

Math 2010 Week 5

Finding limit using polar coordinates

$$(x, y) \longleftrightarrow (r, \theta) \text{ with } \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

Note

$$(x, y) \rightarrow (0, 0) \iff r \rightarrow 0$$

eg Find limits using polar coordinates

$$\textcircled{1} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x^2 + y^2}$$

$$= \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta + r^3 \sin^3 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta}$$

$$= \lim_{r \rightarrow 0} r(\cos^3 \theta + \sin^3 \theta)$$

$$= 0 \quad (\text{Squeeze theorem})$$

Details

$$|\cos^3 \theta + \sin^3 \theta| \leq |\cos^3 \theta| + |\sin^3 \theta| \leq 2$$

$$\therefore |r(\cos^3 \theta + \sin^3 \theta)| \leq 2|r|$$

$$\text{Also } \lim_{r \rightarrow 0} 2|r| = 0 \Rightarrow \lim_{r \rightarrow 0} r(\cos^3 \theta + \sin^3 \theta) = 0$$

$$\textcircled{2} \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + xy}{2(x^2 + y^2)}$$

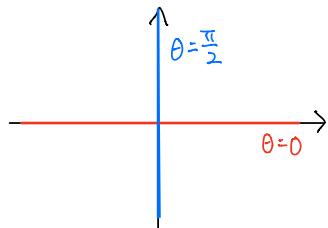
$$= \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta + r^2 \cos \theta \sin \theta}{2r^2}$$

$$= \lim_{r \rightarrow 0} \frac{\cos^2 \theta + \cos \theta \sin \theta}{2}$$

which depends on θ

\therefore limit DNE

$$\text{eg} \quad = \begin{cases} \frac{1}{2} & \text{if } \theta = 0 \\ 0 & \text{if } \theta = \frac{\pi}{2} \end{cases}$$



$$\textcircled{3} \quad \lim_{(x,y) \rightarrow (0,0)} xy \ln(x^2 + y^2)$$

$$= \lim_{r \rightarrow 0} r^2 \cos\theta \sin\theta \ln(r^2)$$

Note $|\cos\theta \sin\theta| \leq 1$,

$r^2 \rightarrow 0, \ln(r^2) \rightarrow -\infty$ as $r \rightarrow 0$

$$\text{i} \quad |r^2 \cos\theta \sin\theta \ln(r^2)| \leq |r^2 \ln(r^2)|$$

$$\text{ii} \quad \lim_{r \rightarrow 0} r^2 \ln(r^2) = \lim_{r \rightarrow 0} \frac{\ln(r^2)}{\frac{1}{r^2}} \quad \left(\frac{-\infty}{\infty} \right)$$

$$= \lim_{r \rightarrow 0} \frac{\frac{2r}{r^2}}{-\frac{2}{r^3}} \quad (\text{L'Hopital's Rule})$$

$$= \lim_{r \rightarrow 0} -r^2 = 0$$

By Squeeze thm, $\lim_{(x,y) \rightarrow (0,0)} xy \ln(x^2 + y^2) = 0$

Iterated Limit (Au 2.7)

$$\textcircled{1} \quad \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y) = \lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x,y) \right)$$

means taking limit with $y \rightarrow 0$ first,
followed by taking limit with $x \rightarrow 0$

$$\textcircled{2} \quad \text{Similar for } \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y)$$

Q Are they equal?

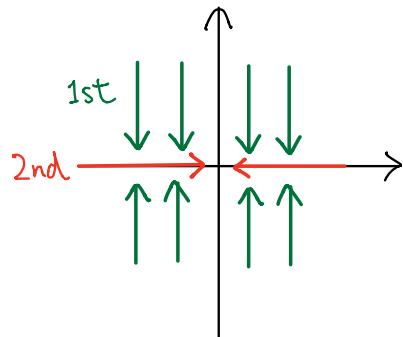
$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y) \stackrel{?}{=} \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y)$$

$$\stackrel{?}{=} \lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

Eg Consider $f(x,y) = \frac{x+y}{x-y}$.

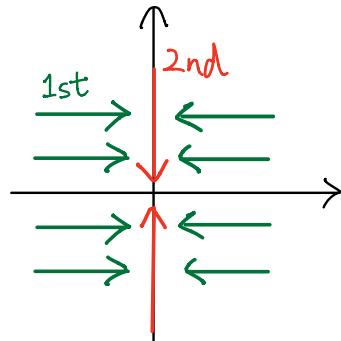
x is fixed, like a constant

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x+y}{x-y} = \lim_{x \rightarrow 0} \frac{x+0}{x-0} = 1$$

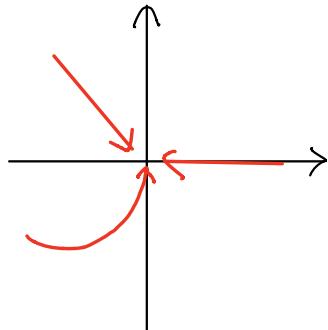


Not unequal

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x+y}{x-y} = \lim_{y \rightarrow 0} \frac{0+y}{0-y} = -1$$



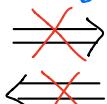
$$\lim_{(x,y) \rightarrow (0,0)} \frac{x+y}{x-y} \text{ DNE}$$



Rmk

① $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y) = \lim_{y \rightarrow 0} \lim_{x \rightarrow 0} f(x,y)$

(Both exist and equal)



$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) \text{ exists}$$

eg. $f(x,y) = \begin{cases} x \cos \frac{1}{y} + y \cos \frac{1}{x} & \text{if } x,y \neq 0 \\ 0 & \text{if } x,y=0 \end{cases}$

② If all the three limits exist, they are equal

Continuity

let $f: A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $\vec{a} \in A$.

Defn 1 (ε - δ) f is said to be continuous at \vec{a} if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $x \in A$ and $\|\vec{x} - \vec{a}\| < \delta$ then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$

Another equivalent definition

Defn 2 f is said to be continuous at \vec{a} if $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})$ exists and equals to $f(\vec{a})$

Defn f is said to be continuous if f is continuous at any point in A .

Eg let $1 \leq k \leq n$. Show $f(x_1, x_2, \dots, x_n) = x_k$ is continuous on \mathbb{R}^n

Pf let $\vec{a} = (a_1, a_2, \dots, a_n)$.

For any $\varepsilon > 0$, take $\delta = \varepsilon$. If $\|\vec{x} - \vec{a}\| < \delta$, then

$$|f(\vec{x}) - f(\vec{a})| = |x_k - a_k| \leq \sqrt{\sum_{i=1}^n (x_i - a_i)^2} = \|\vec{x} - \vec{a}\| < \delta = \varepsilon$$

$$\therefore \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = f(\vec{a}) \text{ for any } \vec{a} \in \mathbb{R}^n$$

$\therefore f$ is a continuous function

Thm If $f, g: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous at \vec{a} , then

① $f(x) \pm g(x)$, $kf(x)$, $f(x)g(x)$ are continuous at \vec{a} ,

where k is a constant

③ $\frac{f(x)}{g(x)}$ is continuous at \vec{a} if $g(\vec{a}) \neq 0$

Pf Follow easily from corresponding properties of limit

From the last 2 results, we have

All polynomials and rational functions (i.e. polynomial/polynomial)
of multi-variables are continuous

e.g. ① $x^3 + 3yz + z^2 - x + 7y$ is continuous on \mathbb{R}^3

② $\frac{x^3 + y^2 + yz}{x^2 + y^2}$ is continuous on

$$\mathbb{R}^3 \setminus \{(x,y,z) : x^2 + y^2 = 0\} = \mathbb{R}^3 \setminus \{z\text{-axis}\}$$

let $Q(\vec{x}) = \frac{p_1(\vec{x})}{p_2(\vec{x})}$ be a rational function

$\vec{\alpha}$ be a zero of $p_2(\vec{x})$ (i.e. $p_2(\vec{\alpha}) = 0$). Then

$Q(\vec{x})$ can be extended to $\Leftrightarrow \lim_{\vec{x} \rightarrow \vec{\alpha}} Q(\vec{x})$
a function continuous at $\vec{\alpha}$ exists

e.g. ① $f(x,y) = \frac{xy + y^3}{x^2 + y^2}$

Note $x^2 + y^2 = 0 \Leftrightarrow (x,y) = (0,0)$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ y=mx}} f(x,y) = \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=mx}} \frac{xy + y^3}{x^2 + y^2}$$

$$= \lim_{x \rightarrow 0} \frac{mx^2 + m^3x^3}{x^2 + m^2x^2}$$

$$= \lim_{x \rightarrow 0} \frac{m + m^3x}{1 + m^2}$$

$$= \frac{m}{1+m^2} = \begin{cases} 0 & \text{if } m=0 \\ \frac{1}{1+m^2} & \text{if } m \neq 0 \end{cases}$$

\therefore Limit various with slope
not equal

$$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y) \text{ DNE}$$

$\therefore f$ cannot be extended to a

continuous function defined on \mathbb{R}^2

$$\textcircled{2} \quad g(x,y) = \frac{x^4 - y^4}{x^2 + y^2}$$

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} g(x,y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^4}{x^2 + y^2} \\&= \lim_{r \rightarrow 0} \frac{r^4 \cos^4 \theta - r^4 \sin^4 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \\&= \lim_{r \rightarrow 0} r^2 (\cos^4 \theta - \sin^4 \theta) \\&= 0 \quad (\text{Sandwich theorem})\end{aligned}$$

Details: $|r^2(\cos^4 \theta - \sin^4 \theta)| \leq 2r^2$,

$$\text{and } \lim_{r \rightarrow 0} 2r^2 = 0$$

$\therefore g(x,y)$ can be extended to a continuous function on whole \mathbb{R}^2 with

$$g(x,y) = \begin{cases} \frac{x^4 - y^4}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Thm If $f: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous at \vec{a} , $g(x)$ is a 1-variable function continuous at $f(\vec{a})$, then $g \circ f(\vec{x})$ is continuous at \vec{a} . Hence

$$\lim_{\vec{x} \rightarrow \vec{a}} g(f(\vec{x})) = g\left(\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})\right) = g(f(\vec{a}))$$

e.g. $g(x) = |x|$, $f(x_1, \dots, x_n) = x_k$ are continuous for $1 \leq k \leq n$

$\Rightarrow g \circ f(x_1, \dots, x_n) = |x_k|$ is a continuous function

e.g. $g(x) = |\ln x|$ is continuous on $(0, +\infty)$

$\Rightarrow g \circ f(x_1, \dots, x_n) = |\ln x_k|$ is continuous on $\{\vec{x} \in \mathbb{R}^n : x_k > 0\}$

Rmk In particular, $\lim_{(x,y) \rightarrow (0,0)} |x| = 0$, $\lim_{(x,y) \rightarrow (1,0)} |\ln x| = 0$

(We used these results in examples of sandwich thm.)

e.g. $\sin(x^2 + yz)$, e^{x-y} , $\cos(\frac{1}{x^2 + y^2})$, $r = \sqrt{x^2 + y^2}$
are continuous (on their domains)

Partial Derivatives

(An 3.2
Thomas 14.3)

Study the rate of change of a function with respect to each variable

Defn Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$, \mathbb{R} open

Then for $i=1, 2, \dots, n$, define the

i -th partial derivative of f at $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}$

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(x_1, \dots, x_n)$$

$$= \lim_{h \rightarrow 0} \frac{f(x_1, \dots, \cancel{x_i+h}, \dots, x_n) - f(x_1, \dots, \cancel{x_i}, \dots, x_n)}{h}$$

$$\text{eg } \frac{\partial f}{\partial x}(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y}(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Other notations:

$$\frac{\partial f}{\partial x} = \partial_1 f = D_1 f = \nabla_1 f = f_x$$

$$\frac{\partial f}{\partial y} = \partial_2 f = D_2 f = \nabla_2 f = f_y$$

eg $f(x, y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = 2x + 0 = 2x \quad (\text{Regard } y \text{ as a constant})$$

$$\frac{\partial f}{\partial y} = 0 + 2y = 2y \quad (\text{Regard } x \text{ as a constant})$$

Note $\frac{\partial f}{\partial x}(1, -1) = 2(1) = 2 > 0$

$$\frac{\partial f}{\partial y}(1, -1) = 2(-1) = -2 < 0$$

$\therefore f$ increases as x increases at $(1, -1)$
decreases as y increases at $(1, -1)$

Rmk Note $f(x, y) = \|(x, y)\|^2$



move closer to origin (f decreases)
 $(1, -1)$ move away from origin (f increases)

e.g. $f(x,y,z) = xy^2 - \cos(xz)$

Then $f_x = y^2 + z \sin(xz)$

$$f_y = 2xy + 0 = 2xy$$

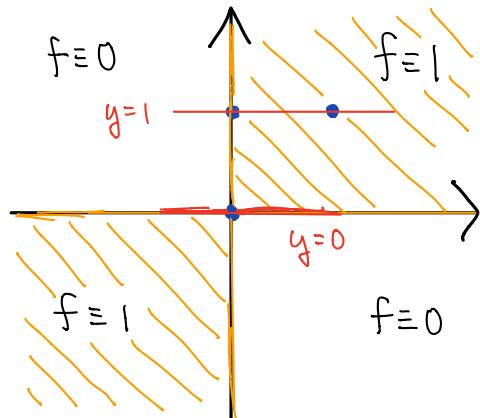
$$f_z = 0 + x \sin(xz) = x \sin(xz)$$

e.g.

$$f(x,y) = \begin{cases} 1 & \text{if } xy \geq 0 \\ 0 & \text{if } xy < 0 \end{cases}$$

Find $\frac{\partial f}{\partial x}(1,1), \frac{\partial f}{\partial x}(0,1), \frac{\partial f}{\partial x}(0,0)$

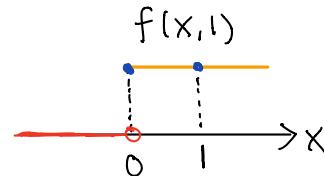
Sol



$\frac{\partial f}{\partial x}$: Fix y , differentiate $f(x,y)$ w.r.t. x

Along $y=1$

$$f(x,1) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$



$$\Rightarrow \frac{\partial f}{\partial x}(1,1) = 0, \quad \frac{\partial f}{\partial x}(0,1) \text{ DNE}$$

Along $y=0$

$$f(x,0) \equiv 1, \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \frac{\partial f}{\partial x}(0,0) = 0$$

Rmk Similarly $\frac{\partial f}{\partial y}(0,0) = 0$

Note f is not continuous at $(0,0)$

$\therefore \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at $(0,0)$ $\cancel{\Rightarrow} f$ is continuous at $(0,0)$

Higher order partial derivative

e.g Consider $f(x,y)$

1st order derivative : $\frac{\partial f}{\partial x} = f_x, \frac{\partial f}{\partial y} = f_y$

2nd order derivatives : $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = (f_x)_x = f_{xx}$

Differentiate w.r.t. x first $\rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = (f_x)_y = f_{xy}$

y first $\rightarrow \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ and $\frac{\partial^2 f}{\partial y^2} = f_{yy}$

3rd order derivatives : $\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \right] = [(f_y)_y]_x = f_{yyy}$
 $\frac{\partial^3 f}{\partial x^3} = f_{xxx}$

Others : $f_{xyy}, f_{yxy}, f_{yyy}, f_{xxy}, f_{xyx}, f_{yxx}$

Rmk Similar for higher order / more variables

e.g f_{xxxx}, g_{xyz} for $g(x,y,z)$

e.g $f(x,y) = x \sin y + y^2 e^{2x}$

Find all 1st, 2nd order partial derivatives

Sol

$$\begin{aligned} f_x &= \sin y + 2y^2 e^{2x} \\ f_y &= x \cos y + 2ye^{2x} \\ f_{xx} &= (f_x)_x = 4y^2 e^{2x} \\ f_{xy} &= (f_x)_y = \cos y + 4ye^{2x} \quad \text{equal?} \\ f_{yx} &= (f_y)_x = \cos y + 4ye^{2x} \quad \text{Coincidence?} \\ f_{yy} &= (f_y)_y = -x \sin y + 2e^{2x} \end{aligned}$$

Q Is it always true that $f_{xy} = f_{yx}$?

A No, a counter example below:

e.g. Compute $f_{xy}(0,0)$, $f_{yx}(0,0)$, where

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

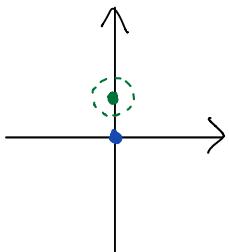
Sol $f_{xy} = (f_x)_y$

$$\Rightarrow f_{xy}(0,0) = \lim_{k \rightarrow 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

Need to find $f_x(0,k)$ for $k \neq 0$ and $f_x(0,0)$

For $k \neq 0$,

$$f = \frac{xy(x^2-y^2)}{x^2+y^2} \text{ near } (0,k)$$



$$f_x = \frac{(x^2+y^2)(3x^2y-y^3)-xy(x^2-y^2)(2x)}{(x^2+y^2)^2} \text{ near } (0,k)$$

$$\Rightarrow f_x(0,k) = \frac{k^2(-k^3) - 0}{k^4} = -k$$

$$\begin{aligned} f_x(0,0) &= \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0 \end{aligned}$$

$$\begin{aligned} f_{xy}(0,0) &= \lim_{k \rightarrow 0} \frac{f_x(0,k) - f_x(0,0)}{k} \\ &= \lim_{k \rightarrow 0} \frac{-k - 0}{k} = -1 \end{aligned}$$

Similar calculation gives $f_{yx}(0,0) = 1$

$$\left. \begin{aligned} &\text{Alternatively, note } f(x,y) = -f(y,x) \\ &\Rightarrow f_{yx}(0,0) = -f_{xy}(0,0) = -1 \end{aligned} \right\}$$

In this example, $f_{xy}(0,0) \neq f_{yx}(0,0)$

Q When do $f_{xy} = f_{yx}$?

Thm (Clairaut's Theorem / Mixed Derivative Thm)

Let $\Omega \subseteq \mathbb{R}^2$ be an open set, $f: \Omega \rightarrow \mathbb{R}$

If f_{xy}, f_{yx} exist and are continuous on Ω ,

then $f_{xy} = f_{yx}$ on Ω

Rmk We will prove a stronger version:

let $\Omega \subseteq \mathbb{R}^2$ be open, $f: \Omega \rightarrow \mathbb{R}$, $\vec{a} \in \Omega$. Suppose

① f_{xy}, f_{yx} exist in an open ball containing \vec{a}

② f_{xy}, f_{yx} are continuous at \vec{a} .

Then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$

f_{xy}, f_{yx} exist on $B_\epsilon(\vec{a})$ for some $\epsilon > 0$

f_{xy}, f_{yx} are continuous at \vec{a}

Tool for the proof:

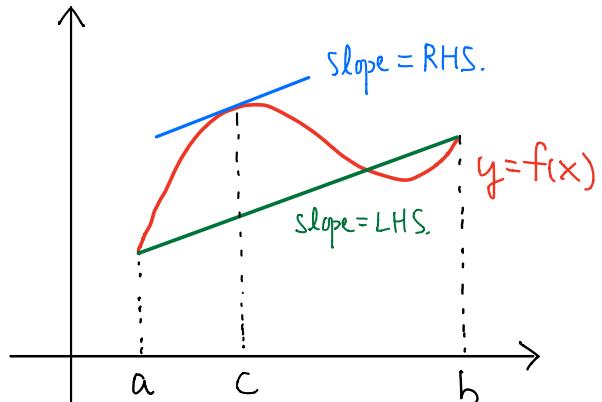
Mean Value Theorem for 1-variable function

Let $f: [a, b] \rightarrow \mathbb{R}$, continuous on $[a, b]$

differentiable on (a, b)

Then $\exists c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



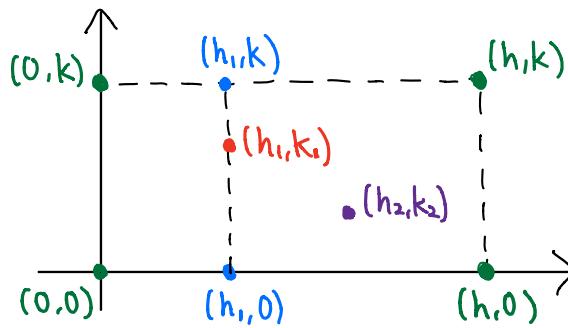
Pf of Clairaut's Theorem

We may assume $\vec{a} = (0, 0) \in S$

Need to show $f_{xy}(0,0) = f_{yx}(0,0)$

Let $h, k > 0$ and $[0, h] \times [0, k] \subseteq S$

$$\alpha = f(h, k) - f(0, k) - f(h, 0) + f(0, 0)$$



$$\text{Let } g(x) = f(x, k) - f(x, 0), \quad 0 \leq x \leq h$$

$$\text{Then } \alpha = g(h) - g(0)$$

$$g'(x) = f_x(x, k) - f_x(x, 0)$$

MVT $\Rightarrow \exists h_i \in (0, h)$ such that

$$\frac{\alpha}{h} = \frac{g(h) - g(0)}{h} = g'(h_i) = f_x(h_i, k) - f_x(h_i, 0)$$

MVT again $\Rightarrow \exists k_i \in (0, k)$ such that

$$\frac{f_x(h_i, k) - f_x(h_i, 0)}{k} = f_{xy}(h_i, k_i) \quad \text{like a MVT for } f(x, y)$$

$$\therefore \alpha = h \left[f_x(h_i, k) - f_x(h_i, 0) \right] = h k f_{xy}(h_i, k_i)$$

Similarly, $\exists (h_2, k_2) \in (0, h) \times (0, k)$ such that

$$\alpha = h k f_{yx}(h_2, k_2)$$

$$\therefore f_{xy}(h_i, k_i) = f_{yx}(h_2, k_2) \quad \textcircled{*}$$

Take $h, k \rightarrow 0^+$, then $(h_i, k_i), (h_2, k_2) \rightarrow (0, 0)$

$\textcircled{*}$ and continuity of f_{xy}, f_{yx} at $(0, 0)$

$$\Rightarrow f_{xy}(0,0) = f_{yx}(0,0)$$