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Preliminary investigation of the anti-fuzzy ring isomorphism theorem

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ABSTRACT

The paper begins with an introduction to fuzzy subsets and their complementary definitions and introduces the concepts of anti-fuzzy subrings and anti-fuzzy ideals. This is followed by a discussion of the many properties of fuzzy ideals and the connection between anti-fuzzy subrings as well as anti-fuzzy ideals. Ultimately we rationalise the isomorphism theorem for the anti-fuzzy ring

Keywords: Anti-fuzzy subrings; anti-fuzzy ideals; isomorphism

1. INTRODUCTION AND REVIEW

L.A. Zadeh proposed fuzzy set theory in 1965[1] and in 1971 A. R. B. Rosenfeld introduced the concept of fuzzy into algebra and studied fuzzy groups, thus beginning the process of developing the canonical group into a fuzzy direction and creating a new branch of mathematics called fuzzy algebra[2]. After this, W. J. Liu extended group theory to ring structures, defining the concepts of fuzzy subrings and fuzzy ideals[3-4], and fuzzy algebra was further developed. As the problem progressed, the paper[5] proposed the relevant concepts of anti-fuzzy groups and studied the relevant properties of anti-fuzzy groups.

In this paper we wish to extend the theory of anti-fuzzy groups to anti-fuzzy rings, introduce the concept of anti-fuzzy ideals, discuss its properties, and give isomorphism theorem for anti-fuzzy rings.

2. DEFINITIONS AND PREPARATORY KNOWLEDGE

The paper begins with an introduction to the definition of fuzzy subsets and their subsets etc. The mathematical notation used in this section, as well as the meaning of the letters, are used generically throughout the paper. The details are as follows

2.1. Definition of fuzzy subsets

Assuming the S is a set, the mapping $\mu : S \rightarrow [0,1]$ is called a fuzzy subset of S

2.2. Definition of the complement of a fuzzy subset

If μ is a Fuzzy subsets of S, then the complement of μ is denoted by μ^c and for $\forall x \in S$ there is

$$\mu^c(x) = 1 - \mu(x)$$

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2.3. Definition and sufficient conditions for anti-fuzzy subrings

If μ is a fuzzy subset of the ring R, for $\forall x, y \in R$ there are:

$$(1)\mu(xy) \le \mu(x) \vee \mu(y)$$
$$(2)\mu(-x) \le \mu(x)$$
$$(3)\mu(x+y) \le \mu(x) \vee \mu(y)$$

then μ is said to be an anti-fuzzy subring of the ring R, or AF subring for short.

Clearly, If μ is a fuzzy subset of the ring R, the sufficient condition of the Proposition: μ is an AF subring of a ring R is that for $\forall x, y \in R$ there are:

$$(1)\mu(xy) \le \mu(x) \vee \mu(y)$$
$$(2)\mu(x-y) \le \mu(x) \vee \mu(y)$$

2.4. Definition of anti-fuzzy ideal and simple related conclusions

If μ is a fuzzy subset of the ring R, for $\forall x, y \in R$ there are:

$$(1)\mu(xy) \le \mu(x) \wedge \mu(y)$$
$$(2)\mu(-x) \le \mu(x)$$
$$(3)\mu(x+y) \le \mu(x) \vee \mu(y)$$

then μ is said to be an anti-fuzzy ideal of the ring R, or AF ideal for short.

Based on the above definition, it is clear that there are: AF ideal must be AF sub-ring

Based on the item (2) of the above definition, we have the following derivation:

$$\mu(-x) \le \mu(x) = \mu(-(-x)) \le \mu(-x) \Rightarrow \mu(-x) = \mu(x)$$

while there we have

$$\mu(0) = \mu(x-x) \le \mu(x) \vee \mu(-x) = \mu(x)$$

thus for anti-fuzzy subring μ ,for $\forall x \in R$, we can conclude that

$$(1)\mu(-x) = \mu(x)$$
$$(2)\mu(0) \le \mu(x)$$

2.5. Definition of isomorphic mapping between AF subrings and fuzzy subsets

Now we can suppose that A and B are AF subrings of the ring R and fuzzy subsets of the ring R, respectively. $f: R \to \overline{R}$ is a homomorphic map of the ring.

If $f(A) = B$, then f is said to be a homomorphic map from A to B and A is said to be homomorphic to B when f is a full projection, denoted as: $A \sim B$

If f is an isomorphic mapping and $f(A) = B$, then f is said to be an isomorphic mapping from A to B, denoted as: $A \cong B$.

2.6. Meaning of main mathematical symbols

Table 1. Meaning of basic symbols in this article

symbol	*meaning*
$\forall$	Arbitrary
$\in$	Belongs to
$\vee$	Select the larger one between two
$\wedge$	Select the smaller one between two
$\Rightarrow$	Deduce
$\subseteq$	Included in
$\Leftrightarrow$	Equivalent to
$\inf$	Lower definite boundary (also known as infimum) e.g. $\inf\{y \mid y = x^2 + 1 \wedge x \in R\} = 1$

Table 2. Meaning of specific and important symbols in this article

symbol	*meaning*
μ	Mapping (or fuzzy set, Anti-fuzzy ideal)
G	Group
R	Ring
μ^c and $\overline{A}$	Complement of μ and A
AF	Anti-fuzzy
A/I	The AF quotient ring of the ring R about I
$A \sim B$	There exists homomorphic functions from A to B
$A \cong B$	There exists isomorphic functions from A to B

3. THE CONNECTION AND PROPERTIES OF ANTI-FUZZY RING AND ANTI-FUZZY IDEAL

3.1. Theorem 1

If μ is a fuzzy subset of the ring R, the sufficient condition of μ being the AF ideal of ring R is that for $\forall x, y \in R$ there exist:

$$(1)\mu(xy) \leq \mu(x) \wedge \mu(y)$$
$$(2)\mu(x-y) \leq \mu(x) \vee \mu(y)$$

To prove, the "$\Rightarrow$" process is obviously established. And for the process "$\Leftarrow$", according to the definition of the AF ideal there is:

$$\mu(xy) \leq \mu(x) \wedge \mu(y)$$

and then:

$$\mu(x-y) = \mu(x+(-y)) \leq \mu(x) \vee \mu(-y) \leq \mu(x) \vee \mu(y)$$

From the above discussion, it is clear that Theorem 1 holds

3.2. Theorem 2

If μ is a fuzzy subset of the ring R, the sufficient condition of μ^c being the AF ideal of ring R is that μ^c is the fuzzy ideal of the ring R.

To prove the theorem, first, for the process "$\Rightarrow$", for $\forall x, y \in R$ there exist:

$$
\begin{aligned}
&(1)\mu(xy) \le \mu(x) \wedge \mu(y) \\
&\Rightarrow 1-\mu^c(xy) \le (1-\mu^c(x)) \wedge (1-\mu^c(y)) \Rightarrow \mu^c(xy) \ge 1-((1-\mu^c(x)) \wedge (1-\mu^c(y))) \\
&\Rightarrow \mu^c(xy) \ge \mu^c(x) \vee \mu^c(y) \\
&(2)\mu(-x) \le \mu(x) \\
&\Rightarrow 1-\mu^c(-x) \le \mu^c(x) \Rightarrow \mu^c(-x) \le \mu^c(x) \\
&(3)\mu(x+y) \le \mu(x) \vee \mu(y) \\
&\Rightarrow 1-\mu^c(x+y) \le (1-\mu^c(x)) \vee (1-\mu^c(y)) \Rightarrow \mu^c(x+y) \ge 1-((1-\mu^c(x)) \vee (1-\mu^c(y))) \\
&\Rightarrow \mu^c(x+y) \ge \mu^c(x) \wedge \mu^c(y)
\end{aligned}
$$

Therefore, Theorem 2 holds

3.3. Proposition 1

If I is the AF ideal of the ring R and $\overline{I}$ is the AF ideal of the ring $\overline{R}$, $f : R \rightarrow \overline{R}$ is a homomorphic mapping, then the following proposition holds

(1) $f(I)$ is the AF ideal of the ring $\overline{R}$,and among them there are:

$$
f(I)(y) = \begin{cases} \inf\{I(x) \mid f(x) = y\}, f^{-1}(y) \ne \phi \\ 0, f^{-1}(y) = \phi \end{cases}
$$

(2) $f^{-1}(\overline{I})$ is the AF ideal of R, and among there is: for $\forall x \in R$, $f^{-1}(\overline{I})(x) = \overline{I}(f(x))$

(3) $f(f^{-1}(\overline{I})) \supseteq \overline{I}$ and the left and right sides are equal when and only when f is epimorphism

(4) $f^{-1}(f(I)) \subseteq I$ and the left and right sides are equal when I is constant in the core $Kerf$.

3.4. Proposition 2

If μ is a fuzzy subset of the ring R, for $\forall x \in R$ we definite that the fuzzy subset $r+\mu$ of the ring R is:

$$
(r+\mu)(x) = \mu(x-r)
$$

From this, we give the following proposition: If μ is a AF subring for the ring R, for $\forall x, y \in R$ there we have:

$$
x+\mu = y+\mu \Leftrightarrow \mu(x-y) = \mu(0)
$$

to prove, first for the process "$\Rightarrow$", if $x+\mu = y+\mu$ holds, then for $\forall x, y \in \mu$ and $\forall r \in R$ there is

$$(x+\mu)(r)=(y+\mu)(r)$$
$$\mu(r-x)=\mu(r-y)$$
$$\mu(x-r)=\mu(y-r)$$

therefor, for the $\mu(0)$

$$\mu(0)\le\mu(x-y)=\mu(x-r+r-y)$$
$$\le\mu(x-r)\vee\mu(r-y)=\mu(x-r)\vee\mu(y-r)=\mu(0)\vee\mu(y-r)$$

especially, we set the value $r=x$, then

$$\mu(0)\le\mu(x-y)\le\mu(0)\Rightarrow\mu(x-y)=\mu(0)$$

For the process "$\Rightarrow$", to prove:for $\forall r\in R$, we have

$$(x+\mu)(r)=\mu(x-r)=\mu(x-y+y-r)\le\mu(x-y)\vee\mu(y-r)=\mu(0)\vee\mu(y-r)$$

thus we have:

$$(x+\mu)(r)\le(y+\mu)(r)$$

the same can be proved:

$$(x+\mu)(r)\ge(y+\mu)(r)$$

therefore:

$$(x+\mu)(r)=(y+\mu)(r)\Rightarrow x+\mu=y+\mu$$

3.5. Theorem 3

If I is the AF ideal of the ring R, then $R/I=\{r+I\mid r\in R\}$ can forms a ring. And among them exists:

$$(1)\,zero\ element\ is\ I-(r+I)=(-r)+I$$
$$(2)(r_1+I)+(r_2+I)=(r_1+r_2)+I$$
$$(3)(r_1+I)(r_2+I)=(r_1r_2)+I$$

To prove, if there are $r_1+I=r_1^{'}+I$; $r_2+I=r_2^{'}+I$, then $I(r_1-r_1^{'})=I(0)$ and $I(r_2-r_2^{'})=I(0)$

$$\Rightarrow I(0)\le I((r_1+r_2)-(r_1^{'}+r_2^{'}))=I((r_1-r_1^{'})+(r_2-r_2^{'}))$$
$$\le I(r_1-r_1^{'})\vee I(r_2-r_2^{'})=I(0)\vee I(0)=I(0)$$

$$\Rightarrow I(0)\le I(r_1r_2-r_1^{'}r_2^{'})=I(r_1r_2-r_1^{'}r_2+r_1^{'}r_2-r_1^{'}r_2^{'})=I((r_1-r_1^{'})r_2+(r_2-r_2^{'})r_1^{'})$$
$$\le I((r_1-r_1^{'})r_2)\vee I((r_2-r_2^{'})r_1)$$
$$\le(I(r_1-r_1^{'})\wedge I(r_2))\vee(I(r_2-r_2^{'})\wedge I(r_1))=(I(0)\wedge I(r))\vee(I(0)\wedge I(r))=I(0)\vee I(0)=I(0)$$

$$\therefore I((r_1+r_2)-(r_1^{'}+r_2^{'}))=I(r_1r_2-r_1^{'}r_2^{'})=I(0)$$
$$\therefore(r_1+r_2)+I=(r_1^{'}+r_2^{'})+I;r_1r_2+I=r_1^{'}r_2^{'}+I$$

So the operation in R/I is reasonable. And since R is a ring, according to the operation in R/I, it's clear that R/I is a ring.

3.6. Definition of AF quotient ring

Now we define the ring R/I in the above theorem the AF quotient ring of the ring R about I. For this, we make the following rigorous definition.

If I is the AF ideal of the ring R, then we define $R/I=\{r+I \mid r\in R\}$ as the AF quotient ring of the ring R about I.

4. ANTI-FUZZY QUOTIENT RING AND ITS ISOMORPHISM THEOREM

4.1. Theorem 4

If A is the AF subring for R, I is the AF ideal for R, then A/I is the AF subring for R/I. Among them there is $A/I(r+I)=\inf\{A(x) \mid r+I=x+I\}$

To prove, for $\forall r_1, r_2\in R$, we have

$$\begin{aligned}(1)A/I((r_1+I)(r_2+I)) &= A/I(r_1r_2+I)=\inf\{A(x) \mid x+I=r_1r_2+I\}\\ &\le \inf\{A(x_1x_2) \mid x_1+I=r_1+I, x_2+I=r_2+I\}\\ &\le \inf\{A(x_1)\vee A(x_2) \mid x_1+I=r_1+I, x_2+I=r_2+I\}\\ &= \inf\{A(x_1) \mid x_1+I=r_1+I\}\vee \inf\{A(x_2) \mid x_2+I=r_2+I\}=A/I(r_1+I)\vee A/I(r_2+I))\end{aligned}$$

$$\begin{aligned}(2)A/I(-(r+I)) &= A/I((-r)+I))=\inf\{A(-x) \mid (-x)+I=(-r)+I\}\\ &\le \inf\{A(x) \mid x+I=r+I\}=A/I(r+I)\end{aligned}$$

Thus A/I is the AF subring for R/I.

4.2. Definition of AF quotient subring

Now we define AF subring A/I in the theorem 4 the AF quotient subring of the ring A about I. The rigorous definition is as followed.

If A is the AF subring for R, I is the AF ideal for R, then we define that A/I is the AF quotient subring for R/I.

4.3. Theorem 5

If I and $\overline{I}$ are both the AF ideal of the ring R, then the $\overline{I}/I$ is the AF ideal of R/I. To prove:

$\because I$ is the AF ideal of the ring R

$\therefore \overline{I}$ is the AF ideal of the ring R

according to the theorem 4, we can know that the $\overline{I}/I$ is the AF subring of R/I. And on the other hand, for $\forall r_1, r_2\in R$, we have

$$\begin{aligned}\overline{I}/I((r_1+I)(r_2+I)) &= \overline{I}/I((r_1r_2)+I)=\inf\{\overline{I}(x) \mid x+I=(r_1r_2)+I\}\\ &\le \inf\{\overline{I}(x_1x_2) \mid (x_1x_2)+I=(r_1r_2)+I\}\\ &\le \inf\{\overline{I}(x_1)\wedge \overline{I}(x_2) \mid x_1+I=r_1+I, x_2+I=r_2+I\}\\ &= \inf\{\overline{I}(x_1) \mid x_1+I=r_1+I\}\wedge \inf\{\overline{I}(x_2) \mid x_2+I=r_2+I\}=\overline{I}/I(r_1+I)\wedge \overline{I}/I(r_2+I))\end{aligned}$$

according to the theorem 1, $\overline{I} / I$ is the AF subring of R / I .

4.4. Theorem 6

If A is the AF subring for R , B is the AF ideal for R , then $A \sim A / I$

To prove, we set $f : R \to R / I$ satisfies $\forall r \in R, f(r) = r + 1$. Then according to the definition of R / I , f is a full isomorphic mapping. And for $\forall r \in R$

$$f(A)(r+1) = \inf\{A(t) \mid f(t) = r + I\} = \inf\{A(t) \mid t + I = r + I\} = A / I(r + I)$$

$$\therefore f(A) = A / I \Rightarrow A \sim A / I$$

4.5. Theorem 7

If A is the AF subring for R , I is the AF ideal for R , $f : R \to \overline{R}$ is a full isomorphic mapping. When I is a constant among in the core $Kerf$, then there is $A / I \cong f(A) / f(I)$.

To prove, according to the proposition 1 and proposition 2,we know that $f(A)$ is the AF subring for the ring $\overline{R}$, and $f(I)$ is the AF ideal for the ring $\overline{R}$. Now we create $g : R / I \to \overline{R} / f(I)$ satisfies

$\forall r \in R, g(r + I) = f(r) + f(I)$. Thus for:

(1) $\forall x, y \in R$,if $x_1 + I = x_2 + I \Rightarrow I(x_1 - x_2) = I(0)$

$$\Rightarrow f(I)(f(x) - f(y)) = f(I)(f(x - x)) = \inf\{I(x) \mid f(x) = f(x_1 - x_2)\} = I(x_1 - x_2) = I(0)$$
$$= \inf\{I(x) \mid f(x) = 0\}$$

$$\Rightarrow f(x) + f(I) = f(x) + f(I)$$

$\therefore g$ is mapping $\xrightarrow{f \text{ is } surjective}$ f is surjective mapping

(2) $\forall y_1, y_2 \in R$ if $y_1 + f(I) = y_2 = f(I)$, and there exists $x_1, x_2 \in R$ satisfies $y_1 = f(x_1), y_2 = f(x_2)$

$$\Rightarrow f(x_1) + f(I) = f(x_2) + f(I)$$

$$\Rightarrow f(I)(f(x_1) - f(x_2)) = f(I)(0) = I(0)$$
$$\Rightarrow f(I)(f(x_1 - x_2)) = I(0)$$
$$\Rightarrow f^{-1}(f(I))(x_1 - x_2) = I(0) \xrightarrow{proposition\ 1} I(x_1 - x_2) = I(0)$$

$$\therefore x_1 + I = x_2 + I$$

$\therefore g : R / I \to \overline{R} / f(I)$ is injective, while g is surjective, obviously g is a maintaining operation

$\therefore g$ is a isomorphic mapping

(3) $\forall y \in R, \exists x \in R$ satisfies $f(x) = y$.

$$
\begin{aligned}
&\Rightarrow g(A/I)(y+f(I)) \\
&\quad =\inf\{A/I(t+I) \mid g(t+I)=y+f(I)\}=\inf\{A/I(t+I) \mid f(x)+f(I)=y+f(I)\} \\
&\quad =\inf\{A/I(t+I) \mid f(I)(f(t-x))=f(I)(0)\}=\inf\{A/I(t+I) \mid I(t-x)=I(0)\} \\
&\quad =\inf\{A/I(t+I) \mid t+I=x+I\}=A/I(x+I)
\end{aligned}
$$

while for $f(A)/f(I)(y+f(I))$

$$
\begin{aligned}
&=\inf\{f(A)(v) \mid v+f(I)=y+f(I)\} \\
&=\inf\{A(u) \mid f(u)+f(I)=f(x)+f(I)\}=\inf\{A(u) \mid f(I)(f(u-x))=f(I)(0)\} \\
&=\inf\{A(u) \mid I(u-x)=I(0)\}=\inf\{A(u) \mid u+I=x+I\} \\
&=A/I(x+1)
\end{aligned}
$$

$$\therefore g(A/I)=f(A)/f(I)$$

$$\therefore A/I \cong f(A)/f(I)$$

5. SUMMARY

This paper starts from introducing the definition of fuzzy subsets and their complements, and introduces the concept of anti-fuzzy subrings as well as anti-fuzzy ideals. The paper then discusses the equivalence conditions of the anti-fuzzy subring and the anti-fuzzy ideal, and argues for the sufficiency of the above equivalence conditions when the fuzzy subset is the anti-fuzzy ideal of the ring

The article also discusses the relationship between the anti-fuzzy in the immediately following ideal when in different state mappings, and the relationship between two ideals shaped like $r+\mu$, respectively, in the form of propositions.

The article finally proves four decision theorems that are closely related to the homology of the inverse fuzzy quotient ring.

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