Differential Equations and Linear Algebra

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All errors, typographical and substantive, and other offenses, are entirely my own.

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2 - Second Order Equations

2.1 - Second Derivatives in Science and Engineering

Question: 2.1.1

Find a cosine and a sine that solve $d^2y/dt^2 = -9y$. This is a second order equation so we expect *two constants* C and D (from integrating twice):

Simple harmonic motion $y(t) = C\cos(\omega t) + D\sin(\omega t)$

What is ω? If the system starts from rest (this means $dy/dt = 0$ at $t = 0$), which constant C or D will be zero?

Differentiating $y(t)$ twice, we get a ω^2 term. Making the necessary substitutions, we can see that $\omega^2 = 9$, implying $\omega = 3$. Thus we have $y = \sin(3t)$ and $y = \cos(3t)$. The constants C and D are determined by the initial conditions. Assuming the system starts at rest, we must have

$$\frac{dy}{dt} = -3C\sin(3t) + 3D\cos(3t) = 0$$

which implies that $dy/dt_{t=0} = 3D = 0$, thus $D = 0$.

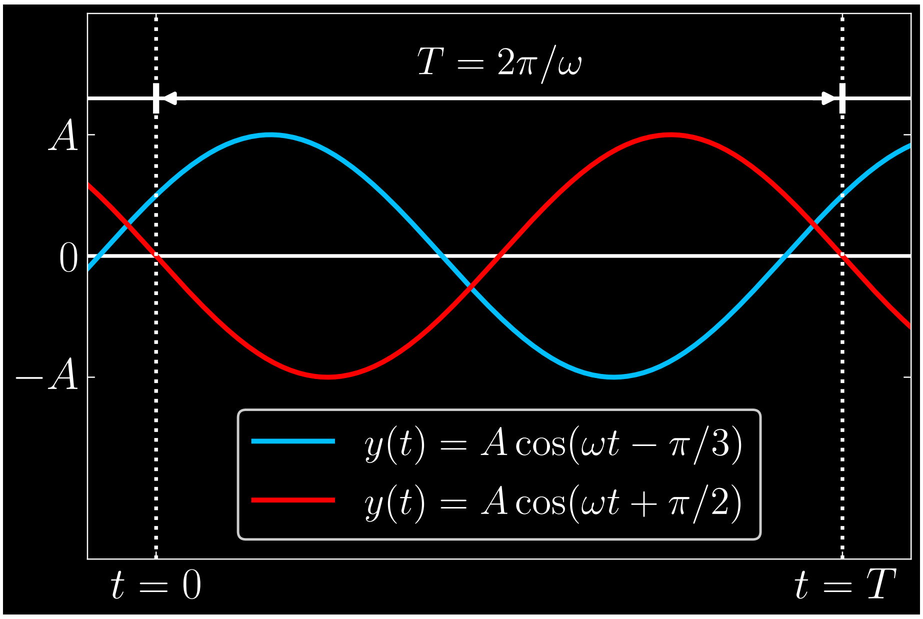
Question: 2.1.2

In Problem 1, which C and D will give the starting values $y(0) = 0$ and $y'(0) = 1$?

We have $y(0) = C = 0$ and $y'(0) = 3D = 1$, or $D = 1/3$.

Question: 2.1.3

Draw Figure 2.3 to show simple harmonic motion $y = A\cos(\omega t - \alpha)$ with phases $\alpha = \pi/3$ and $\alpha = -\pi/2$.



Question: 2.1.4

Suppose the circle in Figure 2.4 has radius 3 and circular frequency $f = 60$ Hertz. If the moving point starts at the angle $-45°$, find its x-coordinate $A\cos(\omega t - \alpha)$. The phase lag is $\alpha = 45°$. When does the point first hit the x-axis?

The circular motion of the point is expressed by the sinusoidal

$$3\cos\left(120\pi t - \frac{\pi}{4}\right)$$

Note that since the frequency is $f = 60$ Hertz, the angular frequency is $2\pi \cdot 60 = 120\pi$ radians s^{-1}. The point hits the x-axis when the argument of the cosine is zero, namely at

$$120\pi t - \frac{\pi}{4} = 0 \qquad \Longrightarrow \qquad t = \frac{1}{480}\mathrm{s}$$

Question: 2.1.5

If you drive at 60 miles per hour on a circular track with radius $R = 3$ miles, what is the time T for one complete circuit? Your circular frequency is $f =$ ____ and your angular frequency is $\omega =$ ____ (with what units?). The period is T.

Using dimensional analysis, the period is

$$T = \frac{1 \text{ hr}}{60 \text{ mi}}(3 \text{ mi}) = \frac{1}{20} \text{ hr}$$

The circular frequency is

$$f = \frac{60 \text{ mi}}{\text{hr}} \frac{1 \text{ hr}}{3600 \text{ s}} \frac{1 \text{ cycle}}{3 \text{ mi}} = \frac{1}{180} \text{ s}^{-1}$$

with angular frequency

$$\omega = 2\pi f = \frac{\pi}{90} \text{ rad s}^{-1}$$

Question: 2.1.6

The total energy E in the oscillating spring-mass system is

$$\begin{aligned} E &= \textbf{kinetic} \text{ energy in mass} + \textbf{potential} \text{ energy in spring} \\ &= \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2 \end{aligned}$$

Compute E when $y = C\cos(\omega t) + D\sin(\omega t)$. The energy is constant!

Given $y(t)$, we have first time-derivative

$$\frac{dy}{dt} = -\omega C\sin(\omega t) + \omega D\cos(\omega t)$$

squaring gives us

$$\left(\frac{dy}{dt}\right)^2 = \omega^2\left(C^2\sin^2(\omega t) - 2CD\sin(\omega t)\cos(\omega t) + D^2\cos^2(\omega t)\right)$$

Lastly, the y^2 term is

$$y^2 = C^2\cos^2(\omega t) + 2CD\sin(\omega t)\cos(\omega t) + D^2\sin^2(\omega t)$$

Combining our ingredients, with $\omega = \sqrt{k/m}$, observe that energy E reduces to a constant:

$$\begin{aligned} E &= \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + \frac{k}{2}y^2 \\ &= \frac{m}{2}\left(\frac{k}{m}\right)\left(C^2\sin^2(\omega t) - 2CD\sin(\omega t)\cos(\omega t) + D^2\cos^2(\omega t)\right) \\ &\quad + \frac{k}{2}\left(C^2\cos^2(\omega t) + 2CD\sin(\omega t)\cos(\omega t) + D^2\sin^2(\omega t)\right) \\ &= C^2 + D^2 \end{aligned}$$

Question: 2.1.7

Another way to show that the total energy E is constant:
Multiply $\boldsymbol{my'' + ky = 0}$ by $\boldsymbol{y'}$. Then integrate $my'y''$ and kyy'.

Take the first term and integrate:

$$m \int y' y''\, dt$$

By integration by parts, let

$$u = \frac{dy}{dt}, \quad \frac{du}{dt} = \frac{d^2y}{dt^2} \quad \implies \quad du = \frac{d^2y}{dt^2} dt$$

Then we can rewrite the first term as

$$m \int u\, du = \frac{m}{2} u^2 + C = \frac{m}{2} (y')^2 + C$$

as for the second term:

$$k \int yy'\, dt = k \int y \frac{dy}{dt}\, dt = k \int y\, dy = \frac{k}{2} y^2 + C$$

Summing the two pieces (sans the constants) restores the original energy function:

$$E = \frac{m}{2} \left(\frac{dy}{dt} \right)^2 + \frac{k}{2} y^2$$

and since the derivative of a constant is zero, it must be the case that E is a constant.

Question: 2.1.8

A **forced oscillation** has another term in the equation and $A \cos(\omega t)$ in the solution:

$$\frac{d^2y}{dt^2} + 4y = F \cos(\omega t) \quad \text{has} \quad y = C \cos(2t) + D \sin(2t) + A \cos(\omega t)$$

(a) Substitute y into the equation to see how C and D disappear (they give y_n. Find the forced amplitude A in the particular solution $y_p = A \cos(\omega t)$.

(b) In case $\omega = 2$ (forcing frequency = natural frequency), what answer does your formula give for A? The solution formula for y breaks down in this case.

The second time-derivative is

$$\frac{d^2y}{dt^2} = -4C \cos(2t) - 4D \sin(2t) - \omega^2 A \cos(\omega t)$$

(a) We have

$$\frac{d^2y}{dt^2} + 4y = \left(4 - \omega^2\right) A\cos(\omega t) = F\cos(\omega t)$$

implying $A = \dfrac{F}{4-\omega^2}$.

(b) When $\omega = 2$, A is undefined.

Question: 2.1.9

Following Problem 8, write down the complete solution $y_n + y_p$ to the equation

$$m\frac{d^2y}{dt^2} + ky = F\cos(\omega t) \quad \text{with} \quad \omega \neq \omega_n = \sqrt{k/m} \quad \text{(no resonance)}$$

The answer y has free constants C and D to match $y(0)$ and $y'(0)$ (A *is fixed* by F.

Per Problem 8, we have solution

$$y = \underbrace{C\cos\left(\sqrt{\frac{k}{m}}t\right) + D\sin\left(\sqrt{\frac{k}{m}}t\right)}_{y_n} + \underbrace{\frac{F}{k - m\omega^2}\cos(\omega t)}_{y_p}$$

All this involves is dividing the equation through by m, understanding that the angular frequency of the sinusoidals in the null solution is the square root of k/m, and making the changes to our formula for A accordingly.

Question: 2.1.10

Suppose Newton's Law $F = ma$ has the force F in the *same* direction as a:

$$my'' = +ky \quad \text{including} \quad y'' = 4y$$

Find two possible choices of s in the exponential solutions $y = e^{st}$. The solution is not sinusoidal and s is real and the oscillations are gone. Now y is unstable.

Substituting in y, we find

$$ms^2e^{st} = ke^{st}$$

This forces

$$s = \pm\sqrt{\frac{k}{m}}$$

Question: 2.1.11

Here is a *fourth* order equation: $d^4y/dt^4 = 16y$. Find *four* values of s that give exponential solutions $y = e^{st}$. You could expect four initial conditions on y: $y(0)$ is given along with what three other conditions?

Equivalently, we find the four complex roots of 16:

$$s^4 = 16$$

which are $s = \pm 2, \pm 2i$.

Question: 2.1.12

To find a particular solution to $y'' + 9y = e^{ct}$, I would look for a multiple $y_p(t) = Ye^{ct}$ of the forcing function. What is that number Y? When does your formula give $Y = \infty$? (Resonance needs a new formula for Y.)

Let $y_p = Ye^{ct}$. Substituting, we find

$$Yc^2e^{ct} + 9Ye^{ct} = e^{ct}$$

Solving for Y yields $Y = \dfrac{1}{c^2 + 9}$. When $c \to \pm 3$, we have resonance, and $Y \to \infty$.

Question: 2.1.13

In a particular solution $y = Ae^{i\omega t}$ to $y'' + 9y = e^{i\omega t}$, what is the amplitude A? The formula blows up when the forcing frequency $\omega =$ what natural frequency?

Substituting, we derive

$$-\omega^2 Ae^{i\omega t} + 9Ae^{i\omega t} = e^{i\omega t}$$

which gives us $A = \dfrac{1}{9 - \omega^2}$. Resonance is when $\omega = 3$.

Question: 2.1.14

Equation (10) says that the tangent of the phase angle is $\tan(\alpha) = y'(0)/\omega y(0)$. First, check that $\tan(\alpha)$ is dimensionless when y is in meters and time is in seconds. Next, if that ratio is $\tan(\alpha) = 1$, should you choose $\alpha = \pi/4$ or $\alpha = 5\pi/4$? Answer:

$$\text{Separately you want } R\cos(\alpha) = y(0) \text{ and } R\sin(\alpha) = y'(0)/\omega$$

If those right hand sides are positive, choose the angle α between 0 and $\pi/2$.
If those right hand sides are negative, add π and choose $\alpha = 5\pi/4$.

Question: If $y(0) > 0$ and $y'(0) < 0$, does α fall between $\pi/2$ and π or between $3\pi/2$ and 2π? If you plot the vector from $(0,0)$ to $(y(0), y'(0)/\omega)$, its angle is α.

As $y(0) > 0$ and $y'(0) < 0$ requires positive cosine and negative sine, α falls between $3\pi/2$ and 2π.

Question: 2.1.15

Find a point on the sine curve in Figure 2.1 where $y > 0$ but $v = y' < 0$ and also $a = y'' < 0$. The curve is sloping down and bending down.

Find a point where $y < 0$ but $y' > 0$ and $y'' > 0$. The point is below the x-axis but the curve is sloping ____ and bending ____.

One area corresponding to the first set of conditions is $\pi/2 < t < \pi$. As for the second, we have $3\pi/2 < t < 2\pi$.

Question: 2.1.16

(a) Solve $y'' + 100y = 0$ starting from $y(0) = 1$ and $y'(0) = 10$. (**This is $\boldsymbol{y_n}$.**)

(b) Solve $y'' + 100y = \cos(\omega t)$ with $y(0) = 0$ and $y'(0) = 0$. (**This can be $\boldsymbol{y_p}$.**)

(a) Let $y = c_1\cos(10t) + c_2\sin(10t)$. Then $y(0) = c_1 = 1$ and $y'(0) = 10c_2 = 10$ implies $c_2 = 1$. Then the null solution is

$$y_n = \cos(10t) + \sin(10t)$$

(b) Let $y_p = R\cos(\omega t)$. Substitute to find

$$-\omega^2 R\cos(\omega t) + 100R\cos(\omega t) = \cos(\omega t)$$

Isolate R to derive

$$R = \frac{1}{100 - \omega^2}$$

The solution to this set of initial conditions requires $y(0) = 0$ and $y'(0) = 0$. Begin with

$$y(t) = c_1 \cos(10t) + c_2 \sin(10t) + \frac{1}{100 - \omega^2} \cos(\omega t)$$

From the first condition, we have $c_1 = -\frac{1}{100-\omega^2}$. From the second, we have $c_2 = 0$. The full solution is

$$y(t) = \frac{1}{100 - \omega^2} (\cos(\omega t) - \cos(10t))$$

Question: 2.1.17

Find a particular solution $y_p = R\cos(\omega t - \alpha)$ to $y'' + 100y = \cos(\omega t) - \sin(\omega t)$.

Substituting y_p gives us

$$\begin{aligned} &-\omega^2 R\cos(\omega t - \alpha) + 100R\cos(\omega t - \alpha) \\ &\quad = \left(100R - \omega^2 R\right)[\cos(\omega t)\cos(\alpha) + \sin(\omega t)\sin(\alpha)] \end{aligned}$$

This implies that

$$\begin{aligned} R\cos(\alpha)\left(100 - \omega^2\right) &= 1 \\ R\sin(\alpha)\left(100 - \omega^2\right) &= -1 \end{aligned}$$

enabling us to conclude that $\alpha = 7\pi/4$, and amplitude

$$R = \frac{\sqrt{2}}{100 - \omega^2}$$

Ergo, the particular solution is

$$y_p = \frac{\sqrt{2}}{100 - \omega^2} \cos\left(\omega t - \frac{7\pi}{4}\right)$$

Question: 2.1.18

Simple harmonic motion also comes from a linear pendulum (like a grandfather clock). At time t, the height is $A\cos(\omega t)$. What is the frequency ω if the pendulum comes back to the start after 1 second? The period does not depend on the amplitude (a large clock or a small metronome or the movement in a watch can all have $T = 1$).

The angular frequency is $2\pi \cdot f = 2\pi$ radians s^{-1}.

Question: 2.1.19

If the phase lag is α, what is the time lag in graphing $\cos(\omega t - \alpha)$?

Put differently, we want to find the value of t' such that we are able to restore ωt as the cosine's argument. If we have

$$t' = t + \alpha/\omega$$

then we get

$$\cos(\omega t' - \alpha) = \cos(\omega(t + \alpha/\omega) - \alpha) = \cos(\omega t)$$

Thus the time lag term is α/ω.

Question: 2.1.20

What is the response $y(t)$ to a delayed impulse if $my'' + ky = \delta(t - T)$?

The full solution will be a factor of the step function, given by:

$$y(t) = \int_0^{t-T} \frac{\sin(\omega_n(t - T - s))}{m\omega_n} \delta(s)\ ds = \frac{\sin(\omega_n(t - T))}{m\omega_n} H(t - T)$$

Intuitively, when $t \leq T$, the right-hand side vanishes. No impulse is imparted, and thus there is no response. But when we are at time $t \geq T$ – after the threshold – the response kicks in.

Question: 2.1.21

(Good challenge) Show that $y = \displaystyle\int_0^t g(t - s)\, f(s)\ ds$ has $my'' + ky = f(t)$.

1. Why is $y' = \displaystyle\int_0^t g'(t - s)\, f(s)\ ds + g(0)\, f(t)$? Notice the two t's in y.
2. Using $g(0) = 0$, explain why $y'' = \displaystyle\int_0^t g''(t - s)\, f(s)\ ds + g'(0)\, f(t)$.
3. Now use $g'(0) = 1/m$ and $mg'' + kg = 0$ to confirm $my'' + ky = f(t)$.

Use the Leibniz integral rule:

$$\frac{d}{dt}\left(\int_0^t g(t - s)\, f(s)\ ds\right) = g(0)\, f(t) + \int_0^t \frac{\partial}{\partial t} g(t - s)\, f(s)\ ds$$

(1) By above, applying the partial derivative inside the second term yields the desired result.

(2) One more application of the Leibniz rule gives us

$$y'' = \int_0^t g''(t - s)\, f(s)\ ds + g(0)\, f'(t) + g'(0)\, f(t)$$

With the premise $g(0) = 0$, the second term vanishes and gives us the expected result.

(3) Derive

$$my'' + ky = \int_0^t [mg''(t-s) + kg(t-s)]\, f(s)\; ds + f(t) = f(t)$$

where the last equality follows by appealing to the nullity of $g(t)$.

Question: 2.1.22

With $f = 1$ (direct current has $\omega = 0$) verify that $my'' + ky = 1$ for this y:

Step response

$$y(t) = \int_0^t \frac{\sin(\omega_n(t-s))}{m\omega_n} \cdot 1\, ds = y_p + y_n = \boldsymbol{\frac{1}{k} - \frac{1}{k}\cos(\omega_n t)}$$

We have second derivative

$$y''(t) = \frac{\omega_n^2}{k}\cos(\omega_n t)$$

Since $\omega_n = \sqrt{k/m}$, $my''(t) = \cos(\omega_n t)$. Ergo, we have

$$my'' + ky = \cos(\omega_n t) + k\left[\frac{1}{k} - \frac{1}{k}\cos(\omega_n t)\right] = 1$$

Question: 2.1.23

(Recommended) For the equation $d^2y/dt^2 = 0$ find the null solution. Then for $d^2g/dt^2 = \delta(t)$ find the fundamental solution (start the null solution with $g(0) = 0$ and $g'(0) = 1$). For $y'' = f(t)$ find the particular solution using formula (16).

Integrating twice, the null solution is

$$y(t) = C_1 t + C_2$$

To find the fundamental solution $g(t)$, imposing the initial conditions $g(0) = 0$ and $g'(0) = 1$ forces $C_1 = 1$ and $C_2 = 0$, thus

$$g(t) = t$$

Lastly, if $y'' = f(t)$, then

$$y_p(t) = \int_0^t (t-s)\, f(s)\; ds$$

Question: 2.1.24

For the equation $d^2y/dt^2 = e^{i\omega t}$ find a particular solution $y = Y(\omega)\, e^{i\omega t}$. Then $Y(\omega)$ is the frequency response. Note the "resonance" when $\omega = 0$ with the null solution $y_n = 1$.

One particular solution is

$$y(t) = -\frac{e^{i\omega t}}{\omega^2}$$

meaning that $Y(\omega) = -1/\omega^2$. When $\omega = 0$, we have resonance, and this particular solution breaks down.

Question: 2.1.25

Find a particular solution $Ye^{i\omega t}$ to $my'' - ky = e^{i\omega t}$. The equation has $-ky$ instead of ky. What is the frequency response $Y(\omega)$? For which ω is Y infinite?

We have

$$\frac{d^2}{dt^2}\left(Ye^{i\omega t}\right) = -Y\omega^2 e^{i\omega t}$$

Substituting and canceling the $e^{i\omega t}$ terms, we get

$$-mY\omega^2 - kY = 1$$

Implying

$$Y(\omega) = -\frac{1}{m\omega^2 + k}$$

If we have

$$\omega = i\sqrt{\frac{k}{m}}$$

then $Y(\omega)$ diverges to infinity – meaning all real frequencies will not lead to this!

2.2 - Key Facts About Complex Numbers

Question: 2.2.1

Mark the numbers $s_1 = 2 + i$ and $s_2 = 1 - 2i$ as points in the complex plane. (The plane has a real axis and an imaginary axis.) Then mark the sum $s_1 + s_2$ and the difference $s_1 - s_2$.