

Module 2A : Probability

Frequentist and Bayesian building blocks

Agenda:

- **Frequentist probability**
 - Venn Diagram & definition of Event
 - Chevalier 'paradox'
 - The Birthday Problem
 - Addition Rule
 - Mutually exclusive, not mutually exclusive
 - Multiplication Rule
 - Independent, not independent
 - Conditional Probability
 - Important differentiation: $P(A \cup B)$, $P(A \cap B)$, $P(A|B)$

Probability:

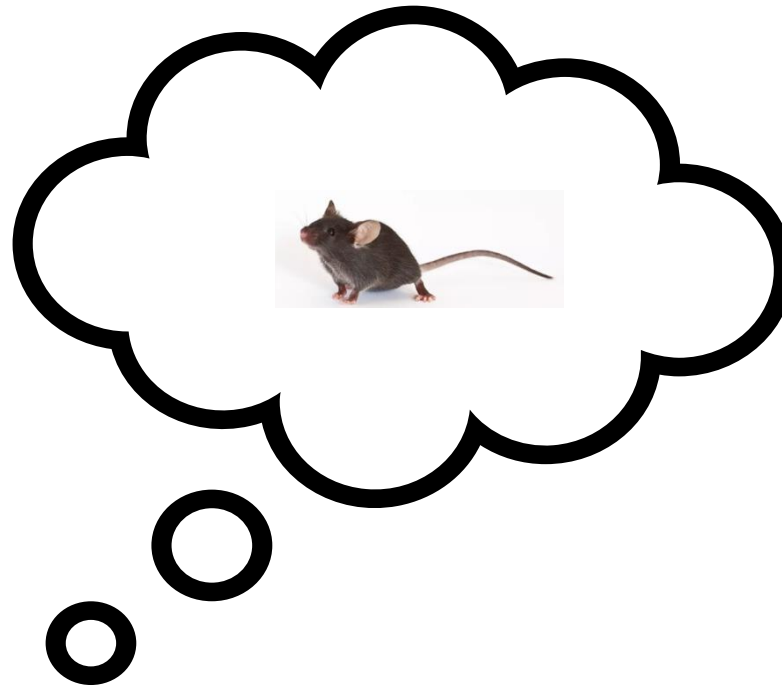
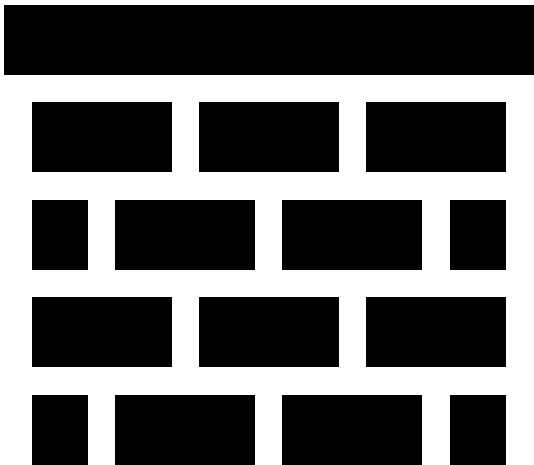
Two major splits in how probability is defined:

Frequency Interpretation:

Frequency of a particular outcome (an event) across many random trials

Subjective (Bayesian) Interpretation:

Subjective belief or opinion of the chance that a particular outcome (an event) will be realized



Random Experiment:

The process of observing the outcome of a chance event

ex - one roll of a die

Sample State Space:

A list of all possible elementary outcomes of an experiment

ex - {1, 2, 3, 4, 5, 6}

Event:

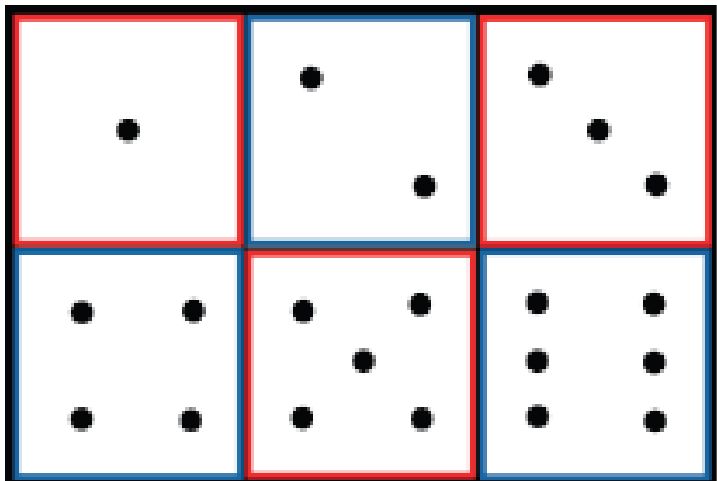
A set made up of elements from the sample space

ex - {1, 2, 3, 4, 5, 6}

ex - “an even number”

Sample Space examples:

One Die Toss



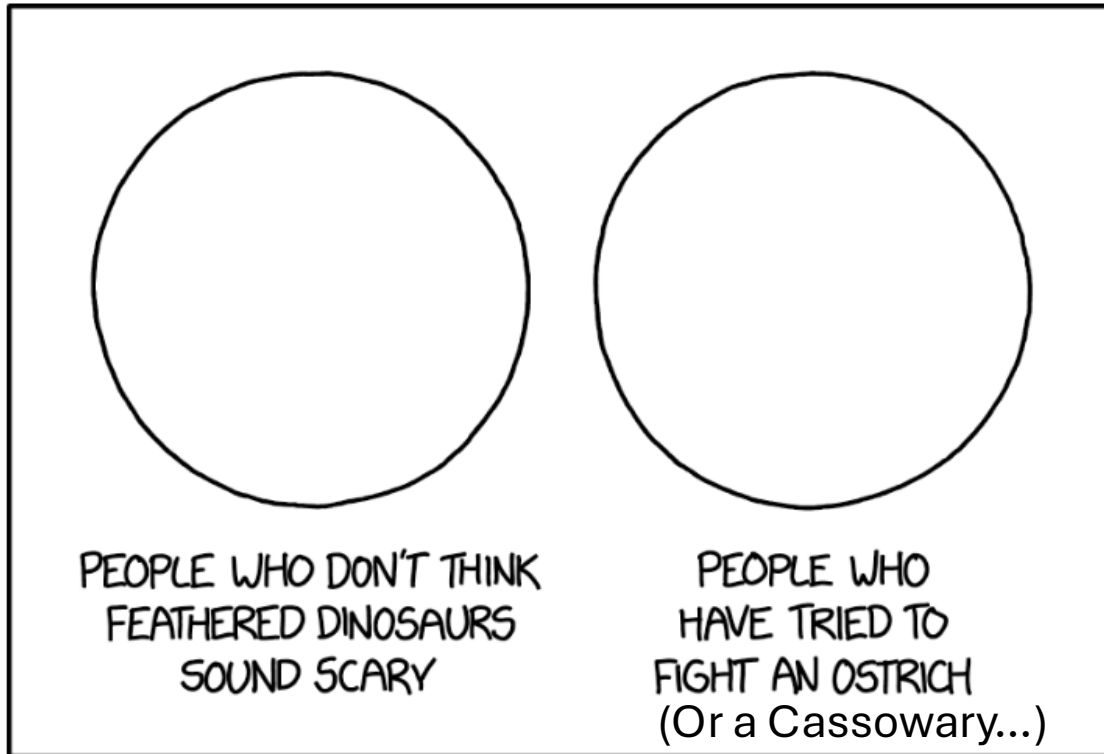
{1,2,3,4,5,6}

One Coin Toss

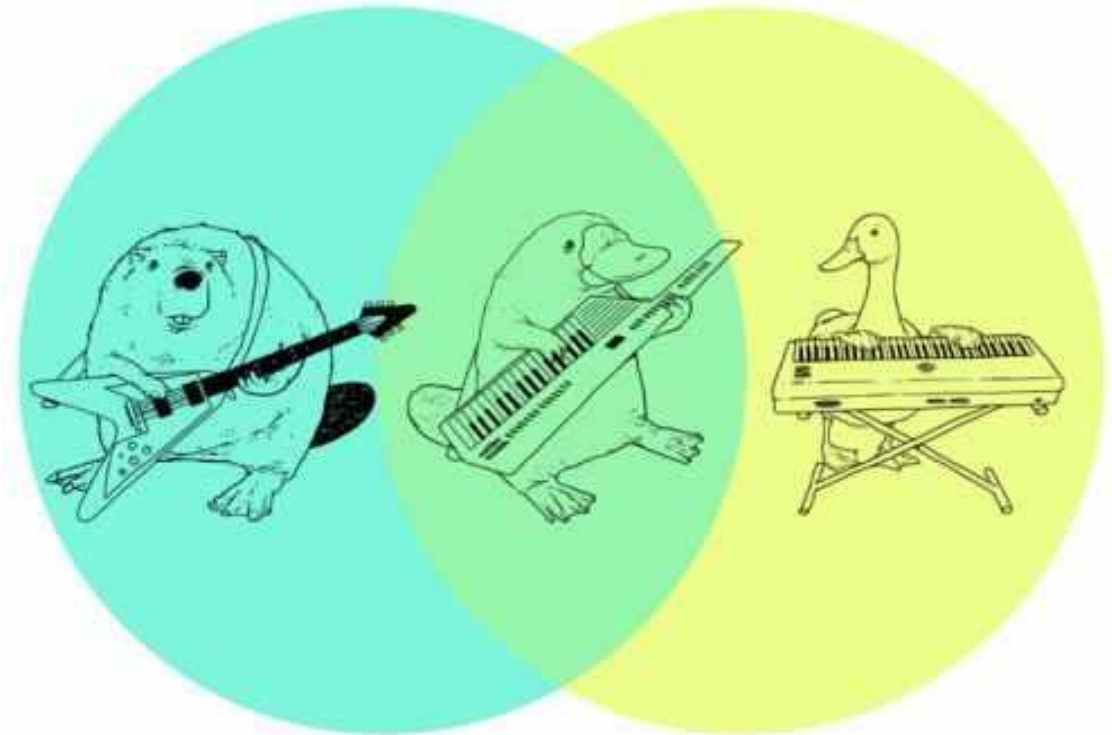


{heads, tails}

FEATHERED DINOSAUR VENN DIAGRAM

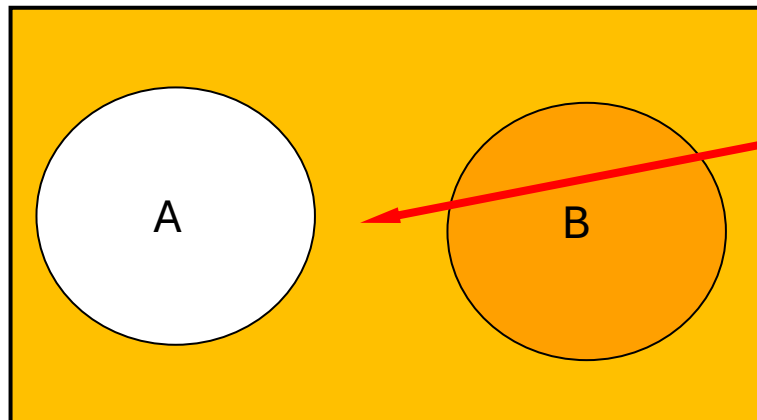


The BEST Venn diagram



NOT A

The event does
not occur (complement);
everything except A

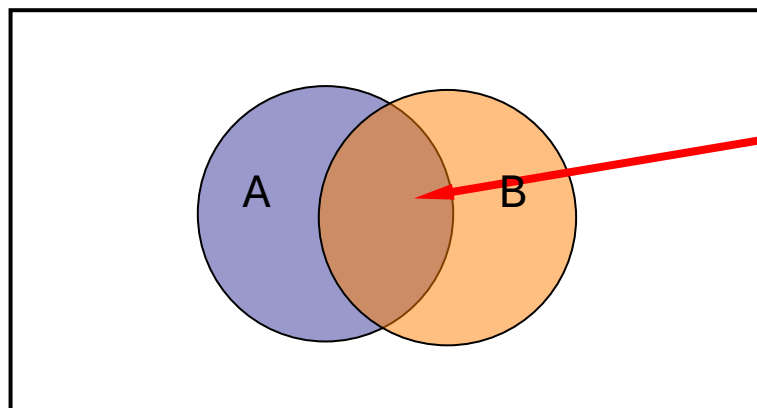


Mutually exclusive

$$P(A \cap B) = 0$$

AND

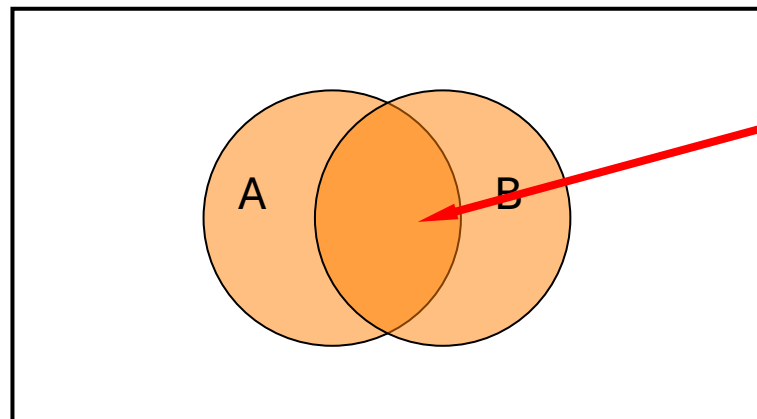
Both events occur
(intersection)



$$P(A \cap B) \neq 0$$

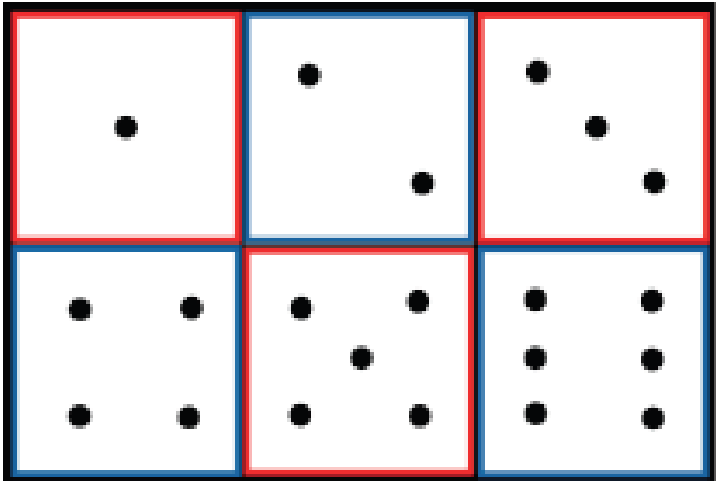
OR

Either events or both
events occur (union)

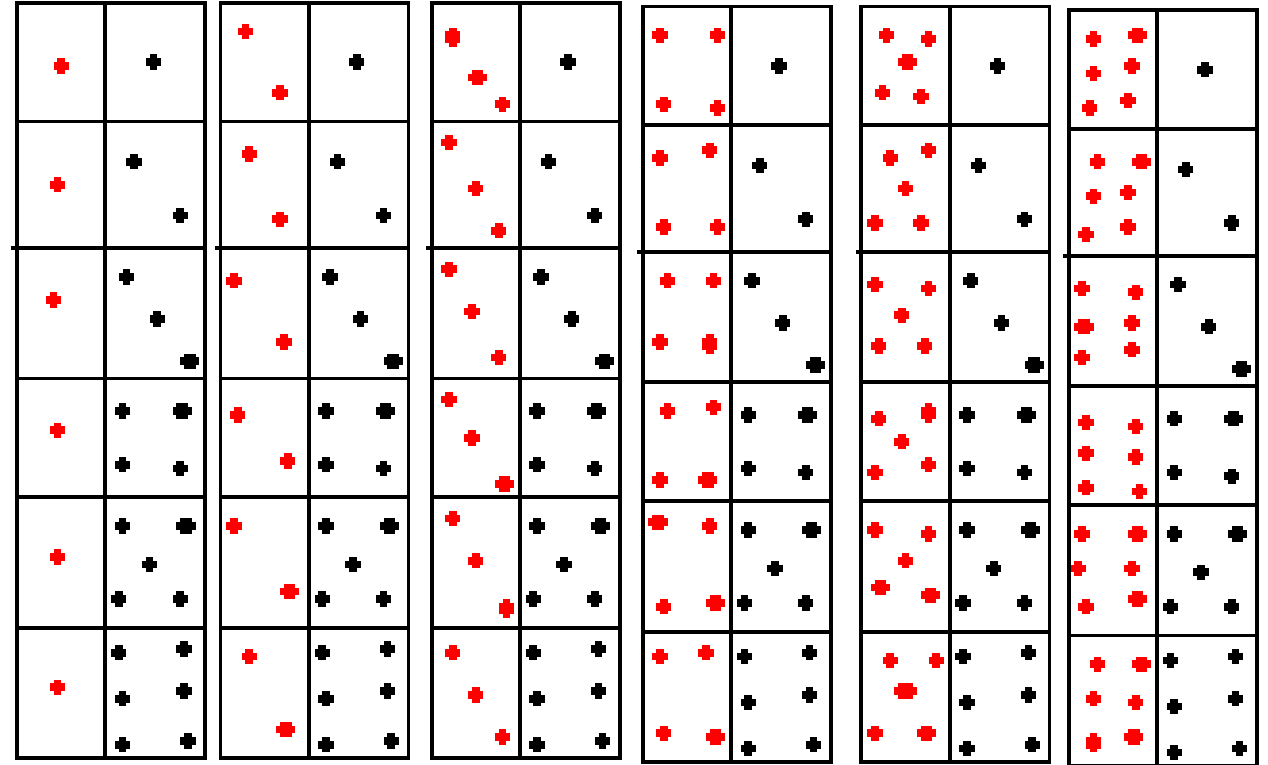


$$P(A \cup B) \neq 0$$

Mutually Exclusive outcomes:



$$P(\text{rolling } 1) = P(\text{rolling } 2) = \dots = P(\text{rolling } 6) = 1/6$$

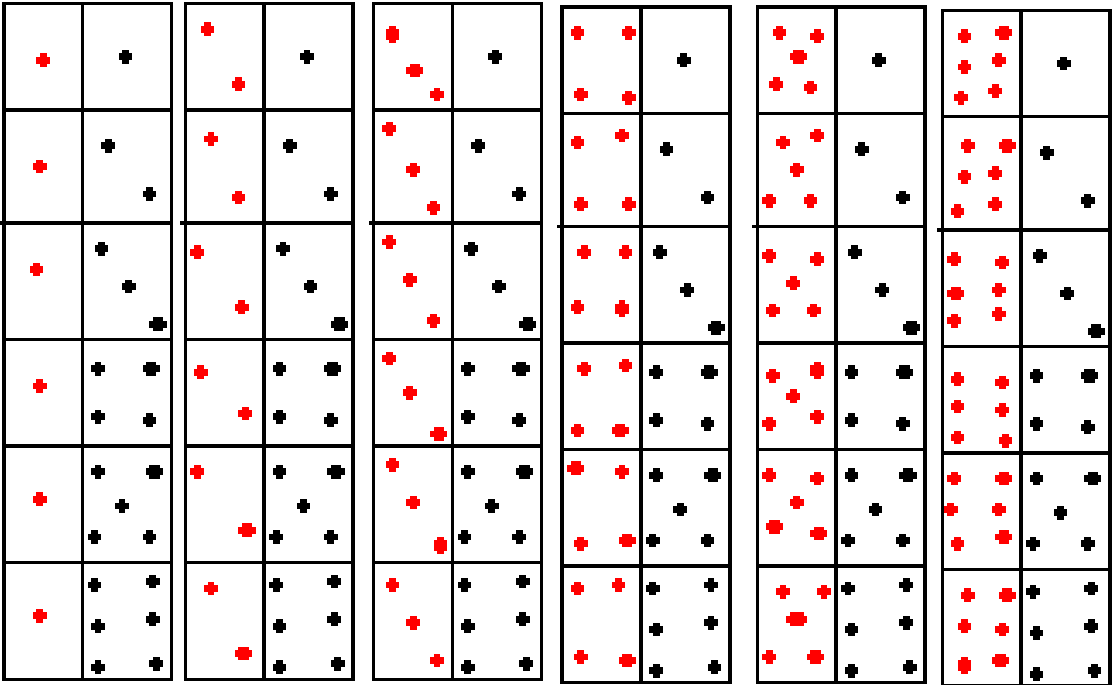


$$P(\text{rolling } 1 \cap \text{rolling } 1) = P(\text{rolling } 1 \cap \text{rolling } 2) = \dots = P(\text{rolling } 1 \cap \text{rolling } 6) = 1/36$$

Q1: Two dice are rolled. The event of interest is the two dice faces add up to 3. What is the probability of this event?

The probability of this event is the sum of the probabilities of the elementary outcomes in this subset.

The full set of elementary outcomes of rolling two dice



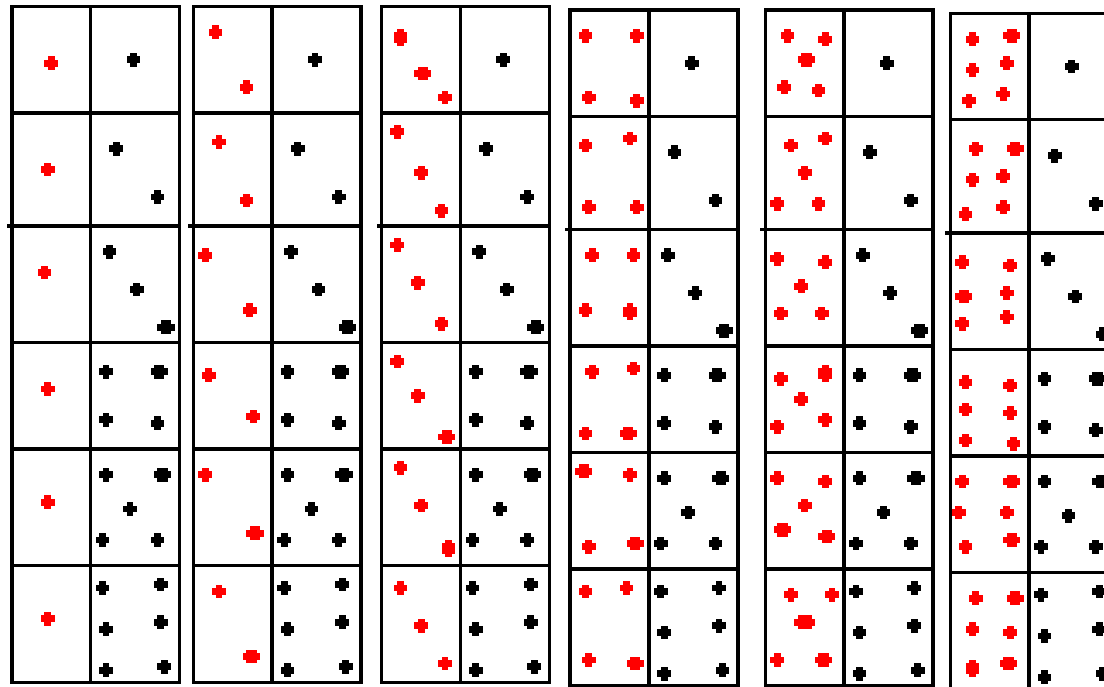
P(showing 3 dots)

Q2: Two dice are rolled, and the red die shows 1. What is the probability of rolling a total of 3?

Q1: Two dice are rolled. The event of interest is the two dice faces add up to 3. What is the probability of this event?

The probability of this event is the sum of the probabilities of the elementary outcomes in this subset.

The full set of elementary outcomes of rolling two dice



P(showing 3 dots)

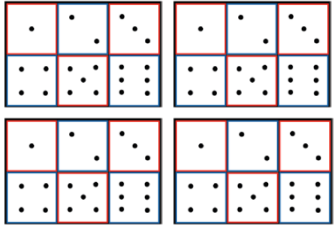
= 2 ways

= $1/36 + 1/36 = 2/36$

Q2: Two dice are rolled, and the red die shows 1, what is the probability of rolling a total of 3?

$P(\text{total } 3 | \text{Die } 1 = 1) = 1/6$

The "paradox" of the Chevalier de Méré



He believed the following:

P(obtaining at least one “6” on 4 single rolls) = $1/6+1/6+1/6+1/6 = 4/6$



P(obtaining at least one “6,6” on 24 double rolls) = $1/36+1/36+1/36+1/36 +1/36+1/36+...+1/36 = 24/36$

So, he thought that the probabilities were the same ($4/6 = 24/36$), **but he was wrong**. Why?

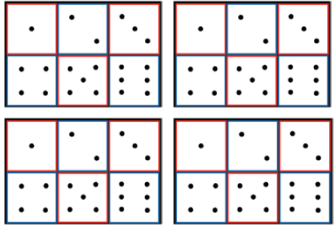
The "paradox" of the Chevalier de Méré

He believed the following:

$P(\text{obtaining at least one "6" on 4 single rolls}) = 1/6 + 1/6 + 1/6 + 1/6 = 4/6$

In actuality:

$P(\text{obtaining } \geq 1 \text{ "6" on 4 rolls}) =$



$P(\text{obtaining at least one "6,6" on 24 double rolls}) = 1/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/36 + \dots + 1/36 = 24/36$

So, he thought that the probabilities were the same ($4/6 = 24/36$), **but he was wrong**. Why?

P(obtaining at least one “6” on 4 single rolls) = $1/6 + 1/6 + 1/6 + 1/6 = 4/6$

$P(\text{obtaining } \geq 1 \text{ "6" on 4 rolls}) = 1 - P(\text{obtaining 0 "6" on 4 rolls}) = 1 - (5/6 * 5/6 * 5/6 * 5/6) = 0.518$
and

Roll 1 | Roll 2 | Roll 3 | Roll 4

Success (1/6) → Success (1/6) → Success (1/6) → Success (1/6) = 1/6⁴

Success (1/6) → Success (1/6) → Success (1/6) → Failure (5/6) = 1/6³ · 5/6

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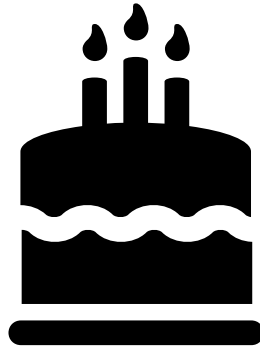
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Failure (5/6) → Failure (5/6) → Failure (5/6) → Failure (5/6) = 5/6⁴ · 1/6² · 5/6²

The Birthday Problem

How many people do you need in a group to have a 50% chance that at least two people will share a birthday?



Resources:

<https://statisticsbyjim.com/fun/birthday-problem/>

<https://betterexplained.com/articles/understanding-the-birthday-paradox/>

https://en.wikipedia.org/wiki/Birthday_problem

The Birthday Problem

How many people do you need in a group to have a 50% chance that at least two people will share a birthday?

When surveyed, most people guess 183. But they are wrong.

The most straightforward intuition is to use NOT logic and independence:

$$\begin{aligned} &P(\text{No one shares a birthday}) \\ &= 1 - P(\text{any two people share a birthday}) \\ &= 1 - P(1 \text{ has a birthday})P(1,2 \text{ distinct birthdays})P(1,2,3 \text{ distinct birthdays})\dots \\ &= 1 - (\mathbf{365/365})(\mathbf{364/365})(\mathbf{363/365})\dots(\mathbf{365-n+1/365}) \end{aligned}$$

The Birthday Problem

How many people do you need in a group to have a 50% chance that at least two people will share a birthday?

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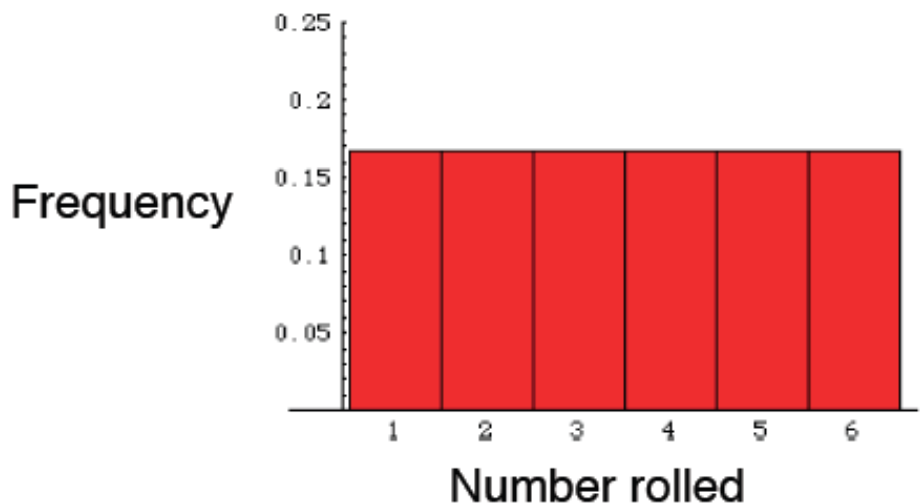
$= 1 - P(\text{any two people share a birthday})$

$= 1 - P(1 \text{ has a birthday})P(1,2 \text{ distinct birthdays})P(1,2,3 \text{ distinct birthdays})\dots$

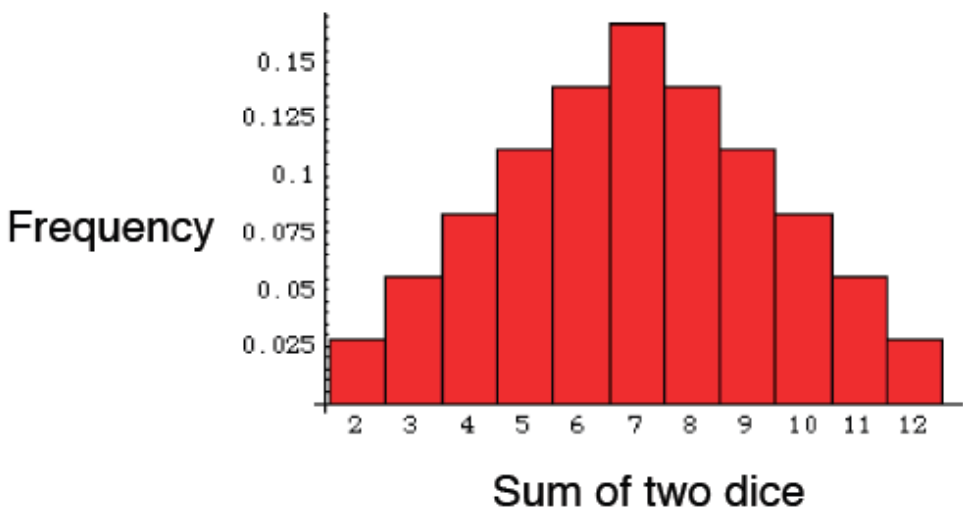
$= 1 - (365/365)(364/365)(363/365)\dots(365-n+1/365)$

- Now you need excel or a simulator to find n (which is **23**)
- You need 57 people to have a 99% chance of any 2 people birthday sharing

Probability distribution for the outcome of a roll of one die:

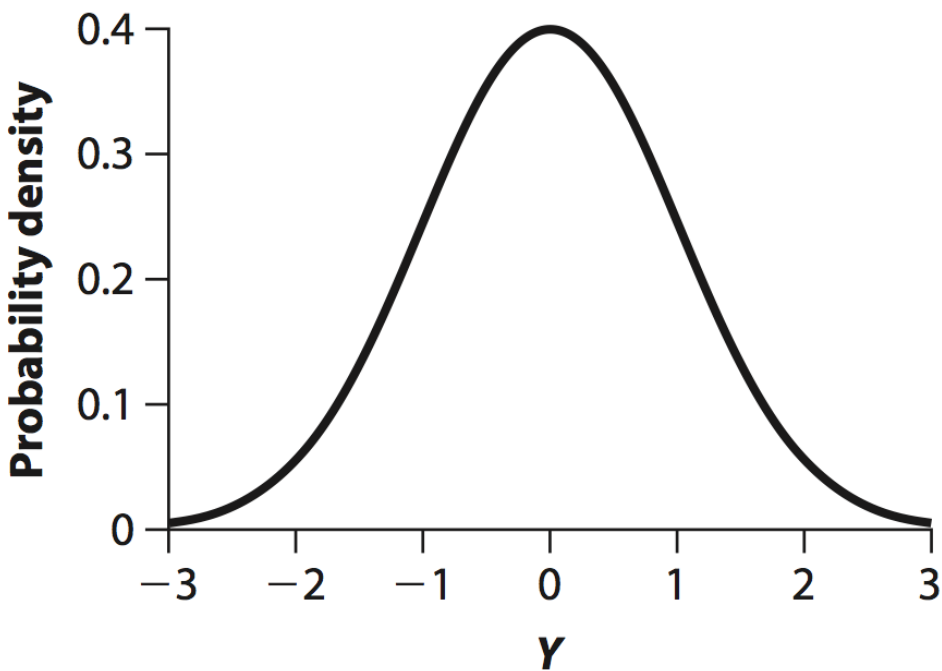


Probability distribution for the sum of a roll of two dice:



Probabilities → Frequency Distributions → Probability Curves

- Probabilities are the foundations for our models of how the world works
- We need distributions to conduct statistical inference



What if a coin is flipped five times and comes up heads each time. Is a tail "due" and therefore more likely than not to occur on the next flip?

<http://onlinestatbook.com/2/probability/gambler.html>



<https://dev.to/josethz00/math-for-devs-gamblers-fallacy-and-monte-carlo-simulation-453a>