MATH 2010 Advanced Calculus
Suggested Solution of Homework 6

Q1. Can the function

$$f(x,y) = \frac{\sin x \sin^3 y}{1-\cos\left(x^2+y^2\right)}$$

be defined at $(0,0)$ in such a way that it becomes continuous there?

Solution: <u>**NO.**</u>
Note that

$$f = \frac{\sin x \sin^3 y}{1-\cos\left(x^2+y^2\right)} = \frac{\sin x \sin^3 y}{2\sin^2\left(\frac{x^2+y^2}{2}\right)} = \frac{\sin x \sin^3 y}{xy^3} \cdot \frac{2\left(\frac{x^2+y^2}{2}\right)^2}{2\sin^2\left(\frac{x^2+y^2}{2}\right)} \cdot \boxed{\frac{xy^3}{2\left(\frac{x^2+y^2}{2}\right)^2}},$$

where the first two terms of the rightmost have limit 1 and the boxed fraction is homogeneous. So there is a good chance that the limit as (x,y) tends to $(0,0)$ does not exist. For example, just consider two different paths $x=y$ and $x=-y$:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f = \lim_{x\to 0}\frac{\sin^4 x}{1-\cos(2x^2)} = \lim_{x\to 0}\frac{\sin^4 x}{2\sin^2(x^2)} = \lim_{x\to 0}\frac{\sin^4 x}{x^4}\frac{2(x^2)^2}{2\sin^2(x^2)}\frac{x^4}{2(x^2)^2} = \frac{1}{2},$$

while

$$\lim_{\substack{(x,y)\to(0,0)\\ x=-y}} f = \lim_{x\to 0}\frac{-\sin^4 x}{1-\cos(2x^2)} = -\lim_{x\to 0}\frac{\sin^4 x}{1-\cos(2x^2)} = -\frac{1}{2},$$

which implies that the limit of f as (x,y) tends to $(0,0)$ does not exist. So the answer is no. $\square$

Q2. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function and let $(a,b)\in\mathbb{R}^2$. We define two single variable functions $g(x)=f(x,b)$ and $h(y)=f(a,y)$.

(a) If $g(x)$ is continuous at $x=a$ and $h(y)$ is continuous at $y=b$, does it follow that f is continuous at (a,b)? Why?

(b) If $f(x,y)$ is continuous at (a,b), does it follow that $g(x)$ is continuous at $x=a$ and $h(y)$ is continuous at $y=b$? Why?

Solution: Part (a) no; part (b) yes.

For part (a), $g(x)$ is continuous at $x = a$ only means that $f(x, y) \to f(a, b)$ along $y = b$ as $x \to a$; and $h(y)$ is continuous at $y = b$ only means that $f(x, y) \to f(a, b)$ along $x = a$ as $y \to b$. However, f continuous at (a, b) requires that $f(x, y) \to f(a, b)$ along any path. So f is not necessarily continuous at (a, b). For counter-example, consider

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

and $a = 0 = b$, then not only $g(x)$ is continuous at $x = a$ and $h(y)$ is continuous at $y = b$, in fact $D_{\vec{u}}f(0,0)$ exists for any $\vec{u}$, but f is not continuous at $(0, 0)$.

For part (b),

$$f \text{ is continuous at } (a,b) \quad \Leftrightarrow \quad \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b);$$

In particular,

$$\lim_{\substack{(x,y)\to(a,b)\\ y=b}} f(x,y) \equiv \lim_{x\to a} g(x) = g(a) \equiv f(a,b) \equiv h(b) = \lim_{y\to b} h(y) \equiv \lim_{\substack{(x,y)\to(a,b)\\ x=a}} f(x,y),$$

by definition, $g(x)$ is continuous at $x = a$ and $h(y)$ is continuous at $y = b$. $\square$

Q3. Compute the gradient for the following functions.

(a) $f(x, y, z) = \sec(x + y) + \dfrac{1}{y + z}$.

(b) $f(x, y) = \dfrac{x + y}{x - y}$.

Solution:

(a) $\nabla f(x,y,z) = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right) = \left(\dfrac{\sin(x+y)}{\cos^2(x+y)}, \dfrac{\sin(x+y)}{\cos^2(x+y)} - \dfrac{1}{(y+z)^2}, -\dfrac{1}{(y+z)^2}\right)$,

or $\nabla f(x,y,z) = \left(\sec(x+y)\tan(x+y), \sec(x+y)\tan(x+y) - \dfrac{1}{(y+z)^2}, -\dfrac{1}{(y+z)^2}\right)$;

(b) $\nabla f(x,y) = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = \left(\dfrac{-2y}{(x-y)^2}, \dfrac{2x}{(x-y)^2}\right)$.

$\square$

Q4. Let $f(x, y, z) = xy + yz + zx$. Using the limit definition, find the directional derivative of f at the point $\boldsymbol{a} = (1, -1, 1)$ along the direction of $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$.

Solution: By definition,

$$\begin{aligned}
D_{\boldsymbol{v}} f(\boldsymbol{a}) &= \lim_{t\to 0} \frac{f(\boldsymbol{a}+t\boldsymbol{v})-f(\boldsymbol{a})}{t|\boldsymbol{v}|} \\
&= \lim_{t\to 0} \frac{(1+t)(-1+2t)+(-1+2t)(1+t)+(1+t)^2-(-1-1+1)}{t\sqrt{6}} \\
&= \lim_{t\to 0} \frac{5t^2+4t}{\sqrt{6}t} \\
&= \lim_{t\to 0} \frac{5t+4}{\sqrt{6}} = \frac{4}{\sqrt{6}} = \frac{2}{3}\sqrt{6}.
\end{aligned}$$

$\square$

Q5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Fix a point (x_0, y_0). Consider a linear approximation of $f(x,y)$ near (x_0, y_0), defined by

$$l(x,y) = f(x_0,y_0) + r(x-x_0) + s(y-y_0),$$

where $r, s \in \mathbb{R}$ are constants. Suppose the error

$$\epsilon(x,y) = f(x,y) - f(x_0,y_0) - r(x-x_0) - s(y-y_0)$$

satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\epsilon(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

Show that the partial derivatives $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist with $r = f_x(x_0,y_0)$ and $s = f_y(x_0,y_0)$.

Proof: By definition, the partial derivative $f_x(x_0,y_0)$ is given by

$$\begin{aligned}
f_x(x_0,y_0) &= \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \\
&= \lim_{h\to 0} \frac{l(x_0+h,y_0)+\epsilon(x_0+h,y_0)-f(x_0,y_0)}{h} \\
&= \lim_{h\to 0} \frac{f(x_0,y_0)+r(x_0+h-x_0)+s(y_0-y_0)+\epsilon(x_0+h,y_0)-f(x_0,y_0)}{h} \\
&= \lim_{h\to 0} \frac{rh+\epsilon(x_0+h,y_0)}{h};
\end{aligned}$$

Since

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\epsilon(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0,$$

then along path $y = y_0$

$$0 = \lim_{\substack{(x,y)\to(x_0,y_0)\\ y=y_0}} \frac{\epsilon(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = \lim_{x\to x_0} \frac{\epsilon(x,y_0)}{|x-x_0|} = \lim_{\substack{h\to 0\\ x-x_0=h}} \frac{\epsilon(x_0+h,y_0)}{|h|},$$

i.e.

$$\lim_{h\to 0} \frac{\epsilon(x_0+h,y_0)}{h} = 0,$$

consequently

$$f_x\,(x_0, y_0) = \lim_{h \to 0} \frac{rh + \epsilon(x_0 + h, y_0)}{h} = \lim_{h \to 0} \frac{rh}{h} + \lim_{h \to 0} \frac{\epsilon(x_0 + h, y_0)}{h} = r + 0 = r.$$

Similarly, along path $x = x_0$,

$$0 = \lim_{\substack{(x,y) \to (x_0, y_0) \\ x = x_0}} \frac{\epsilon(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \lim_{y \to y_0} \frac{\epsilon(x_0, y)}{|y - y_0|} = \lim_{\substack{h \to 0 \\ y - y_0 = h}} \frac{\epsilon(x_0, y_0 + h)}{|h|},$$

i.e.

$$\lim_{h \to 0} \frac{\epsilon(x_0, y_0 + h)}{h} = 0,$$

consequently

$$\begin{aligned}
f_y\,(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \\
&= \lim_{h \to 0} \frac{l(x_0, y_0 + h) + \epsilon(x_0, y_0 + h) - f(x_0, y_0)}{h} \\
&= \lim_{h \to 0} \frac{f(x_0, y_0) + r(x_0 - x_0) + s(y_0 + h - y_0) + \epsilon(x_0, y_0 + h) - f(x_0, y_0)}{h} \\
&= \lim_{h \to 0} \frac{sh + \epsilon(x_0, y_0 + h)}{h} \\
&= \lim_{h \to 0} \frac{sh}{h} + \lim_{h \to 0} \frac{\epsilon(x_0, y_0 + h)}{h} = s + 0 = s.
\end{aligned}$$

$\square$