MATH 2010CD Advanced Calculus I

Suggested Solution of Homework 5

Q1.

$$\begin{aligned}
&u = e^{xy}\sin(4x+y)\\
\implies &u_x = ye^{xy}\sin(4x+y) + 4e^{xy}\cos(4x+y)\\
&u_y = xe^{xy}\sin(4x+y) + e^{xy}\cos(4x+y)\\
\implies &u_{xx} = (y^2-16)e^{xy}\sin(4x+y) + 8ye^{xy}\cos(4x+y)\\
&u_{xy} = (xy-3)e^{xy}\sin(4x+y) + (4+y)e^{xy}\cos(4x+y)\\
&u_y = (x^2-1)e^{xy}\sin(4x+y) + 2xe^{xy}\cos(4x+y)
\end{aligned}$$

Q2. By

$$f_x = 2x\cos(\pi y) + \frac{6}{x^2y^2}$$

$$f_y = -\pi x^2\sin(\pi y) + \frac{12}{xy^3}$$

we have

$$f_x(2,-1) = -\frac{5}{2}, \qquad f_y(2,-1) = -6,$$

hence the equation of the tangent plane at $(2,-1)$ is

$$\begin{aligned}
z &= L(x,y)\\
&= f(2,-1) + f_x(2,-1)(x-2) + f_y(2,-1)(y+1)\\
&= -7 - \frac{5}{2}(x-2) - 6(y+1)\\
&= -8 - \frac{5}{2}x - 6y\\
\iff &\ 5x + 12y + 2z + 16 = 0
\end{aligned}$$

Q3. Firstly,

$$f_y(0,0) = \lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0.$$

Then for $(x,y)\neq(0,0)$,

$$f_y(x,y) = \frac{5xy^4(x^4+y^4) - 4xy^8}{(x^4+y^4)^2} = \frac{x^5y^4 + xy^8}{(x^4+y^4)^2} = \frac{xy^4}{x^4+y^4},$$

so

$$f_{yx}(0,0) = \lim_{t\to 0}\frac{f_y(t,0)-f_y(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0.$$

Q4. We have

$$f_x(0,0) = \lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{1-1}{t} = 0,$$

$$f_y(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \to 0} \frac{1-1}{t} = 0,$$

but

$$\begin{aligned} &\lim_{(x,y)\to(0,0), y=3x^2/2} \frac{\varepsilon(x,y)}{\|(x,y)-(0,0)\|} \\ =& \lim_{(x,y)\to(0,0), y=3x^2/2} \frac{f(x,y) - L(x,y)}{\|(x,y)-(0,0)\|} \\ =& \lim_{(x,y)\to(0,0), y=3x^2/2} \frac{0-1}{\|(x,y)-(0,0)\|} = -\infty, \end{aligned}$$

so f isn't differentiable at $(0,0)$.

Q5. (i) We have

$$\begin{aligned} f_x(0,0) &= \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t \to 0} t \sin \frac{1}{t} = 0, \\ f_y(0,0) &= \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t \to 0} t \sin \frac{1}{t} = 0, \end{aligned}$$

and

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon(0,0)}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} \sin \frac{1}{\sqrt{x^2+y^2}} = 0$$

so f is differentiable hence continuous at $(0,0)$.

(ii) For $(x,y) \neq (0,0)$,

$$\begin{aligned} f_x(x,y) &= 2x \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{x}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}, \\ f_y(x,y) &= 2y \sin \frac{1}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} \cos \frac{1}{\sqrt{x^2+y^2}}. \end{aligned}$$

Then by

$$\begin{aligned} \lim_{(x,y)\to(0,0), y=kx} f_x(x,y) &= 0 - \lim_{(x,y)\to(0,0), y=kx} \frac{1}{\sqrt{k^2+1}} \cos \frac{1}{x\sqrt{k^2+1}} \qquad \text{DNE}, \\ \lim_{(x,y)\to(0,0), y=kx} f_y(x,y) &= 0 - \lim_{(x,y)\to(0,0), y=kx} \frac{k}{\sqrt{k^2+1}} \cos \frac{1}{x\sqrt{k^2+1}} \qquad \text{DNE}, \end{aligned}$$

f_x, f_y are not continuous at $(0,0)$.