Differential Equations and Linear Algebra
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All errors, typographical and substantive, and other offenses, are entirely my own.

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2 - Second Order Equations

2.1 - Second Derivatives in Science and Engineering

> **Question: 2.1.1**
>
> Find a cosine and a sine that solve $d^2y/dt^2 = -9y$. This is a second order equation so we expect *two constants* C and D (from integrating twice):
>
> **Simple harmonic motion** $y(t) = C\cos{(\omega t)} + D\sin{(\omega t)}$
>
> What is ω? If the system starts from rest (this means $dy/dt = 0$ at $t = 0$), which constant C or D will be zero?

Differentiating $y(t)$ twice, we get a ω^2 term. Making the necessary substitutions, we can see that $\omega^2 = 9$, implying $\omega = 3$. Thus we have $y = \sin{(3t)}$ and $y = \cos{(3t)}$. The constants C and D are determined by the initial conditions. Assuming the system starts at rest, we must have

$$\frac{dy}{dt} = -3C\sin{(3t)} + 3D\cos{(3t)} = 0$$

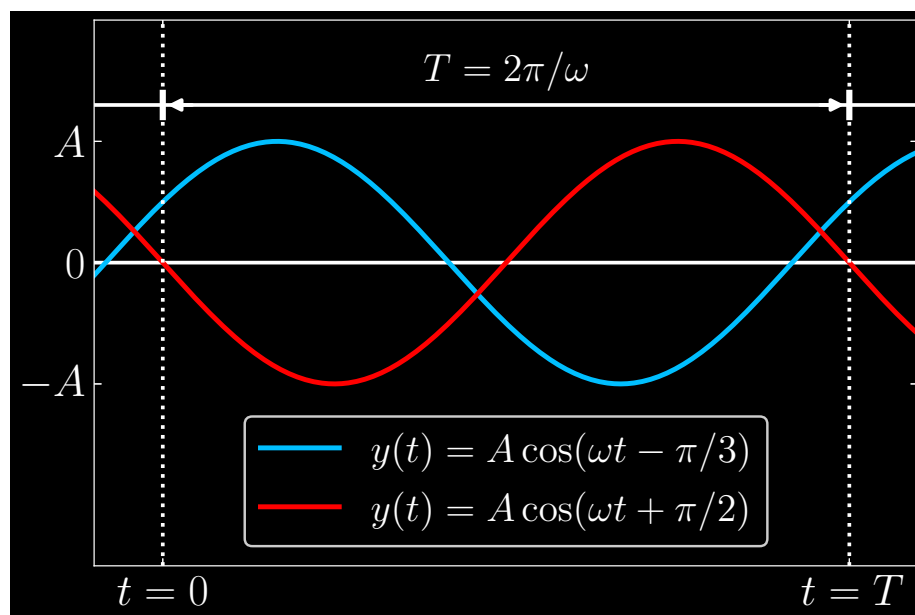which implies that $dy/dt_{t=0} = 3D = 0$, thus $D = 0$.

> **Question: 2.1.2**
>
> In Problem 1, which C and D will give the starting values $y(0) = 0$ and $y'(0) = 1$?

We have $y(0) = C = 0$ and $y'(0) = 3D = 1$, or $D = 1/3$.

Question: 2.1.3

Draw Figure 2.3 to show simple harmonic motion $y = A\cos(\omega t - \alpha)$ with phases $\alpha = \pi/3$ and $\alpha = -\pi/2$.

Question: 2.1.4

Suppose the circle in Figure 2.4 has radius 3 and circular frequency $f = 60$ Hertz. If the moving point starts at the angle $-45°$, find its x-coordinate $A\cos(\omega t - \alpha)$. The phase lag is $\alpha = 45°$. When does the point first hit the x-axis?

The circular motion of the point is expressed by the sinusoidal

$$3\cos\left(120\pi t - \frac{\pi}{4}\right)$$

Note that since the frequency is $f = 60$ Hertz, the angular frequency is $2\pi \cdot 60 = 120\pi$ radians s^{-1}. The point hits the x-axis when the argument of the cosine is zero, namely at

$$120\pi t - \frac{\pi}{4} = 0 \quad \Longrightarrow \quad t = \frac{1}{480}\text{s}$$

Question: 2.1.5

If you drive at 60 miles per hour on a circular track with radius $R = 3$ miles, what is the time T for one complete circuit? Your circular frequency is $f = $ ____ and your angular frequency is $\omega = $ ____ (with what units?). The period is T.

Using dimensional analysis, the period is

$$T = \frac{1 \text{ hr}}{60 \text{ mi}} (3 \text{ mi}) = \frac{1}{20} \text{ hr}$$

The circular frequency is

$$f = \frac{60 \text{ mi}}{\text{hr}} \frac{1 \text{ hr}}{3600 \text{ s}} \frac{1 \text{ cycle}}{3 \text{ mi}} = \frac{1}{180} \text{ s}^{-1}$$

with angular frequency

$$\omega = 2\pi f = \frac{\pi}{90} \text{ rad s}^{-1}$$

Question: 2.1.6

The total energy E in the oscillating spring-mass system is

$$E = \textbf{kinetic} \text{ energy in mass} + \textbf{potential} \text{ energy in spring}$$

$$= \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2$$

Compute E when $y = C\cos(\omega t) + D\sin(\omega t)$. The energy is constant!

Given $y(t)$, we have first time-derivative

$$\frac{dy}{dt} = -\omega C \sin(\omega t) + \omega D \cos(\omega t)$$

squaring gives us

$$\left(\frac{dy}{dt}\right)^2 = \omega^2 \left(C^2 \sin^2(\omega t) - 2CD \sin(\omega t)\cos(\omega t) + D^2 \cos^2(\omega t)\right)$$

Lastly, the y^2 term is

$$y^2 = C^2 \cos^2(\omega t) + 2CD \sin(\omega t)\cos(\omega t) + D^2 \sin^2(\omega t)$$

Combining our ingredients, with $\omega = \sqrt{k/m}$, observe that energy E reduces to a constant:

$$\begin{aligned}
E &= \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2 \\
&= \frac{m}{2}\left(\frac{k}{m}\right)\left(C^2 \sin^2(\omega t) - 2CD \sin(\omega t)\cos(\omega t) + D^2 \cos^2(\omega t)\right) \\
&\quad + \frac{k}{2}\left(C^2 \cos^2(\omega t) + 2CD \sin(\omega t)\cos(\omega t) + D^2 \sin^2(\omega t)\right) \\
&= C^2 + D^2
\end{aligned}$$

Question: 2.1.7

Another way to show that the total energy E is constant:
Multiply $my'' + ky = 0$ by y'. Then integrate $my'y''$ and kyy'.

Take the first term and integrate:

$$m \int y'y'' \, dt$$

By integration by parts, let

$$u = \frac{dy}{dt}, \quad \frac{du}{dt} = \frac{d^2y}{dt^2} \quad \implies \quad du = \frac{d^2y}{dt^2} dt$$

Then we can rewrite the first term as

$$m \int u \, du = \frac{m}{2}u^2 + C = \frac{m}{2}(y')^2 + C$$

as for the second term:

$$k \int yy' \, dt = k \int y\frac{dy}{dt} \, dt = k \int y \, dy = \frac{k}{2}y^2 + C$$

Summing the two pieces (sans the constants) restores the original energy function:

$$E = \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2$$

and since the derivative of a constant is zero, it must be the case that E is a constant.

Question: 2.1.8

A **forced oscillation** has another term in the equation and $A\cos(\omega t)$ in the solution:

$$\frac{d^2y}{dt^2} + 4y = F\cos(\omega t) \quad \text{has} \quad y = C\cos(2t) + D\sin(2t) + A\cos(\omega t)$$

(a) Substitute y into the equation to see how C and D disappear (they give y_n. Find the forced amplitude A in the particular solution $y_p = A\cos(\omega t)$.

(b) In case $\omega = 2$ (forcing frequency = natural frequency), what answer does your formula give for A? The solution formula for y breaks down in this case.

The second time-derivative is

$$\frac{d^2y}{dt^2} = -4C\cos(2t) - 4D\sin(2t) - \omega^2 A\cos(\omega t)$$

(a) We have
$$\frac{d^2y}{dt^2} + 4y = \left(4 - \omega^2\right) A \cos\left(\omega t\right) = F \cos\left(\omega t\right)$$
implying $A = \dfrac{F}{4 - \omega^2}$.

(b) When $\omega = 2$, A is undefined.

Question: 2.1.9

Following Problem 8, write down the complete solution $y_n + y_p$ to the equation

$$m\frac{d^2y}{dt^2} + ky = F \cos\left(\omega t\right) \quad \text{with} \quad \omega \neq \omega_n = \sqrt{k/m} \quad \text{(no resonance)}$$

The answer y has free constants C and D to match $y(0)$ and $y'(0)$ (A is *fixed* by F.

Per Problem 8, we have solution

$$y = \underbrace{C \cos\left(\sqrt{\frac{k}{m}}t\right) + D \sin\left(\sqrt{\frac{k}{m}}t\right)}_{y_n} + \underbrace{\frac{F}{k - m\omega^2} \cos\left(\omega t\right)}_{y_p}$$

All this involves is dividing the equation through by m, understanding that the angular frequency of the sinusoidals in the null solution is the square root of k/m, and making the changes to our formula for A accordingly.

Question: 2.1.10

Suppose Newton's Law $F = ma$ has the force F in the *same* direction as a:
$$my'' = +ky \quad \text{including} \quad y'' = 4y$$
Find two possible choices of s in the exponential solutions $y = e^{st}$. The solution is not sinusoidal and s is real and the oscillations are gone. Now y is unstable.

Substituting in y, we find

$$ms^2 e^{st} = k e^{st}$$

This forces

$$s = \pm\sqrt{\frac{k}{m}}$$

Question: 2.1.11

Here is a *fourth* order equation: $d^4y/dt^4 = 16y$. Find *four* values of s that give exponential solutions $y = e^{st}$. You could expect four initial conditions on y: $y(0)$ is given along with what three other conditions?

Equivalently, we find the four complex roots of 16:

$$s^4 = 16$$

which are $s = \pm 2, \pm 2i$.

Question: 2.1.12

To find a particular solution to $y'' + 9y = e^{ct}$, I would look for a multiple $y_p(t) = Ye^{ct}$ of the forcing function. What is that number Y? When does your formula give $Y = \infty$? (Resonance needs a new formula for Y.)

Let $y_p = Ye^{ct}$. Substituting, we find

$$Yc^2e^{ct} + 9Ye^{ct} = e^{ct}$$

Solving for Y yields $Y = \dfrac{1}{c^2 + 9}$. When $c \to \pm 3$, we have resonance, and $Y \to \infty$.

Question: 2.1.13

In a particular solution $y = Ae^{i\omega t}$ to $y'' + 9y = e^{i\omega t}$, what is the amplitude A? The formula blows up when the forcing frequency $\omega = $ what natural frequency?

Substituting, we derive

$$-\omega^2 Ae^{i\omega t} + 9Ae^{i\omega t} = e^{i\omega t}$$

which gives us $A = \dfrac{1}{9 - \omega^2}$. Resonance is when $\omega = 3$.

Question: 2.1.14

Equation (10) says that the tangent of the phase angle is $\tan(\alpha) = y'(0)/\omega y(0)$. First, check that $\tan(\alpha)$ is dimensionless when y is in meters and time is in seconds. Next, if that ratio is $\tan(\alpha) = 1$, should you choose $\alpha = \pi/4$ or $\alpha = 5\pi/4$? Answer:

Separately you want $R\cos(\alpha) = y(0)$ and $R\sin(\alpha) = y'(0)/\omega$

If those right hand sides are positive, choose the angle α between 0 and $\pi/2$.
If those right hand sides are negative, add π and choose $\alpha = 5\pi/4$.

Question: If $y(0) > 0$ and $y'(0) < 0$, does α fall between $\pi/2$ and π or between $3\pi/2$ and 2π? If you plot the vector from $(0,0)$ to $(y(0), y'(0)/\omega)$, its angle is α.

As $y(0) > 0$ and $y'(0) < 0$ requires positive cosine and negative sine, α falls between $3\pi/2$ and 2π.

Question: 2.1.15

Find a point on the sine curve in Figure 2.1 where $y > 0$ but $v = y' < 0$ and also $a = y'' < 0$. The curve is sloping down and bending down.

Find a point where $y < 0$ but $y' > 0$ and $y'' > 0$. The point is below the x-axis but the curve is sloping _____ and bending _____.

One area corresponding to the first set of conditions is $\pi/2 < t < \pi$. As for the second, we have $3\pi/2 < t < 2\pi$.

Question: 2.1.16

(a) Solve $y'' + 100y = 0$ starting from $y(0) = 1$ and $y'(0) = 10$. (**This is y_n.**)

(b) Solve $y'' + 100y = \cos(\omega t)$ with $y(0) = 0$ and $y'(0) = 0$. (**This can be y_p.**)

(a) Let $y = c_1\cos(10t) + c_2\sin(10t)$. Then $y(0) = c_1 = 1$ and $y'(0) = 10c_2 = 10$ implies $c_2 = 1$. Then the null solution is

$$y_n = \cos(10t) + \sin(10t)$$

(b) Let $y_p = R\cos(\omega t)$. Substitute to find

$$-\omega^2 R\cos(\omega t) + 100R\cos(\omega t) = \cos(\omega t)$$

Isolate R to derive

$$R = \frac{1}{100 - \omega^2}$$

The solution to this set of initial conditions requires $y(0) = 0$ and $y'(0) = 0$. Begin with

$$y(t) = c_1 \cos\left(10t\right) + c_2 \sin\left(10t\right) + \frac{1}{100 - \omega^2} \cos\left(\omega t\right)$$

From the first condition, we have $c_1 = -\dfrac{1}{100 - \omega^2}$. From the second, we have $c_2 = 0$. The full solution is

$$y(t) = \frac{1}{100 - \omega^2}\left(\cos\left(\omega t\right) - \cos\left(10t\right)\right)$$

Question: 2.1.17

Find a particular solution $y_p = R\cos\left(\omega t - \alpha\right)$ to $y'' + 100y = \cos\left(\omega t\right) - \sin\left(\omega t\right)$.

Substituting y_p gives us

$$-\omega^2 R\cos\left(\omega t - \alpha\right) + 100R\cos\left(\omega t - \alpha\right)$$
$$= \left(100R - \omega^2 R\right)\left[\cos\left(\omega t\right)\cos\left(\alpha\right) + \sin\left(\omega t\right)\sin\left(\alpha\right)\right]$$

This implies that

$$R\cos\left(\alpha\right)\left(100 - \omega^2\right) = 1$$
$$R\sin\left(\alpha\right)\left(100 - \omega^2\right) = -1$$

enabling us to conclude that $\alpha = 7\pi/4$, and amplitude

$$R = \frac{\sqrt{2}}{100 - \omega^2}$$

Ergo, the particular solution is

$$y_p = \frac{\sqrt{2}}{100 - \omega^2}\cos\left(\omega t - \frac{7\pi}{4}\right)$$

Question: 2.1.18

Simple harmonic motion also comes from a linear pendulum (like a grandfather clock). At time t, the height is $A\cos\left(\omega t\right)$. What is the frequency ω if the pendulum comes back to the start after 1 second? The period does not depend on the amplitude (a large clock or a small metronome or the movement in a watch can all have $T = 1$).

The angular frequency is $2\pi \cdot f = 2\pi$ radians s^{-1}.

Question: 2.1.19

If the phase lag is α, what is the time lag in graphing $\cos{(\omega t - \alpha)}$?

Put differently, we want to find the value of t' such that we are able to restore ωt as the cosine's argument. If we have

$$t' = t + \alpha/\omega$$

then we get

$$\cos{(\omega t' - \alpha)} = \cos{(\omega(t + \alpha/\omega) - \alpha)} = \cos{(\omega t)}$$

Thus the time lag term is α/ω.

Question: 2.1.20

What is the response $y(t)$ to a delayed impulse if $my'' + ky = \delta(t - T)$?

The full solution will be a factor of the step function, given by:

$$y(t) = \int_0^{t-T} \frac{\sin{(\omega_n(t - T - s))}}{m\omega_n} \delta(s) \ ds = \frac{\sin{(\omega_n(t - T))}}{m\omega_n} H(t - T)$$

Intuitively, when $t \leq T$, the right-hand side vanishes. No impulse is imparted, and thus there is no response. But when we are at time $t \geq T$ − after the threshold − the response kicks in.

Question: 2.1.21

(Good challenge) Show that $y = \int_0^t g(t - s) f(s) \ ds$ has $my'' + ky = f(t)$.

1. Why is $y' = \int_0^t g'(t - s) f(s) \ ds + g(0) f(t)$? Notice the two t's in y.

2. Using $g(0) = 0$, explain why $y'' = \int_0^t g''(t - s) f(s) \ ds + g'(0) f(t)$.

3. Now use $g'(0) = 1/m$ and $mg'' + kg = 0$ to confirm $my'' + ky = f(t)$.

Use the Leibniz integral rule:

$$\frac{d}{dt} \left(\int_0^t g(t - s) f(s) \ ds \right) = g(0) f(t) + \int_0^t \frac{\partial}{\partial t} g(t - s) f(s) \ ds$$

(1) By above, applying the partial derivative inside the second term yields the desired result.

(2) One more application of the Leibniz rule gives us

$$y'' = \int_0^t g''(t - s) f(s) \ ds + g(0) f'(t) + g'(0) f(t)$$

With the premise $g(0) = 0$, the second term vanishes and gives us the expected result.

(3) Derive

$$my'' + ky = \int_0^t \left[mg''(t-s) + kg(t-s) \right] f(s) \, ds + f(t) = f(t)$$

where the last equality follows by appealing to the nullity of $g(t)$.

Question: 2.1.22

With $f = 1$ (direct current has $\omega = 0$) verify that $my'' + ky = 1$ for this y:

Step response

$$y(t) = \int_0^t \frac{\sin\left(\omega_n(t-s)\right)}{m\omega_n} \cdot 1 \, ds = y_p + y_n = \frac{1}{k} - \frac{1}{k}\cos\left(\omega_n t\right)$$

We have second derivative

$$y''(t) = \frac{\omega_n^2}{k} \cos\left(\omega_n t\right)$$

Since $\omega_n = \sqrt{k/m}$, $my''(t) = \cos\left(\omega_n t\right)$. Ergo, we have

$$my'' + ky = \cos\left(\omega_n t\right) + k\left[\frac{1}{k} - \frac{1}{k}\cos\left(\omega_n t\right) \right] = 1$$

Question: 2.1.23

(Recommended) For the equation $d^2y/dt^2 = 0$ find the null solution. Then for $d^2g/dt^2 = \delta(t)$ find the fundamental solution (start the null solution with $g(0) = 0$ and $g'(0) = 1$). For $y'' = f(t)$ find the particular solution using formula (16).

Integrating twice, the null solution is

$$y(t) = C_1 t + C_2$$

To find the fundamental solution $g(t)$, imposing the initial conditions $g(0) = 0$ and $g'(0) = 1$ forces $C_1 = 1$ and $C_2 = 0$, thus

$$g(t) = t$$

Lastly, if $y'' = f(t)$, then

$$y_p(t) = \int_0^t (t - s) f(s) \, ds$$

Question: 2.1.24

For the equation $d^2y/dt^2 = e^{i\omega t}$ find a particular solution $y = Y(\omega)\, e^{i\omega t}$. Then $Y(\omega)$ is the frequency response. Note the "resonance" when $\omega = 0$ with the null solution $y_n = 1$.

One particular solution is

$$y(t) = -\frac{e^{i\omega t}}{\omega^2}$$

meaning that $Y(\omega) = -1/\omega^2$. When $\omega = 0$, we have resonance, and this particular solution breaks down.

Question: 2.1.25

Find a particular solution $Ye^{i\omega t}$ to $my'' - ky = e^{i\omega t}$. The equation has $-ky$ instead of ky. What is the frequency response $Y(\omega)$? For which ω is Y infinite?

We have

$$\frac{d^2}{dt^2}\left(Ye^{i\omega t}\right) = -Y\omega^2 e^{i\omega t}$$

Substituting and canceling the $e^{i\omega t}$ terms, we get

$$-mY\omega^2 - kY = 1$$

Implying

$$Y(\omega) = -\frac{1}{m\omega^2 + k}$$

If we have
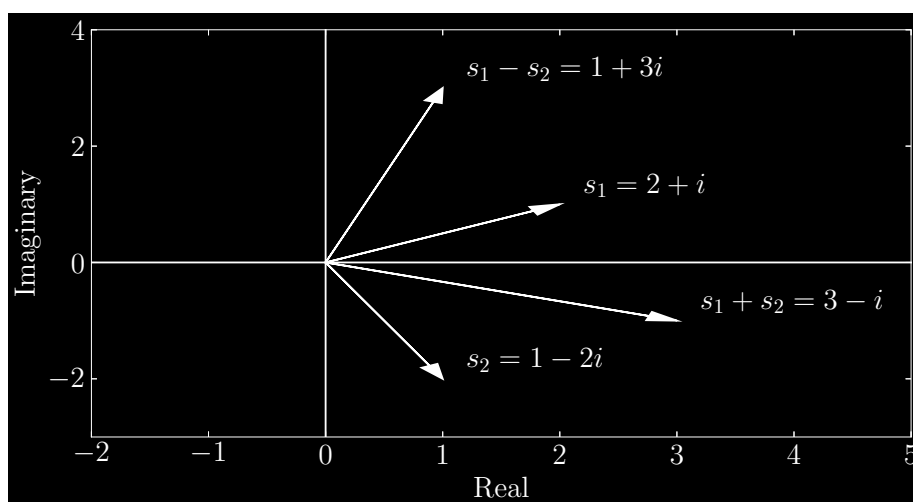
$$\omega = i\sqrt{\frac{k}{m}}$$

then $Y(\omega)$ diverges to infinity – meaning all real frequencies will not lead to this!

2.2 - Key Facts About Complex Numbers

Question: 2.2.1

Mark the numbers $s_1 = 2 + i$ and $s_2 = 1 - 2i$ as points in the complex plane. (The plane has a real axis and an imaginary axis.) Then mark the sum $s_1 + s_2$ and the difference $s_1 - s_2$.

Arithmetic involving complex numbers simply is vector superposition:

Question: 2.2.2

Multiply $s_1 = 2 + i$ times $s_2 = 1 - 2i$. Check absolute values: $|s_1||s_2| = |s_1 s_2|$.

We have
$$(2 + i)(1 - 2i) = 2 - 4i + i + 2 = 4 - 3i$$
with $|s_1| = |s_2| = \sqrt{5}$, $|s_1 s_2| = \sqrt{16 + 9} = 5$, ascertaining $|s_1||s_2| = |s_1 s_2|$.

Question: 2.2.3

Find the real and imaginary parts of $1/(2 + i)$.
Multiply by $(2 - i)/(2 - i)$:

$$\frac{1}{2 + i} \frac{2 - i}{2 - i} = \frac{2 - i}{|2 + i|^2} = \quad ?$$

The denominator is 5, so we have

$$\underbrace{\frac{2}{5}}_{\text{real}} - \underbrace{\frac{i}{5}}_{\text{imaginary}}$$

Question: 2.2.4

Triple angles   Multiply equation (10) by another $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to find formulas for $\cos(3\theta)$ and $\sin(3\theta)$.

Derive

$$
\begin{aligned}
(\cos(\theta) + i\sin(\theta))^3 &= \left(\cos^2(\theta) - \sin^2(\theta) + 2i\cos(\theta)\sin(\theta)\right)(\cos(\theta) + i\sin(\theta)) \\
&= \cos^3(\theta) - \sin^2(\theta)\cos(\theta) + 2i\cos^2(\theta)\sin(\theta) \\
&\quad + i\cos^2(\theta)\sin(\theta) - i\sin^3(\theta) - 2\cos(\theta)\sin^2(\theta) \\
&= \underbrace{\cos^3(\theta) - 3\sin^2(\theta)\cos(\theta)}_{\text{real}} + i\underbrace{\left[3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)\right]}_{\text{imaginary}}
\end{aligned}
$$

Question: 2.2.5

Addition formulas Multiply $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ times $e^{i\phi} = \cos(\phi) + i\sin(\phi)$ to get $e^{i(\theta+\phi)}$. Its real part is $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$. What is its imaginary part $\sin(\theta+\phi)$?

Derive

$$
\begin{aligned}
[\cos(\theta) + i\sin(\theta)][\cos(\phi) + i\sin(\phi)] &= \underbrace{\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)}_{\text{real}} \\
&\quad + i\underbrace{[\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi)]}_{\text{imaginary}}
\end{aligned}
$$

Question: 2.2.6

Find the real part and the imaginary part of each cube root of 1. Show directly that the three roots add to zero, as equation (11) predicts.

The cube roots, in polar form, are given by

$$
\begin{aligned}
e^{i2\pi/3} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\
e^{i4\pi/3} &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \\
&\quad 1
\end{aligned}
$$

which vanish when summed.

> **Question: 2.2.7**
>
> The three cube roots of 1 are z and z^2 and 1, when $z = e^{2\pi i/3}$. What are the three cube roots of 8 and the three cube roots of i? (The angle for i is 90° or $\pi/2$, so the angle for one of its cube roots will be ____. The roots are spaced by 120°.

In polar form, we can express 8 as $8e^{2\pi i}$. Let $z = e^{2\pi i/3}$. Its cube roots are then $2z$, $2z^2$, and 2.

For i, we have polar form $e^{\pi i/2}$. Then its cube roots are $e^{\pi i/6}$, $e^{5\pi i/6}$, and $e^{3\pi i/2}$.

Argue by vector superposition that the cube roots in all cases sum to zero.

> **Question: 2.2.8**
>
> (a) The number i is equal to $e^{\pi i/2}$. Then its ith power i^i comes out equal to a real number, using the fact that $(e^s)^t = e^{st}$. What is that real number i^i?
>
> (b) $e^{i\pi/2}$ is also equal to $e^{5\pi i/2}$. Increasing the angle by 2π does not change $e^{i\theta}$ - it comes around a full circle and back to i. Then i^i has another real value $\left(e^{5\pi i/2}\right)^i = e^{-5\pi/2}$. What are all the possible values of i^i?

(a) We have $\left(e^{\pi i/2}\right)^i = e^{-\pi/2}$.

(b) All possible values of i^i are $e^{(-\pi \pm 4n\pi)/2}$.

> **Question: 2.2.9**
>
> The numbers $s = 3 + i$ and $\bar{s} = 3 - i$ are complex conjugates. Find their sum $s + \bar{s} = -B$ and their product $(s)(\bar{s}) = C$. Then show that $s^2 + Bs + C = 0$ and also $\bar{s}^2 + B\bar{s} + C = 0$. Those numbers s and $\bar{s}$ are the two roots of the quadratic equation $x^2 + Bx + C = 0$.

The sum is $s + \bar{s} = 6$, so $B = -6$. Their product is $s\bar{s} = 10 = C$. Then we have

$$s^2 + Bs + C = s^2 - (s + \bar{s})\, s + s\bar{s} = 0$$
$$\bar{s}^2 + B\bar{s} + C = \bar{s}^2 - (s + \bar{s})\, \bar{s} + s\bar{s} = 0$$

> **Question: 2.2.10**
>
> The numbers $s = a + i\omega$ and $\bar{s} = a - i\omega$ are complex conjugates. Find their sum $s + \bar{s} = -B$ and their product $(s)(\bar{s}) = C$. Then show that $s^2 + Bs + C = 0$. The two solutions of $x^2 + Bx + C = 0$ are s and $\bar{s}$.

The sum is $s + \bar{s} = 2a$, so $B = -2a$. Their product is $s\bar{s} = a^2 + \omega^2$. By the same argument as the previous problem, the solutions of the given polynomial are s and $\bar{s}$.

Question: 2.2.11

(a) Find the numbers $(1+i)^4$ and $(1+i)^8$.

(b) Find the polar form $re^{i\theta}$ of $(1+i\sqrt{3})/(\sqrt{3}+i)$.

(a) We have $1+i = \sqrt{2}e^{i\pi/4}$. Then

$$\left(\sqrt{2}e^{i\pi/4}\right)^4 = 4e^{i\pi} = -4$$

Raising to the eighth power, we simply square our previous answer to get $(1+i)^8 = 16$.

(b) The numerator is $2e^{i\pi/3}$ and the denominator $2e^{i\pi/6}$. The quotient is $e^{i\pi/6}$.

Question: 2.2.12

The number $z = e^{2\pi i/n}$ solves $z^n = 1$. The number $Z = e^{2\pi i/2n}$ solves $Z^{2n} = 1$. How is z related to Z? (This plays a big part in the Fast Fourier Transform.)

They are related by $z = Z^2$.

Question: 2.2.13

(a) If you know $e^{i\theta}$ and $e^{-i\theta}$, how can you find $\sin(\theta)$?

(b) Find all angles θ with $e^{i\theta} = -1$ and all angles ϕ with $e^{i\phi} = -i$.

(a) One way to write $\sin(\theta)$ is

$$\frac{e^{i\theta} - e^{-i\theta}}{2} = \sin(\theta)$$

(b) We have $\theta = \pi \pm 2n\pi$ (which eliminates the imaginary term), and $\phi = -\frac{\pi}{2} \pm 2n\pi$.

Question: 2.2.14

Locate all these points on one complex plane:

(a) $2 + i$
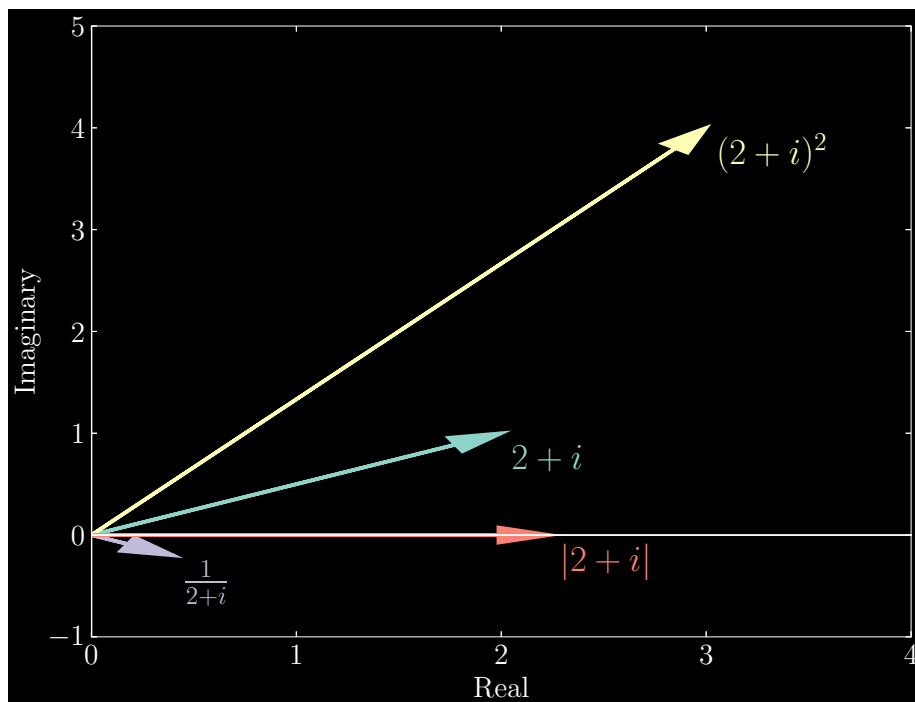
(b) $(2 + i)^2$

(c) $\dfrac{1}{2 + i}$

(d) $|2 + i|$

(a) $2 + i$

(b) $(2 + i)^2 = 3 + 4i$

(c) $\dfrac{1}{2 + i} = \dfrac{2 - i}{5}$

(d) $|2 + i| = \sqrt{5}$

Question: 2.2.15

Find the absolute values $r = |z|$ of these four numbers. If θ is the angle for $6 + 8i$, what are the angles for these four numbers?

(a) $6 - 8i$

(b) $(6 - 8i)^2$

(c) $\dfrac{1}{6 - 8i}$

(d) $8i + 6$

(a) $r = 10$, $\phi = -\theta$, in polar form $z = 10e^{-i\theta}$

(b) $r = 100$, $\phi = -2\theta$, in polar form $z = 100e^{-2\theta}$

(c) $r = 1/10$, $\phi = \theta$, in polar form $z = \dfrac{1}{10}e^{i\theta}$

(d) $r = 10$, $\phi = \theta$, the original number is unchanged

Question: 2.2.16

What are the real and imaginary parts of $e^{a+i\pi}$ and $e^{a+i\omega}$?

The real parts are $-e^a$ and $e^a \cos(\omega)$ and the imaginary parts are 0 and $e^a \sin(\omega)$.

Question: 2.2.17

(a) If $|s| = 2$ and $|z| = 3$, what are the absolute values of sz and s/z?

(b) Find upper and lower bounds in $L \leq |s + z| \leq U$. When does $|s + z| = U$?

(a) $|sz| = |s||z| = 6$, $|s/z| = |s|/|z| = 2/3$

(b) The lower bound is achieved when s and z are antiparallel, meaning the modulus of their vector sum is at best $L = |s + z| = 1$. The upper bound is reached when they are parallel, so $U = |s + z| = 5$.

Question: 2.2.18

(a) Where is the product $(\sin(\theta) + i\cos(\theta))(\cos(\theta) + i\sin(\theta))$ in the complex plane?

(b) Find the absolute value $|S|$ and the polar angle ϕ for $S = \sin(\theta) + i\cos(\theta)$.

This is my favorite problem, because S combines $\cos(\theta)$ and $\sin(\theta)$ in a new way. To find ϕ, you could plot S or add angles in the multiplication of part (a).

(a) Note that $\sin(\theta) = \cos\left(\dfrac{\pi}{2} - \theta\right)$ and $\cos(\theta) = \sin\left(\dfrac{\pi}{2} - \theta\right)$. Then the first term is equal to $\cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right) = e^{i(\pi/2-\theta)}$. Then the product is
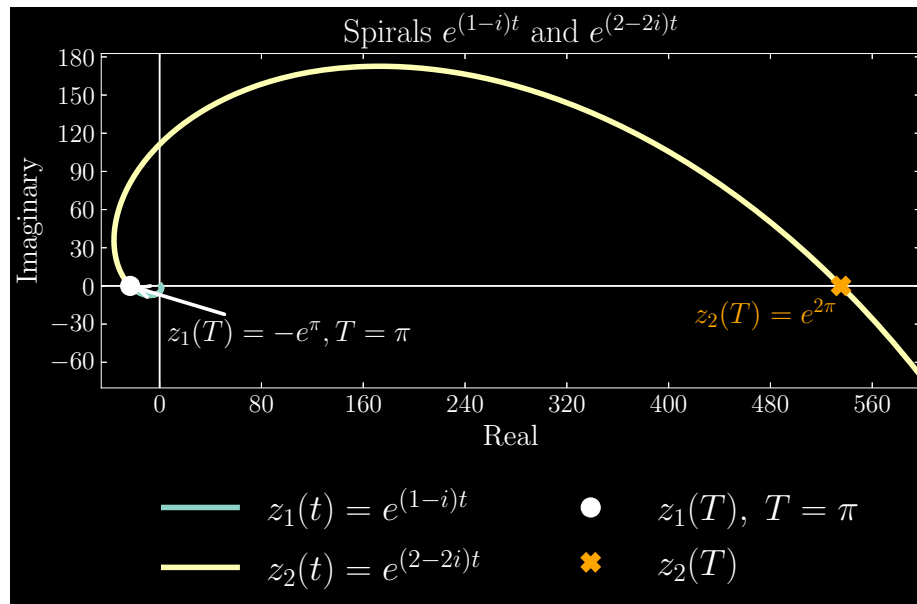
$$e^{i(\pi/2-\theta)}e^{i\theta} = e^{i\pi/2} = i$$

(b) $|S| = 1$ and $\phi = \dfrac{\pi}{2} - \theta$.

Question: 2.2.19

Draw the spirals $e^{(1-i)t}$ and $e^{(2-2i)t}$. Do those follow the same curves? Do they go clockwise or anticlockwise? When the first one reaches the negative x-axis, what is the time T? What point has the second one reached at that time?

As $t \to \infty$, the spirals progress clockwise. When the imaginary part vanishes, we have $T = \pi$, and $e^{(1-i)\pi} = e^{\pi}$. At that point in time, we also have $e^{(2-2i)\pi} = e^{2\pi}$.

Spirals $e^{(1-i)t}$ and $e^{(2-2i)t}$

> **Question: 2.2.20**
>
> The solution to $d^2y/dt^2 = -y$ is $y = \cos(t)$ if the initial conditions are $y(0) =$ _____ and $y'(0) =$ _____. The solution is $y = \sin(t)$ when $y(0) =$ _____ and $y'(0) =$ _____. Write each of those solutions in the form $c_1 e^{it} + c_2 e^{-it}$, to see that real solutions can come from complex c_1 and c_2.

For the case of $y = \cos(t)$, the initial conditions are $y(0) = 1$ and $y'(0) = 0$. In the case of $y = \sin(t)$, we have $y(0) = 0$ and $y'(0) = 1$. In the given form, the initial conditions require

$y = \cos(t)$

$$y(0) = c_1 + c_2 = 1$$
$$y'(0) = i(c_1 - c_2) = 0$$

Implying $c_1 = c_2 = 1/2$.

$y = \sin(t)$

$$y(0) = c_1 + c_2 = 0$$
$$y'(0) = i(c_1 - c_2) = 1$$

Implying $c_1 = 1/2i$ and $c_2 = -1/2i$.

Thus we can write $\cos(t)$ and $\sin(t)$ as

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}, \quad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

The general solution is given by

$$y(t) = C_1 \cos(t) + C_2 \sin(t)$$

which we rewrite as

$$
\begin{aligned}
y(t) &= C_1 \cos(t) + C_2 \sin(t) \\
&= C_1 \left[\frac{e^{it} + e^{-it}}{2} \right] + C_2 \left[\frac{e^{it} - e^{-it}}{2i} \right] \\
&= \left[\frac{C_1 - iC_2}{2} \right] e^{it} + \left[\frac{C_1 + iC_2}{2} \right] e^{-it}
\end{aligned}
$$

> **Question: 2.2.21**
>
> Suppose $y(t) = e^{-t} e^{it}$ solves $y'' + By' + Cy = 0$. What are B and C? If this equation is solved by $y = e^{3it}$, what are B and C?

Note that $y(t) = e^{-t} e^{it} = e^{(-1+i)t}$. Then we have

$$y'' + By' + Cy = (-1+i)^2 e^{(-1+i)t} + B(-1+i) e^{(-1+i)t} + C e^{(-1+i)t} = 0$$

from which we derive

$$(-1+i)^2 + B(-1+i) + C = 0$$

Now wait — this looks suspiciously familiar to the form seen in question 9:

$$s^2 + Bs + C = 0$$

Then we have that $s + \bar{s} = -B$ and $s\bar{s} = C$. Thus

$$s + \bar{s} = -2 = -B \qquad s\bar{s} = 2 = C$$

Therefore, $B = C = 2$, and the equation is $y'' + 2y' + 2y = 0$.
   In the case of $y = e^{3it}$, we have polynomial

$$-9 + 3Bi + C = 0$$

where it follows that $s = 3i$, $s + \bar{s} = 0 = -B$, and $s\bar{s} = 9 = C$. The equation is $y'' + 9y = 0$.

> **Question: 2.2.22**
>
> From the multiplication $e^{iA}e^{-iB} = e^{i(A-B)}$, find the "subtraction formulas" for $\cos(A - B)$ and $\sin(A - B)$.

Derive

$$
\begin{aligned}
e^{iA}e^{-iB} &= [\cos(A) + i\sin(A)][\cos(B) - i\sin(B)] \\
&= [\cos(A)\cos(B) + \sin(A)\sin(B)] + i[\sin(A)\cos(B) - \cos(A)\sin(B)] \\
&= \cos(A - B) + i\sin(A - B) \\
&= e^{i(A-B)}
\end{aligned}
$$

> **Question: 2.2.23**
>
> (a) If r and R are the absolute values of s and S, show that rR is the absolute value of sS. (Hint: Polar form!)
>
> (b) If $\bar{s}$ and $\overline{S}$ are the complex conjugates of s and S, show that $\bar{s}\overline{S}$ is the complex conjugate of sS. (Polar form!)

(a) Write $s = re^{i\theta}$ and $S = Re^{i\phi}$. Then $sS = rRe^{i(\theta+\phi)}$. Then $rR = |sS|$.

(b) Using the aforementioned definitions of s and S, we have

$$\overline{sS} = rRe^{-i(\theta+\phi)} = \left(re^{-i\theta}\right)\left(Re^{-i\phi}\right) = \bar{s}\overline{S}$$

Question: 2.2.24

Suppose a complex number s solves a real equation $s^3 + As^2 + Bs + C = 0$ (with A, B, C real). Why does the complex conjugate $\bar{s}$ also solve this equation? "*Complex solutions to real equations come in conjugate pairs s and $\bar{s}$.*"

We generalize the results of question 23 and claim that the complex conjugate of a product is equal to the product of the complex conjugates. For instance, suppose $s = re^{i\theta}$. Then we prove $\overline{s^n} = \bar{s}^n$:

$$\overline{s^n} = r^n e^{-in\theta} = \prod_{k=1}^{n} re^{-i\theta} = \bar{s}^n$$

We can also prove that the complex conjugate of a sum is the sum of the complex conjugates. Define $z_k = a_k + ib_k$ for $k \in \{1, ..., n\}$. Then

$$\overline{\sum_{k=1}^{n} z_k} = \overline{\sum_{k=1}^{n} (a_k + ib_k)} = \overline{\sum_{k=1}^{n} a_k + i \sum_{k=1}^{n} b_k} = \sum_{k=1}^{n} a_k - i \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \overline{z_k}$$

With these facts, the complex conjugate of the left side simply yields that $\bar{s}$ is too a solution of the real equation.

Question: 2.2.25

(a) If two complex numbers add to $s + S = 6$ and multiply to $sS = 10$, what are s and S? (They are complex conjugates.)

(b) If two numbers add to $s + S = 6$ and multiply to $sS = -16$, what are s and S? (Now they are real.)

(a) $s = 3 + i$, $S = 3 - i$

(b) $s = 8$, $S = -2$

Question: 2.2.26

If two numbers s and S add to $s + S = -B$ and multiply to $sS = C$, show that s and S solve the quadratic equation $s^2 + Bs + C = 0$.
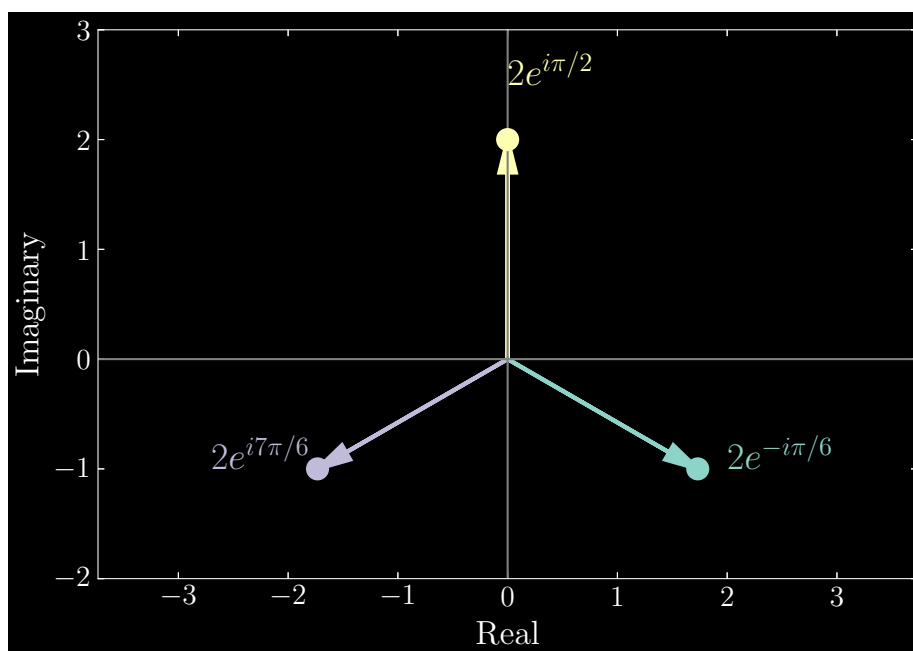
This is an interesting question because we relax the condition that $S = \bar{s}$. Substituting for B and C, we have

$$s^2 - (s + S)s + sS = 0, \qquad S^2 - (s + S)S + sS = 0$$

Question: 2.2.27

Find three solutions to $s^3 = -8i$ and plot the three points in the complex plane. What is the sum of the three solutions?

In polar form, $s^3 = 8e^{-i\pi/2}$. Its cube roots are $2e^{-i\pi/6}$, $2e^{i\pi/2}$, and $2e^{i7\pi/6}$. By vector superposition, the roots sum to zero.

Question: 2.2.28

(a) For which complex numbers $s = a + i\omega$ does e^{st} approach 0 as $t \to \infty$? Those numbers s fill which "half-plane" in the complex plane?

(b) For which complex numbers $s = a + i\omega$ does s^n approach 0 as $n \to \infty$? Those numbers s fill which part of the complex plane? Not a half-plane!

(a) For $a < 0$, $e^{st} = e^{at}e^{i\omega t}$ approaches zero.

(b) In polar form, $s = \sqrt{a^2 + \omega^2}e^{i\tan^{-1}(\omega/a)}$. We must require the modulus to be less than zero, or $\sqrt{a^2 + \omega^2} < 0$, for s^n to tend towards zero as n increases.

2.3 - Constant Coefficients A, B, C

> **Question: 2.3.1**
>
> Substitute $y = e^{st}$ and solve the characteristic equation for s:
>
> (a) $2y'' + 8y' + 6y = 0$
>
> (b) $y'''' - 2y'' + y = 0$

(a) The characteristic equation is $2s^2 + 8s + 6 = 0$ which factorizes into $2(s+3)(s+1) = 0$. Then $s_1 = -3$ and $s_2 = -1$.

(b) The characteristic equation is $s^4 - 2s^2 + 1 = 0$ which factorizes into $\left(s^2 - 1\right)^2 = (s-1)^2(s+1)^2$. It has repeated solutions: $s_1, s_2 = 1$ and $s_3, s_4 = -1$.

> **Question: 2.3.2**
>
> Substitute $y = e^{st}$ and solve the characteristic equation for $s = a + i\omega$:
>
> (a) $y'' + 2y' + 5y = 0$
>
> (b) $y'''' + 2y'' + y = 0$

(a) The characteristic equation is $s^2 + 2s + 5 = 0$, which by the quadratic formula has roots

$$s_1, s_2 = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

(b) The characteristic equation is $s^4 + 2s^2 + 1 = 0$, which by the quadratic formula has double root

$$s^2 = \frac{-2}{2} = -1$$

Rooting once more, we find repeated roots

$$s_1, s_2 = i, \quad s_3, s_4 = -i$$

> **Question: 2.3.3**
>
> Which second order equation is solved by $y = c_1 e^{-2t} + c_2 e^{-4t}$? Or $y = te^{5t}$?

In the first instance, the roots are $s_1 = -2$ and $s_2 = -4$. Then the characteristic polynomial is $(s+2)(s+4) = s^2 + 6s + 8$, corresponding to second order equation $y'' + 6y' + 8y = 0$. In the second case, we have repeated roots $s_1, s_2 = 5$, which has characteristic polynomial $(s-5)^2 = s^2 - 10s + 25$. This corresponds to second order equation $y'' - 10y' + 25y = 0$.

Question: 2.3.4

Which second order equation has solutions $y = c_1 e^{-2t} \cos(3t) + c_2 e^{-2t} \sin(3t)$?

The complex roots are $s_1 = -2 + 3i$ and $s_2 = -2 - 3i$, with characteristic polynomial $s^2 - 4s + 13$. This corresponds to second order equation

$$y'' - 4y' + 13 = 0$$

Question: 2.3.5

Which numbers B give (under)(critical)(over) damping in $4y'' + By' + 16y = 0$?

Underdamping: We must have $B^2 < 4AC$, or in this case, $B^2 < 256$. Then we must have $|B| < 16$.
Critical damping: We have $B^2 = 4AC$, or $B^2 = 256$, implying $|B| = 16$.
Overdamping: We have $B^2 > 4AC$, or $B^2 > 256$, implying $|B| > 16$.

Question: 2.3.6

If you want oscillation from $my'' + by' + ky = 0$, then b must stay below ____.

We must have $b^2 < 4mk$ or $|b| < 2\sqrt{mk}$.

Question: 2.3.7

The roots s_1 and s_2 satisfy $s_1 + s_2 = -2p = -B/A$ and $s_1 s_2 = p^2 + \omega_n^2 = C/A$. Show this two ways:

(a) Start from $A^2 + Bs + C = A(s - s_1)(s - s_2)$. Multiply to see $s_1 s_2$ and $s_1 + s_2$.

(b) Start from $s_1 = -p + i\omega_d$, $s_2 = -p - i\omega_d$

(a) We write

$$A^2 + Bs + C = A(s - s_1)(s - s_2)$$
$$= A\big(s^2 - s(s_1 + s_2) + s_1 s_2\big)$$

Concluding that $B = -A(s_1 + s_2)$ and $C = As_1 s_2$. Then

$$-\frac{B}{A} = s_1 + s_2, \quad \frac{C}{A} = s_1 s_2$$

(b) Let $s_1 = -p + i\omega_d$ and $s_2 = -p - i\omega_d$. Then $s_1 + s_2 = -2p$ and

$$s_1 s_2 = (-p + i\omega_d)(-p - i\omega_d) = p^2 + \omega_d^2$$

Note: There are two errors in the problem statement. The textbook erroneously has the equalities $s_1 + s_2 = -B/2A$ and $s_1 s_2 = p^2 + \omega_n^2$. The corrections have been made in my writing of the problem above.

> **Question: 2.3.8**
>
> Find s and y at the bottom point of the graph of $y = As^2 + Bs + C$. At that minimum point $s = s_{\min}$ and $y = y_{\min}$, the slope is $dy/ds = 0$.

Differentiate with respect to s to find

$$\frac{dy}{ds} = 2As + B = 0 \quad \Longrightarrow \quad s_{\min} = -\frac{B}{2A}$$

The corresponding $y_{\min}$ is

$$
\begin{aligned}
y_{\min} &= A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C \\
&= \frac{B^2}{4A} - \frac{B^2}{2A} + C \\
&= -\frac{B^2}{4A} + C
\end{aligned}
$$

> **Question: 2.3.9**
>
> The parabolas in Figure 2.10 show how the graph of $y = As^2 + Bs + C$ is raised by increasing B. Using Problem 8, show that the bottom point of the graph moves left (change in $s_{\min}$) and down (change in $y_{\min}$) when B is increased by ΔB.

Assume $\Delta B > 0$. By Problem 8, the new $s_{\min}$ becomes

$$s_{\min} = -\frac{(B + \Delta B)}{2A} < -\frac{B}{2A}$$

and the new $y_{\min}$

$$y_{\min} = -\frac{(B + \Delta B)^2}{4A} + C = -\frac{\left(B^2 + 2B\Delta B + \Delta B^2\right)}{4A} + C < -\frac{B^2}{4A} + C$$

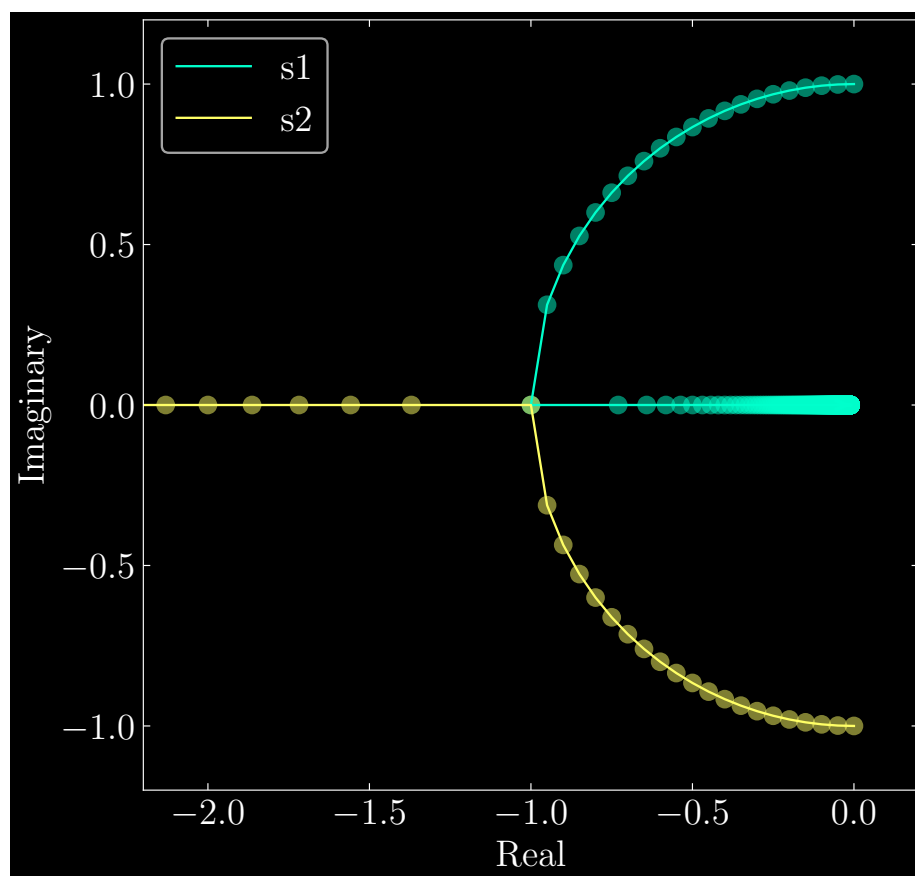both of which correspond to a leftward and downward shift, respectively.

Question: 2.3.10

(recommended) Draw a picture to show the paths of s_1 and s_2 when $s^2 + Bs + 1 = 0$ and the damping increases from $B = 0$ to $B = \infty$. At $B = 0$, the roots are on the __ axis. As B increases, the roots travel on a circle (why?). At $B = 2$, the roots meet on the real axis. For $B > 2$ the roots separate to approach 0 and $-\infty$. *Why is their product $s_1 s_2$ always equal to* 1?

The roots are

$$s = \frac{-B \pm \sqrt{B^2 - 4}}{2}$$

When $B = 0$, the roots are $s_{1,2} = \pm i$, which lie on the imaginary axis. The roots travel on a circle as B grows because they are complex conjugates, and given that the modulus of the roots remains constant, a quarter-circle traced out by the complex vector for one root will be reflected over the real axis for the other root, conforming to the geometry of a circle.

Recall from problem 7 that $s_1 s_2 = C/A = 1$. Another, more intuitive approach is to realize that, $s_1 = \overline{s_2}$, so they are complex conjugates of one another, and their product is the modulus (in this case, unity).
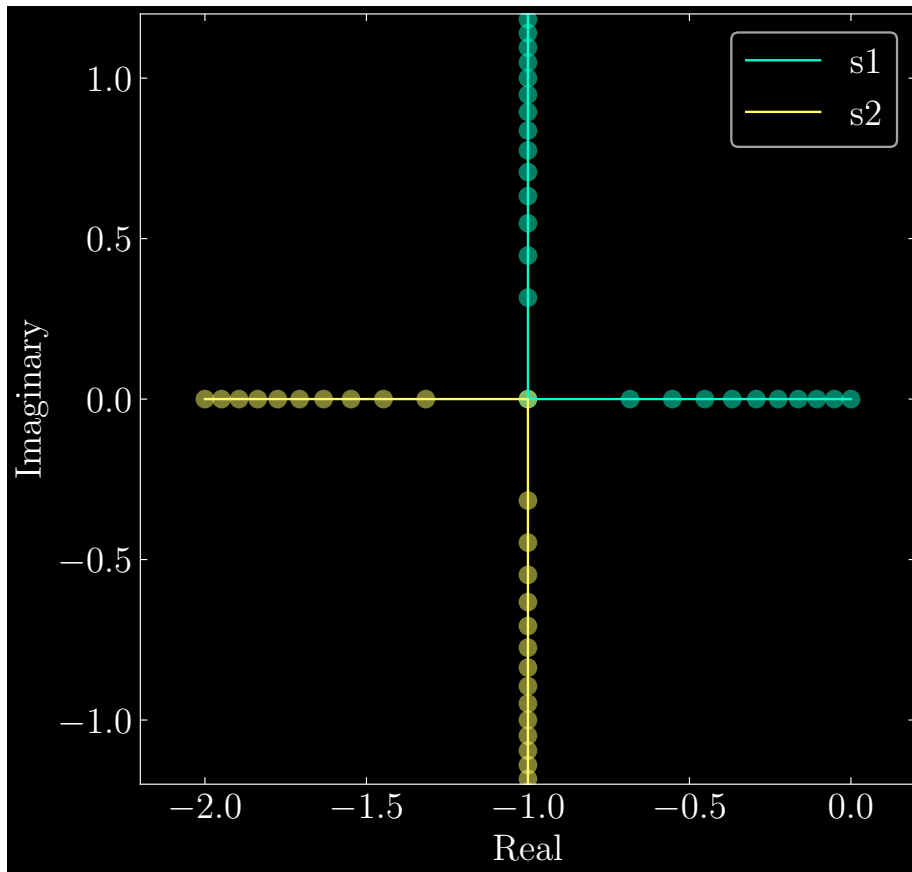
Question: 2.3.11

(this too if possible) Draw the paths of s_1 and s_2 when $s^2 + 2s + k = 0$ and the stiffness increases from $k = 0$ to $k = \infty$. When $k = 0$, the roots are __. At $k = 1$, the roots meet at $s = $ ____. For $k \to \infty$ the two roots travel up/down on a ____ in the complex plane. *Why is their sum $s_1 + s_2$ always equal to -2?*

The roots are
$$s = \frac{-2 \pm \sqrt{2^2 - 4k}}{2} = -1 \pm \sqrt{1 - k}$$

When $k = 0$, the roots are $s_{1,2} = 0, -2$. Once we reach $k = 1$, we have double roots with $s_{1,2} = -1$. As k increases to infinity, the real component remains fixed, while the imaginary component grows in both directions on the vertical axis. Altogether, the trajectory of the roots forms a cross.

As for their sum, observe that the imaginary components of $s_{1,2}$ cancel, and the real parts add to -2.

Question: 2.3.12

If a polynomial $P(s)$ has a double root at $s = s_1$, then $(s - s_1)$ is a double factor and $P(s) = (s - s_1)^2 Q(s)$. Certainly $P = 0$ at $s = s_1$. Show that also $dP/ds = 0$ at $s = s_1$. use the product rule to find dP/ds.

By the product rule, we have

$$\frac{dP}{ds}_{(s=0)} = 2(s - s_1) Q(s) + (s - s_1)^2 \frac{dQ}{ds} = 0$$

Question: 2.3.13

Show that $y'' = 2ay' - \left(a^2 + \omega^2\right) y$ leads to $s = a \pm i\omega$. Solve $y'' - 2y' + 10y = 0$.

The characteristic equation is given by

$$s^2 - 2as + \left(\alpha^2 + \omega^2\right) = 0$$

which has roots
$$s = \frac{2a \pm \sqrt{4a^2 - 4(\alpha^2 + \omega^2)}}{2} = a \pm i\omega$$
For the given equation, we have $a = 1$ and $\omega = 3$. Then we have solution

$$y(t) = e^{(1+3i)t} + e^{(1-3i)t}$$
$$= 2e^t \cos(3t)$$

Question: 2.3.14

The undamped *natural frequency* is $\omega_n = \sqrt{k/m}$. The two roots of $ms^2 + k = 0$ are $s = \pm i\omega_n$ (pure imaginary). With $p = b/2m$, the roots of $ms^2 + bs + k = 0$ are $\boldsymbol{s_1, s_2 = -p \pm \sqrt{p^2 - \omega_n^2}}$. The coefficient $p = b/2m$ has the units of $1 \ / \ $time.
Solve $s^2 + 0.1s + 1 = 0$ and $s^2 + 10s + 1 = 0$ with numbers correct to two decimals.

In the first equation, we have $m = 1$, $b = 0.1$, and $k = 1$. Then $p = 0.05$ and $\omega_n = 1$. This yields roots $s_{1,2} = -0.05 \pm 0.99i$.
The second equation gives us $p = 5$ and $\omega_n = 1$, which has roots $s_{1,2} = -5 \pm \sqrt{24} = -0.10, -9.89$.

Question: 2.3.15

With large overdamping $\boldsymbol{p >> \omega_n}$, the square root $\sqrt{p^2 - \omega_n^2}$ is close to $p - \omega_n^2/2p$. Show that the roots of $ms^2 + bs + k$ are $s_1 \approx \boldsymbol{-\omega_n^2/2p} = $ (small) and $s_2 \approx -2p = \boldsymbol{-b/m}$ (large).

From question 14, the roots are $s_1, s_2 = -p \pm \sqrt{p^2 - \omega_n^2}$. By the overdamping assumption, the roots can then be approximated by $s_1 \approx -\omega_n^2/2p$ and $s_2 \approx -2p - \omega_n^2/2p \approx -2p = -b/m$, appealing to the fact that $p >> \omega_n$.

Question: 2.3.16

With small underdamping $\boldsymbol{p << \omega_n}$, the square root of $p^2 - \omega_n^2$ is approximately $i\omega_n - ip^2/2\omega_n$. Square that to come close to $p^2 - \omega_n^2$. Then the frequency for small underdamping is reduced to $\omega_d \approx \omega_n - p^2/2\omega_n$.

Write

$$\left(i\omega_n - ip^2/2\omega_n\right)^2 = p^2 - \omega_n^2 + \frac{p^4}{4\omega_n^2} \approx p^2 - \omega_n^2$$

Since we have underdamping, the magnitude of the square root of $p^2 - \omega_n^2$ is approximated as $\omega_d \approx \omega_n - p^2/2\omega_n$.

Question: 2.3.17

Here is an 8th order equation with eight choices for solutions $y = e^{st}$:

$$\frac{d^8 y}{dt^8} = y \quad \text{becomes} \quad s^8 e^{st} = e^{st} \quad \text{and} \quad \boldsymbol{s^8 = 1} :$$

Eight roots in Figure 2.6

Find two solutions e^{st} that don't oscillate (s is real). Find two solutions that only oscillate (s is imaginary). Find two that spiral into zero and two that spiral out.

The eighth roots of unity are:

$$1, -1, i, -i, e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$$

The real solutions are e^t and e^{-t} and imaginary are e^{it} and e^{-it}. In order for the solutions to spiral into zero, the real component of s must be negative, ensuring that the outer exponential decays with increasing t. This corresponds to the solutions involving $e^{i3\pi/4}$ and $e^{i5\pi/4}$:

$$e^{t\cos(3\pi/4)}\left[\cos\left(t\sin\left(\frac{3\pi}{4}\right)\right) + \sin\left(t\sin\left(\frac{3\pi}{4}\right)\right)\right]$$

$$e^{t\cos(5\pi/4)}\left[\cos\left(t\sin\left(\frac{5\pi}{4}\right)\right) + \sin\left(t\sin\left(\frac{5\pi}{4}\right)\right)\right]$$

Lastly, the solutions that spiral away from zero involve $e^{i\pi/4}$ and $e^{i7\pi/4}$, and they are

$$e^{t\cos(\pi/4)}\left[\cos\left(t\sin\left(\frac{\pi}{4}\right)\right) + \sin\left(t\sin\left(\frac{\pi}{4}\right)\right)\right]$$

$$e^{t\cos(7\pi/4)}\left[\cos\left(t\sin\left(\frac{7\pi}{4}\right)\right) + \sin\left(t\sin\left(\frac{7\pi}{4}\right)\right)\right]$$

Observe here that the outer exponentials have positive arguments, ensuring growth as t grows.

Question: 2.3.18

$A_n \dfrac{d^n y}{dt^n} + \cdots + A_1 \dfrac{dy}{dt} + A_0 y = 0$ leads to $\boldsymbol{A_n s^n + \cdots + A_1 s + A_0 = 0}$.

The n roots $s_1, ..., s_n$ produce n solutions $y(t) = e^{st}$ (if those roots are distinct). Write down n equations for the constants c_1 to c_n in $y = c_1 e^{s_1 t} + \cdots + c_n e^{s_n t}$ by matching the n initial conditions for $y(0)$, $y'(0)$, ..., $D^{n-1} y(0)$.

Matching the initial conditions, we write

$$y(0) = c_1 + \cdots c_n$$
$$y'(0) = c_1 s_1 + \cdots + c_n s_n$$
$$\vdots$$
$$D^{n-1} y(0) = c_1 s_1^{n-1} + \cdots + c_n s_n^{n-1}$$

Question: 2.3.19

Find two solutions to $\boldsymbol{d^{2015} y / dt^{2015} = dy/dt}$. Describe all solutions to $s^{2015} = s$.

Two solutions are $y = e^t$ and $y = e^{-t}$. The set of all solutions to $s^{2015} = s$ is described by 0 and

$$e^{i 2 k \pi / 2014} \text{ for } k \in \{0, ..., 2013\}$$

Question: 2.3.20

The solution to $y'' = 1$ starting from $y(0) = y'(0) = 0$ is $y(t) = t^2/2$. The fundamental solution to $g'' = \delta(t)$ is $g(t) = t$ by Example 5. Does the integral $\int g(t-s) f(s) \, ds = \int (t-s) \, ds$ from 0 to t give the correct solution $y = t^2/2$?

The forcing function $f(t)$ here simply equals 1, hence the particular solution reduces to solving

$$\int_0^t (t-s) \, ds = -\frac{(t-s)^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Question: 2.3.21

The solution to $y'' + y = 1$ starting from $y(0) = y'(0) = 0$ is $y = 1 - \cos t$. The solution to $g'' + g = \delta(t)$ is $\boldsymbol{g(t) = \sin t}$ by equation (13) with $\omega = 1$ and $A = 1$. Show that $1 - \cos t$ agrees with the integral $\int g(t - s) f(s)\, ds = \int \sin(t - s)\, ds$.

We have

$$\int_0^t \sin(t - s)\ ds = \cos(t - s)\ \Big|_0^t = 1 - \sin(t)$$

Question: 2.3.22

The step function $H(t) = 1$ for $t \geq 0$ is the integral of the delta function. **So the step response $\boldsymbol{r(t)}$ is the integral of the impulse response.** This fact must also come from our basic solution formula:

$$Ar'' + Br' + Cr = 1 \text{ with } r(0) = r'(0) = 0 \text{ has } \boldsymbol{r(t) = \int_0^t g(t - s)\ ds}$$

Change $t - s$ to τ and change ds to $-d\tau$ to confirm that $r(t) = \int_0^t g(\tau)\ d\tau$. Section 2.5 will find two good formulas for the step response $r(t)$.

Let $\tau = t - s$. Then $d\tau/ds = -1$, implying $ds = -d\tau$. Lastly, the change of variables implies the change of integration bounds: $s_{\text{upper}} = t$ goes to $\tau_{\text{upper}} = t - t = 0$ and $s_{\text{lower}} = 0$ goes to $\tau_{\text{lower}} = t - 0 = t$. Thus we have

$$r(t) = -\int_t^0 g(\tau)\ d\tau = \int_0^t g(\tau)\ d\tau$$

2.4 - Forced Oscillations and Exponential Response

Problems 1-4 use the exponential response $y_p = e^{ct}/P(c)$ **to solve** $P(D)\,y = e^{ct}$.

Question: 2.4.1

Solve these constant coefficient equations with exponential driving force:

(a) $y_p'' + 3y_p' + 5y_p = e^t$

(b) $2y_p'' + 4y_p = e^{it}$

(c) $y'''' = e^t$

(a) Let $y_p = Ye^t$. Without using $P(D)\,y = e^{ct}$, we have

$$Ye^t + 3Ye^t + 5Ye^t = e^t$$
$$\implies \qquad 9Y = 1$$
$$\implies \qquad Y = 1/9$$

Thus $y_p = \dfrac{1}{9}e^t$. Note that $P(1) = 1^2 + 3 \cdot 1 + 5 = 9$.

(b) Using $P(D)\,y = e^{ct}$, we have

$$y_p = \frac{e^{it}}{P(i)} = \frac{e^{it}}{-2+4} = \frac{e^{it}}{2}$$

(c) Using $P(D)\,y = e^{ct}$, we have

$$y_p = \frac{e^t}{P(1)} = e^t$$

Question: 2.4.2

These equations $P(D)\,y = e^{ct}$ use the symbol D for d/dt. Solve for $y_p(t)$:

(a) $\left(D^2 + 1\right) y_p(t) = 10e^{-3t}$

(b) $\left(D^2 + 2D + 1\right) y_p(t) = e^{i\omega t}$

(c) $\left(D^4 + D^2 + 1\right) y_p(t) = e^{i\omega t}$

(a) $y_p(t) = \dfrac{10}{\left(-3\right)^2 + 1}e^{-3t} = e^{-3t}$

(b) $y_p(t) = \dfrac{1}{-\omega^2 + 2\omega i + 1}e^{i\omega t} = \dfrac{1}{1 - \omega^2 + 2\omega i}e^{i\omega t}$

(c) $y_p(t) = \dfrac{1}{1 - \omega^2 + \omega^4}e^{i\omega t}$

Question: 2.4.3

How could $y_p = e^{ct}/P(c)$ solve $y'' + y = e^t e^{it}$ and then $y'' + y = e^t \cos t$?

Observe that y_p solves the first equation when $c = (1 + i)\,t$, and then the real part serves as the solution to the second.

$$y_p = \frac{e^{(1+i)t}}{P(1+i)} = \frac{1}{1+2i}e^t e^{it} = \frac{1-2i}{5}e^t[\cos(t) + i\sin(t)]$$

$$= \underbrace{\frac{1}{5}e^t[\cos(t) + 2\sin(t)]}_{\text{Solution to second equation}} + \frac{i}{5}e^t[-2\cos(t) + \sin(t)]$$

Question: 2.4.4

(a) What are the roots s_1 to s_3 and the null solutions to $y_n''' - y_n = 0$?

(b) Find particular solutions to $y_p''' - y_p = e^{it}$ and to $y_p''' - y_p = e^t - e^{i\omega t}$.

(a) The roots are $s_1 = 1$, $s_2 = e^{i2\pi/3} = \cos(2\pi/3) + i\sin(2\pi/3)$, and $s_3 = e^{i4\pi/3} = \cos(4\pi/3) + i\sin(4\pi/3)$. The null solution is

$$y_n = c_1 e^t + c_2 e^{\left(-1/2 + i\sqrt{3}/2\right)t} + c_3 e^{\left(-1/2 - i\sqrt{3}/2\right)t}$$

$$= c_1 e^t + (c_2 + c_3)\,e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + i(c_2 - c_3)\,e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

(b) For the first equation, we have $y_p = \dfrac{1}{-1-i}e^{it}$. In the second case, appeal to the superposition principle to separately calculate the particular solution for each forcing term to find

$$y_p = \frac{te^t}{3} + \frac{1}{1+i\omega^3}e^{i\omega t}$$

Decomposed into real and imaginary components, we can rewrite as:

$$y_p = \frac{te^t}{3} + \frac{\cos(\omega t) + \omega^3\sin(\omega t)}{1+\omega^6} + i\left[\frac{\sin(\omega t) - \omega^3\cos(\omega t)}{1+\omega^6}\right]$$

Question: 2.4.5

Which value of C gives resonance in $y'' + Cy = e^{i\omega t}$? Why do we never get resonance in $y'' + 5y' + Cy = e^{i\omega t}$?

We must have $P(i\omega) = (i\omega)^2 + C = 0$ for some value of C. It immediately follows that we must have $C = \omega^2$. In the second case, observe that the roots of $P(s) = s^2 + 5s + C$ can never be pure imaginary. Thus it is impossible for $s = i\omega$ to ever be a solution to $P(s) = 0$.

Question: 2.4.6

Suppose the third order equation $P(D)\, y_n = 0$ has solutions $y = c_1 e^t + c_2 e^{2t} + c_3 e^{3t}$. What are the null solutions to the sixth order equation $P(D)\, P(D)\, y_n = 0$?

We must have repeated roots, as we can think of $P(D)$ as repeated 'factors' to the polynomial corresponding to the sixth order equation. With repeated roots, we must have

$$y = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} + c_4 t e^t + c_5 t e^{2t} + c_6 t e^{3t}$$

Question: 2.4.7

Complete this table with equations for s_1 and s_2 and y_n and y_p:

Undamped free oscillation	$my'' + ky = 0$	$y_n =$ ____
Undamped forced oscillation	$my'' + ky = e^{i\omega t}$	$y_p =$ ____
Damped free motion	$my'' + by' + ky = 0$	$y_n =$ ____
Damped forced motion	$my'' + by' + ky = e^{ct}$	$y_p =$ ____

Let $\omega_n = \sqrt{k/m}$. In the order of the table:

(a) Solve $s^2 + \omega_n^2 = 0$ to find

$$s_{1,2} = \pm i\omega_n, \quad y_n = c_1 \cos(\omega_n t) + i c_2 \sin(\omega_n t)$$

(b) Divide through by m and $P(i\omega) = (i\omega)^2 + \omega_n = \omega_n - \omega^2$. Then we have

$$y_p = \frac{1}{m(\omega_n^2 - \omega^2)} e^{i\omega t}$$

(c) Solve $s^2 + (b/m)\, s + \omega_n^2 = 0$ to find

$$s = \frac{-b/m \pm \sqrt{\dfrac{b^2}{m^2} - 4\omega_n^2}}{2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

This leads us to null solution

$$y_n = e^{-b/2m}\left(c_1 e^{\left(\sqrt{b^2 - 4mk}\right)t} + c_2 e^{-\left(\sqrt{b^2 - 4mk}\right)t}\right)$$

(d) Divide through by m and $P(c) = c^2 + (b/m)\, c + k/m$. Then we have

$$y_p = \frac{1}{mc^2 + bc + k} e^{ct}$$

Question: 2.4.8

Complete the same table when the coefficients are 1 and $2Z_{\omega_n}$ and ω_n^2 with $Z < 1$.

Undamped and free	$y'' + \omega_n^2 y = 0$	$\boldsymbol{y_n} = $ _____
Undamped and forced	$y'' + \omega_n^2 y = e^{i\omega t}$	$\boldsymbol{y_p} = $ _____
Underdamped and free	$y'' + 2Z\omega_n y' + \omega_n^2 y = 0$	$\boldsymbol{y_n} = $ _____
Underdamped and forced	$y'' + 2Z\omega_n y' + \omega_n^2 y = e^{ct}$	$\boldsymbol{y_p} = $ _____

Let $Z = b/\sqrt{4mk}$ and ω_n be as before. These are similar equations as in the previous question, but using different variable definitions.

(a) Solve $s^2 + \omega_n^2 = 0$ to find roots $s = \pm i\omega_n$. The null solution is

$$y_n = c_1 e^{i\omega_n t} + c_2 e^{-i\omega_n t} = C_1 \cos(\omega_n t) + iC_2 \sin(\omega_n t)$$

(b) Calculate $P(i\omega) = \omega^2 + \omega_n^2$ and divide the right-hand side by $P(i\omega)$ to find

$$y_p = \frac{1}{\omega^2 + \omega_n^2} e^{i\omega t}$$

(c) Solve $s^2 + 2Z\omega_n s + \omega_n^2 = 0$ to find roots

$$s = \frac{-2Z\omega_n \pm \sqrt{4Z^2\omega_n^2 - 4\omega_n^2}}{2} = -Z\omega_n \pm i\omega_n\sqrt{1 - Z^2}$$

with the imaginary term ensuing from the premise that $Z < 1$. This leads us to the null solution

$$y_n = e^{-Z\omega_n}\left[c_1 \cos\left(\omega_n\left(\sqrt{1 - Z^2}\right)t\right) + ic_2 \sin\left(\omega_n\left(\sqrt{1 - Z^2}\right)t\right)\right]$$

(d) Calculate $P(c) = c^2 + 2Z\omega_n c + \omega_n^2$ and divide the right-hand side by $P(c)$ to get

$$y_p = \frac{1}{c^2 + 2Z\omega_n c + \omega_n^2} e^{ct}$$

Question: 2.4.9

What equations $y'' + By' + Cy = f$ have these solutions?

(a) $y = c_1 \cos(2t) + c_2 \sin(2t) + \cos(3t)$

(b) $y = c_1 e^{-t} \cos(4t) + c_2 e^{-t} \sin(4t) + \cos(5t)$

(c) $y = c_1 e^{-t} + c_2 t e^{-t} + e^{i\omega t}$

(a) The first two terms constitute the null solution. One solution is to substitute the null solution in the equation, and deducing that in collecting the sine and cosine terms, B and C must be such that these equations vanish:

$$-4c_1 + 2Bc_2 + Cc_1 = 0$$
$$-4c_2 - 2Bc_1 + Cc_2 = 0$$

We can conclude that $B = 0$ and $C = 4$.

Alternatively, recall from question 2.3.7 that $s_1 + s_2 = B/A$ and $s_1 s_2 = C/A$ (we call these Vieta's formulas, which can be generalized to any polynomial of degree n). From the absence of the damping term, we can surmise that the null solution is undamped and freely oscillates, with roots $s_{1,2} = \pm 2i$. Using Vieta's formulas, we have $s_1 + s_2 = B = 0$ and $s_1 s_2 = C = 4$.

The homogeneous equation is

$$y'' + 4y = 0$$

For the forcing term f, substitute the particular solution to derive

$$-9\cos(3t) + 4\cos(3t) = f = -5\cos(3t)$$

(b) The roots are $s_{1,2} = -1 \pm 4i$, which lead to polynomial $s^2 + 2s + 17$. Intuitively, the damping term tracks with the exponential decay. Our homogeneous equation is

$$y'' + 2y' + 17y = 0$$

For the forcing term, substitute the particular solution to find

$$-25\cos(5t) - 10\sin(5t) + 17\cos(5t) = f = -8\cos(5t) - 10\sin(5t)$$

(c) We have repeated real roots: $s_{1,2} = 1$. This suggests polynomial

$$(s-1)^2 = s^2 - 2s + 1$$

The corresponding homogeneous differential equation, which has critical damping, is

$$y'' - 2y' + y = 0$$

The particular solution arises from forcing term

$$(i\omega)^2 e^{i\omega t} - 2i\omega e^{i\omega t} + e^{i\omega t} = f = e^{i\omega t}\left(1 - \omega^2 - 2i\right)$$

Question: 2.4.10

If $y_p = te^{-6t}\cos(7t)$ solves a second order equation $Ay'' + By' + Cy = f$, what does that tell you about A, B, C, and f?

The particular solution suggests that we simultaneously have resonance and complex roots $s = -6 \pm 7i$: thus $B^2 < 4AC$ and if $P(s) = As^2 + Bs + C$, then

$P(-6 \pm 7i) = 0$. Now, the forcing function must be a linear combination of real terms involving $e^{-6t}\cos(7t)$ or $e^{-6t}\sin(7t)$; the t arises because of resonance, and is not a part of the forcing function. If we wish to determine exactly A, B, C, and f, we must substitute the particular solution into our equation (this is tedious).

Sparing the reader from the work, it turns out A, B, C are $1, 12, 85$, respectively. The homogeneous equation is

$$y'' + 12y' + 85 = 0$$

Incorporating these into the quadratic formula returns the desired roots. Using these values, the forcing function is then

$$f = -14e^{-6t}\sin(7t)$$

The insights:

○ The forcing function, generally, has form

$$f(t) = Y(s = -6 + 7i)\, e^{-6t}(\cos(7t) + i\sin(7t))$$

○ Multiplying through by the complex coefficient $Y(s)$ will yield the form

$$f(t) = e^{-6t}(c_1\cos(7t) + c_2\sin(7t))$$

where c_1 and c_2 are real coefficients.

○ Since the forcing function oscillates at the natural frequency (i.e. has the same roots as the characteristic polynomial implied by the differential equation), this implies that the particular solution y_p will include a t term. This is resonance: intuitively, it suggests that since the forcing frequency is equal to the natural frequency, there will be an amplification in the response as evidenced by the t term.

Question: 2.4.11

(a) Find the steady oscillation $y_p(t)$ that solves $y'' + 4y' + 3y = 5\cos(\omega t)$.

(b) Find the amplitude A of $y_p(t)$ and its phase lag α.

(c) Which frequency ω gives maximum amplitude (maximum gain)?

(a) In rectangular form, let $y_p(t) = M\cos(\omega t) + N\sin(\omega t)$. Deriving M and N in the same fashion as equations (20) and (21) in Strang lead us to

$$M = \frac{5(3 - \omega^2)}{(3 - \omega^2)^2 + (4\omega)^2}$$

$$N = \frac{5(4\omega)}{(3 - \omega^2)^2 + (4\omega)^2}$$

(b) The amplitude A is
$$A = \frac{5}{\sqrt{(3 - \omega^2)^2 + (4\omega)^2}}$$
and its phase lag is
$$\alpha = \tan^{-1}\left(\frac{4\omega}{3 - \omega^2}\right)$$

(c) When the denominator is smallest, the gain is largest. Differentiate the argument of the square root and optimize to find:
$$\frac{d}{d\omega}\left(\left(3 - \omega^2\right)^2 + (4\omega)^2\right) = -4\omega\left(3 - \omega^2\right) + 8(4\omega) = 0 \quad \implies \quad \omega = 0, \pm\sqrt{5}i$$

Taking the second derivative and evaluating at $\omega = 0$ (we cannot have an imaginary frequency in this instance) ascertains that this is a local minimum.
$$\frac{d^2}{d\omega^2}_{(\omega=0)}\left(\left(3 - \omega^2\right)^2 + (4\omega)^2\right) = -12 + 12\omega^2 + 32 = 20$$

Thus the maximum amplitude is $A = 5/3$.

Conceptually, we can conclude that a higher oscillating frequency contributes to increasing damping of the response function. So much so, that the forcing ought not to oscillate at all: $\omega = 0$.

> **Question: 2.4.12**
>
> Solve $y'' + y = \sin(\omega t)$ starting from $y(0) = 0$ and $y'(0) = 0$. Find the limit of $y(t)$ as ω approaches 1, and the problem approaches resonance.

We have $y_n(t) = c_1 \cos(t) + c_2 \sin(t)$. For the particular solution, let $y_p(t) = Y \sin(\omega t)$. Solving for Y, we find
$$Y = \frac{1}{1 - \omega^2}$$
Imposing the initial conditions, we must have
$$y(0) = c_1 = 0$$
$$y'(0) = c_2 + \frac{\omega}{1 - \omega^2} = 0 \quad \implies \quad c_2 = -\frac{\omega}{1 - \omega^2}$$

Therefore, our solution is
$$y(t) = \frac{-\omega \sin(t) + \sin(\omega t)}{1 - \omega^2}$$
Now consider the limit as $\omega \to 1$. Since we have the indeterminate form $0/0$, we apply l'Hôpital's rule to find
$$\lim_{\omega \to 1} y(t) = \lim_{\omega \to 1} \frac{-\sin(t) + t\cos(\omega t)}{-2\omega} = \frac{\sin(t) - t\cos(t)}{2}$$
As expected, resonance happens.

Question: 2.4.13

Does critical damping and a double root $s = -1$ in $y'' + 2y' + y = e^{ct}$ produce an extra factor t in the null solution y_n or in the particular solution y_p (proportional to e^{ct})? What is y_n with constants c_1, c_2? What is $y_p = Ye^{ct}$?

An extra factor t appears in the null solution, which is

$$y_n = c_1 e^{-t} + c_2 t e^{-t}$$

The particular solution is

$$Yc^2 e^{ct} + 2Yc e^{ct} + Y e^{ct} = e^{ct} \quad \implies \quad Y = \frac{1}{c^2 + 2c + 1}$$

$$y_p = \frac{e^{ct}}{c^2 + 2c + 1}$$

Question: 2.4.14

If $c = i\omega$ in Problem 13, the solution y_p to $y'' + 2y' + y = e^{i\omega t}$ is ____. That fraction Y is the transfer function at $i\omega$. What are the magnitude and phase in $Y = Ge^{-i\alpha}$?
By rescaling both t and y, we can reach $A = C = 1$. Then $\omega_n = 1$ and $B = 2Z$. The model problem is $y'' + 2Zy' + y = f(t)$.

The particular solution becomes

$$y_p = \frac{e^{i\omega t}}{1 - \omega^2 + 2i\omega}$$

The gain and phase are calculated by

$$G = \frac{1}{\sqrt{(1 - \omega^2)^2 + 4\omega^2}}, \quad \alpha = \tan^{-1}\left(\frac{2\omega}{1 - \omega^2}\right)$$

Question: 2.4.15

What are the roots of $s^2 + 2Zs + 1 = 0$? Find two roots for $Z = 0, \frac{1}{2}, 1, 2$ and identify each type of damping. The natural frequency is now $\omega_n = 1$.

The roots are

$$s = \frac{-2Z \pm \sqrt{4Z^2 - 4}}{2} = -Z \pm \sqrt{Z^2 - 1}$$

For each value of Z, we have

Z	Roots	Damping
0	$\pm i$	No damping
$\dfrac{1}{2}$	$-\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$	Underdamping
1	-1 (double root)	Critical damping
2	$-2 \pm \sqrt{3}$	Overdamping

Question: 2.4.16

Find two solutions to $y'' + 2Zy' + y = 0$ for every Z except $Z = 1$ and $Z = -1$. Which solution $g(t)$ starts from $g(0) = 0$ and $g'(0) = 1$? What is different about $Z = 1$?

The characteristic polynomial is $s^2 + 2Zs + 1 = 0$, which has roots $s = -Z \pm \sqrt{Z^2 - 1}$, as in the previous question. For all Z except $Z = 1, -1$ (which has repeated roots), we have

$$y_n(t) = c_1 e^{\left(-Z + \sqrt{Z^2-1}\right)t} + c_2 e^{\left(-Z - \sqrt{Z^2-1}\right)t}$$

Under the initial conditions, we must have

$$y_n(0) = c_1 + c_2 = 0 \implies \boxed{c_1 = -c_2}$$

$$y_n'(0) = c_1\left(-Z + \sqrt{Z^2 - 1}\right) + c_2\left(-Z - \sqrt{Z^2 - 1}\right)$$

$$= 2c_1\sqrt{Z^2 - 1} = 1 \implies \boxed{c_1 = \frac{1}{2\sqrt{Z^2 - 1}}}$$

At $Z = 1$, observe that c_1, c_2 are undefined, as we have resonance and the null solution must feature a te^{st} term.

Question: 2.4.17

The equation $my'' + ky = \cos(\omega_n t)$ is exactly at resonance. The driving frequency on the right side equals the natural frequency $\omega_n = \sqrt{k/m}$ on the left side. Substitute $y = Rt\sin\left(\sqrt{k/m}\,t\right)$ to find R. This resonant solution grows in time because of the factor t.

Substituting, we find

$$2R\sqrt{mk}\cos\left(\sqrt{k/m}\,t\right) = \cos\left(\sqrt{k/m}\,t\right)$$

which leads us to conclude that

$$R = \frac{1}{2\sqrt{mk}}$$

Question: 2.4.18

Comparing the equations $Ay'' + By' + Cy = f(t)$ and $4Az'' + Bz' + (C/4)\,z = f(t/4)$, what is the difference in their solutions?

Note that there is a typo in the book; the argument in the forcing term in the z-equation needs to be scaled by $1/4$. Let $y(t)$ be a solution to the y-equation. The relationship between the roots of the y- and z-equations is $s_z = s_y/4$. This is analogous to "scaling" time by $1/4$, which is consistent with the z-forcing function.

Now, we have $y(t/4)$. Its derivatives are

$$y'\left(\frac{t}{4}\right) = \frac{1}{4}y\left(\frac{t}{4}\right), \quad y''\left(\frac{t}{4}\right) = \frac{1}{16}y\left(\frac{t}{4}\right)$$

Substituting in the z-equation gives us

$$\left(\frac{A}{4} + \frac{B}{4} + \frac{C}{4}\right)y\left(\frac{t}{4}\right) = f\left(\frac{t}{4}\right)$$

But since the forcing function here is not itself scaled (only time is scaled), we can throw a 4 in front of $y(t/4)$ to conclude that

$$z(t) = 4y\left(\frac{t}{4}\right)$$

Question: 2.4.19

Find the fundamental solution to the equation $g'' - 3g' + 2g = \delta(t)$.

The characteristic polynomial is $s^2 - 3s + 2 = 0$ which has roots $s_{1,2} = 1, 2$. Then the equation is overdamped (real roots), and the fundamental solution is given by

$$g(t) = e^{2t} - e^t$$

Question: 2.4.20

(Challenge problem) Find the solution to $y'' + By' + y = \cos(t)$ that starts from $y(0) = 0$ and $y'(0) = 0$. Then let the damping constant B approach zero, to reach the resonant equation $y'' + y = \cos(t)$ in Problem 17, with $m = k = 1$.
Show that your solution $y(t)$ is approaching the resonant solution $\frac{1}{2}t\sin(t)$.

The characteristic polynomial is $s^2 + Bs + 1 = 0$ with roots

$$s_{1,2} = \frac{-B \pm \sqrt{B^2 - 4}}{2}$$

Thus the null solution is

$$y_n(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

The particular solution is

$$y_p(t) = \frac{1}{B}\sin(t)$$

which can be deduced by inspection: notice how the y'' and y terms will annihilate each other (made convenient by the fact that $\omega = 1$ here), so the forcing has a response offset by 90 degrees (cosine to sine) and scaled in amplitude by $1/B$. Imposing the initial conditions forces

$$y(0) = c_1 + c_2 = 0 \implies \boxed{c_1 = -c_2}$$

$$y'(0) = c_1 s_1 + c_2 s_2 + \frac{1}{B} = 0 \implies \boxed{c_1 = \frac{1}{B(s_2 - s_1)}}$$

Our full solution is

$$y(t) = \frac{e^{s_1 t} - e^{s_2 t}}{B(s_2 - s_1)} + \frac{1}{B}\sin(t) = \frac{e^{s_2 t} - e^{s_1 t}}{B\sqrt{B^2 - 4}} + \frac{1}{B}\sin(t)$$

There are many terms so it is mildly confusing, but if you set $B = 0$ and algebraically manipulate everything to have common denominators, we end up with $y(t) = 0/0$. The indeterminate form enables us to move forward with l'Hôpital's rule. Unfortunately the algebra will be tedious. Use the following identities:

$$s_{1,2(B=0)} = \boxed{\pm i}$$

$$\frac{ds_{1,2}t}{dB}_{(B=0)} = t\left(-\frac{1}{2} \pm \frac{B}{2\sqrt{B^2 - 4}}\right) = \boxed{-\frac{t}{2}}$$

$$\frac{d}{dB}_{(B=0)}\left(B\sqrt{B^2 - 4}\right) = \sqrt{B^2 - 4} + \frac{B^2}{\sqrt{B^2 - 4}} = \boxed{2i}$$

$$\frac{d}{dB}_{(B=0)}\left(\sqrt{B^2 - 4}\right) = \frac{B}{\sqrt{B^2 - 4}} = \boxed{0}$$

Now, applying l'Hôpital's rule:

$$\lim_{B\to 0} y(t) = \lim_{B\to 0} \frac{e^{s_2 t}\frac{ds_2 t}{dB} - e^{s_1 t}\frac{ds_1 t}{dB} + \frac{d}{dB}\left(\sqrt{B^2 - 4}\right)\sin(t)}{\frac{d}{dB}\left(B\sqrt{B^2 - 4}\right)}$$

$$= \frac{\frac{t}{2}\left(-e^{-it} + e^{it}\right)}{2i}$$

$$= \frac{1}{2}t\sin(t)$$

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Tiresome, but a deeply satisfying conclusion.

> **Question: 2.4.21**
>
> Suppose you know three solutions y_1, y_2, y_3 to $y'' + B(t)\,y' + C(t)\,y = f(t)$. How could you find $B(t)$ and $C(t)$ and $f(t)$?

First determine that $y_1 - y_2$ and $y_2 - y_3$ are in the null space of the equation, and one leading to the forcing term. Define each as u and v respectively. Analogize to solving a system of equations:

$$u'' + B(t)\,u' + C(t)\,u = 0$$

$$v'' + B(t)\,v' + C(t)\,v = 0$$

$$y_3'' + B(t)\,y_3' + C(t)\,y_3 = f(t)$$

Isolate the $B(t)$ and $C(t)$ terms. My solutions for each are

$$B(t) = \frac{u''v - v''u}{v'u - u'v}$$

$$C(t) = \frac{u''v' - v''u'}{vu' - uv'}$$

Now incorporate both into the third equation to derive $f(t)$.

2.5 - Electrical Networks and Mechanical Systems

> **Question: 2.5.1**
>
> (Resistors in parallel) Two parallel resistors R_1 and R_2 connect a node at voltage V to a node at voltage zero. The currents are V/R_1 and V/R_2. What is the total current I between the nodes? Writing R_{12} for the ratio V/I, what is R_{12} in terms of R_1 and R_2?

Since the resistors are in parallel, we sum the currents to find

$$I = I_1 + I_2 = \frac{V}{R_1} + \frac{V}{R_2} = V\left(\frac{R_1 + R_2}{R_1 R_2}\right)$$

Substitute into $R_{12} = V/I$ to find

$$R_{12} = \frac{R_1 R_2}{R_1 + R_2}$$

> **Question: 2.5.2**
>
> (Inductor and capacitor in parallel) Those elements connect a node at voltage $Ve^{i\omega t}$ to a node at voltage zero (grounded node). The currents are $(V/i\omega L)\,e^{i\omega t}$ and $V(i\omega C)\,e^{i\omega t}$. The total current $Ie^{i\omega t}$ between the nodes is their sum. Writing $Z_{12}e^{i\omega t}$ for the ratio $Ve^{i\omega t}/Ie^{i\omega t}$, what is Z_{12} in terms of $i\omega L$ and $i\omega C$?

Deriving the total current, we get

$$Ie^{i\omega t} = \frac{V}{i\omega L}e^{i\omega t} + V(i\omega C)\,e^{i\omega t} = Ve^{i\omega t}\left(\frac{1}{i\omega L} + i\omega C\right)$$

Substituting into the equation for Z_{12} yields

$$Z_{12}e^{i\omega t} = \frac{Ve^{i\omega t}}{Ie^{i\omega t}} = \frac{i\omega L}{1 - \omega^2 CL}$$

> **Question: 2.5.3**
>
> The impedance of an RLC loop is $Z = i\omega L + R + 1/i\omega C$. This impedance Z is real when $\omega = $ _____. This impedance is pure imaginary when _____. This impedance is zero when _____.

The impedance is real when $i\omega L = -1/i\omega C$ (assuming there is a signal, i.e. non-zero ω). Solving for ω, we find $\omega = 1/\sqrt{LC}$. It is pure imaginary when $R = 0$. The impedance is zero when both of the aforementioned conditions hold.

> **Question: 2.5.4**
>
> What is the impedance Z of an RLC loop when $R = L = C = 1$? Draw a graph that shows the magnitude $|Z|$ as a function of ω.
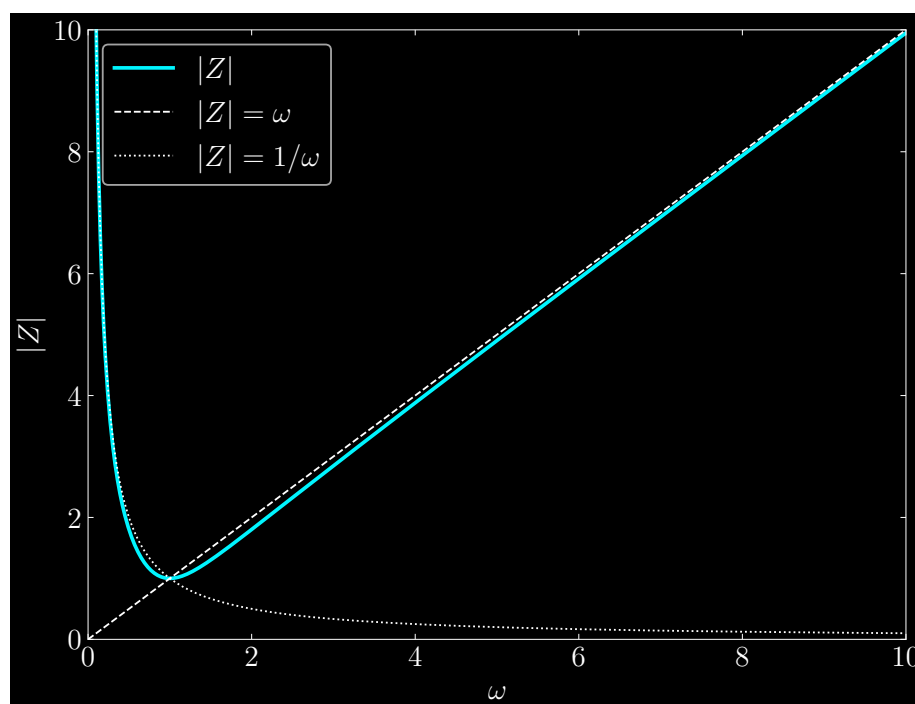
We get

$$Z = 1 + i\omega + \frac{1}{i\omega} = 1 + i\left(\omega - \frac{1}{\omega}\right)$$

The magnitude of the impedance is

$$|Z| = \sqrt{1 + \left(\omega - \frac{1}{\omega}\right)^2}$$

Observe that as $\omega \to \infty$, only the ω dominates and washes out the others, so we can approximate $|Z| \approx \omega$. Conversely, if $\omega \to 0$, then $1/\omega$ dominates, and we have $|Z| \approx 1/\omega$. We can represent this graphically:

Question: 2.5.5

Why does an LC loop with no resistor produce a 90° phase shift between current and voltage? Current goes around the loop from a battery of voltage V in the loop.

Conceptually, think of the impedance Z as the amount by which current is opposed in a circuit. This opposition can come in two forms: resistance or reactance, the real and imaginary components of Z, respectively. Depending on the values of both, the phase difference between current and voltage changes:

$$I(t) = \frac{V}{|Z|}e^{i(\omega t - \alpha)}$$

Now, under the premise that R is zero, we have

$$R \to 0 \quad \implies \quad \tan(\alpha) = \frac{\omega L - 1/\omega C}{R} \to \infty$$

When $\tan(\alpha)$ diverges in this manner, we can conclude that $\alpha = \pi/2$, ascertaining that the phase shift between current and voltage is indeed $90°$.

Question: 2.5.6

The mechanical equivalent of zero resistance is zero damping: $my'' + ky = \cos(\omega t)$. Find c_1 and Y starting from $y(0) = 0$ and $y'(0) = 0$ with $\omega_n^2 = k/m$.

$$y(t) = c_1 \cos(\omega_n t) + Y \cos(\omega t)$$

That answer can be written in two equivalent ways:

$$y = Y(\cos(\omega t) - \cos(\omega_n t)) = 2Y \sin\left(\frac{(\omega_n - \omega)\, t}{2}\right) \sin\left(\frac{(\omega_n + \omega)\, t}{2}\right)$$

Substitute y_p into the equation to find

$$-m\omega^2 Y + kY = 1 \quad \implies \quad \boxed{Y = \frac{1}{k - m\omega^2}}$$

From the first initial condition, we deduce

$$y(0) = c_1 + Y = 0 \quad \implies \quad \boxed{c_1 = -Y}$$

which gives us complete solution

$$y(t) = \frac{\cos(\omega_n t) - \cos(\omega t)}{k - m\omega^2}$$

Question: 2.5.7

Suppose the driving frequency ω is close to ω_n in Problem 6. A fast oscillation $\sin[(\omega_n + \omega)\, t/2]$ is multiplying a very slow oscillation $2Y \sin[(\omega_n - \omega)\, t/2]$. By hand or by computer, draw the graph of $y = \sin(t)\sin(9t)$ from 0 to 2π.
You should see a fast sine curve inside a slow sine curve. This is a **beat**.

Suppose that $\omega \to \omega_n$. Then we can write for the fast oscillation

$$\sin\left[\frac{(\omega_n + \omega)\, t}{2}\right] \approx \sin(\omega_n t)$$
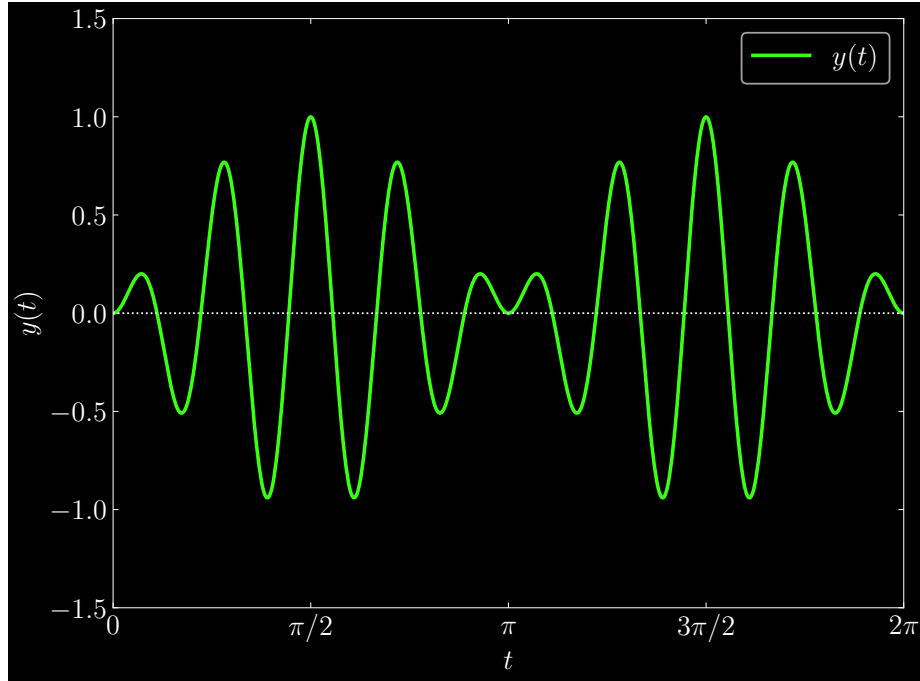
For the slow oscillation, the argument to the sinusoid becomes very small. We can then write the following approximation

$$2Y \sin\left[\frac{(\omega_n - \omega)\, t}{2}\right] \approx Y(\omega_n - \omega)\, t$$
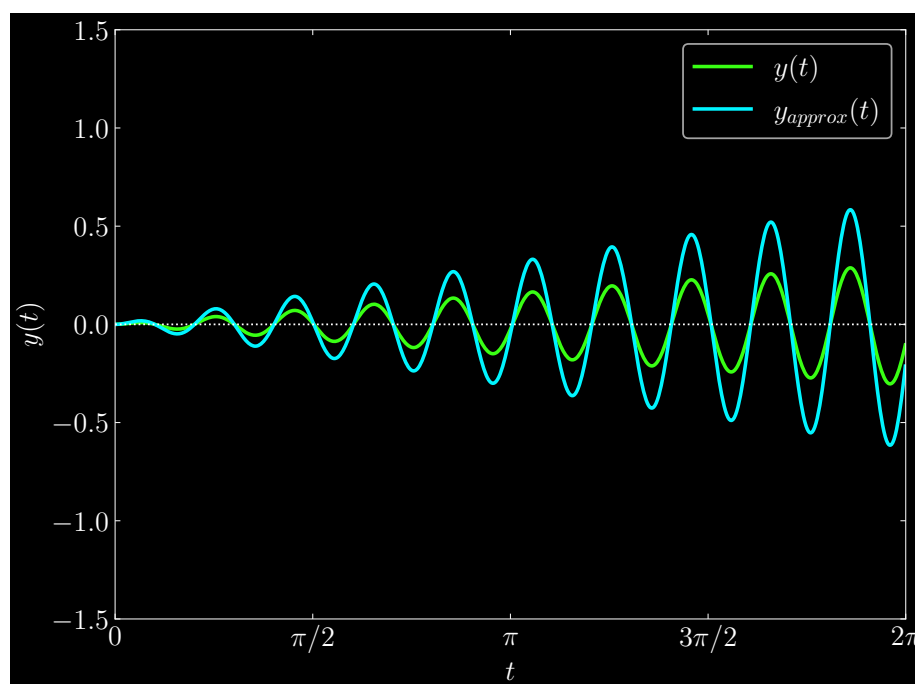
Taking the product of the approximations, we get

$$y \approx Y(\omega_n - \omega)\, t \sin(\omega_n t)$$

In the numerical example, by inspection we can see that $\omega_n = 10$ and $\omega = 8$, which generates the following graph of the product of oscillations:

But to see the resonance effects as $\omega \to \omega_n$, consider $\omega_n = 10$ and $\omega = 9.9$. As the delta between them diminishes, the convergence to the resonance solution (y_{approx}) becomes apparent:

Question: 2.5.8

What m, b, k, F equation for a mass-dashpot-string-force corresponds to Kirchhoff's Voltage Law around a loop? What force balance equation on a mass corresponds to Kirchhoff's Current Law?

Mass m is akin to inductance L, as both are inertial quantities that resist changes in velocity or electric current, respectively. The damping constant b is analogous to circuit resistance R, as both are measures of the subtractive forces acting on a system's total energy. Lastly, the spring constant k serves the function as the reciprocal capacitance $1/C$ – where k quantifies the stiffness of a spring (how hard or easy it is to stretch or compress it), reciprocal capacitance (sometimes called elastance) is a measure of how hard or easy it is to store charge.

In Kirchhoff's Voltage Law, the sum of voltage drops around a closed loop must vanish. It is also the Current Law integrated. The case of the Current Law is perhaps easier to conceptualize: the net sum of current flow at an ode must be zero. In the mechanical analogy, for a multinodal system, every node must have balanced forces.

If we analogize $y(t)$ (displacement) to the integral of the current (total distance the charge propagates):

$$y(t) \sim \int I(t)\ dt$$

Then the Voltage Law corresponds to the mechanical equation

$$my'' + by' + ky = \textbf{applied force}$$

Question: 2.5.9

If you only know the natural frequency ω_n and the damping coefficient b for one mass and one spring, why is that *not enough* to find the damped frequency ω_d? If you know all of m, b, k what is ω_d?

The damped frequency is reliant on the damping ratio $Z = b/2\sqrt{mk}$, namely

$$\omega_d = \omega_n \sqrt{1 - Z^2}$$

Without k, it is impossible to know ω_d.

Question: 2.5.10

Varying the number a in a first order equation $y' - ay = 1$ changes the *speed* of the response. Varying B and C in a second order equation $y'' + By' + Cy = 1$ changes the *form* of the response. Explain the difference.

The solution to the first order equation is

$$y(t) = \frac{1 - e^{-at}}{a}$$

where varying a alters the rate at which $y(t)$ grows while maintaining the specific form of the response. In the case of the second order equation, altering B and C potentially changes the relationship between B^2 and $4AC$, which criticaly determines the existence and degree of damping in the equation. Moreover, the natural frequency $\omega_n = \sqrt{C/A} = \sqrt{C}$ is altered.

Question: 2.5.11

Find the step response $r(t) = y_p + y_n$ for this overdamped system:

$$r'' + 2.5r' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0$$

We find y_n and y_p to be

$$r(t) = 1 + c_1 e^{-2t} + c_2 e^{-t/2}$$

Applying the initial conditions, the coefficients evaluate to

$$r(0) = 1 + c_1 + c_2 = 0 \implies \boxed{c_1 = -(1 + c_2)}$$

$$r'(0) = -2c_1 - \frac{c_2}{2} = 0 \implies \boxed{c_1 = -\frac{c_2}{4}}$$

Leading to $c_1 = 1/3$ and $c_2 = -4/3$, and response

$$r(t) = 1 + \frac{e^{-2t}}{3} - \frac{4e^{-t/2}}{3}$$

Question: 2.5.12

Find the step response $r(t) = y_p + y_n$ for this critically damped system. The double root $s = -1$ produces what form for the null solution?

$$r'' + 2r' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0$$

We have response

$$r(t) = 1 + c_1 e^{-t} + c_2 t e^{-t}$$

which features resonance from the double root. Applying the initial conditions gives us

$$r(0) = 1 + c_1 = 0 \implies \boxed{c_1 = -1}$$
$$r'(0) = 1 - c_1 + c_2 = 0 \implies \boxed{c_2 = -2}$$

This gives us final response

$$r(t) = 1 - e^{-t} - 2t e^{-t}$$

Question: 2.5.13

Find the step response $r(t)$ for this underdamped system using equation (22):
$$r'' + r' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0$$

Using equation (22), we first identify p, ω_n^2, and ω_d. Matching the coefficients in the equation, we have

$$p = \frac{1}{2}, \quad \omega_n^2 = 1, \quad \omega_d = \frac{\sqrt{3}}{2}$$

which corresponds to response

$$r(t) = 1 - \frac{2}{\sqrt{3}} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t + \frac{\pi}{3}\right)$$

Note that the phase shift $\phi = \pi/3$ is calculated from $\phi = \sin^{-1}\left(\sqrt{3}/2\right) = \pi/3$.

Question: 2.5.14

Find the step response $r(t)$ for this undamped system and compare with (22):
$$r'' + r = 1 \text{ with } r(0) = 0 \text{ and } r'(0) = 0$$

Without using equation (22), we identify the null and particular solutions as

$$r(t) = 1 + c_1 e^{it} + c_2 e^{-it}$$

Applying the initial conditions gives us

$$r(0) = 1 + c_1 + c_2 \implies \boxed{c_1 = -(1 + c_2)}$$
$$r'(0) = c_1 i - c_2 i = 0 \implies \boxed{c_1 = c_2}$$

Combined, we have $c_1 = c_2 = -1/2$, and solution

$$r(t) = 1 - \cos(t)$$

Now, using equation (22) gives us parameters $p = 0$, $\omega_n = 1$, $\omega_d = 1$. From this, we calculate $\sin(\phi) = 1 \implies \phi = \pi/2$, giving us the same solution:

$$r(t) = 1 - \sin(t + \pi/2) = 1 - \cos(t)$$

Question: 2.5.15

For $b^2 < 4mk$ (underdamping), what parameter decides the speed at which the step response $r(t)$ rises to $r(\infty) = 1$? Show that the **peak time** is $T = \pi/\omega_d$ when $r(t)$ reaches its maximum before settling back to $r = 1$ at peak time $r'(T) = 0$.

Since we have underdamping (namely, that damping exists), that establishes that $p \neq 0$. Then differentiating equation (22) and optimizing the derivative by setting equal to zero, we end up with the following condition:

$$\omega_d \cos(\omega_d t + \phi) = p \sin(\omega_d t + \phi)$$

Using the fact that $\omega_d/p = \tan(\phi)$, divide both sides by p and the cosine term to derive the clean relation

$$\tan(\phi) = \tan(\omega_d t + \phi)$$

which is true if and only if we have

$$t = \frac{k\pi}{\omega_d}$$

For positive integer k, which presents to us an infinite number of solutions (technically, the oscillation persists forever, even if the amplitude is killed from the exponential decay with time). However, we can only have one solution for the maximum. So let's go with $k = 1$, which will be the earliest time as the outer exponential function decays. The peak time, as expected, is derived as

$$T = \frac{\pi}{\omega_d}$$

The degree to which the response "stabilizes" around $r(\infty) = 1$ is contingent on p, which is the decay parameter in e^{-pt}.

> **Question: 2.5.16**
>
> If the voltage source $V(t)$ in an RLC loop is a unit step function, what resistance R will produce an overshoot to $r_{\max} = 1.2$ if $C = 10^{-6}$ Farads and $L = 1$ Henry? (Problem 15 found the peak time T when $r(T) = r_{\max}$).
> Sketch two graphs of $r(t)$ for $p_1 < p_2$. Sketch two graphs as ω_d increases.

From the Voltage Law, the differential equation we want to solve is
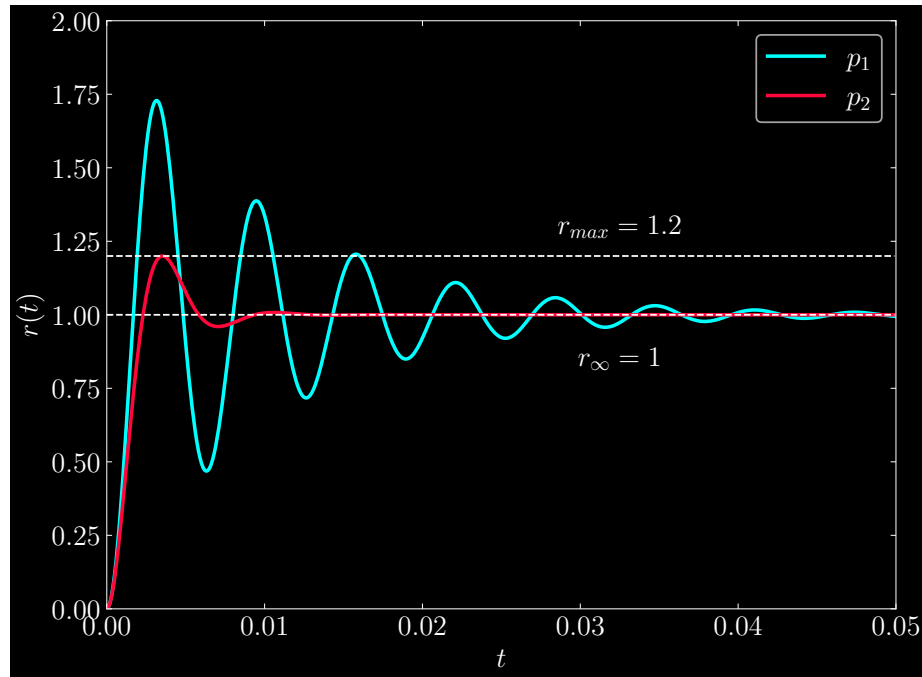
$$y'' + Ry' + 10^6 y = 1$$

and we want $r_{\max} = 1.2$. From the previous exercise, we determined that the corresponding time for this maximum response is $T = \pi/\omega_d$. Writing the response in polar form, we have parameters

$$p = \frac{R}{2}, \quad \omega_n^2 = 10^6, \quad \omega_d^2 = 10^6 - \frac{R^2}{4} \quad \left(\frac{1}{\omega_d^2} = \frac{4}{4 \cdot 10^6 - R^2} \right)$$

Now, we substitute T in our response function and isolate R:

$$r(T) = 1 - \frac{\omega_n}{\omega_d} e^{-R\pi/2\omega_d} \sin(\pi + \phi) = 1 + \frac{\omega_n}{\omega_d} e^{-R\pi/2\omega_d} \frac{\omega_d}{\omega_n} \qquad \left(\text{since } \sin(\phi) = \frac{\omega_d}{\omega_n} \right)$$

$$\implies \qquad\qquad r_{\max} = 1.2 = 1 + e^{-R\pi/2\omega_d}$$

$$\implies \qquad\qquad \ln(0.2) = -\frac{R\pi}{2} \left(\frac{4}{4 \cdot 10^6 - R^2} \right)^{1/2}$$

$$\implies \qquad\qquad R^2 = \frac{\ln(0.2)^2 \cdot 4 \cdot 10^6}{\pi^2 + \ln(0.2)^2}$$

$$\implies \qquad\qquad R \approx \boxed{912 \ \Omega}$$

Graphically, $R_1 < R_2$ implies $p_1 < p_2$, but $\omega_{n,1}$ and $\omega_{n,2}$ are inversely inequal. The lower resistance R features higher amplitudes in the oscillation of the response – less energy is removed from the circuit system than in the case of higher resistance.

Question: 2.5.17

What values of m, b, k will give the step response $r(t) = 1 - \sqrt{2}e^{-t}\sin\left(t + \frac{\pi}{4}\right)$?

We have
$$\frac{\omega_n}{\omega_d} = \sqrt{2}, \quad p = 1, \quad \omega_d = 1$$
These relations in turn imply
$$\omega_n^2 = \frac{k}{m} = 2, \quad \frac{b}{m} = 2$$
One choice of parameters that satisfy these relations is $m = 1$ and $b = k = 2$.

Question: 2.5.18

What happens to the $p - \omega_d - \omega_n$ right triangle as the damping ratio ω_n/p increases to 1 (critical damping)? At that point the damped frequency ω_d becomes ____. The step response becomes $r(t) =$ ____.

We have damped frequency $\omega_d = 0$, which implies that we have repeated roots $s_{1,2} = p$ and $p = \omega_n$. Because of resonance, our solution changes to be
$$r(t) = 1 + c_1 e^{-pt} + c_2 t e^{-pt}$$
Applying the initial conditions $r(0)$ and $r'(0)$ give us coefficients $c_1 = -1$ and $c_2 = -p$. Assembling it all together, our solution is
$$r(t) = 1 - e^{-pt}(1 + pt)$$

Question: 2.5.19

The roots $s_1, s_2 = -p \pm i\omega_d$ are poles of the transfer function $1/(As^2 + Bs + C)$
Show directly that the product of the roots $s_1 = -p + i\omega_d$ and $s_2 = -p - i\omega_d$ is $s_1 s_2 = \omega_n^2$. The sum of the roots is $-2p$. The quadratic equation with those roots is $s^2 + 2ps + \omega_n^2 = 0$.

The product is

$$s_1 s_2 = (-p + i\omega_d)(-p - i\omega_d) = p^2 + \omega_d^2 = \omega_n^2$$

The imaginary components cancel in the sum, giving us $s_1 + s_2 = -2p$. These relations are Vieta's formulas for a quadratic, where the sum of the roots equals $-B/A$ and their product C/A.

Question: 2.5.20

Suppose p is increased while ω_n is held constant. How do the roots s_1 and s_2 move?

The imaginary component ω_d decreases in magnitude, meaning that the roots move along the circle of radius ω_n on the complex plane towards the negative real axis at $-p \to -\omega_n$.

Question: 2.5.21

Suppose the mass m is increased while the coefficients b and k are unchanged. What happens to the roots s_1 and s_2?

All else equal, increasing the mass m means that the roots move inward onto a complex circle of smaller radius, approaching zero in the limit. Since the roots are expressed as

$$s = -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4mk}}{2m}$$

We can see that increasing m will eventually force the second term to be imaginary, even as the real and imaginary components in magnitude diminish. Importantly, $\omega_n^2 = k/m$

Intuitively, we imagine a heavy mass-dashpot-spring system. The natural frequency ω_n will drastically decrease, indicating the increasingly slow oscillation of the mass. Since the damping coefficient is unchanged, the damping per unit mass is increasingly washed out with a heavier mass, indicating a proportionally lower amount of energy loss given an input forcing.

> **Question: 2.5.22**
>
> **Ramp response** How could you find $y(t)$ when $F = t$ is a ramp function?
>
> $$y'' + 2py' + \omega_n^2 y = \omega_n^2 t \text{ starting from } y(0) = 0 \text{ and } y'(0) = 0$$
>
> A particular solution (straight line) is $y_p = $ ____. The null solution still has the form $y_n = $ ____. Find the coefficients c_1 and c_2 in the null solution from the two conditions at $t = 0$.
> This ramp response $y(t)$ can also be seen as the integral of ____.

The particular solution is $y_p = t - 2p/\omega_n^2$. The null solution has form $y_n = c_1 e^{s_1 t} + c_2 e^{s_2 t}$, where the roots are

$$s_{1,2} = -p \pm \sqrt{p^2 - \omega_n^2} = -p \pm i\omega_d$$

This gives us complete solution

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + t - \frac{2p}{\omega_n^2}$$

To find the coefficients, apply the initial conditions and derive

$$y(0) = c_1 + c_2 - \frac{2p}{\omega_n^2} = 0 \implies \boxed{c_1 = \frac{2p}{\omega_n^2} - c_2}$$

$$y'(0) = c_1 s_1 + c_2 s_2 + 1 = 0 \implies \boxed{c_2 = \frac{1 + \frac{2p}{\omega_n^2} s_1}{s_1 - s_2}}$$

which then implies

$$\boxed{c_1 = \frac{1 + \frac{2p}{\omega_n^2} s_2}{s_2 - s_1}}$$

The ramp response (slanted line) is the integral of the step response (horizontal line).

2.6 - Solutions to Second Order Equations

Find a particular solution by inspection (or the method of undetermined coefficients)

Question: 2.6.1

(a) $y'' + y = 4$

(b) $y'' + y' = 4$

(c) $y'' = 4$

(a) $y_p = 4$

(b) $y_p = 4t$

(c) $y_p = 2t^2$

Question: 2.6.2

(a) $y'' + y' + y = e^t$

(b) $y'' + y' + y = e^{ct}$

(a) $y_p = \frac{1}{3} e^t$

(b) Substitute $y_p = Y e^{ct}$ to find

$$c^2 Y + cY + Y = 1 \implies Y = \frac{1}{c^2 + c + 1}$$

Question: 2.6.3

(a) $y'' - y = \cos(t)$

(b) $y'' + y = \cos(2t)$

(c) $y'' + y = t + e^t$

(a) Substitute $y_p = Y \cos(t)$ to find

$$-2Y \cos(t) = \cos(t) \implies Y = -\frac{1}{2}$$

(b) Substitute $y_p = Y \cos(2t)$ to find

$$-3Y \cos(2t) = \cos(2t) \implies Y = -\frac{1}{3}$$

(c) $y_p = t + \frac{1}{2} e^t$

Question: 2.6.4

For these $f(t)$, predict the form of $y(t)$ with undetermined coefficients:

(a) $f(t) = t^3$

(b) $f(t) = \cos(2t)$

(c) $f(t) = t\cos(t)$

(a) $y_p(t) = at^3 + bt^2 + ct + d$

(b) $y_p(t) = a\cos(2t) + b\sin(2t)$

(c) $y_p(t) = (at + b)\cos(t) + (ct + d)\sin(t)$

Question: 2.6.5

Predict the form for $y(t)$ when the right hand side is

(a) $f(t) = e^{ct}$

(b) $f(t) = te^{ct}$

(c) $f(t) = e^t\cos(t)$

(a) $y_p(t) = Ye^{ct}$

(b) $y_p(t) = (at + b)e^{ct}$

(c) $y_p(t) = e^t(a\cos(t) + b\sin(t))$

Question: 2.6.6

For $f(t) = e^{ct}$ when is the prediction for $y(t)$ different from Ye^{ct}?

If e^{ct} is one of the null solutions, we must have a Yte^{ct} term to account for the resonance.

Use the method of undetermined coefficients to find a solution $y_p(t)$.

Question: 2.6.7

(a) $y'' + 9y = e^{2t}$

(b) $y'' + 9y = te^{2t}$

(a) Let $y_p(t) = Ye^{2t}$. Then

$$4Ye^{2t} + 9Ye^{2t} = e^{2t} \implies Y = \frac{1}{13} \implies y_p(t) = \frac{e^{2t}}{13}$$

(b) Let $y_p(t) = (at + b)\, e^{2t}$. Then
$$y_p'(t) = ae^{2t} + 2(at + b)\, e^{2t}$$
$$= (2at + a + 2b)\, e^{2t}$$
$$y_p''(t) = 2ae^{2t} + 2(2at + a + 2b)\, e^{2t}$$
$$= (4at + 4a + 4b)\, e^{2t}$$

Substituting into the equation, we derive
$$(4at + 4a + 4b)\, e^{2t} + 9(at + b)\, e^{2t} = (13at + 4a + 13b)\, e^{2t} = te^{2t}$$

Here, we can deduce that $a = \dfrac{1}{13}$ and $b = -\dfrac{4}{13}\dfrac{1}{13}$. Our particular solution is
$$y_p(t) = \frac{1}{13}\left(t - \frac{4}{13}\right) e^{2t}$$

Question: 2.6.8

(a) $y'' + y' = t + 1$

(b) $y'' + y' = t^2 + 1$

(a) Observe that the roots of the characteristic polynomial are $s = 0, -1$. Then one of the null solutions is simply a constant, so that cannot be present in our particular solution. Let $y_p(t) = at^2 + bt$ (no constant term). Then

$$y'(t) = 2at + b$$
$$y''(t) = 2a$$

Substituting, we get
$$2at + (2a + b) = t + 1$$
Where we can conclude $a = 1/2$ and $b = 0$, and
$$y_p(t) = \frac{1}{2}t^2$$

(b) The equation is the same but the forcing changes, so we have ansatz $y_p(t) = at^3 + bt^2 + ct$ (again, no constant term). Then
$$y'(t) = 3at^2 + 2bt + c$$

$$y''(t) = 6at + 2b$$

We substitute to get
$$3at^2 + (6a + 2b)\, t + 2b + c = t^2 + 1$$

Concluding that $a = 1/3$, $b = -1$, and $c = 2$. This corresponds to particular solution
$$y_p(t) = \frac{1}{3}t^3 - t^2 + 2t$$

Question: 2.6.9

(a) $y'' + 3y = \cos(t)$

(b) $y'' + 3y = t\cos(t)$

(a) Let $y_p(t) = a\cos(t) + b\sin(t)$. Substituting gives us
$$2a\cos(t) + 2b\sin(t) = \cos(t)$$
From which we can deduce $a = 1/2$ and $b = 0$, yielding solution
$$y_p(t) = \frac{1}{2}\cos(t)$$

(b) Let $y_p(t) = (at + b)\cos(t) + (ct + d)\sin(t)$. Then
$$\begin{aligned}
y_p'(t) &= a\cos(t) - (at + b)\sin(t) + c\sin(t) + (ct + d)\cos(t) \\
&= (ct + a + d)\cos(t) + (-at - b + c)\sin(t) \\
y_p''(t) &= c\cos(t) - (ct + a + d)\sin(t) - a\sin(t) + (-at - b + c)\cos(t) \\
&= (-at - b + 2c)\cos(t) - (ct + 2a + d)\sin(t)
\end{aligned}$$
Substituting gives us
$$(2at + 2b + 2c)\cos(t) + (2ct - 2a + 2d)\sin(t) = t\cos(t)$$
We must have $a = d - 1/2$ and $b = c = 0$, which corresponds to particular solution
$$y_p(t) = \frac{1}{2}t\cos(t) + \frac{1}{2}\sin(t)$$

Question: 2.6.10

(a) $y'' + y' + y = t^2$

(b) $y'' + y' + y = t^3$

(a) Let $y_p(t) = at^2 + bt + c$. Then $y_p'(t) = 2at + b$ and $y_p''(t) = 2a$. Substituting yields
$$at^2 + (2a + b)\,t + 2a + b + c = t^2$$
where we can deduce that $a = 1$, $b = -2$, and $c = 0$, and solution
$$y_p(t) = t^2 - 2t$$

(b) Let $y_p(t) = at^3 + bt^2 + ct + d$. Then $y_p'(t) = 3at^2 + 2bt + c$ and $y_p''(t) = 6at + 2b$. Substituting yields
$$at^3 + (3a + b)\,t^2 + (6a + 2b + c)\,t + 2b + c + d = t^3$$
which gives us coefficients $a = 1$, $b = -3$, $c = 0$, and $d = 6$. We have particular solution
$$y_p(t) = t^3 - 3t^2 + 6$$

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Question: 2.6.11

(a) $y'' + y' + y = \cos(t)$

(b) $y'' + y' + y = t\sin(t)$

(a) Let $y_p(t) = a\cos(t) + b\sin(t)$. Then $y_p'(t) = -a\sin(t) + b\cos(t)$ and $y_p''(t) = -a\cos(t) - b\sin(t)$. Substituting gives us

$$-a\sin(t) + b\cos(t) = \cos(t)$$

which has $a = 0$ and $b = 1$, with solution $y_p(t) = \sin(t)$.

(b) Let $y_p(t) = (at + b)\cos(t) + (ct + d)\sin(t)$. Then

$$\begin{aligned} y_p'(t) &= a\cos(t) - (at + b)\sin(t) + c\sin(t) + (ct + d)\cos(t) \\ &= (ct + a + d)\cos(t) + (-at - b + c)\sin(t) \\ y_p''(t) &= c\cos(t) - (ct + a + d)\sin(t) - a\sin(t) + (-at - b + c)\cos(t) \\ &= (-at - b + 2c)\cos(t) + (-ct - 2a - d)\sin(t) \end{aligned}$$

Substituting gives us

$$(ct + a + 2c + d)\cos(t) + (-at - b - 2a + c)\sin(t) = t\sin(t)$$

This forces $c = 0$, $a = -1$, $d = 1$, and $b = 2$, which corresponds to solution

$$y_p(t) = (-t + 2)\cos(t) + \sin(t)$$

Problems 12-14 involve resonance. Multiply the usual form of y_p by t.

Question: 2.6.12

(a) $y'' + y = e^{it}$

(b) $y'' + y = \cos(t)$

(a) Let $y_p(t) = Yte^{it}$. Then $y_p'(t) = Y\left(e^{it} + ite^{it}\right) = Y(1 + it)e^{it}$ and $y_p''(t) = Y\left(ie^{it} + (i - t)e^{it}\right) = Y(2i - t)e^{it}$. Substituting gives us

$$2iYe^{it} = e^{it} \implies Y = -\frac{i}{2}$$

which corresponds to solution

$$y_p(t) = -\frac{i}{2}te^{it}$$

(b) Let $y_p(t) = t(a\cos(t) + b\sin(t))$. Then

$$y_p'(t) = a\cos(t) + b\sin(t) + t(-a\sin(t) + b\cos(t))$$
$$= (bt + a)\cos(t) + (-at + b)\sin(t)$$
$$y_p''(t) = b\cos(t) - (bt + a)\sin(t) - a\sin(t) + (-at + b)\cos(t)$$
$$= (-at + 2b)\cos(t) - (bt + 2a)\sin(t)$$

Substituting gives us

$$2b\cos(t) - 2a\sin(t) = \cos(t)$$

Forcing $a = 0$ and $b = 1/2$, and solution

$$y_p(t) = \frac{1}{2}t\sin(t)$$

Question: 2.6.13

(a) $y'' - 4y' + 3y = e^t$

(b) $y'' - 4y' + 3y = e^{3t}$

(a) Let $y_p(t) = Yte^t$. Then $y_p'(t) = Y(e^t + te^t)$ and $y_p''(t) = Y(2e^t + te^t)$. Substituting gives us

$$-2Ye^t = e^t \implies Y = -\frac{1}{2}$$

which corresponds to solution

$$y_p(t) = -\frac{1}{2}te^t$$

(b) Let $y_p(t) = Yte^{3t}$. Then $y_p'(t) = Y\left(e^{3t} + 3te^{3t}\right)$ and $y_p''(t) = Y\left(6e^{3t} + 9te^{3t}\right)$. Substituting gives us

$$2Ye^{3t} = e^{3t} \implies Y = \frac{1}{2}$$

which corresponds to solution

$$y_p(t) = \frac{1}{2}te^{3t}$$

Question: 2.6.14

(a) $y' - y = e^t$

(b) $y' - y = te^t$

(c) $y' - y = e^t\cos(t)$

(a) Let $y_p(t) = Yte^t$. Then $y_p'(t) = Y(e^t + te^t)$. Then we substitute to derive

$$Ye^t = e^t \implies Y = 1$$

Then we have $y_p(t) = te^t$.

(b) Let $y_p(t) = Yt^2e^t$. Then $y_p'(t) = Y(t^2e^t + 2te^t)$. We substitute to get

$$2Yte^t = te^t \implies Y = \frac{1}{2}$$

Our solution is

$$y_p(t) = \frac{1}{2}t^2e^t$$

(c) Let $y_p(t) = e^t(a\cos(t) + b\sin(t))$. Then

$$y_p'(t) = e^t[(a\cos(t) + b\sin(t)) + (-a\sin(t) + b\cos(t))]$$
$$= e^t[(a+b)\cos(t) + (b-a)\sin(t)]$$

We substitute to get

$$e^t(b\cos(t) - a\sin(t)) = e^t\cos(t) \implies a = 0, b = 1$$

Therefore our solution is

$$y_p(t) = e^t\sin(t)$$

Question: 2.6.15

For $y'' - 4y = e^t\sin(t)$ (exponential times sinusoidal) we have two choices:

   1   (Real)   Substitute $y_p = Me^t\cos(t) + Ne^t\sin(t)$:
                                   determine M and N

   2   (Complex)   Solve $z'' + 4z = e^{(1+i)t}$.
                                   Then y is the imaginary part of z.

Use both methods to find the same $y(t)$ − which do you prefer?

---

1. Let y_p be as given. Then

$$y_p' = Me^t(\cos(t) - \sin(t)) + Ne^t(\sin(t) + \cos(t))$$
$$y_p'' = -2Me^t\sin(t) + 2Ne^t\cos(t)$$

Substituting gives us

$$e^t[(2N - 4M)e^t\cos(t) - (2M + 4N)\sin(t)] = e^t\sin(t)$$

We must have $-4M + 2N = 0$ and $-(2M + 4N) = 1$. Solve the matrix equation:

$$\begin{bmatrix} -4 & 2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Which gives us $M = -1/5$ and $N = 1/10$.

2. In the complex case, let $y_p = Ye^{(1+i)t}$. Then $y_p' = Y(1+i)e^{(1+i)t}$ and $y_p'' = Y(1+i)^2 e^{(1+i)t}$, and our task reduces to isolating Y in the equation

$$Y(1+i)^2 - 4Y(1+i) = 1$$

Doing so yields

$$Y = -\frac{1}{4+2i} = \frac{-4+2i}{20} = \frac{-2+i}{10}$$

Our particular solution is

$$y_p = \left(\frac{-2+i}{10}\right)e^t(\cos(t) + i\sin(t))$$
$$= -\frac{1}{5}e^t\cos(t) + \frac{1}{10}e^t\sin(t) + i\left[\frac{1}{10}e^t\cos(t) - \frac{1}{5}e^t\sin(t)\right]$$

Taking the real component, we see that it matches with what we had in part 1.

Question: 2.6.16

(a) Which values of c give resonance for $y'' + 3y' - 4y = te^{ct}$?

(b) What form would you substitute for $y(t)$ if there is no resonance?

(c) What form would you use when c produces resonance?

(a) The characteristic polynomial is $s^2 + 3s - 4$ which has roots $s_1 = -4$ and $s_2 = 1$, so those values of c will give resonance. This is because our ansatz, when of the form $y_p = (at + b)e^{ct}$, includes the form of the null solutions.

(b) If no resonance, then we can safely put forth our ansatz with the form $y_p = (at + b)e^{ct}$.

(c) Otherwise, use form $y_p = (at^2 + bt)e^{ct}$.

Question: 2.6.17

This is the rule for equations $P(D)y = e^{ct}$ with resonance $P(c) = 0$:
If $P(c) = 0$ and $P'(c) \neq 0$, look for a solution $y_p = Cte^{ct}$ $(m = 1)$
If c is a root of multiplicity m, then y_p has the form _____.

Then y_p has form $Ct^m e^{ct}$.

Question: 2.6.18

(a) To solve $d^4y/dt^4 - y = t^3 e^{5t}$, what form do you expect for $y(t)$?

(b) If the right side becomes $t^3 \cos(5t)$, which 8 coefficients are to be determined?

(a) The characteristic polynomial is $s^4 - 1$, which gives us roots $s = \pm 1, \pm i$. Since $c = 5$, we need not concern ourselves with resonance, and we guess $y_p = (at^3 + bt^2 + ct + d) e^{5t}$.

(b) In this case, we then determine the number of combinations between sine and cosine and the four terms of a third-degree polynomial:

$$t^3 \cos(5t) \quad t^2 \cos(5t) \quad t \cos(5t) \quad \cos(5t)$$
$$t^3 \sin(5t) \quad t^2 \sin(5t) \quad t \sin(5t) \quad \sin(5t)$$

Question: 2.6.19

For $y' - ay = f(t)$, the method of undetermined coefficients is looking for all $f(t)$ so that the usual formula $y_p = e^{at} \int e^{-as} f(s) \, ds$ is easy to integrate. Find these integrals for the "nice functions" $f = e^{ct}$, $f = e^{i\omega t}$, and $f = t$:

$$\int e^{-as} e^{cs} \, ds \qquad \int e^{-as} e^{i\omega s} \, ds \qquad \int e^{-as} s \, ds$$

(a)
$$\int e^{-as} e^{cs} \, ds = \int e^{(-a+c)s} \, ds = \frac{e^{(-a+c)s}}{-a+c} + C$$

(b)
$$\int e^{-as} e^{i\omega s} \, ds = \int e^{(-a+i\omega)s} \, ds = \frac{e^{(-a+i\omega)s}}{-a+i\omega} + C$$

(c) Resort to integration by parts. Let $u = s$ and $dv = e^{-as} ds$. Then $du = ds$ and $v = -e^{-as}/a$.

$$\int e^{-as} s \, ds = -\frac{e^{-as}s}{a} + \frac{1}{a} \int e^{-as} \, ds = -e^{-as} \left(\frac{s}{a} + \frac{1}{a^2} \right)$$

Problems 20-27 develop the method of variation of parameters.

Find two solutions y_1, y_2 to $y'' + 3y' + 2y = 0$. Use those in formula (13) to solve

(a) $y'' + 3y' + 2y = e^t$

(b) $y'' + 3y' + 2y = e^{-t}$

(a) The roots to the characteristic polynomial are $s_{1,2} = -2, -1$. Then we have null solutions $y_1 = e^{-2t}$ and $y_2 = e^{-t}$, and correspondingly $y_1' = -2e^{-2t}$ and $y_2' = -e^{-t}$. The Wronskian is calculated as

$$W(t) = -e^{-3t} + 2e^{-3t} = e^{-3t}$$

By the variation of parameters, the particular solution is

$$y_p(t) = -e^{-2t} \int e^{3t}\, dt + e^{-t} \int e^{2t}\, dt$$
$$= -\frac{e^t}{3} + \frac{e^t}{2}$$
$$= \frac{e^t}{6}$$

(b) In the second case, we have resonance, since $c = -1$ which is one of our roots. The particular solution becomes

$$y_p(t) = -e^{-2t} \int e^t\, dt + e^{-t} \int dt$$
$$= e^{-t}(t - 1)$$

Find two solutions to $y'' + 4y' = 0$ and use variation of parameters for

(a) $y'' + 4y' = e^{2t}$

(b) $y'' + 4y' = e^{-4t}$

(a) The characteristic polynomial has roots $s_{1,2} = 0, -4$. Then we have null solutions $y_1 = 1$ and $y_2 = e^{-4t}$, and correspondingly $y_1' = 0$ and $y_2' = -4e^{-4t}$. The Wronksian is

$$W(t) = -4e^{-4t}$$

Variation of parameters gives us particular solution

$$y_p(t) = \frac{1}{4} \int e^{2t} \, dt - \frac{e^{-4t}}{4} \int e^{6t} \, dt$$
$$= \left(\frac{1}{8} - \frac{1}{24} \right) e^{2t}$$
$$= \frac{e^{2t}}{12}$$

(b) Again, resonance presents itself. Calculate

$$y_p(t) = \frac{1}{4} \int e^{-4t} \, dt - \frac{e^{-4t}}{4} \int dt = -e^{-4t} \left(\frac{t}{4} + \frac{1}{16} \right)$$

Question: 2.6.22

Find an equation $y'' + By' + Cy = 0$ that is solved by $y_1 = e^t$ and $y_2 = te^t$. If the right side is $f(t) = 1$, what solution comes from the VP formula (13)?

The null solutions imply a double root $s = 1$, so we have $B = 2$ and $C = 1$ for

$$y'' + 2y' + 1 = 0$$

The derivatives of the null solutions are $y_1' = e^t$ and $y_2' = e^t(1 + t)$. The Wronskian is

$$W(t) = e^{2t}(1 + t) - te^{2t} = e^{2t}$$

and the particular solution is

$$y_p(t) = -e^t \int te^{-t} \, dt + te^t \int e^{-t} \, dt$$
$$= t + 1 - t$$
$$= 1$$

We could have come to this conclusion on inspection, but good to see that variation of parameters brings us to this result nonetheless.

Question: 2.6.23

$y'' - 5y' + 6y = 0$ is solved by $y_1 = e^{2t}$ and $y_2 = e^{3t}$, because $s = 2$ and $s = 3$ come from $s^2 - 5s + 6 = 0$. Now solve $y'' - 5y' + 6y = 12$ in two ways:
1. Undetermined coefficients (or inspection)
2. Variation of parameters using (13)
The answers are different. Are the initial conditions different?

---

1. By inspection, $y_p = 2$. Our complete solution is

$$y(t) = c_1 e^{2t} + c_2 e^{3t} + 2$$

2. The Wronskian is calculated as

$$W(t) = 3e^{5t} - 2e^{5t} = e^{5t}$$

Our particular solution is

$$y_p(t) = -e^{2t} \int 12e^{-2t}\, dt + e^{3t} \int 12e^{-3t}\, dt = 2$$

Both deriving the particular solution by inspection and variation of parameters lead to the same result, but if we integrate from 0 to t to incorporate initial conditions, we see

$$
\begin{aligned}
y(t) &= -e^{2t} \int_0^t 12e^{-2s}\, ds + e^{3t} \int_0^t 12e^{-3s}\, ds \\
&= 6e^{2t}\left(e^{-2t} - 1\right) - 4e^{3t}\left(e^{-3t} - 1\right) \\
&= 2 - 6e^{2t} + 4e^{3t}
\end{aligned}
$$

Which is the full solution − including the null solutions − implying $y(0) = y'(0) = 0$.

Question: 2.6.24

What are the initial conditions $y(0)$ and $y'(0)$ for the solution (13) coming from the variation of parameters, starting from any y_1 and y_2?

We can see that $y(0) = 0$, as both terms cancel each other out at $t = 0$. To find the time derivative, use the dummy variable s inside the integral and apply Leibniz's rule:

$$
\begin{aligned}
y'(t) &= \frac{d}{dt}\left[-y_1(t) \int_0^t \frac{y_2(s)\,f(s)}{W(s)}\, ds + y_2(t) \int_0^t \frac{y_1(s)\,f(s)}{W(s)}\, ds\right] \\
&= -y_1'(t) \int_0^t \frac{y_2(s)\,f(s)}{W(s)}\, ds - \frac{y_1(t)\,y_2(t)\,f(t)}{W(t)} \\
&\quad + y_2'(t) \int_0^t \frac{y_1(s)\,f(s)}{W(s)}\, ds + \frac{y_1(t)\,y_2(t)\,f(t)}{W(t)} \\
\implies y'(0) &= 0
\end{aligned}
$$

The initial position and velocity are assumed to be zero.

Question: 2.6.25

The equation $y'' = 0$ is solved by $y_1 = 1$ and $y_2 = t$. Use variation of parameters to solve $y'' = t$ and also $y'' = t^2$.

We have derivatives $y_1' = 0$ and $y_2' = 1$, and Wronskian

$$W(t) = 1$$

The particular solution for $y'' = t$ is derived as

$$y_p(t) = -\int t^2\, dt + t \int t\, dt = -\frac{t^3}{3} + \frac{t^3}{2} = \frac{t^3}{6}$$

And for $y'' = t^2$ we have

$$y_p(t) = -\int t^3 \, dt + t \int t^2 \, dt = -\frac{t^4}{4} + \frac{t^4}{3} = \frac{t^4}{12}$$

Question: 2.6.26

Solve $y_s'' + y_s = 1$ for the step response using variation of parameters, starting from the null solutions $y_1 = \cos(t)$ and $y_2 = \sin(t)$.

We have $y_1' = -\sin(t)$ and $y_2' = \cos(t)$, and Wronskian

$$W(t) = \sin^2(t) + \cos^2(t) = 1$$

The particular solution is

$$y_p(t) = -\cos(t) \int \sin(t) \, dt + \sin(t) \int \cos(t) \, dt = \cos^2(t) + \sin^2(t) = 1$$

If we impose initial conditions $y(0) = y'(0) = 0$, then we have

$$y(t) = -\cos(t) \int_0^t \sin(t) \, dt + \sin(t) \int_0^t \cos(t) \, dt$$
$$= \cos^2(t) - \cos(t) + \sin^2(t)$$
$$= 1 - \cos^2(t)$$

Question: 2.6.27

Solve $y_s'' + 3y_s' + 2y_s = 1$ for the step response starting from the null solutions $y_1 = e^{-t}$ and $y_2 = e^{-2t}$.

We have $y_1' = -e^{-t}$ and $y_2' = -2e^{-2t}$, with Wronskian

$$W(t) = -2e^{-3t} + e^{-3t} = -e^{-3t}$$

The particular solution is

$$y_p(t) = e^{-t} \int e^t \, dt - e^{-2t} \int e^{2t} \, dt = 1 - \frac{1}{2} = \frac{1}{2}$$

If we impose initial conditions $y(0) = y'(0) = 0$, then we have

$$y(t) = e^{-t} \int_0^t e^t \, dt - e^{-2t} \int_0^t e^{2t} \, dt$$
$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

> **Question: 2.6.28**
>
> Solve $Ay'' + Cy = \cos(\omega t)$ when $A\omega^2 = C$ (the case of resonance). Example 4 suggests to substitute $y = Mt\cos(\omega t) + Nt\sin(\omega t)$. Find M and N.

We have

$$
\begin{aligned}
y' &= M\cos(\omega t) + N\sin(\omega t) + t(-M\omega\sin(\omega t) + N\omega\cos(\omega t)) \\
&= (M + N\omega t)\cos(\omega t) + (N - M\omega t)\sin(\omega t) \\
y'' &= N\omega\cos(\omega t) - \left(M\omega + N\omega^2 t\right)\sin(\omega t) \\
&\quad - M\omega\sin(\omega t) + \left(N\omega - M\omega^2 t\right)\cos(\omega t) \\
&= \left(2N\omega - M\omega^2 t\right)\cos(\omega t) - \left(2M\omega + N\omega^2 t\right)\sin(\omega t)
\end{aligned}
$$

Substituting and letting $A = C/\omega^2$, we find

$$
C\left[\left(\frac{2N}{\omega} - Mt\right)\cos(\omega t) - \left(\frac{2M}{\omega} + Nt\right)\sin(\omega t)\right]
$$
$$
+ Ct[M\cos(\omega t) + N\sin(\omega t)] =
$$
$$
\implies \qquad 2C\left[\frac{N}{\omega}\cos(\omega t) - \frac{M}{\omega}\sin(\omega t)\right] = \cos(\omega t)
$$

We can conclude that $M = 0$ and $N = \omega/2C$, and our particular solution is

$$
y_p(t) = \frac{\omega}{2C}t\sin(\omega t)
$$

> **Question: 2.6.29**
>
> Put $g(t)$ into the great formulas (17)-(18) to see the equations above them.

(17): We define $g(t)$ as

$$
g(t) = \frac{e^{s_1 t} - e^{s_2 t}}{s_1 - s_2}
$$

Substituting gives us:
Particular solution, constant coefficients

$$
\begin{aligned}
y_p(t) &= \int_0^t \left(\frac{e^{s_1(t-T)} - e^{s_2(t-T)}}{s_1 - s_2}\right) f(T)\ dT \\
&= \frac{e^{s_1 t}}{s_1 - s_2}\int_0^t e^{s_1 T} f(T)\ dT - \frac{e^{s_2 t}}{s_1 - s_2}\int_0^t e^{s_2 T} f(T)\ dT
\end{aligned}
$$

(18): We define $g(t)$ as

$$
g(t) = te^{st}
$$

Substituting gives us:

Particular solution y_p, null solutions e^{st}, te^{st}

$$y_p(t) = \int_0^t (t - T)\, e^{s(t-T)} f(T)\ dT$$

$$= \int_0^t te^{s(t-T)} f(T)\ dT - \int_0^t Te^{s(t-T)} f(T)\ dT$$

$$= te^{st} \int_0^t e^{-sT} f(T)\ dT - e^{st} \int_0^t Te^{-sT} f(T)\ dT$$

2.7 - Laplace Transforms $Y(s)$ and $F(s)$

> **Question: 2.7.1**
>
> Take the Laplace transform of each term in these equations and solve for $Y(s)$, with $y(0) = 0$ and $y'(0) = 1$. Find the roots s_1 and s_2 − the poles of $Y(s)$:
>
> $$\begin{aligned} Undamped && y'' + 0y' \ +16y &= 0 \\ Underdamped && y'' + 2y' \ +16y &= 0 \\ Critically\ damped && y'' + 8y' \ +16y &= 0 \\ Overdamped && y'' + 10y'+16y &= 0 \end{aligned}$$
>
> For the overdamped case use PF2 to write $Y(s) = A/(s - s_1) + B/(s - s_2)$.

1. *Undamped.* The transform in the s-domain is
$$\left(s^2 + 16\right) Y(s) - sy(0) - y'(0) = \left(s^2 + 16\right) Y(s) - 1 = 0$$
   Leading us to transform of $y(t)$
$$Y(s) = \frac{1}{s^2 + 16}$$
   which has roots $s_{1,2} = \pm 4i$.

2. *Underdamped.* The transform in the s-domain is
$$\left(s^2 + 2s + 16\right) Y(s) - (s + 2) y(0) - y'(0) = \left(s^2 + 2s + 16\right) Y(s) - 1 = 0$$
   Leading us to transform of $y(t)$
$$Y(s) = \frac{1}{s^2 + 2s + 16} = \frac{1}{(s + 1)^2 + 15}$$
   which has roots $s_{1,2} = -1 \pm \sqrt{15}i$.

3. *Critically damped.* The transform in the s-domain is
$$\left(s^2 + 8s + 16\right) Y(s) - (s + 8) y(0) - y'(0) = \left(s^2 + 8s + 16\right) Y(s) - 1 = 0$$
   Leading us to transform of $y(t)$
$$Y(s) = \frac{1}{s^2 + 8s + 16} = \frac{1}{(s + 4)^2}$$
   which has roots $s_{1,2} = -4$.

4. *Overdamped.* The transform in the s-domain is
$$\left(s^2 + 10s + 16\right) Y(s) - (s + 10) y(0) - y'(0) = \left(s^2 + 10s + 16\right) Y(s) - 1 = 0$$
   Leading us to transform of $y(t)$
$$Y(s) = \frac{1}{s^2 + 10s + 16} = \frac{1}{(s + 2)(s + 8)} = \frac{1}{6(s + 2)} - \frac{1}{6(s + 8)}$$
   which has roots $s_{1,2} = -2, -8$.

Question: 2.7.2

Invert the four transforms $Y(s)$ in Problem 1 to find $y(t)$.

1. *Undamped.* The inverse transform, with no damping and $\omega = 4$, is

$$y(t) = \frac{1}{4}\sin(4t)$$

2. *Underdamped.* The inverse transform, with underdamping and oscillating frequency $\omega = \sqrt{15}$, is

$$y(t) = \frac{1}{\sqrt{15}}e^{-t}\sin\left(\sqrt{15}t\right)$$

3. *Critically damped.* The inverse transform, with critical damping and repeated roots, is

$$y(t) = te^{-4t}$$

4. *Overdamped.* The inverse transform, with overdamping (and thus no oscillation) is

$$y(t) = \frac{1}{6}\left(e^{-2t} - e^{-8t}\right)$$

Question: 2.7.3

  (a) Find the Laplace transform $Y(s)$ when $y' = e^{at}$ with $y(0) = A$.

  (b) Use PF2 to break $Y(s)$ into two fractions $C_1/(s-a) + C_2/s$.

  (c) Invert $Y(s)$ to find $y(t)$ and check that $y' = e^{at}$ and $y(0) = A$.

(a) Our transform into the s-domain gives us

$$sY(s) - A = \frac{1}{s-a}$$

which gives us transform of $y(t)$

$$Y(s) = \frac{1}{s(s-a)} + \frac{A}{s}$$

(b) By partial fractions, we can rewrite the transform as

$$Y(s) = -\frac{1/a}{s} + \frac{1/a}{s-a} + \frac{A}{s} = \frac{1/a}{s-a} + \frac{(A - 1/a)}{s}$$

(c) From the inverse Laplace transform we get

$$y(t) = \frac{1}{a}e^{at} + A - \frac{1}{a}$$

From which it immediately follows that $y' = e^{at}$ and $y(0) = A$.

> **Question: 2.7.4**
>
> (a) Find the transform $Y(s)$ when $y'' = e^{at}$ with $y(0) = A$ and $y'(0) = B$.
>
> (b) Split $Y(s)$ into $C_1/(s-a) + C_2/(s-a)^2 + C_3/s$.
>
> (c) Invert $Y(s)$ to find $y(t)$. Check $y'' = e^{at}$ and $y(0) = A$ and $y'(0) = B$.

(a) Our transform into the s-domain gives us

$$s^2 Y(s) - sA - B = \frac{1}{s-a}$$

which gives us transform of $y(t)$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{1}{s^2(s-a)}$$

(b) The textbook is incorrect about the form of the partial fractions decomposition; the repeated term is in s and s^2. By partial fractions, we can rewrite the last term as

$$\begin{aligned}
Y(s) &= \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s-a} \\
&= \frac{C_1 s(s-a) + C_2(s-a) + C_3 s^2}{s^2(s-a)} \\
&= \frac{(C_1 + C_3)\,s^2 + (C_2 - aC_1)\,s - aC_2}{s^2(s-a)}
\end{aligned}$$

From which we can derive

$$C_1 = -\frac{1}{a^2}, \qquad C_2 = -\frac{1}{a}, \qquad C_3 = \frac{1}{a^2}$$

And finally, the transform

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} - \frac{1/a^2}{s} - \frac{1/a}{s^2} + \frac{1/a^2}{s-a}$$

(c) By the inverse Laplace transform, we have

$$y(t) = A + Bt - \frac{1}{a^2} - \frac{1}{a}t + \frac{1}{a^2}e^{at}$$

From which it follows that $y'' = e^{at}$, $y(0) = A$, and $y'(0) = B$.

Question: 2.7.5

Transform these differential equations to find $Y(s)$:

(a) $y'' - y' = 1$ with $y(0) = 4$ and $y'(0) = 0$

(b) $y'' + y = \cos(\omega t)$ with $y(0) = y'(0) = 0$ and $\omega \neq 1$

(c) $y'' + y = \cos(t)$ with $y(0) = y'(0) = 0$. What changed for $\omega = 1$?

(a) The transform gives us

$$\left(s^2 - s\right) Y(s) + (1 - s) y(0) - y'(0) = \left(s^2 - s\right) Y(s) + 4(1 - s) = \frac{1}{s}$$

which implies

$$Y(s) = \frac{1}{s^2(s - 1)} + \frac{4}{s}$$

By partial fractions, the first term can be rewritten as

$$\frac{1}{s^2(s-1)} = \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s-1}$$
$$= \frac{C_1 s(s-1) + C_2(s-1) + C_3 s^2}{s^2(s-1)}$$
$$= \frac{(C_1 + C_3)\, s^2 + (C_2 - C_1)\, s - C_2}{s^2(s-1)}$$

Forcing the numerator to unity, we require $C_1 = -1$, $C_2 = -1$, and $C_3 = 1$, which gives us transform of $y(t)$

$$Y(s) = -\frac{1}{s^2} + \frac{3}{s} + \frac{1}{s-1}$$

(b) The transform gives us

$$\left(s^2 + 1\right) Y(s) - sy(0) - y'(0) = \left(s^2 + 1\right) Y(s) = \frac{s}{s^2 + \omega^2}$$

which implies

$$Y(s) = \frac{s}{(s^2 + 1)(s^2 + \omega^2)}$$

(c) We need only set $\omega = 1$ in the previous transform, giving us a double pole in the denominator:

$$Y(s) = \frac{s}{\left(s^2 + 1\right)^2}$$

Question: 2.7.6

Find the Laplace transforms F_1, F_2, F_3 of these functions f_1, f_2, f_3:

$$f_1(t) = e^{at} - e^{bt} \qquad f_2(t) = e^{at} + e^{-at} \qquad f_3(t) = t\cos(t)$$

1. $F_1 = \dfrac{1}{s-a} - \dfrac{1}{s-b} = \dfrac{a-b}{(s-a)(s-b)}$

2. $F_2 = \dfrac{1}{s-a} + \dfrac{1}{s+a} = \dfrac{2s}{s^2 - a^2}$

3. One solution here is to once more exploit the additive property of the Laplace transform by writing

$$f_3(t) = t\cos(t) = \frac{1}{2}t\left(e^{it} + e^{-it}\right)$$

Then the transform becomes

$$F_3 = \frac{1}{2}\left[\frac{1}{(s-i)^2} + \frac{1}{(s+i)^2}\right]$$

$$= \frac{s^2 - 1}{(s^2 + 1)^2}$$

Question: 2.7.7

For any real or complex a, the transform of $f = te^{at}$ is ____. By writing $\cos(\omega t)$ as $\left(e^{i\omega t} + e^{-i\omega t}\right)/2$, transform $g(t) = t\cos(\omega t)$ and $h(t) = te^t\cos(\omega t)$. (*Notice that the transform of h is new.*)

The transform of f is $F(s) = \dfrac{1}{(s-a)^2}$. We can rewrite $g(t)$ as

$$g(t) = t\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) = \frac{te^{i\omega t}}{2} + \frac{te^{-i\omega t}}{2}$$

Then the transform of $g(t)$ is

$$G(s) = \frac{1}{2(s-i\omega)^2} + \frac{1}{2(s+i\omega)^2}$$

$$= \frac{(s+i\omega)^2 + (s-i\omega)^2}{2(s^2 + \omega^2)^2}$$

$$= \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$$

For $h(t)$, we rewrite as

$$h(t) = te^t\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) = \frac{te^{(1+i\omega)t}}{2} + \frac{te^{(1-i\omega)t}}{2}$$

which has corresponding transform

$$
\begin{aligned}
H(s) &= \frac{1}{2(s-(1+i\omega))^2} + \frac{1}{2(s-(1-i\omega))^2} \\
&= \frac{(s-(1-i\omega))^2 + (s-(1+i\omega))^2}{2(s^2-2s+1+\omega^2)^2} \\
&= \frac{2s^2 - 2s(1-i\omega) - 2s(1+i\omega) + (1-i\omega)^2 + (1+i\omega)^2}{2(s^2-2s+1+\omega^2)^2} \\
&= \frac{(s-1)^2 - \omega^2}{\left((s-1)^2 + \omega^2\right)^2}
\end{aligned}
$$

Observe that in relation to $G(s)$, the s-argument is offset by 1.

Question: 2.7.8

Invert the transforms F_1, F_2, F_3 using PF2 and PF3 to discover f_1, f_2, f_3:

$$
F_1(s) = \frac{1}{(s-a)(s-b)} \qquad F_2(s) = \frac{s}{(s-a)(s-b)} \qquad F_3(s) = \frac{1}{s^3-s}
$$

By partial fractions decomposition, we have

$$
F_1(s) = \frac{1}{(s-a)(a-b)} + \frac{1}{(b-a)(s-b)}
$$
$$
\implies f_1(t) = \frac{e^{at}}{a-b} + \frac{e^{bt}}{b-a}
$$

In the case of $F_2(s)$, write

$$
F_2(s) = \frac{c_1}{s-a} + \frac{c_2}{s-b} = \frac{(c_1+c_2)s - (bc_1+ac_2)}{(s-a)(s-b)}
$$

We must have $c_1 + c_2 = 1$ and $bc_1 + ac_2 = 0$. Solving for the unknowns gives us

$$
F_2(s) = \frac{a}{a-b}\frac{1}{s-a} + \frac{b}{b-a}\frac{1}{s-b}
$$

which corresponds to

$$
f_2(t) = \frac{a}{a-b}e^{at} + \frac{b}{b-a}e^{bt}
$$

Lastly, rewrite $F_3(s)$ as

$$
F_3(s) = \frac{1}{s(s-1)(s+1)} = \frac{c_1}{s} + \frac{c_2}{s-1} + \frac{c_3}{s+1}
$$

Solving for the unknowns gives us transform

$$
F_3(s) = -\frac{1}{s} + \frac{1}{2(s-1)} + \frac{1}{2(s+1)}
$$

which corresponds to

$$
f_3(t) = -1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}
$$

> **Question: 2.7.9**
>
> Step 1 transforms these equations and initial conditions. Step 2 solves for $Y(s)$. Step 3 inverts to find $y(t)$:
>
> (a) $y' - ay = t$ with $y(0) = 0$
>
> (b) $y'' + a^2 y = 1$ with $y(0) = 1$ and $y'(0) = 2$
>
> (c) $y'' + 3y' + 2y = 1$ with $y(0) = 4$ and $y'(0) = 5$.
>
> What particular solution y_p to (c) comes from using "undetermined coefficients"?

Transforming each to the s-domain, we find

(a)

$$sY - y(0) - aY = (s - a)Y = \frac{1}{s^2}$$

which implies

$$Y(s) = \frac{1}{s^2(s-a)}$$

Rewrite using partial fractions decomposition as

$$
\begin{aligned}
Y(s) &= \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s-a} \\
&= \frac{c_1 s(s-a) + c_2(s-a) + c_3 s^2}{s^2(s-a)} \\
&= \frac{(c_1 + c_3)s^2 + (-ac_1 + c_2)s - c_2 a}{s^2(s-a)} \\
&= -\frac{1}{a^2 s} - \frac{1}{as^2} + \frac{1}{a^2(s-a)}
\end{aligned}
$$

Transforming back into the t-domain, we have

$$y(t) = -\frac{1}{a^2} - \frac{1}{a}t + \frac{1}{a^2}e^{at}$$

(b)

$$s^2 Y - sy(0) - y'(0) + a^2 Y = \left(s^2 + a^2\right)Y - s - 2 = \frac{1}{s}$$

which implies

$$Y(s) = \frac{1}{s(s^2 + a^2)} + \frac{s}{s^2 + a^2} + \frac{2}{s^2 + a^2}$$

Applying partial fractions decomposition to the first term, we rewrite $Y(s)$ as

$$Y(s) = \frac{1}{a^2 s} + \left(1 - \frac{1}{a^2}\right)\frac{s}{s^2 + a^2} + \frac{2}{a}\frac{a}{s^2 + a^2}$$

Transforming back to the t-domain yields

$$y(t) = \frac{1}{a^2} + \left(1 - \frac{1}{a^2}\right)\cos(at) + \frac{2}{a}\sin(at)$$

(c)

$$s^2 Y - sy(0) - y'(0) + 3(sY - y(0)) + 2Y = \left(s^2 + 3s + 2\right) Y - 4s - 17 = \frac{1}{s}$$

which implies

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2 + 3s + 2)} + \frac{4s + 17}{s^2 + 3s + 2} \\ &= \frac{1}{s(s + 1)(s + 2)} + \frac{4s + 17}{(s + 1)(s + 2)} \end{aligned}$$

Conducting the partial fractions decomposition rewrites the transform as

$$Y(s) = \frac{1}{2s} + \frac{12}{s + 1} - \frac{17}{2(s + 2)}$$

which transforms back into the t-domain as

$$y(t) = \frac{1}{2} + 12e^{-t} - \frac{17}{2}e^{-2t}$$

Solving by way of the Laplace transform is quite horrid. One could solve this far more simply by observing that the roots of the characteristic polynomial are $s_{1,2} = -1, -2$, applying initial conditions, and finding the particular solution by inspection.

Questions 10-16 are about partial fractions.

> **Question: 2.7.10**
>
> Show that PF2 in equation (9) is correct. Multiply both sides by $(s - a)(s - b)$:
> $$(*) \qquad 1 = \underline{\quad} + \underline{\quad}.$$
>
> (a) What do those two fractions in $(*)$ equal at the points $s = a$ and $s = b$?
>
> (b) The equation $(*)$ is correct at those two points a and b. It is the equation of a straight ____. So why is it correct for every s?

(a) I believe the author means multiply both sides by $(s - a)(s - b)$. We have

$$1 = \frac{s - c}{a - c} + \frac{s - a}{c - a}$$

At $s = a$ and $s = c$, both are 1.

(b) Equation $(*)$ is that of a straight line, and it is correct for every s because a line is defined by two points, and since $(a, 1)$ and $(c, 1)$ satisfy the equation, the line must be defined for all s.

> **Question: 2.7.11**
>
> Here is the PF2 formula with numerators. Formula $(*)$ had $K = 1$ and $H = 0$:
>
> $$\textbf{PF2'} \qquad \frac{Hs + K}{(s-a)(s-b)} = \frac{Ha + K}{(s-a)(a-b)} + \frac{Hb + K}{(b-a)(s-b)}$$
>
> To show that PF2' is correct, multiply both sides by $(s-a)(s-b)$. You are left with the equation of a straight ____. Check your equation at $s = a$ and at $s = b$. Now it must be correct for all s, and PF2' is proved.

Multiplying gives us

$$Hs + K = \left(\frac{s-b}{a-b}\right)(Ha + K) + \left(\frac{s-a}{b-a}\right)(Hb + K)$$

Once again, we have the equation of a straight line. At $s = a$ the first term on the right is $Ha + K$, and at $s = b$ we have as our second term $Hb + K$. Two points define a line, and thus this must hold for all s.

> **Question: 2.7.12**
>
> Break these functions into two partial fractions using PF2 and PF2':
>
> (a) $\dfrac{1}{s^2 - 4}$
>
> (b) $\dfrac{s}{s^2 - 4}$
>
> (c) $\dfrac{Hs + K}{s^2 - 5s + 6}$

(a)
$$\frac{1}{s^2 - 4} = \frac{1}{(s-2)(s+2)} = \frac{1}{4(s-2)} - \frac{1}{4(s+2)}$$

(b)
$$\frac{s}{s^2 - 4} = \frac{s}{(s-2)(s+2)} = \frac{1}{2(s-2)} + \frac{1}{2(s+2)}$$

(c)
$$\frac{Hs + K}{s^2 - 5s + 6} = \frac{Hs + K}{(s-2)(s-3)} = \frac{3H + K}{s-3} - \frac{2H + K}{s-2}$$

Question: 2.7.13

Find the integrals of $(a)(b)(c)$ in Problem 12 by integrating each partial fraction. The integrals of $C/(s-a)$ and $D/(s-b)$ are logarithms.

(a)
$$\int \frac{ds}{s^2 - 4} = \frac{1}{4} \ln(s-2) - \frac{1}{4} \ln(s+2) = \frac{1}{4} \ln\left(\frac{s-2}{s+2}\right)$$

(b)
$$\int \frac{s}{s^2 - 4} \, ds = \frac{1}{2} \ln(s-2) + \frac{1}{2} \ln(s+2) = \frac{1}{2} \ln\left(s^2 - 4\right)$$

(c)
$$\int \left(\frac{Hs + K}{s^2 - 5s + 6}\right) ds = (3H + K) \ln(s-3) - (2H + K) \ln(s-2)$$

Question: 2.7.14

Extend PF3 to PF3' in the same way that PF2 extended PF2':

$$\textbf{PF3'} \qquad \frac{Gs^2 + Hs + K}{(s-a)(s-b)(s-c)} = \frac{Ga^2 + Ha + K}{(s-a)(a-b)(a-c)} + \frac{?}{?} + \frac{?}{?}$$

$$\frac{Gs^2 + Hs + K}{(s-a)(s-b)(s-c)} = \frac{Ga^2 + Ha + K}{(s-a)(a-b)(a-c)} + \frac{Gb^2 + Hb + K}{(s-b)(b-a)(b-c)}$$
$$+ \frac{Gc^2 + Hc + K}{(s-c)(c-a)(c-b)}$$

Question: 2.7.15

The linear polynomial $(s-b)/(a-b)$ equals 1 at $s=a$ and 0 at $s=b$. Write down a quadratic polynomial that equals 1 at $s=a$ and 0 at $s=b$ and $s=c$.

Immediately it is apparent that we have roots $s=b,c$, so we have $p(s) = q(s)(s-b)(s-c)$. Since $p(s)$ is a quadratic, we can conclude that $q(s)$ is a constant, with

$$q(s) = \frac{1}{(a-b)(a-c)}$$

Question: 2.7.16

What is the number C so that $C(s - b)(s - c)(s - d)$ equals 1 at $s = a$?

Note A complete theory of partial fractions must allow double roots (when $b = a$). The formula can be discovered from l'Hôpital's Rule (in PF3 for example) when b approaches a. Multiple roots lose the beauty of PF3 and PF3' − we are happy to stay with simple roots a, b, c.

Simply choose

$$C = \frac{1}{(a - b)(a - c)(a - d)}$$

Questions 17 - 21 involve the transform $F(s) = 1$ of the delta function $f(t) = \delta(t)$.

Question: 2.7.17

Find $F(s)$ from its definition $\int_0^\infty f(t)\, e^{-st}\, dt$ when $f(t) = \delta(t - T)$, $T \geq 0$.

We have

$$\int_0^\infty \delta(t - T)\, e^{-st}\, dt = e^{-sT}$$