MATH 2010CD Advanced Calculus I

Suggested Solution of Homework 4

Q1 By

$$\lim_{(x,y)\to(0,0)} e^y = 1, \lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = 1,$$

we have

$$\lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x} = 1.$$

Q2 Use the polar system, we get

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim_{r\to 0} r(\cos^3 x - \cos x \sin^2 y).$$

Since $\cos^3 x - \cos x \sin^2 y$ is bounded, by the sandwich theorem, we get

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} r(\cos^3 x - \cos x \sin^2 y) = 0.$$

Q3 By

$$\left| y \sin \frac{1}{x} \right| \le |y| \qquad \text{and} \qquad \lim_{(x,y)\to(0,0)} |y| = 0,$$

we can use the sandwich theorem to deduce that

$$\lim_{(x,y)\to(0,0)} y \sin \frac{1}{x} = 0.$$

Q4 (a) $\lim_{(x,y)\to(0,1)} f(x,y) = \lim_{(x,y)\to(0,1)} 1 = 1.$

(b) $\lim_{(x,y)\to(2,3)} f(x,y) = \lim_{(x,y)\to(2,3)} 0 = 0.$

(c) By

$$\lim_{(x,y)\to(0,0),x=0} f(x,y) = \lim_{(x,y)\to(0,0)} 1 = 1,$$

and

$$\lim_{(x,y)\to(0,0),y=x^2} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0,$$

we can deduce that the limit doesn't exist.

Q5 (a)

$$\begin{aligned}\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y - xy + 4x^2 - 4x} &= \lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2(y+4) - x(y+4)} \\ &= \lim_{(x,y)\to(2,-4)} \frac{1}{x^2 - x} = \frac{1}{2}\end{aligned}$$

(b) By

$$\left|(2x^2+y^2)\sin\frac{1}{\sqrt{x^2+4y^2}}\right| \le 2x^2+y^2 \qquad \text{and} \qquad \lim_{(x,y)\to(0,0)} 2x^2+y^2 = 0,$$

we can use the sandwich theorem to deduce that

$$\lim_{(x,y)\to(0,0)} (2x^2+y^2)\sin\frac{1}{\sqrt{x^2+4y^2}} = 0.$$

(c) Since

$$\lim_{(x,y)\to(0,0),y=kx^{5/2}} \frac{x^5y^2}{x^{10}-y^4} = \lim_{x\to 0}\frac{k^2x^{10}}{x^{10}(1-k^4)} = \frac{k^2}{1-k^4},$$

the limit doesn't exist.

(d) Since

$$\lim_{(x,y)\to(1,-1),y=-1} \frac{xy+1}{x^2-y^2} = \lim_{x\to 1}\frac{-x+1}{x^2-1} = \lim_{x\to 1} -\frac{1}{x+1} = -\frac{1}{2},$$

$$\lim_{(x,y)\to(1,-1),x=1} \frac{xy+1}{x^2-y^2} = \lim_{y\to -1}\frac{y+1}{1-y^2} = \lim_{y\to -1}\frac{1}{1-y} = \frac{1}{2},$$

the limit doesn't exist.