbf{Z}^+$ to $\mathbf{Z}^+$
  - If the domain is **Z**, $f(x) = x^2$ is not an injection because, for instance, $f(2) = f(-2) = 4$

Surjections

Definition: A function *f* from *A* to *B* is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

A function *f* is called a *surjection* if it is *onto*.

- $\forall y \exists x (f(x) = y)$, where the domain for *x* is the domain of the function and the domain for *y* is the codomain of the function
- All elements in the codomain have a pre-image in the domain
- Co-domain is the same as the range



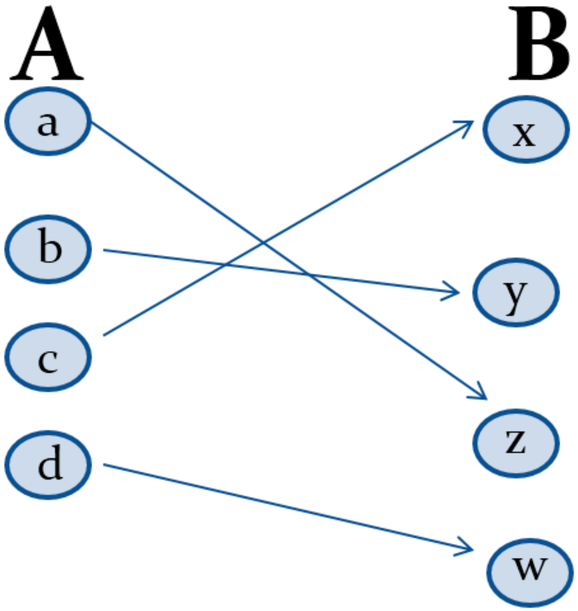
Examples of Surjections

- $f(x) = x + 1$, from **Z** to **Z**
- All enrolled students at a university having an active student ID

Bijections

Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

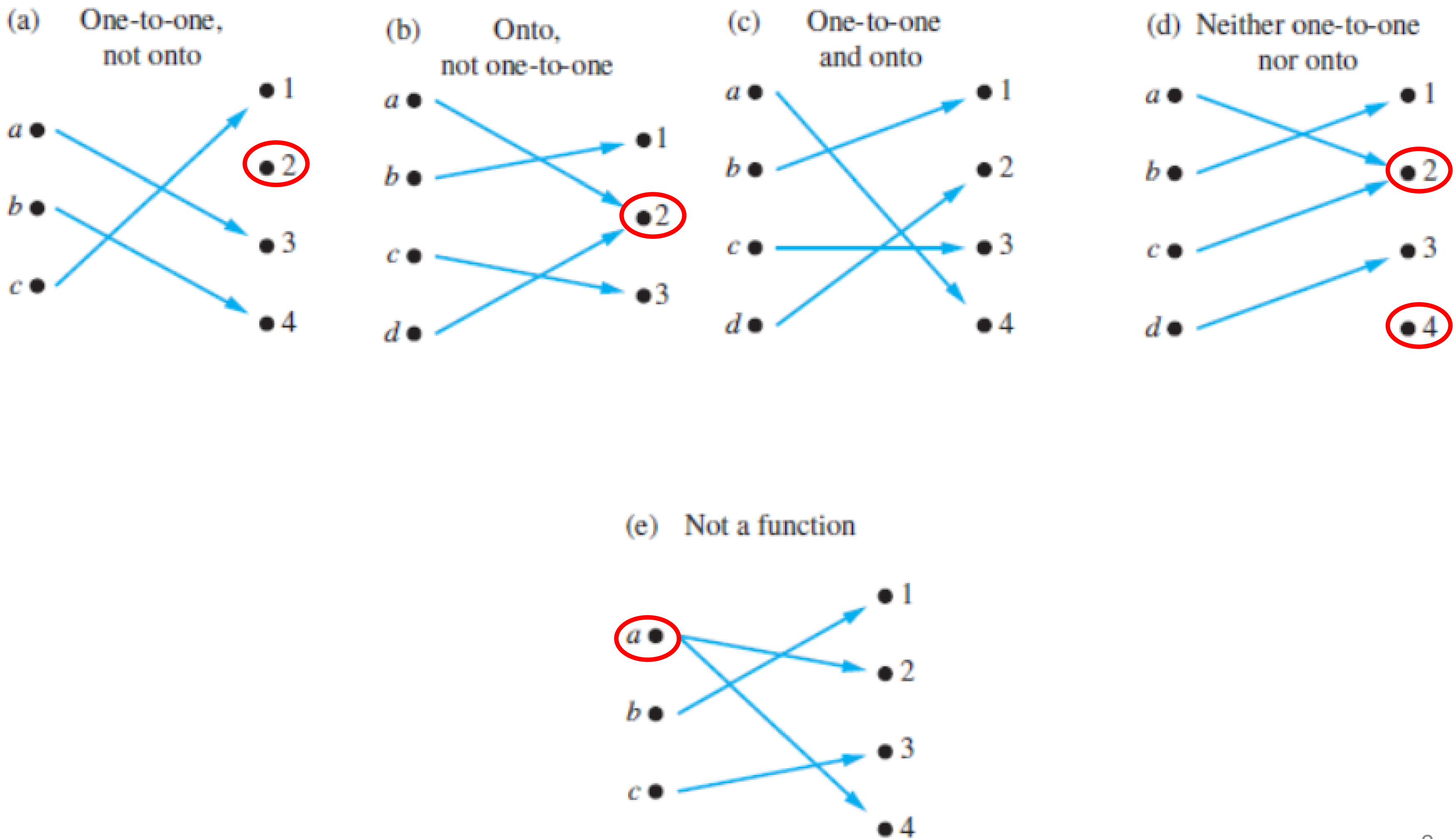
- Every element in the codomain has a unique pre-image
- Co-domain is the same as the range



Examples of Bijections

- f: $\{a, b, c, d\} \to \{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$
- Let A be a set. The *identity function* on A is the function $\iota_A : A \to A$, where $\iota_A(x) = x$ for all $x \in A$
  - identity function assigns each element to itself

Examples of Different Types of Functions



Showing that f is one-to-one or onto [1]

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

To show that f is bijective Show that it is both injective and surjective.

Showing that f is one-to-one or onto 2

Example 1: Is f(x) = 4x -1, from **R** to **R**, injective?

Solution:

Let us pick up two arbitrary numbers a and b such that f(a) = f(b)

By the definition of f

$$4a - 1 = 4b - 1$$

$$4a = 4b$$

$$a = b$$

∴ f(x) = 4x -1, from **R** to **R** is injective

Showing that f is one-to-one or onto

Example 2: Is f(x) = 2x -3, from R to R, surjective?

Solution:

Part 1 (scratch work)

Let us pick an arbitrary number y ∈ **R** in the codomain

If such a number exists,

$2x - 3 = y$

$2x = y + 3$

$x = (y + 3)/2$

Part 2 (actual solution)

Let y ∈ **R** be an arbitrary element from the codomain

Let $x = (y + 3)/2$

$f(x) = 2((y + 3)/2)-3 = y + 3 -3 = y$

An arbitrary element from the codomain has a preimage in the domain

∴ f(x) = 2x -3, from R to R, is surjective

Showing that f is one-to-one or onto

Example 3: Is $f(x) = x^2$ from **Z** to **Z** onto?

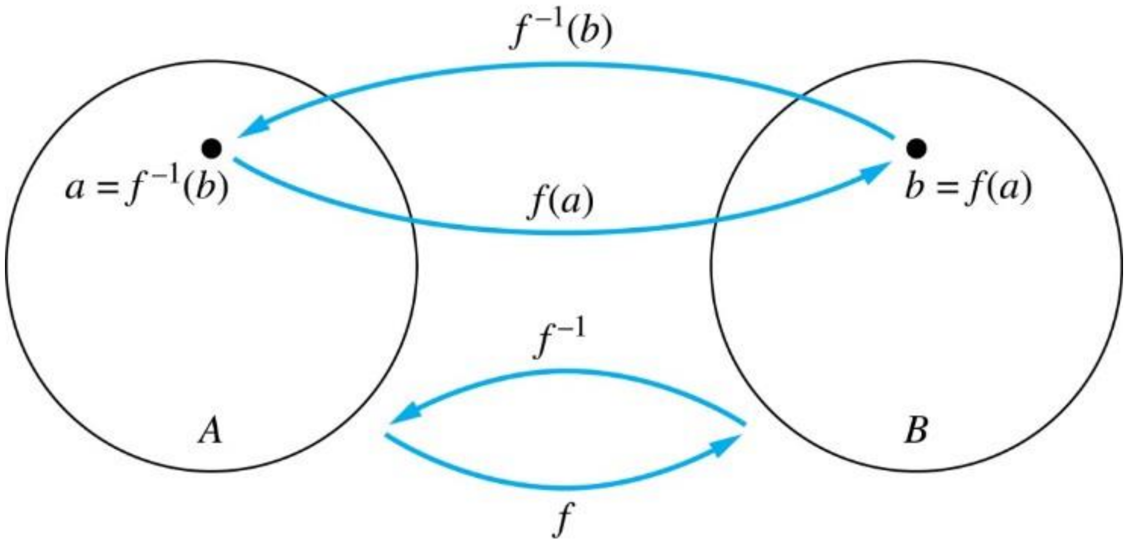
Solution:

Using counterexample

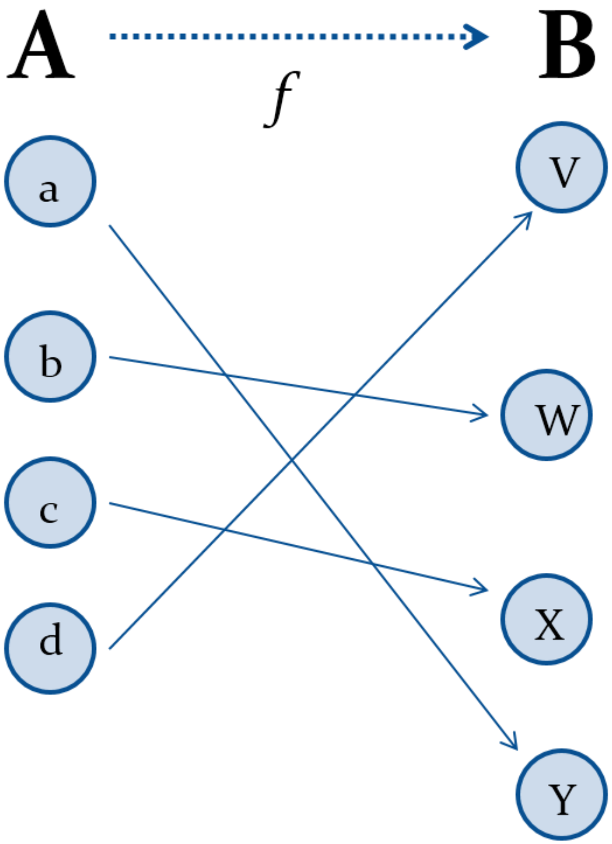
No, f is not onto because there is no integer x with $x^2 = -1$.

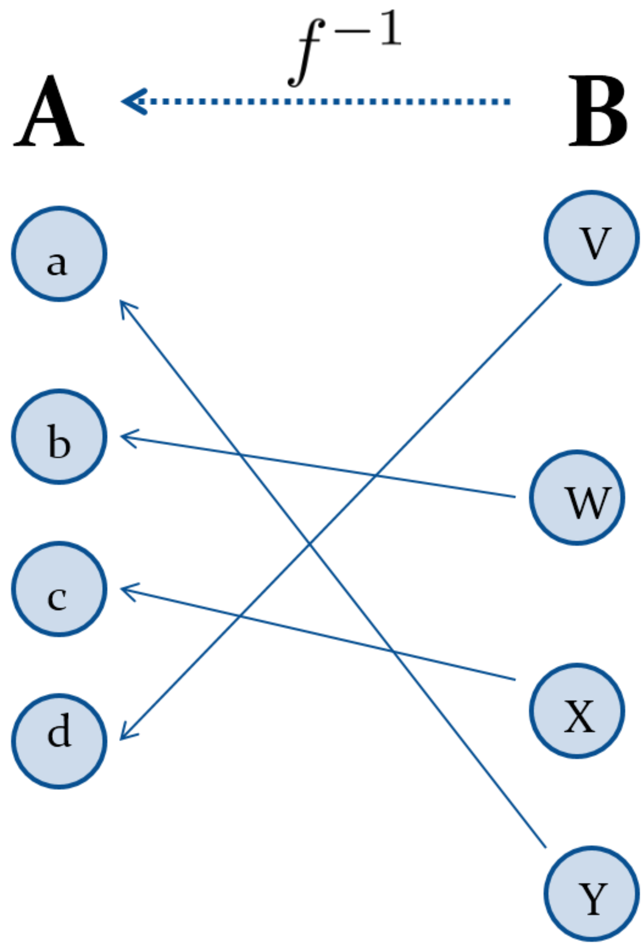
Inverse Functions [1]

Definition: Let f be a bijection from A to B. Then the *inverse* of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$



Inverse Functions 2





Inverse Functions 3

- No inverse function exists unless f is a bijection
  - If f is not a bijection then it is either not an injection or it is not a surjection (or both)
  - If f is not an injection some element in the codomain is the image of more than one element in the domain
  - If f is not a surjection, for some element in the codomain there is no pre-image
- We say that all bijection functions are *invertible*

Questions[1]

Example 1: Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Questions [2]

Example 2: Let f: **Z** $\rightarrow$ **Z** be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

Questions 3

Example 3: Let f: **R** $\rightarrow$ **R** be such that $f(x) = x^2$

Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is neither one-to-one [e.g. –2 and +2 both map to 4] nor onto [no preimages for –ve numbers]. In fact f^{-1} is not even a function.

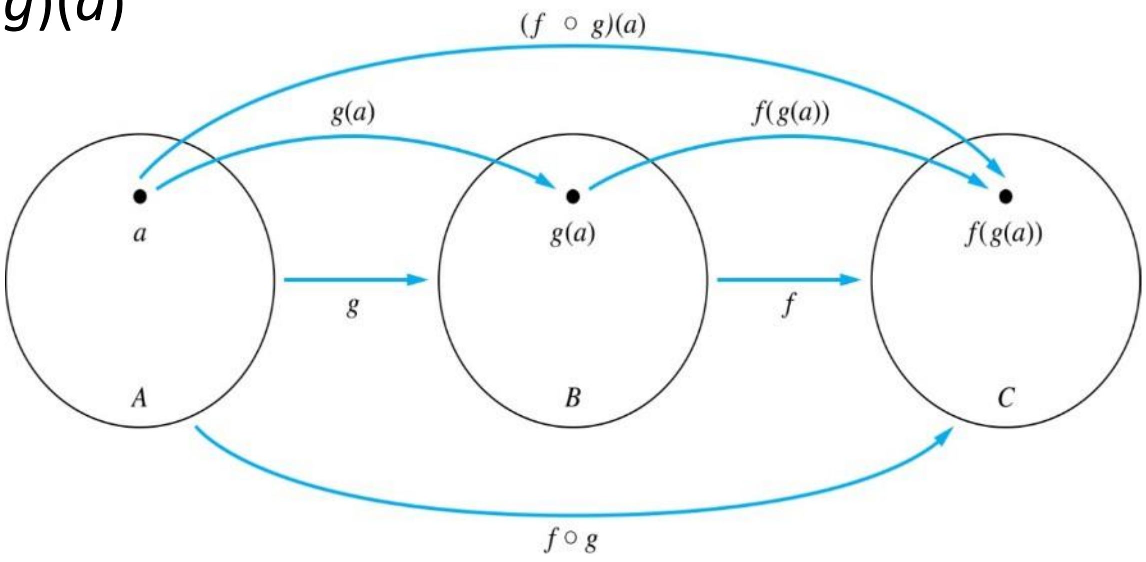
Note:- If we restrict f to be from the set of non-ve real numbers to the set of non-ve real numbers then the function is invertible. In this case $f^{-1}(y) = \sqrt{(x)}$

Composition [1]

Definition: Let f: $B \rightarrow C$, g: $A \rightarrow B$. The *composition of f with g*, denoted $f \circ g$ is the function from A to C defined by

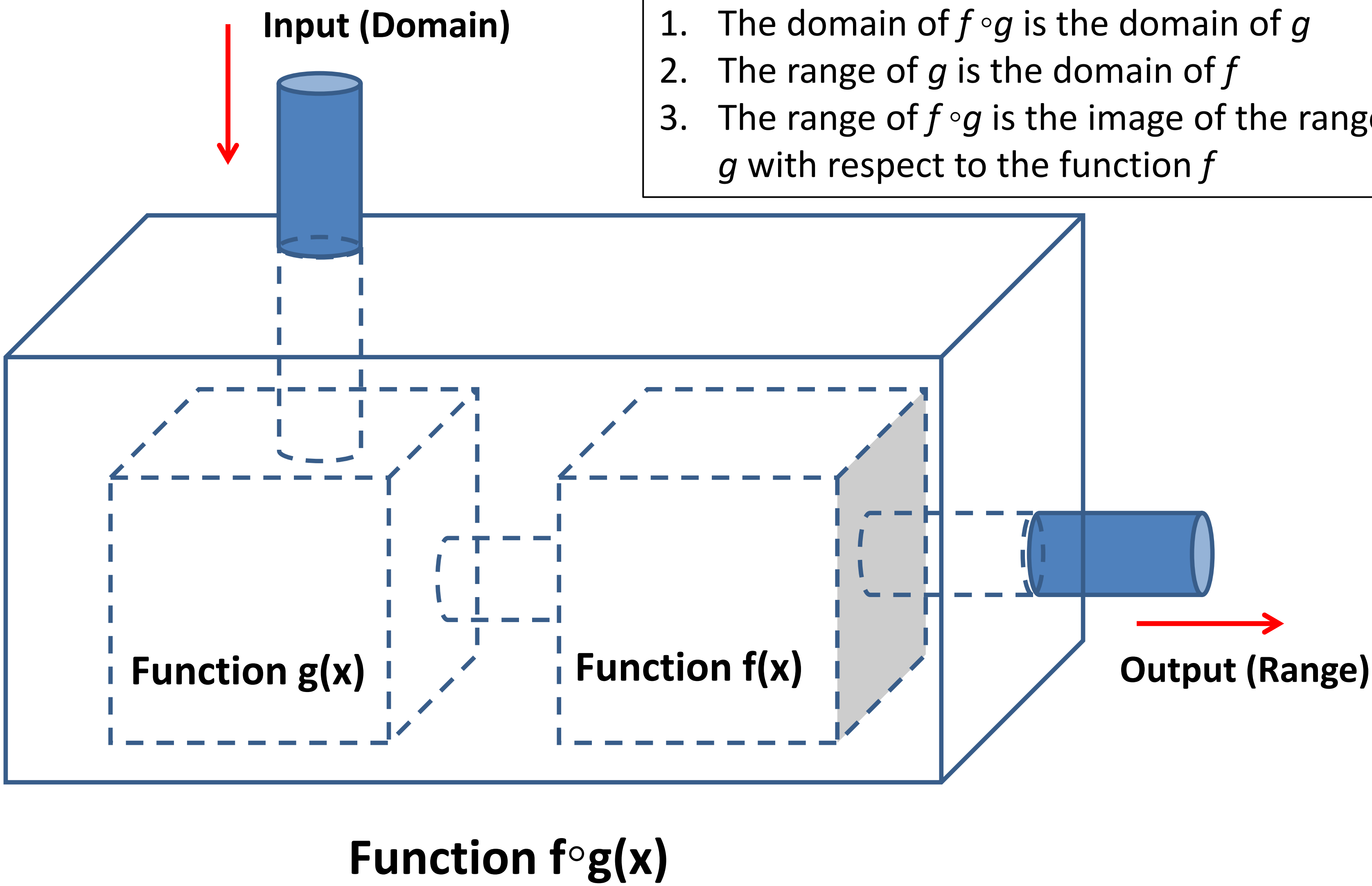
$$f \circ g(x) = f(g(x))$$

To find $(f \circ g)(a)$ we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result $g(a)$ to obtain $(f \circ g)(a)$

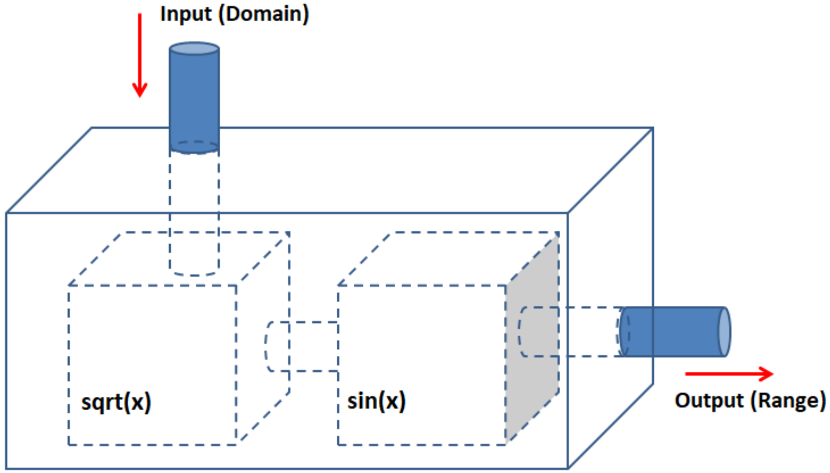


Composition 2

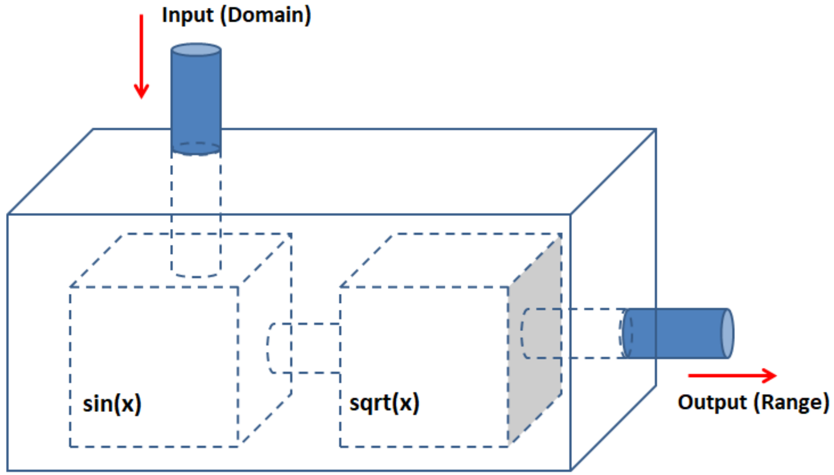
1. The domain of $f \circ g$ is the domain of g
2. The range of g is the domain of f
3. The range of $f \circ g$ is the image of the range of g with respect to the function f



sin∘sqrt(x) vs sqrt∘sin(x)



Function sin∘sqrt(x)

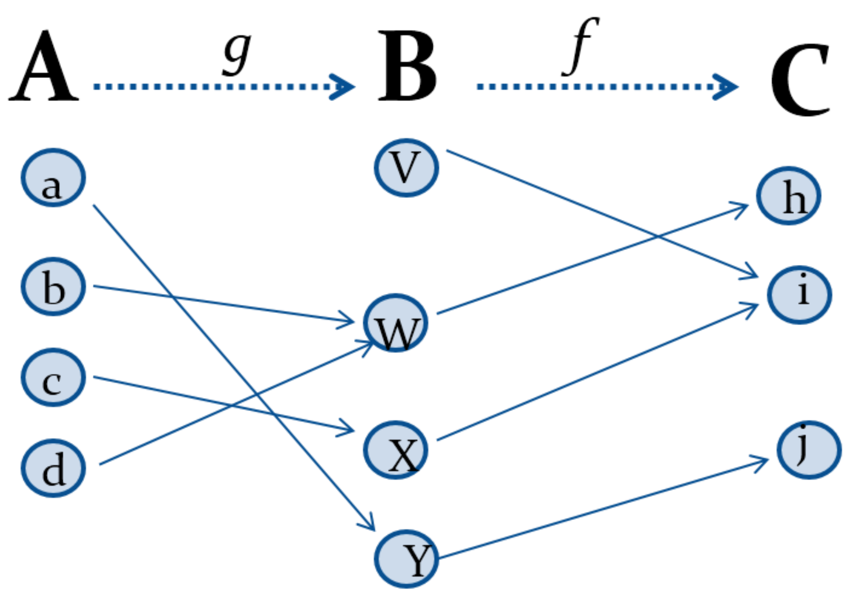


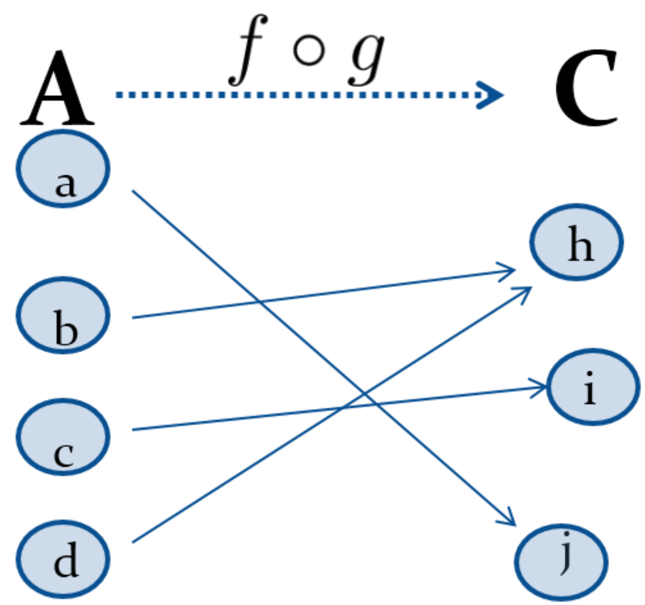
Function sqrt∘sin(x)

```
public static void main(String[] args) {
  final double DEG_45 = Math.PI/4;

  double fogx = Math.sin(Math.sqrt(DEG_45));
  System.out.println("fogx: " + fogx);

  double gofx = Math.sqrt(Math.sin(DEG_45));
  System.out.println("gofx: " + gofx);
}
```

```
fogx: 0.7746914034386123
gofx: 0.8408964152537145
```

Composition$_3$





Composition

Example: If

$$f(x) = x^2 \text{ and } g(x) = 2x + 1,$$

then

$$f(g(x)) = (2x+1)^2$$

and

$$g(f(x)) = 2x^2 + 1$$

Composition Questions [1]

Example 2: Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$.

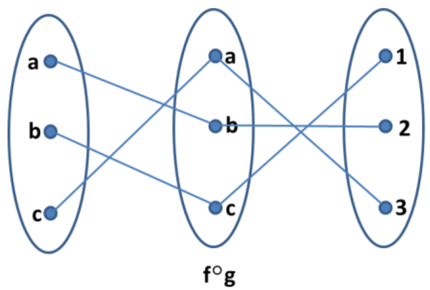
What is the composition of f and g, and what is the composition of g and f.

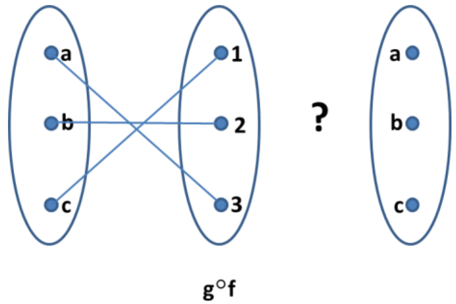
Solution: The composition $f \circ g$ is defined by

$$f \circ g(a) = f(g(a)) = f(b) = 2.$$

$$f \circ g(b) = f(g(b)) = f(c) = 1.$$

$$f \circ g(c) = f(g(c)) = f(a) = 3.$$





Note that $g \circ f$ is not defined, because the range of f {1, 2, 3} is not a subset of the domain of g {a, b, c}.

Composition Questions

Example 2: Let f and g be functions from the set of integers to the set of integers defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

What is the composition of *f* and *g*, and also the composition of *g* and *f* ?

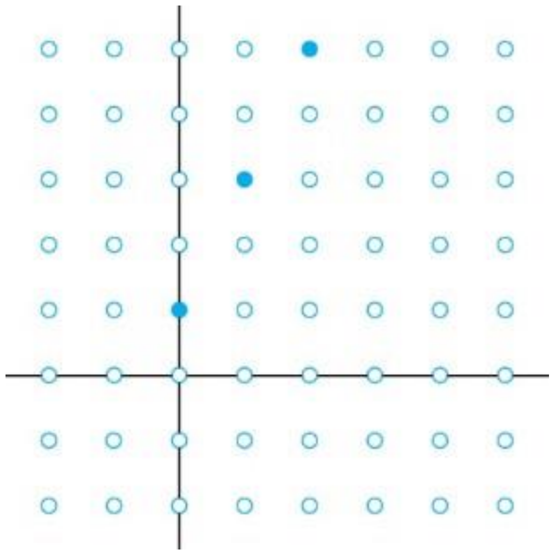
Solution:

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$
$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

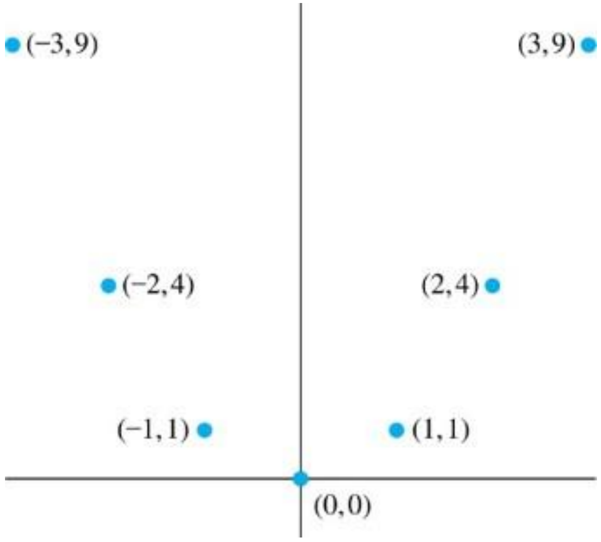
Graphs of Functions

Let *f* be a function from the set *A* to the set *B*. The *graph* of the function *f* is the set of ordered pairs

$$\{(a,b) \mid a \in A \text{ and } f(a) = b\}.$$



Graph of $f(n) = 2n + 1$ from Z to Z



Graph of $f(x) = x^2$ from Z to Z

[Jump to long description]

Some Important Functions

The *floor* function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to *x*.

The *ceiling* function, denoted

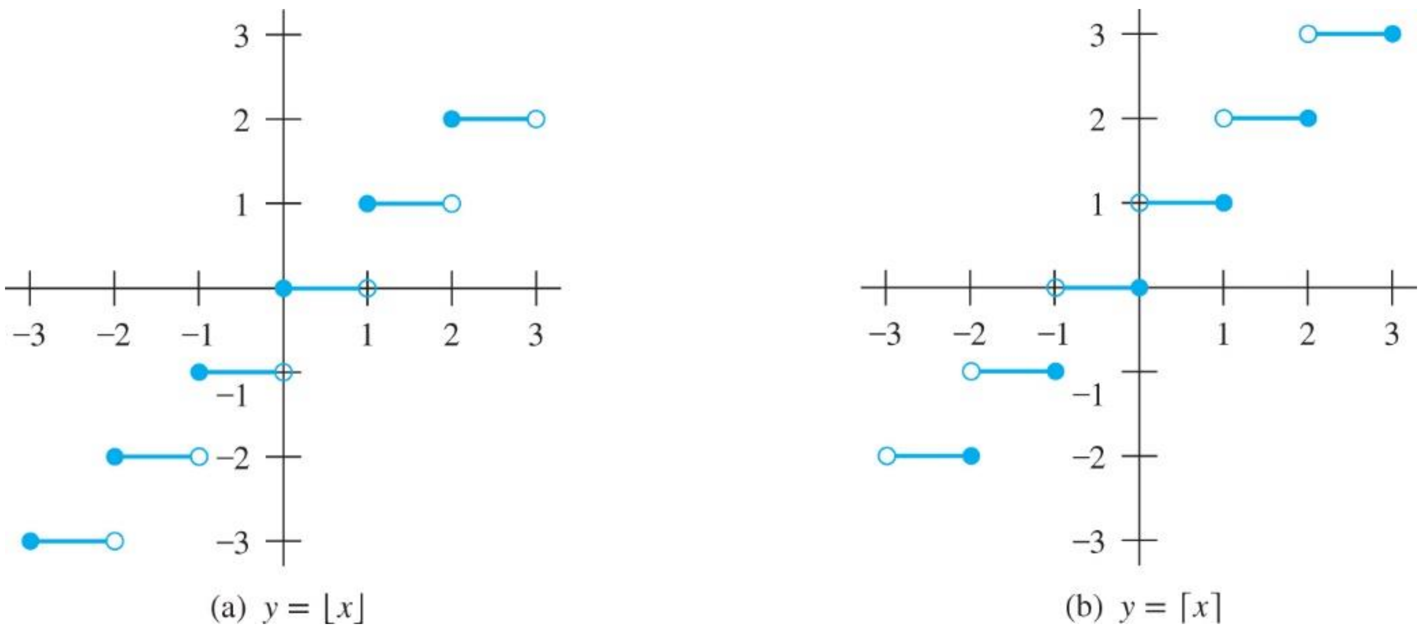
$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to *x*

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions [1]



Graph of (a) Floor and (b) Ceiling Functions