2021final 2760sol - final solution

Probability for Engineers (香港中文大學)

THE CHINESE UNIVERSITY OF HONG KONG

ENGG2760C/D Probability for Engineers

Final Examination

Instructions:

- This is an **EXAMINATION**. You are **NOT ALLOWED** to communicate with any other person in any way during the examination.
- The duration of this examination is **2 hours**.
- Answer **ALL** questions in English. The full mark is **100 marks**. Sufficient intermediate steps should be given to receive full credit.
- Unless otherwise specified, numerical answers should be either **exact** or **corrected to at least 3 significant figures**.

Question 1 (20 marks)

Let A and B be arbitrary events. Which of the following is true? If you answer yes, prove it using the axioms of probability. If you answer no, prove it or provide a counterexample.

(a) (10 marks) $P(A|B) + P(A|\overline{B}) = 1$.

Solution:

- No.
- If A and B are the events of independent six-sided dice rolls, then

$$P(A|B) = P(A|\overline{B}) = P(A) = 1/6.$$

Thus

$$P(A|B) + P(A|\overline{B}) = 1/6 + 1/6 = 1/3.$$

(b) (10 marks) $P(A \cup B|A \cap B) = 1$.

Solution: Yes, because

$$\begin{aligned}
P(A \cup B|A \cap B) &= \frac{P[(A \cup B) \cap (A \cap B)]}{P(A \cap B)} \\
&= \frac{P[(A \cap A \cap B) \cup (B \cap A \cap B)}{P(A \cap B)} \\
&= \frac{P(A \cap B)}{P(A \cap B)} \\
&= 1.
\end{aligned}$$

Question 2 (20 marks)

n independent random numbers are sampled uniformly from the interval $[0, 1]$.

(a) (10 marks) If $n = 10$, what is the probability that exactly 7 of them are greater than 0.7?

> **Solution:** Let X be the number of such random numbers greater than 0.7.
> Then X follows a binomial distribution with $n = 10$ and $p = 0.3$.
> The required probability is
>
> $$\begin{aligned} P(X = 7) &= b(7; 10, 0.3) \\ &= \binom{10}{7} 0.3^7 (1 - 0.3)^{10-7} \\ &\approx 0.009002 \end{aligned}$$

(b) (10 marks) If $n = 70$, use the Central Limit Theorem to estimate the probability that their sum is between 30 and 35 (inclusive).

> **Solution:** Let X_i denote the value of the i-th random number $(i = 1, \ldots, n)$. Then, $X_1, \ldots, X_n$ are independent random variable with $X_i \sim U(0, 1)$, for all i, thus
>
> $$\begin{aligned} \mu &\equiv E(X_i) = 1/2 \\ \sigma^2 &\equiv \mathrm{Var}(X_i) = 1/12 \end{aligned}$$
>
> By the Central Limit Theorem, the required probability is
>
> $$\begin{aligned} &P\left(30 \le \sum_{i=1}^{n} X_i \le 35\right) \\ &= P\left(\frac{30 - 70(1/2)}{\sqrt{70(1/12)}} \le \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \le \frac{35 - 70(1/2)}{\sqrt{70(1/12)}}\right) \\ &\approx P(-2.07 \le Z \le 0) \\ &= \Phi(0) - \Phi(-2.07) \\ &\approx 0.5 - 0.0192 \\ &= 0.4808 \end{aligned}$$

Question 3 (20 marks)

Alice takes T hours to travel to Bob's house, where T is a random variable with PDF

$$f(t) = \begin{cases} 5/t^6, & \text{when } t \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (10 marks) Find the CDF $F(t)$ of T.

Solution: For $t < 1$,

$$F(t) = 0.$$

For $t \geq 1$,

$$\begin{aligned} F(t) &= \int_1^t \frac{5}{x^6}\, dx \\ &= \left[\frac{-1}{x^5}\right]_1^t \\ &= 1 - \frac{1}{t^5} \end{aligned}$$

(b) (10 marks) The distance between Alice's and Bob's house is one mile so that Alice travels at a speed $V = 1/T$ miles per hour. What is Alice's expected speed $E(V)$?

Solution:

$$\begin{aligned} E(V) &= E(1/T) \\ &= \int_{-\infty}^{\infty} \frac{1}{t} \cdot f(t)\, dt \\ &= \int_1^{\infty} \frac{1}{t} \cdot \frac{5}{t^6}\, dt \\ &= \left[\frac{-5}{6t^6}\right]_1^{\infty} \\ &= \frac{5}{6} \end{aligned}$$

Question 4 (20 marks)

Companies A and B produce lightbulbs. Their lifetimes are exponential random variables with mean 2 years for company A and 1 year for company B.

(a) (10 marks) A shop sources $4/5$ of its lightbulbs from company A and the remaining $1/5$ from company B. If a random lightbulb from the shop survived for 2 years, how likely is it to have been produced by company B?

Solution: Let T be the lifetime of the the lightbulb, and A and B be the (complementary) events that the respective company produced it. Then

$$P(T \geq t \,|\, A) = e^{-t/2}$$
$$P(T \geq t \,|\, B) = e^{-t}$$

By Bayes' theorem, the required probability is

$$\begin{aligned} P(B \,|\, T \geq 2) &= \frac{P(T \geq 2 \,|\, B)P(B)}{P(T \geq 2)} \\ &= \frac{P(T \geq 2 \,|\, B)P(B)}{P(T \geq 2 \,|\, A)P(A) + P(T \geq 2 \,|\, B)P(B)} \\ &= \frac{e^{-1}(4/5)}{e^{-1}(4/5) + e^{-2}(1/5)} \\ &\approx 0.915776 \end{aligned}$$

(b) (10 marks) What is the probability that a lightbulb produced by company B outlasts one produced by company A? Assume their lifetimes are independent.

Solution:

- Let X and Y be the lifetimes of the lightbulbs produced by companies A and B, respectively.
- The PDF of X is

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & \text{for } x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

The PDF of Y is

$$f_Y(y) = \begin{cases} e^{-y}, & \text{for } y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

- Since X and Y are independent, the joint PDF of X and Y is

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)f_Y(y) \\ &= \begin{cases} \frac{1}{2}e^{-(x/2)-y}, & \text{for } x > 0 \text{ and } y > 0, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

- The required probability is

$$\begin{aligned} P(X < Y) &= \int_0^\infty \int_x^\infty f_{XY}(x,y)\, dy\, dx \\ &= \int_0^\infty \int_x^\infty \frac{1}{2}e^{-(x/2)-y}\, dy\, dx \\ &= \int_0^\infty \frac{1}{2}e^{-x/2}\Big[-e^{-y}\Big]_x^\infty\, dx \\ &= \int_0^\infty \frac{1}{2}e^{-3x/2}\, dx \\ &= \frac{1}{2}\left[\frac{-2}{3}e^{-3x/2}\right]_0^\infty \\ &= \frac{1}{3} \end{aligned}$$

Question 5 (20 marks)

Let X and Y be two continuously random variables. The (marginal) PDF of Y is

$$f_Y(y) = \begin{cases} \frac{2}{9}(1+7y), & \text{for } 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

The conditional PDF of X given $(Y = y)$ is

$$f_{X|Y}(x|y) = 3\left(\frac{x+2y}{1+7y}\right), \quad \text{for } 0 \le x \le 1,\ 0 \le y \le 1,$$

and $f_{X|Y}(x|y) = 0$ elsewhere.

(a) (3 marks) Find the joint PDF $f_{XY}(x, y)$ of X and Y.

Solution:

$$\begin{aligned} f_{XY}(x,y) &= f_{X|Y}(x|y) \cdot f_Y(y) \\ &= \begin{cases} \frac{2}{3}(x+2y), & \text{for } 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

(b) (5 marks) Find the (marginal) PDF $f_X(x)$ of X.

Solution: For $0 \le x \le 1$,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \\ &= \int_0^1 \frac{2}{3}(x+2y)\, dy \\ &= \frac{2}{3}\Big[xy + y^2\Big]_0^1 \\ &= \frac{2}{3}(x+1) \end{aligned}$$

and $f_X(x) = 0$ elsewhere.

(c) (12 marks) Find $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Cov}(X, Y)$.

Solution:

$$E(X) = \int_0^1 \frac{2}{3}x(x+1)\,dx = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{5}{9}$$

$$E(X^2) = \int_0^1 \frac{2}{3}x^2(x+1)\,dx = \frac{2}{3}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{7}{18}$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{13}{162}$$

$$E(Y) = \int_0^1 \frac{2}{9}y(1+7y)\,dy = \frac{2}{9}\left[\frac{y^2}{2} + \frac{7y^3}{3}\right]_0^1 = \frac{17}{27}$$

$$E(Y^2) = \int_0^1 \frac{2}{9}y^2(1+7y)\,dy = \frac{2}{9}\left[\frac{y^3}{3} + \frac{7y^4}{4}\right]_0^1 = \frac{25}{54}$$

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{97}{1458}$$

$$\begin{aligned}
E(XY) &= \int_0^1 \int_0^1 \frac{2}{3}xy(x+2y)\,dy\,dx \\
&= \frac{2}{3}\int_0^1 \left[\frac{x^2y^2}{2} + \frac{2xy^3}{3}\right]_{y=0}^{y=1} dx \\
&= \frac{2}{3}\int_0^1 \left(\frac{x^2}{2} + \frac{2x}{3}\right) dx \\
&= \frac{2}{3}\left[\frac{x^3}{6} + \frac{x^2}{3}\right]_0^1 \\
&= \frac{1}{3} \\
\mathrm{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\
&= -\frac{4}{243}
\end{aligned}$$

— End of Paper —