Math 2010 Week 9

Application of chain rule:

Level Set (Au 4.4, Thomas 14.6)

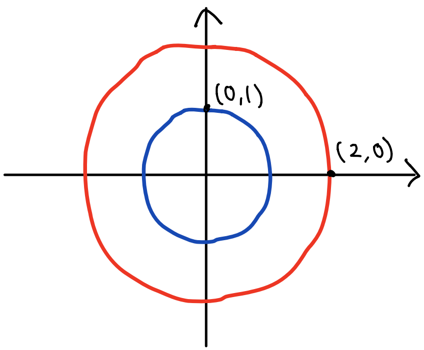
Recall: Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $c \in \mathbb{R}$

$$L_c = f^{-1}(c) = \{x \in \Omega : f(x) = c\}$$

eg Some level sets of $f(x,y) = x^2+y^2$

$$f^{-1}(1) = \{x^2+y^2=1\}$$

$$f^{-1}(4) = \{x^2+y^2=4\}$$



Thm Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, Ω is open, let $c \in \mathbb{R}$, $S = f^{-1}(c)$ and $a \in S$

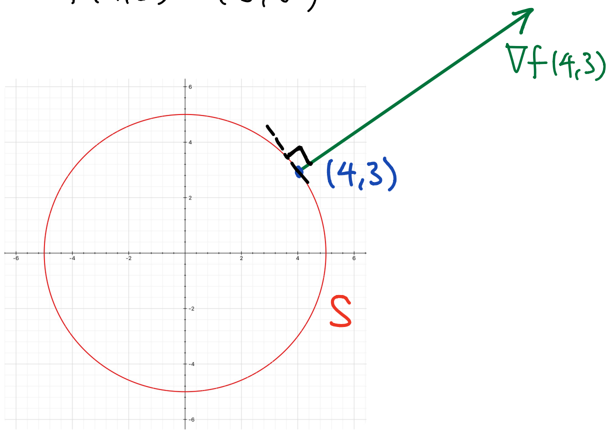
Suppose f is <u>differentiable</u> at a, $\nabla f(a) \neq 0$

Then $\nabla f(a) \perp S$ at a

eg $f(x,y) = x^2+y^2$ $\quad \nabla f = (2x, 2y)$

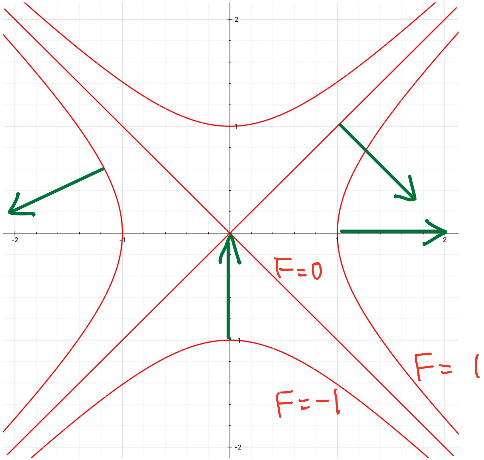
Let $S = f^{-1}(25)$, then $(4,3) \in S$

$$\nabla f(4,3) = (8,6)$$



eg $f(x,y) = x^2 - y^2$

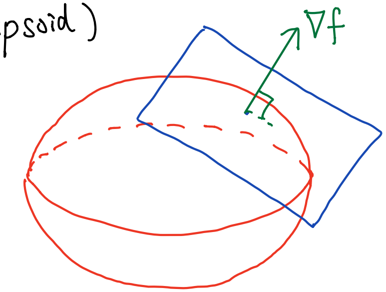
$$\nabla f(x,y) = (2x, -2y)$$



$\longrightarrow$ are directions of ∇f

eg $S: x^2 + 4y^2 + 9z^2 = 22$ (Ellipsoid)

Find equation of tangent plane of S at $(3,1,1)$



Sol

Let $f(x,y,z) = x^2 + 4y^2 + 9z^2$, $S = f^{-1}(22)$

Also $f(3,1,1) = 22$, so $(3,1,1) \in S$

$\nabla f = (2x, 8y, 18z)$

$\nabla f(3,1,1) = (6, 8, 18) \perp S$ at $(3,1,1)$

$\therefore (6,8,18)$ is a normal vector for tangent plane

Equation of tangent plane:

$$[(x,y,z) - (3,1,1)] \cdot (6,8,18) = 0$$

$$6(x-3) + 8(y-1) + 18(z-1) = 0$$

$$3x + 4y + 9z = 22$$

Thm Let $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, Ω is open,
let $c \in \mathbb{R}$, $S = f^{-1}(c)$ and $a \in S$
Suppose f is differentiable at a, $\nabla f(a) \neq 0$
Then $\nabla f(a) \perp S$ at a

Pf Let $\gamma(t)$ be a curve on S, $\gamma(0) = a$

Then $\gamma(t)$ on $S = f^{-1}(c)$

$\Rightarrow$ $f(\gamma(t)) = c$ is a constant

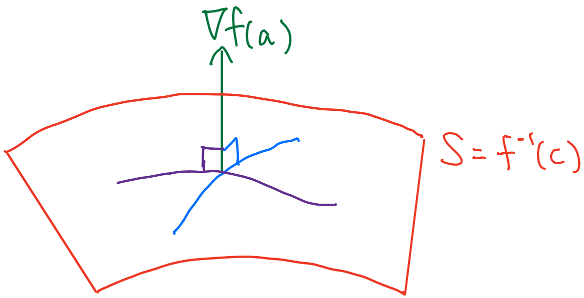
By chain rule,

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = \frac{df}{dt} = 0$$

Put $t=0$, then $\nabla f(a) \cdot \gamma'(0) = 0$

$\therefore$ $\nabla f(a) \perp$ any curve on S at a.

$\therefore$ $\nabla f(a) \perp S$ at a



Rmk

We applied chain rule to $f(\gamma(t))$ above.

Similarly, by considering the curve

$$\alpha(t) = a + tu, \quad \|\vec{u}\| = 1,$$

One can prove

$$D_u f(a) = \frac{d}{dt}\Big|_{t=0} f(\alpha(t))$$

$$= \nabla f(\alpha(0)) \cdot \alpha'(0)$$

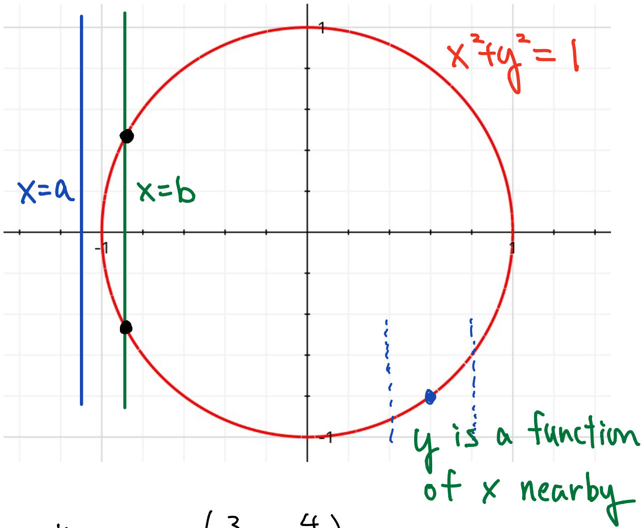
$$= \nabla f(\vec{a}) \cdot u$$

Another application of chain rule:

Implicit differentiation (Au 4.5, Thomas 14.4)

$C: x^2 + y^2 = 1$

Find $\frac{dy}{dx}$ at $\left(\frac{3}{5}, -\frac{4}{5}\right)$



Locally near $\left(\frac{3}{5}, -\frac{4}{5}\right)$,

$$y^2 = 1 - x^2,\ y<0 \Rightarrow y = -\sqrt{1-x^2}$$

$\therefore$ y is a function of x near $\left(\frac{3}{5}, -\frac{4}{5}\right)$

To find $\frac{dy}{dx}$ at $\left(\frac{3}{5}, -\frac{4}{5}\right)$,

Method 1: $\frac{d}{dx}\left(-\sqrt{1-x^2}\right)$

Method 2: Implicit differentiation (chain rule)

$$x^2+y^2=1$$

(Regard x as a variable, y as a function of x)

Take $\frac{d}{dx}$: $2x + 2y\frac{dy}{dx} = 0$

Put $(x,y) = \left(\frac{3}{5}, -\frac{4}{5}\right)$, $2\left(\frac{3}{5}\right) + 2\left(-\frac{4}{5}\right)\frac{dy}{dx} = 0$

$$\therefore \left.\frac{dy}{dx}\right|_{\left(\frac{3}{5}, -\frac{4}{5}\right)} = \frac{3}{4}$$

__Rmk__ __Near__ $(-1,0)$, y is not a function of x

If $a<-1$ a little bit, the vertical line $x=a$ does not intersect with C

If $b>-1$ a little bit, the vertical line $x=b$ intersects with C at two different points eg Consider

$$S: x^3 + z^2 + ye^{xz} + z\cos y = 0 \quad (*)$$

Given that z can be regarded as a function $z = z(x,y)$ of independent variables x, y locally near $(0,0,0)$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(0,0,0)$.

Rmk ① Not easy to express z in terms of x, y

Sol Take $\frac{\partial}{\partial x}$ to $(*)$

$$3x^2 + 2z\frac{\partial z}{\partial x} + ye^{xz}\left(z + x\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial x}\cos y = 0$$

Put $(x,y,z) = (0,0,0)$,

$$0 + 0 + 0 + \frac{\partial z}{\partial x}(1) = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x}(0,0) = 0}$$

Similarly, take $\frac{\partial}{\partial y}$ to $(*)$

$$0 + 2z\frac{\partial z}{\partial y} + e^{xz} + ye^{xz}\left(x\frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial y}\cos y - z\sin y = 0$$

Put $(x,y,z) = (0,0,0)$, then

$$0 + 0 + 1 + 0 + \frac{\partial z}{\partial y}(1) - 0 = 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y}(0,0) = -1}$$

Rmk ② From computations above,

$$\frac{\partial z}{\partial x} = -\frac{3x^2 + yze^{xz}}{2z + xye^{xz} + \cos y}$$

$$\frac{\partial z}{\partial y} = \frac{z\sin y - e^{xz}}{2z + xye^{xz} + \cos y}$$

whenever the denominator is non-zero

Finding Extrema (Maximum & Minimum) (Au Ch 5, Thomas 14.7)

Defn Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$, $a \in A$

① f is said to have <u>global (absolute) maximum</u> at a

if $f(a) \geq f(x)$ for all $x \in A$

② f is said to have <u>local (relative) maximum</u> at a

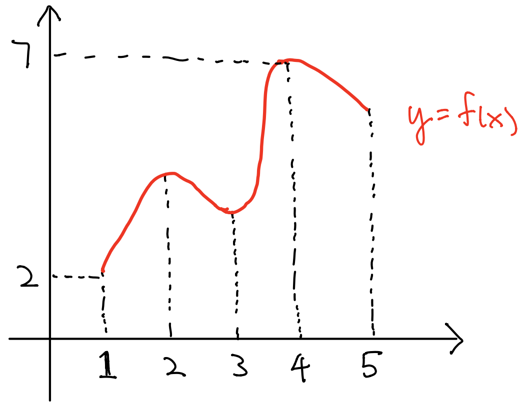
if $f(a) \geq f(x)$ for all $x \in A$ near a,

(i.e. $\exists \varepsilon > 0$ such that $f(x) \leq f(a)$ for all $x \in A \cap B_\varepsilon(a)$)

③ Similar definitions for <u>global (absolute) minimum</u> and <u>local (relative) minimum</u>

Rmk Global extremum (max/min) is also a local extremum.

eg Let $f: [1, 5] \to \mathbb{R}$



Global max at 4

Global min at 1

Local max. at 2, 4

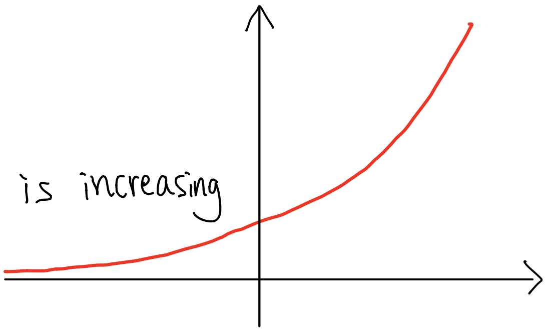
Local min. at 1, 3, 5

Max Value : 7

Min Value : 2

Rmk NOT every function has global max/min!

① $f(x)=e^x$ on $\mathbb{R}$



$\lim_{x\to-\infty} f(x)=0$

But $f(x)>0 \ \forall x\in\mathbb{R}$

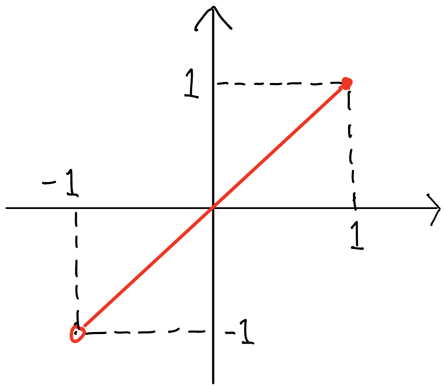
$\therefore$ No global min

$\lim_{x\to\infty} f(x)=\infty$

$\therefore$ No global max

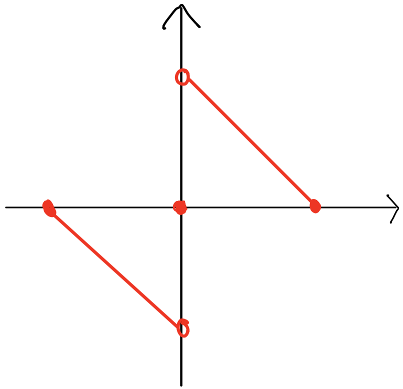
(Domain is not bounded)

② $f(x)=x$ on $(-1,1]$ (Domain is not closed)



f has global max at 1 but no global min

③ $f:[-1,1]\to\mathbb{R}$ $f(x)=\begin{cases}1-x & \text{if } x\in(0,1]\\ 0 & \text{if } x=0\\ -1-x & \text{if } x\in[-1,0)\end{cases}$



f has neither global max or min

(f is not continuous)

Q When must a function have global max/min?

> **Extreme Value Theorem (EVT)**
>
> Let $A \subseteq \mathbb{R}^n$ be closed and bounded,
>
> $f : A \longrightarrow \mathbb{R}$ is continuous.
>
> Then f has global max and min.

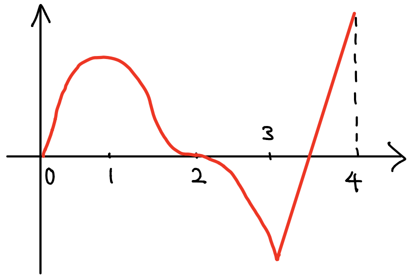
Rmk

① Compact = closed and bounded

② The theorem provides a sufficient condition, but not necessary condition for the existence of global extrema

Q How to locate max/min?

eg $f : A = [0,4] \longrightarrow \mathbb{R}$



A is closed, bounded

f is continuous

$\Rightarrow f$ has global max and min

Recall One variable calculus:

Extrema can only occur at

$$\left.\begin{array}{lll} \text{i} & f'(x) = 0 & : x = 1, 2 \\ \text{ii} & f'(x) \text{ DNE} & : x = 3 \end{array}\right\} \text{Critical points}$$

$$\begin{array}{lll} \text{iii} & x \in \partial A & : x = 0, 4 \end{array} \quad \text{Boundary points}$$

Comparing values of f at these 5 points:

f has global min at 3, global max at 4.

Defn $f: A \to \mathbb{R}$, $a \in \operatorname{int}(A)$.

Then a is called a critical point of f if either

① $\nabla f(a)$ DNE $\left(\text{i.e. } \frac{\partial f}{\partial x_i}(a) \text{ DNE for some } i\right)$

② $\nabla f(a) = \vec{0}$ $\left(\text{i.e. } \frac{\partial f}{\partial x_i}(a) = 0 \text{ for all } i\right)$

Thm (First derivative test)

Suppose $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ attains a local extremum at $a \in \operatorname{int}(A)$, then a is a critical point of f

Pf Suppose f has a local extremum at $a \in \operatorname{int}(A)$

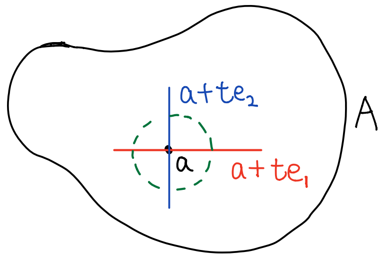
If $\nabla f(a)$ DNE, then a is a critical point.

If $\nabla f(a)$ exists, then all $\frac{\partial f}{\partial x_i}(a)$ exist.

For any $i = 1, \cdots, n$, let

$$g_i(t) = f(a + te_i)$$

Note $a \in \operatorname{int}(A) \Rightarrow g_i(t)$ is defined near $t = 0$



By definition, $g_i'(0) = \frac{\partial f}{\partial x_i}(a)$ exists.

f has local extremum at a

$\Rightarrow g_i$ has local extremum at 0

$\Rightarrow g_i'(0) = 0$ since it exists

$\Rightarrow \frac{\partial f}{\partial x_i}(a) = 0$ (for any $i = 1, 2, \cdots, n$)

$\therefore \nabla f(a) = \vec{0}$

$\therefore a$ is a critical point.

Strategy for finding extrema

Let $f: A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$.

① Find critical points of f in $\text{int}(A)$

② Study f on boundary ∂A:

Find max/min of f on ∂A

③ Comparing values of f at points found in ① and ②

eg Find global max/min of

$$f(x,y) = x^2 + 2y^2 - x + 3 \text{ for } x^2 + y^2 \leq 1$$

Rmk Domain $= A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is closed and bounded

Also, f is polynomial $\Rightarrow$ f is continuous

By EVT, f has global max and min on A

Sol We follow the strategy above:

Step 1: Critical points in $\text{int}(A)$

$\nabla f = (2x-1, 4y)$ exists everywhere

Also, $$\nabla f = \vec{0} \Leftrightarrow \begin{cases} 2x-1=0 \\ 4y=0 \end{cases}$$

$$\Leftrightarrow (x,y) = \left(\tfrac{1}{2}, 0\right)$$

Note $\left(\frac{1}{2}, 0\right) \in \text{int}(A) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

$\therefore$ f has only one critical point $\left(\frac{1}{2}, 0\right)$ in $\text{int}(A)$ with

$$f\left(\tfrac{1}{2}, 0\right) = \left(\tfrac{1}{2}\right)^2 + 0 - \tfrac{1}{2} + 3 = \tfrac{11}{4}$$

Step 2: Study f on $\partial A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$

Parametrize ∂A by

$$(x,y) = (\cos\theta, \sin\theta), \quad \theta \in [0, 2\pi]$$

$$\begin{aligned} f(\cos\theta, \sin\theta) &= \cos^2\theta + 2\sin^2\theta - \cos\theta + 3 \\ &= \cos^2\theta + 2(1-\cos^2\theta) - \cos\theta + 3 \\ &= -\cos^2\theta - \cos\theta + 5 \\ &= -\left(\cos\theta + \tfrac{1}{2}\right)^2 + \tfrac{1}{4} + 5 \\ &= \frac{21}{4} - \left(\cos\theta + \frac{1}{2}\right)^2 \end{aligned}$$

max value of f on $\partial A = \frac{21}{4}$, occurs when

$x = \cos\theta = -\frac{1}{2}$, ie. $(x,y) = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

min value of f on $\partial A = 3$ occurs when

$x = \cos\theta = 1$, ie. $(x,y) = (1,0)$

Step 3: Compare values of f at points from step 1 and 2.

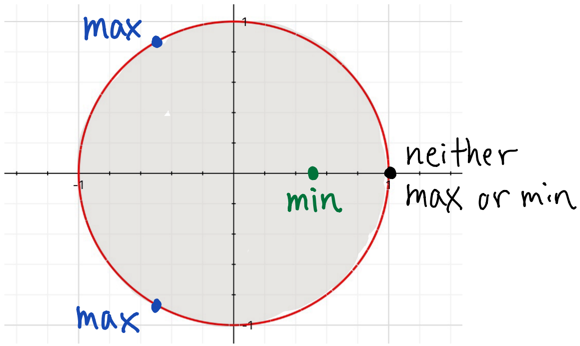
$f(\frac{1}{2}, 0) = \frac{11}{4}$ ← min

$f\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = \frac{21}{4}$ ← max

$f(1,0) = 3$

Max value $= \frac{21}{4}$ at $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

Min value $= \frac{11}{4}$ at $(\frac{1}{2}, 0)$



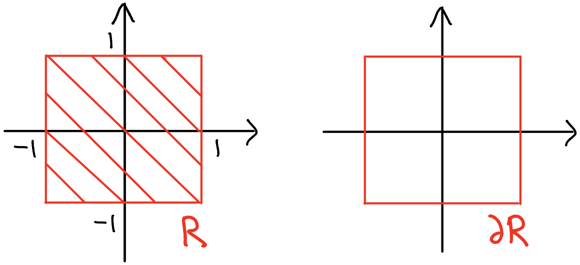
eg 2 Find global max and min of

$$f(x,y) = \sqrt{x^2+y^4} - y$$

on $R = \{(x,y) \in \mathbb{R}^2 : -1 \le x, y \le 1\}$

Sol

R is the square $[-1,1] \times [-1,1] \subseteq \mathbb{R}^2$



R is closed and bounded

Also, f is continuous

By EVT, f has global max and min.

To find them...

Step 1 Study $\text{int}(R) = \{(x,y) \in \mathbb{R}^2, -1 < x, y < 1\}$

Ex Show that $\frac{\partial f}{\partial x}(0,0)$ DNE $\quad \left(f(x,0) = |x| \right)$

For $(x,y) \neq (0,0)$, ∇f exists and

$$\nabla f = \left(\frac{\partial f}{\partial x} \; \frac{\partial f}{\partial y} \right) = \left(\frac{x}{\sqrt{x^2+y^4}}, \frac{2y^3}{\sqrt{x^2+y^4}} - 1 \right)$$

$$\nabla f = (0,0) \iff \begin{cases} \dfrac{x}{\sqrt{x^2+y^4}} = 0 \\ \dfrac{2y^3}{\sqrt{x^2+y^4}} - 1 = 0 \end{cases}$$

$$\therefore x = 0, \quad \frac{2y^3}{y^2} - 1 = 0 \Rightarrow y = \frac{1}{2}$$

$\therefore$ f has two critical points in $\text{int}(A)$

$(0,0)$ and $(0, \frac{1}{2})$

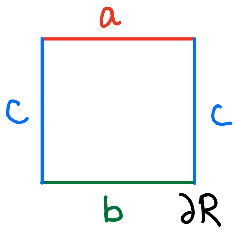
∇f DNE $\qquad \nabla f = \vec{0}$

$$f(0,0) = 0, \quad f(0, \tfrac{1}{2}) = -\tfrac{1}{4}$$

Step 2 Study f on ∂R

$$f(x,y) = \sqrt{x^2+y^4} - y$$

$$\partial R = \{(x,y) : |x|=1,\ -1 \le y \le 1\}$$
$$\cup \{(x,y) : |y|=1,\ -1 \le x \le 1\}$$



Consider different parts of ∂R:

ⓐ $y=1,\ -1 \le x \le 1$

$$f(x,1) = \sqrt{x^2+1} - 1 \Rightarrow 0 \le f \le \sqrt{2} - 1$$

ⓑ $y=-1,\ -1 \le x \le 1$

$$f(x,-1) = \sqrt{x^2+1} + 1 \Rightarrow 2 \le f \le \sqrt{2} + 1$$

ⓒ $|x|=1,\ -1 \le y \le 1$

Not the sharpest bound but good enough for us ⊛

$$f(x,y) = \sqrt{1+y^4} - y \Rightarrow 0 = 1-1 < f \le \sqrt{2}+1$$

$\therefore$ On ∂R, f has min value 0 at $(0,1)$

max value $\sqrt{2}+1$ at $(\pm 1, -1)$

Step 3 Comparing values

$f(0,0) = 0$

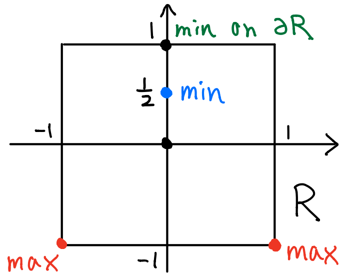
$f(0,\frac{1}{2}) = -\frac{1}{4}$ ← min

$f(0,1) = 0$

$f(\pm 1, -1) = \sqrt{2}+1$ ← max

Max value $= \sqrt{2}+1$ at $(\pm 1, -1)$

Min value $= -\frac{1}{4}$ at $(0, \frac{1}{2})$



⊛ We may find the sharpest bound.

By one variable calculus, easy to see

$\sqrt{1+y^4} - y$ is decreasing on $[-1,1]$

$\Rightarrow$ min value on ⓒ $= \sqrt{1+1^2} - 1 = \sqrt{2} - 1$

Finding extrema on an unbounded region

eg Find global extrema of

$$f(x,y) = x^2 + y^2 - 4x + 6y + 7 \text{ on } \mathbb{R}^2$$

Rmk $\mathbb{R}^2$ is not bounded. So f may not have global extrema. Observe that

$$\lim_{\underbrace{(x,y)\to\infty}_{\text{"(x,y) are far away from origin"}}} f(x,y) = +\infty. \quad \text{Hence}$$

① f has no global maximum on $\mathbb{R}^2$

② Strategy for finding global minimum:

Find a closed and bounded region R s.t. f is "large enough" outside R. Then

$$\min \text{ on } R = \min \text{ on } \mathbb{R}^2$$

Sol Find critical points of f:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x-4, 2y+6) \text{ exists on } \mathbb{R}^2$$

$$\nabla f = (0,0) \iff \begin{cases} 2x-4=0 \\ 2y+6=0 \end{cases}$$

$$\iff (x,y) = (2,-3)$$

$\therefore$ Only one critical point $(2,-3)$, $f(2,-3) = -6$

We want to show f has min at $(2,-3)$:

For $(x,y) \in \mathbb{R}^2$, let $r = \sqrt{x^2+y^2}$

Then
$$\begin{aligned} f(x,y) &= x^2+y^2-4x+6y+7 \\ &\overset{(*)}{\geq} r^2 - 4r - 6r + 7 \\ &= r(r-10)+7 \end{aligned}$$

$(*)$ $r = \sqrt{x^2+y^2} \geq |x|, |y| \Rightarrow \begin{cases} x \leq r \Rightarrow -4x \geq -4r \\ -y \leq r \Rightarrow 6y \geq -6r \end{cases}$

Hence, if $\sqrt{x^2+y^2} = r \geq 10$

then $f(x,y) \geq 7 > f(2,-3)$

Let $R = \overline{B_{10}(0,0)}$, $f|_R$ = restriction of f on R

By EVT, $f|_R$ has global min.

In Int(R), $(2,-3)$ is the only critical point

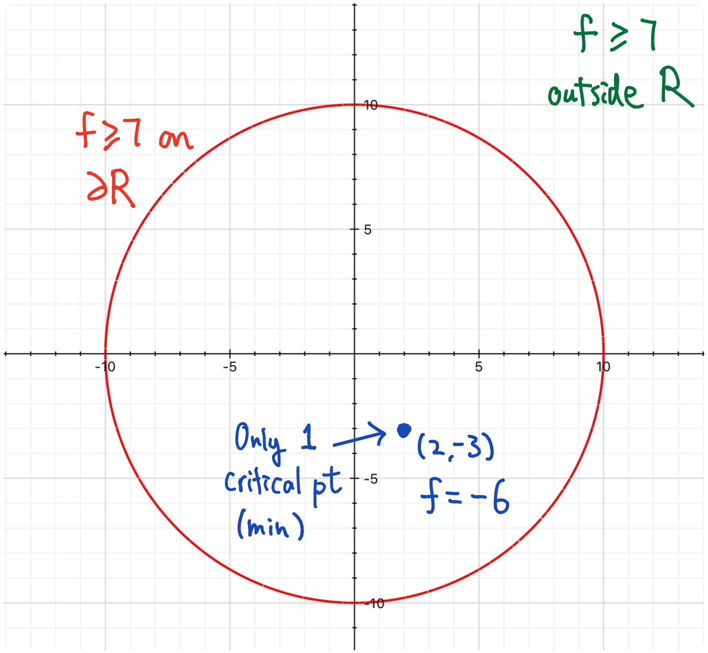
$$f(2,-3) = -6$$

On ∂R, $f(x,y) \geq 7 > f(2,-3)$

$\therefore$ $f|_R$ has global min at $(2,-3)$

For $(x,y) \notin R$, $f(x,y) \geq 7 > f(2,-3)$

$\therefore$ f has no global max and global min. value -6 at $(2,-3)$



Rmk

① Easier to do this question using algebra.

$$f(x,y) = x^2 + y^2 - 4x + 6y + 7$$
$$= (x-2)^2 + (y+3)^2 - 6$$

Answer is clear

② An example without global max or min.

Let $g(x,y) = x^2 - y^2 - 4x + 6y + 7$

When $x=0$, $g(0,y) = -y^2 + 6y + 7$

$$\lim_{y\to\pm\infty} g(0,y) = -\infty \Rightarrow \text{no global min.}$$

When $y=0$, $g(x,0) = x^2 - 4x + 7$

$$\lim_{x\to\pm\infty} g(x,0) = \infty \Rightarrow \text{no global max}$$