MATH2010 Advanced Calculus I, Term 2

Midterm Examination

Suggested Solution

Solution 1. (a) Open but not closed.

(b) Closed but not open.

(c) Neither open nor closed.

Solution 2. Given the plane equation $x+y+z=2$, the normal vector to this plane is $\mathbf{n} = \langle 1, 1, 1\rangle$.

Given the line $x = t$, $y = 1-t$, $z = 2+2t$, its direction vector is $\mathbf{u} = \langle 1, -1, 2\rangle$.

The direction vector $\mathbf{v}$ of the required line is the cross product of $\mathbf{n}$ and $\mathbf{u}$:

$$\mathbf{v} = \mathbf{n} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -1, -2\rangle.$$

The parametric equation of the line through the point $(0, 1, 2)$ with direction vector $\mathbf{v}$ is:

$$\begin{cases} x(t) = 3t, \\ y(t) = 1 - t, \\ z(t) = 2 - 2t. \end{cases}$$

Solution 3. We parameterize C in terms of t:

$$\begin{cases} x(t) = t, \\ y(t) = \frac{t^2}{2}, \\ z(t) = \frac{t^3}{6}. \end{cases}$$

The length of C from $(0, 0, 0)$ to $(6, 18, 36)$ is:

$$L = \int_0^6 \sqrt{1 + t^2 + \frac{t^4}{4}}\, dt = 42.$$

Solution 4. (a)

$$\begin{aligned} \lim_{(x,y)\to(2,2)} \frac{xy-4}{\sqrt{xy}-2} &= \lim_{(x,y)\to(2,2)} \frac{(xy-4)(\sqrt{xy}+2)}{xy-4} \\ &= \lim_{(x,y)\to(2,2)} \sqrt{xy}+2 \\ &= 4 \end{aligned}$$

(b) The limit does not exist.

Along the line $y = 0$, we have:

$$\lim_{(x,0)\to(0,0)} \frac{x^2 \cdot 0}{x^4 + 0^2} = 0$$

Along the line $y = x^2$, we have:

$$\lim_{(x,x^2)\to(0,0)} \frac{x^2 \cdot x^2}{x^4 + x^4} = \frac{1}{2}$$

Since the limits along the two paths are different, the limit of the function as (x, y) approaches $(0, 0)$ does not exist.

(c) Since $|\sin(z)| \leq |z|$ for all $z \in \mathbb{R}$, we have

$$0 \leq \left|\frac{\sin(x^2 y)}{x^2 + y^2}\right| \leq \frac{|x^2 y|}{x^2 + y^2}.$$

As $(x, y) \to (0, 0)$, the right-hand side of the inequality approaches 0. By the squeeze theorem, we deduce that

$$\lim_{(x,y)\to(0,0)} \left|\frac{\sin(x^2 y)}{x^2 + y^2}\right| = 0.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 y)}{x^2 + y^2} = 0.$$

Solution 5. To show that the function $f(x, y) = x - y$ is continuous at the point $(1, 1)$ using the $\epsilon - \delta$ definition, we need to prove that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all points (x, y), if $\sqrt{(x-1)^2 + (y-1)^2} < \delta$, then $|f(x, y) - f(1, 1)| \leq \epsilon$.

Given any $\epsilon > 0$, we pick $\delta = \frac{1}{2}\epsilon$. Then

$$0 < \sqrt{(x-1)^2 + (y-1)^2} < \delta \implies |x-1| \leq \delta = \epsilon/2, \quad |y-1| \leq \delta = \epsilon/2.$$

It follows that

$$\begin{aligned}
|f(x, y) - f(1, 1)| &= |(x - y) - (1 - 1)| \\
&= |(x - 1) - (y - 1)| \\
&\leq |x - 1| + |y - 1| \\
&\leq \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

This proves that $f(x, y) = x - y$ is continuous at $(1, 1)$.

Solution 6.

$$f_y(0, 0) = \lim_{h\to 0} \frac{(0^3 + (0 + h)^3)^{\frac{1}{3}} - 0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

Hence, $f_y(0, 0) = 1$.

Solution 7. (a)

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{0}{h} = 0.$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h^m}{h(h^2+0^2)^{\frac{n}{2}}} = \lim_{h\to 0} \frac{h^{m-1}}{|h|^n}.$$

Case 1: $m = n+1$ and n is even

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{h^{m-1}}{|h|^n} = 1.$$

Case 2: $m = n+1$ and n is odd

$$\lim_{h\to 0^+} \frac{h^{m-1}}{|h^n|} = 1, \quad \lim_{h\to 0^-} \frac{h^{m-1}}{-h^n} = -1.$$

Therefore, $\frac{\partial f}{\partial x}(0,0)$ does not exist.

Case 3: $m \geq n+2$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{h^{m-1}}{|h|^n} = 0.$$

To conclude, we have

$$\frac{\partial f}{\partial x}(0,0) = \begin{cases} 1, & \text{if } m = n+1 \text{ and } n \text{ is even} \\ DNE, & \text{if } m = n+1 \text{ and } n \text{ is odd} \\ 0, & \text{if } m \geq n+2. \end{cases}$$

(b)

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^m}{(x^2+y^2)^{\frac{n}{2}}} &= \lim_{r\to 0} \frac{(r\cos\theta)^m}{r^n} \\ &= \lim_{r\to 0} r^{m-n}(\cos\theta)^m \end{aligned}$$

Since $(\cos\theta)^m$ is bounded between -1 and 1, the function will be continuous at the origin for the smallest integer value of $m-n$ that is greater than 0, which is 1.

(c) Suppose that $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist, then the linearization $L(x,y)$ of the function $f(x,y)$ at $(0,0)$ is defined as

$$L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y.$$

To show that f is differentiable at $(0,0)$, it is equivalent to show that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-L(x,y)}{\sqrt{x^2+y^2}} = 0.$$

It follows from (a) that the linearization $L(x, y)$ of the function $f(x, y)$ at $(0, 0)$ is

$$L(x,y) = \begin{cases} x, & \text{if } m = n+1 \text{ and } n \text{ is even} \\ DNE, & \text{if } m = n+1 \text{ and } n \text{ is odd} \\ 0, & \text{if } m \geq n+2. \end{cases}$$

Since $L(x, y)$ does not exist when $m = n + 1$ for odd n, we consider the remaining cases where $L(x, y)$ exists to determine the differentiability of $f(x, y)$ at $(0, 0)$.

Case 1: $m = n + 1$ and n is even

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{r^{m-n}(\cos\theta)^m - r\cos\theta}{r} \\ &= (\cos\theta)^m - \cos\theta. \end{aligned}$$

Since $(\cos\theta)^m - \cos\theta$ can take on different values for different θ (and is not a constant), this implies that the limit does not exist. Therefore, f is not differentiable at $(0, 0)$.

Case 3: $m \geq n + 2$

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} &= \lim_{r\to 0} \frac{r^{m-n}(\cos\theta)^m - 0}{r} \\ &= \lim_{r\to 0} r^{m-n-1}(\cos\theta)^m \\ &= 0. \end{aligned}$$

This implies that f is differentiable at $(0, 0)$.

We conclude that the smallest possible value of $m - n$ for $f(x, y)$ to be differentiable at $(0, 0)$ is 2.