C18 Computer Vision

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Computer Vision: This time ...

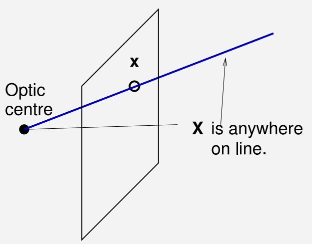
1. Introduction; imaging geometry; camera calibration
2. Salient feature detection – edges, line and corners
3. **Recovering 3D from two images I: epipolar geometry.**
4. Recovering 3D from two images II: stereo correspondence algorithms; triangulation.

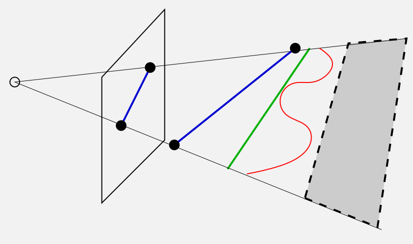
Recovering 3D from two images I: epipolar geometry

3.1 Introduction: Forward and inverse mappings

Geometrical image entities **back-project** to entities of higher dimensionality in the scene.

So, points in the image back-project to **lines** in the scene, lines to planes ...





$$\mathbf{X} = \alpha \mathbf{x} \qquad 0 < \alpha < \infty$$

Introduction: What do single-view ambiguities tell us?

Single views are NOT sufficient to solve geometric problems in data-driven vision.

We need methods of understanding multiple views.

Shape-from-stereo

- different cameras
- different viewpoints
- same time

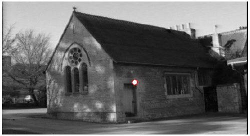
Structure-from-motion

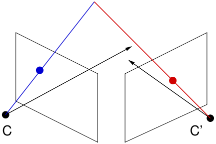
- same camera
- different viewpoints
- different times

Note that single views can be sufficient to solve geometric problems in top-down model-driven vision (but not always).

Introduction: Reconstruction from two views

In principle, recovering 3D structure is straightforward. Find a bit of the scene that is observable in two or more cameras, and backproject the two rays to find their intersection in the world.





Right (camera $\mathcal{C}'$) View Left (camera $\mathcal{C}$) View Backprojection

Arranged for cross-eyed fusion

There are three things to cover:

1. Understanding the geometry — **Epipolar geometry**
2. Determining which points in the images are from the same scene location — **the correspondence problem**
3. Determining the 3D structure by back-projecting rays — **triangulation**

What clues from identical parallel cameras?

Assume two cameras with the same f, separated by t_x in the x-dirn.

Inhomogenous coordinates: Same scene Y, Z but different X ...

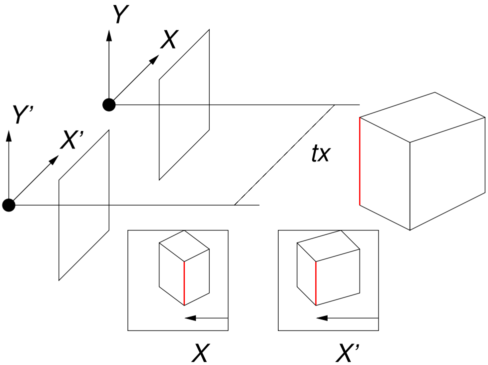
$\Rightarrow Y' = Y$, $Z' = Z$, but

$$X' = X + t_x,$$

As

$x' = fX'/Z \quad x = fX/Z$

$$\Rightarrow \frac{1}{Z} = \frac{1}{ft_x}(x' - x)$$



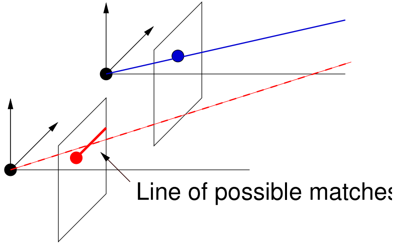
In this case the **reciprocal depth** $1/Z$ is proportional to the horizontal **disparity** $(x' - x)$.

What clues from identical parallel cameras? /ctd

The point $\mathbf{x}$ in the left image can come from any point $\mathbf{X}$ on the backprojected ray from the left camera $\mathcal{C}$
Could the corresponding $\mathbf{x}'$ be anywhere in $\mathcal{C}'$?

$$\begin{aligned} x' &= (f/Z)(X + t_x) \\ \Rightarrow x' &= x + ft_x/Z \end{aligned}$$

$$\begin{aligned} y' &= (f/Z)(Y) \\ \Rightarrow y' &= y \end{aligned}$$



As Z ranges from 0 to ∞, $0 < ft_x/Z < \infty$

So the locus of possible matches appears to be on a straight line.

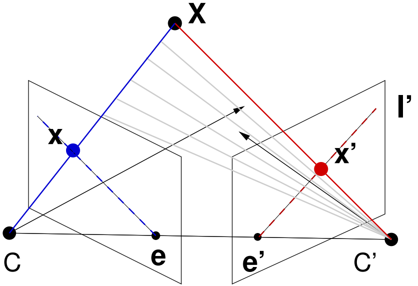
Does this generalize to arbitrary camera geometry? And how?

3.2 Epipolar geometry in arbitrary cameras

The locus of matches is the projection into $\mathcal{C}'$ of the backprojected ray in $\mathcal{C}$.

This is **always a straight line**, and is called the **epipolar line**, labelled $\mathbf{l}'$.

As **X** moves along the ray, the other ray sweeps out the **epipolar plane**. The intersection of the epipolar plane with the image plane is the epipolar line.



Epipolar geometry in arbitrary cameras

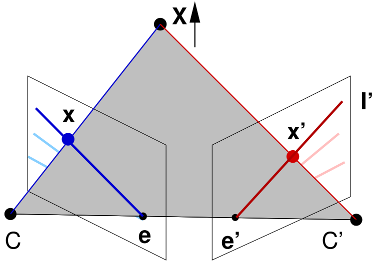
If $\mathbf{x}$ is moved then the entire plane moves too, generating a new epipolar line.

Epipolar planes hinge about the **camera baseline**, forming a pencil of planes.

This means that all the epipolar lines in camera $\mathcal{C}'$ meet at its **epipole** $\mathbf{e}'$, where the baseline pierces the image plane.

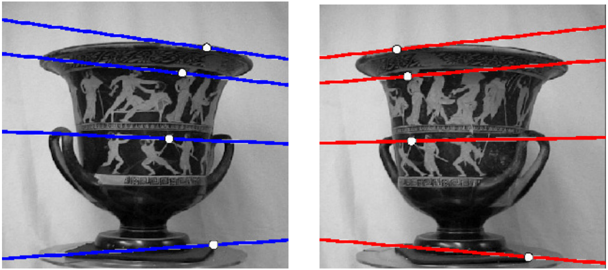
This point is the projection of the optical centre of camera $\mathcal{C}$ into $\mathcal{C}'$.

There are equivalent constructs in camera $\mathcal{C}$. All points on the same epipolar line in $\mathcal{C}$ share the same epipolar line in $\mathcal{C}'$.



Epipolar geometry examples

1. Converging cameras

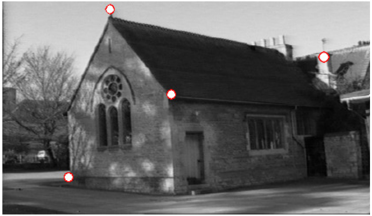


Notice that the epipoles most often lie off the physical image planes

What would be a quick test *you* could carry out to see whether the epipole was on the image plane?

Epipolar geometry examples

2. (Close to) parallel cameras



Epipolar geometry depends only on the relative pose of the cameras (ie the rotation and translation between them) and on the cameras' intrinsic parameters.

It does not depend on the scene structure.
Can you reason qualitatively why not?

3.3 Algebraic representation & the F matrix

There are three bits of preamble to get across:

1. The first explains how homogeneous notation handles points at infinity.
2. The second introduces the homogeneous notation for lines.
   You may notice a duality between lines and points ...
3. The last is the matrix representation of the vector product.

Preamble I: Points at ∞ in homogeneous coordinates

A line of points in 3D through the point **A** with direction **D** is

$$\mathbf{X}(\mu) = \mathbf{A} + \mu\mathbf{D}$$

Writing this in homogeneous notation

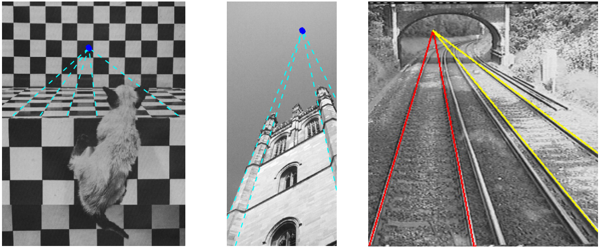
$$\begin{bmatrix} X_1(\mu) \\ X_2(\mu) \\ X_3(\mu) \\ X_4(\mu) \end{bmatrix} \stackrel{P}{=} \begin{bmatrix} \mathbf{A} + \mu\mathbf{D} \\ 1 \end{bmatrix} \stackrel{P}{=} \begin{bmatrix} \mathbf{A} \\ 1 \end{bmatrix} + \mu \begin{bmatrix} \mathbf{D} \\ 0 \end{bmatrix} \stackrel{P}{=} \frac{1}{\mu}\begin{bmatrix} \mathbf{A} \\ 1 \end{bmatrix} + \begin{bmatrix} \mathbf{D} \\ 0 \end{bmatrix}$$

In the limit $\mu \to \infty$ the point on the line is $\begin{bmatrix} \mathbf{D} \\ 0 \end{bmatrix}$.

So, homogeneous vectors with $X_4 = 0$ represent points "at infinity".

- **Points at infinity are equivalent to directions**
- **Parallel lines in the scene meet at the same point at ∞**

Points at ∞ and vanishing points



The projection of a point at ∞ into the image is the **vanishing point**

To find it, simply project the point-at-∞ into the image ...

$$\mathbf{v} = \boldsymbol{K} \left[\ \boldsymbol{R} \mid \mathbf{t} \ \right] \begin{bmatrix} \mathbf{D} \\ 0 \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

Preamble II: Homogeneous notation for lines

Recall that a point $(x, y)^\top$ in 2D is represented by the homogeneous 3-vector

$$\mathbf{x} \stackrel{P}{=} (x_1, x_2, x_3)^\top$$

where $x = x_1/x_3,\ y = x_2/x_3$.

Equivalently $\mathbf{x} = \lambda(x, y, 1)^\top$.

The line $l_1 x + l_2 y + l_3 = 0$ in 2D is represented by the homogeneous 3 -vector

$$\mathbf{l} \stackrel{P}{=} (l_1, l_2, l_3)^\top$$

For example, the line $y = 1$ is written $-y + 1 = 0$, and so $\mathbf{l} = (0, -1, 1)^\top$ is a homogeneous representation — as would $(0, 47, -47)^\top$, etc.

A point x on the line l has

$$\mathbf{l}^\top \mathbf{x} \;=\; \mathbf{x}^\top \mathbf{l} \;=\; 0$$

or equivalently, thinking about scalar products, $\mathbf{l} \cdot \mathbf{x} = 0$.

Preamble II: Homogeneous notation for lines

Reminder: A point $\mathbf{x}$ on the line $\mathbf{l}$ has $\mathbf{l}^\top \mathbf{x} = 0$, $\mathbf{x}^\top \mathbf{l} = 0$, or $\mathbf{l} \cdot \mathbf{x} = 0$.

The line through two points $\mathbf{p}$ and $\mathbf{q}$ is given by

$$\mathbf{l} \stackrel{P}{=} \mathbf{p} \times \mathbf{q}.$$

Proof: Use the properties of the scalar triple product,

$$\mathbf{p} \cdot \mathbf{l} = \mathbf{p} \cdot (\mathbf{p} \times \mathbf{q}) \equiv 0 \quad \mathbf{q} \cdot \mathbf{l} = \mathbf{q} \cdot (\mathbf{p} \times \mathbf{q}) \equiv 0 \ .$$

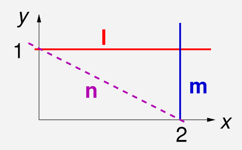
The intersection of two lines is the point

$$\mathbf{x} \stackrel{P}{=} \mathbf{l}_1 \times \mathbf{l}_2$$

Proof: For you to fill in ...

[**] Example ... DIY

♣ Find (i) the point of intersection of the lines **l** and **m**; (ii) the point of intersection of the line **l** and the x-axis; and (iii) the equation of the line **n** joining $(0,1)$ and $(2,0)$



$\mathbf{l} = (0, -1, 1)$ and $\mathbf{m} = (-1, 0, 2)$.

$$\mathbf{x} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$ which is the point $(2,1)$.

The x-axis is $(0,1,0)$ and so it and **l** meet at

$$\mathbf{x} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

which is a 2D point at infinity at $x = \pm\infty$.

$$\mathbf{n} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$ which is the line $x + 2y - 2 = 0$ or

$y = -(1/2)x + 1$

Preamble III: Matrix representation of vector products

The vector product $\mathbf{a} \times \mathbf{b} =$

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\boldsymbol{a}]_\times \mathbf{b}$$

$[\boldsymbol{a}]_\times$ is a 3×3 skew-symmetric matrix and has rank=2.

$\mathbf{a}$ is the kernel of $[\boldsymbol{a}]_\times$. (Why?)

Example: compute the vector product of $\mathbf{l} = (1, 2, 3)$ and $\mathbf{m} = (2, 3, 4)$.
Pseudo-determinant method gives $(-1, 2, 1)^\top$.
Skew-sym method gives

$$\begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

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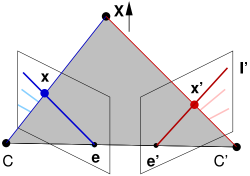
Algebraic representation of Epipolar Geometry

We now know that the epipolar geometry defines the

$\boxed{\text{mapping from point } \mathbf{x} \text{ to line } \mathbf{l}'.}$



The mapping depends only the cameras, not on the structure. This means

$\boxed{\text{The mapping depends on the overall projection matrices } \boldsymbol{P} \text{ and } \boldsymbol{P}'.}$

We will show that the mapping is linear, and can be written as

$\boxed{\mathbf{l}' = \boldsymbol{F}\mathbf{x}, \text{ where } \boldsymbol{F} \text{ is the } \textbf{fundamental matrix}}$

Algebraic representation of Epipolar Geometry /ctd

With no loss of generality we can use the first camera $\mathcal{C}$ to define the world coordinate frame, so that its overall 3×4 projection matrix is

$$\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{I}|\mathbf{0}]$$

We will define the rotation and translation between cameras frames as

$$\mathbf{X}' = \begin{bmatrix} \boldsymbol{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \mathbf{X}$$

So that

$$\boldsymbol{P}' = \boldsymbol{K}'[\boldsymbol{R}|\mathbf{t}]$$

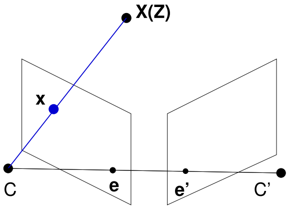
NB, the camera intrinsics can be different.

Algebraic representation of Epipolar Geometry /ctd

Step 1: back project a ray from $\mathcal{C}$

Point $\mathbf{x}$ back-projects to ray $\mathbf{X}(\zeta)$ that satisfies

$$\boldsymbol{P}\mathbf{X}(\zeta) = \boldsymbol{K}[\boldsymbol{I}|\mathbf{0}]\mathbf{X}(\zeta) = \mathbf{x}$$



where we use ζ as a parameter.

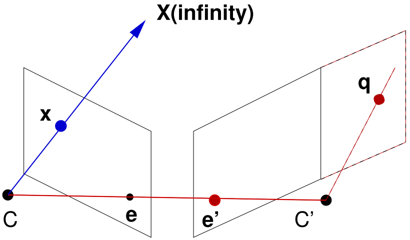
$$\text{Now } \mathbf{x} \overset{P}{=} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boldsymbol{K}[\boldsymbol{I}|\mathbf{0}] \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \boldsymbol{K} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{X}(\zeta) = \begin{bmatrix} \zeta[\boldsymbol{K}]^{-1}\mathbf{x} \\ 1 \end{bmatrix} \text{ and } \Rightarrow \mathbf{X}(\infty) = \begin{bmatrix} [\boldsymbol{K}]^{-1}\mathbf{x} \\ 0 \end{bmatrix}$$

In effect, $[\boldsymbol{K}]^{-1}$ corrects the direction of the ray. Direction $\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$ would be incorrect, because $\mathbf{x}$ was measured in a non-ideal camera.

Algebraic representation of Epipolar Geometry /ctd

Step 2: Choose two points on the ray and project the into the second camera $\mathcal{C}'$



Choose Point (1) with $\zeta = 0$

$\Rightarrow$It is the optical centre $\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$

Choose Point (2) with $\zeta = \infty$

$\Rightarrow$A pt at infinity $\begin{bmatrix} \boldsymbol{K}^{-1}\mathbf{x} \\ 0 \end{bmatrix}$

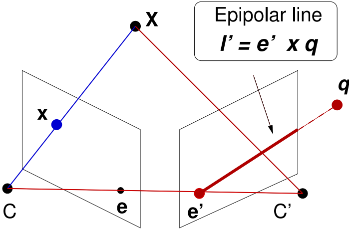
Project these two points into $\mathcal{C}'$

$$(1) \quad \mathbf{e}' = \boldsymbol{K}'[\boldsymbol{R}|\mathbf{t}]\begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} = \boldsymbol{K}'\mathbf{t} \qquad (2) \quad \mathbf{q} = \boldsymbol{K}'[\boldsymbol{R}|\mathbf{t}]\begin{bmatrix} \boldsymbol{K}^{-1}\mathbf{x} \\ 0 \end{bmatrix} = \boldsymbol{K}'\boldsymbol{R}\boldsymbol{K}^{-1}\mathbf{x}$$

Note that the first point is the epipole.

Algebraic representation of Epipolar Geometry /ctd

Step 3: Use vector product to find epipolar line.



$$\mathbf{l}' = (\boldsymbol{K}'\mathbf{t}) \times \boldsymbol{K}'\boldsymbol{R}\boldsymbol{K}^{-1}\mathbf{x}$$

Now tidy up ...

(1) using the general identity $(\boldsymbol{M}\mathbf{a}) \times (\boldsymbol{M}\mathbf{b}) = \boldsymbol{M}^{-\top}(\mathbf{a} \times \mathbf{b})$, where $\boldsymbol{M}^{-\top} = [\boldsymbol{M}^{-1}]^{\top}$, and (2) using the skew symmetric matrix

Hence

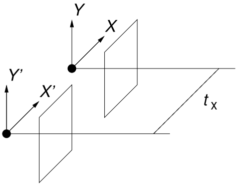
$$\mathbf{l}' = [\boldsymbol{K}']^{-\top}(\mathbf{t} \times \boldsymbol{R}\boldsymbol{K}^{-1}\mathbf{x}) = [\boldsymbol{K}']^{-\top}[\boldsymbol{t}]_{\times}\boldsymbol{R}[\boldsymbol{K}]^{-1}\mathbf{x}$$

So

Epipolar Line is:	$\mathbf{l}' = \boldsymbol{F}\mathbf{x}$
Fundamental matrix is:	$\boldsymbol{F} = [\boldsymbol{K}']^{-\top}[\boldsymbol{t}]_{\times}\boldsymbol{R}\boldsymbol{K}^{-1}$
As $\mathbf{x}'^{\top}\mathbf{l}' = 0$ ...	$\mathbf{x}'^{\top}\boldsymbol{F}\mathbf{x} = 0$

Example: identical parallel cameras

$$\mathbf{K} = \mathbf{K}' = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{R} = \mathbf{I}, \quad \mathbf{t} = (t_x, 0, 0)^\top$$



$$\begin{aligned} \mathbf{F} &= [\mathbf{K}']^{-\top}[\mathbf{t}]_\times[\mathbf{R}][\mathbf{K}]^{-1} \\ &= \begin{bmatrix} 1/f & 0 & 0 \\ 0 & 1/f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -t_x \\ 0 & t_x & 0 \end{bmatrix} [\mathbf{I}] \begin{bmatrix} 1/f & 0 & 0 \\ 0 & 1/f & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \quad \mathbf{x}'^\top \mathbf{F}\mathbf{x} = [x' \; y' \; 1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

which reduces to ....

Example /ctd

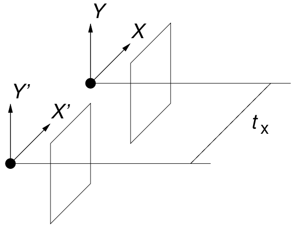
In the exercises you are asked to show that the epipole is in the right nullspace of the fundamental matrix — that is,

$$\boldsymbol{F}\mathbf{e} = \mathbf{0}$$

By inspection, for the parallel identical cameras example:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \qquad \Rightarrow \text{Evidently } \mathbf{e} \stackrel{P}{=} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

What is the geometric interpretation of this result?



Summary of properties

$\boldsymbol{F}$ is a rank 2 matrix with 7 d.o.f.

If $\mathbf{x} \leftrightarrow \mathbf{x}'$ then $\mathbf{x}'^\top \boldsymbol{F}\mathbf{x} = 0$

The epipolar lines in $\mathcal{C}$ and $\mathcal{C}'$ are

$$\mathbf{l} = \boldsymbol{F}^\top \mathbf{x}' \qquad \mathbf{l}' = \boldsymbol{F}\mathbf{x}$$

The epipoles in $\mathcal{C}$ and $\mathcal{C}'$ are obtained from

$$\boldsymbol{F}\mathbf{e} = \mathbf{0} \qquad \boldsymbol{F}^\top \mathbf{e}' = \mathbf{0}$$

(The last is $\mathbf{e}'^\top \boldsymbol{F} = \mathbf{0}^\top$. That is, $\mathbf{e}'$ is the in the left nullspace of $\boldsymbol{F}$.)

For $\boldsymbol{P} = \boldsymbol{K}[\boldsymbol{I}|\mathbf{0}]$ and $\boldsymbol{P}' = \boldsymbol{K}'[\boldsymbol{R}|\mathbf{t}]$ the fundamental matrix is derived as

$$\boldsymbol{F} = [\boldsymbol{K}']^{-\top}[\boldsymbol{t}]_\times[\boldsymbol{R}][\boldsymbol{K}]^{-1}$$

where $-\top$ denotes transpose of the inverse.

Computing F: Algebraic minimizations

The basis for several methods of computing $\boldsymbol{F}$ lies in re-writing the constraint $\mathbf{x}'^{\top}\boldsymbol{F}\mathbf{x} = 0$ for each match $\mathbf{x} \leftrightarrow \mathbf{x}'$ as

$$\begin{bmatrix} x'x & x'y & x' & y'x & y'y & y' & x & y & 1 \end{bmatrix} \begin{bmatrix} F_{11} \\ \vdots \\ F_{33} \end{bmatrix} = 0 \, .$$

Inserting a row for each of M matches builds the system

$$\boldsymbol{A}_{M\times 9}\mathbf{f}_{9\times 1} = \mathbf{0}_{M\times 1}$$

The properties of $\boldsymbol{A}$ and $\mathbf{f}$ are then exploited in various ways.

In Lecture 5 you'll see an 8-point method and a more expensive least squares method using a proper image cost, but here let's look at a 7-point algorithm which minimizes an algebraic cost.

Minimal linear 7-point solution for $\boldsymbol{F}$

With 7 entries, matrix $\boldsymbol{A}_{7\times 9}$ has rank 7 and nullity 2.

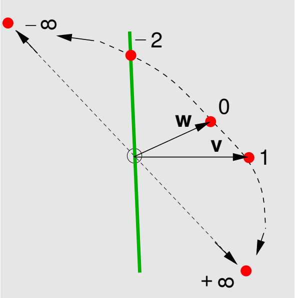
Let vectors $\mathbf{v}$ and $\mathbf{w}$ be two vectors spanning the nullspace of $\boldsymbol{A}$. Every $\mathbf{f} = (\alpha\mathbf{v} + \beta\mathbf{w})$ satisfies $\boldsymbol{A}\mathbf{f} = \mathbf{0}$, but we need to find an $\mathbf{f}$ that give $\det\boldsymbol{F} = 0$.

Surprisingly, we don't need to consider every α and β ...

For any $\mathbf{f}$ which is a solution, any scaling (+ve of -ve) of $\mathbf{f}$ is also a solution. So to explore the entire plane containing $\mathbf{v}$ and $\mathbf{w}$ we need only map out a **path** that traverses a **half plane**.

One such path is $\mathbf{f} = \alpha\mathbf{v} + (1-\alpha)\mathbf{w}$, where α runs from $-\infty$ to $+\infty$.

Eg, $\alpha = -2$ deals with all solutions on the green line.



Notice that $\alpha \to \pm\infty$ generates points $\to \pm\alpha(\mathbf{v} - \mathbf{w})$ which define the half plane division.

7-point solution for $\boldsymbol{F}$

We want to find an α along that path that makes $\det\boldsymbol{F} = 0$.

However, we have to take care!
$\det\boldsymbol{F} = 0$ is a cubic in α

$$\det\boldsymbol{F} = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 = 0 \ .$$

So there are 3 solutions.

The algorithm goes like

- Generate the matches
- Statistically centre all sets of x and y values.
- Build $\boldsymbol{A}$ from seven of the matches.
- Use SVD to find the two vectors $\mathbf{v}$, $\mathbf{w}$ spanning the nullspace.
- Use $\mathbf{v}$, $\mathbf{w}$ to find the coeffs of the cubic
- Solve the cubic, and test which α is best.

Skeleton Matlab code

```
% xc1 and xc2 are 3 x npoint
% matrices of matched and
% centred image points
%
% Choose 7 of them
% Generate a complete A matrix
  for (i=1:7)
    A(i,1) = xc2(1,i) * xc1(1,i);
    A(i,2) = xc2(1,i) * xc1(2,i);
    A(i,3) = xc2(1,i);
    A(i,4) = xc2(2,i) * xc1(1,i);
    A(i,5) = xc2(2,i) * xc1(2,i);
    A(i,6) = xc2(2,i);
    A(i,7) = xc1(1,i);
    A(i,8) = xc1(2,i);
    A(i,9) = 1;
end
% Perform SVD to obtain
% nullspace of A
  [U,S,V] = svd(A);
  v = V(:,8);
  w = V(:,9);
```

```
% find the cubic coeffs
% from a subroutine
  [a0,a1,a2,a3] = ...
            f_cubic_coeffs(v,w);
  coeffs=[a3,a2,a1,a0];

% Solve the roots of polynomial
  alpha=roots(coeffs);

% for alpha(1), (2), (3)
% generate f 9x1 vector
  f = alpha(1)*v +(1-alpha(1))*w;
% convert into 3x3 matrix
  F1 = fvec_to_Fmat(f);

% ... etc ... for F2 and F3



% then test which is best!
```

Cubic coefficients ...

Why cubic? Writing $\mathbf{d} = \mathbf{v} - \mathbf{w}$, then $\mathbf{f} = \alpha\mathbf{d} + \mathbf{w}$, and

$$\det \boldsymbol{F} = \begin{vmatrix} \alpha d_1 + w_1 & \alpha d_2 + w_2 & \alpha d_3 + w_3 \\ \alpha d_4 + w_4 & \alpha d_5 + w_5 & \alpha d_6 + w_6 \\ \alpha d_7 + w_7 & \alpha d_8 + w_8 & \alpha d_9 + w_9 \end{vmatrix}$$

Minutes of fun later, you'll find

$$\begin{aligned} a_0 &= +w_1 w_5 w_9 + w_2 w_6 w_7 + w_3 w_4 w_8 - w_1 w_6 w_8 - w_2 w_4 w_9 - w_3 w_5 w_7 \\ a_1 &= d_1 w_5 w_9 + w_1 d_5 w_9 + w_1 w_5 d_9 + d_2 w_6 w_7 + w_2 d_6 w_7 + w_2 w_6 d_7 + d_3 w_4 w_8 + w_3 d_4 w_8 + w_3 w_4 d_8 \\ &\quad - d_1 w_6 w_8 - w_1 d_6 w_8 - w_1 w_6 d_8 - d_2 w_4 w_9 - w_2 d_4 w_9 - w_2 w_4 d_9 - d_3 w_5 w_7 - w_3 d_5 w_7 - w_3 w_5 d_7 \\ a_2 &= +d_1 d_5 w_9 + d_1 w_5 d_9 + w_1 d_5 d_9 + d_2 d_6 w_7 + d_2 w_6 d_7 + w_2 d_6 d_7 + d_3 d_4 w_8 + d_3 w_4 d_8 + w_3 d_4 d_8 \\ &\quad - d_1 d_6 w_8 - d_1 w_6 d_8 - w_1 d_6 d_8 - d_2 d_4 w_9 - d_2 w_4 d_9 - w_2 d_4 d_9 - d_3 d_5 w_7 - d_3 w_5 d_7 - w_3 d_5 d_7 \\ a_3 &= +d_1 d_5 d_9 + d_2 d_6 d_7 + d_3 d_4 d_8 - d_1 d_6 d_8 - d_2 d_4 d_9 - d_3 d_5 d_7 \end{aligned}$$

Tedious, but easy enough to write a matlab routine

```
function [a0,a1,a2,a3] = f_cubic_coeffs(v,w)
d= v-w;
a0 =   w(1)*w(5)*w(9) + w(2)*w(6)*w(7) + w(3)*w(4)*w(8) ...
     - w(1)*w(6)*w(8) - w(2)*w(4)*w(9) - w(3)*w(5)*w(7);
a1 = ... blah blah blah ...
```

[**For info**] Centering the data

For numerical stability it is essential that the elements of $\boldsymbol{A}$ are not of markedly different orders of magnitude.

The standard method of "statistical centering" an image quantity x involves shifting and scaling the data so that the centred data have $\mu_{x_{\boldsymbol{C}}} = 0$ and standard deviation of $\sigma_{x_{\boldsymbol{C}}} = \sqrt{2}$.

Treating x and y as independent, a centered point is

$$\mathbf{x}^C = \boldsymbol{C}\mathbf{x} = \begin{bmatrix} \sqrt{2}/\sigma_x & 0 & -\sqrt{2}\mu_x/\sigma_x \\ 0 & \sqrt{2}/\sigma_y & -\sqrt{2}\mu_y/\sigma_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and similarly for points $\mathbf{x}'$ in the other image.

Using centered points in $\boldsymbol{A}$, the required matrix is obtained from the $\boldsymbol{F}^C$ recovered via SVD as

$$\boldsymbol{F} = [\boldsymbol{C}']^\top \boldsymbol{F}^C [\boldsymbol{C}]$$

You are just undoing the linear transformation.

A note on the Essential Matrix

The fundamental matrix $\boldsymbol{F} = [\boldsymbol{K}']^{-\top}[\boldsymbol{t}]_\times \boldsymbol{R}\boldsymbol{K}^{-1}$ rrequires $\mathbf{x}'^\top \boldsymbol{F}\mathbf{x} = 0$

If however we know the intrinsic calibrations, we can transform the matching points into their respective ideal images.

Points in the ideal images are related by

$$\mathbf{x}'^\top \boldsymbol{E}\mathbf{x} = 0$$

where the Essential Matrix

$$\boldsymbol{E} = [\boldsymbol{t}]_\times \boldsymbol{R}$$

Properties of the essential matrix were derived (in a variety of forms) in the early 1980's. The broad sweep of methods to be described for computing $\boldsymbol{F}$ have counterparts for $\boldsymbol{E}$, although opportunities (and challenges) arise from its reduced number of degrees of freedom and special form.

Summary

3.1 Introduction

3.2 Epipolar Geometry

- Clues from Parallel Cameras; the Epipolar Plane and lines; Cat's whiskers for converging cameras

3.3 Algebraic Representation and the Fundamental Matrix

- Representing the epipolar plane; linear relationship between point in one image and epipolar line in the other.

3.4 Computing the F matrix: a 7-point method

- A minimal linear method. Simple code available from course page.