

Math 2010 Week 3

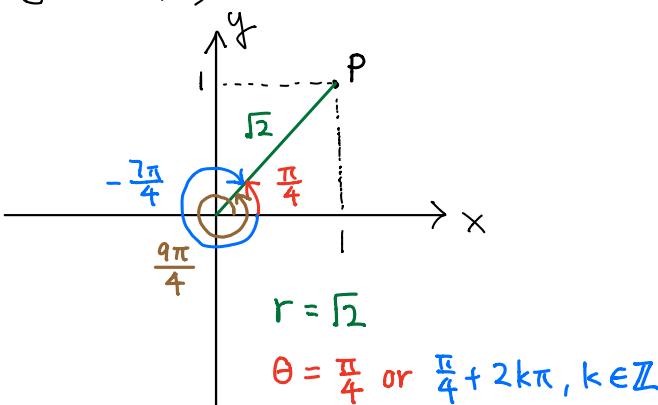
Polar coordinates in \mathbb{R}^2 (Au 1.5.2 Thomas 11.3, 11.4)

A point $P = (x, y) \in \mathbb{R}^2$ can be represented by

$$r = \sqrt{x^2 + y^2} = \text{distance from origin}$$

θ = angle from the positive x-axis to \overrightarrow{OP} in counter-clockwise direction

e.g. $P = (1, 1)$



Rmk

① For $P = (0, 0)$ $\begin{cases} r = 0 \\ \theta \text{ is not (uniquely) defined.} \end{cases}$

② Different conventions for ranges of r and θ

$$r \in [0, \infty) \text{ or } r \in \mathbb{R} \leftarrow \text{our textbook}$$

$$\theta \in [0, 2\pi) \text{ or } \theta \in \mathbb{R}$$

In this course, we usually take

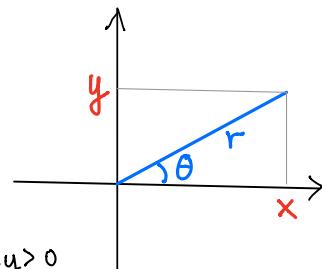
$$r \in [0, \infty) \text{ and } \theta \in \mathbb{R}$$

Change of coordinates formula

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ for } x, y > 0$$



Similar formula for θ in other quadrants

Curves in Polar Coordinates

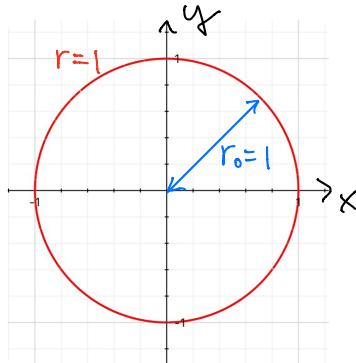
e.g. Circle with radius $r_0 > 0$, centered at origin

Polar Equation

$$r = r_0$$

Parametric form

$$\begin{cases} r = r_0 \\ \theta = t, t \in [0, 2\pi] \end{cases}$$



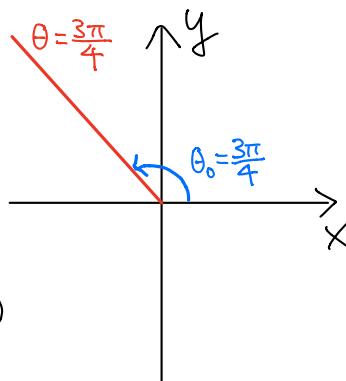
e.g. Half ray from origin

Polar Equation

$$\theta = \theta_0$$

Parametric form

$$\begin{cases} r = t, t \in [0, \infty) \\ \theta = \theta_0 \end{cases}$$



e.g. Archimedes Spiral

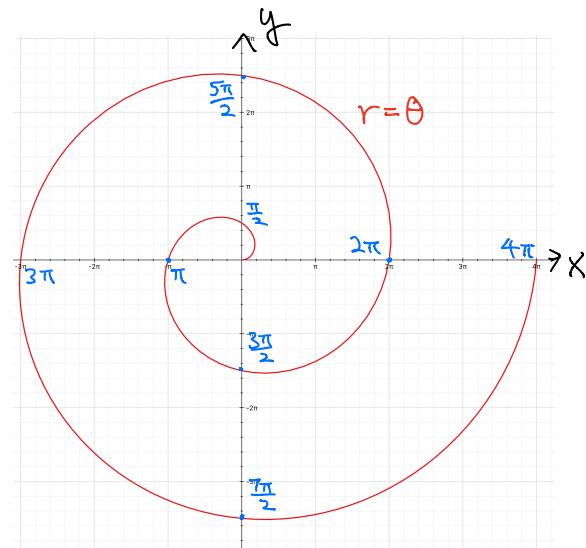
Let $k > 0$ be a constant

Polar Equation

$$r = k\theta$$

Parametric form

$$\begin{cases} r = kt \\ \theta = t \end{cases}, t \in [0, \infty)$$



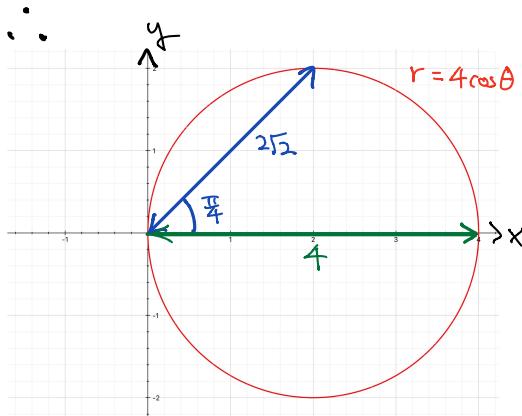
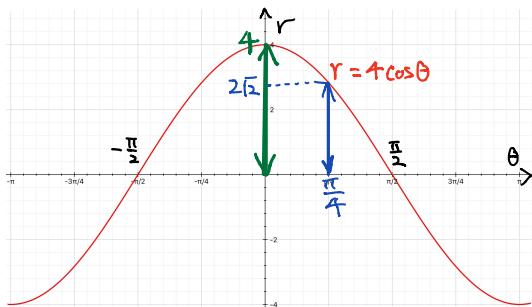
Picture for $t \in [0, 4\pi]$

$$\text{eg } r = 4 \cos \theta$$

Our convention: $r \geq 0 \Rightarrow \cos \theta \geq 0$

$\therefore \theta$ is in quadrant I or II

Take $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$



Observation: It looks like a circle... Yes!

$$\text{Note } r = 4 \cos \theta \Rightarrow r^2 = 4r \cos \theta$$

$$\Rightarrow x^2 + y^2 = 4x$$

$$\Rightarrow (x-2)^2 + y^2 = 2^2 \quad (\text{Circle with radius } 2 \text{ centered at } (2, 0))$$

$$\text{eg } r \cos(\theta - \frac{\pi}{4}) = \sqrt{2}$$

Try $\rightarrow \theta = 0 \Rightarrow r = 2$

$$\text{Note } \sqrt{2} = r \cos(\theta - \frac{\pi}{4})$$

$$\theta = \frac{\pi}{2} \Rightarrow r = 2$$

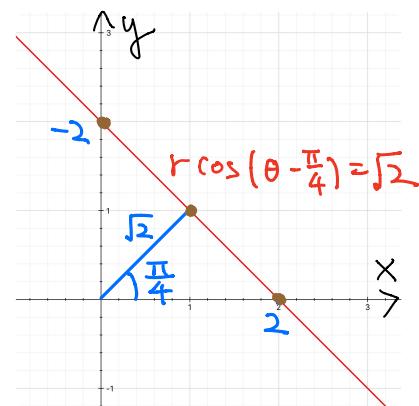
$$= r \left(\cos \theta \cos \frac{\pi}{4} + \sin \theta \sin \frac{\pi}{4} \right)$$

$$= \frac{r}{\sqrt{2}} \cos \theta + \frac{r}{\sqrt{2}} \sin \theta$$

$$= \frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y$$

$$\therefore x + y = 2$$

i.e. a line



$$r \in [0, \infty) \text{ vs } r \in (-\infty, \infty)$$

Our convention

It is sometimes convenient to allow $r < 0$ and interpret

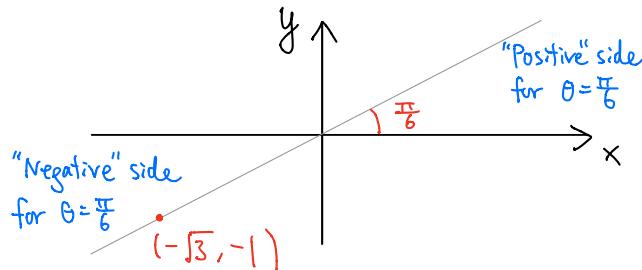
$$(x, y) = (r \cos \theta, r \sin \theta)$$

$$= (-|r| \cos \theta, -|r| \sin \theta)$$

$$= -(|r| \cos \theta, |r| \sin \theta)$$

$$\text{eg } r = -2, \theta = \frac{\pi}{6}$$

$$\Rightarrow (x, y) = \left(-2 \cos \frac{\pi}{6}, -2 \sin \frac{\pi}{6} \right) = (-\sqrt{3}, -1)$$



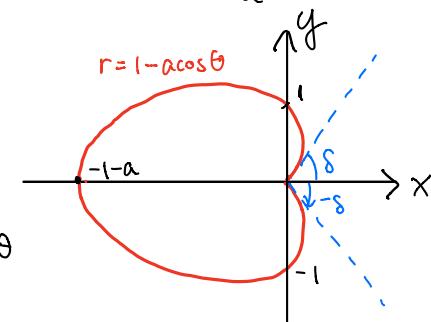
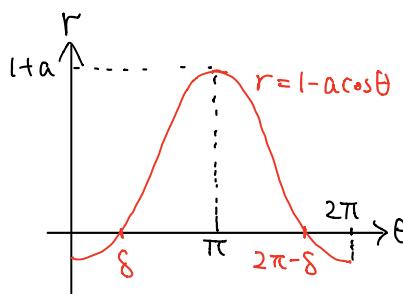
eg $r = 1 - a \cos \theta$, where $a > 1$ is a constant.

Case 1 If we require $r \geq 0$, then

$$1 - a \cos \theta \geq 0 \Rightarrow \cos \theta \leq \frac{1}{a} < 1$$

$\therefore \theta$ cannot run through the whole interval $[0, 2\pi]$

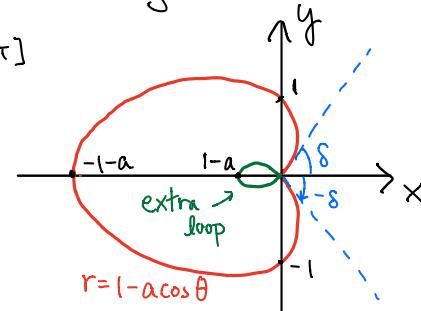
but only $[\delta, 2\pi - \delta]$, where $\delta = \cos^{-1} \frac{1}{a}$



Case 2 If we allow $r \in \mathbb{R}$ to be negative

θ can run through $[0, 2\pi]$

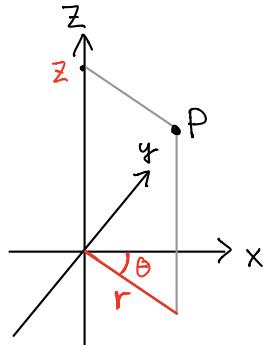
and we get a smooth, self-intersecting curve



Some Coordinates Systems in \mathbb{R}^3

Cylindrical Coordinates

(x, y, z) Express x, y using polar $\rightarrow (r, \theta, z)$



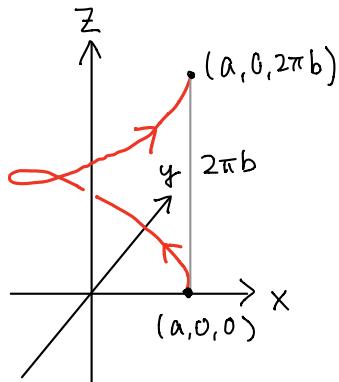
Formulas

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned}$$

e.g (Helix)

$$\begin{cases} r = a \\ \theta = t \\ z = bt \end{cases}$$

$$t \in [0, 2\pi]$$

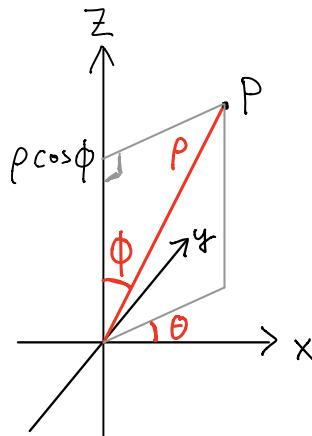


Spherical Coordinates

Describe $P \in \mathbb{R}^3$ by

$$\left\{ \begin{array}{l} p = \text{distance from origin} \\ = \sqrt{x^2 + y^2 + z^2} \\ \theta = \theta \text{ as in cylindrical coordinates} \\ \phi = \text{angle from positive z-axis to } \overrightarrow{OP} \end{array} \right.$$

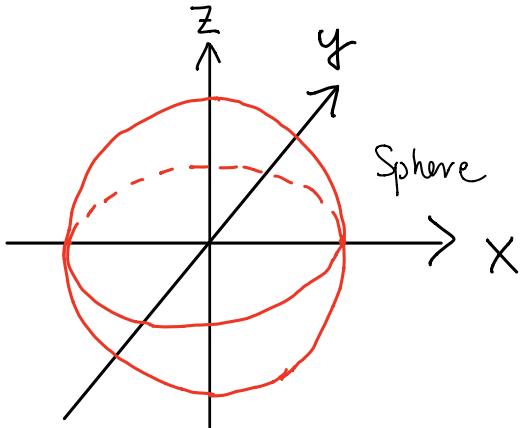
Rmk $\phi \in [0, \pi]$



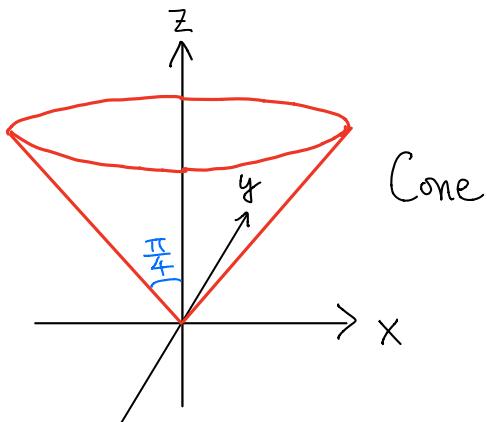
Formulas

$$\begin{aligned} x &= p \sin \phi \cos \theta \\ y &= p \sin \phi \sin \theta \\ z &= p \cos \phi \end{aligned}$$

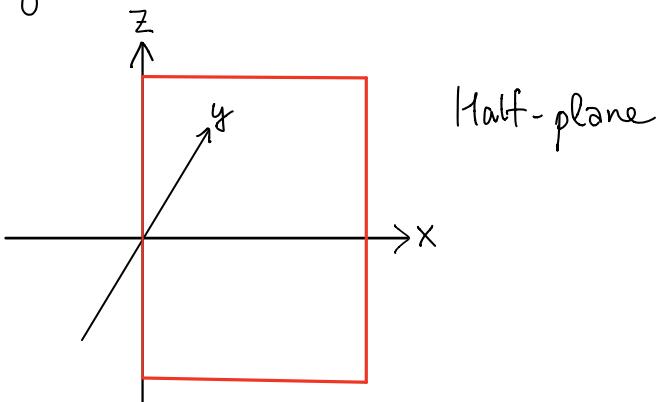
e.g. $\rho = 2$



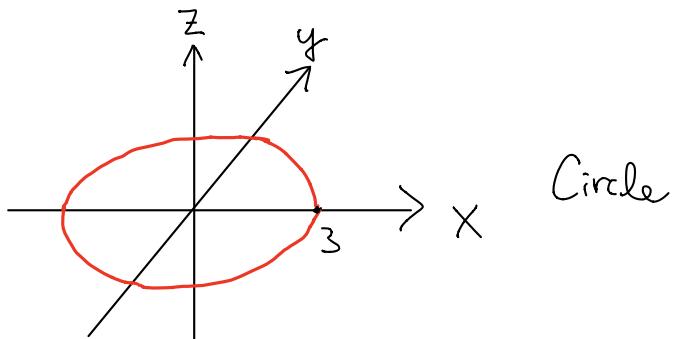
e.g. $\phi = \frac{\pi}{4}$



e.g. $\theta = 0$



e.g.



Equations

$$\begin{cases} \rho = 3 \\ \phi = \frac{\pi}{2} \end{cases}$$

or

Parametric form

$$\begin{cases} \rho = 3 \\ \theta = t, t \in [0, 2\pi] \\ \phi = \frac{\pi}{2} \end{cases}$$

Topological Terminology in \mathbb{R}^n

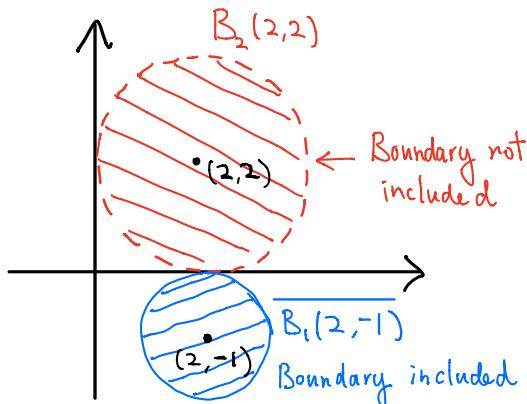
Let $\vec{x}_0 \in \mathbb{R}^n$, $\varepsilon > 0$. Define

$$B_\varepsilon(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < \varepsilon\}$$

= open ball with radius ε
and centered at \vec{x}_0 .

$$\overline{B_\varepsilon(\vec{x}_0)} = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| \leq \varepsilon\}$$

= closed ball with radius ε
and centered at \vec{x}_0 .



Defn Let $S \subseteq \mathbb{R}^n$. Define the following sets :

① The interior of S is the set

$$\text{Int}(S) = \{\vec{x} \in \mathbb{R}^n : B_\varepsilon(x) \subset S \text{ for some } \varepsilon > 0\}$$

Points in $\text{Int}(S)$ are called interior points of S .

② The exterior of S is the set

$$\text{Ext}(S) = \{\vec{x} \in \mathbb{R}^n : B_\varepsilon(x) \subset \mathbb{R}^n \setminus S \text{ for some } \varepsilon > 0\}$$

Points in $\text{Ext}(S)$ are called exterior points of S .

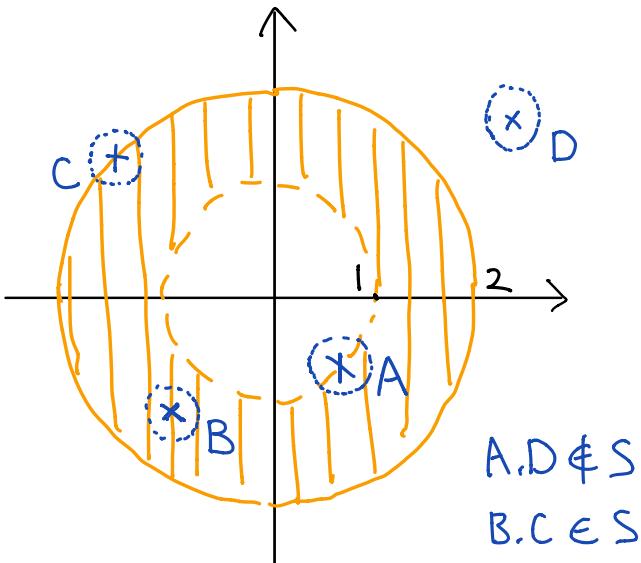
③ The boundary of S is the set

$$\partial S = \left\{ \vec{x} \in \mathbb{R}^n : \begin{array}{l} B_\varepsilon(x) \cap S \neq \emptyset \\ B_\varepsilon(x) \cap \mathbb{R}^n \setminus S \neq \emptyset \end{array} \text{ for any } \varepsilon > 0 \right\}$$

Points in ∂S are called boundary points of S .

eg

$$S = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\} \subseteq \mathbb{R}^2$$



A, C are boundary points of S

B is an interior point of S

D is an exterior point of S

Prop let $S \subseteq \mathbb{R}^n$. Then

① \mathbb{R}^n is the disjoint union of $\text{Int}(S)$, $\text{Ext}(S)$ and ∂S .

② $\text{Int}(S) \subseteq S$, $\text{Ext}(S) \subseteq \mathbb{R}^n \setminus S$

a point in ∂S may or may not be in S

(See A,C in last example)

Defn A subset $S \subseteq \mathbb{R}^n$ is called

① open if $\forall x \in S, \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$

② closed if $\mathbb{R}^n \setminus S$ is open

Equivalent definition A subset $S \subseteq \mathbb{R}^n$ is

① open if $S = \text{Int}(S)$

② closed if $S = \text{Int}(S) \cup \partial S$

Subset $S \subseteq \mathbb{R}^2$	$B_r(0,0)$ $= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$	$\overline{B_r(0,0)}$ $= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$	S' $= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$	\mathbb{R}^2	\emptyset
Int(S)	$B_r(0,0)$	$B_r(0,0)$	\emptyset	\mathbb{R}^2	\emptyset
Ext(S)	$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > r^2\}$ $= \mathbb{R}^2 \setminus \overline{B_r(0,0)}$	$\mathbb{R}^2 \setminus \overline{B_r(0,0)}$	$\mathbb{R}^2 \setminus S'$	\emptyset	\mathbb{R}^2
∂S	S'	S'	S'	\emptyset	\emptyset
Open?	✓	✗	✗	✓	✓
Closed?	✗	✓	✓	✓	✓
Picture					

Rmk

- ① There are exactly two subsets of \mathbb{R}^n which are both open and closed

\mathbb{R}^n and \emptyset

- ② Some subsets of \mathbb{R}^n are neither open or closed

e.g. $\{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\} \subseteq \mathbb{R}^2$

$(0, 1] \subseteq \mathbb{R}$ $\mathbb{Q} \subseteq \mathbb{R}$

- ③ For any $S \subseteq \mathbb{R}^n$

$\text{Int}(S), \text{Ext}(S)$ are open in \mathbb{R}^n

∂S is closed in \mathbb{R}^n

Two more definitions

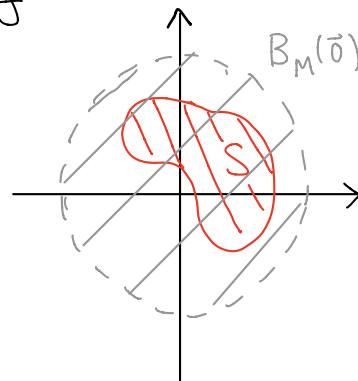
Let $S \subseteq \mathbb{R}^n$ be a subset

- ① S is called bounded if $\exists M > 0$ such that

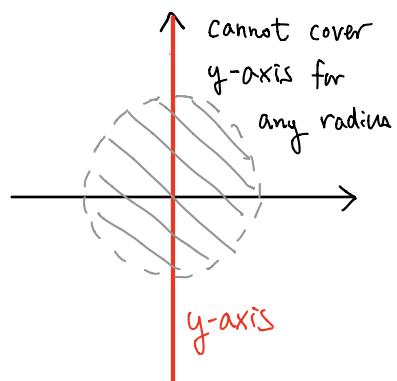
$$S \subseteq B_M(\vec{0}) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < M\}$$

S is called unbounded if S is not bounded

e.g.



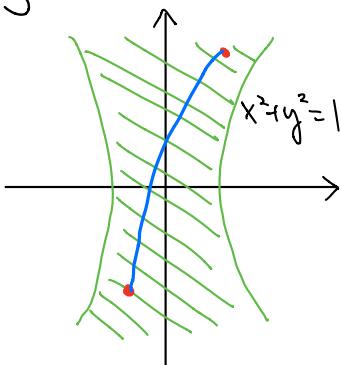
S is bounded



y-axis is unbounded

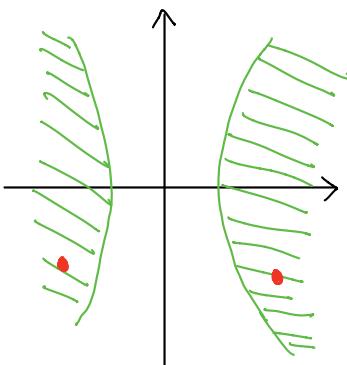
② S is called path-connected if any two points in S can be connected by a curve in S

eg



$$\{(x,y) : x^2 - y^2 \leq 1\}$$

path-connected



$$\{(x,y) : x^2 - y^2 \geq 1\}$$

Not path-connected

Rmk There is also a different notion called connectedness. However, we will not discuss it.

Jordan Curve Theorem

A simple closed curve in \mathbb{R}^2 divides \mathbb{R}^2 into two path-connected components, with one bounded and one unbounded.

eg

