CS 2305: Discrete Mathematics for Computing I

Lecture 28

- KP Bhat

Relations and Their Properties

Section 9.1

Binary Relations[1]

- **Definition:** A *binary relation* R from a set A to a set B is a subset of the Cartesian product of A and B
  - $R \subseteq A \times B$

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.
Hence, $A \times B = \{(a, b) \mid a \in A \land b \in B\}$
Section 2.1

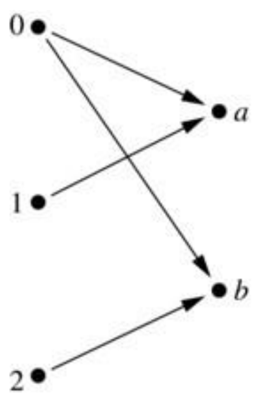
- We use the notation $a\ R\ b$ to denote that $(a, b) \in R$ and $a \not R\ b$ to denote that $(a, b) \notin R$.

Binary Relations$_2$

$$R \subseteq A \times B$$

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

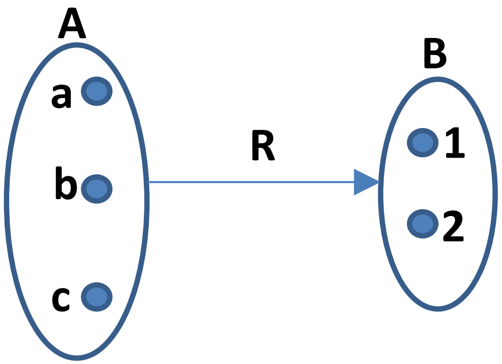
Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

Binary Relations3

A relation R from A to B is a subset of A X B

All subsets of A X B (all possible relations between A and B):

{}	{(a,1)}	{(a,2)}	{(b,1)}
{(b,2)}	{(c,1)}	{(c,2)}	{(a,1),(a,2)}
{(a,1),(b,1)}	{(a,1),(b,2)}	{(a,1),(c,1)}	{(a,1),(c,2)}
{(a,2),(b,1)}	{(a,2),(b,2)}	{(a,2),(c,1)}	{(a,2),(c,2)}
{(b,1),(b,2)}	{(b,1),(c,1)}	{(b,1),(c,2)}	{(b,2),(c,1)}
{(b,2),(c,2)}	{(c,1),(c,2)}	{(a,1),(a,2),(b,1)}	{(a,1),(a,2),(b,2)}
{(a,1),(a,2),(c,1)}	{(a,1),(a,2),(c,2)}	{(a,1),(b,1),(b,2)}	{(a,1),(b,1),(c,1)}
{(a,1),(b,1),(c,2)}	{(a,1),(b,2),(c,1)}	{(a,1),(b,2),(c,2)}	{(a,1),(c,1),(c,2)}
{(a,2),(b,1),(b,2)}	{(a,2),(b,1),(c,1)}	{(a,2),(b,1),(c,2)}	{(a,2),(b,2),(c,1)}
{(a,2),(b,2),(c,2)}	{(a,2),(c,1),(c,2)}	{(b,1),(b,2),(c,1)}	{(b,1),(b,2),(c,2)}
{(b,1),(c,1),(c,2)}	{(b,2),(c,1),(c,2)}	{(a,1),(a,2),(b,1),(b,2)}	{(a,1),(a,2),(b,1),(c,1)}
{(a,1),(a,2),(b,1),(c,2)}	{(a,1),(a,2),(b,2),(c,1)}	{(a,1),(a,2),(b,2),(c,2)}	{(a,1),(a,2),(c,1),(c,2)}
{(a,1),(b,1),(b,2),(c,1)}	{(a,1),(b,1),(b,2),(c,2)}	{(a,1),(b,1),(c,1),(c,2)}	{(a,1),(b,2),(c,1),(c,2)}
{(a,2),(b,1),(b,2),(c,1)}	{(a,2),(b,1),(b,2),(c,2)}	{(a,2),(b,1),(c,1),(c,2)}	{(a,2),(b,2),(c,1),(c,2)}
{(b,1),(b,2),(c,1),(c,2)}	{(a,1),(a,2),(b,1),(b,2),(c,1)}	{(a,1),(a,2),(b,1),(b,2),(c,2)}	{(a,1),(a,2),(b,1),(c,1),(c,2)}
{(a,1),(a,2),(b,2),(c,1),(c,2)}	{(a,1),(b,1),(b,2),(c,1),(c,2)}	{(a,2),(b,1),(b,2),(c,1),(c,2)}	{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)}



There are 3*2 = 6 elements in A X B and 2^6 = 64 subsets of A X B

In this instance there are 64 relations between A and B

Binary Relations$_4$

Example: Let A be the set of cities in the U.S.A., and let B be the set of 50 states in the U.S.A.

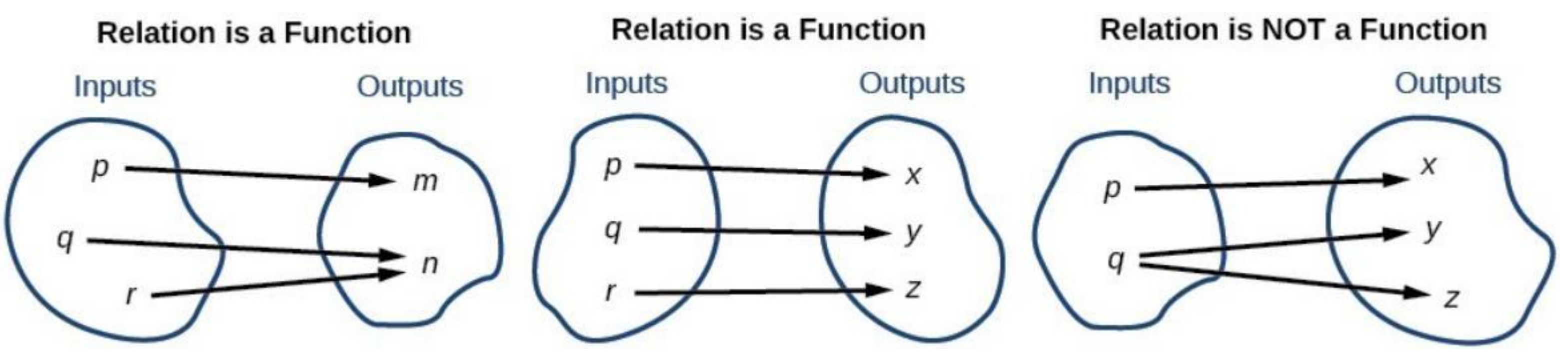
Define relation R as follows

$R = \{(a,b) \mid a \in A, b \in B \wedge \underline{a \text{ is in state } b}\}$

Then (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R

Functions as Relations

- A function *f* from a set A to a set B assigns exactly one element of B to each element of A
- Relations are generalizations of functions and they can be used to express a much wider class of relationships between sets

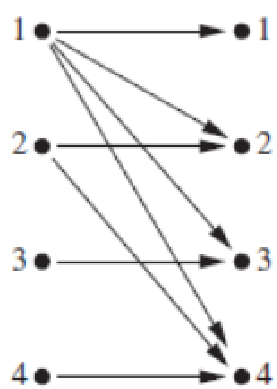


Binary Relations on a Set $_1$

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation
  R $= \{(a,b) \mid a$ divides $b\}$ are
  (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Graphical and tabular representation of the relation

Binary Relations on a Set 2

Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \le b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_5 = \{(a,b) \mid a = b + 1\},$$

$$R_6 = \{(a,b) \mid a + b \le 3\}.$$

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

(1,1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Solution: Checking the conditions that define each relation, we see that the pair (1,1) is in R_1, R_3, R_4 , and R_6: (1,2) is in R_1 and R_6: (2,1) is in R_2, R_5, and R_6: (1, −1) is in R_2, R_3, and R_6 : (2,2) is in R_1, R_3, and R_4.

Binary Relations on a Set

Question: How many relations are there on a set A?

Solution: Because a relation on A is a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, therefore there are 2^{n^2} relations on a set A.

Types of Relations[1]

- Let *R* be a relation on a set *A*. There are several properties that can be used to classify R
- R can be classified as:
  - reflexive
  - symmetric
  - antisymmetric
  - transitive

Types of Relations[2]

To explain the various types of relations, we will use the following illustrative relations:

Let set $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$

Roster Notation

Let $a,b \in Z$

$R_A = \{(a, b) \mid a \leq b\}$,

$R_B = \{(a, b) \mid a > b\}$,

$R_C = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R_D = \{(a, b) \mid a = b\}$,

$R_E = \{(a, b) \mid a = b + 1\}$,

$R_F = \{(a, b) \mid a + b \leq 3\}$.

Set Builder Notation

Types of Relations$_3$

- When classifying a relation based on a property keep in mind that:
  - there should be no violation of the pertinent property
  - there may be additional elements in the relation that do not satisfy the pertinent property
    - e.g. there may be additional elements in a reflexive relation that do not satisfy the reflexive property

Reflexive Relations[1]

Definition: A relation R on a set A is *reflexive* if $(a,a) \in R$ for every element $a \in A$. In terms of quantifiers, $\forall a((a, a) \in R)$, where the universe of discourse is the set of all elements in A.

- Each element in set A is related to itself

Example: The following are examples of reflexive relations:

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$

- Contains all pairs of the form (a, a)

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

- Contains all pairs of the form (a, a)

$R_A = \{(a, b) \mid a \leq b\}$

- Every integer is less than or equal to itself

$R_C = \{(a, b) \mid a = b \text{ or } a = -b\}$

- The first condition is always true: every integer is equal to itself

$R_D = \{(a, b) \mid a = b\}$

- Every integer is equal to itself

Reflexive Relations[2]

Example: Is the “divides” relation on the set of positive integers reflexive?

Solution:

Because $a \mid a$ whenever a is a positive integer, the “divides” relation is reflexive.

Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.

Symmetric Relations[1]

Definition: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. In terms of quantifiers:

$\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$

- An element related to a second element implies the second element is also related to the first element.

Example: The following are examples of symmetric relations:

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$

- (1, 1) is trivially symmetric
- (1, 2) and (2, 1) are both in the relation

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$

- (1, 1), (2, 2), (3, 3) and (4, 4) are trivially symmetric
- (1, 2) & (2, 1), (1, 4) & (4, 1) are both in the relation

$R_C = \{(a, b) \mid a = b \text{ or } a = -b\}$,

- If $a = b$ then $b = a$; If $a = -b$ then $b = -a$

$R_D = \{(a, b) \mid a = b\}$,

- $a = b$ implies that $b = a$.

$R_F = \{(a, b) \mid a + b \leq 3\}$.

- $a + b \leq 3$ implies that $b + a \leq 3$.

Symmetric Relations$_2$

Example: Is the "divides" relation on the set of positive integers symmetric?

Solution:

$a \mid b \not\Rightarrow b \mid a$ (e.g. $1 \mid 2$ but $2 \nmid 1$)

$\therefore$ the relation is not symmetric

Antisymmetric Relations[1]

Definition: A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*. In terms of quantifiers, $\forall a \forall b(((a, b) \in R \land (b, a) \in R) \to (a = b))$

- there are no pairs of distinct elements a and b with a related to b and b related to a.
- for example, $a \le b$ and $b \le a$ implies that $a = b$

Notes:

1. The terms *symmetric* and *antisymmetric* are not opposites, because a relation can have both of these properties or may lack both of them
   - the relation $R_1 = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric
   - the relation $R_2 = \{(1, 3), (3, 1), (2, 3)\}$ is neither symmetric nor antisymmetric
     - $(2, 3) \in R_2$ but $(3, 2) \notin R_2 \therefore$ not symmetric
     - $(1, 3), (3, 1) \in R_2$ but $1 \neq 3 \therefore$ not antisymmetric
2. A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b) in which $a \neq b$

Antisymmetric Relations[2]

Example: The following are examples of antisymmetric relations. In each case there is no violation of the antisymmetric property

- R4 = {(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)}
- R5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}
- R6 = {(3, 4)}

Antisymmetric Relations3

Example: Is the “divides” relation on the set of positive integers antisymmetric?

Solution:

Let us assume that $a \mid b$ and $b \mid a$

Since a and b are positive integers, there exist positive integers m and n such that

$b = am$ and $a = bn$

$b = (bn)m$

$nm = 1$, which yields $n = 1$ & $m = 1$, since m and n are both integers

$\therefore a = b$

Hence the relation is antisymmetric

Transitive Relations[1]

Definition: A relation R on a set A is called *transitive* if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. In terms of quantifiers:

$\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \rightarrow (a, c) \in R)$.

- If the first element is related to the second element, and the second element is related to the third element, then the first element must be related to the third element

Example: The following are examples of transitive relations:

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

- $(3, 2)$, $(2, 1), (3, 1) \in R_4$; $(4, 2), (2, 1), (4, 1) \in R_4$; $(4, 3), (3, 2), (4, 2) \in R_4$; $(4, 3), (3, 1), (4, 1) \in R_4$

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

- $(1, 2)$, $(2, 3), (1, 3) \in R_5$; $(1, 2), (2, 4), (1, 4) \in R_5$; $(1, 3), (3, 4), (1, 4) \in R_5$; $(2, 3), (3, 4), (2, 4) \in R_5$

$R_6 = \{(3, 4)\}$

- No violation of Transitive Relation condition

$R_A = \{(a, b) \mid a \leq b\}$

- $a \leq b$ and $b \leq c$ implies $a \leq c$

$R_B = \{(a, b) \mid a > b\}$

- $a > b$ and $b > c$ implies $a > c$

$R_C = \{(a, b) \mid a = b \text{ or } a = -b\}$

- $a = \pm b$ and $b = \pm c$ implies $a = \pm c$

$R_D = \{(a, b) \mid a = b\}$

- $a = b$ and $b = c$ implies $a = c$

Transitive Relations 2

Example: Is the "divides" relation on the set of positive integers transitive?

Solution:

Let us assume that a | b and b | c

Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c.

It follows that this relation is transitive.

One More Example

Let R be the following relation defined on the set $\{a, b, c, d\}$

$R = \{(a, a), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, b), (c, c), (d, b), (d, d)\}$

Determine whether R is: (a) reflexive. (b) symmetric (c) antisymmetric (d) transitive

Solution:

- R is reflexive because R contains (a, a), (b, b), (c, c) and (d, d)
- R is not symmetric because $(a, c) \in R$ but $(c, a) \notin R$
- R is not antisymmetric because both $(b, c) \in R$ and $(c, b) \in R$ but $b \neq c$
- R is not transitive because, for example, $(a, c) \in R$ and $(c, b) \in R$ but $(a, b) \notin R$

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent.* The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation $R = \{(a,b) \mid a \equiv b \pmod{m}\}$
is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, or $b = a + (-k)\, m$ so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations: $a - c = (a - b) + (b - c) = km + lm = (k + l)\, m$.

$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore, $a \equiv c \pmod{m}$.

Given a≡b (mod m) then
a mod m = b mod m
m | (a – b)
a = b + km

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not symmetric. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

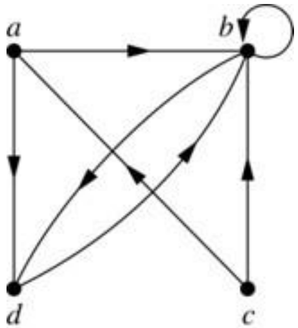
Representing Relations Using Digraphs[1]

- Relations with a manageable number of elements can be represented pictorially, using digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

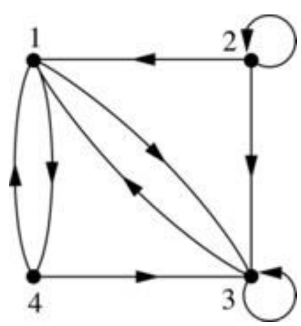
  – An edge of the form (a,a) is called a *loop*.

A drawing of the digraph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is shown here.



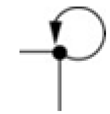
Representing Relations Using Digraphs[2]

- The relation R on a set A can be represented by a digraph that has the elements of A as its vertices and the ordered pairs (a, b), where (a, b) ∈ R, as its edges
- The digraph of the relation R = {(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3) } on set A = {1, 2, 3, 4} is shown below

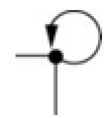
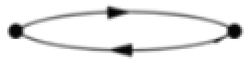


Determining which Properties a Relation has from its Digraph[1]

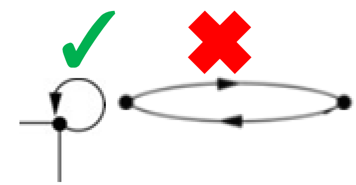
The directed graph representing a relation can be used to determine the type of a relation

Reflexivity: A loop must be present at all vertices in the graph. 

Symmetry: If (x,y) is an edge, then so is (y,x).

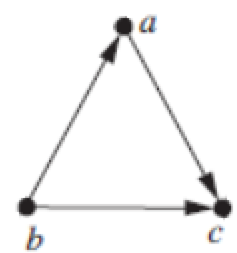
– All edges either loops or “anti-parallel”  

Antisymmetry: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.

– If no “anti-parallel” edges then no violation of anti-symmetry 

Transitivity: If (x,y) and (y,z) are edges, then so is (x,z).

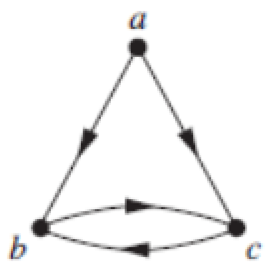
– All path of length 2 (e.g. b-a-c) accompanied by a corresponding path of length 1 (e.g. b-c)



Determining which Properties a Relation has from its Digraph2

Example: Determine the type of relation represented by the digraph

Solution:



No loops on all vertices, therefore not reflexive

Not all edges are anti-parallel, therefore not symmetric

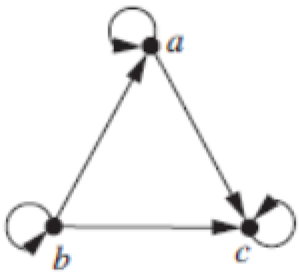
Antiparallel edges between b and c, therefore not antisymmetric

The following paths of length 2 not accompanied by a corresponding path of length 1, therefore not transitive

- (b,c) and (c, b) [(b, b) is missing]
- (c, b) and (b, c) [(c, c) is missing]

Determining which Properties a Relation has from its Digraph[3]

Example: Determine the type of relation represented by the digraph



Solution:

Loops on all vertices, therefore reflexive

No anti-parallel edges, therefore not symmetric

No violation of anti-symmetric property

The following path of length 2 accompanied by a corresponding path of length 1, therefore transitive

– (b,a); (a, c) & (b, c)

Combining Relations

Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.

Example: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2)(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

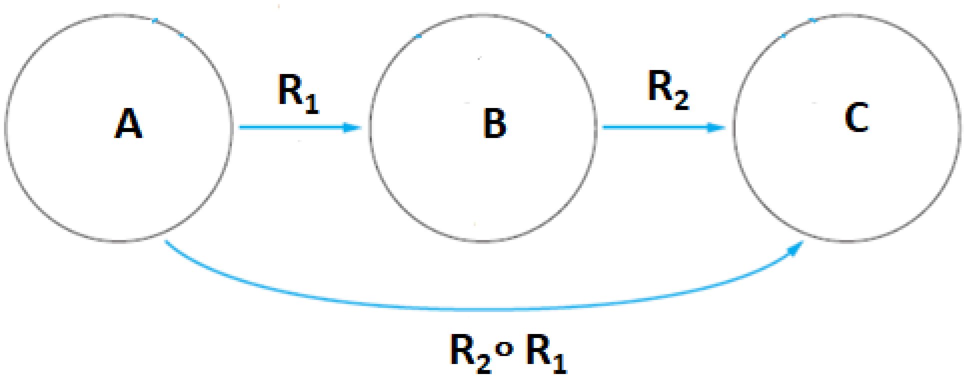
Composition[1]

Definition: Suppose

- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

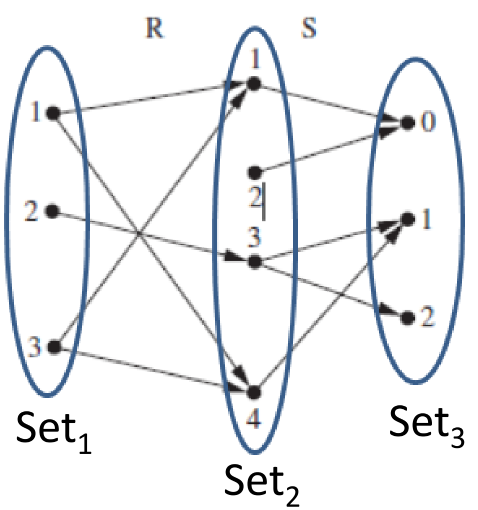


Composition₂

Example: What is the composite of the relations *R* and *S*, where *R* is the relation from {1, 2, 3} to {1, 2, 3, 4} with *R* = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and *S* is the relation from {1, 2, 3, 4} to {0, 1, 2} with *S* = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?

Solution: $S \circ R$ is constructed using all ordered pairs in *R* and ordered pairs in *S*, where the second element of the ordered pair in *R* agrees with the first element of the ordered pair in *S*.

Computing all the ordered pairs in the composite, we find $S \circ R$ = {(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)}



1 → 1 → 0 (1, 0)
1 → 4 → 1 (1, 1)
2 → 3 → 1 (2, 1)
2 → 3 → 2 (2, 2)
3 → 1 → 0 (3, 0)
3 → 4 → 1 (3, 1)