CS 2305: Discrete Mathematics for Computing I

Lecture 26

- KP Bhat

Product Rule: An Illustrative Example

- Assume that a procedure consists of three steps s_1, s_2, s_3
- There are 5 ways to perform s_1: designated as A, B, C, D, E
- There are 4 ways to perform s_2: designated as 1, 2, 3, 4
- There are 3 ways to perform s_3: designated as α, β, γ
- By the Product Rule there are 5 X 4 X 3 = 60 ways to perform the procedure, which are enumerated below

A-1-α, A-1-β, A-1-γ, A-2-α, A-2-β, A-2-γ, A-3-α, A-3-β, A-3-γ, A-4-α, A-4-β, A-4-γ, B-1-α, B-1-β, B-1-γ, B-2-α, B-2-β, B-2-γ, B-3-α, B-3-β, B-3-γ, B-4-α, B-4-β, B-4-γ, C-1-α, C-1-β, C-1-γ, C-2-α, C-2-β, C-2-γ, C-3-α, C-3-β, C-3-γ, C-4-α, C-4-β, C-4-γ, D-1-α, D-1-β, D-1-γ, D-2-α, D-2-β, D-2-γ, D-3-α, D-3-β, D-3-γ, D-4-α, D-4-β, D-4-γ, E-1-α, E-1-β, E-1-γ, E-2-α, E-2-β, E-2-γ, E-3-α, E-3-β, E-3-γ, E-4-α, E-4-β, E-4-γ

Basic Counting Principles: The Product Rule [2]

Example: How many bit strings of length seven are there?

Solution: Since each of the seven bits is either a 0 or a 1 (i.e. 2 choices), the answer is $2^7 = 128$.

Example: A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

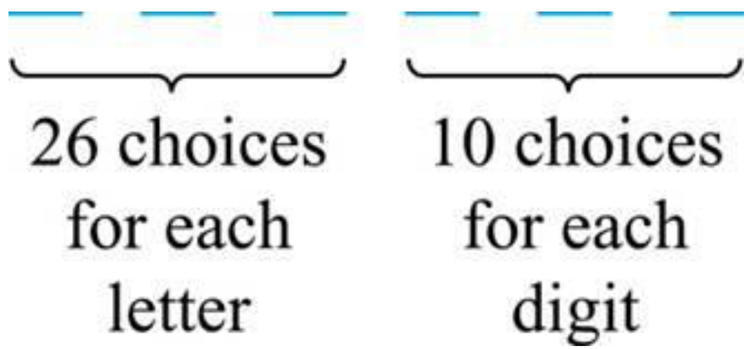
Solution: Assigning an office to Sanchez can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez can be done in 11 ways. So there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

The Product Rule[1]

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule,

there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ different possible license plates.



The Product Rule₂

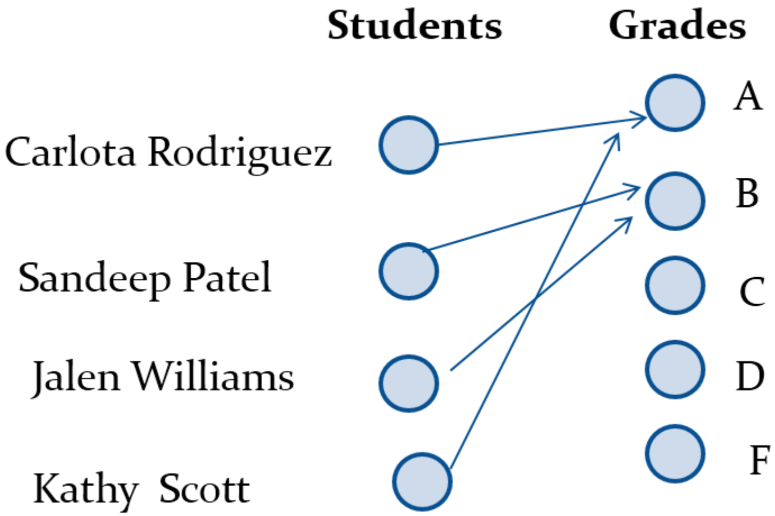
Example: A multiple-choice test contains 10 questions. There are four possible answers for each question. In how many ways can a student answer the questions on the test if the student answers every question?

Solution:

There are 4 ways to answer each question. By the product rule, there are $4 \cdot 4 \cdot \ldots \cdot 4 = 4^{10} = 1{,}048{,}576$ ways for the student to answer the questions.

Refresher on Functions

Definition: Let A and B be nonempty sets. A *function* f from A to B, denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.



Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?
Solution: Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1, n ways to choose the value of a_2, etc. The product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions..

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements?
Solution: Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1, $n-1$ ways to choose the value of a_2, … $n-(k-1)$ ways to choose a_k etc.

The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Telephone Numbering Plan

Example: The *North American numbering plan* (*NANP*) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was *NYX-NNX-XXXX*.
- In the new plan, the format is *NXX-NXX-XXXX*.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format *NYX.*
- There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format *NXX.*
- There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format *NNX.*
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format *XXXX.*

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$.

Number of new plan telephone numbers: $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$.

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$. (*In Section* 5.1, *mathematical induction was used to prove this same result*.)

Solution: Let us assume that the elements of set S are a_1, a_2, a_3, …., a_n. Let us represent the subsets of S by bit strings of length n where the ith bit of a bit string is 1 if a_i belongs to the corresponding subset and 0 if a_i does not belong to the corresponding subset.

By the product rule, there are 2^n such bit strings, and therefore 2^n subsets.

Since n = |S|, there are $2^{|S|}$ subsets.

DNA and Genomes

A *gene* is a segment of a DNA molecule that encodes a particular protein and the entirety of genetic information of an organism is called its *genome*.

DNA molecules consist of two strands of blocks known as nucleotides. Each nucleotide is composed of bases: adenine (A), cytosine (C), guanine (G), or thymine (T).

The DNA of bacteria has between 10^5 and 10^7 links (one of the four bases). Mammals have between 10^8 and 10^{10} links. So, <u>by the product rule there are at least 4^{10^5} different sequences of bases in the DNA of bacteria and 4^{10^8} different sequences of bases in the DNA of mammals</u>.

The human genome includes approximately 23,000 genes, each with 1,000 or more links.

Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

Not on the exam

Basic Counting Principles: The Sum Rule[1]

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

<u>Generalized Form</u>

Suppose that a task can be done in one of n_1 ways, or in one of n_2 ways, … , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \cdots + n_m$.

Sum Rule: An Illustrative Example

- Alternative 1
  - There are 5 ways to perform a task: designated as A, B, C, D, E
- Alternative 2
  - There are 4 ways to perform the task: designated as 1, 2, 3, 4
- Alternative 3
  - There are 3 ways to perform the task: designated as α, β, γ
- By the Sum Rule the number of ways to perform the task is 5 + 4 + 3 = 12, as shown below

$$\overbrace{A,\ B,\ C,\ D,\ E}^{\text{Alternative-1}}$$

$$\overbrace{1,\ 2,\ 3,\ 4}^{\text{Alternative-2}}$$

$$\overbrace{\alpha,\ \beta,\ \gamma}^{\text{Alternative-3}}$$

Basic Counting Principles: The Sum Rule 2

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

Rule of Thumb for Applying the Product Rule or the Sum Rule

- If we must make one choice and then another choice, the product rule applies
- If we must make one choice or another choice, the sum rule applies

Alternatives exist

Example: There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Solution: a) There are 18 choices to choose a mathematics major, and for each of those choices we have 325 choices to choose a computer science major. Hence, by the product rule, we can choose two representatives in 18 * 325 = 5850 ways

b) There are 18 ways to choose a mathematics major and 325 ways to choose a computer science major. Since we can choose the representative from either major, there are a total of 18 + 325 = 343 choices to pick the representative.

Combining the Sum and Product Rule $_1$

Many counting problems cannot be solved using just the sum rule or just the product rule.

However, many complicated counting problems can be solved using both of these rules in combination.

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution:

Number of choices when the label is a single letter = 26

By product rule number of choices when the label is a single letter followed by a digit = 26*10 = 260

∴ by the sum rule number of possible labels = 26 + 260= 286

Combining the Sum and Product Rule[2]

Example: How many strings are there of lowercase letters of length four or less, not counting the empty string?
Solution: Number of strings of length 1 = 26

Number of strings of length 2 = 26^2 = 676 (from the product rule)

Number of strings of length 3 = 26^3 = 17576 (from the product rule)

Number of strings of length 4 = 26^4 = 456976 (from the product rule)

By the sum rule it follows that there are
26 + 676 + 17576 + 456976 = 475254 strings of lowercase letters of length four or less, not counting the empty string

Counting Passwords

Problem: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

  $P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$

  $P_7 = 36^7 - 26^7 =$

  $78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$

  $P_8 = 36^8 - 26^8 =$

  $2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Internet Addresses

Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

Class A Addresses: used for the largest networks, a 0,followed by a 7-bit netid and a 24-bit hostid.
Class B Addresses: used for the medium-sized networks, a 10,followed by a 14-bit netid and a 16-bit hostid.
Class C Addresses: used for the smallest networks, a 110,followed by a 21-bit netid and a 8-bit hostid.

- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Hostids consisting of all 0s and all 1s are not available in any network.

Jump to long description

Counting Internet Addresses[1]

Example: How many different IPv4 addresses are available for computers on the internet?

Solution: Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

By the sum rule x = x_A, x_B, and x_C

- x_A:

  There are $2^7 - 1 = 127$ Class A netids, since netid 1111111 is unavailable

  For each netid, there are $2^{24} - 2 = 16{,}777{,}214$ hostids, since hostids consisting of all 0s and all 1s are unavailable

  By the product law the total number of Class A addresses is 127 * 16,777,214 = 2,130,706,178

- x_B:

  There are $2^{14} = 16{,}384$ Class B netids

  For each netid, there are $2^{16} - 2 = 65{,}534$ hostids, since hostids consisting of all 0s and all 1s are unavailable

  By the product law the total number of Class B addresses is 16,384 * 65,534 = 1,073,709,056

Counting Internet Addresses$_2$

- x_C:

  There are $2^{21} = 2,097,152$ Class C netids

  For each netid, there are $2^8 - 2 = 254$ hostids, since hostids consisting of all 0s and all 1s are unavailable

  By the product law the total number of Class C addresses is $2,097,152 * 254 = 532,676,608$

Hence, the total number of available IPv4 addresses is

$$x = x_A + x_B + Xc$$
$$= 2,130,706,178 + 1,073,709,056 + 532,676,608$$
$$= 3, 737,091,842.$$

Not Enough Today !!
The IPv6 protocol solves the problem of too few addresses.

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

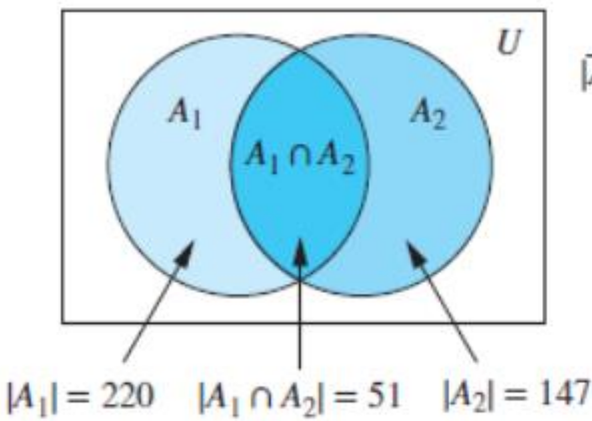
Generalized Principle of Inclusion-Exclusion

- **For two sets**
  - $|A \cup B| = \underbrace{|A|+|B|}_{\textbf{Singletons}} - \underbrace{|A \cap B|}_{\textbf{Pair}}$
- **For three sets**
  - $|A \cup B \cup C| = \underbrace{|A|+|B|+|C|}_{\textbf{Singletons}} - \underbrace{(|A \cap B|+|A \cap C|+|B \cap C|)}_{\textbf{Pairs}} + \underbrace{|A \cap B \cap C|}_{\textbf{Triple}}$
- **For four sets**
  - $|A \cup B \cup C \cup D| = \underbrace{|A|+|B|+|C|+|D|}_{\textbf{Singletons}} -$
    $\underbrace{(|A \cap B|+|A \cap C|+|A \cap D|+|B \cap C|+|B \cap D|+|C \cap D|)}_{\textbf{Pairs}} +$
    $\underbrace{(|A \cap B \cap C|+|A \cap B \cap D|+|A \cap C \cap D|+|B \cap C \cap D|)}_{\textbf{Triple}} - \underbrace{|A \cap B \cap C \cap D|}_{\textbf{Quadruple}}$

Notice the alternating + and - signs

Principle of Inclusion-Exclusion

- A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

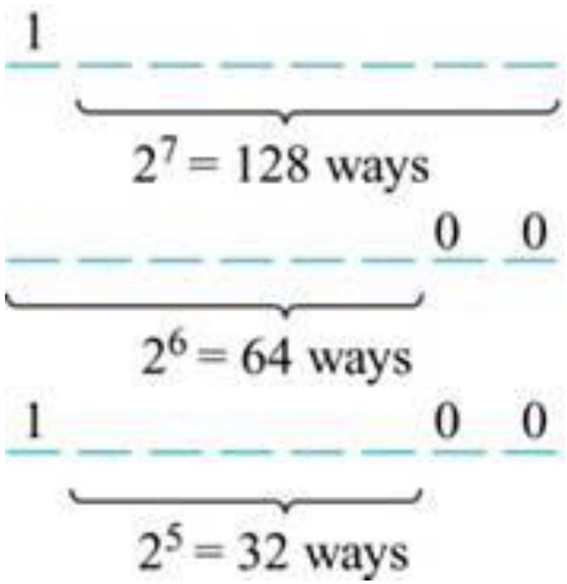


$$
\begin{aligned}
|\overline{A_1 \cup A_2}| &= |U| - |A_1 \cup A_2| \\
&= |U| - (|A_1| + |A_2| - |A_1 \cap A_2|) \\
&= 350 - (220 + 147 - 51) \\
&= 350 - 316 \\
&= 34
\end{aligned}
$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.



- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
    - $1*2^7$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
    - 2^6*1
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$
    - $1*2^5*1$

Hence, the number is $128 + 64 - 32 = 160$.

Counting Patterns

Example: How many seven digit numbers (where leading 0s are acceptable) do not contain the substring 1234?

Solution: Using the product rule, there are 10^7 possible strings.

Let us count how many strings contain the substring "1234".

Here are the choices:

1 2 3 4 _ _ _: $1 * 10^3 = 10^3$ choices

_ 1 2 3 4 _ _: $10 * 1 * 10^2 = 10^3$ choices

_ _ 1 2 3 4 _: $10^2 * 1 * 10 = 10^3$ choices

_ _ _ 1 2 3 4: $10^3 * 1 = 10^3$ choices

Use the subtraction rule, the count of seven digit numbers that do not contain the substring 1234 is:

$10^7 – 4 * 10^3 = 9{,}996{,}000$