Lecture Notes of CSCI5610 Advanced Data Structures

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Contents

Lecture 1: Course Overview and Computation Models

A *data structure*, in general, stores a set of elements and supports certain operations on those elements. From your undergraduate courses, you should have learned two ways by which data structures are useful:

- They alone can be employed directly for information retrieval (e.g., "find all the people whose ages are equal to 25", or "report the number of people aged between 20 and 40").
- They serve as building bricks in implementing algorithms efficiently (e.g., Dijkstra's algorithm would be slow unless it uses an appropriate structure such as the priority queue).

This (graduate) course aims to deepen our knowledge of data structures. Specifically:

- We will study new data structures for solving important problems in computer science with strong performance guarantees (heuristic solutions, which perform well only on some inputs, will not be of interest in this course).
- We will discuss *techniques* for designing and analyzing data structures with non-trivial performance guarantees. Those techniques are *generic* in the sense that they are useful in a great variety of scenarios and may enable you to discover innovative structures of your own.

The word RAM model. Computer science is a subject under mathematics. Before analyzing any algorithms, we need to first define a computation model properly.

Unless otherwise stated, we will be using the standard *word RAM*[1] model. In this model, the *memory* is an infinite sequence of *cells*, where each cell is a sequence of w bits for some integer w and is indexed by an integer *address*. Each cell is also called a *word*; and accordingly, the parameter w is often referred to as the *word length*. The *CPU* has a (constant) number of cells, each of which is called a *register*. The CPU can perform only the following *atomic* operations:

- Set a register to some constant or to the content of another register.
- Compare two numbers in registers.
- Perform $+, -, *, /$ on two numbers in registers.
- Shift the word in a register to the left (or right) by a certain number of bits.
- Perform the AND, OR, XOR on two registers.
- When an address x has been stored in a register, read the content of the memory cell at address x into a register, or conversely, write the content of a register into that memory cell.

[1] Random access machine with word-level parallelism.

The *time* (or *cost*) of an algorithm is measured by the number of atomic operations performed.

The word length w needs to be long enough to encode all the memory addresses! For example, if your algorithm uses n^2 memory cells for some integer n, then the word length will need to have at least $2\log_2 n$ bits.

The real RAM model. In word RAM, (memory/register) cells can store only integers. Next, we will slightly modify the model to deal with real values.

Simply "allowing" each cell to store a real value does not give us a satisfactory model. For example, how many bits would you use for a real value? In fact, even if the number of bits *were* infinite, still we would not be able to represent all the real values even in a short interval like $[0, 1]$ — the set of real values in that interval is *uncountably* infinite! If we cannot even specify the word length for a "real-valued" cell, how to properly define the atomic operations for performing shifts, AND, OR, and XOR?

We can alleviate the issue by introducing the concept of *black box*. We allow a (memory/register) cell c to store a real value x, but in this case the algorithm is forbidden to look *inside* c, that is, the algorithm has no control over the representation of x. In other words, c is now a black box, holding the value x *precisely* (by magic).

A black box remains as a black box after computation. For example, suppose that two registers both contain $\sqrt{2}$. We can multiply them, but the product 2 must be understood as a real value. This is similar to the requirement in C++ that the product of two float numbers remains as a float number.

Now we can formally extend the RAM model as follows:

- Each cell can store either an integer or a real value.
- For operations $+, -, *, /$, if any operand is a real value, the result is a real value.
- Shifting, AND, OR, and XOR cannot be performed on registers storing real values.

We will refer to the new model as the *real RAM* model.

Although the real RAM model is mathematically sound, no one has proven that it is polynomial-time equivalent to Turing machines (it would be surprising if it was). We must be very careful *not to abuse the power of real value computation*. For example, in real RAM, we can compute 2^n in $O(\log n)$ time: once $2^{i/2}$ is ready, 2^i can be obtained in constant time. In Word RAM, 2^n takes n/w words to represent; hence, even writing out 2^n in memory takes $\Omega(n/w)$ time. In this course, we will exercise caution to make sure that every algorithm should run in exactly the same time complexity no matter the input values are real or integer numbers.

Randomness. All the atomic operations are *deterministic* so far. In other words, our models so far do not permit *randomization* which is sometimes important (e.g., hashing).

To fix the issue, we introduce one more atomic operation for both word- and real-RAM. This operation, named *RAND*, takes two non-negative integer parameters x and y, and returns an integer chosen uniformly at random from $[x, y]$. In other words, every integer in $[x, y]$ can be returned with probability $1/(y - x + 1)$. The values of x, y should be in $[0, 2^w - 1]$ because they each need to be encoded in a word.

Math conventions. $\mathbb{R}$ denotes the set of real values and $\mathbb{N}$ denotes the set of integers. For an integer $x \geq 1$, $[x]$ denotes the set $\{1, 2, ..., x\}$; if $x = 0$, then $[x] = \emptyset$.

For a tree T, $root(T)$ represents its root. For a node u in T, $parent(u)$ denotes the parent of u (if $u = root(T)$, $parent(u) =$ nil) and $sub(u)$ denotes the subtree rooted at u.

You should be familiar with the notations of $O(.), \Omega(.)$, $\Theta(.)$, $o(.)$, and $\omega(.)$. We also use $\tilde{O}(f(n_1, n_2, ..., n_x))$ to denote the class of functions that are $O(f(n_1, n_2, ..., n_x) \cdot \text{polylog}(n_1 + n_2 + ... + n_x))$, namely, $\tilde{O}(.)$ hides a polylogarithmic factor.

Lecture 2: The Binary Search Tree and the 2-3 Tree

In this lecture, we will first review the binary search tree (BST) from your undergraduate knowledge and then discuss the 2-3 tree, a replacement of the BST that admits simpler analysis in many situations.

2.1 The binary search tree

2.1.1 The basics

Let S be a set of n real values. A BST on S is a binary tree $\mathcal{T}$ with the properties below.

- Every node u in $\mathcal{T}$ stores an element in S, called the *key* of u and denoted as $key(u)$. Conversely, every element in S is the key of one node in $\mathcal{T}$. This means $\mathcal{T}$ has n nodes.
- For every non-root node u, if u is the left (resp. right) child of $p = parent(u)$, the keys in $sub(u)$ are smaller (resp. larger) than $key(p)$.

$\mathcal{T}$ is *balanced* if its height is $O(\log n)$. Henceforth, all BSTs are balanced unless otherwise stated.

The BST supports many operations efficiently. The following are some examples at the undergraduate level.

- **Insertion/deletion.** We can add/remove an element to/from S in $O(\log n)$ time.
- **Predecessor/successor search.** We can find the predecessor/successor of any $q \in \mathbb{R}$ using $O(\log n)$ time. Recall that the predecessor (resp. successor) is the largest (resp. smallest) element in S at most (resp. least) q.
- **Range reporting.** Given an interval $I = [x, y]$ in $\mathbb{R}$, we can report $I \cap S$ in $O(\log n + k)$ time where $k = |I \cap S|$.
- **Find-min/max.** We can return the smallest/largest element of S in $O(\log n)$ time.

The next two operations are beyond the undergraduate level.

- **Split.** Given an element $x \in S$, we can divide the BST of S into a BST on $S \cap (-\infty, x)$ and a BST on $S \cap [x, \infty)$, all in $O(\log n)$ time.
- **Join.** Let S_1 and S_2 be sets of real values s.t. $x < y$ for any $x \in S_1$ and $y \in S_2$. Assuming a BST on S_1 and a BST on S_2, we can produce a BST on $S_1 \cup S_2$ in $O(\log |S_1 \cup S_2|)$ time.

We will not explain the details of splits and joins on the BST (they are a bit sophisticated) but we will do so on the 2-3 tree in Section 2.2.

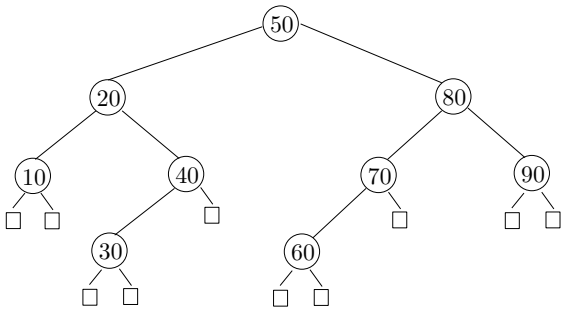


Figure 2.1: A BST (every square is a conceptual leaf)

2.1.2 Slabs

Next, we introduce the notion of *slab* which will appear very often in this course.

Let $\mathcal{T}$ be a BST on S. We will regard each empty child pointer in $\mathcal{T}$ as a *conceptual* leaf; see Figure 2.1. You should not confuse this with a (regular) leaf node z of $\mathcal{T}$ (every z has two conceptual leaves as its "children"). The number of conceptual leaf nodes is $n+1$. Henceforth, we will use the term *regular node* to refer to a "real" node that is not a conceptual leaf.

For a regular/conceptual node u in $\mathcal{T}$, its *slab* — denoted as $slab(u)$ — is defined recursively as follows:

- If $u = root(\mathcal{T})$, $slab(u) = (-\infty, \infty)$.
- Otherwise, let $p = parent(u)$. If u is the left child of p, $slab(u) = slab(p) \cap (-\infty, key(p))$; otherwise, $slab(u) = slab(p) \cap [key(p), \infty)$.

Example. Figure 2.1 shows a BST on $S = \{10, 20, ..., 90\}$. The slab of node 40 is $[20, 50)$, while that of its right conceptual leaf is $[40, 50)$. □

Proposition 2.1. *For any regular/conceptual nodes u and v in $\mathcal{T}$, we have:*

- *if u is an ancestor of v, then $slab(v)$ is covered by $slab(u)$;*
- *if u and v have no ancestor-descendant relationship, then $slab(u)$ is disjoint with $slab(v)$.*

Proposition 2.2. *The slabs of all the conceptual leaves partition $\mathbb{R}$.*

Lemma 2.3 (Canonical Partitioning Lemma)**.** *Given any interval $q = [x, y)$ with $x, y \in S \cup \{-\infty, \infty\}$, we can partition it into $O(\log n)$ disjoint slabs.*

Proof. We will prove the lemma in the special case where q has the form $[x, \infty)$; the general case is left to you as an exercise. Consider the algorithm below:

canonical-partition (q)
/* condition: q has the form $[x, \infty)$ */
1. $\Sigma \leftarrow \emptyset$, $u \leftarrow root(\mathcal{T})$
2. **if** $key(u) = x$ **then**
3. add to Σ the slab of the right child of u and **return** Σ
4. **elseif** $key(u) < x$ **then**
5. $u \leftarrow$ the right child of u and **goto** Line 2
6. **else**
7. add to Σ the slab of the right child of u
8. $u \leftarrow$ the left child of u and **goto** Line 2

Σ takes at most one slab from each level of $\mathcal{T}$ and hence has a size $O(\log n)$. □

We will refer to the slabs promised by the above lemma as the *canonical slabs* of q.

Example. In Figure 2.1, the interval $q = [30, 90)$ is partitioned by its canonical slabs $[30, 40)$, $[40, 50)$, $[50, 80)$, $[80, 90)$. □

2.1.3 Augmenting a BST

The BST's power can be further enhanced by associating nodes with additional information. For example, we can store at each node u of $\mathcal{T}$ a *count* that equals the number of keys in $sub(u)$. We will call the resulting structure the *count BST*. The count BST supports all the operations in Section 2.1 with the same performance guarantees. In addition, it also supports:

- **Range counting.** Given an interval $q = [x, y]$ with $x, y \in \mathbb{R}$, we can report $|q \cap S|$ (note: the output is a single integer) in $O(\log n)$ time.

2.2 The 2-3 tree

In a binary tree, every internal node has a *fanout* (i.e., number of child nodes) either 1 or 2. In this section, we will see a variant of the BST where the fanout can be 2 or 3. This variant, called the 2-3 tree, retains all the BST's performance guarantees. We will explain how to support the split and join operations in Section 2.1.1 on the 2-3 tree.

2.2.1 Structure Description

A *2-3 tree* on a set S of n real values is a tree $\mathcal{T}$ with the following properties.

- Every internal node has 2 or 3 child nodes. All the leaf nodes are at the same level[1].
- Every element of S is stored in one leaf. Every leaf stores 2 or 3 elements of S unless $n = 1$, in which case $\mathcal{T}$ has a single leaf containing the only element of S.
- If an internal node u has child nodes $v_1, ..., v_f$ where $f = 2$ or 3, u stores a *routing element* e_i ($i \in [f]$) that is the smallest element stored in the leaf nodes of $sub(v_i)$. Furthermore, all the elements stored in $sub(v_i)$ are less than e_{i+1}, for each $i \in [f-1]$.

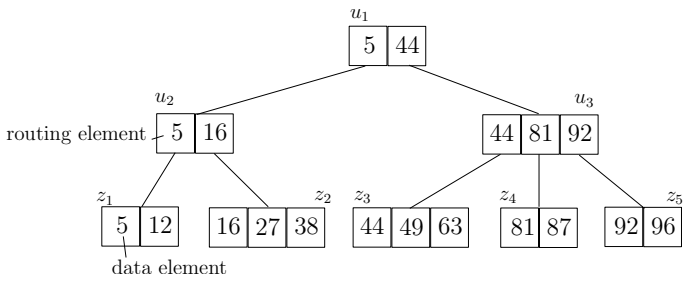


Figure 2.2: A 2-3 tree example

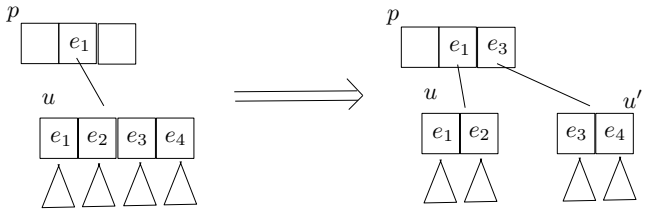


Figure 2.3: Treating an overflow

The height of $\mathcal{T}$ is $O(\log n)$. Its space is $O(n)$ because the number of nodes decreases by a factor of 2 per level as we descend from the leaves.

Example. Figure 2.2 shows a 2-3 tree on $S = \{5, 12, 16, 27, 38, 44, 49, 63, 81, 87, 92, 96\}$. □

2.2.2 Overflows and underflows

In in subsection assumes $n \geq 2$. In a 2-3 tree, an internal or leaf node *overflows* if it contains 4 elements or *underflows* if it contains only 1 element.

Treating overflows. We consider the case where the overflowing node u is not $root(\mathcal{T})$ (the opposite case is left to you). Suppose that u contains elements $e_1, e_2, ..., e_4$ in ascending order; let $p = parent(u)$. We create another node u', move e_3 and e_4 from u to u', and add a routing element e_3 to p for u'; see Figure 2.3. The steps so far take constant time. At this moment, p may be overflowing, which is then treated in the same manner. Since the overflow may propagate to the root, in the worst case we spend $O(\log n)$ time overall.

Treating underflows. Again, we consider the case where the underflowing u is not $root(\mathcal{T})$ (leaving the opposite to you). Suppose that the only element in u is e; let $p = parent(u)$. As p has at least two child nodes, u definitely has a *sibling* u'; due to symmetry, we will discuss only the situation where u' is the right sibling of u.

- If u' has 2 elements, move all the elements of u into u', delete u' from the tree, and remove the routing element in p for u'' see Figure 2.4(a). If p underflows, we remedy it in the same manner. Since the underflow may propagate to the root, in the worst case we spend $O(\log n)$ time overall.
- If u' has 3 elements e_1, e_2, e_3, move e_1 from u' into u and modify the routing element in p for u'; see Figure 2.4(b).

[1]Recall that a node's *level* is the number of edges on its path to the root.

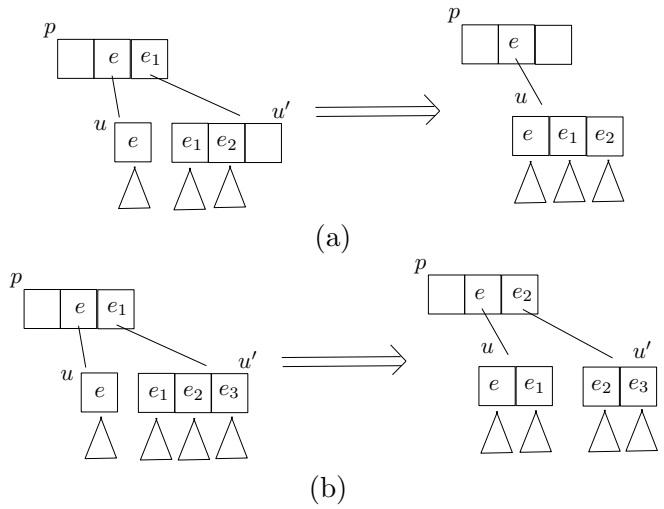


Figure 2.4: Treating an underflow

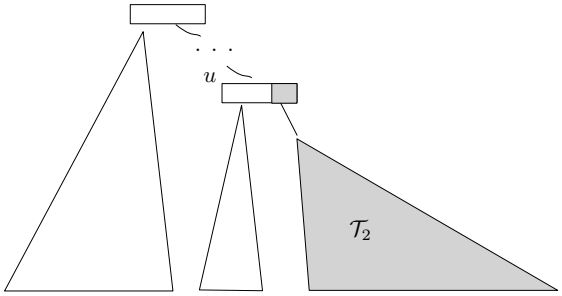


Figure 2.5: Join

The underflow/overflow treating algorithms imply that an insertion or a deletion can be supported in $O(\log n)$ time.

2.2.3 Splits and joins

Recall that our main purpose for discussing the 2-3 tree is to seek a (relatively) easy way to support the split and join operations, re-stated below:

- **Split**: Given a real value $x \in S$, we want to split S into $S_1 = (-\infty, x)$ and $S_2 = S \setminus S_1$. Assuming a 2-3 tree on S, we want to produce a 2-3 tree on S_1 and a 2-3 tree on S_2, all in $O(\log n)$ time.
- **Join**: Let S_1 and S_2 be sets of real values s.t. $x < y$ for any $x \in S_1, y \in S_2$. We want to merge them into $S = S_1 \cup S_2$. Assuming a 2-3 tree on each of S_1 and S_2, we want to produce a 2-3 tree on S in $O(\log |S|)$ time.

Join. Let us discuss joins first because the algorithm is needed in splits. Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the 2-3 tree on S_1 and S_2, respectively. Let h_1 and h_2 be the heights of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Assume, w.l.o.g., $h_1 \geq h_2$. Set $\Delta = h_1 - h_2$

- If $h_1 = h_2$, create a root u which has $\mathcal{T}_1$ as the left subtree and $\mathcal{T}_2$ as the right subtree.

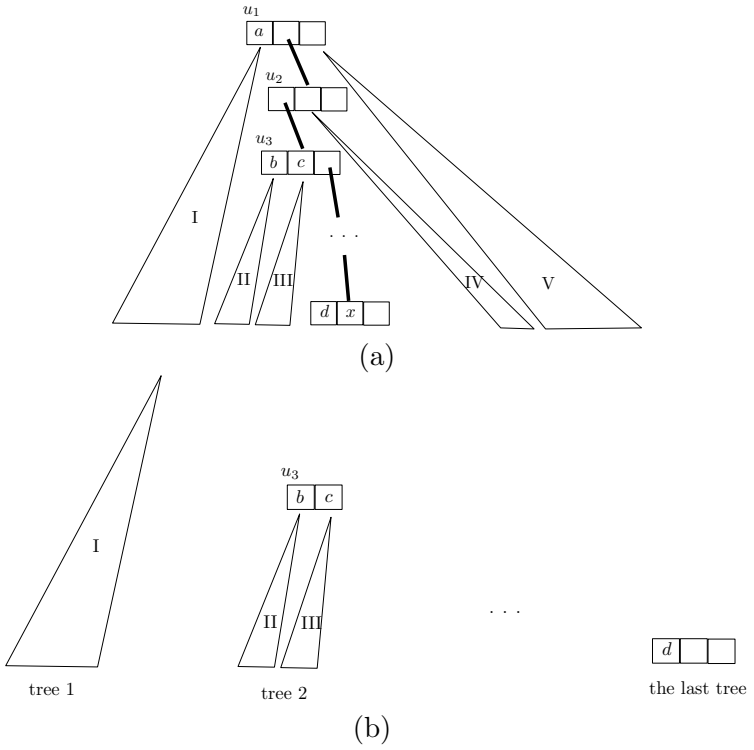


Figure 2.6: Split

- Otherwise, let u be the level-$(\Delta-1)$ node on the rightmost path of $\mathcal{T}_1$. Add $\mathcal{T}_2$ as the *rightmost* subtree of u. See Figure 2.5. This may leave u overflowing, which is then treated in the way explained earlier.

Overall, a join can be performed in $O(1+\Delta)$ time, which is $O(\log n)$.

Split. Due to symmetry, we will explain only how to produce the 2-3 tree of S_1. Let $\mathcal{T}$ be the 2-3 tree on S. First, find the path Π in $\mathcal{T}$ from the root to the leaf containing the value x (used for splitting). It suffices to focus on the part of $\mathcal{T}$ "on the left" of Π. We will partition this part into a set Σ of $t = O(\log n)$ 2-3 trees.

Example. Consider Figure 2.6(a) where Π is indicated by the bold edges. We can ignore subtrees labeled as IV and V because they are "on the right" of Π. Now, let us focus on the part on the left. At the root u_1 (level 0), Π descends from the 2nd routing element; the subtree labeled as I is added to Σ. At the level-1 node u_2, Π descends from the 1st routing element; no tree is added to Σ. At the level-2 node u_3, Π descends from the 3rd routing element; the 2-3 tree added to Σ has u_3 as the root, but only two subtrees labeled as II and III, respectively. The same idea applies to every level. At the leaf level, what is added to Σ is a 2-3 tree with only one node. Note how all the 2-3 trees shown in Figure 2.6(b) together cover all the elements of S_1. □

Formally, we generate Σ by including at most one 2-3 tree at each level ℓ. Let u be the level-ℓ node on Π. Let $e_1, ..., e_f$ ($f = 2$ or 3) the elements in u sorted in ascending order.

- If Π descends from e_1, add no tree to Σ.
- If Π descends from e_2, we add the subtree referenced by e_1 to Σ.

- If Π descends from e_3, we add the subtree rooted at u to Σ, after removing e_3 and its subtree.

Suppose that the procedure adds t trees. Refer to those trees as $\mathcal{T}_1', \mathcal{T}_2', ..., \mathcal{T}_t'$. Denote by h_i the height of $\mathcal{T}_i'$ for each $i \in [t]$. Arrange the trees so that

$$h_1 \geq h_2 \geq ... \geq h_t.$$

We can now join all the trees together to obtain the 2-3 tree on S_1. To achieve $O(\log n)$ time, we do the joins in *descending* order of i:

1. **for** $i = t$ to 2
2. $\quad \mathcal{T}_{i-1}' \leftarrow$ the join of $\mathcal{T}_{i-1}'$ and $\mathcal{T}_i'$

The final $\mathcal{T}_1'$ is the 2-3 tree on S_1. The cost of all the joins is:

$$\sum_{i=1}^{t} O(1 + h_{i-1} - h_i) = O(t + h_1) = O(\log n).$$

2.3 Remarks

The BST and the 2-3 tree are covered in most textbooks on data structures, e.g., [14]. Their inventors are controversial; see `https://en.wikipedia.org/wiki/Segment_tree` and `https://en.wikipedia.org/wiki/Binary_search_tree`.

Exercises

Problem 1. Complete the proof of Lemma 2.3.

Problem 2 (range max). Let S be a set n people. We know the age and salary of each person in S. Design a data structure of $O(n)$ space to answer the following query in $O(\log n)$ time: find the maximum salary of all the people aged between x and y, where $x, y \in \mathbb{R}$.

Problem 3. Let S is a set of n real values. Given a count BST on S, explain how to answer following query in $O(\log n)$ time: given $k \in [1, n]$, find the k-th largest element in S.

Problem 4. Let $\mathcal{T}$ be a 2-3 tree on a set S of n real values. Given any $x \leq y$, describe an algorithm to obtain in $O(\log n)$ time a 2-3 tree on the set $S \setminus [x, y]$ (the set of elements in S outside $[x, y]$).

Problem 5* (meldable heap). Design a data structure of $O(n)$ space to store a set S of n real values to satisfy the following requirements:

- An element can be inserted to S in $O(\log n)$ time.
- The smallest element in S can be deleted in $O(\log n)$ time.
- Let S_1 and S_2 be disjoint sets of real values. Given a data structure (that you have designed) on S_1 and another on S_2, you can obtain a data structure on $S_1 \cup S_2$ in $O(\log(|S_1| + |S_2|))$ time. Note that here we do not have the constraint that the values in S_2 should be larger than those in S_1.

Problem 6. Modify the 2-3 tree into a *count 2-3 tree* that supports range counting in $O(\log n)$ time. Also explain how to maintain the count 2-3 tree in $O(\log n)$ time under insertions, deletions, (tree) splits, and joins.

Lecture 3: Structures for Intervals

In this lecture, we will discuss the *interval tree* and the *segment tree*, which are two different approaches to manage a set S of intervals. The ideas behind both approaches are also useful in designing structures on intervals, segments, rectangles, etc. (this will be clear later in the course). Set $n = |S|$. The interval tree uses $O(n)$ space, whereas the segment tree uses $O(n \log n)$ space. We will discuss the *stabbing query*: given a search value $q \in \mathbb{R}$, such a query returns all the intervals $\sigma \in S$ satisfying $q \in \sigma$. Both the interval tree and the segment tree answer a query in $O(\log n + k)$ time, where k is the number of intervals reported. Although the interval tree supersedes the segment tree for stabbing queries, there are other types of queries where the segment tree is more useful (we will see examples in the exercises).

3.1 The interval tree

In this section, we will consider that each interval in S has the form $[x, y]$ where x and y are real values.

3.1.1 Structure

Given an interval $[x, y]$, we call x and y its *left* and *right endpoints*, respectively. Denote by P the set of endpoints of the intervals in S. Create a BST $\mathcal{T}$ (Section 2.1) on P. For each node u in $\mathcal{T}$, define:

$$stab(u) \quad = \quad \{\sigma \in S \mid \ u \text{ is the } \textit{highest} \text{ node in } \mathcal{T} \text{ satisfying } key(u) \in \sigma.\}$$

We will refer to $stab(u)$ as the *stabbing set* of u. Store the intervals of $stab(u)$ in two lists: the first (resp. second) is sorted by left (resp. right) endpoint. This completes the construction of the interval tree.

Example. Let $S = \{[1, 2], [3, 7], [4, 12], [5, 9], [6, 11], [8, 15], [10, 14], \ [13, 16]\}$. Figure 3.1 shows a BST on $P = \{1, 2, ..., 16\}$. The stabbing set of node 9 is $\{[6, 11], [4, 12], [5, 9], [8, 15]\}$. The stabbing set of node 13, on the other hand, is $\{[10, 14], [13, 16]\}$. □

Each interval of S belongs to the stabbing set of *exactly one* node. The space consumption is therefore $O(n)$.

3.1.2 Query

To answer a stabbing query with search value q, we start from the root u of $\mathcal{T}$. Assume, w.l.o.g. due to symmetry, $q < key(u)$. We can ignore the intervals in the (stabbing sets of the nodes in the) right subtree of u because all those intervals $[x, y]$ must satisfy $x \geq key(u) > q$ and thus cannot

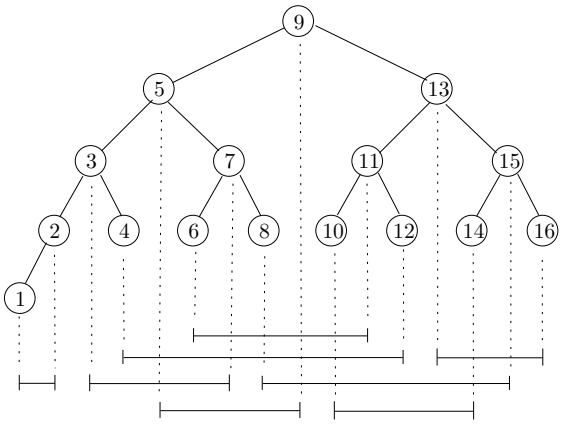


Figure 3.1: An interval tree

cover q. Searching the left subtree of u can be done by recursion because that subtree is an interval tree itself. It remains to explain how to report the intervals in $stab(u)$ that cover q. This can be done in $O(1 + k_u)$ time if there are k_u such intervals. Recall that all the intervals $[x, y] \in stab(u)$ *must* contain $key(u)$. Therefore, $[x, y]$ contains q if and only if $x \leq q$. Thus, we scan the intervals of $stab(u)$ in ascending order of *left* endpoint and stop as soon as finding an interval $[x, y]$ with $x > q$.

The discussion gives rise to the following algorithm. Descend a root-to-leaf path Π of $\mathcal{T}$ to reach the conceptual leaf (Section 2.1.2) whose slab covers q. For each node u on Π, if $q < key(u)$, scan $stab(u)$ in ascending order of left endpoint; otherwise, scan $stab(u)$ in ascending order of right endpoint. The query time is

$$\sum_{u \in \Pi} O(1 + k_u) \quad = \quad O(\log n + k)$$

noticing that every interval is reported at exactly one node on Π.

3.2 The segment tree

In this section, we will consider that each interval in S has the form $[x, y)$ (open on the right).

3.2.1 Structure

As before, create a BST $\mathcal{T}$ on P. Recall from Lemma 2.3 that every interval $\sigma \in S$ can be divided into $O(\log n)$ canonical intervals, each of which is the slab of a regular/conceptual node u in $\mathcal{T}$. We assign σ to every such u. Define S_u as the set of intervals assigned to u; we store S_u in a linked list. This finishes the construction of the segment tree.

Example. Consider $S = \{[1, 2), [3, 7), [4, 12), [5, 9), [6, 11), [8, 15), [10, 14), [13, 16)\}$. Figure 3.2 shows a BST on $P = \{1, 2, ..., 16\}$ with the conceptual leaves indicated. Interval $[4, 12)$, for example, is partitioned into canonical intervals $[4, 5), [5, 9), [9, 11), [11, 12)$ and hence is assigned to 4 nodes: the right conceptual leaf of node 4, node 7, node 10, and the left conceptual leaf of node 12. For u = node 10, $S_u = \{[4, 12), [6, 11)\}$. □

Since every interval in S has $O(\log n)$ copies, the total space is $O(n \log n)$.

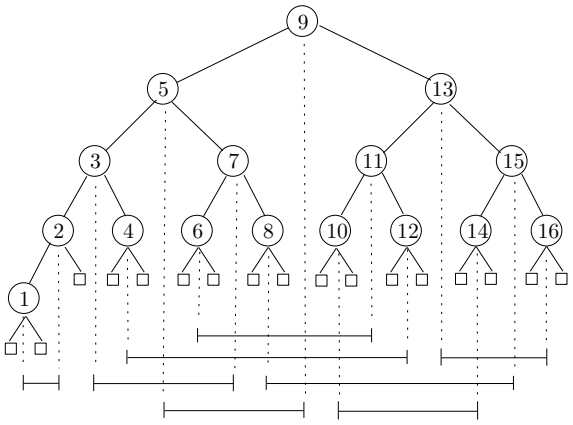


Figure 3.2: A segment tree (each box is a conceptual leaf)

3.2.2 Query

A stabbing query with search value q can be answered as follows:

1. Π = the set of regular/conceptual nodes whose slabs contain q
2. **for** each node $u \in \Pi$ **do**
3. report S_u

The proof of the next fact is left to you as an exercise:

Proposition 3.1. *The algorithm reports every $\sigma \in S$ covering q once and exactly once.*

The query cost is clearly $O(\log n + k)$.

3.3 Remarks

The interval tree was independently proposed by Edelsbrunner [17] and McCreight [31], while the segment tree is due to Bentley [6].

Exercises

Problem 1. Describe how to construct an interval tree on n intervals in $O(n \log n)$ time.

Problem 2. Describe how to construct a segment tree on n intervals in $O(n \log n)$ time.

Problem 3. Prove Proposition 3.1.

Problem 4. In Section 3.2, we assumed that each interval in S has the form $[x, y)$. Modify the segment tree to achieve the same performance guarantees on intervals of the form $[x, y]$.

Problem 5. Let S be a set of n intervals in $\mathbb{R}$. Design a structure of $O(n)$ space to answer the following query efficiently: given an interval $q = [x, y]$ in $\mathbb{R}$, report all the intervals $\sigma \in S$ such that $\sigma \cap q \neq \emptyset$. Your query time needs to be $O(\log n + k)$, where k is the number of reported intervals.

Problem 6 (stabbing max). Let S be a set of n intervals, each associated with a real-valued *weight*. Given a value $q \in \mathbb{R}$, a *stabbing max* query returns the interval with the largest weight (you can assume that all the weights are distinct). Design a structure of $O(n)$ space to answer any such query in $O(\log n)$ time.

(Hint: segment tree.)

Problem 7 (2D stabbing max). Let S be a set of n axis-parallel rectangles in $\mathbb{R}^2$ (i.e., each rectangle in S has the form $[x_1, x_2] \times [y_1, y_2]$). Each rectangle $r \in S$ is associated with a real-valued *weight*. Describe a structure of $O(n \log n)$ space that answers the following query in $O(\log^2 n)$ time: given a point $q \in \mathbb{R}^2$, report the maximum weight of the rectangles $r \in S$ satisfying $q \in r$.

(Hint: build a segment tree on x-coordinates; for each node u, build another segment tree on S_u.)

Problem 8. Let S be a set of n horizontal segments of the form $[x_1, x_2] \times y$ in $\mathbb{R}^2$. Given a vertical segment $q = x \times [y_1, y_2]$, a query reports all the segments $\sigma \in S$ that intersect q. Design a data structure to store S in $O(n \log n)$ space such that every query can be answered in $O(\log^2 n + k)$ time, where k is the number of segments reported.

Lecture 4: Structures for Points

This lecture will discuss several structures on points in $\mathbb{R}^d$ where the dimensionality $d \geq 2$ is a constant. Our discussion will focus on $d = 2$, while in the exercises you will be asked to obtain structures of higher dimensionalities.

Central to our discussion is *orthogonal range reporting*. Let S be a set of points in $\mathbb{R}^d$. Given an axis-parallel rectangle $q = [x_1, y_1] \times [x_2, y_2] \times ... \times [x_d, y_d]$, an *orthogonal range reporting* query returns $q \cap S$. The structures in this lecture will provide different tradeoffs between space and query time. For simplicity, we will assume that the points of S are in *general position*: no two points in S have the same x- or y-coordinate. This assumption allows us to focus on the most important ideas, and can be easily removed with standard tie breaking techniques (as we will see in an exercise). For simplicity, we will refer to a query simply as a "range query".

4.1 The kd-tree

This structure stores S in $O(n)$ space and answers a 2D range query in $O(\sqrt{n} + k)$ time, where k is the number of points in $q \cap S$.

4.1.1 Structure

We describe the kd-tree in a recursive manner.

$n = 1$. If S has only a single point p, the kd-tree has only a single node storing p.

$n \geq 2$. Let ℓ be a *vertical* line that divides P as evenly as possible, that is, there are at most $\lceil n/2 \rceil$ points of P on each side of ℓ. Create a node u which stores ℓ (i.e., the x-coordinate of ℓ). Let P_1 (resp., P_2) be the set of points in P that are on the left (resp., right) of ℓ.

Consider P_1. If $|P_1| = 1$, create a left child v_1 of u storing the only point in P_1. Next, we assume $|P_1| \geq 2$. Let ℓ_1 be a *horizontal* line that divides P_1 as evenly as possible. Create a left child v_1 of u storing ℓ_1. Let P_{11} (resp., P_{12}) be the set of points in P_1 that are below (resp., above) of ℓ_1. Recursively, create a kd-tree $\mathcal{T}_{11}$ on P_{11} and a kd-tree $\mathcal{T}_{12}$ on P_{12}. Make $\mathcal{T}_{11}$ and $\mathcal{T}_{12}$ the left and right subtrees of v_1, respectively.

The processing of P_2 is similar. If $|P_2| = 1$, create a right child v_2 of u storing the only point in P_2. Otherwise, let ℓ_2 be a horizontal line that divides P_2 as evenly as possible. Create a right child v_2 of u storing ℓ_2. Let P_{21} (resp., P_{22}) be the set of points in P_2 that are below (resp., above) of ℓ_2. Recursively, create a kd-tree $\mathcal{T}_{21}$ on P_{21} and a kd-tree $\mathcal{T}_{22}$ on P_{22}. Make $\mathcal{T}_{21}$ and $\mathcal{T}_{22}$ the left and right subtrees of v_2, respectively.

The kd-tree is a binary tree where every internal node has two children. The points of S are stored at the leaves. The total number of nodes is $O(n)$.

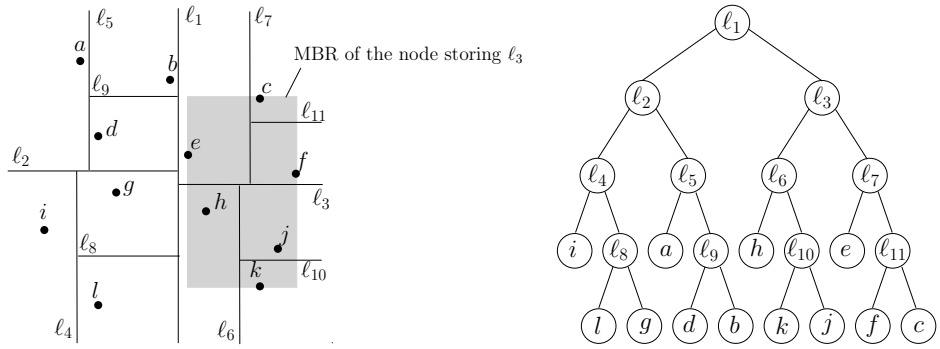


Figure 4.1: A kd-tree

For each node u in the tree, we store its *minimum bounding rectangle* (MBR) which is the *smallest* axis-parallel rectangle covering all the points in $sub(u)$. Note that the MBR of an internal node u can be obtained from those of its children in $O(1)$ time.

Example. Figure 4.1 shows a kd-tree on 12 points. The shaded rectangle illustrates the MBR of the node storing ℓ_3. □

4.1.2 Range reporting

Let $\mathcal{T}$ be a kd-tree on S. A range query is answered by visiting all the nodes in $\mathcal{T}$ whose MBRs intersect with the search rectangle q. At a leaf, we report the point p stored there if $p \in q$. We will prove that the query cost is $O(\sqrt{n} + k)$. For this purpose, we divide the nodes u accessed into two categories:

- type 1: the MBR of u intersects with a boundary edge of q;
- type 2: the MBR of u is fully contained in q.

We will prove that there are $O(\sqrt{n})$ nodes of type 1. In an exercise, you will be asked to prove that the number of type-2 nodes is bounded by $O(k)$. It will then follow that the query cost is $O(\sqrt{n} + k)$.

Lemma 4.1. *Any vertical line ℓ can intersect with the MBRs of $O(\sqrt{n})$ nodes.*

Proof. It suffices to prove the lemma for the case where n is a power of 2 (think: why?). Fix any ℓ. We say that a node is *ℓ-intersecting* if its MBR intersects with ℓ. Let $f(n)$ be the maximum number of ℓ-intersecting nodes in any kd-tree storing n points.

Now consider the kd-tree $\mathcal{T}$ we constructed on S. Let $\hat{u}$ be the root of $\mathcal{T}$; recall that $\hat{u}$ stores a vertical line ℓ_1. Due to symmetry, assume that ℓ is on the right of ℓ_1. Denote by u the right child of $\hat{u}$; note that the line ℓ_2 stored in u is horizontal. Let v_1 and v_2 be the left and right child nodes of u, respectively. See Figure 4.2 for an illustration.

The ℓ-intersecting nodes in $\mathcal{T}$ consists of $\hat{u}$, u, and the ℓ-intersecting nodes in $sub(v_1)$ and $sub(v_2)$. Since each of $sub(v_1)$ and $sub(v_2)$ contains $n/4$ points, we have:

$$f(n) \leq 2 + 2 \cdot f(n/4).$$

Solving the recurrence gives $f(n) = O(\sqrt{n})$. □

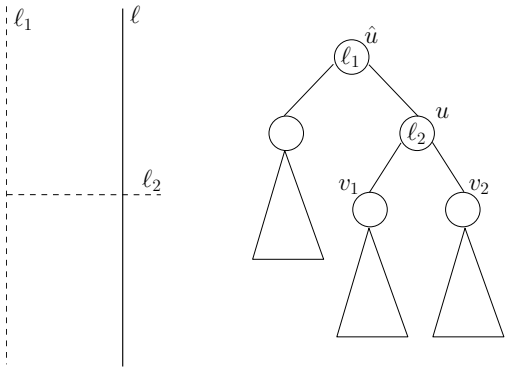


Figure 4.2: Proof of Lemma 4.1

An analogous argument shows that any *horizontal* line can intersect with the MBR of $O(\sqrt{n})$ nodes, too. The MBR of any type-1 node must intersect with at least one of the following 4 lines: the two vertical lines passing the left and right edges of q, and the two horizontal lines passing the lower and upper edges of q. It thus follows that there can be $O(\sqrt{n})$ type-1 nodes.

4.2 Bootstrapping

This section will present a technique to obtain a structure that uses $O(n)$ space, and answers any range query in $O(n^\epsilon + k)$ time, where $\epsilon > 0$ can be any small constant (for the kd-tree, $\epsilon = 1/2$). There are several ways to achieve the purpose. We will discuss an approach that illustrates an interesting *bootstrapping* phenomenon.

Lemma 4.2. *Suppose that a structure Υ can store n points in $\mathbb{R}^2$ in at most $F(n)$ space and answer a range query in at most $Q(n) + O(k)$ time. For any integer $\lambda \in [2, n/2]$, there exists a structure that uses at most $\lambda \cdot F(\lceil n/\lambda \rceil) + O(n)$ space and answers a range query in at most $2 \cdot Q(\lceil n/\lambda \rceil) + \lambda \cdot O(\log(n/\lambda)) + O(k)$ time.*

Proof. Let S be the set of n points. Find $\lambda - 1$ vertical lines $\ell_1, ..., \ell_{\lambda-1}$ to satisfy the following requirements:

- No point of S falls on any line.
- Let $x_1, ..., x_{\lambda-1}$ be the x-coordinates of $\ell_1, ..., \ell_{\lambda-1}$, respectively. Define λ *slabs* as follows:
  - Slab 1 includes all the points of $\mathbb{R}^2$ with x-coordinates less than x_1;
  - Slab $i \in [2, \lambda - 1]$ includes all the points of $\mathbb{R}^2$ with x-coordinates in $[x_{i-1}, x_i)$;
  - Slab λ includes all the points of $\mathbb{R}^2$ with x-coordinates at least $x_{\lambda-1}$.

  We require that S should have at most $\lceil n/\lambda \rceil$ points in each slab.

For each $i \in [1, \lambda]$, define S_i to be the set of points in S covered by slab i. For each S_i, create two structures:

- The structure Υ (as stated in the lemma) on S_i; represent the structure as $\mathcal{T}_i$.
- A BST B_i on the y-coordinates of the points in S_i.

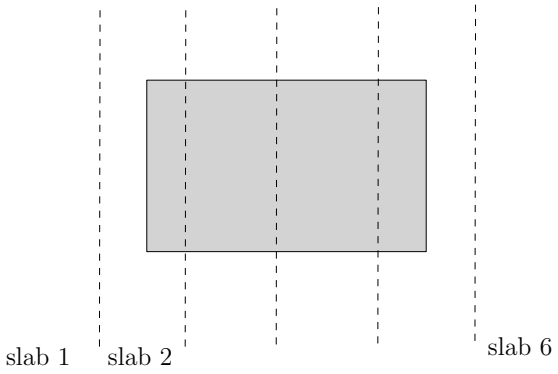


Figure 4.3: Proof of Lemma 4.2

The space consumption is $\lambda \cdot F(\lceil n/\lambda \rceil) + O(n)$.

Let us now discuss how to answer a range query with search rectangle q. If q falls entirely in some slab $i \in [1, \lambda]$, we answer the query using $\mathcal{T}_i$ directly in $Q(\lceil n/\lambda \rceil) + O(k)$ time.

Consider now the case where q intersects at least two slabs. Denote by q_i the intersection of q with slab i, for each $i \in [1, \lambda]$. Each q_i is one of the following types:

- type 1: empty, i.e., q is disjoint with slab i;
- type 2: the x-range of q_i equals that of slab i (i.e., the x-range of q spans that of slab i);
- type 3: the x-range of q_i is non-empty but is shorter than that of slab i.

Figure 4.3 shows an example where q is the shaded rectangle and $\lambda = 6$. Rectangles q_1 and q_6 are of type 1, q_3 and q_4 are of type 2, while q_2 and q_5 are of type 3.

For type 1, we do nothing. For type 3, we deploy $\mathcal{T}_i$ to find $q_i \cap S_i$ in $Q(\lceil n/\lambda \rceil) + O(k_i)$ time, where $k_i = |q_i \cap S_i|$. There can be at most two rectangles of type 3; so we spend at most $2 \cdot Q(\lceil n/\lambda \rceil) + O(k)$ time.

For each type-2 rectangle q_i, we can ignore the x-dimension: a point $p \in S_i$ falls in q_i if and only if the y-coordinate of p is covered by the y-range of q_i. We can therefore find all the points of $q_i \cap S_i$ using B_i in $O(\log(n/\lambda) + k_i)$ time. As there can be λ rectangles of Type 2, we end up spending at most $\lambda \cdot O(\log(n/\lambda)) + O(k)$ time. □

The above lemma is *bootstrapping* because once we have obtained a structure for range reporting, it may allow us to *improve* ourselves "automatically". For example, with the kd-tree, we have already achieved $F(n) = O(n)$ and $Q(n) = O(\sqrt{n})$. Thus, by Lemma 4.2, for any $\lambda \in [2, n/2]$ we immediately have a structure of $\lambda \cdot F(\lceil n/\lambda \rceil) = O(n)$ space whose query time is

$$O(\sqrt{n/\lambda}) + \lambda \cdot O(\log n)$$

plus the linear output time $O(k)$. Setting λ to $\Theta(n^{1/3})$ makes the query time $O(n^{1/3} \log n + k)$; note that this is a polynomial improvement over the kd-tree. But we can do even better! Now that we have achieved $F(n) = O(n)$ and $Q(n) = O(n^{1/3} \log n)$, for any $\lambda \in [2, n/2]$ Lemma 4.2 yields another structure of $O(n)$ space with query time

$$\tilde{O}((n/\lambda)^{1/3}) + \lambda \cdot O(\log n)$$

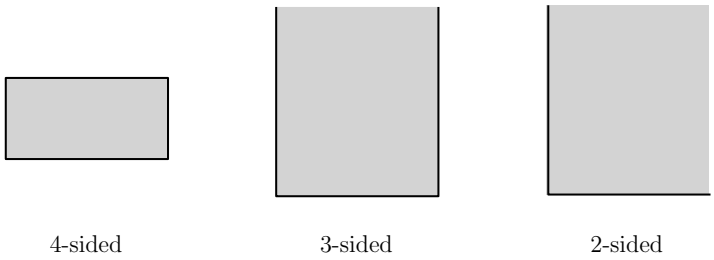


Figure 4.4: Different types of axis-parallel rectangles

plus the linear output time $O(k)$. Setting λ to $\Theta(n^{1/4})$ makes the query time $\tilde{O}(n^{1/4}) + O(k)$, thus achieving another polynomial improvement!

Repeating this roughly $1/\epsilon$ times produces a structure of $O(n/\epsilon) = O(n)$ space and query time $O(n^{\epsilon} + k)$, where ϵ can be any positive constant.

4.3 The priority search tree

A 2D range query is *4-sided* because the query rectangle q is "bounded" on all sides. If we write $q = [x_1, x_2] \times [y_1, y_2]$, all the values x_1, x_2, y_1, and y_2 are finite. Such queries are difficult: no linear-size structures known today can guaranteed a query time of $O(\log n + k)$.

If one of x_1, x_2, y_1, and y_2 is infinite (either $-\infty$ or ∞), q is said to be *3-sided*. More specially, if (i) two of x_1, x_2, y_1, and y_2 are infinite and (ii) those two values are on different dimensions, q is said to be *2-sided*. See Figure 4.4 for an illustration.

We will introduce a structure called the *priority search tree* which uses linear space and answers a 3-sided query in $O(\log n + k)$ time, where k is the number of points reported.

4.3.1 Structure

Due to symmetry, we consider that q has the form $[x_1, x_2] \times [y, \infty)$. Given a point $p \in \mathbb{R}^2$, we denote by x_p and y_p its x- and y-coordinate, respectively.

To build a priority search tree on S, first create a BST $\mathcal{T}$ on the x-coordinates of the points in S. Each regular/conceptual node u in $\mathcal{T}$ stores a *pilot point* — denoted as $pilot(u)$ — defined recursively as follows:

- If $u = root(\mathcal{T})$, $pilot(u)$ is the *highest* point in S.
- Otherwise, $pilot(u)$ is the highest one among all the points $p \in S$ satisfying
  - $x_p \in slab(u)$,[1] and
  - p is not the pilot point of any proper ancestor of u.

  If no such p exists, $pilot(u)$ is empty.

This finishes the construction of the priority search tree. Every point in S is the pilot point of *exactly* one node (which can be conceptual). The space is $O(n)$.

[1] See Section 2.1.2 for the definition of slab.

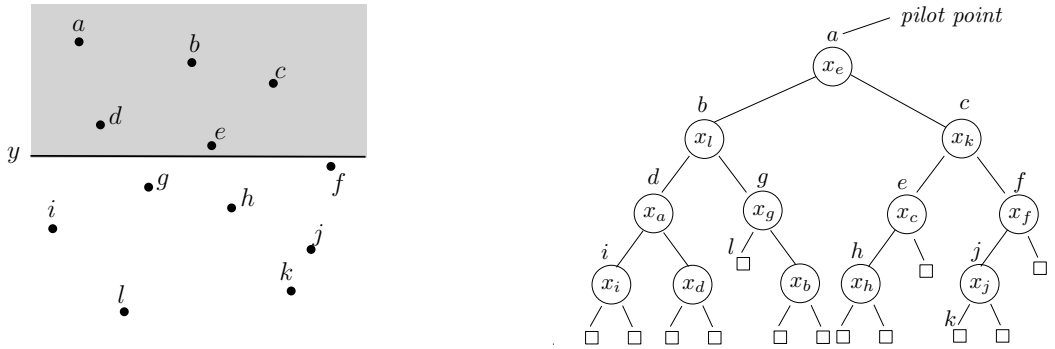


Figure 4.5: A priority search tree

Example. Figure 4.5 shows a priority search tree on the point set $\{a, b, ..., l\}$. □

Remark. Observe that the priority search tree is simultaneously a *max heap* on the y-coordinates of the points in S. For this purpose, the structure is also known by the name *treap*.

4.3.2 Answering a 3-sided query

Let us first consider a (very) special query: the search rectangle q has the form $(-\infty, \infty) \times [y, \infty)$ (a "1-sided" rectangle). Equivalently, this is to ask how we can use the priority search tree to report all the points in S whose y-coordinates are at least y.

Lemma 4.3. *Given a search rectangle $q = (-\infty, \infty) \times [y, \infty)$, we can find all the points in $S \cap q$ in $O(1 + k)$ time, where $k = |S \cap q|$.*

Proof. We answer the query using the following algorithm (setting u to the root of $\mathcal{T}$ initially):

report-subtree(u, y)
/* u is a regular/conceptual node in $\mathcal{T}$ */
1. **if** u has no pilot point **or** its pilot point p has y-coordinate $< y$ **then return**
2. report p
3. **if** u is a conceptual leaf **then return**
4. **report-subtree**(v_1, y) where v_1 is the left child of u (v_1 is possibly conceptual)
5. **report-subtree**(v_2, y) where v_2 is the right child of u (v_2 is possibly conceptual)

The correctness follows from the fact that *pilot*(u) is the *highest* among all the pilot points stored in *sub*(u). Next, we analyze the query cost. Each node u visited can be divided into two types: (1) *pilot*$(u) \in q$ and (2) *pilot*$(u) \notin q$. There are exactly k type-1 nodes. As the parent of a type-2 node must be of type 1, the number of type-2 nodes is at most $2k$. The total cost is therefore $O(1 + k)$. □

Example. If q is the shaded region in Figure 4.5, the query algorithm visits nodes $x_e, x_l, x_a, x_i, x_d, x_g$, x_k, x_c, x_h, and x_f. □

We are ready to explain how to answer a general 3-sided query with $q = [x_1, x_2] \times [y, \infty)$. W.l.o.g., we can assume that x_1 and x_2 are the x-coordinated of some points in S (think: why?). Find

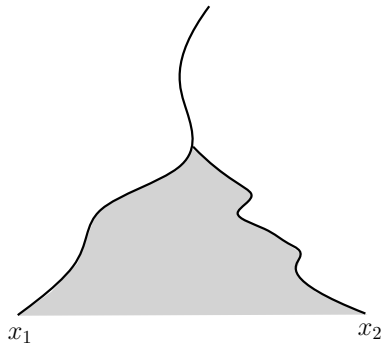


Figure 4.6: Search paths Π_1 and Π_2 and the portion in between

- the path Π_1 in $\mathcal{T}$ from the root to the conceptual leaf whose slab covers x_1;
- the path Π_2 in $\mathcal{T}$ from the root to the conceptual leaf whose slab covers x_2.

Figure 4.6 illustrates how Π_1 and Π_2 look like in general: they descend from the root and diverge at some node. We are interested only in the nodes u that

- are in $\Pi_1 \cup \Pi_2$, or
- satisfy $slab(u) \subseteq [x_1, x_2]$ (such are in the shaded portion in Figure 4.6).

For every other node v (violating both conditions above), $slab(v)$ must be disjoint with $[x_1, x_2]$; and therefore, $pilot(v)$ must be outside q. This motivates the following the query algorithm:

1. find Π_1 and Π_2
2. **for** every node $u \in \Pi_1 \cup \Pi_2$ **do**
3. report $pilot(u)$ if $pilot(u) \in q$
4. find $\Sigma = \{\text{node } u \mid slab(u) \text{ is a canonical slab of } [x_1, x_2]\}$
5. **for** every node $u \in \Sigma$ **do**
6. **report-subtree**(u, y)

For every node $u \in \Sigma$, Line 5 reports all the qualifying pilot points in $sub(u)$ because (i) $sub(u)$ is a max heap and (ii) we can ignore the x-range $[x_1, x_2]$ of q in exploring $sub(u)$. By Lemma 4.3, the cost of **report-subtree**(u, y) is $O(1 + k_u)$ where k_u is the number of points reported from $sub(u)$. The total query cost is therefore bounded by

$$O\left(|\Pi_1| + |\Pi_2| + \sum_{u \in \Sigma}(1 + k_u)\right) = O(\log n + k).$$

The filtering technique. Given a query time complexity such as $O(\log n + k)$, we would often interpret the $O(\log n)$ term as the "search time" and the $O(k)$ term as the "report time". By rewriting $O(\log n + k)$ as $O(\log n + k + k)$, we can allow ourselves a higher search time of $O(\log n + k)$! Indeed, this is the case with the priority search tree: notice that the algorithm may access $O(\log n + k)$ nodes without reporting anything. The idea of charging the search time on the output is generally known as *filtering*.

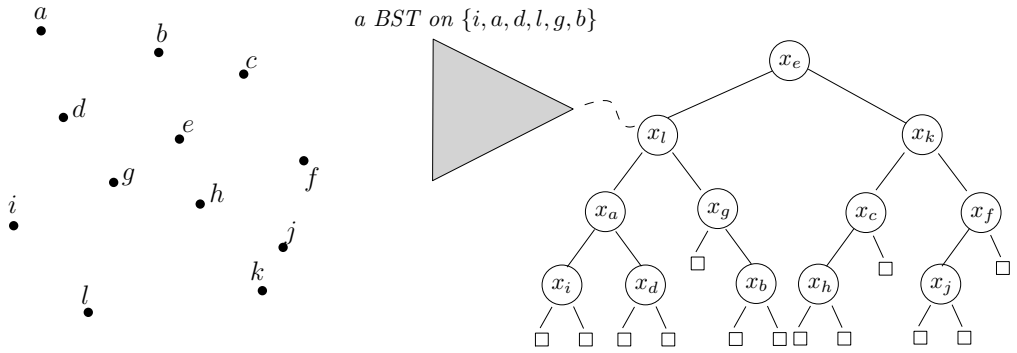


Figure 4.7: A range tree (the shaded triangle illustrates the secondary BST of node x_l)

4.4 The range tree

Returning to 4-sided queries, we will introduce the *range tree* which consumes $O(n \log n)$ space and answers a query in $O(\log^2 n + k)$ time.

4.4.1 Structure

Build a BST $\mathcal{T}$ on the x-coordinates of the points in S. For each regular/conceptual node u in $\mathcal{T}$, denote by S_u the set of points $p \in S$ with $x_p \in slab(u)$ (recall that x_p is the x-coordinate of p). Associate u with a *secondary* BST $\mathcal{T}_u$ on the y-coordinates of the points in S_u. Every $p \in S_u$ is stored at the node in $\mathcal{T}_u$ whose key equals y_p.

Example. Figure 4.7 shows the BST $\mathcal{T}$ for the point set $\{a, b, ..., l\}$. If u is the node x_l, $S_u = \{i, a, d, l, g, b\}$. The secondary BST of u is created on those points' y-coordinates. Point b is stored in the secondary BSTs of the right conceptual child of node x_b, node x_b itself, node x_g, node x_l, and node x_e. □

Proposition 4.4. *For each $p \in S$, x_p appears in the slabs of $O(\log n)$ nodes.*

Proof. By Proposition 2.1, if the slabs of two nodes u and v in $\mathcal{T}$ intersect, one of u and v must be an ancestor of the other. Thus, all the nodes whose slabs contain x_p must be on a single root-to-leaf path in $\mathcal{T}$. The proposition follows from the fact that the height of $\mathcal{T}$ is $O(\log n)$. □

The space consumption is therefore $O(n \log n)$.

4.4.2 Range reporting

We answer a range query $q = [x_1, x_2] \times [y_1, y_2]$ as follows (assuming w.l.o.g. that x_1 and x_2 are the x-coordinates of some points in S):

1. find the set Σ of nodes in $\mathcal{T}$ whose slabs are the canonical slabs of $[x_1, x_2]$
2. **for** each node $u \in \Sigma$ **do**
3. use $\mathcal{T}_u$ to report $\{p \in S_u \mid y_p \in [y_1, y_2]\}$

Proposition 4.5. *Every point p in $q \cap S$ is reported exactly once.*

Proof. Clearly, $x_p \in [x_1, x_2]$. Therefore, x_p appears in *exactly* one canonical slab of $[x_1, x_2]$ (by Lemma 2.3, the canonical slabs form a partition of $[x_1, x_2]$). Let u be the node whose $slab(u)$ is that canonical slab. Thus, $p \in S_u$ and will be reported *only* there. □

The proof of the next proposition is left to you as an exercise:

Proposition 4.6. *The query time is* $O(\log^2 n + k)$.

4.5 Pointer-machine structures

A *pointer machine structure* is a directed graph G satisfying the following conditions:

- There is a special node r in G called the *root*.
- Every node in G stores a constant number of words.
- Every node in G has a constant number of outgoing edges (but may have an arbitrary number of incoming edges).
- Any algorithm that accesses G must follow the rules below:
  - The first node visited must be r.
  - The algorithm is permitted to access a non-root node u in G only if it has already accessed an in-neighbor of u. This implies that the algorithm must have found a path from r to u in G.

All the structures discussed so far are pointer-machine structures. A simple example of a non-pointer-machine structure is the array. Given an array A of size n, we can access directly $A[i]$ for any $i \in [1, n]$ in constant time, without following any "path".

Pointer-machine structures bear unique importance in computer science because they are applicable in scenarios where it is not possible to perform any meaningful calculation on *addresses*. One such scenario arises from distributed computing where each "node" is a machine (e.g., your cell phone). A pointer to a node u is the IP address of machine u. No "arrays" can be implemented in such a scenario because, to enable constant time access to $A[i]$, you need to calculate the address of $A[i]$ by adding i to the starting address of A — something not possible in distributed computing (adding i to an IP address tells you essentially nothing).

4.6 Remarks

The kd-tree was first described by Bentley [5]. The priority search was invented by McCreight [32]. The range tree was independently developed by several works that appeared almost the same time, e.g., [7, 28, 29, 42].

Range reporting on pointer machines has been well understood. In 2D space, any pointer-machine structures achieving $O(\text{polylog}\, n + k)$ query time — let alone $O(\log n + k)$ — must consume $\Omega(n \frac{\log n}{\log \log n})$ space [13]. A structure matching this lower bound and attaining $O(\log n + k)$ query time has been found [11]. Similar results also hold for higher dimensionalities, where the space and query complexities increase by $O(\text{polylog}\, n)$ factors; see [1, 13].

By leveraging the power of the RAM model (address calculation and manipulating bits within a word), it is possible to design structures with better complexities *outside* the pointer-machine class. For example, in 2D space, it is possible to achieve $O(\log n + k)$ time using $O(n \log^{\epsilon} n)$ space, where $\epsilon > 0$ can be any small constant [2, 12]. See also [10] for results of higher dimensionalities.

Exercises

Problem 1. Prove that there can be $O(k)$ nodes of Type 2 (as defined in Section 4.1.2).

Problem 2. Describe an algorithm to build the kd-tree on n points in $O(n \log n)$ time.

Problem 3. Explain how to remove the general position assumption for the kd-tree. That is, you still need to retain the same space and query complexities even if the assumption does not hold.

Problem 4. Let S be a set of points in $\mathbb{R}^d$ where $d \geq 2$ is a constant. Extend the kd-tree to obtain a structure of $O(n)$ space that answers any d-dimensional range reporting query in $O(n^{1-1/d} + k)$ time, where k is the number of points reported.

Problem 5. What is the counterpart of Lemma 4.2 in 3D space?

Problem 6*. Improve the query time in Lemma 4.2 to $2 \cdot Q(\lceil n/\lambda \rceil) + O(\log n + \lambda + k)$.

(Hint: one way to do so is to use the interval tree and stabbing queries.)

Problem 7. Consider the stabbing query discussed in Lecture 3 on a set S of n intervals in $\mathbb{R}$. Show that you can store S in a priority search tree such that any stabbing query can be answered in $O(\log n + k)$ time, where k is the number of intervals reported.

(Hint: turn the query into a 2-sided range reporting query on a set of n points converted from S.)

Problem 8. Prove Proposition 4.6.

Problem 9. Let S be a set of points in $\mathbb{R}^d$ where d is a constant. Design a data structure that stores S in $O(n \log^{d-1} n)$ space, and answers any orthogonal range reporting query on S in $O(\log^d n + k)$ time, where k is the number of reported points.

Problem 10 (range counting). Let S be a set of n points in $\mathbb{R}^2$. Given an axis-parallel rectangle q, a *range count* query reports $|q \cap S|$, i.e., the number of points in S that are covered by q. Design a structure that stores S in $O(n \log n)$ space, and answers a range count query in $O(\log^2 n)$ time.

Problem 11*. Let S be a set of n horizontal segments of the form $[x_1, x_2] \times y$ in $\mathbb{R}^2$. Given a vertical segment $q = x \times [y_1, y_2]$, a query reports all the segments $\sigma \in S$ that intersect q. Design a data structure to store S in $O(n)$ space such that every query can be answered in $O(\log^2 n + k)$ time, where k is the number of segments reported. (This improves an exercise in Lecture 3.)

(Hint: use the interval tree as the base tree, and the priority search tree as secondary structures.)

Problem 12. Prove: on a pointer-machine structure G with n nodes, the longest path from the root to a node in G has length $\Omega(\log n)$. (This implies that $O(\log n + k)$ is the best query bound one can hope for range reporting using pointer-machine structures.)

(Hint: suppose that each node has an outdegree of 2. Starting from the root, how many nodes can you reach within x hops?)

Lecture 5: Global Rebuilding and Charging Arguments

All the structures we have seen — except the BST and 2-3 trees — so far are *static*, namely, they do not support insertions and deletions on the underlying set of elements. In general, a structure is *semi-dynamic* if it allows insertions but not deletions, or *fully dynamic* if it allows both. We will devote several lectures to generic techniques that can be used to turn a static structure into a semi/fully-dynamic one. Today, we will learn *global rebuilding*, which is a simple idea at the core of many other dynamization techniques (e.g., the ones in the next two lectures). We will also learn about *charging arguments*, a powerful method to analyze amortized cost.

5.1 Amortized cost

Recall that the BST supports *every* update on a set S of values in $O(\log n)$ time where n is the size of S at the moment the update is performed. Next, we will introduce a weaker type of guarantees known as *amortized* bounds.

Given a function $U : \mathbb{N} \to \mathbb{N}$, we say that a semi-dynamic structure has *amortized insertion cost* $U(n)$ if it can process any sequence of m insertions in $\sum_{i=1}^{m} U(|S_i|)$ time, where S_i the set of elements before the i-th insertion. We can also extend the above notions to a fully dynamic structure. Let U_1 and U_2 be functions from $\mathbb{N}$ to $\mathbb{N}$. We say that the structure has *amortized insertion cost* U_1 and *amortized deletion cost* U_2 if it can process any mixed sequence of m updates (each can independently be an insertion or deletion) in

$$\sum_{i=1}^{m} U_{op_i}(|S_i|)$$

where S_i the set of elements before the i-th update and $op_i = 1$ if the i-th update is insertion, or 2, otherwise.

More generally, consider a structure that supports $\ell \geq 1$ operations, labeled as $1, 2, ..., \ell$, respectively. Let U_i ($i \in [\ell]$) be a function $\mathbb{N}$ to $\mathbb{N}$. We say that the structure has *amortized* cost U_i on operation i for each $i \in [\ell]$ if it can process a sequence of m arbitrarily mixed operations in $\sum_{j=1}^{m} U_{op_j}(|S_j|)$ time, where S_j the set of elements before the j-th operation, and $op_j = i$ if the j-th operation has label i.

A per-update bound implies an amortized bound but not the vice versa. For example, since the BST supports every update in $O(\log n)$ time, it also guarantees an $O(\log n)$ amortized update time. On the other hand, even if a structure can ensure $O(\log n)$ amortized time, it does not necessarily handle every update in $O(\log n)$ time. Indeed, an amortized bound of $O(\log n)$ implies a bound only on the total time of any m operations, rather than a bound on every operation.

5.2 Charging arguments

Suppose that a structure needs to support m operations $op_1, op_2, ..., op_m$. Let C_i be the cost of op_i for $i \in [m]$. To argue for a small amortized bound, we need to prove that $\sum_{i=1}^{m} C_i$ is small. The rationale behind a charging argument is to assign a "fake cost" $\bar{C}_i$ to each op_i, which will serve as the operation's *amortized* cost. The assignment must guarantee:

$$\sum_{i=1}^{m} C_i \quad \leq \quad \sum_{i=1}^{m} \bar{C}_i. \tag{5.1}$$

If we can prove every $\bar{C}_i$ is small, then we obtain an amortized bound on the whole operation sequence. For example, if $\bar{C}_i = O(\log |S_i|)$ where S_i the set of elements before the i-th operation, then (5.1) yields an $O(\log n)$ amortized bound.

A charging argument works by setting all $\bar{C}_1, ..., \bar{C}_m$ to 0 in the beginning and then increasing them gradually as we process the sequence. At each operation $i \in [m]$, we will break the cost C_i into small portions and *charge* each portion on an appropriate $j \leq i$ (namely, adding the portion to $\bar{C}_j$). The charging must be done judiciously to make sure all $\bar{C}_1, ..., \bar{C}_m$ remain small in the end.

5.3 Dynamic arrays

An array A of size s is a sequence of s consecutive memory cells. In an operating system, accesses to A are limited to the cells allocated (e.g., reading $A[s+1]$ will incur a "segmentation fault" under Linux). For this reason, the array size is often considered fixed.

Next, we will partially remedy this drawback. We want to design a *dynamic array* that stores a set S of n elements under insertions. The new structure should satisfy the following requirements:

- The elements in S must be stored in n consecutive cells. Furthermore, the i-th cell stores the i-th inserted element, for each $i \in [n]$.
- The structure uses $O(n)$ space.
- The structure supports insertions in $O(1)$ amortized time.

Initially, $n = 0$ and A is empty. We perform each insertion as follows:

insert (e)
1. $n \leftarrow n + 1$
2. **if** n is a power of 2 **then**
3. create a new array A' of length $2n$
4. copy all the elements of A into A'
5. destroy A and $A \leftarrow A'$
6. $A[n] = e$

It is obvious that A fulfills our first two requirements (A has size at most $2n$ at all times). Next, we will analyze the insertion cost. If n is not a power of 2, an insertion finishes in constant time; otherwise, an insertion takes $O(n)$ time. Thus, the total cost of n insertions is $O(\sum_{i=1}^{n} 1) + O(2^1 + 2^2 + ... + 2^{\lfloor \log_2 n \rfloor}) = O(n)$. Therefore, the structure guarantees $O(1)$ amortized insertion cost.

Next, we will see how to use the charging argument to arrive at the same conclusion. At receiving the i-th insertion $i \in [n]$, we initialize its amortized cost $\bar{C}_i$ to be 0. If i is not a power

of 2, we charge constant time to $\bar{C}_i$ — which now becomes $O(1)$ — to account for the insertion's cost. Now consider i to be a power of 2. The insertion needs to perform "global rebuilding" (Lines 3-6) which requires $O(i)$ cost. We charge the cost on $i/2$ insertions: insertion j for $j \in [i/2+1, i]$. This way, each insertion j is amortized a cost of $O(\frac{i}{i/2}) = O(1)$ and hence $\bar{C}_j$ increases by $O(1)$.

With the above charging strategy, we have accounted for the cost of all the insertions. As each insertion $i \in [n]$ is charged for only one global rebuilding (think: why?), the final $\bar{C}_i = O(1)$. We thus conclude that our structure ensures $O(1)$ amortized insertion time.

Exercises

Problem 1 (stacks with dynamic arrays). Implement a stack with the following requirements:

- At any moment, store the stack in an array A where $A[i]$ is the i-th least recently inserted among all the elements remaining in the stack.
- The space is $O(n)$ where n is the current stack size.
- Each stack operation (i.e., push and pop) is performed in $O(1)$ amortized time.

Problem 2 (Exercise 17.3-7 of [14]). Suppose that we want to implement the following two operations on a set S of integers (S is empty at the beginning):

- Insert(e): Add a new integer e into S (you are assured that e is not already in S).
- Delete-Half: Delete the $\lceil |S|/2 \rceil$ smallest elements from S.

Describe a data structure that consumes $O(|S|)$ space, and supports each operation in $O(\log |S|)$ time amortized.

Problem 3* (Priority Queue with Attrition). Let S be a dynamic set of integers. At the beginning S is empty. We want to support the following operations:

- `Insert-with-Attrition`(e): First removes all integers in S that are greater than e, and then adds e to S.
- `Delete-Min`: Removes and returns the smallest integer of S.

For example, suppose we perform the following sequence of operations:

1. `Insert-with-Attrition`(83)
2. `Insert-with-Attrition`(5)
3. `Insert-with-Attrition`(10)
4. `Insert-with-Attrition`(15)
5. `Insert-with-Attrition`(12)
6. `Delete-Min`
7. `Delete-Min`

After Operation 3, $S = \{5, 10\}$ (note that 83 has been deleted by Operation 2). After Operation 5, $S = \{5, 10, 12\}$. After Operation 6, $S = \{10, 12\}$.

Describe a data structure with the following guarantees:

- At all times, the space consumption is $O(|S|)$.
- Any sequence of n operations (each being an `insert-with-attrition` or `delete-min`) is processed with $O(n)$ time, i.e., $O(1)$ amortized time per operation.

Lecture 6: The Logarithmic Method

Today, we will learn a technique called the *logarithmic method* for turning a static structure semi-dynamic. We will use the kd-tree (Section 4.1) to illustrate the technique. Indeed, the kd-tree serves as an excellent example because it may seem exceedingly difficult to modify the structure for updates. For example, the first cut in a kd-tree — let us recall — ought to be a vertical line that divides the point set as evenly as possible. Unfortunately, even a single point insertion would throw off the balance and thus destroy the whole tree. It may be surprising to you that later we will make the kd-tree semi-dynamic without changing the structure at all.

6.1 Decomposable problems

A query is *decomposable* if the following holds for any *disjoint* sets S_1 and S_2: given the query answers on S_1 and S_2, respectively, the answer on $S_1 \cup S_2$ can be obtained in constant time.

Consider, for example, orthogonal range reporting on 2D points. Given an axis-parallel rectangle q, the query answer on S_1 (or S_2) is the set Σ_1 (or Σ_2) of points therein covered by q. Clearly, $\Sigma_1 \cup \Sigma_2$ is the answer of the same query on $S_1 \cup S_2$. In other words, once Σ_1 and Σ_2 are available, we have already obtained the answer on $S_1 \cup S_2$ (nothing needs to be done). Hence, the query is decomposable.

As another example, consider range counting on a set of real values. Given an interval $q \subseteq \mathbb{R}$, the query answer on S_1 (or S_2) is the number c_1 (or c_2) of values therein covered by q. Clearly, $c_1 + c_2$ is the answer of the same query on $S_1 \cup S_2$. In other words, once c_1 and c_2 are available, we can obtain the answer on $S_1 \cup S_2$ in constant time. Hence, the query is decomposable.

You can verify that all the queries we have seen so far are decomposable: predecessor/successor, find-min/max, range reporting, range counting/max, stabbing, etc.

6.2 The logarithmic method

This section serves as a proof of the following theorem:

Theorem 6.1. *Suppose that there is a static structure Υ that*

- *stores n elements in $F(n)$ space;*
- *can be constructed in $n \cdot U(n)$ time;*
- *answers a decomposable query in $Q(n)$ time (plus, if necessary, a cost linear to the number of reported elements).*

Set $h = \lceil \log_2 n \rceil$. There is a semi-dynamic structure Υ' that

- *stores n elements in $\sum_{i=0}^{h} F(2^i)$ space;*
- *supports an insertion in $O\left(\sum_{i=0}^{h} U(2^i)\right)$ amortized time;*
- *answers a decomposable query in $O(\log n) + \sum_{i=0}^{h} Q(2^i)$ time (plus, if necessary, a cost linear to the number of reported elements)*

Before delving into the proof, let us first see its application on the kd-tree. We know that the kd-tree consumes $O(n)$ space, can be constructed in $O(n \log n)$ time (see an exercise of Lecture 4), and answers a range reporting query in $O(\sqrt{n}+k)$ time, where k is the number of reported elements. Therefore:

$$\begin{aligned} F(n) &= O(n) \\ U(n) &= O(\log n) \\ Q(n) &= O(\sqrt{n}). \end{aligned}$$

Theorem 6.1 immediately gives a semi-dynamic structure that uses

$$\sum_{i=0}^{\lceil \log_2 n \rceil} O(2^i) = O(n)$$

space, supports an insertion in

$$\sum_{i=0}^{\lceil \log_2 n \rceil} O\left(\log 2^i\right) = O(\log^2 n)$$

amortized time, and answers a query in

$$\sum_{i=0}^{\lceil \log_2 n \rceil} O\left(\sqrt{2^i}\right) = O(\sqrt{n})$$

plus $O(k)$ time.

6.2.1 Structure

Let S be the input set of elements; let $n = |S|$ and $h = \lceil \log_2 n \rceil$. At all times, we divide S into disjoint subsets $S_0, S_1, ..., S_h$ (some of which may be empty) satisfying:

$$|S_i| \leq 2^i. \tag{6.1}$$

Create a structure of Υ on each subset; denote by Υ_i the structure on S_i. Then, $\Upsilon_1, \Upsilon_2, ..., \Upsilon_h$ together constitute our semi-dynamic structure. The space usage is bounded by $\sum_{i=0}^{h} F(2^i)$.

Before receiving any updates, $S_0 = S_1 = ... = S_{h-1} = \emptyset$ and $S_h = S$. Accordingly, $\Upsilon_0, ..., \Upsilon_{h-1}$ are empty and Υ_h stores the entire S.

6.2.2 Query

To answer a query q, we simply search all of $\Upsilon_1, ..., \Upsilon_h$. Since the query is decomposable, we can obtain the answer on S from the answers on $S_1, ..., S_h$ in $O(h)$ time. The overall query time is

$$O(h) + \sum_{i=0}^{h} Q(2^i) = O(\log n) + \sum_{i=0}^{h} Q(2^i).$$

6.2.3 Insertion

To insert an element e_{new}, we first identify the smallest $i \in [0, h]$ satisfying:

$$1 + \sum_{j=0}^{i} |S_j| \quad \leq \quad 2^i. \tag{6.2}$$

We now proceed as follows:

- If i exists, we destroy $\Upsilon_0, \Upsilon_1, ..., \Upsilon_i$ and move all the elements in $S_0, S_1, ..., S_{i-1}$, together with e_{new}, *into* S_i. After this, $S_0, S_1, ..., S_{i-1}$ are empty and S_i contains all their elements, the elements that were already in S_i before, and e_{new}. Rebuild Υ_i on the S_i from scratch.
- If i does not exist, we destroy $\Upsilon_0, \Upsilon_1, ..., \Upsilon_h$, and move all the elements in $S_0, S_1, ..., S_h$, together with e_{new}, *into* S_{h+1}. Build Υ_{h+1} on S_{h+1} from scratch. The value of h then increases by 1.

Let us now analyze the amortized insertion cost with a charging argument. Each time Υ_i ($i \geq 0$) is rebuilt, we spend

$$O(|S_i|) \cdot U(|S_i|) \quad = \quad O(2^i) \cdot U(2^i) \tag{6.3}$$

cost (recall that, as stated in Theorem 6.1, a structure on n elements can be built in $n \cdot U(n)$ time). The lemma below gives a crucial observation:

Lemma 6.2. *Every time when Υ_i is rebuilt, at least $1 + 2^{i-1}$ elements are added to S_i (i.e., every such element was in some S_j with $j < i$).*

Proof. Set $\lambda = i$. By our choice of i, the inequality (6.2) does not hold for $i = \lambda - 1$. This means:

$$1 + \sum_{j=0}^{\lambda-1} |S_j| \quad \geq \quad 1 + 2^{\lambda-1}.$$

This proves the claim because all the elements in $S_1, ..., S_{\lambda-1}$, as well as e_{new}, are added to S_λ. □

We can therefore charge the cost of rebuilding Υ_i — namely the cost in (6.3) — on the at least 2^{i-1} elements added to S_i, such that each of those elements bears only

$$\frac{O(2^i)}{2^{i-1}} \cdot U(2^i) \quad = \quad O(U(2^i))$$

cost.

In other words, every time an element e moves to new S_i, it bears a cost of $O(U(2^i))$. Note that an element never moves from S_i to an S_j with $j < i$. Therefore, e can be charged at most $h + 1$ times with a total cost of

$$O\left(\sum_{i=0}^{h} U(2^i).\right)$$

We have proved that any sequence of m insertions can be processed in

$$O\left(m \cdot \sum_{i=0}^{h} U(2^i)\right)$$

time.

6.3 Remarks

The logarithmic method was developed by Bentley and Saxe [8]. There are standard *de-amortization* techniques (see [34]) that convert a structure with small amortized update time into a structure achieving a small time bound on *every* update. By applying those techniques, we can turn our modified kd-tree into a structure that ensures $O(\log^2 n)$ time on every insertion.

Exercises

Problem 1*. In Section 6.2, we applied Theorem 6.1 to argue that the kd-tree can support an insertion in $O(\log^2 n)$ amortized time. Strictly speaking, Theorem 6.1 only tells us that any sequence of n insertions can be processed in $O(n \log^2 n)$ time. In order to claim an $O(\log^2 n)$ amortized bound, we must show that any sequence of n insertions can be processed in $O(\sum_{i=1}^{n} \log^2 i$ time (because there are only $i-1$ elements before insertion i). Explain how the issue can be fixed.

(Hint: There are many approaches; here we outline an easy one (which is not the fastest for practical implementation). Recall that the set S at the beginning of the logarithmic method need not be empty. Apply the logarithmic method until the size of S doubles. Reset with global rebuilding.)

Problem 2. Design a semi-dynamic data structure that stores a set of n intervals in $O(n)$ space, answers a stabbing query in $O(\log^2 n + k)$ time (where k is the number of intervals reported), and supports an insertion in $O(\log^2 n)$ amortized time.

Problem 3.** Let S be a set of n points in $\mathbb{R}^2$ that have been sorted by x-coordinate. Design an algorithm to build the priority search tree on S in $O(n)$ time.

(Hint: how to construct a max heap on n real values in $O(n)$ time?)

Problem 4. Design a semi-dynamic data structure that stores a set of n 2D points in $O(n)$ space, answers a 3-sided range reporting query in $O(\log^2 n + k)$ time (where k is the number of points reported), and supports an insertion in $O(\log n)$ amortized time.

(Hint: Problem 3.)

Lecture 7: Weight Balancing

In this lecture, we will discuss a technique called *weight balancing* that allows us to (fully) dynamize sophisticated structures such as the interval tree, the priority search tree, the range tree, and so on. These structures are "multi-layered" because they associate each node of a BST with a secondary structure. To dynamize such structures, we need a more powerful version of the BST where nodes seldom become imbalanced during updates.

7.1 BB[α]-trees

Let $\mathcal{T}$ be a BST on a set S of n real values. Given a BST $\mathcal{T}$, we denote by $|\mathcal{T}|$ the number of nodes in $\mathcal{T}$. Given a node u in $\mathcal{T}$, we define its *weight* $w(u)$ as the number of nodes in $sub(u)$ and its *balance factor* as:

$$\rho(u) \quad = \quad \frac{\min\{|\mathcal{T}_1|, |\mathcal{T}_2|\}}{w(u)}$$

where $\mathcal{T}_1$ (or $\mathcal{T}_2$, resp.) is the left (or right, resp.) subtree of u.

Let α be a real-valued constant satisfying $0 < \alpha \leq 1/5$. A node u in $\mathcal{T}$ is said to be *α-balanced* in either situation below: $|w(u)| \leq 4$ or $\rho(u) \geq \alpha$. In other words, for an α-balanced u, either $sub(u)$ has very few nodes, or each subtree of u has roughly the same size (up to a constant factor). When neither condition holds, we say that u is *α-imbalanced*.

$\mathcal{T}$ is a *BB[α]-tree* if every node is α-balanced (where BB stands for *bounded balanced*). We associate each node u with its weight $w(u)$. This allows us to compute its balance factor from its weight and those of its child nodes. The space consumption of $\mathcal{T}$ remains $O(n)$.

Lemma 7.1. *The height of a BB[α]-tree $\mathcal{T}$ is $O(\log n)$, where the big-O hides a constant factor dependent on α.*

Proof. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the left and right subtree of $root(\mathcal{T})$, respectively. By definition of BB[α], we know $|\mathcal{T}_1| \leq (1-\alpha)|\mathcal{T}|$ and $|\mathcal{T}_2| \leq (1-\alpha)|\mathcal{T}|$, namely, the subtree size drops by a constant factor every time we descend a level. Hence, we can descend only $O(\log n)$ times. □

Lemma 7.2. *If S has been sorted, a BB[α]-tree $\mathcal{T}$ can be constructed in $O(n)$ time.*

Proof. Take the median element $e \in S$ (i.e., the $\lceil n/2 \rceil$-smallest in S). Create a node u with $key(u) = e$ and make u the root of $\mathcal{T}$. Each subtree of u has at least $n/2 - 1$ nodes. If $n \geq 4$, the balance factor $\rho(u) \geq \frac{n/2-1}{n} = 1/2 - 1/n \geq 1/4 > \alpha$. Hence, u is α-balanced. Construct the left subtree of u recursively on $\{e' < e \mid e' \in S\}$ and the right subtree of u recursively on $\{e' > e \mid e' \in S\}$. The construction time is left to you as an exercise. □

Corollary 7.3. *After the construction of $\mathcal{T}$ in Lemma 7.2, every node has a balance factor at least 1/4 as long as its weight is at least 4 .*

Proof. Follows immediately from the proof of Lemma 7.2. □

7.2 Insertion

To insert a value e_{new} in S, descend $\mathcal{T}$ to the conceptual leaf z whose slab (Section 2.1.2) covers e_{new}. Replace z with a regular leaf with key e_{new}. The insertion, however, may cause some nodes to be α-imbalanced. Such nodes can appear only on the path Π from the root to z (think: why?). Let u be the *highest* α-imbalanced node. The cost so far is $O(\log n)$ by Lemma 7.1.

If u does not exist, the insertion finishes. Otherwise, use Lemma 7.2 to rebuild the entire $sub(u)$. The keys in $sub(u)$ can be collected from $sub(u)$ in sorted order using $O(w(u))$ time (depth first traversal). The construction of $sub(u)$ takes $O(w(u))$ time. The insertion cost is $O(\log n + w(u))$, which can be terribly large. However, as shown later, subtree rebuilding occurs infrequently such that each update is amortized only $O(\log n)$ time.

7.3 Deletion

To delete a value e_{old} from S, first find the node v with $key(v) = e_{old}$. We will discuss only the case where v is a leaf (the opposite case is left as an exercise). Delete v from $\mathcal{T}$. The cost so far is $O(\log n)$.

The deletion may cause some nodes to become α-imbalanced. These nodes can appear only on the root-to-v path Π. Let u be the *highest* α-imbalanced node. If u exists, rebuild $sub(u)$ in the same way as in insertion. The deletion cost is $O(\log n + w(u))$. Again, we will show that each update is amortized only $O(\log n)$ time.

7.4 Amortized analysis

The lemma below explains a key property of weight balancing:

Lemma 7.4. *Suppose that $sub(u)$ has just been reconstructed. Let $w^* = w(u)$ at this moment. Then, the next reconstruction of $sub(u)$ can happen only after $w^*/24$ elements have been inserted or deleted in $sub(u)$.*

Proof. If $w^* \leq 24$, the lemma holds because trivially at least $1 \geq w^*/24 = \Omega(w^*)$ update is needed. Focus now on $w^* \geq 24$. By Corollary 7.3, $\rho(u) \geq 1/4$.

We argue that at least $w^*/24$ updates must occur in $sub(u)$ before $\rho(u)$ drops below $\alpha \leq 1/5$. Let n_1 be the number of nodes in the left subtree $\mathcal{T}_1$ of u at the moment; $n_1 \geq w^*/4$ (Corollary 7.3). Suppose that after x updates in $sub(u)$, $|\mathcal{T}_1|/w(u) \leq 1/5$. We will prove that $x \geq w^*/24$.

After x updates, we must have $w(u) \leq w^* + x$ and $|\mathcal{T}_1| \geq n_1 - x$. Therefore, $|\mathcal{T}_1|/w(u) \geq \frac{n_1 - x}{w^* + x}$. For the ratio to be at most 1/5, we need:

$$\begin{aligned} \frac{n_1 - x}{w^* + x} &\leq 1/5 \Rightarrow \\ 6x &\geq 5n_1 - w^* \geq w^*/4 \Rightarrow \\ x &\geq w^*/24. \end{aligned}$$

A symmetric argument shows that at least $w^*/24$ updates are needed to make $|\mathcal{T}_2|/w(u) \leq 1/5$ to happen, where $|\mathcal{T}_2|$ is the right subtree of u. This completes the proof. □

The constant 24 in the lemma can be made much smaller with a more sophisticated analysis. In this course, we aim at presenting the core ideas with arguments that are as simple as possible.

Theorem 7.5. *The BB[α]-tree supports any sequence of n updates (mixture of insertions and deletions) in $O(n \log n)$ time, namely, $O(\log n)$ amortized time per update.*

Proof. It suffices to concentrate on the cost of subtree reconstruction. By Lemma 7.4, whenever a subtree $sub(u)$ is rebuilt, we can charge the $O(w(u))$ rebuilding cost on the $\Omega(w(u))$ insertions/deletions that have taken place in $sub(u)$ since the last reconstruction of $sub(u)$. Each of those updates bears only $O(1)$ cost. How many times can we charge an update this way? The answer is $O(\log n)$ because each insertion or deletion can affect only the (subtrees of the) $O(\log n)$ nodes on the update path. □

7.5 Dynamization with weight balancing

The weight balancing technique can dynamize nearly all the structures in Lecture 3 and 4. Those structures are "two-layered" meaning that:

- they use a BST $\mathcal{T}$ as the *primary* structure, and
- every node in $\mathcal{T}$ is associated with a *secondary* structure.

Let us first understand why updates are costly if we implement $\mathcal{T}$ as an "undergraduate" BST such as the AVL-tree. Recall that the AVL-tree performs rotations to keep the tree balanced. When a node u is involved in a rotation, $sub(u)$ changes, thus forcing us to reconstruct the secondary structure of u. Such a reconstruction can be very expensive because the secondary structure can be very large. The "graduate-level" BST — the BB[α]-tree — remedies the issue by ensuring that subtree reconstructions occur only occasionally (Lemma 7.4).

Next, we will explain how to deploy weight balancing to dynamize a two-layered structure.

Structure. Let $\mathcal{T}$ be a BB[α]-tree on a set S of n real values. For each node u in $\mathcal{T}$, denote by S_u the set of keys in $sub(u)$. Associate u with a secondary structure $\mathcal{T}_u$ created on S_u. We do not care about what $\mathcal{T}_u$ is but we assume it supports an insertion and a deletion in $O(\log |S(u)|) = O(\log n)$ time (this implies that the structure can be built in $O(|S_u| \log |S_u|) = O(w(u) \log n)$ time). We will show how to support an update in $O(\log^3 n)$ amortized time.

Insertion. To insert a value e_{new}, we first create a new leaf z in $\mathcal{T}$ with $key(z) = e_{new}$ in $O(\log n)$ time by following a root-to-z path Π. For every node u on Π, add e_{new} to $\mathcal{T}_u$ in $O(\log n)$ time. The cost so far is $O(\log^2 n)$.

The insertion finishes if no subtree reconstruction occurs in $\mathcal{T}$. Now, consider that we need to reconstruct the subtree of a node u on Π. For this purpose, reconstruct the secondary structures of all the nodes in $sub(u)$, which takes $O(|S_u| \log^2 |S_u|) = O(w(u) \log^2 n)$ time (think: why?). By Lemma 7.4, $\Omega(w(u))$ updates must have taken place in $sub(u)$ since the last reconstruction of $sub(u)$. We charge the construction cost over those updates, each of which is bears an additional cost of $O(\log^2 n)$.

Deletion. To delete a value e_{old}, first find the node u in $\mathcal{T}$ with $key(u) = e_{old}$. We will discuss only the case where u is a leaf (the opposite case is left to you). Let Π be the root-to-u path. Delete e_{old} from $\mathcal{T}$ and from $\mathcal{T}_u$ for every node u on Π in $O(\log n)$ time. The cost so far is $O(\log^2 n)$.

The deletion finishes if no subtree reconstruction occurs. Suppose that we need to reconstruct the subtree of some node on Π. The reconstruction algorithm and its analysis are exactly the same as in the insertion case.

Overall. Each update can be charged only $O(\log n)$ times for subtree reconstructions and thus has amortized cost $O(\log^3 n)$.

7.6 Remarks

Our definition is one of the many ways to describe the BB[α] tree. See [33] for the original proposition. The BB[α]-tree can actually be updated in $O(\log n)$ on *every* insertion/deletion. Whenever a node u becomes α-imbalanced, we can fix it in constant time by performing a rotation (in a manner similar to the AVL-tree). Even better, after the fix, u can become α-imbalanced only after $\Omega(w(u))$ updates have taken place in $sub(u)$. The details can also be found in [9].

Exercises

Problem 1. Prove the construction time in Lemma 7.2.

Problem 2. Complete the deletion algorithm in Section 7.3 and 7.5 for the case where e_{old} is the key of an internal node.

(Hint: Convert it to deleting a leaf, as in the AVL-tree.)

Problem 3. Explain how to support an insertion and a deletion on the interval tree (Section 3.1) in $O(\log^2 n)$ amortized time, where n is the number of intervals. Your structure must still be able to answer a stabbing query in $O(\log n + k)$ time, where k is the number of intervals reported.

Problem 4. Explain how to support an insertion and a deletion on the priority search tree (Section 4.3) in $O(\log^2 n)$ amortized time, where n is the number of points. Your structure must still be able to answer a 3-sided range query in $O(\log n + k)$ time, where k is the number of points reported.

Problem 5.** Improve the update time in the previous problem to $O(\log n)$.

Problem 6. Explain how to support an insertion and a deletion on the range tree (Section 4.4) in $O(\log^3 n)$ amortized time, where n is the number of points. Your structure must still be able to answer a 4-sided range query in $O(\log^2 n + k)$ time, where k is the number of points reported.

Lecture 8: Partial Persistence

A dynamic data structure is *ephemeral* because, once updated, its previous version is lost. Consider, for example, n insertions into an initially empty BST. In the end, we have a BST with n nodes (the final version). In history, $n-1$ other versions have ever been created and lost.

Wouldn't it be nice to retain all the versions so that we can "travel back in time" and search an arbitrary past version? One naive way to do so is to store a separate copy of each historical version, which requires $O(n^2)$ space. We will learn a powerful technique called *partial persistence* that allows us to achieve the purpose in just $O(n)$ space (clearly optimal). The technique is applicable to any pointer-machine structure (Section 4.5) where each node has a constant in-degree (for the BST, the in-degree is 1). This includes most of the structures you already know: the linked list, the priority queue, all the structures in Lectures 3 and 4, and so on (but not dynamic arrays).

The technique has implications beyond history preservation. It allows us to solve difficult problems using surprisingly primitive structures. One example is the 3-sided range query that we tackled with the priority search tree (PST) in Section 4.3: as you will see in an exercise, we can achieve the space and query complexities of the PST by simply making the BST partially persistent.

8.1 The potential method

Let us first introduce a new method for amortized analysis called the *potential method*. Consider M operations on a data structure, the i-th ($i \in [M]$) of which has cost C_i. As discussed in Section 5.2, we can assign a non-negative integer $\bar{C}_i$ to operation i as its amortized cost as long as

$$\sum_{i=1}^{M} C_i \quad \leq \quad \sum_{i=1}^{M} \bar{C}_i.$$

Define Φ — called the *potential function* — as a function that maps the current structure to a real value. Let T_0 be the initial structure before all operations and T_i be the structure after operation i.

Lemma 8.1. *If $\Phi(T_M) \geq \Phi(T_0)$, the amortized cost of operation i is at most $C_i + \Phi(T_i) - \Phi(T_{i-1})$.*

Proof. It suffices to prove $\sum_{i=1}^{M} C_i \leq \sum_{i=1}^{n} (C_i + \Phi(T_i) - \Phi(T_{i-1}))$. This is obvious because

$$\sum_{i=1}^{M} \Phi(T_i) - \Phi(T_{i-1}) \quad = \quad \Phi(T_M) - \Phi(T_0) \geq 0.$$

□

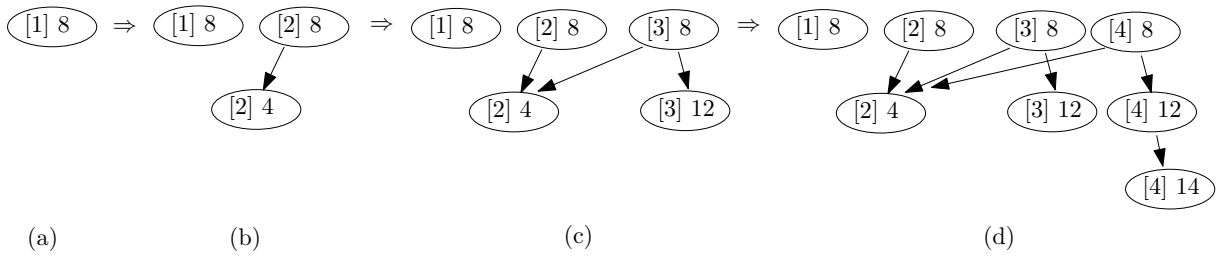


Figure 8.1: Illustration of naive copying on the insertion sequence of 8, 4, 12, 14.

To get familiar with the method, we will apply it to obtain yet another analysis of the dynamic array's amortized insertion cost. As discussed in Section 5.3, we double the array size whenever n reaches a power of 2. Define the potential function Φ as:

$$\Phi \quad = \quad c \cdot \text{the number of insertions after the previous doubling}$$

for some constant c chosen later. Consider insertion $i \in [M]$. If the insertion triggers no doubling, $C_i = O(1)$, Φ increases by c, and thus $C_i + \Phi(T_i) - \Phi(T_{i-1}) = C_i + c = O(1)$. Otherwise, $C_i = O(n)$, Φ drops by $c(\frac{n}{2} - 1)$ (think: why), and thus $C_i + \Phi(T_i) - \Phi(T_{i-1}) = O(n) - cn/2$, which is negative if we choose c sufficiently large. Thus, we always have $C_i + \Phi(T_i) - \Phi(T_{i-1}) = O(1)$, yielding the conclusion that each insertion has constant amortized cost.

8.2 Partially persistent BST

Starting with an empty BST $\mathcal{T}_0$, we will process a sequence of n updates (mixture of insertions and deletions). The i-th ($1 \leq i \leq n$) update is said to happen at *time* i. Denote by $\mathcal{T}_i$ the BST after the update, which is said to be of *version* i. Our goal is to retain the BSTs of all versions. We will call the BST of the latest version the *live* BST and denote it as $\mathcal{T}$ (i.e., $\mathcal{T} = \mathcal{T}_i$ after i updates). Denote by $\mathcal{A}$ the update algorithm of the BST, which can be any implementation of the BST, e.g., the AVL-tree, the red-black tree, the BB[α]-tree, etc.

8.2.1 The first attempt

Our first idea is to enforce the following principle: *whenever $\mathcal{A}$ needs to change a node u, make a copy of u and apply the changes only on the new copy.*

Example. Consider the update sequence that inserts 8, 4, 12, and 14. As shown in Figure 8.1(a), $\mathcal{T}_1$ contains a single node, whose label "[i] k" indicates that it is created at time i with key k.

The second insertion creates node "[2] 4" as the left child of "[1] 8" in the live BST. By the aforementioned principle, we do not alter "[1] 8", but copy it to node "[2] 8" and make "[2] 4" the left child of "[2] 8". As can be seen in Figure 8.1(b), both BSTs $\mathcal{T}_1$ and $\mathcal{T}_2$ are explicitly stored.

To insert 12, we create "[3] 12", copy "[2] 8" to "[3] 8", and make "[3] 12" the right child of "[3] 8". The structure at this moment is in Figure 8.1(c). Note that the left child of "[3] 8" is still "[2] 4" (which was not affected by the current update). Observe how Figure 8.1(c) stores 3 BSTs $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$.

Figure 8.1(d) presents the final structure after inserting 14, which encodes BSTs $\mathcal{T}_1, ..., \mathcal{T}_4$. □

We will refer to the above method *naive copying*. As each update on the live BST accesses $O(\log n)$ nodes, naive copying can create $O(\log n)$ nodes per update in the persistent structure. The overall space consumption is therefore $O(n \log n)$. Any BST in the past can be found and searched efficiently: for any $i \in [1, n]$, first find the root of $\mathcal{T}_i$ can be identified in $O(\log n)$ time[1], after which the search can then be performed within $\mathcal{T}_i$ as if the other versions did not exist.

Naive copying sometimes duplicates a node that is not modified by an update. In Figure 8.1(d), for example, although the only node modified by the update is "[3] 12", the method duplicates all its ancestors. We will remedy the drawback with a new approach in the next subsection.

8.2.2 An improved method

Our new idea is to introduce a *modification field* in each node u. When $\mathcal{A}$ needs to change a pointer of u, we record the change in the field and perform node copying only when the field is full. It turns out that a constant-size field suffices to reduce the space to $O(n)$.

Each node now takes the form $\{([i]\, k, ptr_1, ptr_2), mod\}$ where

- $([i]\, k, ptr_1, ptr_2)$ indicates that the node is created at version i with key k and pointers ptr_1 and ptr_2 (which may be NULL);
- *mod* is the modification field, which is empty when the node is created and can record one pointer change.

Example. We will first insert 8, 4, 12, 14, 2, and then delete 2, 14. Figure 8.2(a) shows the structure after the first insertion. Here, the ptr_1 and ptr_2 of node I are both NULL. The empty space on the right of the vertical bar indicates an empty *mod*.

To insert 4, we create node II and make it the left child of node I. This means redirecting the left pointer of node I to node II at time 2. This pointer change is described in the *mod* of node I; see Figure 8.2(b). Observe how the current structure encodes both $\mathcal{T}_1$ and $\mathcal{T}_2$.

The insertion 12 creates node III, which should be the right child of node I. As the *mod* of node I is already full, we cannot write the pointer change inside node I and thus need to do node copying. As shown in Figure 8.2(c), this spawns node IV, which stores "[3] 8" and has ptr_1 and ptr_2 referencing nodes II and III, respectively. The current structures encodes $\mathcal{T}_1, \mathcal{T}_2$, and $\mathcal{T}_3$. Figures 8.2(d) and (e) illustrate the insertion of 14 and 2, respectively.

The next operation deletes 2. Accordingly, we should set the pointer of node II to NULL. Since node II's *mod* is full, we copy it to node VII. This, in turn, requires changing the left pointer of node IV; the change is recorded in its *mod*. The current structure in Figure 8.2(f) encodes $\mathcal{T}_1, ..., \mathcal{T}_6$.

Finally, the deletion of 14 requires nullifying the right pointer of node III. As Node III's *mod* is full, we copy it to node VIII, which further triggers node IV to be copied to node IX. Figure 8.2(g) gives the final structure which encodes $\mathcal{T}_1, ..., \mathcal{T}_7$. □

In general, $\mathcal{A}$ can change the live BST with two operations:

- C-operation: creating a new node, which happens only in an insertion for storing the key inserted;

[1] By creating a separate BST on the root versions.

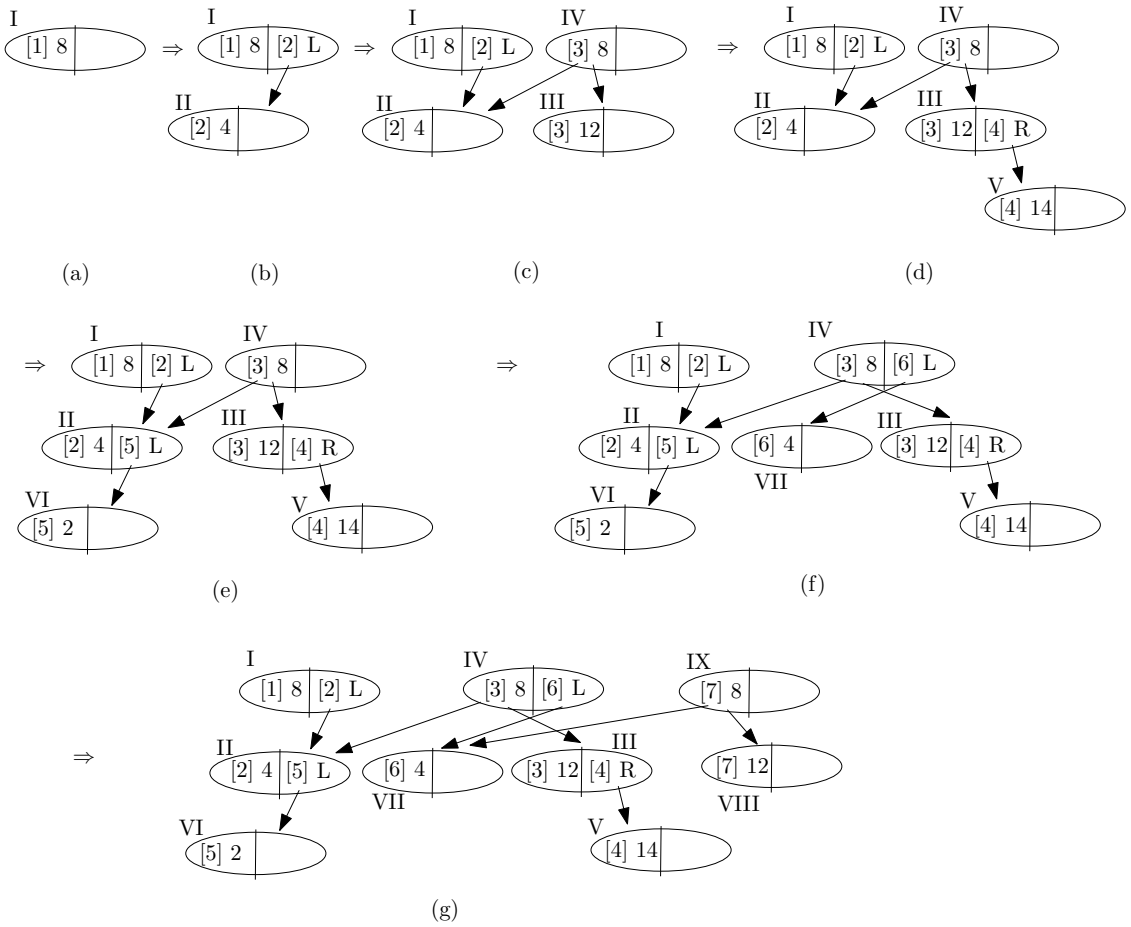


Figure 8.2: Illustration of the improved method on the update sequence of inserting 8, 4, 12, 14, 2 followed by deleting 2 and 14.

- P-operation: modifying a pointer in some node u.

We modify each operation for the persistent structure as follows:

- C-operation: create a new node $\{([i]\, k, \text{NULL}, \text{NULL}), \emptyset\}$ by filling in i and k appropriately;
- P-operation: to modify a pointer in a node u, we invoke:

ptr-update(u)
1. **if** the *mod* of u is empty **then**
2. record the pointer modification in *mod*
3. **return**
4. **else** /* *mod* full */
5. copy u to node v and modifies the pointer in v
6. **if** u has a parent $\hat{u}$ in the live BST
7. call **ptr-update**($\hat{u}$) to add a pointer from $\hat{u}$ to v

Note that Line 7 recursively invokes **ptr-update** and can induce multiple node copies.

The time to build a persistent BST is clearly $O(n \log n)$ (the proof is left to you). As in Section 8.2.1, we can identify the root of any $\mathcal{T}_i$ ($1 \leq i \leq n$) in $O(\log n)$ time, after which $\mathcal{T}_i$ can then be search as a normal BST. We will analyze the space consumption in the next subsection.

8.2.3 Space

Denote by m_i ($1 \leq i \leq n$) the number of C/P-operations that $\mathcal{A}$ performs on the live tree in processing the i-th update. We will prove:

Lemma 8.2. *The algorithm in Section 8.2.2 creates $O(\sum_{i=1}^{n} m_i)$ nodes in the persistent tree.*

The lemma immediately implies:

Theorem 8.3. *Given a sequence of n updates on an initially empty BST, we can build a persistent BST of $O(n)$ space in $O(n \log n)$ time.*

Proof. The red-black tree performs at most one C-operation and $O(1)$ P-operations in each insertion/deletion. □

Proof of Lemma 8.2. Set

$$M = \sum_{i=1}^{n} m_i$$

namely, M is the total number of C/P-operations performed by $\mathcal{A}$. These operations happen in succession, and hence, can be listed as operation 1, 2, ..., M, respectively. Let C_j ($1 \leq j \leq M$) be the number of nodes created by the j-th operation in the persistent tree. We will prove $\sum_{j=1}^{M} C_j = O(M)$, or equivalently, each operation creates $O(1)$ nodes amortized.

Denote by S_j ($1 \leq j \leq M$) the set of nodes in the live tree after operation j. Define specially S_0 as the empty set. Introduce a potential function Φ that maps S_j to a real value as follows:

$$\Phi(S_j) \quad = \quad \text{the number of nodes in } S_j \text{ whose } \textit{mod} \text{ fields are } \textit{non}\text{-empty.} \tag{8.1}$$

Clearly, $\Phi(S_M) \geq \Phi(S_0) = 0$. By Lemma 8.1, operation j creates at most

$$C_j + \Phi(S_j) - \Phi(S_{j-1}) \tag{8.2}$$

nodes after amortization. Next, we will show that the above is at most 1 for every j, which will complete the proof of Lemma 8.2.

If operation j is a C-operation, it creates a node with empty *mod* and finishes. Hence, $C_j = 1$, $S_j = S_{j-1}$, and hence (8.2) equals 1.

Now, consider operation j as a P-operation. Every new node is created by copying (Line 5 of **ptr-update**). Each time this happens, we lose a node with non-empty *mod* (i.e., node u in **ptr-update**), create a node with empty *mod* (i.e., v in the pseudocode), and possibly fill in the *mod* of one node (i.e., $\hat{u}$, if it exists). Therefore, $\Phi(S_j) - \Phi(S_{j-1})$ is at most $-C_j + 1$ such that (8.2) can never exceed 1.

8.3 General pointer-machine structures

The following result generalizes Theorem 8.3:

Theorem 8.4 ([16]). *Consider any pointer-machine structure defined in Section 4.5 where every node has a constant in-degree. Suppose that $\mathcal{A}$ is an algorithm used to process a sequence of n updates (mixture of insertions and deletions) with amortized update cost $U(n)$. Let m_i be the number of nodes created/modified by $\mathcal{A}$ in processing the i-th update ($1 \leq i \leq n$). Then, we can create a persistent structure that records all the historical versions in $O(n \cdot U(n))$ time. The structure consumes $O(\sum_{i=1}^{n} m_i)$ space. The root of every version can be identified in $O(\log n)$ time.*

For example, if the structure is the linked list, then $U(n) = O(1)$ and $m_i = O(1)$. Therefore, we can construct a persistent linked list of $O(n)$ space in $O(n)$ time. The head node of the linked list of every past version can be identified in $O(\log n)$ time.

The theorem can be established using the modification-logging approach in Section 8.2.2, except that the modification field should be made sufficiently large (but still have a constant size). We omit the details which can be found in [16].

8.4 Remarks

The methods in this lectures were developed by Driscoll, Sarnak, Dominic, and Tarjan in [16].

Exercises

Problem 1. Prove the construction time in Theorem 8.3.

Problem 2. Let S be a set of n horizontal rays in $\mathbb{R}^2$, each having the form $[x, \infty) \times y$. Explain how to store S in a persistent BST of $O(n)$ space such that, given any vertical segment $q = x \times [y_1, y_2]$, we can report all the rays in S intersecting q using $O(\log n + k)$ time, where k is the number of rays reported.

Problem 3. Let P be a set of n points in $\mathbb{R}^2$. Explain how to store P in a persistent BST of $O(n)$ space such that any 3-sided range query of the form $(-\infty, x] \times [y_1, y_2]$ can be answered in $O(\log n + k)$ time, where k is the number of points reported.

(Hint: Problem 2.)

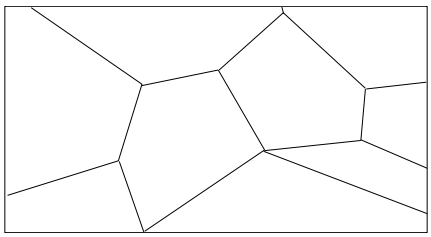
Problem 4. Let P be a set of n points in $\mathbb{R}^2$. Given an axis-parallel rectangle q, a *range count* query reports the number of points in P that are covered by q. Design a structure that stores P in $O(n \log n)$ space that can answer a range count query in $O(\log n)$ time.

Remark: this improves an exercise in Lecture 4.
(Hint: persistent count BST.)

Problem 5. Prove Theorem 8.4 for the linked list.

Remark: the persistent linked list is a way to store all the past versions of a document that has undergone a sequence of edits (regard a document as a sequence of characters).

Problem 6* (point location). A polygonal subdivision of $\mathbb{R}^2$ is a set of interior-disjoint convex polygons whose union is $\mathbb{R}^2$. The following shows an example (for clarity, the boundary of $\mathbb{R}^2$ is represented as a rectangle).



Given a point q in $\mathbb{R}^2$, a *point location* query reports the polygon that contains q (if q falls on the boundary of more than one polygon, any such polygon can be reported). Let n be the number of segments in the subdivision. Design a structure of $O(n)$ space that can answer any point location query in $O(\log n)$ time.

(Hint: persistent BST.)

Lecture 9: Dynamic Perfect Hashing

In *dictionary search*, we want to store a set S of n integers in a data structure to answer the following queries: given an integer q, report whether $q \in S$ (the output is boolean: yes or no). At the undergraduate level, we have learned that the problem can be tackled with *hashing*. Specifically, we can store S in a hash table of $O(n)$ space which answers a query in $O(1)$ expected time. In practice, we may not be satisfied with $O(1)$ *expected* query cost because it implies that the actual search time can be large occasionally. Ideally, we would like to build a *perfect* hash table that guarantees $O(1)$ query cost in the *worst* case.

This lecture will introduce a technique called *cuckoo hashing* which can maintain a perfect hash table of $O(n)$ size with $O(1)$ *amortized expected* time per update (what this means will be defined formally later in Section 9.2). We will, however, establish only a weaker bound of $O(\log n)$ amortized expected; as a benefit in return, this illustrates nicely how data structures can arise from graph theory.

9.1 Two random graph results

Let U and V each be a set of $c \cdot n \geq 2$ vertices, for some integers $c > 0, n > 0$. We generate a random bipartite graph G by repeating the **gen-edge** operation n times:

gen-edge
1. pick a vertex $u \in U$ uniformly at random
2. pick a vertex $v \in V$ uniformly at random
3. connect u, v with an edge (there can be multiple edges between two vertices)

We will label the n operations as $1, 2, ..., n$, respectively. Given any $I \subseteq [cn]$, we use OP_I to denote the set of operations with labels in I.

Let us define a *cycle* in G as a sequence of vertices $w_1, w_2, ..., w_\ell, w_1$ such that

- $w_1, w_2, ..., w_\ell$ are distinct;
- an edge exists between every two consecutive vertices in the sequence;
- all the ℓ edges are distinct (i.e., created by different **gen-edge** operations).

The vertex set $W = \{w_1, w_2, ..., w_\ell\}$ is said to *induce* the cycle.

Example. Consider $U = \{1, 2\}$ and $V = \{a, b\}$. Suppose that we perform only one **gen-edge** which gives edge $\{1, a\}$. The vertex sequence $1, a, 1$ is not a cycle. Suppose that we perform another **gen-edge** which again gives edge $\{1, a\}$. Now, the vertex sequence $1, a, 1$ becomes a cycle. In other words, $\{1, a\}$ induces a cycle. □

Lemma 9.1. *When $c \geq 8e^2$, it holds with probability at least 7/8 that G contains no cycles (here $e \approx 2.718$ represents the base of natural logarithm).*

Proof. Fix an arbitrary integer $\ell \in [2, n]$. We will prove an upper bound on the probability that G has a cycle of length ℓ. Clearly,

$$\Pr[\text{a cycle of length } \ell] \quad = \quad \sum_{W \subseteq U \cup V : |W| = \ell} \Pr[W \text{ induces a cycle in } G]. \tag{9.1}$$

Consider an arbitrary $W = \{u_1, u_2, ..., u_\ell\}$. W inducing a cycle means that there exist ℓ distinct edges each between two vertices in W. In other words, there is at least a subset $I \subseteq [n]$ with size $|I| = \ell$ such that, for each $i \in I$, the i-th **gen-edge** operation picks two vertices from W. We thus have

$$\begin{aligned} &\Pr[W \text{ induces a cycle in } G] \\ \leq \quad &\sum_{I \subseteq [n] : |I| = \ell} \Pr[\text{each operation in } \mathrm{OP}_I \text{ picks two vertices in } W]. \end{aligned} \tag{9.2}$$

Procedure **gen-edge** creates an edge on W with probability at most $(\frac{\ell}{cn})^2$ because both u and v it chooses must fall in W. It follows that

$$\Pr[\text{each operation in } \mathrm{OP}_I \text{ creates an edge on two vertices in } W] \quad \leq \quad \left(\frac{\ell}{cn}\right)^{2\ell}$$

and hence by (9.2)

$$\Pr[W \text{ induces a cycle in } G] \quad \leq \quad \binom{n}{\ell} \cdot \left(\frac{\ell}{cn}\right)^{2\ell}$$

with which (9.1) gives

$$\begin{aligned} \Pr[\text{a cycle of length } \ell] \quad &\leq \quad \binom{2cn}{\ell} \cdot \binom{n}{\ell} \cdot \left(\frac{\ell}{cn}\right)^{2\ell} \\ (\text{by FactA.1}) \quad &\leq \quad \left(\frac{e \cdot 2cn}{\ell}\right)^{\ell} \cdot \left(\frac{e \cdot n}{\ell}\right)^{\ell} \cdot \left(\frac{\ell}{cn}\right)^{2\ell} \\ &= \quad \left(\frac{2e^2}{c}\right)^{\ell} \leq (1/4)^{\ell}. \end{aligned}$$

We can now prove the lemma with

$$\begin{aligned} \Pr[G \text{ has a cycle}] \quad &\leq \quad \sum_{\ell=2}^{n} \Pr[\text{a cycle of length } \ell] \\ &\leq \quad \sum_{\ell=2}^{n} (1/4)^{\ell} < 1/8. \end{aligned}$$

□

An almost identical argument establishes:

Lemma 9.2. *When $c \geq 4e^3$, it holds with probability at least $1 - \frac{c}{n^2}$ that G has no simple path longer than $4 \log_2 n$ edges (a path is simple if it passes no vertex twice).*

The proof is left as an exercise.

9.2 Amortized expected update cost

Suppose that a structure processes m updates. As mentioned in Section 5.1, we can claim that the i-th ($i \in [m]$) update has *amortized* cost $\bar{C}_i$ if

$$\sum_{i=1}^{m} C_i \leq \sum_{i=1}^{m} \bar{C}_i,$$

where C_i is the actual cost of the i-th update.

Now consider that structure is randomized such that each C_i is a random variable. In this case, we say that the i-th ($i \in [m]$) update has *amortized expected* cost $\bar{C}_i$ if $\mathbf{E}[\sum_{i=1}^{m} C_i] \leq \sum_{i=1}^{m} \bar{C}_i$, which means

$$\sum_{i=1}^{m} \mathbf{E}[C_i] \quad \leq \quad \sum_{i=1}^{m} \bar{C}_i.$$

For example, if the structure has $O(1)$ amortized expected update time, it processes any m updates in $O(m)$ expected total time.

9.3 Cuckoo hashing

9.3.1 Hash functions

Denote by $\mathbb{D}$ the domain from which the elements of S are drawn. A *hash function* h maps $\mathbb{D}$ to a set of integers $\{1, 2, ..., N\}$ for some $N \geq 1$. The output $h(e)$ is the *hash value* of $e \in \mathbb{D}$. We will assume *uniform hashing*, which means:

- for any element $e \in \mathbb{D}$, $\Pr[h(e) = i] = 1/N$ for any $i \in [1, N]$;
- the above holds regardless of the hash values of the other elements in $\mathbb{D}$.

9.3.2 The hash table, query, and deletion

We maintain two arrays A, B each of size $N = O(n)$ where the concrete value of N will be chosen later. There are two hash functions g and h, both mapping $\mathbb{D}$ to $\{1, ..., N\}$. We enforce

Invariant: Each element $e \in S$ is stored at either $A[g(e)]$ or $B[h(e)]$.

This makes queries and deletions very simple:

- **Query:** Given an element q, report yes if $A[g(e)]$ or $B[h(e)] = q$; otherwise, report no.
- **Deletion:** To delete an element $e \in S$, erase $A[g(e)]$ or $B[h(e)]$ whichever equals e.

Clearly, both operations finish in $O(1)$ worst-case time.

9.3.3 Insertion

Next, we explain how to insert an element e_{new}. If $A[g(e_{new})]$ is empty, we store e_{new} at $A[g(e_{new})]$ and finish; otherwise, if $B[h(e_{new})]$ is empty, we store e_{new} at $B[h(e_{new})]$ and finish.

If both $A[g(e_{new})]$ and $B[h(e_{new})]$ are occupied, we launch a *bumping process* which can be intuitively understood as follows. Remember every element $e \in S$ has two "nests": $A[g(e)]$ and $B[h(e)]$. If e is evicted from one nest, we are obliged to store it in the other. With this mindset, let us place e_{new} at $A[g(e_{new})]$ and evict the element e originally stored there. Thus, e must go into its other nest in B, thereby evicting another element there. The process goes on until all the elements have been placed properly. There is a chance that this may not be possible, in which case we declare failure.

Formally, we perform the bumping process as follows:

bump(e)
1. turn = g; $cnt = 0$
2. **while** $cnt \leq 4 \log_2 n$ **do**
3. cnt++
4. **if** turn = g **then**
5. **if** $A[g(e)]$ empty **then**
6. place e at $A[g(e)]$; **return** success
else
7. swap e and $A[g(e)]$; turn = h
else
/* turn = h */
8. **if** $B[h(e)]$ empty **then**
9. place e at $B[h(e)]$; **return** success
else
10. swap e and $B[h(e)]$; turn = g
11. **return** failure

Note that functions g and h are used in a round-robin fashion.

Example. Set $N = 4$. The first insertion adds element e_1 to S; suppose $g(e_1) = 2$ and $h(e_1) = 3$. The insertion finishes by storing e_1 in $A[2]$.

The second insertion adds e_2, for which we assume $g(e_2) = 2$ and $h(e_2) = 4$. As $A[2]$ is occupied but $B[4]$ is empty, the algorithm stores e_2 at $B[4]$. Now $A = (-, e_1, -, -)$ and $B = (-, -, -, e_2)$.

The next element inserted is e_3; let $g(e_3) = 2$ and $h(e_3) = 4$. As $A[2]$ and $B[4]$ are both occupied, a bumping process starts. The process places 35 at $A[2]$ and evicts e_1 which was originally stored at $A[2]$. For element e_1, we find $B[h(e_1)] = B[3]$ empty and thus puts e_1 there. The insertion finishes with $A = (-, e_3, -, -)$ and $B = (-, -, e_1, e_2)$.

Consider one more insertion e_4 with $g(e_4) = 2$ and $h(e_4) = 4$. As $A[2]$ and $B[4]$ are occupied, the bumping process replaces e_3 with e_4 at $A[2]$. Currently, $A = (-, e_4, -, -)$. As $h(e_3) = 4$, the process replaces e_2 with e_4 at $B[4]$, after which $B = (-, -, e_1, e_3)$. The process then puts e_2 at $A[g(e_2)] = A[2]$ and removes e_4 originally there, after which $A = (-, e_2, -, -)$. The process continues in this manner and eventually declares failure. □

If the bumping process fails, we simply rebuild the whole structure:

rebuild
1. choose another two hash functions g and h
2. insert the elements of S one by one, and stop if failure declared (due to bumping)
3. **if** Line 2 fails **then** repeat from Line 1

9.3.4 Global Rebuilding

We ensure the following constraint on the array size N:

$$2e^3 \cdot n \leq N \leq 8e^3 \cdot n. \tag{9.3}$$

This can be achieved with global rebuilding. Initially, for the first insertion in S, we store the only element in an array of size $N = 4e^3$; call this a *checkpoint moment*. In general, after $N/(8e^3)$ updates since the previous checkpoint, we reconstruct the structure by calling **rebuild** (Section 9.3.3) with array size $N = 4e^3 \cdot |S|$; call this another *checkpoint moment*.

Lemma 9.3. *Equation* (9.3) *holds at all times.*

Proof. Let n_{old} be the size of S at the previous checkpoint; thus, $N = 4e^3 \cdot n_{old}$. There can be at most $n_{old} + N/(8e^3) = 1.5n_{old}$ elements in S at any moment until the next checkpoint. Hence, it holds at all times that $2e^3 \cdot |S| \leq 2e^3 \cdot 1.5n_{old} < N$. On the other hand, there must be at least $n_{old} - N/(8e^3) = 0.5n_{old}$ elements in S till the next check point. Hence, it holds at all times that $|S| \geq 0.5n_{old} = N/(8e^3)$. □

9.4 Analysis

This section will prove:

Theorem 9.4. *Fix any sequence of n updates (mixture of insertions and deletions). The above algorithm maintains a perfect hash table under the updates in $O(n \log n)$ total expected time.*

The core of the proof is to establish:

Lemma 9.5. *Consider any checkpoint. Let N be the array size set at the checkpoint. The total cost of the following tasks is $O(N \log N)$ expected:*

- *rebuilding the structure at the checkpoint;*
- *performing the next $N/(8e^3)$ updates (i.e., until the next checkpoint).*

The lemma implies Theorem 9.4. To see why, notice that there are $\Omega(N)$ updates between the previous and the current checkpoint. Therefore, we can charge the $O(N \log N)$ cost on those updates such that each is amortized $O(\log N)$ expected.

9.4.1 Proof of Lemma 9.5

We will prove only the first bullet because a similar argument applies to the second bullet (left as an exercise).

We start by establishing a vital connection between cuckoo hashing and random graphs. Set $U = V = \{1, 2, ..., N\}$. For each an element $e \in S$, create an edge between vertex $g(e) \in U$ and vertex $h(e) \in V$. Let G be the bipartie graph obtained. As $g(e)$ (or $h(e)$, resp.) chooses each vertex in U (or V, resp.) with the same probability, G is a random graph obtained in Section 9.1.

Corollary 9.6. *With probability at least* $1/2$*,* G *has both the properties below:*

- G *has no cycles.*
- G *has no simple path of longer than* $4\log_2 n$ *edges.*

Proof. Consider first $n = |S| \leq 16$. In this case, G obviously cannot have a simple path of length $4\log_2 n = 16$ because such a path needs 17 vertices. By Lemma 9.1, G has the first property with probability at least $7/8$.

Consider now $n > 16$. Lemma 9.1 shows that the first property can be violated with probability at most $1/8$. Since (9.3) always holds, Lemma 9.2 indicates that the second property can be violated with probability at most $c/n^2 = (4e^3)/n^2 \leq (4e^3)/16^2 < 1/3$ (at the check point we choose the array size $N = 4e^3 \cdot |S|$; hence, the value c in Lemma 9.2 is $4e^3$). Hence, the probability for at least one property to be violated is no more than $1/8 + 1/3 < 1/2$ (union bound; see Lemma A.2). □

Lemma 9.7. *Line 2 of* **rebuild** *(Section 9.3.3) takes* $O(n \log n)$ *time.*

Proof. Line 2 performs n insertions at Line 2. Each insertion takes $O(1)$ time if no bumping process is required. Otherwise, it takes $O(\log n)$ time before declaring success or failure. □

Lemma 9.8. *If* G *has both properties in Corollary 9.6, Line 2 of* **rebuild** *successfully builds the entire structure.*

Proof. We will prove that, with the two properties, the bumping process will *never* fail. This will establish the lemma.

When the bumping process evicts an element e from one nest to the other — say from $A[g(e)]$ to $B[h(e)]$ — we cross an edge in G from vertex $g(e) \in U$ to $h(e) \in V$. Therefore, if the process fails, we must have traveled on a path Π of more than $4\log_2 n$ edges. As G has no cycles, Π cannot pass two identical vertices. This means that Π must be a *simple* path, which yields a contradiction. □

We can now put together Corollary 9.6, Lemmas 9.7 and 9.8 to prove that **rebuild** finishes in $O(n \log n)$ expected time. Let X be the number of times that Line 2 is executed. By Corollary 9.6 and Lemma 9.8, every time Line 2 is executed, it fails with probability at most $1/2$, which indicates that $\Pr[X = t] \leq (1/2)^{t-1}$. Lemma 9.7 implies that the total cost of **rebuild** is $O(X \cdot n \log n)$. Therefore, the expected cost is

$$\sum_{t=1}^{\infty} O(t \cdot n \log n) \cdot \Pr[X = t] \quad = \quad \sum_{t=1}^{\infty} O(t \cdot n \log n) \cdot \left(\frac{1}{2}\right)^{t-1} = O(n \log n).$$

This completes the proof of the first bullet of Lemma 9.5.

9.5 Remarks

Our discussion of cuckoo hashing emphasized on its relationships to random graphs. As mentioned, cuckoo hashing actually achieves $O(1)$ amortized expected time per update, about which the interested student may refer to [35] for a proof.

Our assumption of uniform hashing can also be relaxed. As shown in [35], $O(\log n)$-wise independent hashing (i.e., intuitively this means that any $O(\log n)$ hash values are guaranteed to be independent; our assumption is essentially n-wise independence) is good enough, but the analysis would have to deviate significantly from the two lemmas in Section 9.1. It is worth noting that there exist $O(\log n)$-wise independent hash functions that can be evaluated in constant expected time; see [38].

Exercises

Problem 1. Prove Lemma 9.2.

Problem 2. Prove the second bullet of Lemma 9.5 assuming all those $N/(8e^3)$ updates are insertions.

(Hint: pretend all those insertions were given at the checkpoint and include them in the argument for proving the first bullet.)

Problem 3. Prove the second bullet of Lemma 9.5 in general (i.e., allowing deletions).

Problem 4 (a uniform hashing function requires lots of space to represent). Let $\mathbb{D}$ be the set of integers from 1 to D where $D \geq 1$ is an integer.

(a) How many different functions are there mapping $\mathbb{D}$ to $\{1, 2, ..., N\}$ where $N \geq 1$ is an integer?

(b) Prove: at least $D \log_2 N$ bits are required to represent all the above functions, regardless of how the functions are encoding in binary form.

(c)* Prove: any uniform-hashing function from $\mathbb{D}$ to $\{1, 2, ..., N\}$ requires $D \log_2 N$ bits to represent.

(Hint: such a hash function must be a random variable. What are the possible values of this random variable?)

Remark: this means that uniform hashing may not be a fair assumption for practical applications.

The next two exercises would help you gain intuition as to why cuckoo hashing guarantees $O(1)$ expected amortized update time.

Problem 5. Consider a checkpoint rebuild where $N = 4e^3 n$ and $n = |S|$. Recall that the **rebuild** algorithm (Section 9.3.3) inserts the elements of S one by one. Let $e \in S$ be the last element inserted. Prove: when e is inserted, $A[g(e)]$ is occupied with probability at most $\frac{1}{4e^3}$.

(Hint: for any $e' \neq e$, $\Pr[g(e) = g(e')] = 1/N$.)

Remark: this means the insertion of e requires no bumping process with probability at least $1 - \frac{1}{4e^3} > 98\%$.

Problem 6. Same settings as in Problem 5. Suppose that the insertion of e launches the bumping process. Recall that the process evicts a sequences of elements; let the sequence be $e_1, e_2, ..., e_\ell$.

(a) Prove: $\Pr[\ell > 1] \leq \frac{1}{4e^3}$.

(b) Assume that e, e_1, e_2 are distinct. Prove: $\Pr[\ell > 2] \leq \left(\frac{1}{4e^3}\right)^2$.

(c) Assume that $e, e_1, ..., e_t$ are distinct. Prove: $\Pr[\ell > t] \leq \left(\frac{1}{4e^3}\right)^t$.

(Hint: if you can solve (a), you can solve the rest. For (a), think of something similar to Problem 5.)

Lecture 10: Binomial and Fibonacci Heaps

In your undergraduate study, you should have learned that a *heap* (a.k.a. a *priority queue*) supports the following operations on a set S of n elements from an ordered domain.

- **Insertion:** add a new element in S;
- **Delete-min:** find and remove the smallest element in S.

It is easy to design a structure of $O(n)$ space that supports both operations in $O(\log n)$ time (e.g., the BST).

This lecture will introduce two new heap implementations. The first one, called the *binomial heap*, achieves $O(1)$ amortized cost per insertion and $O(\log n)$ amortized cost per delete-min. Thus, any mixture of n_1 insertions and n_2 delete-mins can be processed in $O(n_1 + n_2 \cdot \log n_1)$ time, which is much better than the "undergraduate heap" if $n_1 \gg n_2$.

In the second part, we will modify the binomial heap to support an additional "decrease-key" operation (to be defined in Section 10.2.5) in $O(1)$ amortized time. The modification yields the *Fibonacci heap*, which allows us to improve the running time of several fundamental graph algorithms (e.g., Dijkstra's and Prim's), compared to their implementation using the "undergraduate heap".

We need to clarify some jargon about trees. Usually, we do not assume any ordering on the children of a node in a tree. However, such orderings are important to Binomial and Fibonacci heaps. In every "tree" to appear in this lecture, the child nodes of a node u are always ordered. If node v is the i-th ($i \geq 1$) child of u, we say that v has *child rank* i. Accordingly, $sub(v)$ (the subtree rooted at v or simply the subtree of v) is said to be *the i-th proper subtree* of u. Every node has a child rank, with the root being the only exception.

Furthermore, we will use the term "heap" in a broad sense. Let $\mathcal{T}$ be a tree where each node u stores an integer *key*, denoted as $key(u)$. We call $\mathcal{T}$ a *heap* if, for any node u in $\mathcal{T}$, $key(u)$ is the smallest in $sub(u)$.

10.1 The binomial heap

10.1.1 Binomial trees

We now introduce the binomial tree (which is to be distinguished from binomial heap).

Definition 10.1. A *binomial tree of order 0* is a single node. Inductively, a *binomial tree of order k* is a tree where

- the root has k child nodes;
- the i-th ($i \in [k]$) proper subtree of the root is a binomial tree of order $i - 1$.

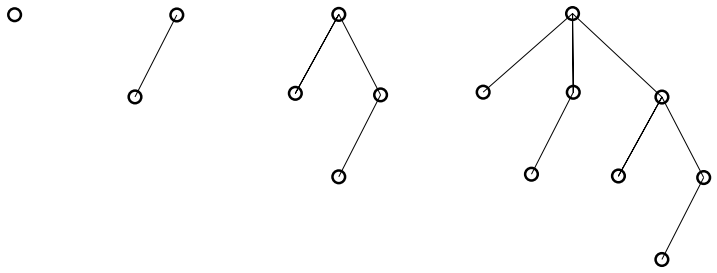


Figure 10.1: Binomial trees of orders 0, 1, 2, and 3

□

The definition implies that if a non-root node u has child rank i, u itself must have exactly $i-1$ child nodes. See Figure 10.1 for an illustration. The next proposition is easy to prove.

Proposition 10.2. *A binomial tree of order k has 2^k nodes.*

10.1.2 The structure of a binomial heap

We are ready to define the binomial heap.

Definition 10.3. Let S be a set of values in $\mathbb{R}$. A *binomial heap* on S is a set Σ of binomial trees such that

- each node of a tree in Σ stores an element of S as the *key*;
- every element in S is the key of exactly one node (counting all the trees in Σ);
- every tree in Σ is a heap.

The binomial heap is *clean* if no two trees in Σ have the same order; otherwise, it is *dirty*. □

Proposition 10.4. *A binomial heap on S uses space $O(n)$, and every binomial tree in Σ has order $O(\log n)$, where $n = |S|$.*

Proof. The space bound follows directly from Definition 10.3. The claim on the order follows from Proposition 10.2. □

10.1.3 Insertion

To insert an element e_{new}, we first make an order-0 binomial tree B where the (only) node stores e_{new} as the key and then add B into Σ. The cost is $O(1)$. It is worth mentioning that an insertion may leave the binomial heap in a dirty state.

10.1.4 Delete-min

Denote by m the current size of Σ. To perform a delete-min, we find in $O(m)$ time the tree B in Σ whose root has the smallest key (which must be the smallest in S). Next, we remove $root(B)$ and, for each child node u of $root(B)$, add $sub(u)$ to Σ. The cost is $O(m + f) = O(m + \log n)$ so far.

Finally, a *cleanup process* is launched to convert the binomial heap to a clean state. To start, we create $O(\log n)$ linked lists L_i, $0 \leq i = O(\log n)$, where L_i stores a pointer to every binomial

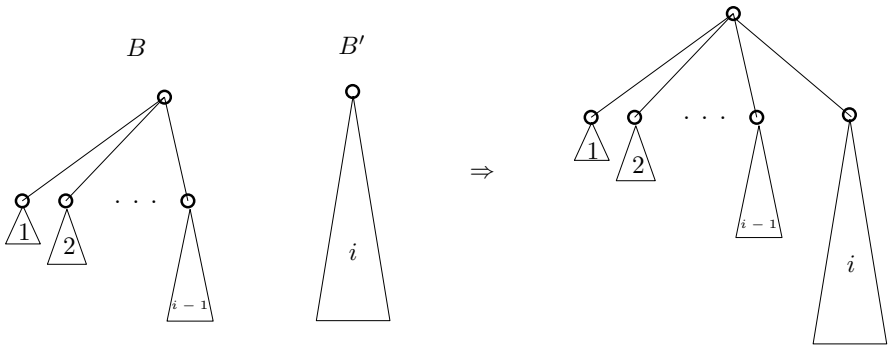


Figure 10.2: Illustration of merge

tree in Σ with order i. Then, we process i in ascending order as follows: as long as L_i has two trees B and B', merge them by making $root(B')$ the last child of $root(B)$ (assume, w.l.o.g., that $root(B)$ has a smaller key than $root(B')$). See Figure 10.2 for an illustration. B becomes an order-$(i+1)$ tree and moves from L_i to L_{i+1}. B', on the other hand, disappears from L_i. The number of trees in Σ decreases by one after each merge. Therefore, the cleanup process takes $O(m + \log n)$ time in total (Σ has at most $m + O(\log n)$ trees when the process starts).

10.1.5 Amortization

We will use the potential method (Section 8.1) to prove that an insertion and delete-min have $O(1)$ and $O(\log n)$ amortized cost, respectively. Define a potential function

$$\Phi(\Sigma) \quad = \quad c \cdot |\Sigma|$$

where c is a sufficiently large constant to be decided later.

Each insertion takes constant time and increases the potential by c. By Lemma 8.1, the insertion has amortized cost $O(1) + c = O(1)$.

We now turn attention to delete-min. At the beginning, the potential is $c \cdot m$ where m is the size of Σ at that moment. After the operation, the binomial heap is clean, meaning that $|\Sigma| = O(\log n)$. Hence, the potential function changes by $-c \cdot m + O(\log n)$. Given that the operation incurs $O(m + \log n)$ actual time, it is amortized

$$O(m + \log n) - c \cdot m + O(\log n)$$

cost, which is $O(\log n)$ by choosing c sufficiently large.

10.2 The Fibonacci heap

The Fibonacci heap is similar to the Binomial heap except that it adopts a relaxed version of the binomial tree.

10.2.1 Relaxed binomial trees

Definition 10.5. A *relaxed binomial tree of (RBT) order 0* is a tree of a single node. Inductively, an *RBT of order k* is a tree where

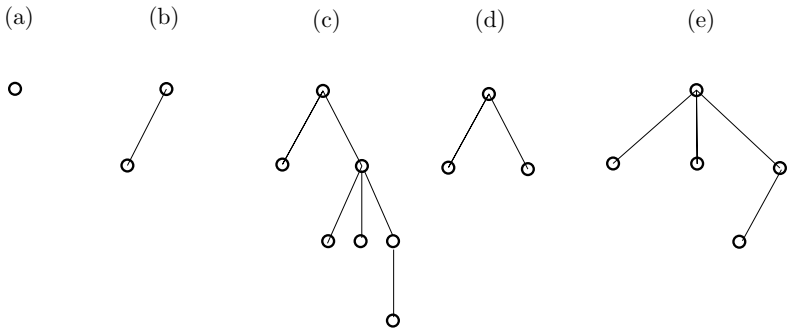


Figure 10.3: From (a) to e: relaxed binomial trees of orders 0, 1, 2, 2, and 3, respectively

- the root has k child nodes;
- the i-th ($i \in [k]$) proper subtree of the root is an RBT with order at least $\max\{0, i-2\}$.

□

Comparing the above to Definition 10.1, one can see that the relaxation is in the second bullet: we no longer require the root's i-th proper subtree to have a *specific* order; instead, we place a lower bound on the order. See Figure 10.3 for an illustration. Note that, for $k \geq 1$, order-k RBTs are not unique; in fact, the number of possible order-k RBTs is infinite. Furthermore, observe that Definition 10.5 implies:

> **Order invariant:** If a node u has child rank i, u itself has at least $\max\{0, i-2\}$ child nodes.

We now prove an important lemma.

Lemma 10.6. *An RBT of order k has at least $(\frac{1+\sqrt{5}}{2})^{k-2}$ nodes if $k \geq 2$.*

Proof. Define $f(k)$ to be the smallest number of nodes in an RBT of order k. Clearly, $f(0) = 1$ and $f(1) = 2$. By definition, an order-k RBT with $k \geq 2$ must have at least $1 + f(0) + \sum_{i=2}^{k} f(i-2)$ nodes where the term 1 counts the root, $f(0)$ is the minimum size of the root's first proper subtree, and $\sum_{i=2}^{k} f(i-2)$ is the minimum size of the root's other proper subtrees. It follows that

$$f(k) \quad \geq \quad 1 + f(0) + \sum_{i=2}^{k} f(i-2) = 2 + \sum_{i=0}^{k-2} f(i). \tag{10.1}$$

Fibonacci numbers $F_0, F_1, F_2, ...$ are defined as

$$F_k \quad = \quad \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2 \end{cases}$$

and have the following well-known properties (proof omitted):

Proposition 10.7. *$F_{k+2} \geq (\frac{1+\sqrt{5}}{2})^k$ and $F_{k+2} = 1 + \sum_{i=0}^{k} F_i$.*

Next, we will prove

Claim: $f(k) \geq F_k$ for $k \geq 0$

which together with Proposition 10.7 will establish Lemma 10.6. It is easy to verify the claim for $k = 0$ and $k = 1$. Fix an arbitrary $t \geq 2$. Assuming inductively the correctness for any $k \leq t - 1$, next we prove the claim for $k = t$.

$$\begin{aligned} \text{(10.1) and the inductive assumption} \Rightarrow f(t) &\geq 2 + \sum_{i=0}^{t-2} F_i \\ \text{(by Proposition 10.7)} &= 1 + F_t \\ &> F_t. \end{aligned}$$

We now complete the proof of Lemma 10.6. □

10.2.2 The structure of a Fibonacci heap

Definition 10.8. Let S be a set of values in $\mathbb{R}$. A *Fibonacci heap* on S is a set Σ of RBTs such that

- every node of each tree in Σ stores an element of S as the *key*;
- every element in S is the key of exactly one node (counting all the trees in Σ);
- every tree in Σ is a heap.

The Fibonacci heap is *clean* if no two RBTs in Σ have the same order; otherwise, it is *dirty*. □

Proposition 10.9. *A Fibonacci heap on S uses $O(n)$ space and every RBT in Σ has order $O(\log n)$ where $n = |S|$.*

Proof. The space bound follows directly from Definition 10.8. The claim on the order follows from Proposition 10.6. □

Every node u (of each tree in Σ) has a *color*, denoted as $color(u)$, which can be *white* or *black*. At all times, we enforce the following invariant for every *non-root* node:

> **Color invariant:** Suppose that a non-root node u has child rank i. If u is white, $sub(u)$ must be an RBT of order at least $i - 1$. If u is black, $sub(u)$ must be an RBT of order $\max\{0, i - 2\}$.

No color constraints are placed on the roots of the trees in Σ.

Proposition 10.10. *Let T be an RBT in Σ of order k. Let u be a non-root internal node in T with $color(u) = white$, and v be an arbitrary child of u. T is still an RBT of order k even after we remove $sub(v)$ from T.*

Proof. Suppose that $sub(u)$ is an RBT of order $k_u \geq 1$ before the removal of $sub(v)$. After the removal, because the child rank can only decrease for each remaining child of u, $sub(u)$ must be an RBT of order $k_u - 1$.

Let $p = parent(u)$. Suppose that $sub(v)$ is an RBT of order k_p before the removal. Next, we prove that, after the removal, $sub(p)$ is still an RBT of order k_p. Let i be the child rank of u. It suffices to prove that, after the removal, $k_u - 1 \geq \max\{0, i - 2\}$. This is true because u is white and hence we must have $k_u \geq \max\{1, i - 1\}$ before the removal.

The proposition now follows. □

One can intuitively understand the colors' meanings as follows. A white u with child rank i has at least $i-1$ child nodes currently. It is "safe" in the sense that it can afford to lose a child (Proposition 10.10). However, if u is black, there is a chance that it may have exactly $\max\{0, i-2\}$ child nodes currently. In that case, it cannot afford to lose any child (see the order invariant in Section 10.2.1).

10.2.3 Insertion

To insert an element e_{new}, we make an order-0 RBT-tree R where the (only) node has key e_{new} and is colored white. Then, add R into Σ. The total cost is $O(1)$. An insertion may leave the Fibonacci heap in a dirty state.

10.2.4 Delete-min

Denote by m the size of Σ at the beginning of a delete-min. The operation finds in $O(m)$ time the tree $R \in \Sigma$ whose root has the smallest key (which must be the smallest in S). Then, remove $root(R)$ and, for each child node u of $root(R)$, color u white and add $sub(u)$ to Σ. The cost is $O(m + \log n)$ (Proposition 10.6 shows that $root(R)$ has $O(\log n)$ child nodes).

Launch a *cleanup process* to convert the Fibonacci heap to a clean state. First, create $O(\log n)$ linked lists L_i, $0 \leq i = O(\log n)$, where L_i stores a pointer to every Fibonacci tree in Σ with order i. Then, process i in ascending order as follows: as long as L_i has two trees R and R', merge them as follows:

merge(R, R')
/* R and R' have the same order */
/* w.o.l.g., assume that $root(R)$ has a smaller key than $root(R')$*/
1. color $root(R)$ and $root(R')$ white
2. make $root(R')$ the last child of $root(R)$

Proposition 10.11. *After the merge, R is an RBT of order $i+1$. Furthermore, the color invariant still holds.*

Proof. Let $r = root(R')$. R' has order i and becomes the $(i+1)$-th proper subtree of $root(R)$. Hence, R is an RBT of order $i+1$ after the merge. The color invariant holds after the merge because r is white, consistent with the fact that r now has child rank $i+1$ and $sub(r)$ has order i. □

After the merge, R moves from L_i to L_{i+1} and R' disappears from L_i. The cleanup process finishes in $O(m + \log n)$ time (same analysis as in Section 10.1.4).

10.2.5 Decrease-key

A salient functionality of the Fibonacci heap is the following operation:

> **Decrease-key**(u, x_{new})**:** Parameter u is a node in some tree of Σ and needs to satisfy $key(u) > x_{new}$. The operation deletes $key(u)$ from S and adds x_{new} to S.

Let $R \in \Sigma$ be the tree containing u. If $u = root(R)$, the operation simply sets $key(u) = x_{new}$ and finishes (think: why correct?). Consider now $u \neq root(R)$; let $p = parent(u)$. If $x_{new} > key(p)$, we carry out the modification and finish (think: why correct?).

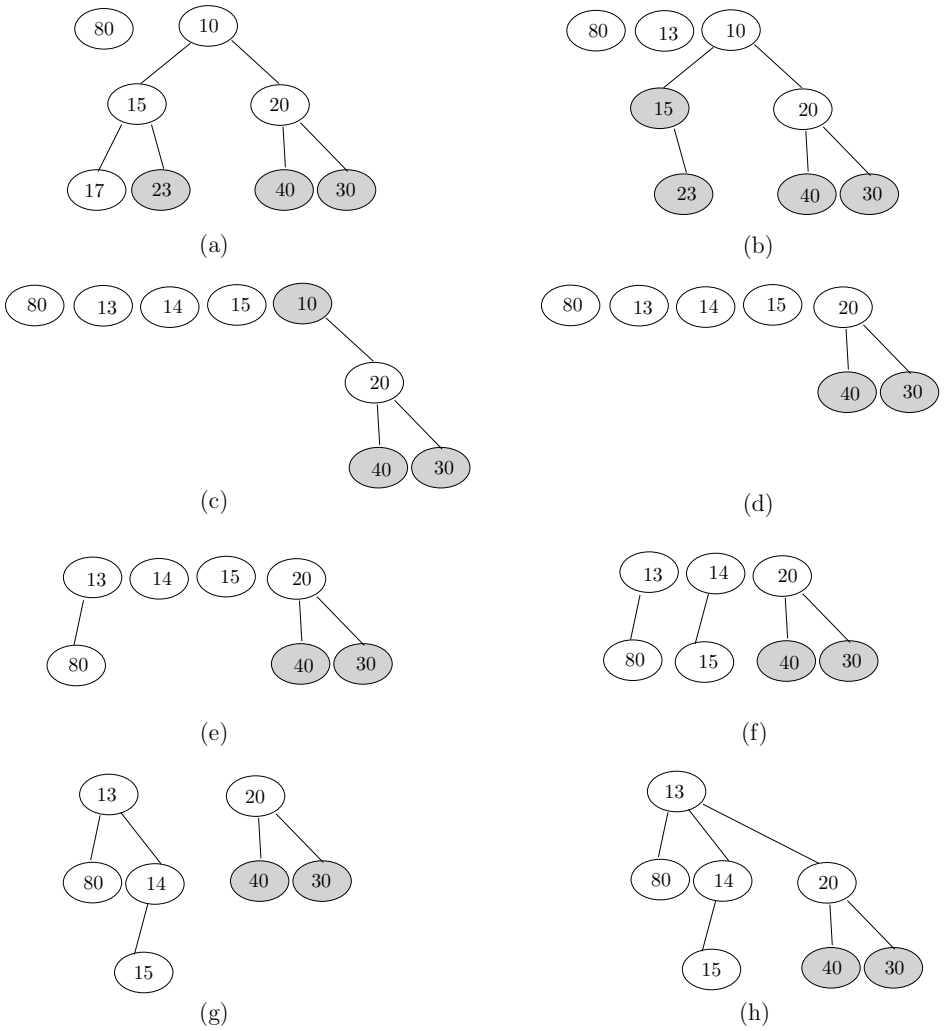


Figure 10.4: (a) shows a Fibonacci heap. (b) is the heap after decreasing 17 to 13, and (c) after decreasing 23 to 14. (d) shows the heap during a delete-min operation; here the smallest element 10 has been removed. Continuing the delete-min, (e) merges the trees of 13 with 80, (f) merges the trees of 14 with 15, (g) merges the trees of 13 with 14, and (h) merges the trees of 13 with 20.

In case $x_{new} \leq key(p)$, we remove $sub(u)$ from R, color u white, and add $sub(u)$ to Σ. Then, we process p using the algorithm below.

fix-parent(p)
/* p just lost a child */
1. **if** $color(p)$ = white **then** color p black and **return**
   /* next consider $color(p)$ = black */
2. remove $sub(p)$ from the RBT containing p, color p white, and add $sub(p)$ to Σ
3. **if** p has a parent **then fix-parent**($parent(p)$)

If Line 2 executes on p, we say that p is *repaired*. Note that **fix-parent** can propagate to ancestors. The total cost of decrease-key is $O(1 + g)$ where g is the number of nodes repaired.

Example. Figure 10.4(a) shows a Fibonacci heap where each node is labeled with its key. Σ has

two RBTs with order 0 and 2, respectively. Consider a decrease-key which reduces 17 to 13. The operation first modifies 17 to 13, which, however, is less than the parent 15. Thus, we remove the subtree of 13 and add it to Σ (which has 3 trees now). Node 15 turns black. Figure 10.4(b) gives the current heap. The next decrease-key reduces 23 to 14. It modifies 23 to 14, removes its subtree, and adds it to Σ. The color of node 14 changes from black to white. As node 15 is black, we repair it by removing its subtree, adding its subtree to Σ, and coloring it white. Node 10 then turns black. The current heap is presented in Figure 10.4(c).

A delete-min operation at this moment goes through the roots of all the 5 trees in Σ to identify the smallest element 10. After 10 is deleted, its (only) subtree is added to Σ, giving rise to Figure 10.4(d). A cleanup process is launched to merge the trees of the same order. Figure 10.4(e) merges 13 and 80 into a tree of order 1, and similarly, Figure 10.4(f) merges 14 with 15. The trees of 13 and 14 are then merged, yield a tree of order 2 as shown in Figure 10.4(g). Finally, the trees of 13 and 20 are merged, producing the final Fibonacci heap in Figure 10.4(h). □

The proof of the next proposition is left as an exercise.

Proposition 10.12. *After a decrease-key, all the trees in Σ are still RBTs. Furthermore, the color invariant still holds.*

10.2.6 Amortization

We will prove that an insertion, delete-min, and decrease-key have amortized cost $O(1)$, $O(\log n)$, and $O(1)$, respectively. Define a potential function:

$$\Phi(\Sigma) \quad = \quad c_1 \cdot |\Sigma| + c_2 \cdot (\text{number of black nodes})$$

where c_1 and c_2 are constants to be decided later.

Each insertion takes constant time and increases the potential by c_1. By Lemma 8.1, the insertion is amortized a cost of $O(1) + c_1 = O(1)$.

Consider a delete-min. Let m be the size of Σ before the operation. When the delete-min finishes, the Fibonacci heap is clean, meaning that $|\Sigma| = O(\log n)$. The clean-up process can only decrease the number of black nodes. Therefore, the operation decreases the potential by at least $c_1 \cdot m - O(\log n)$. Given that the delete-min is processed in $O(m + \log n)$ time, its amortized cost is at most

$$O(m + \log n) - (c_1 \cdot m - O(\log n))$$

which is $O(\log n)$ when c_1 is larger than the hidden constant in the first big-O.

Finally, consider a decrease-key performed on node u. If **fix-parent** is not invoked, the cost is constant. Next, we assume that **fix-parent** is called. Denote by g the number of nodes repaired. Let us make some observations.

- Every node repaired is black before the operation but white afterwards. On the other hand, **fix-parent** can turn at most one node from white to black (this can happen only at Line 1). Thus, the number of black nodes drops by $g - 1$ (if $g = 0$, then $g - 1 = -1$ and "dropping" by -1 means increasing by 1).
- Because $sub(u)$ and the subtree of every repaired node are inserted into Σ, $|\Sigma|$ increases by $g + 1$.

Hence, the decrease-key increases the potential by $c_1(g+1) - c_2 \cdot (g-1)$. As the decrease-key incurs $O(1+g)$ actual computation time, its amortized cost is

$$O(1+g) + c_1(g+1) - c_2 \cdot (g-1)$$

which is $O(1)$ as long as c_2 is greater than the sum of c_1 and the hidden constant of the first big-O.

10.3 Remarks

The binomial heap was proposed by Vuillemin [41], while the Fibonacci heap by Fredman and Tarjan [20].

Exercises

Problem 1. Prove Proposition 10.12.

(Hint: First, prove that all the trees in Σ are still RBTs. For this purpose, use two facts. First, if $sub(u)$ is an RBT of order k, it must be an RBT of order $k-1$ after u loses a child, regardless of $color(u)$. Second, if u is black, $sub(u)$ is removed immediately after u loses a child; this makes sure that the subtree of $parent(p)$ is still an RBT. Second, prove that the color invariant holds because after a white node loses a child, it satisfies the requirement of black node.)

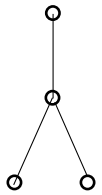
Problem 2. Suppose that we want to support an extra operator called **find-min** which reports the smallest key in S, but does not remove it. Explain how to adapt the binomial heap to support this operation in $O(1)$ worst-case time, without affecting the performance guarantees of insertion and delete-min.

Problem 3*. Explain how to modify the binomial heap's algorithm to support delete-min in $O(\log n)$ time (no amortization) and (as before) an insertion in $O(1)$ amortized time.

(Hint: keep the binomial heap clean *at all times*).

Problem 4. Prove or disprove: a relaxed binomial tree of n nodes has height $O(\log n)$.

Problem 5. Give a sequence of insert, delete-min, and decrease-key operations on an initially empty set such that the Fibonacci heap after all the operations has a single tree that looks like:



Problem 6 (meld). Let S_1 and S_2 be two disjoint sets. Given a Fibonacci heap Σ_1 on S_1 and a Fibonacci heap Σ_2 on S_2, explain how to obtain a Fibonacci heap on $S_1 \cup S_2$ in constant time.

Problem 7. Implement Dijkstra's algorithm on a graph of n nodes and m edges in $O(m + n \log n)$ time.

Lecture 11: Union-Find Structures

This lecture will discuss the *disjoint set problem*. Let V be a set of n integers. $\mathcal{F}$ is a collection of disjoint sets such that each set in $\mathcal{F}$ is a non-empty subset of V, and the union of all the sets in $\mathcal{F}$ is V. We want to support the operations below:

- **makeset**(e): Given an integer $e \notin V$, add e to V and add a singleton set $\{e\}$ to $\mathcal{F}$.
- **find**(e): Given an $e \in V$, report which set $S \in \mathcal{F}$ contains $e \in V$.
- **union**(e, e'): In this operation, we are given two elements e and e' in V. Suppose that S and S' are the sets in $\mathcal{F}$ that contain e and e', respectively. We want to remove S and S' from $\mathcal{F}$ and add $S \cup S'$ to $\mathcal{F}$. The operation essentially combines S and S' into one set.

The output of **find**(e) can be anything that can uniquely identify the set containing e. However, the same identifier must be used for the same set, i.e., if e and e' belong to the same set, the outputs of **find**(e) and **find**(e') must be identical. For simplicity, we assume that V is empty in the beginning (before any operation is performed).

Data structures solving the disjoint set problem are called *union-find structures*.

11.1 Structure and algorithms

Structure. We store each set $S \in \mathcal{F}$ in a tree T where

- T has $|S|$ nodes;
- every element $e \in S$ is stored at a distinct node u in T;
- each node u stores an integer $rank(u)$, referred to as the *rank* of u.

Note that the number of trees is $|\mathcal{F}|$. The space consumption is obviously $O(n)$.

Makeset(e)**.** Create a tree with a single node storing e whose rank is 0. The operation finishes in $O(1)$ time.

Union(e, e')**.** Denote by T (or T', resp.) the tree that contains e (or e', resp.). We will proceed by making the following assumption.

Assumption 1: We are given the roots r and r' of T and T', respectively.

The assumption's removal is easy and will be left as an exercise. The union operation is performed as follows:

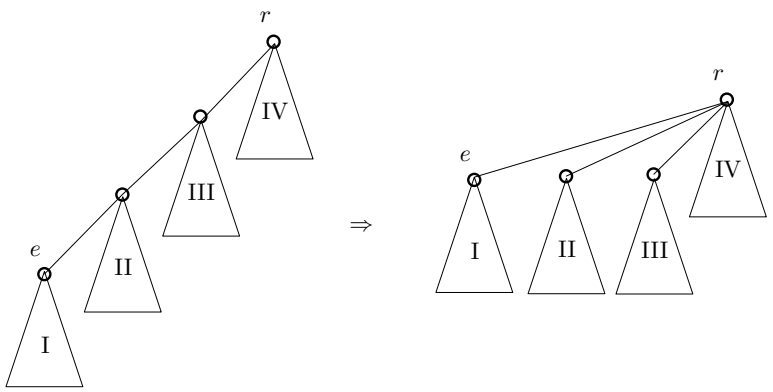


Figure 11.1: Illustration of **find**

union(r, r')
/* assume, w.o.l.g., that $rank(r) \geq rank(r')$ */
1. make r' a child of r
2. **if** $rank(r) = rank(r')$ **then**
3. increase $rank(r)$ by 1

The operation again finishes in $O(1)$ time.

Find(e)**.** For this operation, we need to make another assumption.

Assumption 2: The node where e is stored is given.

The operation proceeds as follows:

find(e)
/* let T be the tree where e is stored */
1. $\Pi \leftarrow$ the path from node e to the root r of T
2. **for** each node u on Π **do**
3. set $parent(u) \leftarrow r$
4. **return** r

See Figure 11.1 for an illustration. Note that r is used as the identifier of the set stored in T. The running time is $O(|\Pi|)$ where $|\Pi|$ gives the number of nodes on Π. This may appear large, but as the rest of the lecture will discuss, the amortized cost of **find** turns out to be very small.

11.2 Analysis 1

We will prove an amortized cost of $O(\log^* n)$ for all operations, where $\log^* n$ is the smallest t satisfying $\underbrace{\log_2 \log_2 ... \log_2}_{t} n < 2$. For example, $\log^* 16 = 3$ because $\log_2 \log_2 16 = 2$ while $\log_2 \log_2 \log_2 16 = 1$. It is worth mentioning that $\log^* n \leq 5$ for all $n \leq 2^{65536}$, which is already larger than the total number of atoms on earth. The $O(\log^* n)$ bound will be subsumed by another result to be established in Section 11.3. However, the argument in this section is (much) simpler, and illustrates some properties that will also be useful in Section 11.3.

11.2.1 Basic properties

Let us start with several basic facts about the node ranks (Section 11.1).

Proposition 11.1. *Once a node u stops being a root, $rank(u)$ is fixed forever.*

Proof. The rank of u is modified only in **union** and only when u is a root. Once u becomes a non-root, it will never be a root again. □

Proposition 11.2. *For any non-root node u, $rank(parent(u)) > rank(u)$.*

Proof. Easy to show by induction on the operations performed. □

Proposition 11.3. *Consider an arbitrary node u in the structure. Every time* **find** *changes $parent(u)$, the new $parent(u)$ must have a larger rank than the old $parent(u)$.*

Proof. Let $p = parent(u)$ at the current moment. If **find** modifies $parent(u)$, p cannot be the root r of the tree where u belongs. By Proposition 11.2, $rank(p) < rank(r)$. The claim follows from the fact that $r = parent(u)$ after the **find** operation. □

Proposition 11.4. *A root u with rank ρ has at least 2^ρ nodes in its subtree.*

Proof. This is obviously true for $\rho = 0$. Inductively, assuming the claim's correction on $\rho = i - 1$, we will prove its correctness on $\rho = i$. The rank of u can increase from $i-1$ to i only when u takes a new child v with rank $i - 1$ in a **union** operation. Before the operation, both $sub(u)$ and $sub(v)$ must have 2^{i-1} nodes (inductive assumption). It follows that, after the operation, $sub(u)$ has 2^i nodes after the operation. □

Corollary 11.5. *The rank of a node is $O(\log n)$.*

Proof. Directly follows from Proposition 11.4. □

Lemma 11.6. *At most $n/2^\rho$ nodes have rank ρ.*

Proof. When the rank of a node u increases to ρ in a **union** operation, we conceptually *assign* to u all the nodes in $sub(u)$; remember that u must be a root at the moment (due to the way **union** runs). We argue that, for any two nodes u_1 and u_2 with rank ρ, the set of nodes assigned to u_1 is disjoint with that to u_2. This will complete the proof because, by Proposition 11.4, at least 2^ρ nodes are assigned to u when $rank(u)$ reaches ρ.

Let v be a node assigned to u when $rank(u)$ reaches ρ. We will show that v will not be assigned to any other node. Suppose, on the contrary, that v is later assigned to a different node u' when $rank(u')$ reaches ρ. When this happens, u' must be the root of the tree T containing v. Thus, u is also in T (if two nodes are in the same tree, they will remain so forever). Hence, u' is a proper ancestor of u. However, Propositions 11.1 and 11.2 suggest that $rank(u') > rank(u) \geq \rho$, contradicting $rank(u') = \rho$. □

Corollary 11.7. *At most $n/2^{\rho-1}$ nodes have ranks at least ρ.*

Proof. By Lemma 11.6, the number of such nodes is at most

$$\sum_{i=\rho}^{\infty} \frac{n}{2^\rho} \quad < \quad \frac{n}{2^{\rho-1}}.$$

□

11.2.2 An $O(\log \log n)$ bound

In this subsection, we will prove an amortized bound of $O(\log \log n)$ per operation. We will prove a slightly weaker claim: any sequence of operations can be processed in $O(n \log \log n)$ total time, where n is the number of elements in V at the end of the sequence (think: why this is a weaker claim?). It is easy to adapt the proof to prove an $O(\log \log n)$ amortized bound.

We divide the nodes with positive ranks into *groups*. Specifically, group $g \geq 0$ includes all the nodes u satisfying

$$rank(u) \in [2^g, 2^{g+1}). \tag{11.1}$$

Because of Corollary 11.5, the number of groups is $O(\log \log n)$.

Now consider a **find**(e) operation. Recall that it finishes in $O(|\Pi|)$ time, where Π is the path from the root r to the node e. We account for the cost by looking at each node $u \in \Pi$:

- Case 1: If u has rank 0, charge $O(1)$ cost on **find**.
- Case 2: If $u = r$ or $parent(u) = r$, charge $O(1)$ cost on **find**.
- Case 3: If u and $parent(u)$ are in different groups, charge $O(1)$ cost on **find**.
- Case 4: Otherwise, charge $O(1)$ cost on u.

Thus, all the $O(|\Pi|)$ time has been amortized on either **find** or individual nodes.

Proposition 11.8. *Cases 1-3 charge $O(\log \log n)$ time on each* **find.**

Proof. Cases 1 and 2 obviously charge only $O(1)$ time on **find.** Consider Case 3. By Proposition 11.2, as we ascend Π, the node rank monotonically increases. Thus, if Case 3 applies x times, we can find x nodes on Π with increasingly larger group numbers. The claim follows because x is at most the number of groups, which is $O(\log \log n)$. □

Lemma 11.9. *Case 4 can charge $O(n \log \log n)$ cost in total for all the* **find** *operations.*

Proof. Case 4 can happen only on a node u whose group number g has already been forever fixed. To see why, first note that, by definition of Case 4, u must be a non-root node. By Proposition 11.1, $rank(u)$ has already been fixed forever and, hence, so has g.

As every node in group g has rank at least 2^g, Corollary 11.7 shows that group g can have at most $n/2^{2^g-1} = O(n/2^{2^g})$ nodes. Later, we will argue that every node in group g can be charged at most 2^g times. This will indicate that Case 4 charges a cost of

$$O\left(\frac{n}{2^{2^g}} \cdot 2^g\right) \quad = \quad O(n) \tag{11.2}$$

on group-g nodes in total. The lemma will then follow from the fact that there are $O(\log \log n)$ groups.

It remains to prove that a node u in group g can be charged at most 2^g times. Every time u is charged by a **find** operation, $parent(u)$ must also be in group g (otherwise, Case 3 would have applied). The operation changes $parent(u)$ in the end because $parent(u) \neq r$ before the operation (otherwise, Case 2 would have applied; recall that r is the root of the tree containing u) while $parent(u) = r$ afterwards. By Proposition 11.3, the new $parent(u)$ has a larger rank than the old $parent(u)$. As there are only 2^g distinct ranks in group g, u can be charged in Case 4 at most 2^g times before the rank of $parent(u)$ moves out of group g. □

We amortize the $O(n \log\log n)$ bound in Lemma 11.9 over the n **makeset** operations that created the n nodes in V. Each operation therefore bears $O(\log\log n)$ cost.

11.2.3 An $O(\log^* n)$ bound

Let us change the definition of group g to

$$rank(u) \in [2^{2^g}, 2^{2^{g+1}}). \tag{11.3}$$

The number of groups drops to $O(\log\log\log n)$. Repeating the above analysis gives an amortized bound of $O(\log\log\log n)$, as is left as an exercise. To push the power of the argument to the extreme, let us adopt the following definition:

$$rank(u) \in \Big[\underbrace{2^{2^{\cdots^2}}}_{g}, \underbrace{2^{2^{\cdots^2}}}_{g+1}\Big). \tag{11.4}$$

The number of groups is now $O(\log^* n)$. The same argument yields an amortized bound of $O(\log^* n)$ and is also left as an exercise.

11.3 Analysis 2*

In this section, we will prove an amortized bound of $O(\alpha(n))$ for each operation, where $\alpha(n)$ is the inverse of the Ackermann's function. As n grows, $\alpha(n)$ increases extremely slowly, e.g., $\alpha(n) \leq 5$ for $n = \underbrace{2^{2^{\cdots^2}}}_{2^{2048}}$. We start with an introduction to this bizarre-looking function.

11.3.1 Ackermann's function and its inverse

What we will discuss is one of the many variants of Ackermann's function. Denote by $\mathbb{N}_{\geq 0}$ the set of positive integers. Given a function $f : \mathbb{N}_{\geq 0} \to \mathbb{N}$, we define for $k \geq 1$

$$f^{(k)}(n) \quad = \quad \underbrace{f(f(...f}_{k}(n)...)).$$

For example, $\log_2^{(2)} n = \log_2 \log_2 n$ and should not be confused with $\log_2^2 n = (\log_2 n)^2$.

Now, we introduce a family of functions from $\mathbb{N}_{\geq 0}$ to $\mathbb{N}$:

$$\begin{aligned} A_0(x) &= x + 1 \\ A_k(x) &= A_{k-1}^{(x+1)}(x) \text{ for } k \geq 1. \end{aligned} \tag{11.5}$$

To see how quickly these functions grow, consider some small values of k:

$$\begin{aligned} A_1(x) = A_0^{(x+1)}(x) = \underbrace{A_0(A_0(...A_0}_{x+1}(x)...)) = 2x + 1 &> 2x \\ A_2(x) = A_1^{(x+1)}(x) = \underbrace{A_1(A_1(...A_1}_{x+1}(x)...)) > x2^x &\geq 2^x \\ A_3(x) = A_2^{(x+1)}(x) = \underbrace{A_2(A_2(...A_2}_{x+1}(x)...)) &> \underbrace{2^{2^{\cdots^2}}}_{x}. \end{aligned}$$

If we define $2 \uparrow x = \underbrace{2^{2^{\cdots^2}}}_{x}$, then

$$A_4(x) = A_3^{(x+1)}(x) \quad > \quad \underbrace{2 \uparrow (2 \uparrow (...(2 \uparrow 2)...))}_{x \uparrow\text{'s}}$$

Calling $A_k(2)$ *Ackermann's function* (which is a function of k), we define the *inverse* of Ackermann's function as

$$\alpha(n) \quad = \quad \text{the smallest } k \text{ satisfying } A_k(1) \geq n. \tag{11.6}$$

11.3.2 An $O(\alpha(n))$ bound

For every non-root node u with $rank(u) \geq 1$, define $k(u)$ as the largest integer $k \geq 0$ satisfying

$$rank(parent(u)) \quad \geq \quad A_k(rank(u)) \tag{11.7}$$

where $A_k(.)$ is given in (11.5). The definition is sound because, by Proposition 11.2, $rank(parent(u)) \geq rank(u) + 1 = A_0(rank(u))$. Note that, as k grows, $A_k(rank(u))$ increases very rapidly (Section 11.3.1). The value of $k(u)$ captures the largest k such that $A_k(rank(u)) \leq rank(parent(u))$.

It is important to note that, even though $rank(u)$ has been fixed forever (by Proposition 11.1 and the fact u is non-root), $k(u)$ can still grow. This is because $parent(u)$ may change due to **find** operations, and every time it happens, $rank(parent(u))$ increases, which may bump up $k(u)$. On the other hand, it is easy to see that $k(u)$ can never decrease.

We divide the non-root nodes into *groups*, but in a way different from Section 11.2.2. Specifically, group $g \geq 0$ includes all the non-root nodes u with $k(u) = g$. As mentioned earlier, as $k(u)$ can increase over time, u may move to groups of higher numbers.

Proposition 11.10. *$0 \leq k(u) \leq \alpha(n)$, namely, there are at most $1 + \alpha(n)$ groups.*

Proof. Lemma 11.6 implies that every node has rank at most $O(\log n)$. The claim follows from the definition in (11.6).[1] □

Consider a **find**(e) operation, which finishes in $O(|\Pi|)$ time where Π is the path from the root r to the node e. We account for the cost by looking at each node $u \in \Pi$:

- Case 1: If $rank(u) = 0$ or u is a root, charge $O(1)$ cost on **find**.
- Case 2: If u has a proper non-root ancestor v such that $k(v) = k(u)$, charge $O(1)$ cost on u (note: v can be, but is not necessarily, the parent of u).
- Case 3: Otherwise, charge the cost on **find**.

Thus, the $O(|\Pi|)$ time of **find** has been amortized on either the operation itself or individual nodes.

Proposition 11.11. *Case 1 can apply at most twice on each* **find.**

Proof. There are only one rank-0 node and one root on Π. □

[1] You would probably ask why not $O(\alpha(\log n))$. In fact, it *is* $O(\alpha(\log n))$, except that this is not very helpful because we can prove $\alpha(n) = O(\alpha(\log n))$.

Proposition 11.12. *Case 3 charges $O(\alpha(n))$ time on each* **find.**

Proof. If Case 3 applies x times on a **find**, we can find x nodes u on Π with *distinct* $k(u)$. The claim follows then from Proposition 11.10. □

The rest of the section serves as a proof for:

Lemma 11.13. *Case 2 can charge $O(n \cdot \alpha(n))$ cost in total for all the* **find** *operations.*

We amortize the above cost over the n **makeset** operations that created the n nodes in V. Each operation therefore bears $O(\alpha(n))$ cost.

11.3.3 Proof of Lemma 11.13

We will prove later:

> **Claim 1:** A non-root node with rank ρ can be charged a total cost of $O(\rho \cdot \alpha(n))$ in Case 2, summing over all the **find** operations.

Since there are $O(n/2^\rho)$ nodes with rank ρ (Lemma 11.6), it follows that the total time charged by Case 2 is bounded by

$$O\left(\sum_{\rho=1}^{\infty} \frac{n}{2^\rho} \cdot \rho \cdot \alpha(n)\right) \quad = \quad O(n \cdot \alpha(n)).$$

which will complete the proof of Lemma 11.13.

Claim 1, on the other hand, is implied by:

> **Claim 2:** For each $g \in [0, \alpha(n)]$, when node u stays in group g, Case 2 can charge u at most $rank(u)$ times.

The rest of the discussion will focus on proving Claim 2.

When u belongs to group g, we have $g = k(u)$. Thus, by definition of $k(u)$ in (11.7):

$$rank(parent(u)) \quad \geq \quad A_g(rank(u)) = A_g^{(1)}(rank(u))$$

while

$$rank(parent(u)) \quad < \quad A_{g+1}(rank(u)) = A_g^{(rank(u)+1)}(rank(u)).$$

Consider an arbitrary **find** operation that charges u in Case 2. Let i be the largest integer in $[1, rank(u) + 1)$ satisfying

$$rank(parent(u)) \quad \geq \quad A_g^{(i)}(rank(u)) \tag{11.8}$$

before the operation.

Lemma 11.14. *After the* **find** *operation, it must hold that $rank(parent(u)) \geq A_g^{(i+1)}(rank(u))$.*

Proof. Let v be the proper non-root ancestor of u in Case 2. Thus, $k(v) = k(u) = g$.

Let r be the root of the tree where u belongs. Since v is a non-root node, it is a proper descendant of r. We have:

$$\begin{aligned} rank(r) &\geq rank(parent(v)) \quad \text{(by Proposition 11.2)} \\ &\geq A_g(rank(v)) \quad \text{(by def. of } k(v)) \\ &\geq A_g(rank(parent(x))) \quad \text{(by monotonicity of } A_g(.)) \\ &\geq A_g(A_g^{(i)}(rank(u))) \quad \text{(by (11.8))} \\ &= A_g^{(i+1)}(rank(u)). \end{aligned}$$

The lemma then follows from the fact that $parent(u) = r$ after the **find** operation. □

The lemma implies Claim 2, because after $rank(u)$ applications of the Lemma 11.14, it must hold that

$$rank(parent(u)) \quad \geq \quad A_g^{(rank(u)+1)}(rank(u)) = A_{g+1}(rank(u)). \tag{11.9}$$

This indicates that u will then move up to a group numbered at least $g + 1$.

11.4 Remarks

The union-find structure we described is due to Tarjan [39]. The amortized bound in Section 11.3 was proved to be tight by Fredman and Saks [19]. In other words, Tarjan's structure is already asymptotically optimal. Analysis 1 was adapted from the lecture notes at `http://people.seas.harvard.edu/~cs125/fall16/lec3.pdf` and those at `https://people.eecs.berkeley.edu/~daw/teaching/cs170-s03/Notes/lecture12.pdf`. Analysis 2 was adapted from the book [26] of Kozen.

Exercises

Problem 1. Prove an $O(\log\log\log n)$ amortized bound when the group is defined using (11.3).

Problem 2. Prove an $O(\log^* n)$ amortized bound when the group is defined using (11.4).

Problem 3. Show that Assumption 1 can be removed without affecting the amortized bound.

(Hint: what does the **find** operation return?)

Problem 4*. Prove: each **find** operation finishes in $O(\log n)$ worst-case time.

(Hint: for each node u, prove that its subtree has height at most $rank(u)$.)

Problem 5*. Describe a union-find structure that processes any sequence of n_1 **makeset** operations, n_2 **find** operations, and m **union** operations in $O(n_1 + n_2 + m\log n_1)$ time. Note that this is better than the claim in Section 11.3 if $m \leq n_1/\log n_1$.

(Hint: store each set of $\mathcal{F}$ in a linked list.)

For the following two exercises, we assume the availability of a union-find structure that can support any sequence of t operations in $O(t \cdot T(t))$ time.

Problem 6 (dynamic connectivity). Consider an undirected graph $G = (V, E)$. Set $n = |V|$. Initially, E is empty (i.e., no edges). Design a structure to support the following operations:

- insert(u, v): add an edge between vertices $u, v \in V$ to E;
- query(u, v): given two vertices $u, v \in V$, report *whether* they belong to the same connected component in G.

Your structure must consume $O(n)$ space at all times (regardless of $|E|$), and support each operation in $O(T(n))$ amortized time.

Problem 7 (minimum spanning tree). Consider a weighted undirected graph $G = (V, E)$, where each edge in E is associated with a positive weight. Suppose that the edges in E have been sorted by weight. Describe an algorithm to obtaining a minimum spanning tree of G in $O(m \cdot T(n))$ time, where $n = |V|$ and $m = |E|$.

(Hint: implement Kruskal's algorithm with a union-find structure.)

Lecture 12: Dynamic Connectivity on Trees

Define $V = \{1, 2, ..., n\}$ where each element is called a *vertex*. F is a *forest* (i.e., a set of trees) such that

- for each tree in F, each node corresponds to a vertex from V;
- every vertex in V corresponds to exactly one node, counting the nodes of all the trees in F.

We want to store F in a data structure to support the following operations:

- **insert**(u, v) where vertices u and v belong to different trees in F: add an edge $\{u, v\}$, which effectively merges two trees (and hence, $|F|$ decreases by 1).
- **delete**(u, v) where u and v belong to the same tree $T \in F$: remove an edge $\{u, v\}$ from T, which effectively breaks T into two trees (and hence, $|F|$ increases by 1);
- **connected**(u, v): return whether $u, v \in V$ are in the same tree.

We will refer to the above problem as *dynamic connectivity on trees*. This lecture will introduce the *Euler-tour structure* which consumes $O(n)$ space, and performs all operations in $O(\log n)$ time. Note that if no deletions are allowed, the problem can be settled with the union-find structure of Lecture 11.

In the second part of the lecture, we will extend the functionality of the Euler-tour structure beyond the above operations. Our final version of the structure will make a powerful tool for the next lecture where we study the dynamic connectivity problem on graphs.

Notations. Given a tree T, we use $|T|$ to represent the number of vertices in T.

12.1 Euler tour

Focusing on *one* tree T, this section will introduce a generic method for "linearizing" the vertices of T.

12.1.1 Rooting a tree

Recall that a *tree* T, in general, is defined as an undirected, connected, graph without cycles. It does not automatically have a "root", without which concepts such as "parents", "children", and "subtrees" are undefined.

Suppose that an arbitrary vertex r has been designated as the *root* of T. A vertex u *parents* another vertex v if (i) $\{u, v\}$ is a tree edge, and (ii) u is closer to r than v. Accordingly, v is a *child* of u. Removing the edge $\{u, v\}$ breaks T into two connected components (CCs):

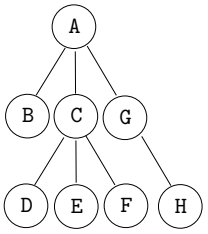


Figure 12.1: An Euler tour: ABACDCECFCAGHGA

- $T^u_{u,v}$: the CC containing u;
- $T^v_{u,v}$: the CC containing v.

We refer to $T^v_{u,v}$ as the *subtree* of v. Specially, the *subtree* of r is the entire T.

Sometimes we will emphasize on the existence of a root by calling T a *rooted tree*.

12.1.2 Euler tour on a rooted tree

Given a rooted tree T, we define an *Euler tour* as a sequence Σ of vertices output by:

euler-tour(T)
1. $r \leftarrow$ root of T
2. append r to the output sequence
3. **for** each child u of r **do**
4. **euler-tour**(the subtree of u)
5. append r to the output

Example. Figure 12.1 shows a tree rooted at A. The figure's caption is an Euler tour, but so is ACECFCDCABAGHGA (there are many more). Note that both Euler tours have the same length. □

12.1.3 A cyclic view

Let $m = |T| - 1$ be the number of edges in T. Conceptually, replace each (undirected) edge $\{u, v\}$ in T with two directed edges (u, v) and (v, u). This creates $2m$ directed edges.

Did you notice that Σ always had length $|\Sigma| = 2m + 1$ in the earlier example? This is not a coincidence. Denote the vertex sequence in Σ as: $u_1, u_2, ..., u_{|\Sigma|}$. For each $i \in [1, |\Sigma| - 1]$, interpret the consecutive vertices u_i, u_{i+1} as enumerating a directed edge (u_i, u_{i+1}). By how **euler-tour** runs, each of the $2m$ directed edges is enumerated exactly once, implying that $|\Sigma| = 2m + 1$. Let Q be the sequence of directed edges $(u_1, u_2), (u_2, u_3), ..., (u_{2m}, u_{2m+1})$, which is a cycle because $u_1 = u_{2m+1}$.

Example. In Figure 12.1, the cycle Q is (A, B), (B, A), (A, C), (C, D), (D, C), (C, E), (E, C), (C, F), (F, C), (C, A), (A, G), (G, H), (H, G), (G, A). □

The reverse is also true:

Proposition 12.1. *Let Q be any permutation of the $2m$ directed edges $(u_1, v_1), (u_2, v_2), ..., (u_{2m}, v_{2m})$ satisfying*

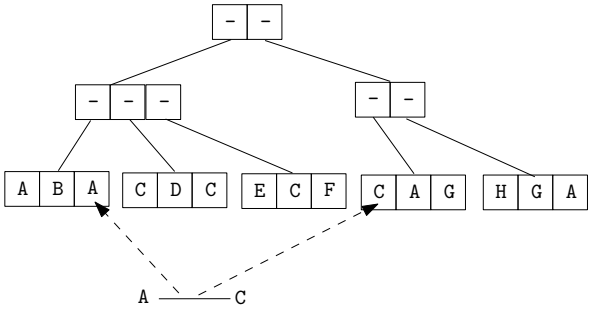


Figure 12.2: An Euler-tour structure for the tree in Figure 12.1 (for clarity, only the pointers of edge $\{\texttt{A}, \texttt{C}\}$ is shown)

- $v_i = u_{i+1}$ *for* $i \in [1, 2m-1]$*;*
- $v_{2m} = u_1$

defines an Euler tour $u_1u_2u_3...u_{2m}u_1$ *of* T *when* T *is rooted at* u_1*.*

The proof is left to you as an exercise.

12.2 The Euler-tour structure

Let T be a rooted tree with an Euler tour Σ. We store Σ in a 2-3 tree Υ where all the routing elements are left empty[1]. It follows from Section 12.1.3 that Υ has space $O(|T|)$.

For each edge $\{u, v\}$ in T, we store two pointers:

- one referencing the the occurrence of u that corresponds to the directed edge (u, v);
- the other referencing the occurrence of v that corresponds to the directed edge (v, u);

The resulting structure is called an *Euler-tour structure* (ETS) of T. See Figure 12.2 for an illustration. The following subsections will discuss several operations supported by Υ.

12.2.1 Cut

The $cut(u, v)$ operation removes an edge $\{u, v\}$ from a rooted T — assume, w.o.l.g., that u parents v — which breaks T into two trees:

- T_1: the subtree rooted at v;
- T_2: the tree obtained by removing T_1 from T.

The operation produces an ETS for T_1 and T_2, respectively.

Let Σ be the Euler tour of T stored in Υ. Identify the subsequence Σ_1 of Σ that starts from the first occurrence of v, and ends at the last occurrence of v. These two occurrences of v can be identified using the pointers associated with the edge $\{u, v\}$. Denote by Σ_2 the sequence obtained

[1] Alternatively, you can assume all the routing elements to have the same dummy key "0", breaking ties as follows: for two routing elements e_1 and e_2 in the same node, e_1 ranks before e_2 if the subtree of e_1 is to the left of the subtree of e_2.

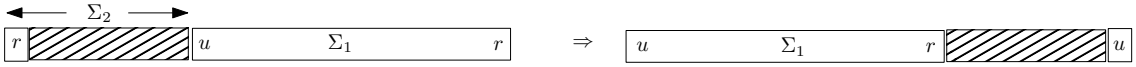


Figure 12.3: Changing the Euler tour in a re-root

by removing Σ_v from Σ. Then, Σ_1 must be an Euler tour of T_1. At this moment, Σ_2 has two consecutive occurrences of u; if we remove one of those occurrences, the resulting Σ is an Euler tour of T_2.

Example. Consider deleting the edge {A, C} from Figure 12.1(a). Σ_1 = CDCECFC is the Euler tour of T_1 (the subtree of C). Σ_2 = ABAAGHGA. There are two consecutive occurrences of A in Σ_2. After removing one of them, Σ_2 = ABAGHGA becomes an Euler tour of T_2 (what remains in Figure 12.1 after trimming T_1). □

The ETS's of T_1 and T_2 can be obtained using the split and join operations of 2-3 trees (Section 2.2.3).

Lemma 12.2. *A cut operation can be performed in $O(\log |T|)$ time.*

The proof is easy and left as an exercise.

12.2.2 Re-root

Remember that the ETS of T depends on the root r. Given any node $u \neq r$, the *re-root*(u) operation roots T at u and produces an ETS consistent with the new root.

Let Σ be the current Euler tour of T (rooted at r). We can obtain a new Euler tour Σ_{new} (rooted at u) as follows:

1. Identify an arbitrary occurrence of u. Let Σ_1 be the subsequence of Σ from that occurrence to the end. Let Σ_2 be the subsequence obtained by trimming Σ_1 from Σ.
2. Delete the first vertex of Σ_2 (which must be r).
3. $\Sigma_{new} = \Sigma_1 : \Sigma_2$, where ":" denotes concatenation.
4. Append u to Σ_{new}.

See Figure 12.3 for an illustration. The correctness follows from the cyclic view explained in Section 12.1.3, and makes a good exercise for you.

Example. . Consider re-rooting the tree of Figure 12.1 at u = C. Before the operation, Σ = ABACDCECFCAGHGA. If Σ_1 = CFCAGHGA, then Σ_2 = ABACDCE. The procedure outputs Σ_{new} = CFCAGHGABACDCEC, which is indeed an Euler tour of T rooted at C. □

Lemma 12.3. *A re-root operation can be supported in $O(\log |T|)$ time.*

The proof is obvious and omitted.

12.2.3 Link

Let T_1 and T_2 be two trees whose roots are u and v, respectively. The *link*(u, v) operation makes u a child of v by adding an edge $\{u, v\}$, which coalesces T_1 and T_2 into a single tree T. The operation produces an ETS for T.

Let Σ_1 (or Σ_2, resp.) be the Euler tour of T_1 (or T_2, resp.). An Euler tour of Σ of T can be derived in two steps:

1. $\Sigma = \Sigma_1 : \Sigma_2$ (concatenation of Σ_1 and Σ_2).
2. Append another occurrence of u at the end of Σ.

Lemma 12.4. *A link operation can be supported in* $O(\log(|T_1| + |T_2|))$ *time.*

The proof should have become obvious, and is omitted.

12.3 Dynamic connectivity

We can now (easily) solve the dynamic connectivity problem on trees. Build an ETS on every tree of F, and support each operation as follows.

Insert(u, v)**.** First, identify the accommodating tree $T_1 \in F$ of u, and similarly T_2 for v. Let the ETS of T_1 (or T_2) be Υ_1 (or Υ_2, resp.). Re-root Υ_1 at u, re-root Υ_2 at v, and then perform a *link*(u, v) operation. The cost is $O(\log n)$ by Lemma 12.3 and 12.4.

Deletion(u, v)**.** Let $T \in F$ be the tree containing the edge $\{u, v\}$. Simply perform *cut*(u, v) on the ETS of T. The cost is $O(\log n)$ by Lemma 12.2.

Connected(u, v)**.** Let $T_1 \in F$ be the tree containing u. Identify a leaf node in the ETS Υ_1 of T_1 which contains an arbitrary occurrence of u. Ascend from that leaf to the root r_1 of Υ_1. In the same manner, find the root r_2 of the ETS Υ_2 of the tree $T_2 \in F$ containing v. Declare "u connected to v" if and only if $r_1 = r_2$. The cost is $O(\log n)$ because every ETS has height $O(\log n)$.

12.4 Augmenting an ETS

Recall that we obtained the count BST (in Section 2.1.3) by augmenting the BST with aggregate information at internal nodes. In this section, we will apply the same type of augmentation to the ETS to enhance its power.

12.4.1 Weighted vertices and trees

Commutative monoids. In discrete mathematics, a *commutative monoid* is a pair $(W, \oplus)$ where

- W is a set of elements called the *domain*;
- $\oplus$ is an operation closed on W (i.e., for any $w_1, w_2 \in W$, $w_1 \oplus w_2 \in W$);
- $\oplus$ is commutative (i.e., $w_1 \oplus w_2 = w_2 \oplus w_1$) and associative (i.e., $w_1 \oplus w_2 \oplus w_3 = w_1 \oplus (w_2 \oplus w_3)$);
- W has an *identity element* I satisfying $w \oplus I = w$ for any $w \in W$.

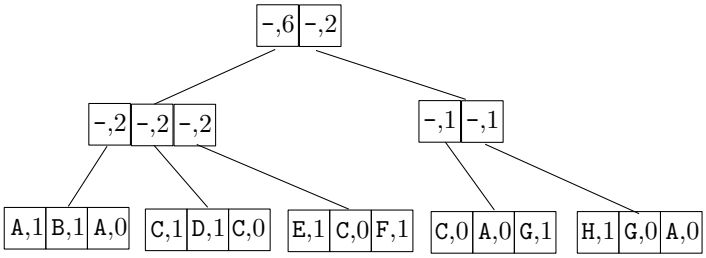


Figure 12.4: An augmented ETS (edge pointers omitted)

The following are some semi-groups commonly encountered in practice:

- $(\mathbb{R}, +)$: addition is closed on real numbers; $I = 0$.
- $(\mathbb{R}, \min)$: minimization is closed on real numbers; $I = \infty$;
- $(\{0, 1\}, \vee)$: OR is closed on $\{0, 1\}$; $I = 0$.

For any subset $S \subseteq W$, we refer to

$$\bigoplus_{w \in S} w$$

as the *sum* of the elements in S. For all the monoids in our discussion, we assume that

- each element in W can be stored in one cell;
- each evaluation of $\oplus$ takes constant time.

Vertex/tree weights. Fix a monoid $(W, \oplus)$. Suppose that T is *weighted* in the sense that every vertex u in the tree T is associated with a *weight* $w(u)$ drawn from W. The *weight* of T is defined as

$$\bigoplus_{u \text{ in } T} w(u).$$

By choosing $(W, \oplus)$ appropriately, we endow the weight of T with various semantics. For instance, if $(W, \oplus) = (\mathbb{R}, +)$ and every vertex is associated with weight 1, the weight of T is precisely the number of nodes in T. As another example, if $(\{0, 1\}, \vee)$ and every vertex is associated with weight either 1 (black) or 0 (white), the weight of T indicates *whether* T has any black nodes.

12.4.2 Maintaining and querying weights

Let T be a weighted tree. Suppose that, in addition to the operations in Section 12.1, we want to support:

- **weight-update**(u, x) where u is a vertex in T and $x \in W$: change $w(u)$ to x.
- **tree-weight**: report the weight of T.

We can achieve the purpose by slightly augmenting the ETS Υ of T. Let Σ be the underlying Euler tour. For every vertex u in T, we

- store $w(u)$ at the leaf element in Υ corresponding to an arbitrary occurrence of u in Σ;

- store I (the identical element of the monoid; see Section 12.4.1) at the leaf elements corresponding to all the other occurrences of u;
- record (say, in a separate array) a pointer to the occurrence carrying $w(u)$.

Also, at every routing element e of Υ, we store the sum of the weights in all the leaf entries underneath e.

Example. Suppose that the monoid is $(\mathbb{R}, +)$ and that each vertex in the tree of Figure 12.1 is associated with weight 1. Figure 12.4 augments the structure in Figure 12.2. A leaf element is in the form "u, w" where u is a vertex and w a weight. A non-leaf element is in the form "$-, w$", where $-$ is a routing element (which is empty) and w a weight. □

Lemma 12.5. *All the statements below are true:*

- *After augmentation, the ETS still retains the performance in Lemmas 12.2-12.3.*
- *Each* **weight-update** *can be performed in* $O(\log |T|)$ *time.*
- *Each* **tree-weight** *can be performed in* $O(1)$ *time.*

The proof is left as an exercise.

12.5 Remarks

The Euler-tour structure we described is an adaptation of the structure developed by Henzinger and King in [22].

Exercises

Problem 1. Prove Proposition 12.1.

Problem 2. Prove Lemma 12.2.

Problem 3. Prove the correctness of the *re-root* algorithm in Section 12.2.2.

(Hint: Proposition 12.1.)

Problem 4. Prove Lemma 12.5.

(Hint: review an exercise in Lecture 2 about the "count 2-3 tree".)

Problem 5 (colored vertices). Same settings as in the dynamic connectivity problem. Suppose that each vertex is colored black or white. Design a data structure to satisfy all the requirements below:

- **insert**, **delete**, and **connected** still in $O(\log n)$ time.
- given a vertex $u \in V$, change its color in $O(\log n)$ time.
- given a vertex $u \in V$, find in $O(\log n)$ time the number of black vertices in the tree of F containing u.

Problem 6*. The same settings as in Problem 4, but one more requirement:

- given a vertex $u \in V$, find in $O(\log n)$ time an (arbitrary) black vertex in the tree of F containing u or declare that the tree has no black vertices.

(Hint: top-down search in a 2-3 tree.)

Problem 7*. Let T be a tree where each vertex is colored black or white. Describe how to store T in an augmented ETS to support the following operation in $O(\log |T|)$ time:

- given an edge $\{u, v\}$ in T, find the number of black vertices in $T^u_{u,v}$ (defined in Section 12.1.1).

(Hint: you can achieve the purpose using *cut*, *tree-weight*, and *link* as black boxes.)

Problem 8* (colored edges). Same settings as in the dynamic connectivity problem. Suppose that each edge is colored black or white. Design a data structure to satisfy all the requirements below:

- **insert**, **delete**, and **connected** still in $O(\log n)$ time.
- given an edge $\{u, v\}$ in the forest, change its color in $O(\log n)$ time.
- given a vertex $u \in V$, find in $O(\log n)$ time the number of black edges in the tree of F containing u.
- given a vertex $u \in V$, find in $O(\log n)$ time an (arbitrary) black edge in the tree of F containing u or declare that the tree has no black edges.

(Hint: convert the problem to one with colored vertices.)

Lecture 13: Dynamic Connectivity on a Graph

This lecture will tackle the *dynamic connectivity problem* in its general form. Specifically, we want to store an undirected graph $G = (V, E)$ in a data structure that supports the following operations:

- **insert**(u, v): add an edge $\{u, v\}$ into E;
- **delete**(u, v): remove an edge $\{u, v\}$ from E;
- **connected**(u, v): return whether vertex $u \in V$ is *connected* to vertex $v \in V$ (namely, whether a path exists between them).

We consider that G has no edges at the beginning.

If no deletions are allowed, the problem can be settled with the union-find structure of Lecture 11. Intuitively, insertions are easy because adding an edge $\{u, v\}$ always makes u and v connected. Removing $\{u, v\}$, however, does *not* necessarily disconnect them. Supporting deletions requires new ideas.

Set $n = |V|$. Naively, each insertion/deletion can be supported in $O(|E|)$ time while ensuring constant time for **connected**. In this lecture, we will describe a structure [23] of $\tilde{O}(n)$ space that performs all operations in $\tilde{O}(1)$ amortized time. Recall that $\tilde{O}(.)$ hides polylog n factors; we will not be concerned with such factors in this lecture (our primary goal is to improve the $O(|E|)$ update bound).

Notations: For simplicity, we will assume that n is a power of 2. Set $h = \log_2 n$. For a tree T, $|T|$ represents the number of nodes in T. If u is a vertex, $u \in T$ indicates that u belongs to T.

13.1 An edge leveling technique

13.1.1 Spanning trees, spanning forests, and Kruskal's algorithm

If G is connected, a *spanning tree* of G is a tree made of $|V| - 1$ edges in E (such a tree includes all the vertices in V). If G is not connected, then a *spanning forest* of G is a set F of trees, where each tree in F is a spanning tree of a different connected component (CC) of G.

We will preserve the connectivity of G by maintaining a spanning forest F. Two vertices $u, v \in V$ are connected if and only if they appear in the same tree in F. Remember that we have learned a powerful tool for managing trees, i.e., the Euler-tour structure (ETS). We will store each tree of F in an ETS, which processes any **connected**(u, v) operation in $\tilde{O}(1)$ time.

The challenge is to update F along with edge insertions and deletions. For this purpose, we need to be careful in choosing the F to maintain. Our strategy will be closely related to Kruskal's algorithm for finding a *minimum spanning forest* (MSF). More specifically, we will give each edge

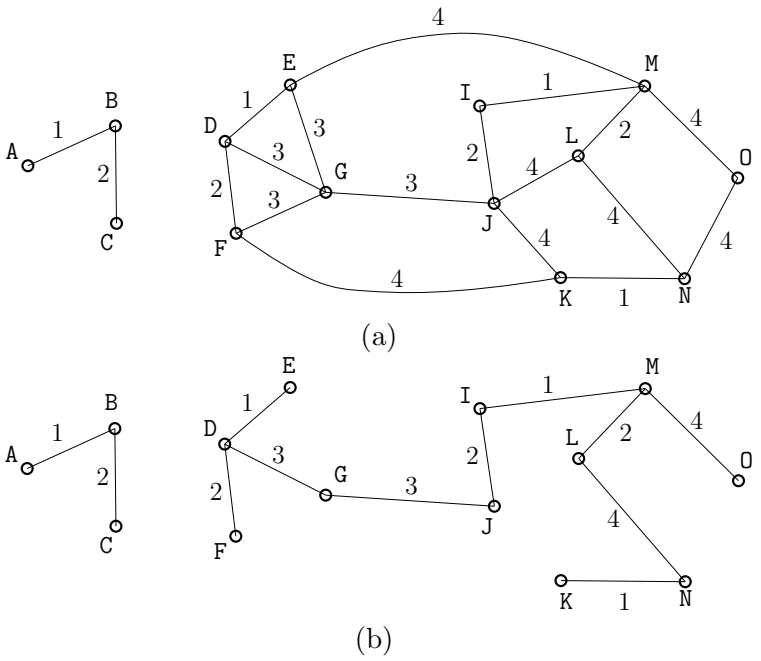


Figure 13.1: (a) shows a weighted graph, and (b) gives an MSF.

a *weight* which is a non-negative integer. The *weight* of F is the sum of weights of all the edges therein. F is an MSF if its weight is the minimum among all the spanning forests. Kruskal gave the following algorithm for finding an MSF:

Kruskal
1. $F \leftarrow$ the set of vertices, each regarded as a tree (of size 1)
2. **while** $\exists$ edge $\{u, v\}$ where u, v are in different trees in F **do**
   /* call $\{u, v\}$ a *cross edge* */
3. $\quad e \leftarrow$ a cross edge with the smallest weight
4. $\quad$ merge two trees in F with e
5. **return** F

We will maintain an F that can be thought of as having been picked by the above algorithm.

Example. Figure 13.1.1(a) shows a graph where the number next to each edge indicates its weight. Figure 13.1.1(b) is one possible MSF that can be output by **Kruskal**. □

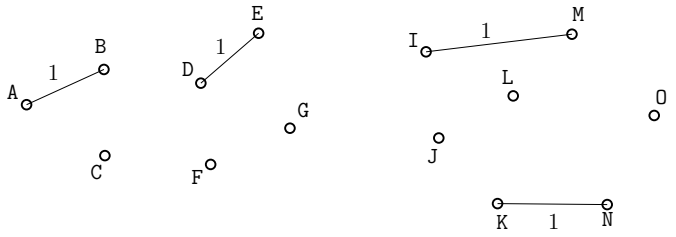
The following is a useful fact (from the undergraduate level) that will be useful:

> **MSF property:** Let F be an arbitrary spanning forest of G (not necessarily the minimum one) and e be an edge that is not in F. Adding e to F creates a cycle. We call e a *short-cut edge* if the weight of e is strictly less than the weight of another edge in the cycle. The MSF property says that F is an MSF *if and only if* no short-cut edges exist.

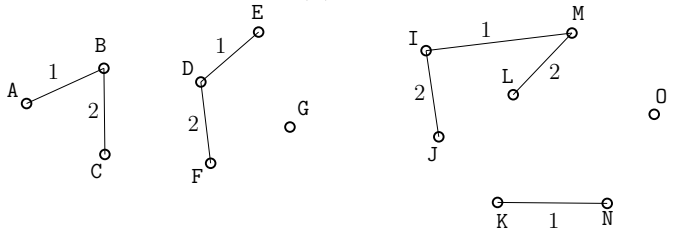
13.1.2 Edge leveling

We assign each edge $e \in E$ a *level* (a.k.a. its weight) — denoted as *level*(e) — which is an integer between 1 and h. Define for each $i \in [1, h]$:

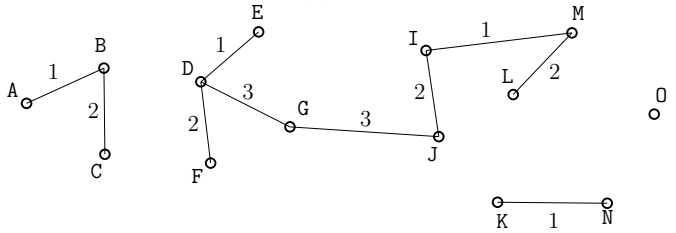
$$E_i \quad = \quad \text{the set of edges in } E \text{ with level } \textit{at most } i.$$



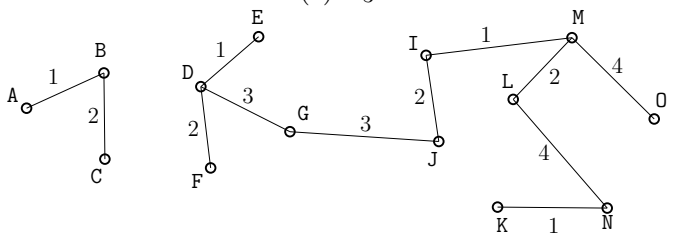
(a) F_1



(b) F_2



(c) F_3



(d) $F_4 = F$

Figure 13.2: Spanning forests for the graph in Figure 13.1.1(a)

Clearly:

$$E_1 \subseteq E_2 \subseteq ... \subseteq E_{\log_2 n} = E.$$

Accordingly, define:

$$G_i \quad = \quad \text{the graph } (V, E_i). \tag{13.1}$$

We demand:

Invariant 1: Each CC of G_i has at most 2^i vertices.

We maintain a spanning forest F_i of G_i for every $i \in [1, h]$, and make sure:

Invariant 2: For $i \in [1, h-1]$, all the edges in F_i must also be present in F_{i+1}.

Note that F_h must be a spanning forest of G (because E_h contains all the edges in G). For this reason, we will also denote F_h as F.

Example. Consider that G is the graph in Figure 13.1.1(a), where the level of each edge is indicated next to it. Here, $h = 4$. Figure 13.2 illustrates the spanning forests $F_1, F_2, ..., F_4$. □

13.1.3 Connections between edge leveling and Kruskal's

Let F_1, F_2, ..., F_h be arbitrary spanning forests satisfying Invariant 1 and 2. We now give a crucial observation.

Lemma 13.1. *Consider any tree $T \in F_i$ (of any i) and any edge e in T. Remove e from T which disconnects T into trees T_1 and T_2. Then, any other edge connecting a node in T_1 with a node in T_2 must have level at least level(e).*

Proof. Assume the existence of an edge $e' = \{u, v\}$ of level $j < level(e)$ such that $u \in T_1$ and $v \in T_2$. Hence, u and v are in the same CC of G_j. Thus, there must exist a path Π from u to v in F_j. By Invariant 2, all the edges in Π must be in F_i because $j < level(e) \leq i$; this means that Π must be in T. However, because Π cannot contain e, we have found two different edges between T_1 and T_2 (i.e., e and some edge on Π), contradicting the fact that T is a tree. □

Corollary 13.2. *F_i is an MSF of G_i for each $i \in [1, h]$.*

Proof. Immediate from the previous lemma and the MSF property (Section 13.1.1). □

13.2 Dynamic connectivity

For each vertex u in G and each level $i \in [1, h]$, we store a linked list for:

$$L_i(u) \quad = \quad \{\text{the set of level-}i \text{ edges incident to } u\}. \tag{13.2}$$

For each $i \in [1, h]$, build an ETS (Lecture 12) on each tree $T \in F_i$, denoted as $\Upsilon(T)$.

The subsequent discussion will concentrate on maintaining the graphs $G_1, ..., G_h$ and their spanning forests $F_1, ..., F_h$. Once this is clear, generating the necessary operations on the linked lists and ETS's becomes elementary exercises.

13.2.1 Connected

Handling a **connected**(u, v) operation amounts to finding out whether u and v belong to the same tree in F. We can do so in $\tilde{O}(1)$ time (Lecture 12) using the ETS's.

13.2.2 Insertion

To perform an **insert**(u, v), we set the level of the new edge $\{u, v\}$ to h, and add it to G_h. If u and v are not connected, we link up with $\{u, v\}$ the trees in F containing u and v, respectively. This also takes $\tilde{O}(1)$ time (Section 13.2.1 and Lecture 12). It is obvious that Invariants 1 and 2 are still satisfied.

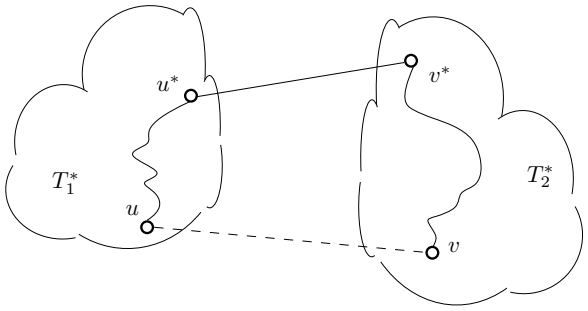


Figure 13.3: Proof of Lemma 13.4

13.2.3 Deletion

Consider the deletion of an edge $e_{old} = \{u^*, v^*\}$. Set $\ell = level(e_{old})$. If e_{old} is not in F, no F_i of any i needs to be altered; and we finish by deleting e_{old} from $G_\ell, G_{\ell+1}, ..., G_h$. The subsequent discussion considers the opposite.

Replacement edges. Removing e_{old} from its tree $T^* \in F$ disconnects T^* into trees T_1^* and T_2^*. Our goal is to find a *replacement edge* between T_1^* and T_2^* to connect them back into one tree in F. Of course, such a replacement edge may not exist, in which case T_1^*, T_2^* are now spanning trees of two different CCs.

Proposition 13.3. *A replacement edge must have level at least ℓ.*

Proof. Immediate from Lemma 13.1. □

Lemma 13.4. *If $\{u, v\}$ of level $i \geq \ell$ is a replacement edge, then u, v, u^*, v^* are all in the same CC of G_i.*

Proof. Since $e = \{u, v\}$ is not in T^*, adding it to T^* creates a cycle passing u, v, u^*, v^* (MSF property). See Figure 13.3. Furthermore, e must have the largest level (a.k.a. weight) in the cycle (MST property). Therefore, the four vertices are connected by a path consisting of edges with weight at most i. □

Algorithm. First remove e_{old} from all of $G_\ell, G_{\ell+1}, ..., G_h$ and $F_\ell, F_{\ell+1}, ..., F_h$. Next, we aim to find a replacement edge whose level is *as low as possible*, starting with $i = \ell$:

replacement(i)

/* find a replacement edge of level i, if exists */
/* let T be the tree in F_i used to contain e_{old}; deleting e_{old} disconnects T into trees T_1 and T_2; w.o.l.g., assume $|T_1| \leq |T_2|$ */

1. **for** each edge $e = \{u, v\}$ in T_1 of level i **do**
2. reduce $level(e)$ to $i - 1$, and add e to G_{i-1}
   /* $i \geq 2$ by Invariant 1 (otherwise T cannot have two edges e_{old} and e) */
3. connect two trees in F_{i-1} with e
   /* Proposition 13.6 will prove that u, v must be in different CCs before the addition of e to G_{i-1} */
4. **while** T_1 has a vertex u on which there is an edge $e = \{u, v\}$ of level i **do**
5. **if** e is a replacement edge **then**
6. **return** e
   else
7. set $level(e)$ to $i - 1$, and add e to G_{i-1}
8. **return** failure

If **replacement** returns failure, we increase i by 1 and try again, until i has exceeded h. If, on the other hand, a replacement edge e is found, we add e to $F_i, F_{i+1}, ..., F_h$.

Example. Suppose that we want to delete the edge $e_{old} = \{\texttt{G}, \texttt{J}\}$ in Figure 13.1.1(a), assuming $F_1, ..., F_4$ as in Figure 13.2. Thus, $\ell = 3$.

Consider the execution of **replacement**(3). Figure 13.4(a) shows the current G after deleting e_{old}, while Figure 13.4(b) illustrates T_1 (the left tree) and T_2 (the right); note that T_1 and T_2 are what remains after removing e_{old} from the largest spanning tree in Figure 13.2(c). The algorithm attempts to find a replacement edge of level 3 to reconnect T_1 and T_2. Lines 1-3 push all the level-3 edges in T_1 to level 2 (only one such edge $\{\texttt{D}, \texttt{G}\}$), yielding the situation in Figures 13.4(c) and (d). Lines 4-8 enumerate every level-3 edge incident to a vertex in T_1 (i.e., $\{\texttt{E}, \texttt{G}\}$ and $\{\texttt{F}, \texttt{G}\}$). Since no such edges make a replacement edge, their levels are reduced to 2. Figures 13.4(e) and (f) illustrate the situation at this moment. The procedure **replacement**(3) returns failure.

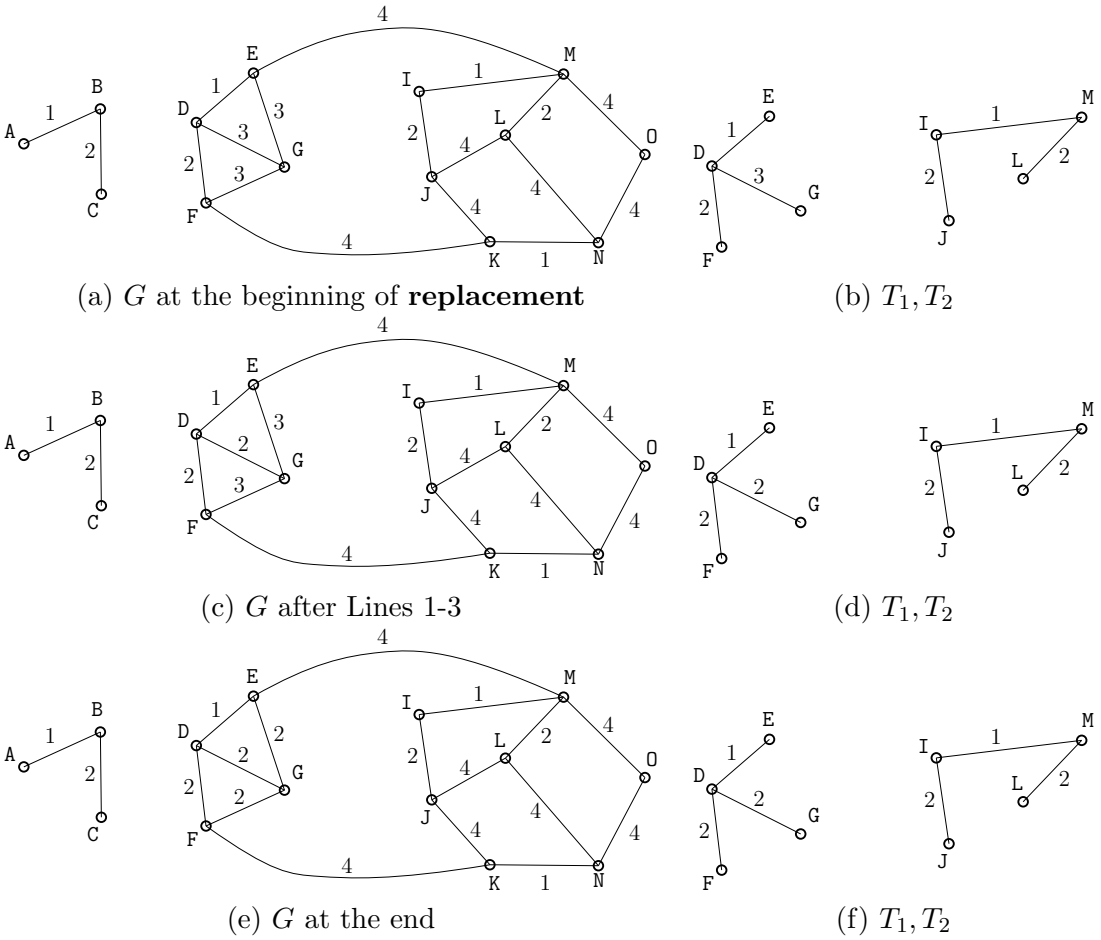
Next, we execute **replacement**(4), in an attempt to find a cross edge of weight 4. The current graph is as shown in Figure13.5(a) (same as Figure 13.4(e)). Figure 13.4(b) illustrate T_1 (the left tree) and T_2 (the right). Compare them to the right spanning tree in Figure 13.2(d) and understand what has caused the differences. The algorithm finds a replacement edge $\{\texttt{G}, \texttt{J}\}$ of level 4. No more changes are done to G (Figure 13.5(c)), but we use $\{\texttt{G}, \texttt{J}\}$ to link up T_1 and T_2 (Figure 13.5(d)), which yields a spanning tree in $F_4 = F$. □

Proposition 13.5. *Invariant 2 holds at all times.*

Proof. For each line in **replacement**, it is easy to prove that if Invariant 2 holds before the line, this is still true after the line. □

To prove the algorithm's correctness, we still need to show:

- **Claim 1:** If **replacement**(h) returns failure, no replacement edge exists.
- **Claim 2:** For each $i \in [1, h]$, F_i is still a spanning forest of G_i.



(a) G at the beginning of **replacement** (b) T_1, T_2

(c) G after Lines 1-3 (d) T_1, T_2

(e) G at the end (f) T_1, T_2

Figure 13.4: Illustration of **replacement**(3)

- **Claim 3:** Invariant 1 still holds after the algorithm finishes.

Claim 1 is in fact a corollary of Lemma 13.4, and left as an exercise for you to prove. We will prove the other two claims in the following subsections.

13.2.4 Proof of Claim 2

We will establish the claim by proving a series of facts about **replacement**.

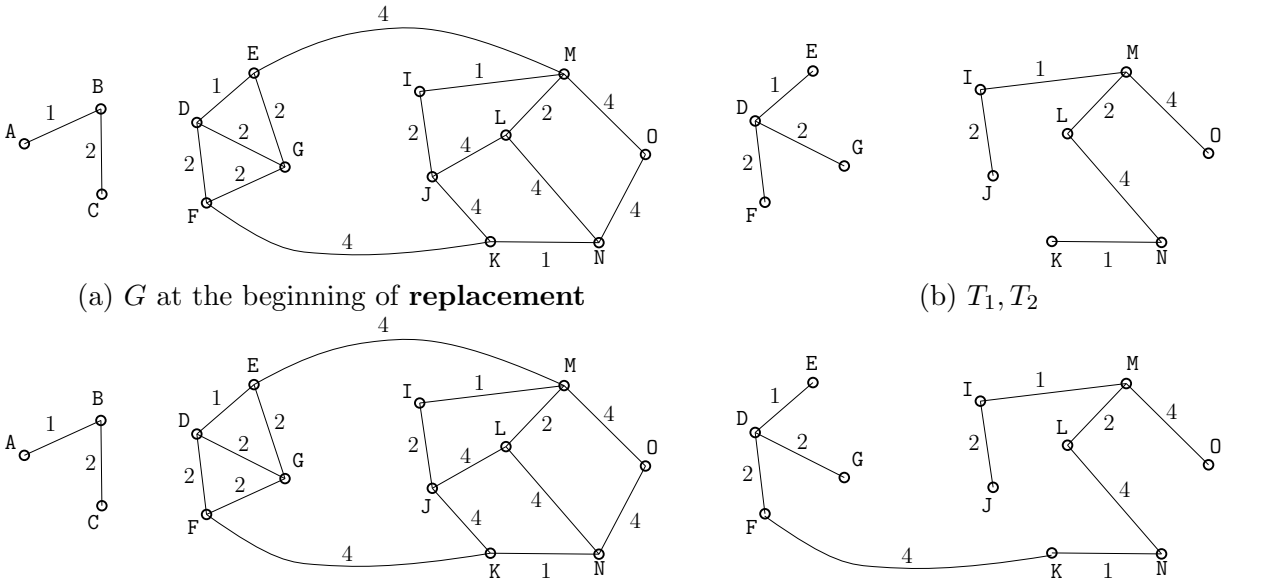
Proposition 13.6. *Consider one iteration of Lines 2-3. If F_{i-1} is a spanning forest of G_{i-1} before Line 2, it remains so after Line 3.*

Proof. It suffices to show that, for the edge $e = \{u, v\}$ identified by the iteration at Line 1, the vertices u and v must be in different CCs of G_{i-1}.

Suppose that this is not true. Consider the moment before $level(e)$ is decreased at Line 2. There exists a path Π in F_{i-1} connecting u and v. All the edges in Π must belong to F_i (Proposition 13.5). But then Π and e make a cycle in F_i, giving a contradiction. □

Proposition 13.7. *F_{i-1} remains as a spanning forest of G_{i-1} after each time Line 7 is executed.*

Proof. True because, right before the line, u and v must be connected in T_1 by a path in G_{i-1} (they must be connected in T_1, and all the edges in T_1 now have level at most $i - 1$). □



(a) G at the beginning of **replacement** (b) T_1, T_2

(c) G at the end (d) T_1, T_2 merged by replacement edge $\{\texttt{F}, \texttt{K}\}$

Figure 13.5: Illustration of **replacement**(4)

Proposition 13.8. *If* **replacement**(i) *returns failure, F_i is a spanning forest of G_i.*

Proof. Consider the connected component C of G_i represented by T before the removal of e_{old}. No new vertex can join C because edge levels can only decrease. Every vertex of C is in either T_1 or T_2. The edges in T_1, which have level at most i, indicate that the vertices in T_1 are connected in G_i. Same for T_2. That **replacement** returns failure indicates that no edges of level i exist between T_1 and T_2. By Lemma 13.1, no edges of any level less than i can exist between T_1 and T_2, either. Therefore, T_1 and T_2 are now spanning trees of two CCs in G_i. □

Proposition 13.9. *After adding the replacement edge e to F_j where $j \geq i$, F_j is a spanning forest of G_j.*

Proof. Consider the connected component C of G_j represented by T before the removal of e_{old}. No new vertex can join C because edge levels can only decrease. Every vertex of C is in either T_1 or T_2. The edges in T_1, which have level at most i, must belong to G_j and, thus, indicate that the vertices in T_1 are connected in G_j. Same for T_2. The discovery of the edge e ascertains that every vertex in T_1 is connected to a vertex in T_2 by a path of edges with level at most j. The tree obtained by coalescing T_1 and T_2 with e is therefore a spanning tree of C. □

This completes the proof of Claim 2.

13.2.5 Proof of Claim 3

Fix an $i \in [1, h]$, and consider the execution of **replacement**(i). The following fact should have become easy to prove (left as an exercise):

Proposition 13.10. *After* **replacement**(i)*, the tree T_1 is the only new spanning tree in F_{i-1}, merging possibly several spanning trees originally in F_{i-1}.*

Hence, to prove Claim 3, it suffices to show that $|T_1| \leq 2^{i-1}$. For this purpose, note first that $|T| \leq 2^i$ due to Invariant 1 because T was a spanning tree in G_i before e_{old} disappeared. Thus, $|T_1| \leq 2^{i-1}$ follows from the fact that $|T_1| \leq |T_2|$ and $|T_1| + |T_2| = |T|$.

13.2.6 Implementation

Replacement can be efficiently implemented using ETS's:

- Obtain the size of a tree in F_i. See Section 12.4.
- At Line 1, the level-i edge e (of T_1) can be found in $\tilde{O}(1)$ time. This was an exercise in Lecture 12 (colored edges; hint: give a special color to each level-i edge).
- Line 2 is easy.
- Line 3 takes $\tilde{O}(1)$ time. See the same exercise as in the 1st bullet.
- At Line 4, an edge e can be found in $\tilde{O}(1)$ time. This was an exercise in Lecture 12 (colored vertices; hint: give a vertex a special color if it has level-i edges).
- The if-condition Line 5 can be checked in $\tilde{O}(1)$ time (a **connected** operation on trees).
- Line 5 takes $\tilde{O}(1)$ time. See the same exercise as in the 4th bullet.
- Line 7 is easy.

We also need to update the linked lists on all the $L_i(u)$'s (see (13.2)) whenever an edge moves from G_i to G_{i-1} for some $i \geq 2$. This can be trivially done in $O(1)$ time per move.

13.2.7 Amortization

Next, we will prove that the total cost of all the deletions is $\tilde{O}(m)$, where m is the number of edges that have ever existed in G. Since every edge must be added by an insertion, we can amortize the $\tilde{O}(m)$ cost over all the insertions such that each insertion bears only $\tilde{O}(1)$ cost.

By implementing our structure as in Section 13.2.6, we know that each deletion takes $O(1)+\tilde{O}(x)$ time, where x is the number of times we *demote* an edge, i.e., decreasing its level by 1. What is the largest possible number of demotions of all deletions? The answer is clearly $mh = \tilde{O}(m)$ because there are h levels, and edge levels never increase. We thus conclude that all deletions require $\tilde{O}(m)$ time.

13.3 Remarks

The dynamic connectivity algorithm discussed in this lecture is based on an approach developed by Holm, Lichtenberg, and Thorup in [23]. That paper also gives the precise polylog n factors we omitted.

Exercises

Problem 1. Prove the MSF property (Section 13.1.1).

Problem 2. Prove Claim 1.

Problem 3. Prove Proposition 13.10.

Problem 4. Verify all the bullets in Section 13.2.6.

Problem 5. Suppose that we want to support one more operation in the dynamic connectivity problem:

- **CC-size**(u): return the number of nodes in the CC that contains the given vertex $u \in V$.

Explain how to extend our structure to support the above operation in $\tilde{O}(1)$ amortized time, while retaining the same performance on **insert**, **delete**, and **connected**.

Problem 6. Same settings as in the dynamic connectivity problem, except that every vertex in G is colored black or white. Besides **insert**, **delete**, and **connected**, we also want to support:

- **blackest-CC**: return any node in a CC with the largest number of black vertices.

Describe a structure that supports all operations in $\tilde{O}(1)$ amortized time.

Lecture 14: Range Min Queries (Lowest Common Ancestor)

This lecture discusses the *range min query* (RMQ) problem, where we want to preprocess an array A of n real values to support:

Range min query: Given integers x, y satisfying $1 \leq x \leq y \leq n$, report $\min_{i=x}^{y} A[i]$.

The problem can be easily solved by an augmented BST (Section 2.1.3) which uses $O(n)$ space, and answers a query in $O(\log n)$ time. Today, we will learn an optimal structure that uses $O(n)$ space and answers a query in $O(1)$ time.

Closely related is the *lowest common ancestor* (LCA) problem where we want to preprocess a rooted tree T to support:

LCA query: Given two nodes u, v in T, return their lowest common ancestor in T.

As you will explore in exercises, the RMQ and LCA problems turn out to be equivalent. We will focus on RMQ in the lecture.

We will consider that the elements in A are distinct (this assumption does not lose any generality; why?). For any x, y satisfying $1 \leq x \leq y \leq n$, define

$$minindex_A(x, y) \quad = \quad \arg\min_{i=x}^{y} A[i].$$

In other words, if $k = minindex_A(x, y)$, then $A[k]$ is the smallest in $A[x], A[x+1], ..., A[y]$. The goal of an RMQ is to find k.

Notations. Given any $x, y \in [1, n]$, $A[x : y]$ is the subarray of A that starts from $A[x]$ and ends at $A[y]$. Specially, if $x > y$, $A[x : y]$ denotes the empty set.

14.1 How many different inputs really?

At first glance, there seems to be an infinite number of inputs because each element in A can be an arbitrary real number. This pessimistic view hardly touches the essence of the problem.

Let us define the *rank permutation* of A as a permutation R of $\{1, 2, ..., n\}$ such that, for each $i \in [1, n]$, $R[i]$ equals j if $A[i]$ is the j-th smallest element in A. What matters for RMQ are not the actual values in A, but instead, is its rank permutation. This is because, regardless of the content of A, we always have:

$$minindex_A(x, y) \quad = \quad minindex_R(x, y).$$

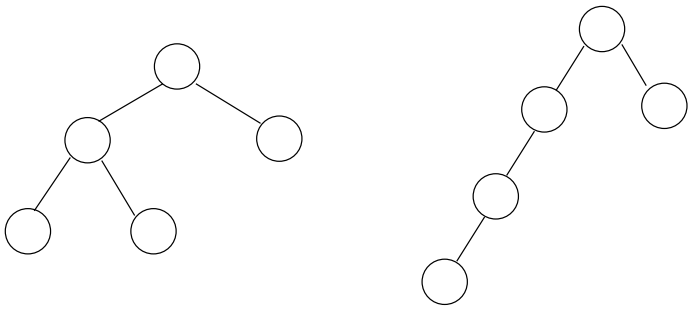


Figure 14.1: The left is the cartesian tree for $A = (4, 2, 5, 1, 3)$, and the right for $A = (4, 3, 2, 1, 5)$.

Example. Suppose that $n = 5$, $A_1 = (16, 7, 20, 2, 10)$ and $A_2 = (25, 11, 58, 3, 12)$. $R = (4, 2, 5, 1, 3)$ is the rank permutation of both A_1 and A_2. □

It thus follows that there are at most $n!$ inputs that are "really" different. However, even this is a serious over-estimate! The following example allows you to see the reason intuitively:

Example. Suppose that $n = 5$, $R_1 = (4, 2, 5, 1, 3)$ and $R_2 = (3, 2, 4, 1, 5)$. For any x, y satisfying $1 \leq x \leq y \leq n$, we always have $minindex_{R_1}(x, y) = minindex_{R_2}(x, y)$. □

Formally, two arrays A_1 and A_2 of size n are said to be *identical* if $minindex_{A_1}(x, y) = minindex_{A_2}(x, y)$ holds for any legal x, y. Next, we will show that the number of distinct inputs is no more than 4^n (which is considerably smaller than $n!$).

Let us define the *cartesian tree* T on A recursively:

- If $n = 0$, then T is empty.
- If $n = 1$, then T has a single node.
- Otherwise, let $k = minindex_A(1, n)$. T is a binary tree where the root's left subtree is the cartesian tree on $A[1 : k-1]$, and the root's right subtree is the cartesian tree on $A[k+1 : n]$.

See Figure 14.1 for an illustration.

Lemma 14.1. *Arrays A_1 and A_2 are identical if and only if their cartesian trees are equivalent.*

The proof is left to you as an exercise.

It is well known that there are no more than 4^n different binary trees with n nodes (you will prove this in an exercise). Therefore, at most 4^n distinct inputs exist.

14.2 Tabulation for short queries

Fix any s satisfying $\Omega(\log n) = s \leq \frac{1}{2} \log_4 n$. Assume, w.o.l.g., that n is a multiple of s. We break A into *chunks* of size s, namely, the first chunk is $A[1 : s]$, the second $A[s + 1 : 2s]$, and so on. In this section, we consider only *short* queries where the indexes x and y fall into the *same* chunk. We will describe a structure of $O(n)$ space that answers all such queries in constant time.

Each chunk can be regarded as an array B of size s. How many different queries are there for chunk B? The answer is $s(s + 1)/2$, which is the number distinct pairs (x, y) satisfying $1 \leq x \leq y \leq s$. As a brute-force approach, we can store the answers for *all* possible queries. This takes

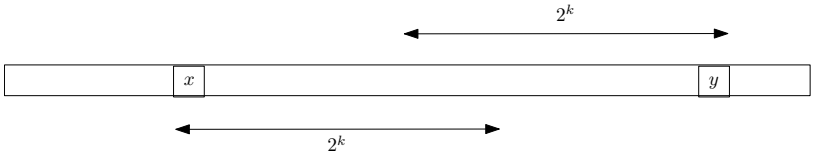


Figure 14.2: Using two pre-computed answers to cover a query

$O(s^2)$ space per chunk and, hence, $O(\frac{n}{s}s^2) = O(n \log n)$ space overall. Unfortunately, this exceeds our linear space budget.

But wait! If two chunks have the same cartesian tree, they are identical (for RMQ), and hence, can share the same set of pre-computed answers! We only need at most 4^s pre-computed answer sets, because there are no more than 4^s different cartesian trees with s nodes (Section 14.1). The total amount of space required is bounded by

$$O(4^s \cdot s^2) = O\left(4^{\frac{1}{2}\log_4 n} \cdot s^2\right) = O(\sqrt{n} \cdot \log^2 n)$$

which is *significantly* less than our $O(n)$ budget!

Each pre-computed answer set is an array of size $O(s^2)$ length, referred to as an *answer array*. We store all (no more than) 4^s such arrays. For each chunk, we associate it with the starting address of its answer array. The total space is $O(n)$.

Given a query with interval $[x, y]$, we can identify the chunk covering $[x, y]$ in $O(1)$ time (think: how?), after which $minindex_A(x, y)$ can be easily acquired from the chunk's answer array in $O(1)$ time.

Remark. The method of pre-computing the answers of all queries in a small domain is known as the *tabulation technique*.

14.3 A structure of $O(n \log n)$ space

This section will describe a structure of $O(n \log n)$ space that answers an (arbitrary) RMQ in constant time.

Structure. For each $i \in [1, n]$, we store

- $minindex_A(i, j)$ for every $j = i + 1$, $i + 2^2 - 1$, ..., $i + 2^\lambda - 1$ where λ is the largest integer satisfying $i + 2^\lambda - 1 \leq n$;
- $minindex_A(j, i)$ for every $j = i - 1$, $i - 2^2 + 1$, ..., $i - 2^\lambda + 1$ where λ is the largest integer satisfying $i - 2^\lambda + 1 \geq 1$.

In other words, for each $i \in [1, n]$, we pre-compute the answers of all queries whose ranges $[x, y]$ satisfy two requirements:

- $[x, y]$ covers a number of elements that is a power of 2;
- it starts or ends at i.

The number of such queries is $O(\log n)$. Therefore, the total space is $O(n \log n)$.

Query. Let $[x, y]$ be the search interval; note that it covers $y - x + 1$ elements. Set

$$\lambda \quad = \quad \lfloor \log_2(y - x + 1) \rfloor \tag{14.1}$$

If $y - x + 1$ is a power of 2, the query answer has explicitly been pre-computed, and can be retrieved in constant time.

Proposition 14.2. *If $y - x + 1$ is not a power of 2, $[x, y]$ is covered by the union of $[x, x + 2^\lambda - 1]$ and $[y - 2^\lambda + 1, y]$.*

The proof is obvious and hence omitted. See Figure 14.2 for an illustration. We can therefore obtain from the pre-computed answers $i = \mathit{minindex}_A(x, x+2^\lambda-1)$ and $j = \mathit{minindex}_A(y-2^\lambda+1, y)$, and then return the smaller between $A[i]$ and $A[j]$. The time required is $O(1)$.

To achieve $O(1)$ query time overall, however, we must be able to compute λ in (14.1) in constant time. This can be achieved with proper preprocessing, as you will explore in an exercise.

14.4 Remarks

Have we obtained the promised structure with $O(n)$ space and $O(1)$ query time? Well, not explicitly, but almost. All we need to do is to combine the solutions in Sections 14.2 and 14.3. This will be left as an exercise.

The elegant structure we discussed was designed by Bender and Farach-Colton [4]. It is worth pointing out that the first optimal LCA (and hence RMQ) structure is due to Harel and Tarjan [21].

Exercises

Problem 1. Prove Lemma 14.1.

Problem 2. Let T be a (rooted) binary tree of n nodes, where each internal node has two child nodes. Let $\Sigma = (u_1, u_2, ..., u_{2n-1})$ be the Euler tour obtained using the algorithm in Section 12.1.2. Σ decides a 0-1 sequence Σ' of length $2n-2$ as follows: for each $i \in [1, 2n-2]$, $\Sigma'[i] = 1$ if u_i is the parent of u_{i+1}, or 0 otherwise.

Prove: no two binary trees of n nodes can produce the same 0-1 sequence.

Problem 3. Prove: there are less than 2^{2n} different binary trees of n nodes.

(Hint: Problem 2.)

Problem 4. Describe a structure of $O(n)$ space such that, given any integer $x \in [1, n]$, we can calculate $\lfloor \log_2 x \rfloor$ in constant time.

Problem 5. Design an optimal RMQ structure of $O(n)$ space and $O(1)$ query time.

Problem 6. Construct an optimal RMQ structure in $O(n \log \log n)$ time.

Problem 7.** Given an array A of size n, describe an algorithm to construct its cartesian tree in $O(n)$ time.

(Hint: scan A from left to right, and build the tree incrementally.)

Problem 8. Construct an optimal RMQ structure in $O(n)$ time.
(Hint: Problem 7.)

Problem 9* (RMQ implies LCA). For the LCA problem, describe a structure of $O(n)$ space and $O(1)$ query time, where n is the number of nodes in the input tree T.

(Hint: use an Euler tour of T.)

Problem 10 (LCA implies RMQ). Suppose that you know how to build an LCA structure of $O(n)$ space and $O(1)$ query time. Show that you can obtain an optimal structure for the RMQ problem.

Lecture 15: The van Emde Boas Structure (Y-Fast Trie)

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This lecture revisits the *predecessor search* problem, where we want to store a set S of n elements drawn from an ordered domain to support:

- **Predecessor query:** given an element q (which may not be in S), return the *predecessor* of q, namely, the largest element in S that does not exceed q.

We already know that the binary search tree (BST) solves the problem with $O(n)$ space and $O(\log n)$ query time.

Our focus in the lecture will be the scenario where the domain of the elements has a finite size U. W.o.l.g., we will assume that all the elements in S are integers in $\{1, 2, ..., U\}$. We will learn the *van Emde Boas structure* (vEBS) which uses $O(n)$ space and answers a query in $O(\log \log U)$ time. Note that for practical scenarios where U is a polynomial of n, the query time is $O(\log \log U) = O(\log \log n)$, improving that of the BST.

The structure to be described also draws ideas from the *y-fast trie* (see Section 15.3 for more details).

For simplicity, we will assume that $\log_2 \log_2 U$ is an integer, namely, $U = 2^{2^x}$ for some integer $x \geq 1$ (think: why is this a fair assumption?). Also, we will assume, again w.o.l.g., that S contains the integer 1 so that the predecessor of any $q \in \{1, ..., U\}$ always exists.

15.1 A structure of $O(n \log U)$ space

If $U = 4$ (which implies $n = O(1)$) or $n = O(1)$, we define the vEBS simply as a linked list storing S (which ensures constant space and query time). Next, we will consider $U \geq 16$.

15.1.1 Structure

We divide the domain $[1, U]$ into $\sqrt{U}$ disjoint *chunks* of size $\sqrt{U}$, namely, Chunk 1 is $[1, \sqrt{U}]$, Chunk 2 is $[\sqrt{U} + 1, 2\sqrt{U}]$, and so on. Note that $\sqrt{U}$ is an integer. For each $i \in [1, \sqrt{U}]$, define

$$S_i \quad = \quad S \cap \text{ Chunk } i.$$

Chunk i is *empty* if $S_i = \emptyset$.

Let P be the set of ids of all the non-empty chunks. Clearly, $|P| \leq \min\{n, \sqrt{U}\}$. Store P in a *perfect* hash table H (Section 9), which we can use to check in constant time whether Chunk i is empty for any $i \in [1, \sqrt{U}]$.

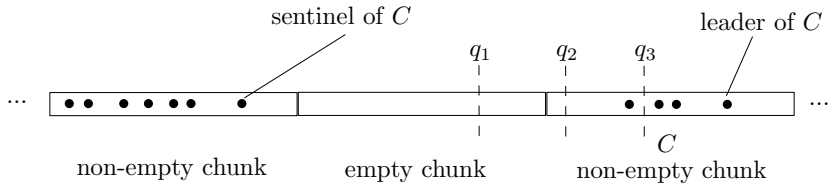


Figure 15.1: Each box shows a chunk, where points represent integers in ascending order from left to right.

Consider a non-empty chunk of id $i \in [1, \sqrt{U}]$. Recall that it corresponds to the range $[(i-1)\sqrt{U}+1, i\sqrt{U}]$. We define for the chunk:

- its *leader* as the largest element in S_i;
- its *sentinel* as the predecessor of $q = (i-1)\sqrt{U} - 1$, which is essentially the greatest leader from Chunks $1, 2, ..., i-1$.

See Figure 15.1 for an illustration.

We are now ready to recursively define the vEBS on S as the collection of:

- hash table H;
- the leader and sentinel of every non-empty chunk;
- a vEBS Υ_P on P;
- a vEBS Υ_i on each non-empty S_i ($i \in [1, \sqrt{U}]$).

Note that Υ_P and each Υ_i are in a domain of size $\sqrt{U}$.

Let us analyze the space consumption. Denote by $f(n, U)$ the space of a vEBS on n integers in a domain of size U. We have:

$$f(n,U) \quad \leq \quad O(n) + f(n, \sqrt{U}) + \sum_{i=1}^{\sqrt{U}} f(|S_i|, \sqrt{U}).$$

Clearly, $f(0, U) = 0$, and $f(n, U) = O(1)$ when $1 \leq n = O(1)$ or $U \leq 4$. In an exercise, you will be asked to prove:

Lemma 15.1. $f(n, U) = O(n \log U)$.

15.1.2 Query

To find the predecessor of an integer $q \in [1, U]$, we first obtain the id $\lambda = \lfloor q/\sqrt{U} \rfloor + 1$ of the chunk that contains q. The following observations are obvious:

- If Chunk λ is empty, the predecessor of q is the leader of the first non-empty chunk to the left of Chunk λ.
- Otherwise, the predecessor of q is either the sentinel of Chunk λ or the predecessor of q in S_i.

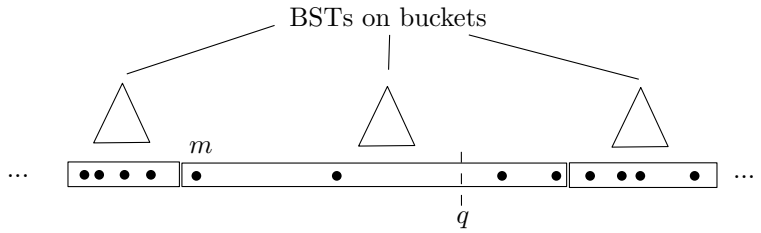


Figure 15.2: Each box shows a bucket, each of which has at most s points ($s = 4$ in this example).

Queries q_1, q_2, and q_3 in Figure 15.1 illustrates three different cases with queries q_1, q_2, and q_3.

The above naturally leads to the following algorithm. First, use H to decide in constant time whether Chunk λ is empty. If so, find the predecessor λ' of λ in P by searching Υ_P, and return the leader of Chunk λ'. Now, consider that Chunk λ is not empty. In this case, find the predecessor x of q in S_i by searching Υ_λ. If x exists, we return x as the final answer; otherwise, return the sentinel of Chunk λ.

Next, we prove that the query time is $O(\log\log U)$. Denote by $g(U)$ the query time of a vEBS when the domain has size U. No matter whether Chunk λ is empty or not, we always search a vEBS (i.e., Υ_P or Υ_i) created for a domain of size $\sqrt{U}$. Therefore:

$$g(U) \quad \leq \quad O(1) + g(\sqrt{U}).$$

Clearly, $g(4) = O(1)$. It thus follows that $g(U) = O(\log\log U)$.

15.2 Improving the space to $O(n)$

In this section, we will combine the structure of the previous section with a bucketing idea to reduce the space to linear while retaining the query time $O(\log\log U)$.

Structure. Set $s = \log_2 U$. Divide the input set S into *buckets*, each of which contains at most s elements of S. Specifically, sort S in ascending order, and then, group the first s elements into the first bucket, the next s elements into the second bucket, and so on. The total number of buckets is $O(n/s)$.

Collect the smallest element in each bucket into a set M. Build a vEBS on M, which consumes $O(|M|\log U) = O(\frac{n}{\log U}\log U) = O(n)$ space. Finally, for each bucket, create a BST (i.e., there are $O(n/s)$ BSTs). All the BSTs consume $O(n)$ space in total.

Query. Given a predecessor query q, we first find the predecessor m of q in M, which takes $O(\log\log U)$ time using the vEBS on M. The predecessor of q in the overall S must be the predecessor of q in the bucket of m, which can be found using the BST on that bucket in $O(\log s) = O(\log\log U)$ time. See Figure 15.2 for an illustration.

15.3 Remarks

The original ideas behind the vEBS are due to Boas [40], but the structure in [40] does not achieve $O(n)$ space. The version we described in this lecture is similar to what Willard [43] called the *y-fast trie*. Patrascu and Thorup [37] proved that no structure of $O(n \operatorname{polylog} n)$ space can achieve

query time strictly better than $O(\log\log U)$ (note: the BST improves the $O(\log\log U)$ query time *sometimes* — when $n \ll U$ — but not always; indeed, when $U = n^{O(1)}$, the vEBS is faster than the BST). In other words, the vEBS is worst-case optimal for the predecessor search.

Exercises

Problem 1. Prove Lemma 15.1.

Problem 2. Let S be a set of n integers in $\{1, ..., U\}$. Design a data structure of $O(n)$ space that answers a range reporting query (Section 2.2.1) on S in $O(\log \log U + k)$ time, where k is the number of integers reported.

Problem 3. Let S be a set of n integers in $\{1, ..., U\}$. Each integer in S is associated with a real-valued *weight*. Given an interval $q = [x, y]$ with $1 \leq x \leq y \leq U$, a *range min query* returns the smallest weight of the integers in $S \cap [x, y]$. Design a data structure of $O(n)$ space that answers a range min query in $O(\log \log U)$ time.

Problem 4. Describe how to support an insertion/deletion on the structure of Section 15.1.1 in $O(\log U)$ expected amortized time. You can assume that a perfect hash table can be updated in constant expected amortized time.

(Hint: think recursively. At the level of domain size U, you are making two insertions each into a domain size of $\sqrt{U}$.)

Problem 5.** Describe how to support an insertion/deletion on the structure of Section 15.2 in $O(\log \log U)$ expected amortized time.

(Hint: buckets can be split and merged periodically.)

Lecture 16: Leveraging the Word Length $w = \Omega(\log n)$ (2D Orthogonal Range Counting)

In all the structures discussed so far, we were never concerned about the length w of a word (a.k.a., a cell), i.e., the number of bits in a word. In the RAM model (Lecture 1), if the input set requires at least n cells to store, then $w \geq \log_2 n$ because this is the least number of bits needed to encode a memory address. Interestingly, this feature can often be used to improve data structures. We will see an example in this lecture.

We will discuss *orthogonal range counting* in 2D space. Let S be a set of n points in $\mathbb{R}^2$. Given an axis-parallel rectangle $q = [x_1, x_2] \times [y_1, y_2]$, a *range count query* reports $|S \cap q|$, namely, the number of points in S covered by q. At this stage of the course, you should know at least two ways to solve the problem. First, you can use the range tree (Section 4.4) to achieve $O(n \log n)$ space and $O(\log^2 n)$ query time (this was an exercise of Lecture 4). Second, by resorting to partial persistence, you can improve the query time to $O(\log n)$ although the space remains $O(n \log n)$ (an exercise of Lecture 8).

Today we will describe a structure with $O(n)$ space consumption and $O(\log n)$ query time. Our structure is essentially just the range tree, but incorporates *bit compression* to reduce the space by a factor of $\Theta(\log n)$.

It suffices to consider that every range count query is *2-sided*, namely, with search rectangle of the form $q = (-\infty, x] \times (-\infty, y]$ (this is known as *dominance counting*). Every general range count query can be reduced to four 2-sided queries (think: how?). We will assume that n is a power of 2; if not, simply add some dummy points to make it so. Finally, we will make the general position assumption that the points in S have distinct x- and y-coordinates (the assumption's removal was an exercise in Lecture 4).

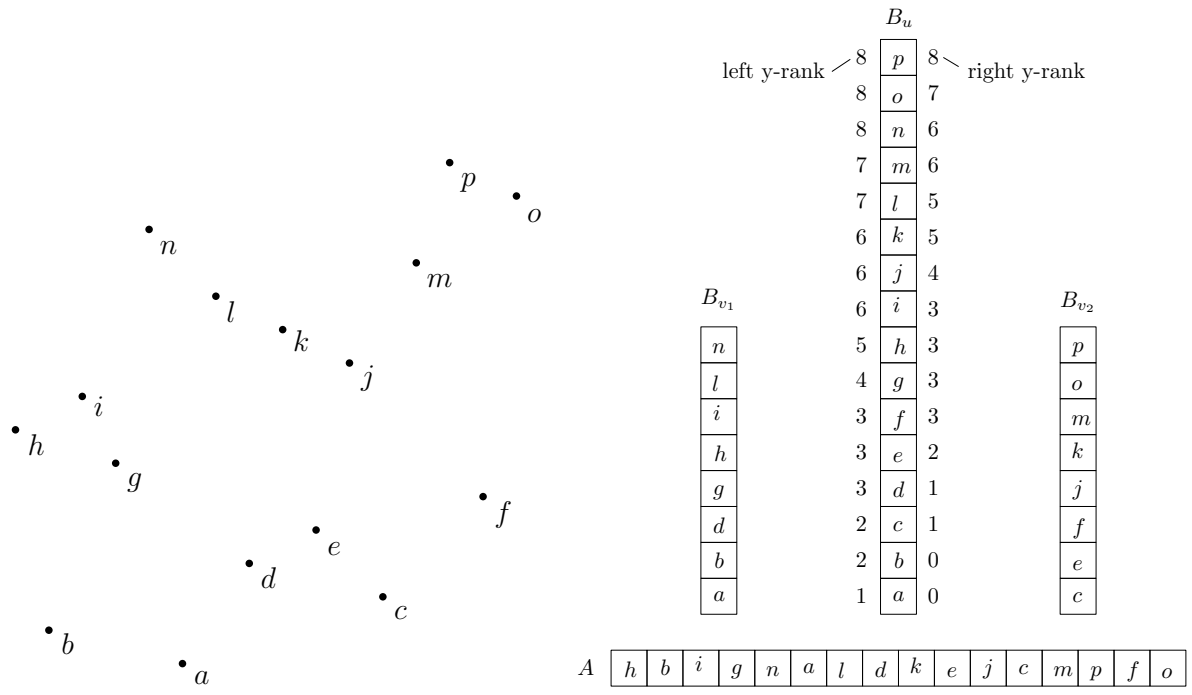
Notations. Given a point $p \in \mathbb{R}^2$, we denote by x_p and y_p its x- and y-coordinates, respectively. Given an array A of length ℓ, and any $i, j \in [1, \ell]$, we will denote by $A[i : j]$ the subarray that starts from $A[i]$ and ends at $A[j]$.

16.1 The first structure: $O(n \log n)$ space and $O(\log n)$ query time

We will first explain how to achieve $O(n \log n)$ space and $O(\log n)$ query time. Our structure can be regarded as a fast implementation of the range tree.

A real number $\lambda \in \mathbb{R}$ is said to have

- *x-rank* r in S, if S has r points p satisfying $x_p \leq \lambda$;
- *y-rank* r in S, if S has r points p satisfying $y_p \leq \lambda$.



(a) The input set

(b) Array A, and the B-arrays of the root u and its child nodes v_1 and v_2

Figure 16.1: The first structure

Structure. Sort the points of S in ascending order of x-coordinate, and store the ordering in an array A of size n.

Construct a binary tree T as follows. First, create the root node which corresponds to $A[1:n]$. In general, given a node u which corresponds to $A[a:b]$ for some integers $a < b$, create its left and right child nodes v_1, v_2 which correspond to $A[a:\frac{b-a+1}{2}]$ and $A[\frac{b-a+1}{2}+1:b]$, respectively. On the other hand, if $a = b$, u is a leaf of T. In any case, denote by S_u the set of points in $A[a:b]$.

Consider an arbitrary internal node u with left child v_1 and right child v_2. Note that $S_{v_1} \cup S_{v_2} = S_u$ and $S_{v_1} \cap S_{v_2} = \emptyset$. We associate u with an array B_u which sorts S_u in ascending of y-coordinate. Along with each $p \in S_u$, we store two integers:

- *left y-rank*: the y-rank of y_p in S_{v_1};
- *right y-rank*: the y-rank of y_p in S_{v_2}.

The B-arrays of all the nodes at the same level of T consume $O(n)$ space in total. Since T has $O(\log n)$ levels, the overall space of our structure is $O(n \log n)$.

Example. Figure 16.1(a) gives a set S of 16 points. The array A is shown at the bottom of Figure 16.1(b). Suppose that node u is the root of T, whose left and right child nodes are v_1 and v_2, respectively. Figure 16.1(b) also shows B_u, B_{v_1}, and B_{v_2}. The left and right y-ranks of each point in B_u are also indicated. □

Query. Let $q = (-\infty, x] \times (-\infty, y]$ be the search region. We assume that we are given the x-rank λ_1 of x in S, and the y-rank λ_2 of y in S (why is the assumption fair?).

Let us deal with a more general subproblem. Suppose that we are at a node u of T, and want to find out how many points in S_u are covered by q (if u is the root of T, $S_u = S$; and hence, the answer is precisely the final query result). We are told:

- λ_1: the x-rank of x in S_u;
- λ_2: the y-rank of y in S_u.

If u is a leaf node, S_u has only a single point; the answer can be found in constant time. Next, we consider that u has left child v_1 and right child v_2. We consider $\lambda_2 \geq 1$ because otherwise the answer is clearly 0.

Let p^* be the point at $B_u[\lambda_2]$ (p^* is the λ_2-th highest point in S_u). We distinguish two scenarios:

- Case 1: $\lambda_1 > |S_u|/2$. This means that all the points in S_{v_1} (note: $|S_{v_1}| = |S_u|/2$) have x-coordinates less than x. Thus, the number of points in S_{v_1} covered by q is exactly the left y-rank of p^*, which has already been pre-computed, and can be retrieved in constant time. However, we still need to find out how many points in S_{v_2} are covered by q. For this purpose, it suffices to solve the same subproblem recursively at v_2. But to do so, we need to prepare the x-rank of x in S_{v_2}, and the y-rank of y in S_{v_2}. Both can be easily obtained in constant time: the former equals $\lambda_1 - |S_{v_1}| = \lambda_1 - |S_u|/2$, while the latter is simply the right y-rank of p^*.
- Case 2: $\lambda_1 \leq |S_u|/2$. It suffices to find the number of points in S_{v_1} covered by q. We do so by recursively solving the subproblem at v_1. For this purpose, we need to prepare the x-rank of x in S_{v_1}, and the y-rank of y in S_{v_1}. Both can be obtained directly: the former is just λ_1, while the latter is the left y-rank of p^*.

In summary, we answer a range count query by descending a single root-to-leaf path in T, and spend $O(1)$ time at each node on the path. The query time is therefore $O(\log n)$.

16.2 Improving the space to $O(n)$

What is the culprit that makes the space complexity $O(n \log n)$? The B-arrays! For each node u in T, the array B_u has length $|S_u|$ and thus require $\Theta(|S_u|)$ words to store. Next, we will compress B_u into $O(1 + |S_u|/\log n)$ words. Accordingly, the overall space is reduced from $O(n \log n)$ to $O(n)$.

Henceforth, let s be an integer satisfying $\Omega(\log n) = s \leq \frac{1}{2}\log_2 n$. We will need:

Lemma 16.1. *With $o(n)$ pre-processing time, we can build a structure of $o(n)$ space to support the following operation in $O(1)$ time: given any bit vector of length s and any integer $t \in [1, s]$, return the number of 0's in the vector's first t bits.*

The proof is left to you as an exercise with hints.

Compressing B_u. We divide B_u into *chunks* of length s, except possibly for one chunk. Specifically, Chunk 1 includes the first s points of B_u, Chunk 2 the next s points, and so on. The last chunk may have less than s points if $|S_u|$ is not a multiple of s.

Let v_1 and v_2 be the left and right child nodes of u, respectively. For each chunk, we store two integers:

- left y-rank: the y-rank of y_p in S_{v_1}, where p is the highest point in the chunk;

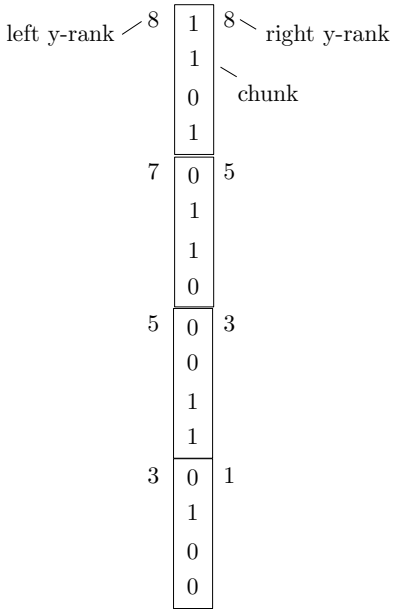


Figure 16.2: The compressed version of the array B_u in Figure 16.1

- right y-rank: the y-rank of y_p in S_{v_2}.

The total space to store the left/right y-ranks of all the chunks is $O(\lceil |S_u|/s \rceil) = O(1 + |S_u|/s)$ words, which is $O(w + \frac{w \cdot |S_u|}{s})$ bits.

For every chunk of σ points ($1 \leq \sigma \leq s$), we also store a *bit vector* of length σ. To explain, let the points in the chunk be $p_1, p_2, ..., p_\sigma$ in ascending order of y-coordinate. The i-th ($i \in [1, \sigma]$) bit in the bit-vector equals

- 0, if p_i comes from S_{v_1};
- 1, otherwise (i.e., p_i from S_{v_2}).

The bit vectors of all the chunks have precisely $|S_u|$ bits.

Example. Continuing the example of Figure 16.1, again let u be the root node. Figure 16.2 illustrates the compressed form of B_u. Here, $s = 4$, and B_u is cut into 4 chunks. The left and right y-ranks of each chunk are indicated outside the boxes. The bit vector of a chunk is given inside the boxes. For instance, the bit vector of the first (i.e., bottom-most) chunk is 0010, that of the second chunk is 1100, and so on. □

Other than the chunks' left/right y-ranks and bit vectors, we store nothing else for u (in particular, B_u is no longer necessary). The space required for u is therefore:

$$O\left(w + \frac{w \cdot |S_u|}{s} + |S_u|\right) \text{ bits} \quad = \quad O\left(1 + \frac{|S_u|}{s} + \frac{|S_u|}{w}\right) \text{ words } = O\left(1 + \frac{|S_u|}{\log n}\right) \text{ words}$$

where the last equality used the fact $w \geq \log_2 n$.

Consider any point $p \in S_u$. Recall that, in the structure of Section 16.1, we stored the left and right y-ranks of p explicitly. This is no longer the case in our new structure. Nevertheless, the lemma below shows that this information is implicitly captured:

Lemma 16.2. *If we know that p is the r-th highest point in S_u (for some $r \in [1, |S_u|]$), we can obtain the left and right y-ranks of p in $O(1)$ time.*

Proof. Since the left and right y-ranks of p add up to r, it suffices to explain how to find the left y-rank in constant time.

Let $i = \lfloor r/s \rfloor + 1$ be the id of the chunk that contains p. If $i \geq 2$, denote by r_{prefix} the left y-rank of Chunk $i-1$; otherwise, define $r_{prefix} = 0$. The value of r_{prefix} has been pre-computed and can be fetched in $O(1)$ time. Set $j = r - s(k-1)$; point p is the j-th highest point within Chunk k. Denote by $\boldsymbol{v}$ the bit-vector of the Chunk k. Use Lemma 16.1 to retrieve in $O(1)$ time the number r_{chunk} of 0's in the first j-th bits of $\boldsymbol{v}$. The left y-rank of p equals $r_{prefix} + r_{chunk}$. □

Example. Continuing the previous example, suppose that we want to find out the left y-rank of point j (see Figure 16.1) in B_u, knowing that j is the 10-th highest point in S_u. We first obtain the id 3 of the chunk containing j. Thus, $r_{prefix} = 5$, which is the left y-rank of Chunk 2. Within Chunk 3, point j is the second highest. In the bit-vector 0110 of the chunk, there is only $r_{chunk} = 1$ zero in the first 2 bits. Therefore, we conclude that the left y-rank of j must be $r_{prefix} + r_{chunk} = 6$. □

Space. We leave it as an exercise for you to prove that the overall space consumption of structure is now $O(n)$ words.

Query. Recall that the core in solving a range count query $q = (-\infty, x] \times (-\infty, y]$ is to tackle the following subproblem where, standing at an internal node u of T, we want to find out $|S_u \cap q|$, assuming that the following are known:

- λ_1: the x-rank of x in S_u;
- $\lambda_2 \geq 1$: the y-rank of y in S_u.

The algorithm in Section 16.1 spends $O(1)$ time at u before recursing into a child node of u. Lemma 16.2 allows us to obtain the left and right y-ranks of p^* in constant time, where p^* is the λ_2-th highest point in S_u. With this, the algorithm of Section 16.1 can still be implemented to run in $O(1)$ time at u.

The overall query time is therefore still $O(\log n)$.

16.3 Remarks

The structure we described is due to Chazelle [12]. When the x- and y-coordinates of all the points are integers, JaJa, Mortensen, and Shi [25] showed that the $w = \Omega(\log n)$ feature can even be used to improve the query time: they developed a structure of $O(n)$ space and $O(\log n / \log\log n)$ query time. Patrascu [36] showed that $O(\log n / \log\log n)$ query time is the best possible for any structure of $O(n \operatorname{polylog} n)$ space.

Exercises

Problem 1. Prove Lemma 16.1.

(Hint: tabulation; see Lecture 14.)

Problem 2. Prove that the structure of Section 16.2 uses $O(n)$ space.

Problem 3. Describe an algorithm to construct the structure of Section 16.2 in $O(n \log n)$ time.

Problem 4*. Make the structure of Section 16.2 fully dynamic to support each insertion and deletion in $O(\log^2 n)$ amortized time. The space consumption should still be $O(n)$. The structure must answer a range count query in $O(\log^2 n)$ time.

(Hint: logarithmic rebuilding + global rebuilding.)

Lecture 17: Approximate Nearest Neighbor Search 1: Doubling Dimension

We define a *metric space* as a pair $(U, dist)$ where

- U is a non-empty set (possibly infinite), and
- *dist* is a function mapping $U \times U$ to $\mathbb{R}_{\geq 0}$ (where $\mathbb{R}_{\geq 0}$ is the set of non-negative real values) satisfying:
  - $dist(e, e) = 0$ for any $e \in U$;
  - $dist(e_1, e_2) \geq 1$ for any $e_1, e_2 \in U$ such that $e_1 \neq e_2$;
  - symmetry, i.e., $dist(e_1, e_2) = dist(e_2, e_1)$ for any $e_1, e_2 \in U$;
  - the triangle inequality, i.e., $dist(e_1, e_2) \leq dist(e_1, e_3) + dist(e_3, e_2)$ for any $e_1, e_2, e_3 \in U$.

We will refer to each element in U as an *object*, and to *dist* as a *distance function*. For any $e_1, e_2 \in U$, $dist(e_1, e_2)$ is the *distance* between the two objects.

This lecture will discuss *nearest neighbor search*. The input is a set S of n objects in U. Given an object $q \in U \setminus S$, a *nearest neighbor query* reports an object $e^* \in S$ with the smallest distance to q, namely:

$$dist(q, e^*) \quad = \quad \min_{e \in S} dist(q, e).$$

The object e^* is a *nearest neighbor* of q.

Ideally, we would like to preprocess S into a data structure such that all nearest neighbor queries can be answered efficiently, no matter what the metric space is. Unfortunately, this is impossible: n distances must be calculated in the worst case, regardless of the preprocessing (we will discuss this in Section 17.4). In other words, the trivial algorithm which simply computes the distances from q to all the objects in S is already optimal. In fact, this problem is not easy even in the *specific* metric space where $U = \mathbb{N}^3$ and *dist* is the Euclidean distance; see the remarks in Section 17.4

We therefore resort to approximation. Fix some constant $c > 1$. If $e^* \in S$ is a nearest neighbor of an object $q \in U \setminus S$, an object $e \in S$ is a *c-approximate nearest neighbor* of q if

$$dist(q, e) \quad \leq \quad c \cdot dist(q, e^*).$$

Accordingly, a *c-approximate nearest neighbor* (c-ANN) *query* returns an arbitrary c-approximate nearest neighbor of q (note: even nearest neighbors may not be unique, let alone c-ANNs). Unfortunately, this problem is still hopelessly difficult: calculating n distances is still necessary in the "hardest" metric space (Section 17.4).

Fortunately, the metric spaces encountered in practice may not be so hard, such that by pre-computing a structure of near-linear space we can answer c-ANN queries efficiently. For example, this is possible for $U = \mathbb{N}^d$ with a constant dimensionality d, and *dist* being the Euclidean distance. In this lecture, we will learn a structure for $c = 3$ that works for many metric spaces, and is *generic* because it treats objects and the function *dist* as *black boxes*. It does not matter whether the objects are multi-dimensional points or DNA sequences, or whether *dist* is the Euclidean distance (for points) or the edit distance (for DNA sequences); our structure works in exactly the same way.

Crucial to the structure is the concept of *doubling dimension* which allows us to measure how hard a metric space is. The performance of our structure is established with respect to the doubling dimension. Our structure is efficient when the doubling dimension is small (i.e., the metric space is easy), but is slow when the dimension is large (the metric space is hard). Even better, the concept is *data dependent*. More specifically, even if the metric space $(U, dist)$ is hard, the input set S may still allow c-ANN queries to be answered efficiently, if the metric space $(S, dist)$ has a small doubling dimension. This is useful in practice: even though c-ANN search under the edit distance may be difficult for arbitrary DNA sequences, it is possible to do much better on a *particular* set S of sequences.

We need to be clear how to measure the space and query time of a structure (remember: we will treat objects and the distance function as black boxes):

- The space of a structure is the number of memory cells occupied, plus the number of objects stored. For example, "$O(n)$ space" means not only the occupation of $O(n)$ memory, but also the storage of $O(n)$ objects.
- The query time will be measured as the sum of two terms: (i) the number of atomic operations of the RAM model, and (ii) the number of times that *dist* is invoked. For example, "$O(\log n)$ time" means that the algorithm performs $O(\log n)$ atomic operation *and* calculates $O(\log n)$ distances.

We define the *aspect ratio* of S as

$$\Delta(S) = \left(\sup_{e_1,e_2 \in S} dist(e_1, e_2)\right) / \left(\inf_{\text{distinct } e_1,e_2 \in S} dist(e_1, e_2)\right) \tag{17.1}$$

namely, the ratio between the maximum and minimum pair-wise distances in S.

Notations. We will reserve e, x, y, z for objects, and X, Y for sets of objects.

17.1 Doubling dimension

Consider an arbitrary metric space $(U, dist)$. We will formalize its doubling dimension in three definitions:

Definition 17.1. Let X be a non-empty subset of U. The **diameter** of X — denoted as $diam(X)$ — is the maximum distance of two objects in X, or formally:

$$\sup_{e_1,e_2 \in X} dist(e_1, e_2).$$

□

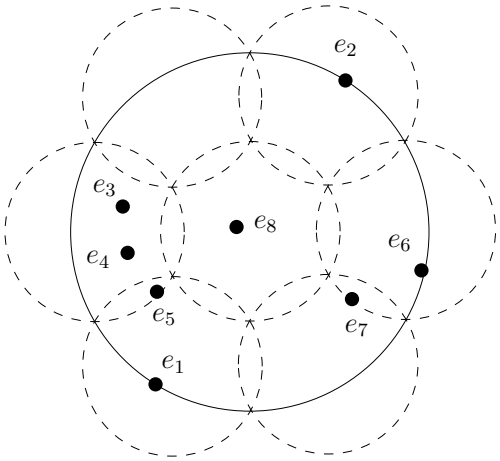


Figure 17.1: When $U = \mathbb{N}^2$ and *dist* is the Euclidean distance, any set X of points can be divided into 7 disjoint subsets whose diameters are at most $\frac{1}{2}diam(X)$.

Definition 17.2. A non-empty $X \subseteq U$ can be 2^λ**-partitioned** (where $\lambda \geq 0$ is a real value) if X can be divided into (disjoint) subsets $X_1, X_2, ..., X_m$ such that

- $m \leq 2^\lambda$
- every X_i ($i \in [m]$) has diameter at most $\frac{1}{2}diam(X)$.

□

Definition 17.3. The **doubling dimension** of the metric space $(U, dist)$ is the smallest real value λ such that every finite non-empty $X \subseteq U$ can be 2^λ-partitioned. □

Example. Let us look at a specific metric space where $U = \mathbb{N}^2$ and *dist* is the Euclidean distance. We will show that $(\mathbb{N}^2, \text{Euclidean})$ has a doubling dimension less than 3.

Suppose that we are given any set X of points in $\mathbb{N}^2$ with $|X| \geq 2$; Figure 17.1 shows an example where X is a finite set of 8 points. Denote by D the smallest disc covering X; in Figure 17.1, D is enclosed by the circle in solid line. The diameter of D is at most $diam(X)$ (think: why?). We can always find 7 discs $D_1, ..., D_7$ of diameter $\frac{1}{2}diam(D)$ such that they together cover D (the proof requires only high-school geometry, and is left as an exercise); in the figure, those discs are indicated in dashed lines. Now, assign each point $e \in X$ to a disc that covers it; if e is covered by more than one disc, assign it to an arbitrary disc (but only one disc). For each $i \in [1, 7]$, define X_i as the set of points assigned to disc D_i. Thus, $X_1, ..., X_7$ partition X; and each X_i has diameter at most $\frac{1}{2}diam(D) \leq \frac{1}{2}diam(X)$.

It thus follows that X can be $2^{\log_2 7}$-partitioned. Therefore, the metric space has a doubling dimension of $\log_2 7 < 3$. □

The fact below follows immediately from Definition 17.3:

Proposition 17.4. *For any non-empty subset $X \subseteq U$, the doubling dimension of $(X, dist)$ is no more than that of $(U, dist)$.*

17.2 Two properties in the metric space

Balls. Recall that, in $\mathbb{R}^d$, a "ball" is the set of points inside a d-dimensional sphere (a 2D ball is a disc). Next, we generalize the concept to metric spaces:

Definition 17.5. For any object $e \in U$ and real value $r \geq 0$, the **ball** $B(e,r)$ includes all the objects $e' \in U$ such that $dist(e,e') \leq r$. The object e is the **center** of the ball, while the value r is the **radius**. □

In $\mathbb{R}^d$, a d-dimensional ball of radius r can be covered by $2^{O(d)}$ balls of radius $\Omega(r)$. A similar result holds for metric spaces too.

Lemma 17.6. *Let λ be the doubling dimension of the metric space $(U, dist)$, and $c \geq 1$ be a constant. Then, any ball $B(e,r)$ can be covered by at most $2^{O(\lambda)}$ balls of radius r/c, namely, there exist objects $e_1, ..., e_m \in U$ such that*

- $m \leq 2^{O(\lambda)}$;
- $B(e,r) \subseteq \bigcup_{i=1}^{m} B(e_i, r/c)$.

Proof. Set $X = B(e,r)$. The triangle inequality implies that $diam(X) \leq 2r$ (think: why?).

Let us first prove the theorem for $c = 2$. By definition of λ, we can divide X into subsets $X_1, ..., X_{m'}$ ($m' \leq 2^{\lambda}$) all of which have diameter at most r. In turn, each X_i ($i \in [m]'$) can be divided into at most 2^{λ} subsets, each of which has diameter at most $r/2$, and hence, can be covered by a ball with radius $r/2$ (think: why?). It thus follows that X can be covered by at most $2^{\lambda} \cdot 2^{\lambda} = 2^{2\lambda}$ balls of radius $r/2$.

The proof for the case $c \neq 2$ is left to you as an exercise. □

Constant aspect-ratio object sets. In $\mathbb{R}^d$, you can place at most $2^{O(d)}$ points in a sphere of radius 1 while ensuring the distance of any two points to be at least $1/2$. The next lemma generalizes this to any metric space:

Lemma 17.7. *Suppose that the metric space $(X, dist)$ has doubling dimension λ, and that the aspect ratio of X is bounded by a constant. Then, X can have no more than $2^{O(\lambda)}$ objects.*

Proof. If $|X| = 1$, the lemma is vacuously true. Next, we consider $|X| \geq 2$. Define:

$$dist_{min} \quad = \quad \inf_{\text{distinct } x_1, x_2 \in X} dist(x_1, x_2)$$

Thus, $diam(X)/dist_{min} = \Delta(X) = O(1)$. This means $diam(X) \leq O(1) \cdot dist_{min}$.

Set

$$c \quad = \quad 4 \cdot \frac{diam(X)}{dist_{min}} = 4 \cdot \Delta(X) = O(1).$$

Take any object $x \in X$. Clearly, the entire $X \subseteq B(x, diam(X))$. By Lemma 17.6, $B(x, diam(X))$ is covered by $m \leq 2^{O(\lambda)}$ balls of radius $diam(X)/c = \frac{1}{4} dist_{min}$. Denote those balls as $B_1, ..., B_m$.

Each B_i ($i \in [m]$) can cover exactly one object in X. To see this, assume that e is the center of B_i. If B_i contains two objects $x, y \in X$, it must hold that $dist(x,y) \leq dist(x,e) + dist(e,y) \leq \frac{1}{2} dist_{min}$, which contradicts the definition of $dist_{min}$. □

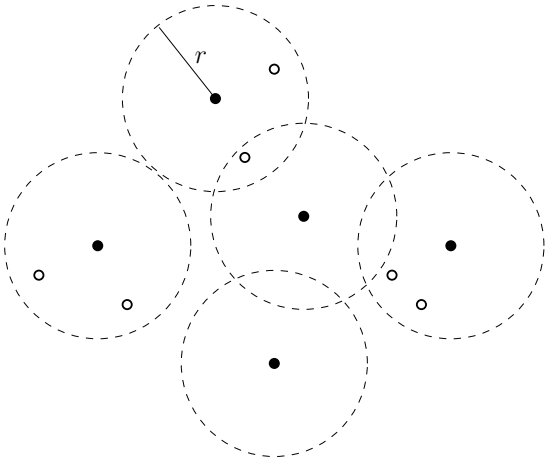


Figure 17.2: A sample net example: X is the set of all points shown, and Y the set of black points.

17.3 A 3-approximate nearest neighbor structure

We are now ready to introduce the promised 3-ANN structure. As before, denote by $(U, dist)$ the underlying metric space, and by $S \subseteq U$ the input set of $n \geq 2$ objects. Set

$$h \quad = \quad \lceil \log_2 diam(S) \rceil \tag{17.2}$$

where $diam(S)$ is the diameter of S (Definition 17.1). Denote by λ the doubling dimension of $(S, dist)$; note that λ can be smaller than the doubling dimension of $(U, dist)$ (Proposition 17.4).

We aim to establish:

Theorem 17.8. *There is a structure of $2^{O(\lambda)} \cdot n \cdot h$ space that answers a 3-ANN query in $2^{O(\lambda)} \cdot h$ time.*

When $\lambda = O(1)$, the space is $O(nh)$ and the query time is $O(h)$. In Section 17.4, we will discuss a number of scenarios where this is true.

17.3.1 Sample nets

Definition 17.9. Consider any $X \subseteq S$ and any real value $r > 0$. A non-empty $Y \subseteq X$ is an **r-sample net** of X if the following two conditions hold:

- for any distinct objects $y_1, y_2 \in Y$, $dist(y_1, y_2) > r$;
- $X \subseteq \bigcup_{y \in Y} B(y, r)$.

□

Note that the second bullet indicates that, for any object $x \in X$, Y has an object y such that $dist(x, y) \leq r$. See Figure 17.2 for an example for the metric space ($\mathbb{N}^2$, Euclidean).

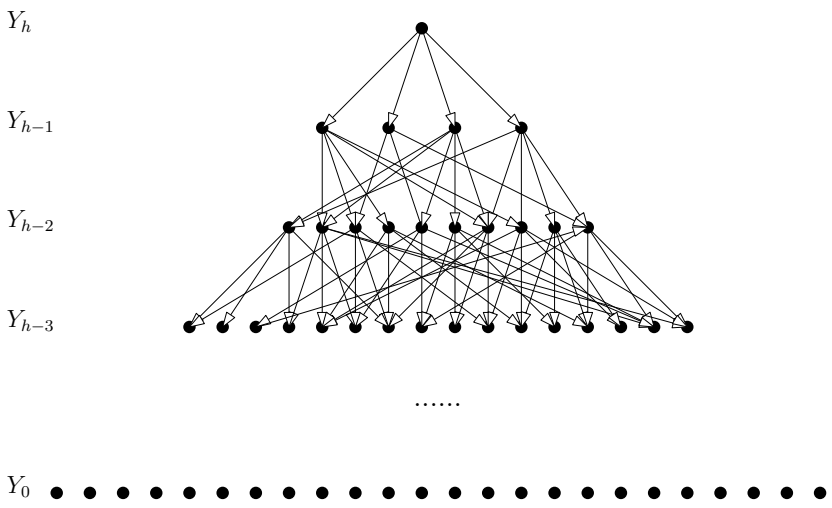


Figure 17.3: Illustration of G

17.3.2 Structure

Our strategy is to gradually "sparsify" the input set S. Define for each $i \in [0, h]$:

$$Y_i \quad = \quad 2^i\text{-sample net of } S.$$

The following facts are obvious:

- Y_h has a single object, noticing that $2^h \geq diam(S)$ (see (17.2));
- $|Y_i| \leq n$ for all i.

It thus follows that the total size of $Y_0, Y_1, ..., Y_h$ is $O(n \cdot h)$.

We will build a directed graph G as follows. The vertices of G form $h+1$ layers 0, 1, ..., h, where the i-th layer ($1 \leq i \leq h$) contains a vertex for each object in Y_i. Edges of G exist only between two *consecutive* layers. Specifically, an object y (a.k.a. vertex) in Y_i ($i \geq 1$) has an out-going edge to an object z (a.k.a. vertex) in Y_{i-1} if and only if

$$dist(y, z) \quad \leq \quad 7 \cdot 2^i. \tag{17.3}$$

See Figure 17.3 for an illustration.

For each object $y \in Y_i$, we denote by $N_i^+(y)$ the set of out-neighbors of y. Each node in $N_i^+(y)$ will be referred to as a *child* of y.

Lemma 17.10. $|N_i^+(y)| = 2^{O(\lambda)}$.

Proof. Due to Lemma 17.7, it suffices to show that $N_i^+(y)$ has a constant aspect ratio. Clearly, $N_i^+(y) \subseteq Y_{i-1}$; hence, any two distinct objects $z_1, z_2 \in N_i^+(y)$ must have distance at least 2^{i-1}. On the other hand, $dist(z_1, z_2) \leq dist(z_1, y) + dist(y, z_2) \leq 7 \cdot 2^i + 7 \cdot 2^i = 14 \cdot 2^i$. Therefore, $N_i^+(y)$ has an aspect ratio at most 28. □

The G constitutes our data structure. It is clear that the space consumption is $2^{O(\lambda)} \cdot n \cdot h$.

17.3.3 Query

Given a query object $q \in U \setminus S$, we descend a single path π in G as follows:

- The first node visited is the root of G, namely, the sole vertex in Y_h.
- Suppose that π contains an object (i.e., vertex) $y \in Y_i$ for some $i \geq 1$. Then, we add to π the child z of y with the smallest $dist(q, z)$, breaking ties arbitrarily.

Then, we return the object in π closest to q as our final answer.

The query time is clearly $2^{O(\lambda)} \cdot h$ (Lemma 17.10). In the rest of the section, we will prove that our answer is a 3-ANN of q.

Denote by e^* the (exact) nearest neighbor of q. Let $y_h, y_{h-1}, ..., y_0$ be the objects in π, where y_i belongs to Y_i for each $i \in [0, h]$. It suffices to prove that at least one of $y_h, y_{h-1}, ..., y_0$ has distance to q at most $3 \cdot dist(q, e^*)$.

Let j be the largest integer satisfying

$$dist(q, y_j) \quad > \quad 3 \cdot 2^j. \tag{17.4}$$

Note that j may not exist. Indeed, our argument proceeds differently depending on whether it does.

Case 1: j does not exist. This means $d(q, y_0) \leq 3 \cdot 2^0 = 3 \leq 3 \cdot dist(q, e^*)$, where the last inequality used the fact that $dist(q, e^*) \geq 1$ (recall that $q \notin S$).

Case 2: $j = h$. In other words, $dist(q, y_h) > 3 \cdot 2^h \geq 3 \cdot diam(S)$. We have:

$$\begin{aligned} dist(q, e^*) &\geq dist(q, y_h) - dist(e^*, y_h) \\ &\geq 3 \cdot diam(S) - diam(S) = 2 \cdot diam(S) \end{aligned} \tag{17.5}$$

which intuitively means that q is far away from the entire S. We can further derive:

$$\begin{aligned} dist(q, y_h) &\leq dist(q, e^*) + dist(e^*, y_h) \\ &\leq dist(q, e^*) + diam(S) \\ &\leq 1.5 \cdot dist(q, e^*) \end{aligned}$$

where the last inequality used (17.5).

Case 3: $j < h$. It thus follows that $dist(q, y_{j+1}) \leq 3 \cdot 2^{j+1}$. Next, we will argue that y_{j+1} is a 3-ANN of q.

Recall that Y_j is a 2^j-sample net of S. Hence, there must exist an object $z \in Y_j$ such that $dist(e^*, z) \leq 2^j$.

Lemma 17.11. *z is a child of y_{j+1}.*

Proof.

$$\begin{aligned} dist(z, y_{j+1}) &\leq dist(z, e^*) + dist(e^*, y_{j+1}) \\ &\leq dist(z, e^*) + dist(q, e^*) + dist(q, y_{j+1}) \\ (e^* \text{ is the nearest neighbor}) \quad &\leq dist(z, e^*) + 2dist(q, y_{j+1}) \\ &\leq 2^j + 2 \cdot 3 \cdot 2^{j+1} < 7 \cdot 2^{j+1}. \end{aligned}$$

□

Also recall that y_j is the child of y_{j+1} *closest* to q, which means:

$$dist(q, z) \geq dist(q, y_j) \geq 3 \cdot 2^j.$$

We now have:

$$\begin{aligned} dist(q, e^*) &\geq dist(q, z) - dist(e^*, z) \\ &\geq 3 \cdot 2^j - 2^j = 2^{j+1}. \end{aligned}$$

Therefore, $dist(q, y_{j+1}) \leq 3 \cdot dist(q, e^*)$.

We now complete the whole proof of Theorem 17.8.

17.4 Remarks

The above structure, which is due to Krauthgamer and Lee [27], is efficient when the underlying metric space $(U, dist)$ has a small doubling dimension λ. This is true when $U = \mathbb{N}^d$ for a constant dimensionality d, and $dist$ is the Euclidean distance. It can be proved [3] that $(\mathbb{N}^d, \text{Euclidean})$ has a doubling dimension of $O(d) = O(1)$ (you will be asked to prove a somewhat weaker statement in an exercise). It immediately follows from Theorem 17.8 (with an improvement you will see in the exercises) that, we can store a set S of n points in $\mathbb{N}^d$ in a structure of $O(n \cdot \log \Delta(S))$ space such that a 3-ANN of any query point can be found in $O(\log \Delta(S))$ time. In comparison, for *exact* nearest neighbor search in $\mathbb{N}^3$ (Euclidean distance), no known structure can achieve $n \operatorname{polylog} n$ space and $\operatorname{polylog} n$ query time simultaneously, even if $\Delta(S)$ is a polynomial of n.

Given a point $p \in \mathbb{N}^d$, let us use $p[i]$ to denote the coordinate of p on dimension i. For any real value $t > 0$, the so-called L_t*-norm* between two points p and q is:

$$\left(\sum_{i=1}^{d} \left| p[i] - q[i] \right|^t \right)^{1/t}.$$

The Euclidean distance is simply the L_2 norm. It is known that the metric space $(\mathbb{N}^d, L_t\text{-norm})$ also has doubling dimension $O(d)$, regardless of t. When d is a constant, we can once again obtain an efficient 3-ANN structure using Theorem 17.8.

What if λ is large? In this case, the metric space is "hard"; and Theorem 17.8 does not work for all inputs S. In the next lecture, we will introduce another technique that permits us to deal with some hard metric spaces (but not all). On the other hand, note that the λ in Theorem 17.8 pertains *only* to the metric space $(S, dist)$, as opposed to $(U, dist)$. Hence, if the input set S is "easy", the theorem still yields a good structure, even though the underlying metric space is hard.

Let us also briefly discuss lower bounds. Remember that our goal is to design a *generic* data structure that treats objects and distance functions as black boxes. In that case, a simple adversary argument suffices to show that no structure can avoid calculating n distances in answering a query if the *exact* nearest neighbor is desired. For this purpose, simply define a set S of n objects where the distance between any two distinct objects is 4. Now, issue a query with an object $q \notin S$. Design the distances in such a way that $dist(q, e^*) = 1$ for exactly one object $e^* \in q$ while $dist(q, e) = 4$ for all other $e \in S \setminus \{e^*\}$. The design clearly satisfies the requirements of a metric space. The trick, however, is that the adversary decides which object in S is e^* by observing how the query algorithm $\mathcal{A}$ runs. Specifically, whenever $\mathcal{A}$ asks for the distance $dist(q, x)$ for some $x \in S$, the

adversary answers 4. The only exception happens when x is the last object in S whose distance to q has not been calculated; in this case, the adversary answers $dist(q, x) = 1$, i.e., setting $e^* = x$. Therefore, $\mathcal{A}$ cannot terminate before all the n distances have been calculated (think: what could go wrong if $\mathcal{A}$ terminates, say, after computing $n - 1$ distances?).

The same argument also shows that n distances must be calculated even if our goal is to return a 3-ANN (think: why?). Krauthgamer and Lee [27] presented a stronger lower bound argument. They showed that if λ is the doubling dimension of the metric space $(S, dist)$, $2^{\Omega(\lambda)} \log |S|$ distances must be calculated to answer c-ANN queries with constant c.

Finally, it is worth mentioning that Krauthgamer and Lee [27] developed a more sophisticated structure that uses $O(n)$ space and answers any $(1 + \epsilon)$-ANN query in $2^{O(\lambda)} \log \Delta(S) + (1/\epsilon)^{O(\lambda)}$ time, where λ is the doubling dimension of $(S, dist)$ and $\epsilon > 0$ is an arbitrary real value.

Exercises

Problem 1*. Prove: in $\mathbb{R}^2$, any disc of radius 1 can be covered by 7 discs of radius 1/2.

(Hint: observe the intersection points made by the $7 + 1 = 8$ circles in Figure 17.1.)

Problem 2. Finish the proof of Lemma 17.6.

(Hint: for $c < 2$, manually increase c to 2. To prove $c = 4$, apply the argument in the proof of Lemma 17.6 twice.)

Problem 3. Given an algorithm to find an r-sample net of S in $O(n^2)$ time where $n = |S|$.

Problem 4. Consider the metric space $(U, dist)$ where $dist(e, e') = 1$ for any distinct $e, e' \in U$. If U has a finite size, what is the doubling dimension of $(U, dist)$?

Problem 5*. Prove: the metric space $(\mathbb{N}^d, \text{Euclidean})$ has doubling dimension $O(d \log d)$.

(Hint: in 2D space, a disc of radius 1 is covered by a square of side length 2, and covers a square of side length $\sqrt{2}$. Extend this observation to $\mathbb{N}^d$.)

Problem 6. Let w be the word length. Let $\mathbb{N}_w$ be the set of integers from 0 to $2^w - 1$. Let P be a set of n points in $\mathbb{N}_w^d$ where d is a fixed constant. The value of n satisfies $w = \Theta(\log n)$. Describe a structure of $O(n \log n)$ space such that, given any point $q \in \mathbb{N}_w^d$, we are able to find a 3-ANN of q in P using $O(\log n)$ time. The distance metric is the Euclidean distance.

Problem 7*. Improve the structure of Theorem 17.8 to achieve $2^{O(\lambda)} \cdot O(n \cdot \log \Delta(S))$ space and $2^{O(\lambda)} \cdot O(\log \Delta(S))$ query time.

(Hint: $Y_i = S$ until i becomes sufficiently large).

Lecture 18: Approximate Nearest Neighbor Search 2: Locality Sensitive Hashing

This lecture continues our discussion on the c-approximate nearest neighbor (c-ANN) search problem. We will learn a technique called *locality sensitive hashing* (LSH). If $(U, dist)$ is a metric space with a constant doubling dimension λ, LSH usually performs worse than the structure of Theorem 17.8. However, the power of LSH is reflected in its ability to deal with "hard" metric spaces with large λ.

For example, consider the metric space $(U, dist) = (\mathbb{N}^d, \text{Euclidean})$, where the dimensionality d should *not* be regarded as a constant. The metric space has doubling dimension $\Theta(d)$. Theorem 17.8 yields a structure that calculates $2^{\min\{\log_2 n, \Omega(d)\}}$ distances, which is already n even for $d = \Omega(\log n)$! In fact, for a difficult problem like this, it is challenging even just to beat the naive query algorithm (which computes n distances) by a polynomial factor, while consuming a polynomial amount of space; e.g., $O((dn)^2)$ space and $O(dn^{0.99})$ query time would make a great structure. LSH allows us to achieve the purpose.

When the objects in U and *dist* are treated as black boxes, we will measure the space and query time of a structure in a more careful manner compared to the last lecture:

- The space of a structure is expressed with two terms: (i) the number of memory cells occupied, and (ii) the number of objects stored.
- The query time is also expressed with two terms: (i) the number of atomic operations performed, and (ii) the number of distances calculated.

Notations and math preliminaries. We will reserve e, x for objects, and Z for sets of objects. Given a set $Z \subseteq U$, we denote by $diam(Z)$ the *diameter* of Z, defined in the same way as in Definition 17.1.

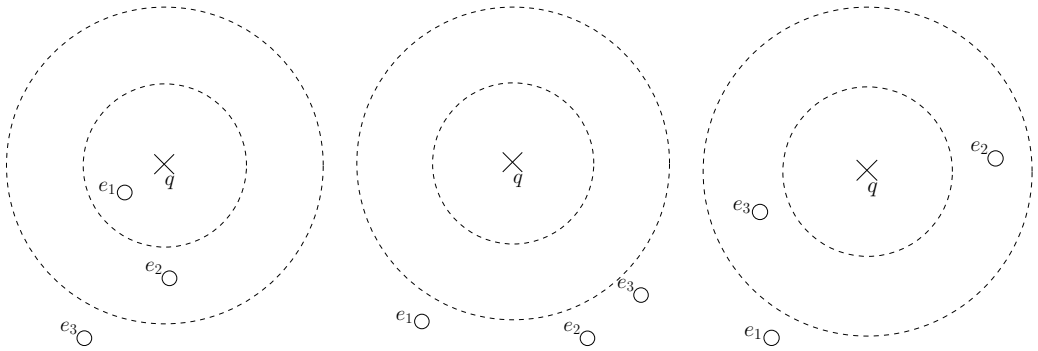
If Z_1, Z_2 are two sets of objects, their *multi-set* union is the collection of all the objects in Z_1 and Z_2, with duplicates retained.

If x is a point in $\mathbb{N}^d$, $x[i]$ denotes its coordinate on the i-th dimension ($i \in [1, d]$).

We will reserve X for random variables. If $X \geq 0$ is a real-valued random variable, we must have for any $t \geq 1$

$$\Pr\left[X \geq t \cdot \mathbf{E}[X]\right] \leq \frac{1}{t} \tag{18.1}$$

which is known as Markov's inequality.



(a) Case 1 (b) Case 2 Case 3

Figure 18.1: Illustrate of $(r, 2)$-near neighbor queries

18.1 (r, c)-near neighbor search

We will define a problem called (r, c)-*near neighbor search*, where $r \geq 1$ and $c > 1$ are real values. Let S be a set of n objects in U. Given an object $q \in U$, an (r, c)-*near neighbor query* — abbreviated as (r, c)-NN query — returns:

- **Case 1:** an object with distance at most cr to q, if S has an object with distance at most r to q;
- **Case 2:** nothing, if S has no object with distance at most cr to q;
- **Case 3:** either nothing or an object with distance at most cr to q, otherwise.

Example. Suppose that $U = \mathbb{N}^2$ and *dist* is the Euclidean distance. Figure 18.1(a) illustrates Case 1, where the inner and outer circles have radii r and $2r$, respectively. $S = \{e_1, e_2, e_3\}$. The cross point q indicates an $(r, 2)$-NN query. Since $dist(q, e_1) \leq r$, the query must return an object, but the object can be either e_1 or e_2. In Figure 18.1(b), however, the query *must not* return anything because all the objects in S have distances to q greater than $2r$ (Case 2). Figure 18.1(c) demonstrates Case 3, where the query may or may not return something; however, if it does, the object returned must be either e_2 or e_3. □

Lemma 18.1. *Suppose that, for any $r \geq 1$ and constant $c > 1$, we know how to build a structure on S that answers (r, c)-NN queries. By building $O(\log diam(S))$ such structures, we can answer any c^2-ANN query on S by issuing $O(\log diam(S))$ (r, c)-NN queries with the same c but different r.*

The proof is left to you as an exercise. In the rest of the lecture, we will focus on (r, c)-NN search.

18.2 Locality sensitive hashing

A *random function* h as a function that is drawn from a family H of functions according to a certain distribution.

Definition 18.2. Consider a metric space $(U, dist)$. Let r, c, p_1, and p_2 be real values satisfying:

- $r \geq 1$, $c > 1$;
- $0 < p_2 < p_1 \leq 1$.

A random function $h : U \to \mathbb{N}$ is an (r, cr, p_1, p_2)-**locality sensitive hash function** if:

- for any objects $x, y \in U$ satisfying $dist(x, y) \leq r$, it holds that $\Pr[h(x) = h(y)] \geq p_1$;
- for any objects $x, y \in U$ satisfying $dist(x, y) > cr$, it holds that $\Pr[h(x) = h(y)] \leq p_2$.

□

We will abbreviate 'locality sensitive hash function" as "LSH function". Given a (r, cr, p_1, p_2)-LSH function h, we define

$$\rho = \frac{\ln(1/p_1)}{\ln(1/p_2)} \tag{18.2}$$

as the *log-ratio* of h. Note that $\rho < 1$.

Lemma 18.3. (The amplification lemma) *Suppose that we know how to obtain an (r, cr, p_1, p_2)-LSH function h. Then, for any integer $\ell \geq 1$, we can build an $(r, cr, p_1^\ell, p_2^\ell)$-LSH function g such that for any object x:*

- *$g(x)$ can be computed in cost $O(\ell)$ times higher than $h(x)$;*
- *$g(x)$ can be stored in $O(\ell)$ space.*

Proof. Take ℓ independent (r, cr, p_1, p_2)-LSH functions $h_1, h_2, ..., h_\ell$. Design $g(x)$ to be the string that concatenates $h_1(x), h_2(x), ..., h_\ell(x)$. For any objects x and y, $g(x) = g(y)$ if and only if $h_i(x) = h_i(y)$ for all $i \in [1, \ell]$. □

Example. We will describe how to obtain an (r, cr, p_1, p_2)-LSH function for $(\mathbb{N}^d, \text{Euclidean})$. First, generate d independent random variables $\alpha_1, \alpha_2, ..., \alpha_d$ each of which follows the normal distribution (i.e., mean 0 and variance 1). Let $\beta > 0$ be a real value that depends on c, and γ a real value generated uniformly at random in $[0, \beta]$. For any point $x \in \mathbb{N}^d$, define:

$$h(x) = \left\lfloor \frac{\gamma + \sum_{i=1}^{d} (\alpha_i \cdot x[i]/r)}{\beta} \right\rfloor. \tag{18.3}$$

Lemma 18.4 ([15])**.** *For any $r \geq 1$ and any constant $c > 0$, the function in* (18.3) *is an (r, cr, p_1, p_2)-LSH function satisfying:*

- *p_2 is a constant;*
- *the log-ratio ρ of the function is at most $1/c$.*

The proof is non-trivial and not required in this course. □

18.3 A structure for (r, c)-NN search

We will now describe a structure for answering (r, c)-NN queries on a set S of n objects in U, assuming the ability to build (r, cr, p_1, p_2)-LSH functions with a log-ratio ρ (see (18.2)). Denote by t_{lsh} the time needed to evaluate the value of an (r, cr, p_1, p_2)-LSH function (e.g., $t_{lsh} = O(d)$ for the function in (18.3)).

Our goal is to prove:

Theorem 18.5. *There is a structure using $O(n^{1+\rho} \cdot \log_{1/p_2} n)$ memory cells and storing $O(n^{1+\rho})$ objects that can answer one single (r, c)-NN query correctly with probability at least $1/10$. The query time is $O(n^{\rho} \cdot \log_{1/p_2} n \cdot t_{lsh})$, plus the cost of calculating $O(n^{\rho})$ distances.*

You may be disappointed: the structure can answer only *one* query with a low success probability. Don't be! Using standard techniques, we can improve the structure to support an arbitrary number of queries with high probability (e.g., $1 - 1/n^{100}$), by increasing the space and query time only by a logarithmic factor; you will explore this in an exercise.

18.3.1 Structure

Let $\ell \geq 1$ and $L \geq 1$ be integers to be determined later. Use Lemma 18.3 to obtain L independent $(r, cr, p_1^{\ell}, p_2^{\ell})$-LSH function $g_1, g_2, ..., g_L$. For each $i \in [1, L]$, define a *bucket* as a maximal set of objects $x \in S$ with the same $g_i(x)$. A *hash table* T_i collects all the non-empty buckets.

The hash tables $T_1, ..., T_L$ constitute our structure. The space consumption is $O(n \cdot L \cdot \ell)$ memory cells plus $O(n \cdot L)$ objects.

18.3.2 Query

Consider an (r, c)-NN query with search object q. For each $i \in [1, L]$, let b_i be the bucket of $g_i(q)$.

We take a collection Z of $2L + 1$ *arbitrary* objects from the *multi-set* union of $b_1, ..., b_L$. In the special case where $\sum_{i=1}^{L} |b_i| \leq 4L+1$, Z collects all the objects in those buckets. We find the object e in Z closest to q, breaking ties arbitrarily. Return e if $dist(q, e) \leq cr$, or nothing, otherwise.

The query time is $O(t_{lsh} \cdot \ell \cdot L)$ atomic operations, plus the cost of computing $O(L)$ distances.

18.3.3 Analysis

We now choose the values of ℓ and L:

$$\ell = \log_{\frac{1}{p_2}} n \tag{18.4}$$

$$L = n^{\rho}. \tag{18.5}$$

Clearly, the space and query time of our structure match the claims in Theorem 18.5. We still need to prove that the query algorithm succeeds with probability at least $1/10$. It suffices to consider that S contains an object e^* with $dist(q, e^*) \leq r$; otherwise, the algorithm is obviously correct (think: why?).

An object $x \in S$ is *good* if $dist(q, x) \leq cr$, or *bad* otherwise. Note that we succeed only if a good object is returned.

Lemma 18.6. *The query is answered correctly if the following two conditions hold:*

- **C1:** *e^* appears in at least one of $b_1, ..., b_L$;*
- **C2:** *there are at most $2L$ bad objects in the multi-set union of $b_1, ..., b_L$.*

Proof. If the multi-set union of $b_1, ..., b_L$ has a size at most $2L$, then **C1** ensures $e^* \in Z$. Otherwise, by **C2**, Z must contain at least a good object. □

Lemma 18.7. C1 *fails with probability at most 1/e.*

Proof.

$$\begin{aligned}
\Pr\left[e^* \notin \bigcup_{i=1}^{L} b_i\right] &= \prod_{i=1}^{L} \Pr[e^* \notin b_i] \\
&= \prod_{i=1}^{L} \Big(1 - \Pr[g_i(e^*) = g_i(q)]\Big) \\
(g_i \text{ is an } (r, cr, p_1^\ell, p_2^\ell)\text{-LSH function}) &\leq \prod_{i=1}^{L} \Big(1 - p_1^\ell\Big) \\
&= \Big(1 - p_1^\ell\Big)^L. \qquad (18.6)
\end{aligned}$$

By (18.4), we know

$$\begin{aligned}
p_1^\ell &= p_1^{\log_{1/p_2} n} \\
&= \Big((1/p_2)^{\log_{1/p_2} p_1}\Big)^{\log_{1/p_2} n} \\
&= n^{\log_{1/p_2} p_1} \\
&= (1/n)^{\log_{1/p_2}(1/p_1)} \\
&= n^{-\rho}.
\end{aligned}$$

Therefore:

$$(18.6) = \Big(1 - n^{-\rho}\Big)^L \leq \exp\left(-n^{-\rho} \cdot L\right) = 1/e$$

where the "$\leq$" used the fact $(1 + z) \leq e^z$ for all $z \geq 0$, and the last equality used (18.5). □

Lemma 18.8. C2 *fails with probability at most 1/2.*

Proof. Let X be the total number of bad objects in the multi-set union of $b_1, ..., b_L$. Fix an arbitrary $i \in [1, L]$. Since g_i is an $(r, cr, p_1^\ell, p_2^\ell)$-LSH function, a bad object has probability at most $p_2^\ell = 1/n$ to fall in the same bucket as q. Hence, in expectation, there is at most 1 bad object in b_i. This means $\mathbf{E}[X] \leq L$. By Markov's inequality (18.1), $Pr[X \geq 2L] \leq 1/2$. □

Therefore, **C1** and **C2** hold simultaneously with probability at least $1 - (1/e + 1/2) > 0.1$. This completes the proof of Theorem 18.5.

18.4 Remarks

The LSH technique was proposed by Indyk and Motwani [24]. Today, effective LSH functions have been found for a large variety of spaces $(U, dist)$, making the technique applicable to many distance functions. The function (18.3) is due to Datar, Immorlica, Indyk, and Mirrokni [15]. The function requires generating only $d+1$ real values: $\alpha_1, ..., \alpha_d$, and γ. This is not a problem in practice, but we must exercise care in theory. Consider, for example, γ, which is a real value in $[0, \beta]$. In the RAM model, we simply cannot generate γ because the only random atomic operation — RAND (Lecture 1) — has only finite precision. The same issue exists for $\alpha_1, ..., \alpha_d$ whose distributions are even more complex. To remedy the issue, we must carefully analyze the amount of precision required to attain a sufficiently accurate version of Lemma 18.4, which is rather difficult (and tedious). We will not delve into that in this course.

Exercises

Problem 1. Prove Lemma 18.1.

Problem 2. Prove the following stronger version of Theorem 18.5: there is a structure using $O(n^{1+\rho} \cdot \log_{1/p_2} n \cdot \log n)$ memory cells and storing $O(n^{1+\rho} \cdot \log n)$ objects that can answer one single (r, c)-NN query correctly with probability at least $1 - 1/n^{100}$. The query time is $O(n^{\rho} \cdot \log_{1/p_2} n \cdot t_{lsh} \cdot \log n)$, plus the cost of calculating $O(n^{\rho} \cdot \log n)$ distances.

(Hint: build $O(\log n)$ independent structures of Theorem 18.5; a query succeeds if it succeeds in any of those structures.)

Problem 3. Prove an even stronger statement: there is a structure using $O(n^{1+\rho} \cdot \log_{1/p_2} n \cdot \log n)$ memory cells and storing $O(n^{1+\rho} \cdot \log n)$ objects that, with probability at least $1 - 1/n^2$, can answer n^{98} (r, c)-NN queries correctly. The query time is $O(n^{\rho} \cdot \log_{1/p_2} n \cdot t_{lsh} \cdot \log n)$, plus the cost of calculating $O(n^{\rho} \cdot \log n)$ distances.

(Hint: if each query fails with probability at most $1/n^{100}$, the probability of answering all n^{98} queries correctly is at least $1 - 1/n^2$.)

Problem 4. Let w be the word length. Let $\mathbb{N}_w$ be the set of integers from 0 to $2^w - 1$. Let P be a set of n points in $\mathbb{N}_w^d$ where $d \geq 1$ should not be regarded as a constant. The value of n satisfies $w = \Theta(\log n)$. Given a point $q \in \mathbb{N}_w^d$, a *query* returns a 4-ANN of q in P. Describe a structure of $\tilde{O}(dn^{1.5})$ space that can answer one query in $\tilde{O}(d\sqrt{n})$ time with probability at least $1 - 1/n^{100}$. The distance metric is the Euclidean distance.

Problem 5 (LSH for the hamming distance). Consider $U = \{0, 1\}^d$ where $d \geq 1$ is an integer. Call each element in U a *string* (i.e., a bit sequence of length d). Given a string e, use $e[i]$ to denote its i-th bit, for $i \in [1, d]$. Given two strings e_1, e_2, $dist(e_1, e_2)$ equals the number of indexes at which e_1 and e_2 differ, or formally $|\{i \in [1, d] \mid e_1[i] \neq e_2[i]\}|$.

Design a function family H where each function maps a string $x \in \{0, 1\}^d$ to $\{0, 1\}$. Specifically, H has exactly d functions $h_1, ..., h_d$ where

$$h_i(x) \quad = \quad x[i].$$

A random function h is drawn uniformly at random from H. For any integers $r \geq 1$ and $c \geq 2$, prove: h is a $(r, cr, \frac{d-r}{d}, \frac{d-cr}{d})$-LSH function.

Lecture 19: Pattern Matching on Strings

In this lecture, we will discuss data structures on *strings*. Denote by Σ an *alphabet* which can be an arbitrarily large set (possibly infinite) where each element is called a *character*. A *string* σ is defined as a finite sequence of characters; denote by $|\sigma|$ the *length* of σ. Specially, define an empty sequence — denoted as $\emptyset$ — as a string of length 0. We will use $\sigma[i]$ $(1 \leq i \leq |\sigma|)$ to represent the i-th character of σ, and $\sigma[i:j]$ $(1 \leq i \leq j \leq |\sigma|)$ to represent the *substring* of σ which concatenates $\sigma[i]$, $\sigma[i+1]$, ..., $\sigma[j]$. We will assume that each character in Σ can be stored in a cell.

Suppose that we are given a (long) string σ^* of length n. Given a string q, we say that a substring $\sigma^*[i:j]$ is an *occurrence* of q if

- $j - i + 1 = |q|$, and
- $q[x] = q[i + x - 1]$ for every $x \in [1, |q|]$.

A *pattern matching query* reports the starting positions of all the occurrences of q, namely, all $i \in [1, |\sigma|]$ such that $\sigma^*[i : i + |q| - 1]$ is an occurrence of q.

Example. Suppose that $\sigma^* =$ `aabcaabcabc`. Given $q =$ `abc`, the query should return 2, 6, and 9, whereas given $q =$ `aabca`, the query should return 1 and 5. □

We want to store Σ in a data structure such that all pattern matching queries can be answered efficiently. We will refer to this as the *pattern matching problem*. Our goal is to prove:

Theorem 19.1. *There is a data structure that consumes $O(n)$ space, and answers any pattern matching query with a non-empty search string q in $O(|q| + occ)$ time, where occ is the number of occurrences of q.*

Both the space usage and the query time are optimal.

19.1 Prefix matching

Consider two strings q and σ with $|q| \leq |\sigma|$. We say that q is a *prefix* of σ if $q = \sigma[1 : |q|]$. For example, `aabc` is a prefix of `aabcaab`. The empty string $\emptyset$ is a prefix of any string.

Our discussion will mainly concentrate on a different problem called *prefix matching*. Let S be a set of n distinct non-empty strings $\sigma_1, \sigma_2, .., \sigma_n$. The subscript $i \in [1, n]$ will be referred to as the *id* of σ_i. We are not responsible for *storing* S; to make this formal, we assume that there is an *oracle* which, given any $i \in [1, n]$ and any $j \in [1, |\sigma_i|]$, tells us the character $\sigma_i[j]$ in constant time. Given a query string q, a *prefix matching query* reports all the ids $i \in [1, n]$ such that q is a prefix of σ_i. We want to design a data structure such that any such query can be answered efficiently.

Example. Suppose that S consists of 11 strings as shown in Figure 19.1. Given $q =$ `abc`, the query should return 3, 6, and 10, whereas given $q =$ `aabca`, the query should return 7 and 11. □

σ_1	`c`
σ_2	`bc`
σ_3	`abc`
σ_4	`cabc`
σ_5	`bcabc`
σ_6	`abcabc`
σ_7	`aabcabc`
σ_8	`caabcabc`
σ_9	`bcaabcabc`
σ_{10}	`abcaabcabc`
σ_{11}	`aabcaabcabc`

Figure 19.1: An input set S of strings for the prefix matching problem

We will prove:

Theorem 19.2. *There is a data structure that consumes $O(n)$ space, and answers any prefix matching query with a non-empty search string q in $O(|q| + k)$ time, where k is the number of ids reported.*

Sections 19.2 and 19.3 together serve as a proof of the above lemma.

19.2 Tries

Let us append a special character $\perp$ to each string in S; e.g., σ_2 in Figure 19.1 now becomes `bc`$\perp$. The distinctness of the (original) strings in S ensures that, with $\perp$ appended, now no string in S is a prefix of another.

In this section, we will introduce a simple structure — which is called the *trie* — that is able to achieve the query time in Theorem 19.2, but consumes more space than desired.

We define a *trie* on S as a tree T satisfying all the properties below:

- Every edge of T is labeled with a character in Σ.
- Concatenating the characters on any root-to-leaf path in Σ gives a string in S.
- There do not exist distinct nodes u, v in T such that, concatenating the characters on the root-to-u path gives the same string as concatenating the characters on the root-to-v path.

The second bullet implies that the number of leaf nodes of T is precisely n (i.e., the number of strings in Σ).

Example. Figure 19.2 shows a trie T on the set S of strings in Figure 19.1. The right most path of T, for example, corresponds to the string `cabc`$\perp$. □

We answer a prefix-matching query q as follows. At the beginning, set $i = 0$ and u to the root of T. Iteratively, assuming $i < |q|$, we carry out the steps below:

1. Check whether u has a child v such that the edge (u, v) is labeled with $q[i]$.
2. If not, terminate the algorithm by returning nothing.
3. Otherwise, set u to v, and increment i by 1.
4. If $i < |q|$, repeat from Step 1.

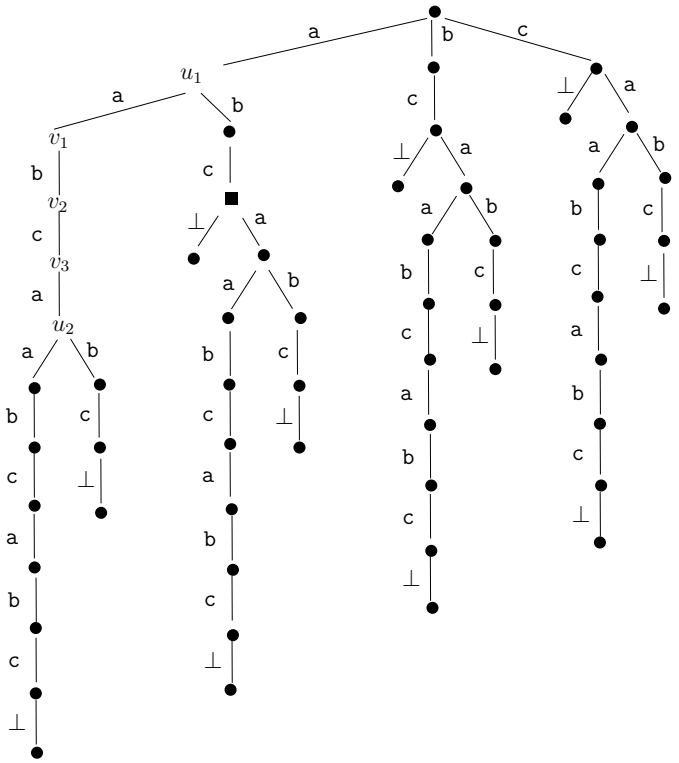


Figure 19.2: A trie

5. Otherwise, report the ids of strings corresponding to the leaves underneath u.

Example. Consider answering a query with $a = \texttt{abc}$ on the trie of Figure 19.2. The query algorithm descends to the node marked as a black square. There are three leaves under that node, corresponding to strings σ_3, σ_6, and σ_{10}, respectively. □

The correctness of the algorithm is obvious. To implement the algorithm in $O(|q| + k)$ time, we need to (i) find v at Step 1 or declare its absence in $O(1)$ time, and (ii) report all the k leaves underneath the final u in $O(k)$ time. Achieving these purposes is easy and left to you as an exercise.

T, however, can have $\Omega(\sum_{i=1}^{n} |\sigma_i|)$ nodes, and thus, may consume more than $O(n)$ space. However, we have not utilized a crucial property stated in the prefix-matching problem: we are *not* responsible for storing S! In the next section, we will show how to leverage the property to reduce the space to $O(n)$ without affecting the query time.

19.3 Patricia Tries

A trie may have many nodes that have only one child (see Figure 19.2). Intuitively, such nodes waste space because they do not help to distinguish the strings in S. Our improved structure — called the *Patricia trie* — saves space by compressing such nodes.

Let the first define the *longest common prefix* (LCP) of S as the longest string that is a prefix of all the strings in S.

Example. For example, if S is the set of strings in Figure 19.1, then the LCP is $\emptyset$. On the other

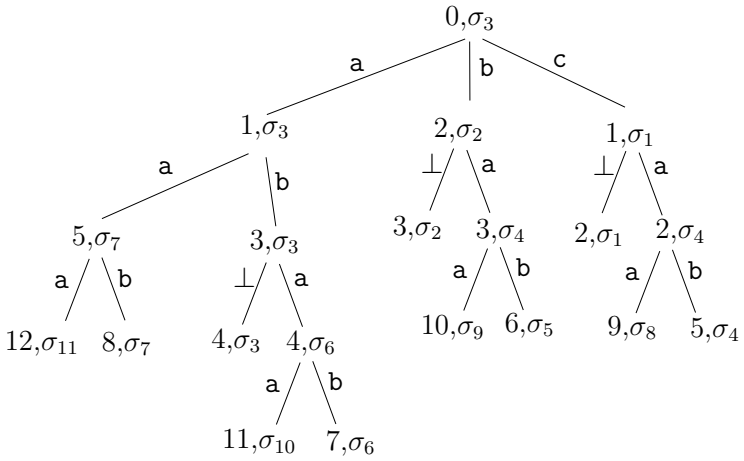


Figure 19.3: A Patricia trie

hand, if S consists of only σ_3, σ_6, and σ_{10}, then the LCP is `abc`. If we add also σ_{11} to S, then the LCP becomes `a`. □

Given a string π, we define $S_\pi = \{\sigma \in S \mid \pi \text{ is a prefix of } \sigma\}$.

Example. Let S be the set of strings in Figure 19.1. $S_{\texttt{a}} = \{\sigma_3, \sigma_6, \sigma_7, \sigma_{10}, \sigma_{11}\}$ and $S_{\texttt{aabca}} = \{\sigma_7, \sigma_{11}\}$. □

Now consider π to be the the LCP of S. Given a character $x \in \Sigma$, we denote by $\pi \circ x$ the string obtained by appending x to π. We call x an *extension character* of S if $|S_{\pi \circ x}| \geq 1$. Note that π being an LCP implies $S_{\pi \circ x}$ is a *proper* subset of S.

Proposition 19.3. *If $|S| \geq 2$, then S has at least two extension characters.*

The proof is easy and left to you.

Example. Let S be the set of strings in Figure 19.1. Its LCP is $\emptyset$. Characters `a`, `b`, and `c` are all extension characters. For $S_{\texttt{aa}} = \{\sigma_7, \sigma_{11}\}$, the LCP is `aabca`. Character `c` is not an extension character of $S_{\texttt{aa}}$ because $S_{\pi \circ \texttt{c}}$ is empty. The extension characters of $S_{\texttt{aa}}$ are `a` and `b`. □

We are ready to define the Patricia trie T on a non-empty S recursively:

- If S has only a single string σ, T is a tree with only one node, labeled as $(|\sigma|, id(\sigma))$ where $id(\sigma)$ is the id of σ.
- Consider now $|S| \geq 2$. Let π be the LCP of S, and X be the set of extension characters of S. T is a tree where
  - the root is labeled as $(|\pi|, id(\sigma))$, where σ is an arbitrary string in S;
  - for every extension character $x \in X$, the root has a subtree which is the Patricia trie on $S_{\pi \circ x}$.

Example. Figure 19.3 shows the Patricia trie on the set S of strings in Figure 19.1. Recall that each string in Figure 19.1 has been appended with the special character $\perp$. □

We leave the proof of the following lemma to you as an exercise:

Lemma 19.4. *The patricia trie on S has n leaves and at most $n-1$ internal nodes.*

As mentioned earlier, the Patricia trie is merely a compressed version of the trie. We illustrate this using an example:

Example. Compare the Patricia trie in Figure 19.3 to the trie in Figure 19.2. It is easy to see that nodes $(1, \sigma_3)$ and $(5, \sigma_7)$ in Figure 19.3 correspond to nodes u_1 and u_2 in Figure 19.2, respectively. As explained next, whenever needed, the entire path $u_1 \xrightarrow{\mathsf{a}} v_1 \xrightarrow{\mathsf{b}} v_2 \xrightarrow{\mathsf{c}} v_3 \xrightarrow{\mathsf{a}} u_2$ in Figure 19.2 can be reconstructed based on the integer 5 and string σ_7.

Denote by S' the set of strings corresponding to the leaves in the left subtree of node u_2 in Figure 19.2 ($S' = \{\sigma_7, \sigma_{11}\}$ but we do not need this in the following discussion). By how the Patricia trie was constructed, from $(5, \sigma_7)$ we know that S' must have an LCP π of length 5. As can be inferred from $(1, \sigma_3)$, for constructing the path $u_1 \xrightarrow{\mathsf{a}} v_1 \xrightarrow{\mathsf{b}} v_2 \xrightarrow{\mathsf{c}} v_3 \xrightarrow{\mathsf{a}} u_2$, it suffices to derive the last $5-1=4$ characters of π, i.e., $\pi[2], \pi[3], \pi[4]$, and $\pi[5]$. This is easy: $\pi[i]$ is simply $\sigma_7[i]$ for each $2 \leq i \leq 5$, and thus, can be obtained from the oracle in constant time. □

The proof for the next lemma is left as an exercise.

Lemma 19.5. *The patricia trie on S can be used to answer any prefix matching query with a non-empty search string q in $O(|q| + k)$ time, where k is the number of ids reported. .*

Theorem 19.2 thus follows from Lemmas 19.4 and 19.5.

19.4 The suffix tree

We now return to the pattern matching problem. Recall that the input is a string σ^* of length n. For each $i \in [1, n]$, define

$$\sigma_i \quad = \quad \sigma^*[i : n].$$

Note that σ_i is a suffix of σ^*. The next fact is immediate:

Proposition 19.6. *For any non-empty string q and any $i \in [1, n]$, $\sigma^*[i : i + |q| - 1]$ is an occurrence of q if and only if q is a prefix of σ_i.*

Create a structure of Theorem 19.2 on $S = \{\sigma_i \mid i \in [1, n]\}$. The structure — called the *suffix tree* on S — achieves the performance guarantees in Theorem 19.1. The proof is left to you as an exercise (think: what is the oracle?).

19.5 Remarks

The suffix tree is due to McCreight [30]. Farach [18] developed a (rather sophisticated) algorithm for constructing the tree in $O(n)$ time.

Exercises

Problem 1. Complete the query algorithm in Section 19.2 to achieve the time complexity of $O(|q| + k)$.

Problem 2. Complete the proof of Lemma 19.4.

(Hint: Proposition 19.3 implies that every internal node has at least two children.)

Problem 3. Complete the proof of Lemma 19.5.

Problem 4. Complete the proof of Theorem 19.1 in Section 19.4.

Problem 5. Let σ^* be a string of length n. Design a data structure of $O(n)$ space such that, given any non-empty string q, we can report the number of occurrences of q in σ^* in $O(|q|)$ time.

Problem 6*. Let S be a set of n strings $\sigma_1, \sigma_2, ..., \sigma_n$. Define $m = \sum_{i=1}^{n} |\sigma_i|$. Given a non-empty string q, an *occurrence* of q is defined by a pair (i, j) such that $\sigma_i[j : j + |q| - 1] = q$. A *general pattern matching query* reports all such pairs. Design a data structure of $O(m)$ space that can answer any query in $O(|q| + occ)$ time, where occ is the number of occurrences of q.

Appendix A: Basic Mathematical Facts

Fact A.1. *For any integers n and k satisfying $1 \leq k \leq n$, it holds that $(\frac{n}{k})^k \leq \binom{n}{k} \leq (\frac{en}{k})^k$.*

Lemma A.2 (Union Bound)**.** *For any probabilistic events $E_1, E_2, ..., E_t$, it holds that*

$$\Pr[E_1 \vee E_2 \vee ... \vee E_t] \quad \leq \quad \sum_{i=1}^{t} \Pr[E_t].$$

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