THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2010C/D Advanced Calculus I 2022-2023

Midterm Examination

- Total points: 100 points
- Time allowed: 90 min
- Answer all questions.
- Please provide justifications for your answers unless specified.

1. (a) Let $\gamma(t) = (\cos^3 t, \sin^3 t)$, for $t \in [0, \pi/2]$, be a curve in $\mathbb{R}^2$. Find the length of the curve γ.

   (b) Let $\mathbf{a}$ be a nonzero vector in $\mathbb{R}^3$ and let α, β and γ be the angles between $\mathbf{a}$ and $\mathbf{i}$, between $\mathbf{a}$ and $\mathbf{j}$ and between $\mathbf{a}$ and $\mathbf{k}$ respectively. Show that
$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

(14 points)

2. Let $S := \{(x, y) \in \mathbb{R}^2 : -1 < x + y \leq 1\}$ be a subset in $\mathbb{R}^2$. Draw the following sets without justification.

   (a) The set of interior points $\mathrm{Int}(S)$.

   (b) The set of boundary points ∂S.

   (c) The set of exterior points $\mathrm{Ext}(S)$.

(6 points)

3. For each of the following statements, state whether it is true or false and explain your answers.

   (a) For any $\mathbf{v} \in \mathbb{R}^3$, $k \in \mathbb{R}$, $\|k\mathbf{v}\| = k\|\mathbf{v}\|$.

   (b) For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

   (c) There exists a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\dfrac{\partial f}{\partial x} = x - y$ and $\dfrac{\partial f}{\partial y} = x + y$.

   (d) If two straight lines in $\mathbb{R}^3$ do not intersect, then they must be parallel.

(8 points)

4. Find each of the following limits or explain why the limit does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$

(12 points)

5. Let $f(x, y) = e^{\sin(2x+y)}$.

   (a) Find the linearization of f at $(0, 0)$.

   (b) Approximate $f(-0.1, 0.1)$ using the linearization in (a).

(8 points)

6. Let $f(x, y) = \dfrac{2xy}{x^2 + y^2}$.

   (a) Show that
   $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = C$$
   for some constant C, and also find the value of C.

   (b) Draw the level set of the function $f(x, y)$ corresponding to the value 1.

(12 points)

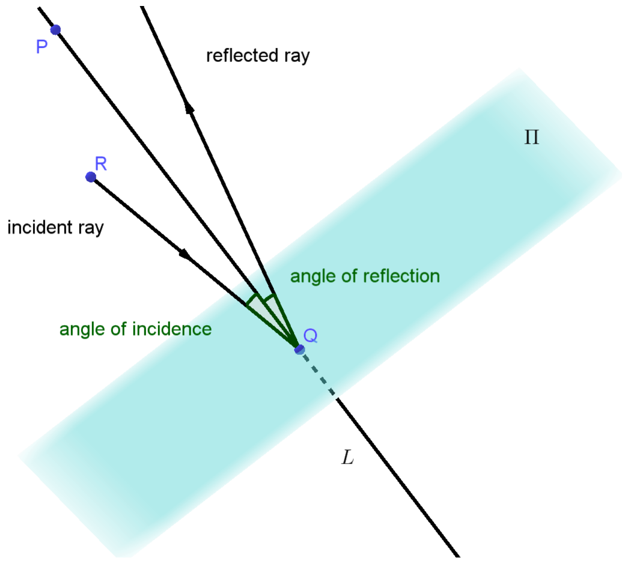
7. Let $\Pi : 2x + y + 3z = 4$ be a plane and let $P = (8, 0, 10)$ be a point in $\mathbb{R}^3$.

   (a) Let L be the straight line orthogonal to Π which passes through P and let Q be the intersection point of L and Π. Find a parametric equation of L and the coordinates of Q.

   (b) Let $R = (7, 0, 6)$ be a point in $\mathbb{R}^3$. Find the projection of R on L.

   (c) Suppose that $\overrightarrow{RQ}$ is a light ray that strikes the mirror Π. Find a parametric equation of the straight line that contains the reflected ray.

   (Hint: The incident ray, the reflected ray and L are lying on the same plane. The angle of incidence equals the angle of reflection.)



(20 points)

8. Let
$$f(x, y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0); \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

   (a) Compute $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for both the cases $(x, y) = (0, 0)$ and $(x, y) \neq (0, 0)$.

   (b) Is f differentiable at $(0, 0)$? Prove your assertion.

   (c) Evaluate $\dfrac{\partial^2 f}{\partial x \partial y}(0, 0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0, 0)$.

(20 points)

End of Paper