MATH2010 Advanced Calculus I, 2023-2024 Term 2

Solution to HOMEWORK 3

Q1

Solution. Note that $\mathbf{r}(t) = (\sqrt{2}\cos t, \sqrt{2}\sin t, t)$ and so $\mathbf{r}\left(\frac{\pi}{4}\right) = \left(1, 1, \frac{\pi}{4}\right)$. Also, we have $\mathbf{r}'(t) = (-\sqrt{2}\sin t, \sqrt{2}\cos t, 1)$ and hence $\mathbf{r}'\left(\frac{\pi}{4}\right) = (-1, 1, 1)$. Therefore, the tangent line to $\mathbf{r}(t)$ at the point where $t = \frac{\pi}{4}$ is

$$\mathbf{T}(s) = \mathbf{r}\left(\frac{\pi}{4}\right) + s\mathbf{r}'\left(\frac{\pi}{4}\right) = \left(1 - s,\ 1 + s,\ \frac{\pi}{4} + s\right),$$

where $s \in \mathbb{R}$.

□

Q2

Solution. Since $\mathbf{u}$ is a unit vector, we have $\|\mathbf{u}\| = 1$. Therefore, the arc length of the curve $\mathbf{r}(t)$ is

$$\int_0^1 \|\mathbf{r}'(t)\| dt = \int_0^1 \|2t\mathbf{u}\| dt = \int_0^1 2t\|\mathbf{u}\| dt = \int_0^1 2t dt = [t^2]_0^1 = 1.$$

□

Q3

Solution. From $r = \dfrac{k}{1 + e\cos\theta}$, we have

$$r + er\cos\theta = k.$$

Now, since $x = r\cos\theta$ and $y = r\sin\theta$, we have $x^2 + y^2 = r^2$. Therefore, we have

$$\begin{aligned}
\sqrt{x^2 + y^2} + ex &= k \\
x^2 + y^2 &= (k - ex)^2 \\
x^2 + y^2 &= k^2 - 2kex + e^2x^2 \\
(1 - e^2)x^2 + y^2 + 2kex - k^2 &= 0.
\end{aligned}$$

□

Q4

Solution.

(i) From $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$ and $z = \rho\cos\varphi$, we have $x^2 + y^2 + z^2 = \rho^2$. Now, we have

$$\begin{aligned}
\rho\cos\varphi\sin\varphi\sin\theta &= 3 \\
(\rho\cos\varphi)(\rho\sin\varphi\sin\theta) &= 3\rho \\
zy &= 3\sqrt{x^2 + y^2 + z^2} \\
9x^2 + 9y^2 + 9z^2 - y^2z^2 &= 0.
\end{aligned}$$

(ii) From $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$ and $z = \rho \cos\varphi$, we have $x^2 + y^2 + z^2 = \rho^2$. Now, we have

$$\rho - \cos\varphi = 2 + \cos^2\varphi$$

$$\rho - \frac{z}{\rho} = 2 + \frac{z^2}{\rho^2}$$

$$\rho^3 - z\rho = 2\rho^2 + z^2$$

$$\sqrt{x^2 + y^2 + z^2}(x^2 + y^2 + z^2 - z) = 2x^2 + 2y^2 + 3z^2.$$

□

Q5

Solution.

(i) Not open; Closed; Bounded; Connected

(ii) Not open; Closed; Not bounded; Not connected

(iii) Open; Not Closed; Not bounded; Connected

□