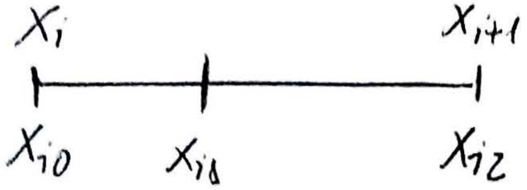
Métodos de Runge-Kutta (RK_{22})

$p=2$, $q=2$.



$x_{i0} = x_i$

$x_{i1} = x_i + \theta_1 h$.

$x_{i2} = x_{i+1}$

Consideramos el esquema:

•) $y_{i0} = y_i$

•) $y_{i1} = y_i + hA_{10} f(x_{i0}, y_{i0}) = y_i + hA_{10} f(x_i, y_i)$ —— ①

•) $y_{i2} = y_i + h\left[A_{20} f(x_{i0}, y_{i0}) + A_{21} f(x_{i1}, y_{i1})\right]$

$= y_i + h\left[A_{20} f(x_i, y_i) + A_{21} f(x_{i1}, y_{i1})\right]$ —— ②

Para determinar los coeficientes A_{10}, A_{20}, A_{21} se considera el desarrollo de Taylor al rededor de x_i de la solución del P.V.I. local $\tilde{y}(x)$

$$\begin{cases} \tilde{y}'(x) = f(x, \tilde{y}(x)), & x \in [x_i, x_{i+1}] \\ \tilde{y}(x_i) = y_i \end{cases}$$

y se compara hasta el término h^2 con ① y ②

Para A_{10}

$$\tilde{y}(\cancelto{x_{i1}}{x_i+\theta_1 h}) = \cancelto{y_i}{\tilde{y}(x_i)} + \theta_1 h\, \cancelto{f(x_i,\cancelto{y_i}{\tilde{y}(x_i)})}{\tilde{y}'(x_i)} + \frac{\theta_1^2 h^2}{2}\tilde{y}''(x_i) + \cdots$$

$$\Rightarrow \tilde{Y}(x_{i1}) = \underbrace{y_i + \theta_1 h\, f(x_i,y_i)}_{3} + \frac{\theta_1^2 h^2}{2}\tilde{y}''(x_i) + \cdots$$

Comparando (3) con (1) tenemos que $\underline{\underline{A_{10}=\theta_1}}$ y por lo tanto:

$$\boxed{y_{i1} = y_i + h\theta_1 f(x_i,y_i)} \quad \text{—— (4)}$$

Para A_{20} y A_{21}

$$\tilde{y}(\cancelto{x_{i+1}}{x_i+h}) = \cancelto{y_i}{\tilde{y}(x_i)} + h\, \cancelto{f(x_i,\cancelto{y_i}{\tilde{y}(x_i)})}{\tilde{y}'(x_i)} + \frac{h^2}{2}\tilde{y}''(x_i) + \frac{h^3}{6}\tilde{y}'''(x_i) + \cdots$$

$$\Rightarrow \tilde{y}(x_{i+1}) = y_i + h f(x_i,y_i) + \frac{h^2}{2}\tilde{y}''(x_i) + \frac{h^3}{6}\tilde{y}'''(x_i) + \cdots \quad \text{—— (5)}$$

Para la segunda derivada $\tilde{y}''$ consideramos la regla de la cadena

$$f \to x, \qquad f \to y \to x.$$

$$\tilde{y}''(x) = [\tilde{y}'(x)]' = [f(x,\tilde{y}(x))]' = \frac{\partial f}{\partial x}(x,\tilde{y}(x)) + \frac{\partial f}{\partial \tilde{y}}(x,\tilde{y}(x))\,\tilde{y}'(x)$$

$$= \frac{\partial f}{\partial x}(x,\tilde{y}(x)) + \frac{\partial f}{\partial \tilde{y}}(x,\tilde{y}(x)) \cdot f(x,\tilde{y}(x))$$

Reemplazando en (5) tenemos:

$$\tilde{y}(x_{i+1}) = y_i + hf(x_i,y_i) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x}(x_i,y_i) + \frac{\partial f}{\partial \tilde{y}}(x_i,y_i)\cdot f(x_i,y_i)\right] + \frac{h^3}{6}y'''(x_i) + \ldots \quad (6)$$

Recordamos el teorema de Taylor para 2 variables

$$f(a+h, b+k) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial \tilde{y}}\right)^n f(a,b)$$

en nuestro caso, considerando $K = f(x_i, y_i)$

tenemos

$$f(\underbrace{x_i+\theta_1 h}_{x_{i1}}, \underbrace{y_i+\theta_1 hk}_{y_{i1}\ (\text{por } (4))}) = f(x_i,y_i) + \theta_1 h\frac{\partial}{\partial x}f(x_i,y_i) + \theta_1 hk\frac{\partial}{\partial \tilde{y}}f(x_i,y_i) + \ldots$$

$$\Rightarrow f(x_{i1}, y_{i1}) - f(x_i,y_i) = \theta_1 h\frac{\partial}{\partial x}f(x_i,y_i) + \theta_1 h\overset{f(x_i,y_i)}{\cancel{k}}\frac{\partial}{\partial \tilde{y}}f(x_i,y_i) + \ldots$$

$$\Rightarrow \frac{f(x_{i1},y_{i1}) - f(x_i,y_i)}{\theta_1 h} = \frac{\partial}{\partial x}f(x_i,y_i) + \frac{\partial}{\partial \tilde{y}}f(x_i,y_i)\cdot f(x_i,y_i) + \ldots$$

$$\Rightarrow \frac{\partial f}{\partial x}(x_i,y_i) + \frac{\partial f}{\partial \tilde{y}}(x_i,y_i)\cdot f(x_i,y_i) = \frac{f(x_{i1},y_{i1}) - f(x_i,y_i)}{\theta_1 h} - \ldots$$

Reemplazando en (6) tenemos:

$$\tilde{y}(x_{i+1}) = y_i + hf(x_i,y_i) + \frac{h^2}{2}\left[\frac{f(x_{i1},y_{i1}) - f(x_i,y_i)}{\theta_1 h} - \ldots\right] + \frac{h^3}{6}\tilde{y}'''(x_i) + \ldots$$

$$\tilde{y}(x_{i+1}) = y_i + h f(x_i, y_i) + \frac{h}{2\theta_1} f(x_{i1}, y_{i1}) - \frac{h}{2\theta_1} f(x_i, y_i) - \cdots + \frac{h^3}{6}\tilde{y}'''(x_i) + \cdots$$

$$\Rightarrow \tilde{y}(x_{i+1}) = \underbrace{y_i + h\left(1 - \frac{1}{2\theta_1}\right) f(x_i, y_i) + h \frac{1}{2\theta_1} f(x_{i1}, y_{i1})}_{(1)} + \frac{h^3}{6}\tilde{y}'''(x_i) + \cdots$$

Comparando (1) con (2) tenemos que $A_{20} = \left(1 - \frac{1}{2\theta_1}\right)$ y $A_{21} = \frac{1}{2\theta_1}$ y por lo tanto

$$\boxed{y_{12} = y_i + h\left(1 - \frac{1}{2\theta_1}\right) f(x_i, y_i) + h \frac{1}{2\theta_1} f(x_{i1}, y_{i1}) = y_{i+1}}$$