MATH2010 Advanced Calculus I, 2023-2024 Term 2

Solution to HOMEWORK 1

Q1

Solution. By triangle inequality, we have

$$\|\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\| \geq \|\mathbf{b}+\mathbf{a}-\mathbf{b}\| = \|\mathbf{a}\| \Longrightarrow \|\mathbf{a}\| - \|\mathbf{b}\| \leq \|\mathbf{a}-\mathbf{b}\|.$$

Also, by triangle inequality, we have

$$\|\mathbf{a}\| + \|\mathbf{b}-\mathbf{a}\| \geq \|\mathbf{a}+\mathbf{b}-\mathbf{a}\| = \|\mathbf{b}\| \Longrightarrow -\|\mathbf{b}-\mathbf{a}\| \leq \|\mathbf{a}\| - \|\mathbf{b}\| \Longrightarrow -\|\mathbf{a}-\mathbf{b}\| \leq \|\mathbf{a}\| - \|\mathbf{b}\|.$$

Therefore, we have

$$-\|\mathbf{a}-\mathbf{b}\| \leq \|\mathbf{a}\| - \|\mathbf{b}\| \leq \|\mathbf{a}-\mathbf{b}\|.$$

□

Q2

Solution.

(i) We have $\overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OR} + \overrightarrow{OS}) = \frac{1}{2}(2-4, -1+3) = (-1, 1)$

(ii) We have $\overrightarrow{AB} = (2-1, 0-(-1)) = (1,1)$ and $\overrightarrow{CD} = (-2-(-1), 2-3) = (-1,-1)$.
Therefore, $\overrightarrow{AB} + \overrightarrow{CD} = (1-1, 1-1) = (0,0)$.

(iii) For unit vectors in $\mathbb{R}^2$, the vectors are $\left(\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\cos\frac{-2\pi}{3}, \sin\frac{-2\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.
For unit vectors in $\mathbb{R}^3$, let $\mathbf{u} = (a,b,c)$. We have

$$\cos\frac{2\pi}{3} = \frac{\mathbf{u}\cdot(1,0,0)}{\|\mathbf{u}\|\|(1,0,0)\|} \Longrightarrow -\frac{1}{2} = \frac{a}{1} \Longrightarrow a = -\frac{1}{2}.$$

Now, we have $a^2+b^2+c^2 = 1 \Longrightarrow b^2+c^2 = \frac{3}{4} \Longrightarrow c = \pm\sqrt{\frac{3}{4}-b^2}$.
Therefore, the vectors are $\left(-\frac{1}{2}, b, \pm\sqrt{\frac{3}{4}-b^2}\right)$ where $b \in \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$.

□

Q3

Solution. Note that the indicated diagonal is $\mathbf{u}+\mathbf{v}$. Denote the angle between $\mathbf{u}$ and the diagonal as θ_1 and the angle between $\mathbf{v}$ and the diagonal as θ_2.

We have

$$\cos\theta_1 = \frac{\mathbf{u}\cdot(\mathbf{u}+\mathbf{v})}{\|\mathbf{u}\|\|\mathbf{u}+\mathbf{v}\|} = \frac{\|\mathbf{u}\|^2 + \mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{u}+\mathbf{v}\|}$$

and

$$\cos\theta_2 = \frac{\mathbf{v}\cdot(\mathbf{u}+\mathbf{v})}{\|\mathbf{v}\|\|\mathbf{u}+\mathbf{v}\|} = \frac{\|\mathbf{v}\|^2 + \mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{u}+\mathbf{v}\|}.$$

If $\|\mathbf{u}\| = \|\mathbf{v}\|$, we have

$$\cos\theta_1 = \frac{\|\mathbf{u}\|^2 + \mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{u}+\mathbf{v}\|} = \frac{\|\mathbf{v}\|^2 + \mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|\|\mathbf{u}+\mathbf{v}\|} = \cos\theta_2.$$

Therefore, the indicated diagonal makes equal angles with $\mathbf{u}$ and $\mathbf{v}$. $\square$

Q4

Solution.

1. We have

$$\|\mathrm{Proj}_{\mathbf{v}}\mathbf{u}\| = \left\|\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right\| = |\mathbf{u}\cdot\mathbf{v}|\frac{\|\mathbf{v}\|}{\|\mathbf{v}\|^2} = \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{v}\|}.$$

   Now, by Cauchy-Schwarz inequality, we have $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$. Therefore, we have

$$\|\mathrm{Proj}_{\mathbf{v}}\mathbf{u}\| = \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{v}\|} \le \frac{\|\mathbf{u}\|\|\mathbf{v}\|}{\|\mathbf{v}\|} = \|\mathbf{u}\|.$$

2. The equality in Part 1 holds if and only if the equality in the Cauchy-Schwarz inequality holds, i.e. $\mathbf{u} = r\mathbf{v}$ for some $r \in \mathbb{R}$.

$\square$

Q5

Solution. We have

$$\begin{aligned}
(\mathbf{u} - \mathrm{Proj}_{\mathbf{v}}\mathbf{u})\cdot\mathrm{Proj}_{\mathbf{v}}\mathbf{u} &= \left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right)\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right)\\
&= \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right) - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right)\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right)\\
&= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}(\mathbf{u}\cdot\mathbf{v}) - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{v}\|^4}(\mathbf{v}\cdot\mathbf{v})\\
&= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}(\mathbf{u}\cdot\mathbf{v}) - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{v}\|^4}(\|\mathbf{v}\|^2)\\
&= \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{v}\|^2} - \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{v}\|^2}\\
&= 0.
\end{aligned}$$

$\square$

Q6

Solution. Note that if vectors $\mathbf{u}$ and $\mathbf{v}$ determine a parallelogram, then the length of the two diagonals are $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}-\mathbf{v}\|$.

Therefore, the two diagonals are equal in length if and only if

$$\begin{gathered}
\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2\\
\iff (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\
\iff \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u}\cdot\mathbf{v})\\
\iff \mathbf{u}\cdot\mathbf{v} = 0,
\end{gathered}$$

which holds if and only if $\mathbf{u}$ is perpendicular to $\mathbf{v}$, i.e. the parallelogram is a rectangle. $\square$