MATH2010 Advanced Calculus I, 2023-2024 Term 2

Solution to HOMEWORK 2

Q1

Solution. (i) We have $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Therefore,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 4 & -1 & 2 \end{vmatrix} = -\mathbf{i} + 2\mathbf{k}.$$

(ii) The volume of the tetrahedron is

$$\frac{1}{6}(\overrightarrow{OA} \times \overrightarrow{OB}) \cdot \overrightarrow{OC} = \frac{1}{6}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 5 & 1 & 3 \end{vmatrix} = \frac{1}{6}(1) = \frac{1}{6}.$$

□

Q2

Solution. We have $\overrightarrow{AB} = (2,0,-1)$, $\overrightarrow{AP} = (1,4,0)$, and $\overrightarrow{AP} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 2 & 0 & -1 \end{vmatrix} = (-4,1,-8)$. Now, denote θ as the angle between $\overrightarrow{AP}$ and $\overrightarrow{AB}$, then we have $\|\overrightarrow{AP} \times \overrightarrow{AB}\| = \|\overrightarrow{AP}\|\|\overrightarrow{AB}\|\sin\theta$. Therefore, the distance from the point to the line is

$$\|\overrightarrow{AP}\|\sin\theta = \frac{\|\overrightarrow{AP} \times \overrightarrow{AB}\|}{\|\overrightarrow{AB}\|} = \frac{\|(-4,1,-8)\|}{\|(2,0,-1)\|} = \frac{9}{\sqrt{5}} = \frac{9\sqrt{5}}{5}.$$

□

Q3

Solution. A normal vector of the plane is

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 0 \end{vmatrix} = (-1,-2,-2).$$

Now, denote θ as the angle between $\overrightarrow{AP}$ and $\mathbf{n}$, then we have $|\overrightarrow{AP} \cdot \mathbf{n}| = \|\overrightarrow{AP}\|\|\mathbf{n}\||\cos\theta|$. Therefore, the distance from the point to the plane is

$$\|\overrightarrow{AP}\||\cos\theta| = \frac{|\overrightarrow{AP} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|(1,4,0) \cdot (-1,-2,-2)|}{\|(-1,-2,-2)\|} = \frac{9}{3} = 3.$$

□

Q4

Solution. A normal vector of the plane is $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. Now, note that $\mathbf{v}$ should be the sum of the projection of $\mathbf{v}$ onto the plane and the projection of $\mathbf{v}$ onto the normal vector. Therefore, we have

$$\mathbf{Proj}_P\mathbf{v} = \mathbf{v} - \mathbf{Proj}_{\mathbf{n}}\mathbf{v} = (1,1,1) - \frac{(1,1,1)\cdot(1,2,6)}{\|(1,2,6)\|^2}(1,2,6) = (1,1,1) - \frac{9}{41}(1,2,6) = \left(\frac{32}{41}, \frac{23}{41}, \frac{-13}{41}\right).$$

□

Q5

Solution. Note that the length of $\vec{x}(t)$ is

$$\text{length}(\vec{x}(t)) = \int_{t_0}^{t_1} \|\vec{x}'(t)\| dt$$

and the length of $\vec{y}(s)$ is

$$\text{length}(\vec{y}(s)) = \int_{s_0}^{s_1} \|\vec{y}'(s)\| ds.$$

Now, we have

$$\begin{aligned}
\text{length}(\vec{x}(t)) &= \int_{t_0}^{t_1} \|\vec{x}'(t)\| dt \\
&= \int_{t_0}^{t_1} \left\| \frac{d}{dt}(\vec{y}(s(t))) \right\| dt \qquad (\text{since } \vec{y}(s(t)) = \vec{x}(t)) \\
&= \int_{t_0}^{t_1} \left\| \frac{d\vec{y}}{ds}\frac{ds}{dt} \right\| dt \qquad (\text{by chain rule}) \\
&= \int_{t_0}^{t_1} \left\| \frac{d\vec{y}}{ds} \right\| \frac{ds}{dt} dt \qquad \left(\text{note that } \frac{ds}{dt} > 0 \text{ since } s(t) \text{ is a strictly increasing bijection}\right) \\
&= \int_{s(t_0)}^{s(t_1)} \|\vec{y}'(s)\|\, ds \qquad (\text{by change of variable}) \\
&= \int_{s_0}^{s_1} \|\vec{y}'(s)\|\, ds \qquad (\text{since } s(t_0) = s_0 \text{ and } s(t_1) = s_1) \\
&= \text{length}(\vec{y}(s)).
\end{aligned}$$

Therefore, $\vec{x}(t)$ and $\vec{y}(s)$ are of the same length.

□