

Math 2010 Week 10

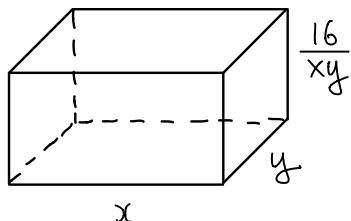
Another example of extrema on unbounded region

e.g. Make a box (without top) with volume = 16

Cost:

Base \$2/unit area

Side \$0.5/unit area



Q How to minimize cost?

Sol Want to minimize

$$\begin{aligned} C(x,y) &= 2xy + \left(\frac{16}{xy}x + \frac{16}{xy}y \right) (2)(0.5) \\ &= 2xy + \frac{16}{x} + \frac{16}{y} \end{aligned}$$

on the domain $\mathcal{S}\mathcal{L} = \{(x,y) \in \mathbb{R}^2 : x, y > 0\}$

- $\mathcal{S}\mathcal{L}$ is neither closed nor bounded.
- \therefore EVT cannot be applied directly
- C is large if x or y is small or large.

Strategy: Find a rectangle R s.t.
 $C > \min. \text{ of } C|_R$ on ∂R and outside R .

Step 1 Find critical points

$$\nabla C = \left(2y - \frac{16}{x^2}, 2x - \frac{16}{y^2} \right) \text{ exists everywhere}$$

$$\nabla C = \vec{0} \iff \begin{cases} 2y - \frac{16}{x^2} = 0 \\ 2x - \frac{16}{y^2} = 0 \end{cases}$$

$$\therefore y = \frac{8}{x^2}, x = \frac{8}{y^2} = \frac{8}{\frac{64}{x^4}} = \frac{x^4}{8}, x > 0$$

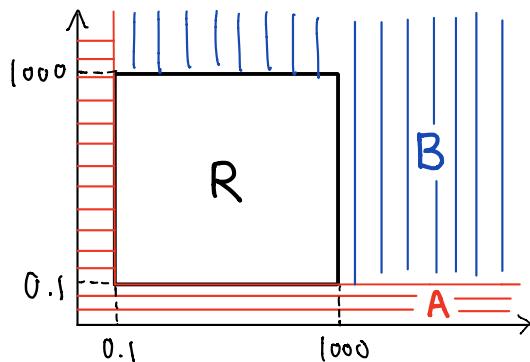
$$\Rightarrow x^3 = 8, x = 2, y = 2$$

\therefore Only one critical point $(2,2)$, $C(2,2) = 24$

Step 2 Choose R s.t. $C > 24$ on ∂R and outside R .

$$C(x,y) = 2xy + \frac{16}{x} + \frac{16}{y}$$

One possible choice: $R = [0.1, 1000] \times [0.1, 1000]$



(A) If $x \leq 0.1$ or $y \leq 0.1$

$$\text{then } C > \frac{16}{X} + \frac{16}{Y} > \frac{16}{0.1} = 160 > 24$$

(B) If $(x \geq 1000, y \geq 1000)$ or $(y \geq 1000, x \geq 1000)$,

$$\text{then } C > 2(0.1)(1000) = 200 > 24$$

Step 3 Analysis

- R is closed and bounded, C is continuous

By EVT, $C|_R$ has minimum

- C has only one critical point $(2,2) \in \mathcal{L}$
 $(2,2) \in \text{int}(R)$, $C(2,2) = 24$

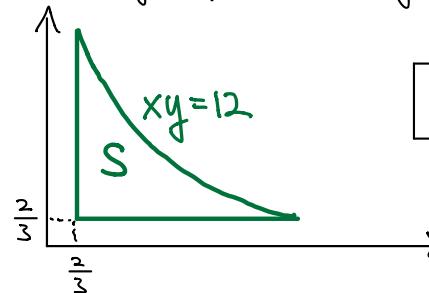
$C > 24$ on ∂R

$\Rightarrow C|_R$ has min value 24 at $(2,2)$

- $C > 24$ outside R

$\Rightarrow C$ has min value 24 at $(2,2)$ on \mathcal{L}

Rmk One may replace R by S below.



Exercise

Taylor Series Expansion (Thomas 14.9)

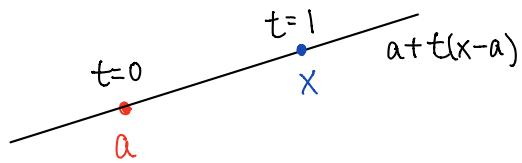
Recall Taylor expansion for 1-variable function $g(t)$ at $t=0$ up to order k

$$g(t) = g(0) + g'(0)t + \frac{1}{2!}g''(0)t^2 + \dots$$

$$+ \frac{1}{k!}g^{(k)}(0)t^k + \text{remainder } \textcircled{*}$$

We want a similar formula for a multi-variable function $f(x)$ defined near a , where $x = (x_1, \dots, x_n)$ $a = (a_1, \dots, a_n)$

Let $g(t) = f(a + t(x-a))$



If $\|x-a\|$ is small, then for $|t| \leq 1$,

$\|t(x-a)\| = |t|\|x-a\| \leq \|x-a\|$ is small
and $g(t)$ is defined

By $\textcircled{*}$,

$$f(a + t(x-a)) = g(0) + g'(0)t + \frac{1}{2!}g''(0)t^2 + \dots$$

$$+ \frac{1}{k!}g^{(k)}(0)t^k + \text{remainder}$$

Put $t=1$,

$$f(x) = g(0) + g'(0) + \frac{1}{2!}g''(0) + \dots + \frac{1}{k!}g^{(k)}(0)$$

$$+ \text{remainder}$$

Next, express $g^{(k)}(0)$ in terms of f :

$$g(0) = f(a + t(x-a)) = f(a)$$

$$g'(t) = \nabla f(a + t(x-a)) \cdot \frac{d}{dt}(a + t(x-a))$$

$$= \nabla f(a + t(x-a)) \cdot (x-a)$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a + t(x-a))(x_i - a_i)$$

$$\Rightarrow g'(0) = \nabla f(a) \cdot (x-a)$$

$$= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

$$g''(t) = \frac{d}{dt} g'(t)$$

$$= \sum_{i=1}^n \frac{d}{dt} \left[\frac{\partial f}{\partial x_i}(a + t(x-a))(x_i - a_i) \right]$$

$$= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(a + t(x-a))(x_j - a_j)(x_i - a_i)$$

$$g''(0) = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j \partial x_i}(a)(x_j - a_j)(x_i - a_i)$$

\therefore Taylor Expansion at a up to order 2 is

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

$$+ \frac{1}{2!} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + \text{remainder}$$

e.g If $n=2$, i.e. $f = f(x, y)$, $a = (x_0, y_0)$

f is C^2 (so $f_{xy} = f_{yx}$), then

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$+ \frac{1}{2} \left[f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 \right]$$

+ remainder.

Similarly, the general term is

$$g^{(k)}(0) = \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k})$$

Thm (Taylor Theorem)

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be open, $f: \mathcal{S} \rightarrow \mathbb{R}$ be C^k .

Then for any $x, a \in \mathcal{S}$,

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2!} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + \dots$$

$$+ \frac{1}{k!} \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k}) + \varepsilon_k(x, a)$$

with $\lim_{x \rightarrow a} \frac{\varepsilon_k(x, a)}{\|x - a\|^k} = 0$

Defn

$$P_k(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \dots$$

$$+ \frac{1}{k!} \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k})$$

is called the k -th order Taylor polynomial of f at a

Rmk

① $P_1(x) = L(x)$

= Linearization of f at a

② P_k and f have equal partial derivatives up to order K at a

$$\underline{\text{Ex}} \quad f(x,y) = e^x \cos y$$

Find 2nd order Taylor polynomial at $a=(0,0)$

$$\underline{\text{Sol}} \quad f_x = e^x \cos y \quad f_y = -e^x \sin y$$

$$f_{xx} = e^x \cos y \quad f_{yx} = -e^x \sin y$$

$$f_{xy} = -e^x \sin y \quad f_{yy} = -e^x \cos y$$

$$\Rightarrow f(0,0) = 1,$$

$$f_x(0,0) = 1, \quad f_y(0,0) = 0$$

$$f_{xx}(0,0) = 1 \quad f_{yy}(0,0) = -1$$

$$f_{xy}(0,0) = f_{yx}(0,0) = 0$$

$$P_2(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y$$

$$+ \frac{1}{2!} (f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2)$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2$$

How about $P_3(x,y)$ at $(0,0)$?

$$P_3(x,y) = P_2(x,y) + \underbrace{\frac{1}{3!} g^{(3)}(0)}_{\text{3rd order terms}}$$

$$f_{xxx} = e^x \cos y \quad \text{3rd order terms}$$

$$f_{xxy} = f_{xyx} = f_{yyx} = -e^x \sin y$$

$$f_{xyy} = f_{yxy} = f_{ygy} = -e^x \cos y$$

$$f_{yyy} = e^x \sin y$$

$$\Rightarrow f_{xxx}(0,0) = 1 \quad f_{xxy}(0,0) = 0$$

$$f_{xyy}(0,0) = -1 \quad f_{yyy}(0,0) = 0$$

$$\begin{aligned} g^{(3)}(0) &= f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y \\ &\quad + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3 \\ &= x^3 - 3xy^2 \end{aligned}$$

$$P_3(x,y) = P_2(x,y) + \frac{1}{3!} (x^3 - 3xy^2)$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{6}x^3 - \frac{1}{2}xy^2$$

Q If $f = f(x, y, z)$ is C^6 , then

Coefficient of xy^2z^3 in $P_6(x, y, z)$ at $(0, 0, 0)$

is $\alpha f_{x,y,z,z,z}(0, 0, 0)$. $\alpha = ?$

Rmk A General Taylor's formula for $f(x, y)$ is given on P822 of Thomas' Calculus.

Matrix form for 2nd order Taylor Polynomial

Defn Let $S \subseteq \mathbb{R}^n$ be open, $f: S \rightarrow \mathbb{R}$ be C^2 .

Then the Hessian matrix of f at $a \in S$ is

$$Hf(a) = \begin{bmatrix} f_{x_1 x_1}(a) & \cdots & f_{x_1 x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(a) & \cdots & f_{x_n x_n}(a) \end{bmatrix}$$

Rmk

① $Hf(a)$ is a symmetric $n \times n$ matrix by the mixed derivatives theorem.

② In Thomas' Calculus, Hessian of f is defined to be the determinant of our Hessian matrix

With the Hessian matrix, the 2nd order Taylor polynomial of f at a can be written as

$$P_2(x) = f(a) + \nabla f(a)(x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a)$$

$1 \times 1 \quad 1 \times 1 \quad 1 \times n \quad n \times 1 \quad 1 \times n \quad n \times n \quad n \times 1$

where $x, a \in \mathbb{R}^n$ are written as column vectors

$$\begin{aligned}(x-a)^T &= \text{Transpose of } x-a \\ &= [x_1 - a_1, \dots, x_n - a_n]\end{aligned}$$

Rmk

$$(x-a)^T H f(a) (x-a) = [x_1 - a_1, \dots, x_n - a_n] \begin{bmatrix} f_{x_1 x_1}(a) & \cdots & f_{x_1 x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_n x_1}(a) & \cdots & f_{x_n x_n}(a) \end{bmatrix} \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}$$

$$= [x_1 - a_1, \dots, x_n - a_n] \begin{bmatrix} f_{x_1 x_1}(a)(x_1 - a_1) + \cdots + f_{x_1 x_n}(a)(x_n - a_n) \\ \vdots \\ f_{x_n x_1}(a)(x_1 - a_1) + \cdots + f_{x_n x_n}(a)(x_n - a_n) \end{bmatrix}$$

$$= f_{x_1 x_1}(a)(x_1 - a_1)(x_1 - a_1) + \cdots + f_{x_1 x_n}(a)(x_1 - a_1)(x_n - a_n)$$

+ ...

⋮

$$+ f_{x_n x_1}(a)(x_1 - a_1)(x_n - a_n) + \cdots + f_{x_n x_n}(a)(x_n - a_n)(x_n - a_n)$$

$$= \sum_{i,j=1}^n f_{x_i x_j}(a)(x_i - a_i)(x_j - a_j)$$

$$= g^{(2)}(0)$$

eg Same $f(x,y) = e^x \cos y$.

Find $p_2(x,y)$ at $\alpha = (0,0)$

using matrix form.

Sol $f(0,0) = 1$

$$\nabla f(0,0) = (1, 0)$$

$$Hf(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned}
 p_2(x,y) &= f(0,0) + \nabla f(0,0) \begin{bmatrix} x-0 \\ y-0 \end{bmatrix} \\
 &\quad + \frac{1}{2} [x-0 \ y-0] Hf(0,0) \begin{bmatrix} x-0 \\ y-0 \end{bmatrix} \\
 &= 1 + [1 \ 0] \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} [x \ y] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
 &= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2
 \end{aligned}$$

eg $g(x,y) = \frac{\ln x}{1-y}$. Find $p_2(x,y)$ at $(1,0)$

Sol $g(1,0) = 0$

$$\nabla g = [g_x, g_y] = \left[\frac{1}{x(1-y)}, \frac{\ln x}{(1-y)^2} \right]$$

$$Hg = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{2\ln x}{(1-y)^3} \end{bmatrix}$$

$$\nabla g(1,0) = [1 \ 0] \quad Hg(1,0) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$p_2(x,y)$

$$= g(1,0) + \nabla g(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2} [x-1 \ y] Hg(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= 0 + [1 \ 0] \begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2} [x-1 \ y] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y$$

Application to local max/min.

If f is C^2 , a is a critical point of f

Then $\nabla f(a) = \vec{0}$. For x near a ,

$$f(x) \approx p_2(x)$$

$$= f(a) + \nabla f(a)(x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a)$$

$$= f(a) + \underbrace{\frac{1}{2}(x-a)^T Hf(a)(x-a)}$$

This term determines whether
 $f(x) > f(a)$ or $f(x) < f(a)$

Rmk For $n=1$, i.e. f is 1-variable.

$$\frac{1}{2}(x-a)^T Hf(a)(x-a) = \frac{1}{2} f''(a)(x-a)^2$$

If $f'(a)=0$, then

$$\begin{cases} f''(a) > 0 \Rightarrow \text{local min at } a \\ f''(a) < 0 \Rightarrow \text{local max at } a \end{cases} \quad \begin{matrix} \left(\begin{smallmatrix} 2^{\text{nd}} \text{ derivative} \\ \text{test} \end{smallmatrix} \right) \end{matrix}$$

For $n=2$, the 2nd order term is

$$\frac{1}{2} [x-x_0 \ y-y_0] \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

$f \text{ is } C^2 \Rightarrow \text{Symmetric}$

To understand nature of critical points, we study quadratic forms of 2 variables.

$$q(x,y) = [x \ y] \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= Ax^2 + 2Bxy + Cy^2$$

Does $q(x,y)$ have a definite sign (always positive or always negative) for $(x,y) \neq (0,0)$?

We can determine it by completing square.

Eg 1 $g(x,y) = 2xy$

Note $g(x,y) = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2$

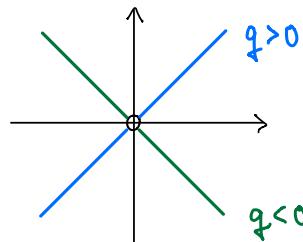
Along $x+y=0$, i.e. $y=-x$,

$$g(x,-x) = -2x^2 < 0 \quad \text{for } x \neq 0$$

Along $x-y=0$, i.e. $y=x$

$$g(x,x) = 2x^2 > 0 \quad \text{for } x \neq 0$$

$\therefore g$ has no definite sign, i.e. indefinite



Clearly $(0,0)$ is a critical point of $g(x,y)$

but neither local max nor min. Such a critical point is called a saddle point.

Eg 2 $g(x,y) = 17x^2 - 12xy + 8y^2$. Definite sign?

Soln

$$\begin{aligned} g(x,y) &= 17 \left[x^2 - \frac{2 \cdot 6}{17} xy + \left(\frac{6}{17} \right)^2 y^2 \right] + \left(8 - \frac{36}{17} \right) y^2 \\ &= 17 \left(x - \frac{6}{17} y \right)^2 + \frac{100}{17} y^2 \end{aligned}$$

$$\therefore g(x,y) > 0 = g(0,0) \quad \text{for } (x,y) \neq (0,0)$$

\therefore The critical point $(0,0)$ is a local min.
also global min of $g(x,y)$

Rmk Expression like $\textcircled{*}$ is called diagonalization of quadratic form. It is not unique! For

$$\text{example } g(x,y) = 5 \left(\frac{x+2y}{\sqrt{5}} \right)^2 + 20 \left(\frac{2x-y}{\sqrt{5}} \right)^2$$

is another diagonalization.

$\swarrow \uparrow$
"Orthogonal" change
of coordinates

Higher dimension example

eg 3 $q(x, y, z) = xy + yz + zx$

Definite sign for $(x, y, z) \neq (0, 0, 0)$?

Sol $q = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 + z(x+y)$

Let $u = \frac{x+y}{2}$ $v = \frac{x-y}{2}$. Then

$$\begin{aligned} q &= u^2 - v^2 + 2uz \\ &= (u^2 + 2uz + z^2) - v^2 - z^2 \\ &= (u+z)^2 - v^2 - z^2 \\ &= \left(\frac{x+y}{2} + z\right)^2 - \left(\frac{x-y}{2}\right)^2 - z^2 \\ &= \frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2 \end{aligned}$$

↑ positive ↑ negative ↑

On the plane $x+y+2z=0$, i.e. $z = -\frac{x+y}{2}$

$$q = q\left(x, y, -\frac{x+y}{2}\right)$$

$$= -\frac{1}{4}(x-y)^2 - \frac{1}{4}(x+y)^2 < 0 \text{ for } (x, y, z) \neq (0, 0, 0)$$

Along the line $x-y=z=0$, i.e. $y=x, z=0$

$$q(x, y, z) = q(x, x, 0)$$

$$= x^2 > 0 \text{ for } x \neq 0$$

∴ The critical point $(0, 0, 0)$ is a saddle point.

For general theory, need linear algebra :

Diagonalization of quadratic form, eigenvalues ...

Defn Let A be a $n \times n$ symmetric matrix.

Then A is said to be

① positive definite if $X^T A X > 0$

for all column vectors $X \in \mathbb{R}^n \setminus \{\vec{0}\}$

② negative definite if $X^T A X < 0$

for all column vectors $X \in \mathbb{R}^n \setminus \{\vec{0}\}$

③ indefinite if \exists column vectors $x, y \in \mathbb{R}^n \setminus \{\vec{0}\}$

such that $X^T A X > 0$ and $y^T A y < 0$

e.g

$$\textcircled{1} [x \ y] \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4y^2 > 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is positive definite

$$\textcircled{2} [x \ y] \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 4y^2 < 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ is negative definite

$$\textcircled{3} [x \ y] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + 4y^2$$

$$[1 \ 0] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1 < 0$$

$$[0 \ 1] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4 > 0$$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ is indefinite

Rmk These are not all the possible cases:

There are symmetric matrix which is not positive definite, negative definite nor indefinite.

$$\textcircled{4} \quad [x \ y] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 \geq 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

$$[0 \ 1] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \Rightarrow \text{not positive definite}$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is neither positive/negative definite
nor indefinite

$$\begin{aligned}\textcircled{5} \quad [x \ y] & \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= x^2 + 4xy + 5y^2 \\ &= (x^2 + 4xy + 4y^2) + y^2 \\ &= (x + 2y)^2 + y^2 > 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}\end{aligned}$$

$\therefore \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite.