04-G

栈与队列

栈混洗

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Stack Permutation

❖ 考查栈 $\mathcal{A} = \langle\, a_1,\ a_2,\ a_3,\ \ldots,\ a_n\,]$

$\mathcal{B} = \mathcal{S} = \varnothing$

❖ 只允许

- 将$\boxed{\text{A}}$的顶元素弹出并压入$\boxed{\text{S}}$，或
- 将$\boxed{\text{S}}$的顶元素弹出并压入$\boxed{\text{B}}$

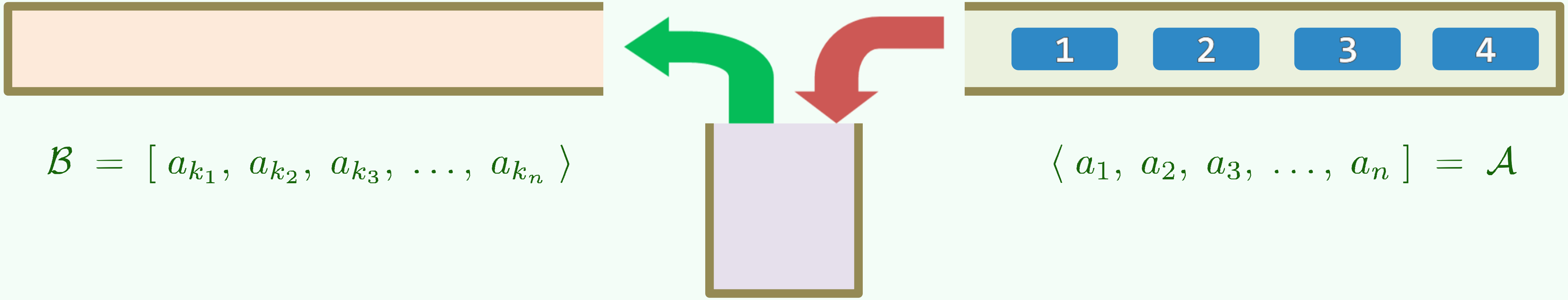
❖ 亦即 $\mathcal{S}.\text{push}(\mathcal{A}.\text{pop}())$

$\mathcal{B}.\text{push}(\mathcal{S}.\text{pop}())$

❖ 若经一系列以上操作后，A中元素全部转入B中

$\mathcal{B} = [\, a_{k_1},\ a_{k_2},\ a_{k_3},\ \ldots,\ a_{k_n}\, \rangle$

则称为A的一个**栈混洗**



$\mathcal{B} = [\, a_{k_1},\ a_{k_2},\ a_{k_3},\ \ldots,\ a_{k_n}\, \rangle$

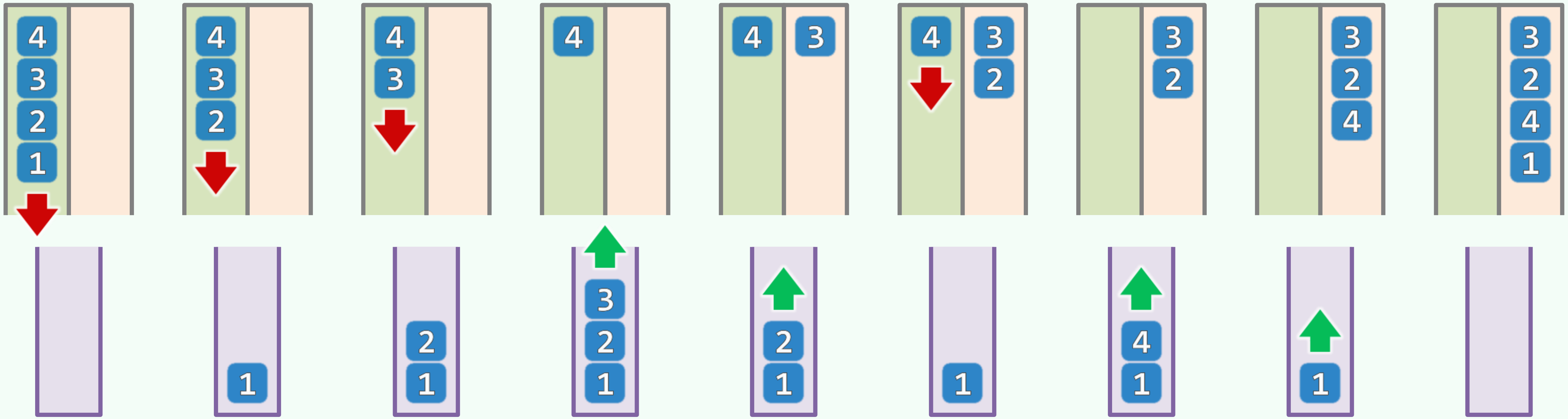
S

$\langle\, a_1,\ a_2,\ a_3,\ \ldots,\ a_n\,] = \mathcal{A}$

计数：SP(n)

❖ 同一输入序列，可有多种栈混洗：[1, 2, 3, 4 >, [4, 3, 2, 1 >, [3, 2, 4, 1 > ...

❖ 一般地，对于长度为n的序列，混洗总数SP(n) = ?



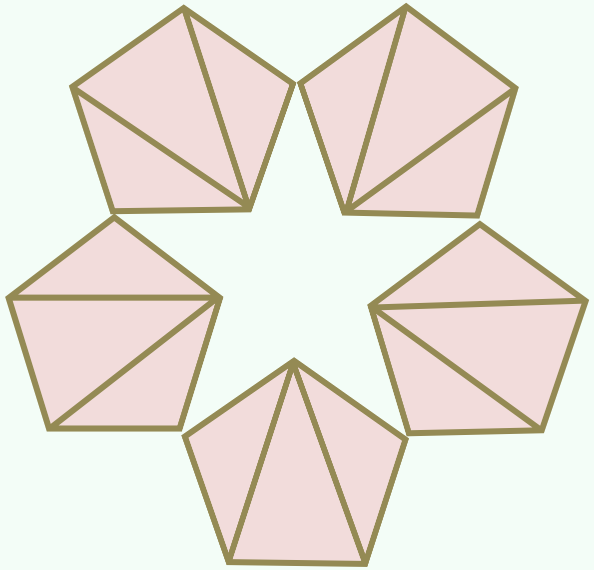
❖ 显然，SP(n) <= n!；更准确地呢？

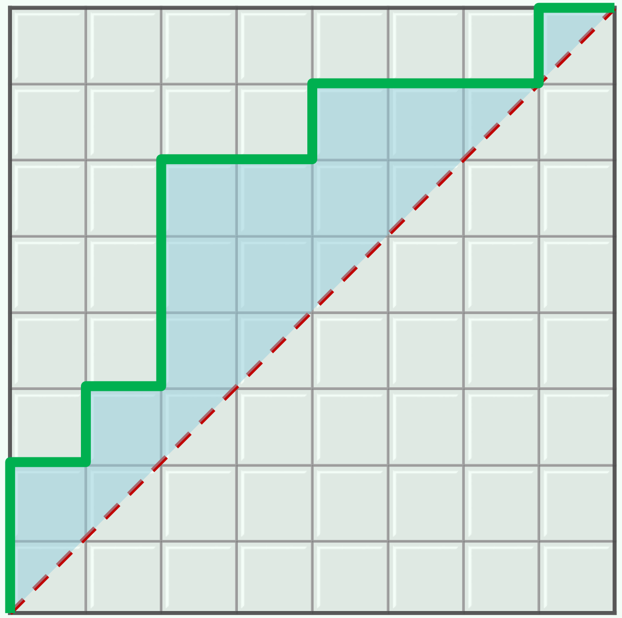
计数：catalan(n)

- $SP(1) = 1$
- 考查S**再度**变空（A**首元素**从S中**弹出**）的时刻，无非n种情况：

$$SP(n) = \sum_{k=1}^{n} SP(k-1) \cdot SP(n-k)$$

$$= catalan(n) = \frac{(2n)!}{(n+1)! \cdot n!}$$



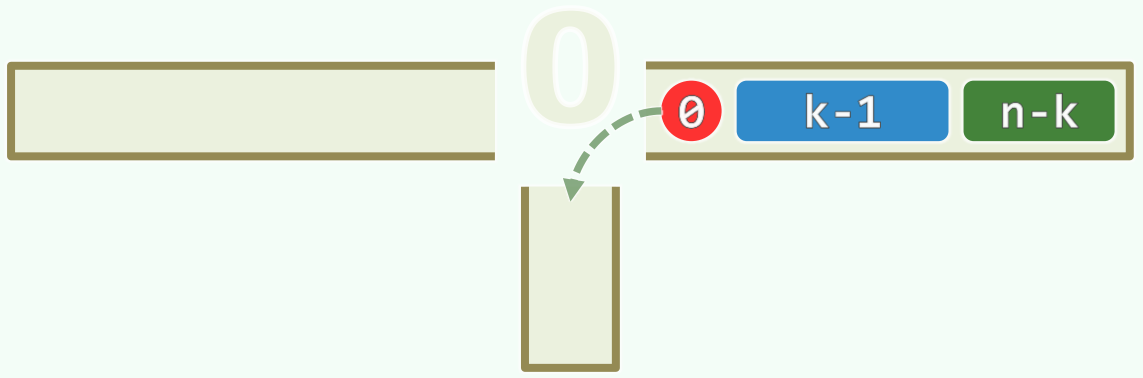


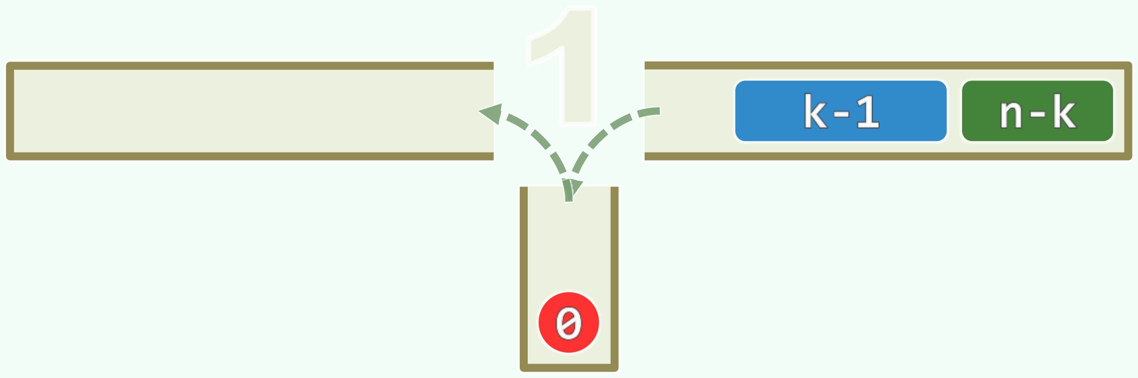
$$SP(2) = 4!/3!/2! = 2$$

$$SP(3) = 6!/4!/3! = 5$$

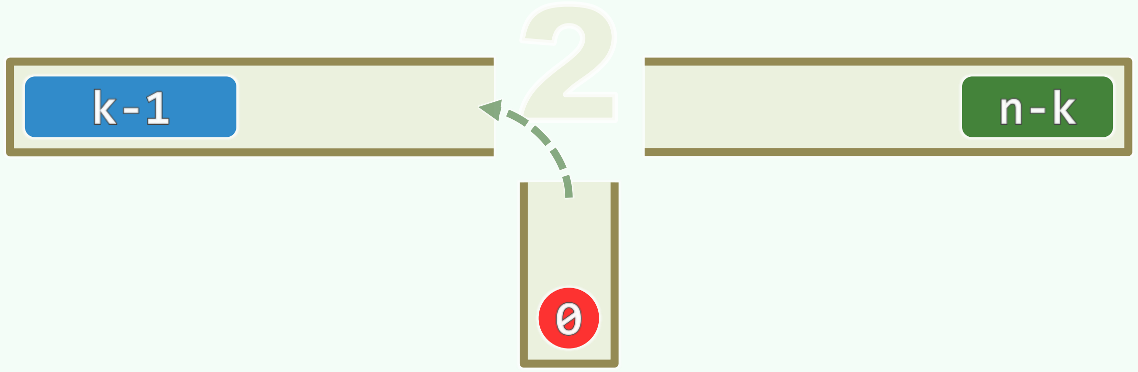
$$\dots \qquad \dots \qquad \dots$$

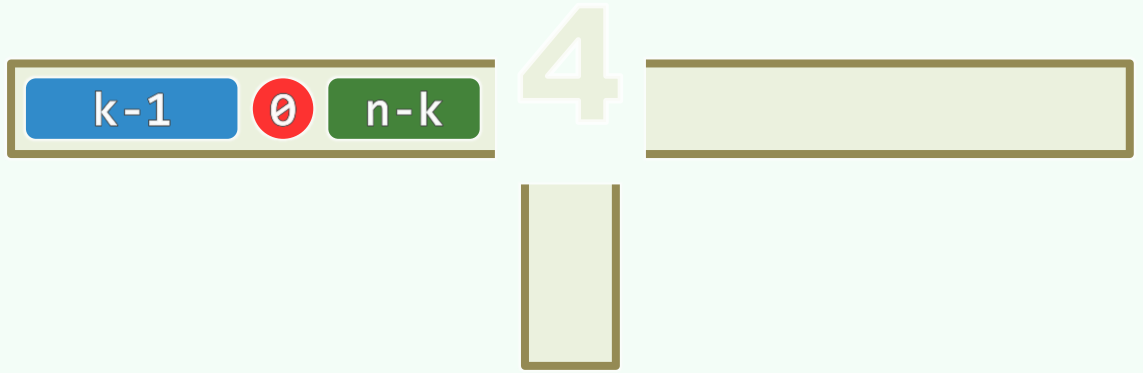
$$SP(6) = 12!/7!/6! = 132$$

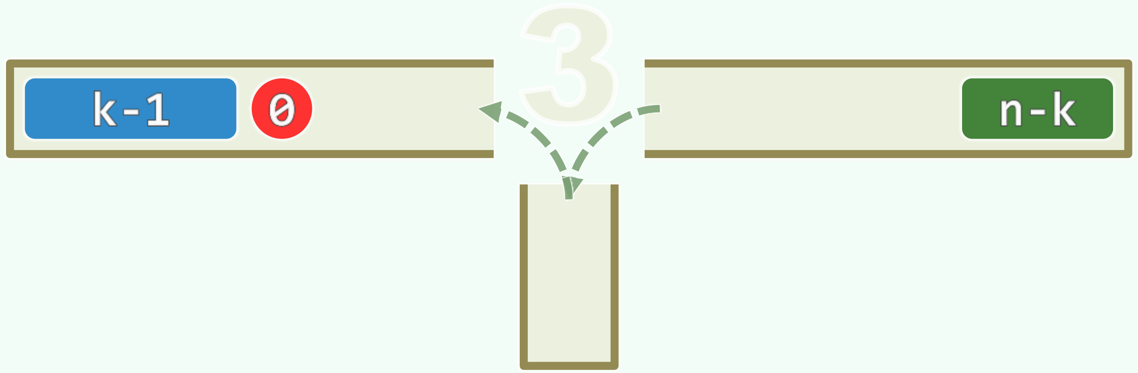




$$SP(n) = \sum_{k=1}^{n} \boxed{SP(k-1)} \cdot \boxed{SP(n-k)}$$







甄别：检测禁形

❖ 输入序列< 1, 2, 3, ..., n]的任一排列[p_1, p_2, p_3, ..., p_n >是否为栈混洗?

❖ 先考查简单情况：n = 3, A = < 1, 2, 3]

- 栈混洗共 6! / 4! / 3! = 5 种；全排列共 3! = 6 种 //少了一种...

❖ $\boxed{[\ 3,\ 1,\ 2\ >}$ //为什么是它?

❖ 观察：任意三个元素能否按某相对次序出现于混洗中，与其它元素无关 //故可推而广之...

❖ 禁形：对任何1 ≤ i < j < k ≤ n, [..., $\boxed{k}$, ..., $\boxed{i}$, ..., $\boxed{j}$, ... > **必非**栈混洗

❖ 反过来，不存在“$\boxed{312}$”模式的序列，一定是栈混洗吗?

甄别：直接模拟

❖ 充要性： A permutation is a stack permutation **iff**

(Knuth, 1968) it does NOT involve the permutation **312** //习题[4-3]

❖ 如此，可得一个$\mathcal{O}(n^3)$的甄别算法 //进一步地...

❖ [p_1, p_2, p_3, ..., p_n >是< 1, 2, 3, ..., n]的栈混洗，**当且仅当**

对于任意i < j，不含模式[..., **j+1**, ..., **i**, ..., **j**, ... >

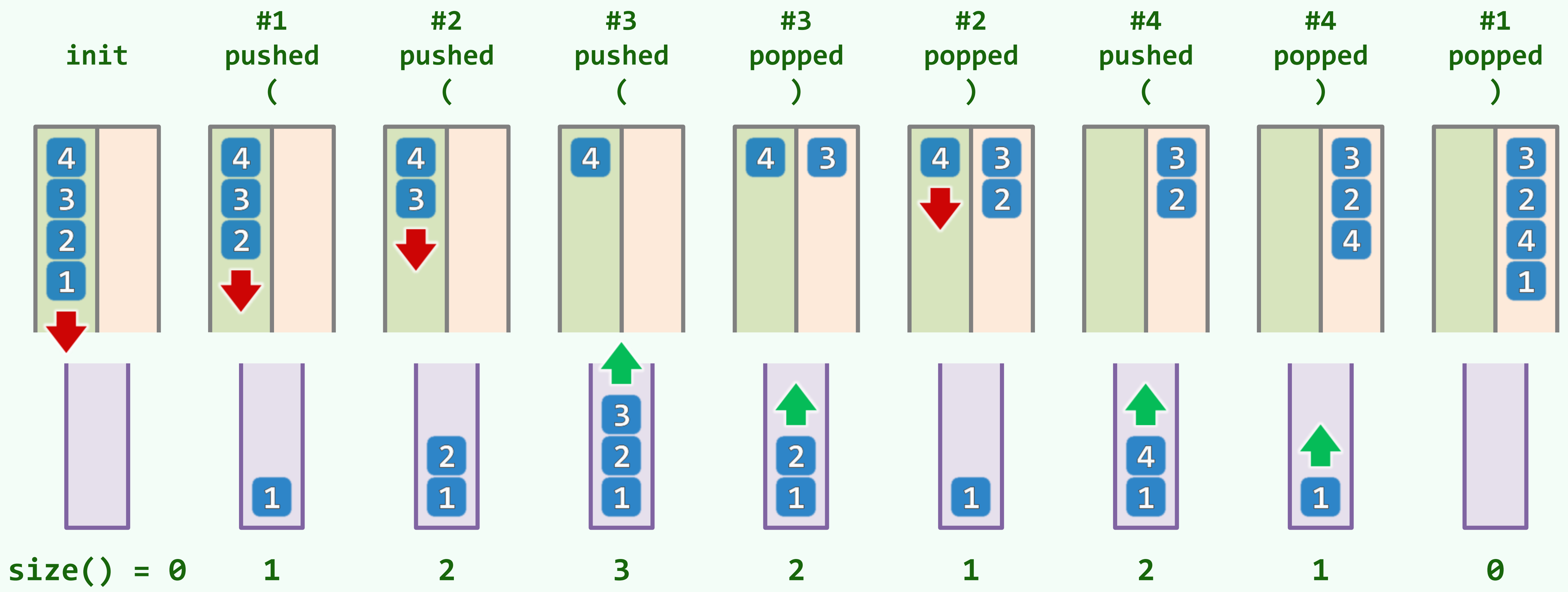
❖ 如此，可得一个$\mathcal{O}(n^2)$的甄别算法 //再进一步地...

❖ $\mathcal{O}(n)$算法： 逐个检视各目标元素： //借助栈A、B和S，直接模拟混洗过程

若在（或能够腾挪至）S顶部，则...；否则（在S中，却非栈顶）...

括号匹配

❖ 观察：每一栈混洗，都对应于栈S的n次push与n次pop操作构成的某一序列；反之亦然



❖ n个元素的栈混洗，等价于n对括号的匹配；二者的组合数，也自然相等