Introductory Probability and Statistical Applications, Second Edition
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Solutions to Chapter 5: Functions of Random Variables

5.1 **Suppose that X is uniformly distributed over $(-1, 1)$. Let $Y = 4 - X^2$. Find the pdf of Y, say $g(y)$, and sketch it. Also verify that $g(y)$ is a pdf.**

By uniform distribution, the pdf of X is $f(x) = 1/2, -1 < x < 1$. The task is to now find a corresponding pdf for Y. Given $Y = H(X) = 4 - x^2$, we can derive the cdf of Y, namely

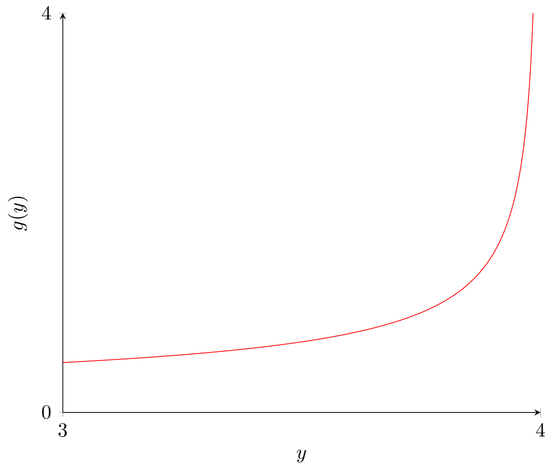
$$
\begin{aligned}
G(y) = P(Y \leq y) &= P(4 - X^2 \leq y) \\
&= P(X \leq -\sqrt{4-y}, \sqrt{4-y} \leq x) \\
&= 1 - P(-\sqrt{4-y} \leq x \leq \sqrt{4-y}) \\
&= 1 - \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \frac{1}{2} dx = \left(1 - \frac{x}{2}\right)_{-\sqrt{4-y}}^{\sqrt{4-y}} \\
&= -\frac{\sqrt{4-y}}{2} - \frac{\sqrt{4-y}}{2} = -\sqrt{4-y}
\end{aligned}
$$

We derive the pdf of Y by differentiating the cdf $G(y)$:

$$
G'(y) = g(y) = \frac{1}{2}(4-y)^{-1/2} = \boxed{\frac{1}{2\sqrt{4-y}}}
$$

Which is distributed over $3 < y < 4$, since X is distributed over $-1 < x < 1$, which maps to $3 < y < 4$ under $H(x)$. To verify $g(y)$ is indeed a pdf, it is clear that $g(y) \geq 0$ for the given domain. All that remains to ascertain is whether $\int_3^4 \frac{1}{2\sqrt{4-y}} dy = 1$. Integrate by u-substitution: let $u = 4 - y$, $du = -1 \cdot dy$. Then

$$
-\frac{1}{2} \int_{u=1(y=3)}^{u=0(y=4)} \frac{1}{\sqrt{u}} du = -\sqrt{u}\Big|_1^0 = \boxed{1}
$$



5.2 **Suppose that X is uniformly distributed over $(1, 3)$. Obtain the pdf of the following random variables:**

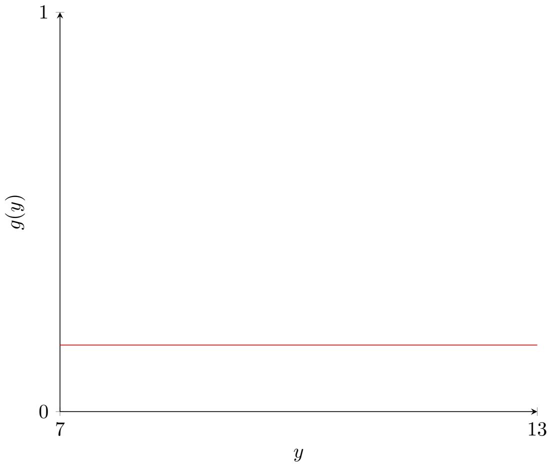
By uniform distribution, the pdf $f(x) = 1/2, 1 < x < 3$.

(a) **$Y = 3X + 4$**

Finding the cdf of Y, we get

$$\begin{aligned} G(y) = P(Y \leq y) &= P(3X + 4 \leq y) \\ &= P\Big(X \leq \frac{y-4}{3}\Big) \\ &= \int_1^{\frac{y-4}{3}} \frac{1}{2} dx = \frac{x}{2}\Big|_1^{\frac{y-4}{3}} \\ &= \frac{y-4}{6} - \frac{1}{2} = \frac{y-7}{6} \end{aligned}$$

Differentiating with respect to y, we get $G'(y) = g(y) = \boxed{1/6}$. Clearly $g(y)$ is positive. For $Y = 3X + 4$ such that $1 < x < 3$, we can deduce $7 < y < 13$. Then $\int_7^{13} 1/6 \, dy = \boxed{1}$.

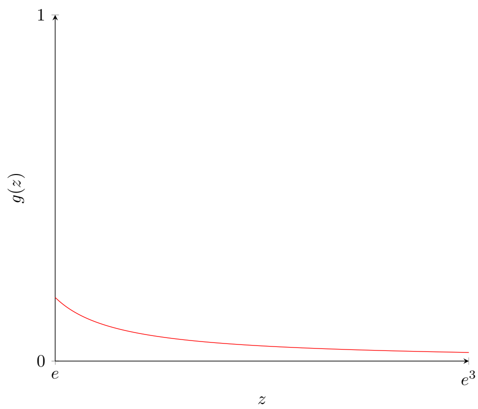


(b) **$Z = e^X$**

Finding the cdf of Z, we get

$$\begin{aligned} G(z) = P(Z \leq z) &= P(e^X \leq z) \\ &= P(X \leq \ln z) \\ &= \int_1^{\ln z} \frac{1}{2} \, dx = \frac{x}{2}\Big|_1^{\ln z} \\ &= \frac{\ln z}{2} - \frac{1}{2} \end{aligned}$$

Observe that since the natural logarithm is a strictly increasing function, the inequality direction is preserved. Therefore, $G'(z) = g(z) = \boxed{1/2z}$. For $Z = e^X, 1 < x < 3$, we get the domain $e < z < e^3$ for $Z = H(X)$. Therefore, $g(z) \geq 0$ across this domain. Moreover, $\frac{1}{2} \int_e^{e^3} \frac{1}{z} \, dz = \frac{1}{2} \ln z|_e^{e^3} = \boxed{1}$.



5.3 **Suppose that the continuous random variable X has pdf $f(x) = e^{-x}, x > 0$. Find the pdf of the following random variables:**

(a) $\boldsymbol{Y = X^3}$

The cdf of Y is derived as

$$\begin{aligned} G(y) = P(Y \le y) &= P(X^3 \le y) \\ &= P(X \le y^{1/3}) \\ &= \int_0^{y^{1/3}} e^{-x} dx = -e^{-x}\Big|_0^{y^{1/3}} = 1 - e^{-y^{1/3}} \end{aligned}$$

The pdf of Y then follows:

$$G'(y) = g(y) = \boxed{\frac{1}{3} y^{-2/3} e^{-y^{1/3}}}$$

For $Y = X^3, x > 0$, it follows that $y > 0$. Therefore, $g(y) \ge 0$ for $y > 0$. Numerically evaluating $\frac{1}{3}\int_0^{+\infty} y^{-2/3} e^{-y^{1/3}} dy$ gives us $\boxed{1}$.

(b) $\boldsymbol{Z = 3/(X+1)^2}$

Here, since $Z = \frac{3}{(X+1)^2}$ is monotonic over the interval $0 < X < +\infty$, we may apply Theorem 5.1 and find $g(z) = f(x)\left|\frac{dx}{dz}\right|$. The inverse function $X(z)$ has two branches, $X(z) = \sqrt{3/z} - 1$ and $X(z) = -\sqrt{3/z} - 1$. However, because X is distributed over the positive reals, we choose the branch $X(z) = \sqrt{3/z} - 1$, defined on $0 < z < 3$. Then $\frac{dx}{dz} = \frac{1}{2}\left(\frac{3}{z}\right)^{-1/2}\left(-\frac{3}{z^2}\right)$, therefore $\left|\frac{dx}{dz}\right| = \frac{1}{2}\frac{3^{1/2}}{z^{3/2}}$. By Theorem 5.1, $g(z) = \boxed{\frac{3^{1/2}}{2} e^{-(\sqrt{3/z}-1)} \frac{1}{z^{3/2}}}$. For $0 < z < 3, g(z) \ge 0$. Moreover, numerical evaluation of $\int_0^3 \frac{3^{1/2}}{2} e^{-(\sqrt{3/z}-1)} \frac{1}{z^{3/2}} dz$ gives us $\boxed{1}$.

5.4 **Suppose that the discrete random variable X assumes the values 1, 2, and 3 with equal probability. Find the probability distribution of $Y = 2X + 3$.**

Each of the outcomes $X = 1, 2, 3$ has probability $P(X = 1) = P(X = 2) = P(X = 3) = 1/3$. Therefore, $H(X = 1) = 5, H(X = 2) = 7, H(X = 3) = 9$, and $P(Y = 5) = P(Y = 7) = P(Y = 9) = 1/3$.

5.5 **Suppose that X is uniformly distributed over the interval $(0, 1)$. Find the pdf of the following random variables:**

By uniform distribution, $f(x) = 1$ for $0 < X < 1$.

(a) $\boldsymbol{Y = X^2 + 1}$

First we find the cdf of Y:

$$\begin{aligned} G(y) = P(Y \le y) &= P(X^2 + 1 \le y) \\ &= P(X^2 \le y - 1) \\ &= P(0 \le X \le \sqrt{y-1}) \\ &= \int_0^{\sqrt{y-1}} dx = \sqrt{y-1} \end{aligned}$$

Differentiating with respect to y gives us the pdf:

$$G'(y) = g(y) = \boxed{\frac{1}{2}(y-1)^{-1/2}}$$

First, note that $Y = X^2+1, 0 < X < 1$ implies $1 < Y < 2$. Then $g(y) \leq 0$ for $1 < Y < 2$. Secondly, $\int_1^2 \frac{1}{2}(y-1)^{-1/2}dy = (y-1)^{1/2}\big|_1^2 = \boxed{1}$.
We may alternatively apply Theorem 5.1 as $Y = X^2+1$ is monotonic on the interval $0 < X < 1$. Then $X(y) = \sqrt{y-1}$ and $\frac{dx}{dy} = \left|\frac{dx}{dy}\right| = \frac{1}{2}(y-1)^{-1/2}$. Therefore $g(y) = \frac{1}{2}(y-1)^{-1/2}$, as expected.

(b) **$Z = 1/(X+1)$**

First we find the cdf of Z:

$$\begin{aligned} G(z) = P(Z \leq z) &= P\Big(\frac{1}{X+1} \leq z\Big) \\ &= P\Big(\frac{1}{z} - 1 \leq X \leq 1\Big) \\ &= \int_{1/z-1}^{1} dx = 1 - (1/z - 1) = 2 - 1/z \end{aligned}$$

Then we derive $g(z)$ as follows:

$$G'(z) = g(z) = \boxed{1/z^2}$$

Note that $Z = 1/(X+1)$ over $0 < X < 1$ implies $1/2 < Z < 1$. Then $g(z) \geq 0$ over $1/2 < Z < 1$. Moreover, $G(1) - G(1/2) = \boxed{1}$.
Alternatively, we can apply Theorem 5.1 because $Z = 1/(X+1)$ is monotonic over $1/2 < Z < 1$. Then $X(z) = 1/z - 1$ and $\frac{dx}{dz} = -1/z^2$, then $\left|\frac{dx}{dz}\right| = 1/z^2$. Therefore $g(z) = 1/z^2$, as expected.

5.6 **Suppose that X is uniformly distributed over the interval $(-1, 1)$. Find the pdf of the following random variables:**

By uniform distribution, $f(x) = \frac{1}{1-(-1)} = 1/2$ for $-1 < X < 1$.

(a) **$Y = \sin(\pi X/2)$**

We first derive the cdf of Y. Here, note that the inverse sine function is increasing over the given interval:

$$\begin{aligned} G(y) = P(Y \leq y) &= P(\sin(\pi X/2) \leq y) \\ &= P(X \leq (2/\pi)\sin^{-1}(y)) \\ &= \int_{-1}^{(2/\pi)\sin^{-1}(y)} \frac{1}{2}\, dx = \frac{1}{\pi}\sin^{-1}(y) + \frac{1}{2} \end{aligned}$$

And now we derive the pdf of Y by differentiating $G(y)$ with respect to y. But first we must determine the derivative of the inverse sine function.

Theorem.
$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

Proof. First observe that $\sin(\sin^{-1}(x)) = x$. Then $\frac{d}{dx}\sin(\sin^{-1}(x)) = \frac{d}{dx}x$, and it immediately follows that $\cos(\sin^{-1}(x))\frac{d}{dx}\sin^{-1}(x) = 1 \implies \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))}$. Now, using the fact that $\sin^2 y + \cos^2 y = 1$, and from that deriving $\cos y = \sqrt{1-\sin^2 y}$, we can write $\cos(\sin^{-1}(x)) = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2}$ by virtue of inverses. Therefore, $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$. $\square$

Proceeding, we derive:

$$G'(y) = g(y) = \frac{d}{dy}\Big(\frac{1}{\pi}\sin^{-1}(y) + \frac{1}{2}\Big) = \boxed{\frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}}$$

For $Y = \sin(\pi X/2)$ on $-1 < X < 1$, it follows that $-1 < Y < 1$. Then $g(y) \geq 0$ for $-1 < Y < 1$. Additionally, $\int_{-1}^{1} \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}dy = \boxed{1}$, confirming $g(y)$ is a pdf.

Alternatively, because $Y = \sin(\pi X/2), -1 < X < 1$ is monotonic, we may apply Theorem 5.1. Finding $X = (2/\pi)\sin^{-1}(y)$ as before, it follows that $\frac{dx}{dy} = |\frac{dx}{dy}| = \frac{2}{\pi\sqrt{1-y^2}}$, and $g(y) = \frac{1}{2}\cdot\frac{2}{\pi\sqrt{1-y^2}} = \frac{1}{\pi\sqrt{1-y^2}}$, as expected.

(b) $\boldsymbol{Y = \cos(\pi X/2)}$

First we derive the cdf of Z. Here, note that the inverse cosine function is strictly decreasing on the given interval:

$$\begin{aligned}
G(z) = P(Z \le z) &= P(\cos(\pi X/2) \le z) \\
&= P\Big(X \ge \frac{2}{\pi}\cos^{-1}(z), X \le -\frac{2}{\pi}\cos^{-1}(z)\Big) \\
&= \int_{-1}^{-\frac{2}{\pi}\cos^{-1}(z)} \frac{1}{2}\,dx + \int_{\frac{2}{\pi}\cos^{-1}(z)}^{1} \frac{1}{2}\,dx \\
&= -\frac{1}{\pi}\cos^{-1}(z) + \frac{1}{2} - \frac{1}{\pi}\cos^{-1}(z) + \frac{1}{2} \\
&= -\frac{2}{\pi}\cos^{-1}(z) + 1
\end{aligned}$$

Analogously, we must determine the derivative of the inverse cosine function before proceeding.

Theorem.
$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

Proof. Since $\cos(\cos^{-1}(x)) = x$, it follows that $\frac{d}{dx}\cos(\cos^{-1}(x)) = \frac{d}{dx}x$, implying $-\sin(\cos^{-1}(x))\frac{d}{dx}\cos^{-1}(x) = 1$. Then $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sin(\cos^{-1}(x))}$. Since $\sin^2 y + \cos^2 y = 1$ implies $\sin y = \sqrt{1-\cos^2 y}$, we have $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-\cos^2(\cos^{-1}(x))}} = -\frac{1}{\sqrt{1-x^2}}$. □

Differentiating $G(z)$ with respect to z yields the pdf of Z:

$$G'(z) = g(z) = \frac{d}{dz}\Big(-\frac{2}{\pi}\cos^{-1}(z) + 1\Big) = \boxed{\frac{2}{\pi}\frac{1}{\sqrt{1-z^2}}}$$

For $Z = \cos(\pi X/2)$ on $-1 < X < 1$, the distribution of Z is over $0 < Z < 1$. Then $g(z) \ge 0$ on that interval. Moreover, $\int_0^1 \frac{2}{\pi}\frac{1}{\sqrt{1-z^2}}\,dz = \boxed{1}$, ascertaining $g(z)$ is a pdf.

Because $Z = \cos(\pi X/2)$ is not monotonic over $-1 < X < 1$, we cannot apply Theorem 5.1 to derive the pdf of Z.

(c) $\boldsymbol{W = |X|}$

In particular, W is defined as:

$$W = \begin{cases} X, & 0 \le X < 1 \\ -X, & -1 < X < 0 \end{cases}$$

Beginning with the derivation of the cdf of W:

$$\begin{aligned}
G(w) = P(W \le w) &= P(|X| \le w) \\
&= P(X \le w, -w \le X) \\
&= \int_0^w \frac{1}{2}\,dx + \int_{-w}^0 \frac{1}{2}\,dx \\
&= \frac{w}{2} + \frac{w}{2} = w
\end{aligned}$$

Differentiating with respect to w yields $G'(w) = g(w) = \boxed{1}$. Since $W = |X|, -1 < X < 1$ implies $0 < W < 1$, clearly $g(w) \ge 0$ on that interval. Moreover, $\int_0^1 w\,dw = \boxed{1}$, confirming $g(w)$ is a pdf.

5.7 **Suppose that the radius of a sphere is a continuous random variable. (Due to inaccuracies of the manufacturing process, the radii of different spheres may be different.) Suppose that the radius R has pdf $f(r) = 6r(1-r), 0 < r < 1$. Find the pdf of the volume V and the surface area S of the sphere.**

Volume. The volume of a sphere is $V(r) = \frac{4}{3}\pi r^3$. First finding the cdf of the volume v:

$$
\begin{aligned}
G(v) = P(V \le v) &= P\Big(\frac{4}{3}\pi r^3 \le v\Big) \\
&= P\Big(r \le \Big(\frac{3}{4\pi}v\Big)^{1/3}\Big) \\
&= \int_0^{(\frac{3}{4\pi}v)^{1/3}} 6r(1-r)\ dr = 3r^2 - 2r^3\Big|_0^{(\frac{3}{4\pi}v)^{1/3}} \\
&= 3\Big(\frac{3}{4\pi}v\Big)^{2/3} - \frac{3}{2\pi}v
\end{aligned}
$$

Differentiating with respect to v gives us the pdf of V:

$$
\begin{aligned}
G'(v) = g(v) &= 2\Big(\frac{3}{4\pi}v\Big)^{-1/3}\Big(\frac{3}{4\pi}\Big) - \frac{3}{2\pi} \\
&= \boxed{\frac{3}{2\pi}\Big(\Big(\frac{3}{4\pi}v\Big)^{-1/3} - 1\Big)}
\end{aligned}
$$

For $V(r) = \frac{4}{3}\pi r^3$ on $0 < r < 1$, we have $0 < V < \frac{4}{3}\pi$. It follows that $g(v) \ge 0$ on this interval, and $\int_0^{\frac{4}{3}\pi} \frac{3}{2\pi}\Big(\Big(\frac{3}{4\pi}v\Big)^{-1/3} - 1\Big)\ dv = \boxed{1}$, ascertaining that $g(v)$ is a pdf.

Surface Area. The surface area of a sphere is given by $A(r) = 4\pi r^2$. First deriving the cdf of A:

$$
\begin{aligned}
G(a) = P(A \le a) &= P(4\pi r^2 \le a) \\
&= P\Big(r \le \Big(\frac{1}{4\pi}a\Big)^{1/2}\Big) \\
&= \int_0^{(\frac{1}{4\pi}a)^{1/2}} 6r(1-r)\ dr = 3r^2 - 2r^3\Big|_0^{(\frac{1}{4\pi}a)^{1/2}} \\
&= 3\Big(\frac{1}{4\pi}a\Big) - 2\Big(\frac{1}{4\pi}a\Big)^{3/2}
\end{aligned}
$$

And now differentiating with respect to a to find the pdf of A:

$$
\begin{aligned}
G'(a) = g(a) &= \frac{3}{4\pi} - 3\Big(\frac{1}{4\pi}a\Big)^{1/2}\Big(\frac{1}{4\pi}\Big) \\
&= \boxed{\frac{3}{4\pi}\Big(1 - \Big(\frac{1}{4\pi}a\Big)^{1/2}\Big)}
\end{aligned}
$$

For $A(r) = 4\pi r^2$ over $0 < r < 1$, we have $0 < A(r) < 4\pi$. Then $g(a) \ge 0$ over this interval, and $\int_0^{4\pi} \frac{3}{4\pi}\Big(1 - \Big(\frac{1}{4\pi}a\Big)^{1/2}\Big)\ da = \boxed{1}$, ascertaining that $g(a)$ is a pdf.

5.8 **A fluctuating electric current I may be considered as a uniformly distributed random variable over the interval $(9, 11)$. If this current flows through a 2-ohm resistor, find the pdf of the power $P = 2I^2$.**

By uniform distribution, $f(i) = \frac{1}{11-9} = \frac{1}{2}, 9 < I < 11$, where I is the random variable for current and i a specific outcome of current. Let P^* be the random variable for power and p^* be a specific outcome of power. Deriving the cdf of P^* gives us:

$$
\begin{aligned}
G(p^*) = P(P^* \le p^*) &= P(2I^2 \le p^*) \\
&= P\Big(I \le \Big(\frac{p^*}{2}\Big)^{1/2}\Big) \\
&= \int_9^{(\frac{p^*}{2})^{1/2}} \frac{1}{2}\ di = \frac{i}{2}\Big|_9^{(\frac{p^*}{2})^{1/2}} \\
&= \frac{1}{2}\Big(\Big(\frac{p^*}{2}\Big)^{1/2} - 9\Big)
\end{aligned}
$$

Deriving the pdf of P^*:

$$
\begin{aligned}
G'(p^*) = g(p^*) &= \frac{1}{2}\left(\frac{1}{2}\left(\frac{p^*}{2}^{-1/2}\left(\frac{1}{2}\right)\right)\right) \\
&= \frac{1}{8}\left(\frac{p^*}{2}\right)^{-1/2} = \boxed{\frac{1}{8}\left(\frac{2}{p^*}\right)^{1/2}}
\end{aligned}
$$

For $P^* = 2I^2, 9 < I < 11$, we have $162 < P^* < 242$. Then $g(p^*) \geq 0$ and $\int_{162}^{242} \frac{1}{8}\left(\frac{2}{p^*}\right)^{1/2} dx = \boxed{1}$, ascertaining that $g(p^*)$ is a pdf.

5.9 **The speed of a molecule in a uniform gas at equilibrium is a random variable V whose pdf is given by $f(v) = av^2e^{-bv^2}, v > 0$, where $b = m/2kT$ and k, T, and m denote Boltzmann's constant, the absolute temperature, and the mass of the molecule, respectively.**

(a) **Evaluate the constant a (in terms of b).**

We proceed by integration by parts. Let $u = v, du = dv, dw = ave^{-bv^2}dv$, and $w = -\frac{a}{2b}e^{-bv^2}$. Then

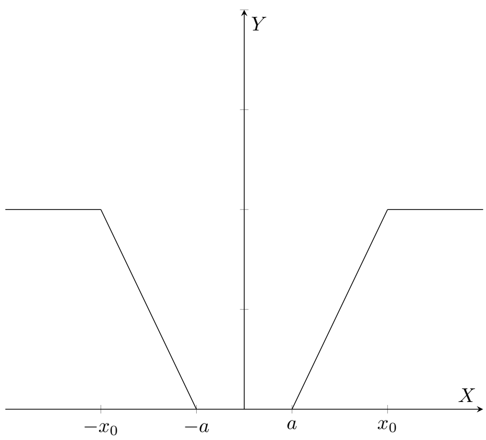
$$
\begin{aligned}
\int_0^{+\infty} av^2e^{-bv^2}\,dv &= -\frac{a}{2b}ve^{-bv^2}\Big|_0^{+\infty} + \frac{a}{2b}\int_0^{+\infty} e^{-bv^2}\,dv \\
&= \frac{a}{2b}\frac{1}{2}\sqrt{\frac{\pi}{b}} = 1 \\
&\implies \boxed{a = \frac{4b^{3/2}}{\sqrt{\pi}}}
\end{aligned}
$$

(b) **Derive the distribution of the random variable $W = mv^2/2$, which represents the kinetic energy of the molecule.**

Without needing to deal with error functions, because $W = mv^2/2$ is monotonic for $v > 0$, we may make use of Theorem 5.1. Then $V = \left(\frac{2w}{m}\right)^{1/2}$, and $\frac{dv}{dw} = \left|\frac{dv}{dw}\right| = \frac{1}{2}\left(\frac{2}{m}\right)\left(\frac{2w}{m}\right)^{-1/2}$. With $f(v) = \frac{4b^{3/2}}{\sqrt{\pi}}v^2e^{-bv^2}$, we can derive the pdf of the kinetic energy W:

$$
\begin{aligned}
g(w) &= \frac{4b^{3/2}}{\sqrt{\pi}}\left(\frac{2w}{m}\right)e^{-b\left(\frac{2w}{m}\right)}\left(\frac{1}{m}\right)\left(\frac{2w}{m}\right)^{-1/2} \\
&= \boxed{\frac{2}{(kT)^{3/2}\pi^{1/2}}w^{1/2}e^{-(w/kT)}, W > 0}
\end{aligned}
$$

5.10 **A random voltage X is uniformly distributed over the interval $(-k, k)$. If X is the input of a nonlinear device with the characteristics shown in Fig. 5.12, find the probability distribution of Y in the following three cases:**



By uniform distribution, $f(x) = \frac{1}{2k}, -k < X < k$.

(a) **$k < a$**

Since the event that $Y = 0$ is equivalent to the event that $X \in (-k, k)$, we may simply calculate $P(Y = 0) = \int_{-k}^{k} \frac{1}{2k}\, dx = \boxed{g(0) = 1}$. Therefore, $\boxed{g(y) = 0}, y \neq 0$.

(b) **$a < k < x_0$**

Here, we define Y piecemeal as follows:

$$Y = \begin{cases} -\dfrac{y_0}{x_0 - a}X - \dfrac{ay_0}{x_0 - a}, & -k < x < -a \\ 0, & -a \leq x \leq a \\ \dfrac{y_0}{x_0 - a}X - \dfrac{ay_0}{x_0 - a}, & a < x < k \end{cases}$$

Therefore, we must find the probability distribution function of Y such that

$$P(0 \leq Y \leq y) = \int_0^y g(y)\, dy + P(Y = 0) = 1$$

Or equivalently:

$$\int_{0(x=-a)}^{y(x=-k)} g(y)\, dy + \int_{x=-a}^{x=a} f(x)\, dx + \int_{0(x=a)}^{y(x=k)} g(y)\, dy = 1$$

First we find $G(y)$ over the X interval $(-k, -a)$:

$$\begin{aligned} G(y) = P(Y \leq y) &= P\Big(-\frac{y_0}{x_0 - a}X - \frac{ay_0}{x_0 - a} \leq y\Big) \\ &= P\Big(X \geq -\Big(\frac{x_0 - a}{y_0}\Big)\Big(y + \frac{ay_0}{x_0 - a}\Big)\Big) \\ &= \int_{-(\frac{x_0-a}{y_0})y - a}^{-a} \frac{1}{2k}\, dx = -\frac{a}{2k} + \frac{(x_0 - a)y}{2ky_0} + \frac{a}{2k} = \frac{(x_0 - a)y}{2ky_0} \end{aligned}$$

Then $G'(y) = g(y) = \boxed{\dfrac{x_0 - a}{2ky_0}, 0 < y < \dfrac{y_0(k - a)}{x_0 - a}}$.

Next we find $G(y)$ over the X interval $(-a, a)$. Because $Y = 0$ over this interval, we may interpret the events $Y = 0$ and $X \in (-a, a)$ to be equivalent. Then we may simply write

$$P(Y = 0) = \int_{-a}^{a} \frac{1}{2k}\, dx = \frac{a}{2k} + \frac{a}{2k} = \boxed{\frac{a}{k}}$$

Lastly, we drive $G(y)$ over the X interval (a, k):

$$\begin{aligned} G(y) = P(Y \leq y) &= P\Big(\frac{y_0}{x_0 - a}X - \frac{ay_0}{x_0 - a} \leq y\Big) \\ &= P\Big(X \leq \Big(\frac{x_0 - a}{y_0}\Big)y - a\Big) \\ &= \int_{a}^{(\frac{x_0-a}{y_0})y - a} \frac{1}{2k}\, dx = \frac{(x_0 - a)y}{2ky_0} - \frac{a}{k} \end{aligned}$$

Then $G'(y) = g(y) = \boxed{\dfrac{x_0 - a}{2ky_0}, 0 < y < \dfrac{y_0(k - a)}{x_0 - a}}$. Since both events $X \in (-k, -a)$ and $X \in (a, k)$ are equivalent to $Y \in (0, y)$, we need only sum the corresponding probability distribution functions of Y over those respective intervals. In particular, we have:

$$\begin{aligned} g(y) = \frac{d}{dy}G(y) &= \frac{d}{dy}\left[\int_{-(\frac{x_0-a}{y_0})y - a}^{-a} \frac{1}{2k}\, dx + \int_{a}^{(\frac{x_0-a}{y_0})y - a} \frac{1}{2k}\, dx\right] \\ &= \frac{x_0 - a}{ky_0}, 0 < y < \frac{y_0(k - a)}{x_0 - a} \end{aligned}$$

Namely, we can conclude that $\boxed{g(y) = \dfrac{x_0 - a}{ky_0}, 0 < y < \dfrac{y_0(k - a)}{x_0 - a}}$ and $\boxed{g(y) = \dfrac{a}{k}, y = 0}$.

(c) **$k > x_0$**

By part (b), we know the probability distribution functions over the X range spaces $(-a, a), (-x_0, -a)$, and (a, x_0); for the latter two pdfs, the interval over Y for which they are defined is $0 < y < y_0$. All that remains is to determine the probability distribution function of Y for when $y = y_0$. Simply, because Y is a constant value for which the equivalent events are $X \in (-k, -x_0)$ and $X \in (x_0, k)$, we may write

$$\begin{aligned} P(Y = y_0) &= \int_{x_0}^{k} \frac{1}{2k}\, dx + \int_{-k}^{-x_0} \frac{1}{2k}\, dx \\ &= \frac{k - x_0}{2k} + \frac{k - x_0}{2k} = \frac{k - x_0}{k} = 1 - x_0/k \end{aligned}$$

Therefore, the probability distribution function here is:

$$g(y) = \begin{cases} a/k, & y = 0 \\ \dfrac{x_0 - a}{k y_0}, & 0 < y < y_0 \\ 1 - \dfrac{x_0}{k}, & y = y_0 \end{cases}$$

5.11 **The radiant energy (in Btu/hr/ft^2) is given as the following function of temperature T (in degree Fahrenheit): $E = 0.173(T/100)^4$. Suppose that the temperature T is considered to be a continuous random variable with pdf**

$$f(t) = \begin{cases} 200t^{-2}, & 40 \leq t \leq 50 \\ 0, & \text{elsewhere} \end{cases}$$

Find the pdf of the radiant energy E.

Let E be the random variable for radiant energy and e a specific outcome of E. First we derive the cdf of E:

$$\begin{aligned} G(e) = P(E \leq e) &= P(0.173(t/100)^4 \leq e) \\ &= P\Big(t \leq 100\Big(\frac{e}{0.173}\Big)^{1/4}\Big) \\ &= 200 \int_{40}^{100(\frac{e}{0.173})^{1/4}} t^{-2}\, dt = -2\Big(\frac{e}{0.173}\Big)^{-3/4} + 5 \end{aligned}$$

Then the pdf of E is:

$$\begin{aligned} G'(e) = g(e) &= \frac{1}{2}\Big(\frac{e}{0.173}\Big)^{-3/4}\Big(\frac{1}{0.173}\Big) \\ &= 2.89\Big(\frac{e}{0.173}\Big)^{-5/4} = \boxed{0.322e^{-5/4}, 0.0044 \leq E \leq 0.0108} \end{aligned}$$

5.12 **To measure air velocities, a tube (known as Pitot static tube) is used which enables one to measure differential pressure. This differential pressure is given by $P = \frac{1}{2}dV^2$, where d is the density of the air and V is the wind speed (mph). If V is a random variable uniformly distributed over $(10, 20)$, find the pdf of P.**

By uniform distribution, $f(v) = \frac{1}{10}, 10 < X < 20$. We first determine the cdf of X:

$$\begin{aligned} G(x) = P(X \leq x) &= P\Big(\frac{1}{2}dV^2 \leq x\Big) \\ &= P\Big(V \leq \Big(\frac{2X}{d}\Big)^{1/2}\Big) \\ &= \int_{10}^{(\frac{2X}{d})^{1/2}} \frac{1}{10}\, dv = \frac{1}{10}\Big(\frac{2X}{d}\Big)^{1/2} - 1 \end{aligned}$$

The pdf of X is then:

$$G'(x) = g(x) = \frac{1}{20}\Big(\frac{2}{d}\Big)\Big(\frac{2X}{d}\Big)^{-1/2} = \boxed{\frac{1}{10d}\Big(\frac{2X}{d}\Big)^{-1/2}, 50d < X < 200d}$$

Which is clearly positive for $X \in (50d, 200d)$, and it also follows that $\int_{50d}^{200d} \frac{1}{10d}\left(\frac{2X}{d}\right)^{-1/2} dx = \boxed{1}$.

5.13 **Suppose that $P(X \leq 0.29) = 0.75$, where X is a continuous random variable with some distribution defined over $(0,1)$. If $Y = 1 - X$, determine k so that $P(Y \leq k) = 0.25$.**

By premise, $P(X \leq 0.29) = \int_0^{0.29} f(x)\ dx = 0.75$. Since it must be the case that $P(0 \leq X \leq 1) = \int_0^1 f(x)\ dx = 1$, it immediately follows that $\int_{0.29}^1 f(x)\ dx = 0.25$. Lastly, in finding the cdf of Y, we determine

$$\begin{aligned} G(y) = P(Y \leq y) &= P(1 - X \leq y) \\ &= P(X \geq 1 - y) \\ &= \int_{1-y}^{1} f(x)\ dx \end{aligned}$$

Now, suppose $y = k$, then $\int_{1-y}^1 f(x)\ dx = 0.25$. Therefore, $1 - y = 0.29$ and $\boxed{y = 0.71}$. Observe that determining the pdf of X, $f(x)$, was completely unnecessary.