SURROGATED ASSISTED BAYESIAN NEURAL NETWORK FOR GEOLOGICAL MODELS

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Thanks go to Robert Taggart for allowing his thesis style to be shamelessly copied.

Sean Luo, Day Month Year.

Abstract

This thesis is an investigation of the ?????????

Contents

Chapter 1

Introduction

Given a sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space $\mathfrak{X}$, the following natural questions may be asked.

(Q1) Which elements of the space $\mathfrak{X}$ can be expressed in the form $\sum_{n=1}^{\infty} a_n x_n$ for some unique sequence $\{a_n\}_{n=1}^{\infty}$ of scalars?

(Q2) Supposing $x \in \mathfrak{X}$ can be written as $\sum_{n=1}^{\infty} a_n x_n$, does it matter in what order the terms of the series are summed?

(Q3) In the setting of (Q2), will the series $\sum_{n=1}^{\infty} \varepsilon_n a_n x_n$ converge for any choice of $\varepsilon_n = \pm 1$?

(Q4) Given a scalar sequence $\{\phi_n\}_{n=1}^{\infty}$, does it give rise to a bounded linear mapping $\sum_{n=1}^{\infty} a_n x_n \mapsto \sum_{n=1}^{\infty} \phi_n a_n x_n$?

Answering these and similar questions for function spaces whose members act on the circle group $\mathbb{T} \cong [0, 2\pi]$ is the main subject of this thesis.

In Chapter 1, we give a general picture of the situation, introducing the concepts of Schauder bases and conditional convergence of sequences in Banach spaces. Standard examples show that not every Banach space with a basis $\{x_n\}_{n=1}^{\infty}$ has the property that the terms in each expansion $\sum_{n=1}^{\infty} a_n x_n$ can be freely rearranged.

The above questions have been the subject of much study for the trigonometric sequence $\{x_n\}_{n=-\infty}^{\infty}$ contained in $\mathfrak{X} = L^p(\mathbb{T})$ and given by $x_n(t) = e^{int}$. If $1 < p < \infty$, then it is known that $\{x_n\}_{n=-\infty}^{\infty}$ is a basis for $\mathfrak{X}$. Moreover, each $f \in \mathfrak{X}$ has the expansion

$$f = \sum_{n=-\infty}^{\infty} a_n x_n \tag{1.0.1}$$

where the series converges in $\mathfrak{X}$ and for each $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-ins} f(s)\, ds. \tag{1.0.2}$$

In Chapter 2 we give an account of this theory in as much as it applies to answering questions similar to those above. In particular, we recall three classical results from harmonic analysis — the M. Riesz Conjugacy Theorem, the Littlewood–Paley Theorem and the Strong Marcinkiewicz Multiplier Theorem. These demonstrate that the expansion (1.0.1) is valid for each $f \in L^p(\mathbb{T})$ (for $1 < p < \infty$) and give sequences $\{\phi_n\}_{n=1}^{\infty}$ for which (Q4) has a positive answer.

In Chapter 3 we present some results of Bourgain which generalise these classical theorems to the $L^p(\mathbb{T}, \mathfrak{Y})$ spaces — those L^p spaces whose functions take values in a Banach space $\mathfrak{Y}$. When $\mathfrak{Y}$ is a Banach space that has the so-called UMD property and when $1 < p < \infty$, analogues of the M. Riesz, Littlewood–Paley and

Marcinkiewicz Theorems hold. In particular, equation (1.0.1) is valid if f belongs to any of these function spaces.

After examining these vector-valued function spaces, we look at functions on $\mathbb{T}$ which are operator-valued. A certain class of functions, whose members are strongly continuous group homorphisms of $\mathbb{T}$ into $\mathfrak{B}(\mathfrak{Y})$, yields analogues of the M. Riesz, Littlewood–Paley and Marcinkiewicz Theorems if $\mathfrak{Y}$ has the UMD property. These analogues were first announced in the 1980s and 1990s by Berkson and Gillespie (see [2] and [3]). They are stated in Chapter 4, where it is also shown how they specialise to the classical theorems. In particular, given a strongly continuous representation $R : \mathbb{T} \to \mathfrak{B}(\mathfrak{Y})$, its 'Fourier series'

$$R = \sum_{n=-\infty}^{\infty} P_n x_n,$$

converges. The nth 'Fourier coefficient' P_n of R is a projection on $\mathfrak{Y}$ given by the formula

$$P_n = \frac{1}{2\pi} \int_0^{2\pi} e^{ins} R(s)\, ds.$$

The similarity between the above two equations and equations (1.0.1) and (1.0.2) is striking.

Proving the theorems stated in Chapter 4 is the main objective of Chapters 5 and 6. Contained in the proofs are a wide range of techniques taken from harmonic analysis and Banach space operator theory. The parts of the proofs of the two main theorems are dispersed across the literature with significant portions being found in [1], [2], [3], [4] and [7]. These chapters provide what is most likely the only unified account of these proofs. Chapter 6 ends with an example that demonstrates the scope of these theorems. Applying the Strong Marcinkiewicz analogue for strongly continuous representations, a wide class of multiplier projections acting on the von Neumann Schatten $\mathcal{C}_p$ spaces are given. This class was recently discovered by Doust and Gillespie (see [16]).

This thesis is a coherent presentation of a quest to generalise three classical theorems that were discovered in the 1920s, 1930s and 1940s. Their analogues are the product of a conglomeration of ideas that straddle the 1980s and 1990s and the application of these new results brings the story into the twenty-first century.

Chapter 2

Convergence and Bases on Banach Spaces

In this chapter we outline the general context in which our study of multiplier theory in Banach spaces takes place. We begin by revising the elementary concepts associated with functional analysis in Banach spaces. Then Section 2.3 introduces the notions of bases, multiplier transforms and unconditional convergence of sequences, setting the scene for the rest of the thesis. The chapter ends by introducing a technique which is frequently used to obtain results pertaining to the unconditionality of sequences.

2.1 Banach, Operator and Dual Spaces

We recall some basic definitions and concepts from Banach space theory and functional analysis. A good introductory reference for this material is [13].

A *Banach space* $\mathfrak{X}$ is a vector space V over a field $\mathbb{F}$ that is equipped with a norm $\|\cdot\|_{\mathfrak{X}}$ that makes V complete with respect to the metric d given by $d(x, x') = \|x - x'\|_{\mathfrak{X}}$. In this thesis we shall always assume that the underlying field $\mathbb{F}$ of scalars is $\mathbb{C}$. Suppose $\mathfrak{Y}$ is also a Banach space with norm $\|\cdot\|_{\mathfrak{Y}}$. A linear operator $T : \mathfrak{X} \to \mathfrak{Y}$ is said to be *bounded* if $\sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} = 1\} < \infty$. The collection of $\mathfrak{Y}$-valued bounded linear operators on $\mathfrak{X}$ is denoted $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$, and is itself a Banach space when equipped with the *operator norm* $\|T\| = \sup\{\|Tx\|_{\mathfrak{Y}} : x \in \mathfrak{X}, \|x\|_{\mathfrak{X}} \leq 1\}$. A $\mathfrak{Y}$-valued linear operator on $\mathfrak{X}$ is bounded if and only if it is continuous with respect to the topologies on $\mathfrak{X}$ and $\mathfrak{Y}$ induced by their respective norms. Furthermore, if $T \in \mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ then $\|Tx\|_{\mathfrak{Y}} \leq \|T\| \, \|x\|_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$.

Henceforth, the norm of a Banach space $\mathfrak{X}$ will be denoted $\|\cdot\|_{\mathfrak{X}}$ (except when $\mathfrak{X}$ is an L^p space — see the example below), while the operator norm $\|\cdot\|$ will contain no subscript because it is clear from the operator's definition what spaces its norm is induced from. The space $\mathfrak{B}(\mathfrak{X}, \mathfrak{X})$ of bounded linear operators sending elements of $\mathfrak{X}$ into $\mathfrak{X}$ will be denoted $\mathfrak{B}(\mathfrak{X})$. An important class of operators included in $\mathfrak{B}(\mathfrak{X})$ are the *projections* on $\mathfrak{X}$, those bounded linear maps P satisfying $P^2 = P$.

There are many examples of Banach spaces. We give one example that will feature regularly throughout the thesis.

Example 2.1.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. Let $\mathfrak{X} = L^p(\Omega, \mathcal{A}, \mu)$ be the space of all equivalence classes of $\mathbb{C}$-valued $\mathcal{A}$-measurable functions f on Ω with the property that $\int_\Omega |f(x)|^p \, d\mu(x) < \infty$. We shall usually blur the distinction between the equivalence classes of $L^p(\Omega, \mathcal{A}, \mu)$ and the representatives of these classes. The norm on $\mathfrak{X}$ is given by $\|f\|_{\mathfrak{X}} = (\int_\Omega |f(x)|^p \, d\mu(x))^{1/p}$ and makes $\mathfrak{X}$ a Banach space. When the underlying σ-algebra and measure is standard (for example, when $\Omega = \mathbb{R}$, $\mathcal{A}$ is the Borel σ-algebra and μ is Lebesgue measure on $\mathbb{R}$), we shall denote $L^p(\Omega, \mathcal{A}, \mu)$ by $L^p(\Omega)$. Because it is notationally less cumbersome* *to write $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(\Omega,\mathcal{A},\mu)}$, we shall do so whenever the context permits no ambiguity. A particularly useful fact is that if $(\Omega,\mathcal{A},\mu)$ is a finite measure space and $1 \leq p < r < q \leq \infty$, then $L^p(\Omega) \cap L^q(\Omega) \subset L^r(\Omega)$ and $L^q(\Omega) \subset L^p(\Omega)$. A quick source of information on these spaces is [23, §6.4].*

Example 2.1.2. *We give a few of the sequence spaces.*

1. *For $1 \leq p < \infty$, define ℓ^p to be the space of all functions $f : \mathbb{N} \to \mathbb{C}$ such that $\sum_{n=1}^{\infty} |f(n)|^p < \infty$, and equip it with the norm $\|f\|_{\ell^p} = (\sum_{n=1}^{\infty} |f(n)|^p)^{1/p}$. Thus ℓ^p may be identified with $L^p(\mathbb{N})$, where the underlying measure is counting measure on $\mathbb{N}$.*
2. *The space ℓ^∞ is the set of all bounded functions $f : \mathbb{N} \to \mathbb{C}$ with norm $\|f\|_{\ell^\infty} = \sup\{|f(n)| : n \in \mathbb{N}\}$.*
3. *The subset c of ℓ^∞ of all convergent sequences has norm given by $\|f\|_c = \|f\|_{\ell^\infty}$ for $f \in c$.*
4. *The subset c_0 of ℓ^∞ of all bounded functions converging to zero has norm given by $\|f\|_{c_0} = \|f\|_{\ell^\infty}$ for $f \in c_0$.*

These are all Banach spaces.

To every Banach space is associated another Banach space known as its dual. Suppose $\mathfrak{X}$ is a Banach space. A *linear functional* on $\mathfrak{X}$ is a linear map from $\mathfrak{X}$ into $\mathbb{C}$. The set of all bounded linear functionals on $\mathfrak{X}$, denoted by $\mathfrak{X}^*$, is made a vector space via pointwise operations. We equip $\mathfrak{X}^*$ with a norm given by $\|x^*\|_{\mathfrak{X}^*} = \sup\{|x^*(x)| : x \in \mathfrak{X}, \|x\| \leq 1\}$. Thus $\mathfrak{X}^*$ coincides with $\mathfrak{B}(\mathfrak{X},\mathbb{C})$ and is itself a Banach space. We call $\mathfrak{X}^*$ the *dual space* of $\mathfrak{X}$. Given $x^* \in \mathfrak{X}^*$ and $x \in \mathfrak{X}$, it is standard to write $\langle x, x^*\rangle$ for $x^*(x)$.

It can be shown that for $1 < p < \infty$, the dual space of $\mathfrak{X} = L^p(\Omega,\mathcal{A},\mu)$ is isometrically isomorphic to $\mathfrak{X} = L^q(X,\Omega,\mu)$, where q satisfies $1/p + 1/q = 1$. Thus $(\ell^p)^*$ is isomorphic to ℓ^q as a Banach space. It is also known that $c_0^* = \ell^1$ and $(\ell^1)^* = \ell^\infty$. See [13, Chapter III, §5 and §11] for more details.

Given a Banach space $\mathfrak{X}$, we may take its dual $\mathfrak{X}^*$. This is also a Banach space and hence has a dual $(\mathfrak{X}^*)^*$, called *the second dual of* $\mathfrak{X}$ and written $\mathfrak{X}^{**}$. Continuing in this way we may construct a sequence $\mathfrak{X}, \mathfrak{X}^*, \mathfrak{X}^{**}, \mathfrak{X}^{***}, \ldots$ of Banach spaces. For which spaces $\mathfrak{X}$ does this list contain any repetitions? This question leads us to consider an important class of Banach spaces, known as the reflexive Banach spaces.

Let $\mathfrak{X}$ be Banach space. To each $x \in \mathfrak{X}$ we may associate a unique element $\widehat{x} \in \mathfrak{X}^{**}$ by the rule $\widehat{x}(x^*) = \langle x, x^*\rangle$ for all $x^* \in \mathfrak{X}$. The map $x \mapsto \widehat{x}$ from $\mathfrak{X}$ into $\mathfrak{X}^{**}$ is called the *natural map* of $\mathfrak{X}$ into its second dual.

A Banach space $\mathfrak{X}$ is said to be *reflexive* if $\mathfrak{X}^{**} = \{\widehat{x} : x \in \mathfrak{X}\}$. If $\mathfrak{X}$ is reflexive then its second dual $\mathfrak{X}^{**}$ is isometrically isomorphic to $\mathfrak{X}$. However, the converse does not hold (see [13, III.11]). It is a consequence of the Riesz Representation Theorem [13, I.3.4] that every Hilbert space is reflexive. The spaces $\mathfrak{X} = L^p(X,\Omega,\mu)$ and ℓ^p are also reflexive for $1 < p < \infty$, but by the above remarks c_0 is not. Since $c_0^{**} = \ell^\infty$, it is clear that $c_0 \subset c_0^{**}$. In fact, c_0 is isometrically embedded into its second dual ℓ^∞. This fact generalises to all Banach spaces. The space $C[0,1]$ of continuous functions on $[0,1]$ with supremum norm is another example of a non-reflexive Banach space.

A *Banach algebra* $\mathfrak{A}$ is an algebra over $\mathbb{F}$ that has a norm $\|\cdot\|_{\mathfrak{A}}$ relative to which $\mathfrak{A}$ is a Banach space and

$$\|ab\|_{\mathfrak{A}} \leq \|a\|_{\mathfrak{A}} \|b\|_{\mathfrak{A}}$$

for all $a, b \in \mathfrak{A}$. If $\mathfrak{A}$ has an identity e, then it is assumed that $\|e\|_{\mathfrak{A}} = 1$. In this thesis we shall always take $\mathbb{F}$ to be $\mathbb{C}$. An example of a Banach algebra is the space $C[0,1]$ equipped with the supremum norm, with multiplication of elements in $C[0,1]$ defined pointwise.

2.2 Topologies in Banach Spaces

A Banach space $\mathfrak{X}$ is easily made a topological space with the topology induced by its norm, called the *norm topology* of $\mathfrak{X}$. We say a sequence of vectors *converges in* $\mathfrak{X}$ if it converges in the norm topology of $\mathfrak{X}$. Another topology is the *weak topology* of $\mathfrak{X}$. A sequence $\{x_n\}_{n=1}^{\infty}$ of vectors in $\mathfrak{X}$ converges to $x \in \mathfrak{X}$ in the weak topology of $\mathfrak{X}$ if $\lim_{n\to\infty}\langle x_n - x, x^*\rangle = 0$ for all $x^* \in \mathfrak{X}^*$. As their names suggest, convergence in the norm topology implies convergence in the weak topology, or in other words, if a sequence converges then it also converges weakly.

We shall consider three topologies in the Banach space $\mathfrak{B}(\mathfrak{X})$. Suppose $\{T_n\}_{n=1}^{\infty}$ is a sequence of operators in $\mathfrak{B}(\mathfrak{X})$. We say it converges to T *in norm* if $\|T - T_n\| \to 0$, *strongly* if $\|Tx - T_nx\|_{\mathfrak{X}} \to 0$ for all $x \in \mathfrak{X}$, and *weakly* if $\langle Tx - T_nx, x^*\rangle \to 0$ for all $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$. The topologies induced are called the norm operator topology, the strong operator topology and the weak operator topology of $\mathfrak{B}(\mathfrak{X})$ respectively. A sequence which converges strongly will converge weakly, and in turn, convergence in the norm topology implies strong convergence. All three topologies have unique limits and distinguish points.

The following proposition is easy to verify and is included for familiarisation with the concepts discussed thus far. It shall also be used to establish later results.

Proposition 2.2.1. *Let $\mathfrak{X}$ be a Banach space and suppose $\{T_n\}_{n=1}^{\infty}$ is a sequence of operators uniformly bounded in norm by some $K > 0$ and converging to T in the strong operator topology of $\mathfrak{B}(\mathfrak{X})$. Then $\|T\| \leq K$.*

Proof. We want to show $\|Tx\|_{\mathfrak{X}} \leq K\|x\|_{\mathfrak{X}}$ for all $x \in \mathfrak{X}$. Fix $x \in \mathfrak{X}$ and $\epsilon > 0$ and choose an $n \in \mathbb{N}$ such that $\|Tx - T_nx\|_{\mathfrak{X}} < \epsilon$. Then

$$\|Tx\|_{\mathfrak{X}} \leq \|Tx - T_nx\|_{\mathfrak{X}} + \|T_nx\|_{\mathfrak{X}} < \epsilon + K\|x\|_{\mathfrak{X}},$$

and the proposition follows. □

2.3 Bases in Banach spaces

The definition and usefulness of a basis in a finite dimensional vector space is well-known. It is natural then to want an analogous concept for Banach spaces. The most useful approach is found in the notion of a Schauder basis. A comprehensive introduction to such bases is [22, pp. 1–52], upon which most of the material in this section is based.

Definition 2.3.1. *A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space $\mathfrak{X}$ is called a* Schauder basis *of $\mathfrak{X}$ if, for every $x \in \mathfrak{X}$, there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ such* *that* $x = \sum_{n=1}^{\infty} a_n x_n$*. A sequence* $\{x_n\}_{n=1}^{\infty}$ *which is a Schauder basis for its closed linear span is called a* basic sequence.

A Banach space $\mathfrak{X}$ is called *separable* if it has a countable dense subset. Thus if $\mathfrak{X}$ has a Schauder basis then $\mathfrak{X}$ is separable. In general the converse is not true. Since the class of Schauder bases is the only type of bases considered in this thesis, it will henceforth be implicit that any discussion about bases refers only to Schauder bases.

Example 2.3.2. *The ordered set of unit vectors* $e_n = (0, 0, 0, \ldots, 1, 0, \ldots)$*, where the* 1 *occurs in the* n*th coordinate, forms a basis in each of the spaces* c*,* c_0 *and* ℓ^p*. Another example of a basis in* c *is given by* $x_1 = (1, 1, 1, \ldots)$ *and* $x_n = e_{n-1}$ *for* $n > 1$*. If* $x = (b_1, b_2, \ldots) \in c$*, then the expansion of* x *with respect to this basis is*

$$x = (\lim_{k\to\infty} a_k)x_1 + (a_1 - \lim_{k\to\infty} a_k)x_2 + (a_2 - \lim_{k\to\infty} a_k)x_3 + \cdots .$$

If a Banach space $\mathfrak{X}$ has a basis, we may consider $\mathfrak{X}$ as a sequence space via an identification of each element $x = \sum_{n=1}^{\infty} a_n x_n$ in $\mathfrak{X}$ with the unique sequence of coefficients $(a_1, a_2, a_3, \ldots)$. To do this note that it is essential to describe the basis as an ordered sequence rather than merely as a set. The following example highlights a deeper reason for why we must specify the ordering of the vectors in a basis.

Example 2.3.3. *Consider the sequence of vectors* $\{x_n\}_{n=1}^{\infty}$ *in* c *given by*

$$x_n = (\underbrace{0, 0, \ldots, 0}_{n-1}, 1, 1, \cdots),$$

for $n \in \mathbb{Z}^+$*. It is not hard to see that this is a basis for* c*. It is called the* summing basis *in* c*. Consider the sequence* b *defined by*

$$b_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k}$$

for $n \in \mathbb{Z}^+$*. By Leibniz' test for alternating series,* b_n *has a limit as* $n \to \infty$*, so* $b \in c$*. The expansion of* b *with respect to the summing basis is* $b = \sum_{n=1}^{\infty} a_n x_n$ *where* $a_n = (-1)^{n+1}\frac{1}{n}$*. Now consider the following rearrangement of the series. Define* $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ *by*

$$\pi(n) = \begin{cases} 2k & \text{if } n = 3k \\ 4k+1 & \text{if } n = 3k+1 \\ 4k+3 & \text{if } n = 3k+2. \end{cases}$$

Then π *is clearly a permutation of* $\mathbb{Z}^+$*. However, the expansion*

$$b' \equiv \sum_{n=1}^{\infty} a_{\pi(n)} x_{\pi(n)}$$

does not lie in c since

$$b'_{3n} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} + \frac{1}{2k+1} - \frac{1}{2k} \right) \geq 2 \sum_{k=1}^{n} \frac{1}{k}$$

diverges as $n \to \infty$.

Not every basis has the defect that the convergence of an expansion with respect to the basis is dependent on the order of summation of the expansion. For example, expansions in finite dimensional spaces can be summed in any order without altering convergence. The following proposition gives various methods for checking whether a sum of vectors can be freely rearranged while still respecting convergence.

Proposition 2.3.4. *[22, Proposition 1.c.1] Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Banach space $\mathfrak{X}$. Then the following are equivalent.*

(i) The series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of the integers.

(ii) The series $\sum_{k=1}^{\infty} x_{n_k}$ converges for every choice of $n_1 < n_2 < n_3 \ldots$.

(iii) The series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ converges for every choice of signs $\varepsilon_n = \pm 1$.

(iv) For every $\epsilon > 0$ there exists an integer n_0 such that $\| \sum_{j \in J} x_j \|_{\mathfrak{X}} < \epsilon$ for every finite set of integers J which satisfies $\min\{j \in J\} > n_0$.

With the aid of the proposition, there are now easier ways to verify that the series $\sum_{n=1}^{\infty} a_n x_n$ given in Example 2.3.2.3.3 is dependent on the order of summation. Simply observe that $\sum_{n=1}^{\infty} a_{2n-1} x_{2n-1}$ does not converge in c and hence statement (ii) of the proposition fails to hold.

Definition 2.3.5. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Banach space $\mathfrak{X}$. A series $\sum_{n=1}^{\infty} x_n$ which satisfies any of the conditions (i), (ii), (iii) or (iv) in Proposition 2.3.2.3.4 is said to be* unconditionally convergent. *A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space $\mathfrak{X}$ is* unconditional *if for every $x \in \mathfrak{X}$, its expansion $x = \sum_{n=1}^{\infty} a_n x_n$ converges unconditionally. Otherwise the basis is said to be* conditional.

Thus the standard ordered bases $\{e_n\}_{n=1}^{\infty}$ for c, c_0 and ℓ^p, $1 \leq p < \infty$ are unconditional bases. This means that any reordering of $\{e_n\}_{n=1}^{\infty}$ also yields unconditional bases in these spaces. The summing basis for c is clearly a conditional basis.

Given a basis $\{x_n\}_{n=1}^{\infty}$, one might ask what complex sequences $\{\phi_n\}_{n=1}^{\infty}$ give rise to a bounded multiplier transform? That is, is there a constant $C > 0$ such that for all $x \in \mathfrak{X}$,

$$\left\| \sum_{n=1}^{\infty} \phi_n a_{n,x} x_n \right\|_{\mathfrak{X}} \leq C \, \|x\|_{\mathfrak{X}} ,$$

where $\sum_{n=1}^{\infty} a_{n,x} x_n$ is the expansion of each $x \in \mathfrak{X}$ with respect to the basis? If that basis is unconditional, then the space of such sequences is just ℓ^{∞} (see [22, Proposition 1.c.7]).

We have seen that conditional bases do exist, and some of the resulting difficulties that arise when working with a conditional basis. Is there any way in which we can avoid such problems in separable Banach spaces? In other words, can we always find an unconditional basis for a separable Banach space? The answer is no. The space $L^1[0,1]$ has no unconditional basis. In fact, $L^1[0,1]$ cannot even be embedded in a space with an unconditional basis [22, Propostion 1.d.1].

There are many open problems regarding unconditional bases. For example, suppose $\mathfrak{X}$ is a Banach space with an unconditional basis, and let $\mathfrak{Y}$ be a complemented subspace of $\mathfrak{X}$. Does $\mathfrak{Y}$ have an unconditional basis? One outstanding problem has recently been solved (see [21]). Does every infinite dimensional Banach space $\mathfrak{X}$ contain an unconditional basic sequence? The answer is no. We shall not pursue such broad problems in this thesis. Instead, we shall use tools from harmonic analysis and spectral theory to examine whether or not particular sequences in certain Banach spaces are conditional, and whether or not they form bases for those spaces.

Chapter 3

Classical Harmonic Analysis

We now leave the broad discussion about bases and conditionality in Banach spaces and instead focus our attention to some particular classes of the L^p Banach spaces. The theory of classical harmonic analysis provides a good starting point from which to tackle problems relating to bases and conditionality in this setting.

In this chapter we give an overview of some basic results from classical harmonic analysis and Fourier theory. We begin by introducing the fundamental concepts of harmonic analysis on locally compact Abelian groups. The Fourier transform and Fourier multipliers will be defined. Connections between multiplier transforms and convolution operators will be drawn in Section 3.1, as well as important examples given. In Section 3.1 we discuss some of the classical results of M. Riesz, Littlewood–Paley and Marcinkiewicz.

The general theory given in this chapter can be used to consider the the set of functions $\{\varphi_n\}_{n\in\mathbb{Z}} \subset L^p(\mathbb{T})$, where $\varphi_n : t \mapsto e^{int}$. Is this a basis for the Banach space $L^p(\mathbb{T})$? Is it an unconditional basic sequence? If not, how close is it to being an unconditional sequence? The results stated in Section 3.1 answer such questions.

3.1 Harmonic Analysis on Locally Compact Abelian Groups

In this section we give the necessary background from which we can begin to answer the questions from above. The theory mentioned here can be found in [19] and is quite standard. In what follows, if G is a group, the inverse of an element $x \in G$ will be denoted by $-x$, and the group operation from $G \times G$ into G will be denoted by $(x, y) \mapsto x + y$.

Definition 3.1.1. *A* locally compact Abelian group *(or an LCA group) G is an Abelian group which is also a locally compact Hausdorff space such that the group operations $x \mapsto -x$ from G onto G and $(x, y) \mapsto x + y$ from $G \times G$ onto G are continuous.*

The most-studied examples of LCA groups are the integers $\mathbb{Z}$, the circle group $\mathbb{T}$ and real line $\mathbb{R}$ with their usual topologies. It is easy to verify that any Abelian group can be made into an LCA group when endowed with the discrete topology. In this thesis the circle group $\mathbb{T}$ will feature most often. It can be modelled by the unit circle $\{\omega \in \mathbb{C} : |\omega| = 1\}$ in the complex plane, or as the quotient group $\mathbb{R}/2\pi\mathbb{Z}$. We shall usually adopt the latter model, thinking of $\mathbb{T}$ as the interval $[0, 2\pi]$ with endpoints identified and addition as the group operation.

A good reason to study LCA groups is that we can use them to construct measure spaces that are translation invariant.

Definition 3.1.2. *Let G be a locally compact Abelian group. A* Haar measure *on G is a positive regular Borel measure μ having the following two properties:*

(i) $\mu(E) < \infty$ if $E \subseteq G$ is compact; and

(ii) $\mu(E + x) = \mu(E)$ for all measurable $E \subseteq G$ and all $x \in G$.

Theorem 3.1.3. *[19, Chapter VII, §2] Let G be an LCA group. Then a Haar measure on G exists and is unique up to multiplication of a positive constant.*

Hence one often speaks of *the* Haar measure. For $G = \mathbb{T}$, Haar measure is usually taken to be normalised Lebesgue measure, $(2\pi)^{-1}dt$. If G is discrete and infinite, Haar measure is usually normalised to have mass one at each point. We denote the L^p space of functions on G with respect to Haar measure by $L^p(G)$. As mentioned in Example 2.1.2.1.1, we will not usually distinguish between functions defined on G and their L^p equivalence classes.

Using Haar measure, we may turn $L^1(G)$ into a Banach algebra.

Theorem 3.1.4. *Let G be an LCA group with Haar measure dy and suppose $f, g \in L^1(G)$. Then for almost all $x \in G$, the function $y \mapsto f(x-y)g(y)$ for $y \in G$ is integrable on G. If we write*

$$h(x) = \int_G f(x-y)g(y)dy,$$

then $h \in L^1(G)$ and $\|h\|_1 \leq \|f\|_1 \|g\|_1$.

Definition 3.1.5. *Let G, f, g be as in Theorem 3.1.3.1.4. Then the* convolution *of f and g, denoted $f * g$, is given by*

$$(f * g)(x) = \int_G f(x-y)g(y)dy$$

for almost all $x \in G$

Corollary 3.1.6. *Let G be an LCA group. Then $L^1(G)$ is a Banach Algebra under convolution and pointwise addition.*

Our next aim is to define the Fourier transform of a function in $L^1(G)$. We start by introducing the characters of G.

Definition 3.1.7. *A* character *on an LCA group G is a continuous mapping ξ from G into $\mathbb{C}$ such that $|\xi(x)| = 1$ and $\xi(x+y) = \xi(x)\xi(y)$ for all $x, y \in G$. The set of all characters of G is denoted $\widehat{G}$.*

Thus a character is a continuous homomorphism of G into $\mathbb{T}$. It can be shown that the set of characters on $\mathbb{T}$ is the set $\{\varphi_n\}_{n\in\mathbb{Z}}$, where $\varphi_n(t) = e^{int}$ for all $n \in \mathbb{Z}$ and all $t \in [0, 2\pi]$. Every character of $\mathbb{Z}$ has the form $n \mapsto e^{int}$ for some $t \in [0, 2\pi]$. The characters of $\mathbb{R}$ are all of the form $x \mapsto e^{ixy}$ for some $y \in \mathbb{R}$.

For any LCA group G, the set of characters $\widehat{G}$ can be made an Abelian group. We shall denote its group operation by $+$ (in what follows, this should cause no confusion with the group operation $+$ for G) and define it by $(\xi_1+\xi_2)(x) = \xi_1(x)\xi_2(x)$ for all $x \in G$. We write $\langle x, \xi\rangle$ for $\xi(x)$.

A topology is defined on $\widehat{G}$ by specifying that $\{\xi_n\}_{n=1}^{\infty} \subset \widehat{G}$ converges to $\xi \in \widehat{G}$ if $\{\xi_n\}_{n=1}^{\infty} \subset \widehat{G}$ converges uniformly to ξ on all compact subsets of G. It can be shown that this topology turns $\widehat{G}$ into a locally compact Abelian group (see [19, Chapter VII, §3]). We say that $\widehat{G}$ is the *dual group* of G.

The *Pontryagin Duality Theorem* (see [19, p.189]) asserts that, for $x \in G$ and $\xi \in \widehat{G}$, the mapping $\xi \mapsto \langle x, \xi \rangle$ is a character on $\widehat{G}$, and every character on $\widehat{G}$ is of this form. Moreover, the topology induced by uniform convergence on compact subsets of $\widehat{G}$ coincides with the original topology on G. In otherwords, if $\widehat{G}$ is the dual group of G, then G is the dual of $\widehat{G}$.

We illustrate the Pontryagin Duality Theorem for the LCA group $\mathbb{T}$. We saw that there exists a bijection between $\widehat{\mathbb{T}}$ and $\mathbb{Z}$, since given any $\xi \in \widehat{\mathbb{T}}$, there is a unique $n \in \mathbb{Z}$ such that $\langle t, \xi \rangle = e^{int}$ for all $t \in \mathbb{T}$. Moreover, the topology $\widehat{T}$ is the discrete topology. Thus $\widehat{\mathbb{T}} \simeq \mathbb{Z}$. One may similarly show that $\widehat{\mathbb{Z}} \simeq \mathbb{T}$. This duality may be described (by abuse of notation) as $\langle n, t \rangle = e^{int} = \langle t, n \rangle$ for all $t \in \mathbb{T}$ and all $n \in \mathbb{Z}$. (On the left hand side, we regard n as an element of $\mathbb{Z}$ while on the right hand side we regard it as a function on $\mathbb{T}$.)

We now have all the tools needed to define the Fourier transform of an integrable function on an LCA group.

Definition 3.1.8. *Let G be an LCA group. Then the* Fourier transform *of a function $f \in L^1(G)$ is defined by*

$$\widehat{f}(\xi) = \int_G \overline{\langle y, \xi \rangle} f(y) dy$$

for all $\xi \in \widehat{G}$, where dy is Haar measure on G.

Thus the Fourier transform of a function $f \in L^1(\mathbb{Z})$ is

$$\widehat{f}(t) = \sum_{n=-\infty}^{\infty} e^{-int} f(n)$$

for all $t \in \mathbb{T}$, since Haar measure on $\mathbb{Z}$ is unit point-mass measure. Similarly, the Fourier transform of a function $f \in L^1(\mathbb{T})$ is

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(t) dt$$

for all $n \in \mathbb{Z}$, where dt is Lebesgue measure. The *Fourier series* of f is the corresponding formal expression

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) \varphi_n$$

where $\varphi_n(t) = e^{int}$ for all $t \in \mathbb{T}$ and all $n \in \mathbb{Z}$.

We aim to use the Fourier transform to construct a large class of operators which act on the L^p space of a given LCA group. We may do this via *Plancherel's Theorem.*

Theorem 3.1.9. *[19, Chapter VII, §4]* Plancherel's Theorem. *Let G be a locally compact Abelian group. Then the Fourier transform on $L^1(G)$ is an isometry of $L^1 \cap L^2(G)$ onto a dense subspace of $L^2(\widehat{G})$. Hence it can be extended to an isometry of $L^2(G)$ onto $L^2(\widehat{G})$.*

For the circle group $\mathbb{T}$, Plancherel's theorem takes the following form.

Theorem 3.1.10. *[19, Theorem I.5.5]*

(i) Let $f \in L^2(\mathbb{T})$. Then

$$\sum_{n=\infty}^{\infty} |\widehat{f}(n)|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \, dt,$$

that is, $\|\widehat{f}\|_{L^2(\mathbb{Z})} = \|f\|_{L^2(\mathbb{T})}$.

(ii) For each $f \in L^2(\mathbb{T})$,

$$f = \lim_{N \to \infty} \sum_{n=-N}^{N} \widehat{f}_n \varphi_n$$

in $L^2(\mathbb{T})$ norm, where $\varphi_n : t \mapsto e^{int}$.

(iii) Given any sequence $\{a_n\}_{n=-\infty}^{\infty}$ of complex numbers in $L^2(\mathbb{Z})$, then there exists a unique $f \in L^2(\mathbb{T})$ such that $a_n = \widehat{f}(n)$.

Thus the correspondence $f \leftrightarrow \{\widehat{f}(n)\}_{n=-\infty}^{\infty}$ is an isometry between $L^2(\mathbb{T})$ and $L^2(\mathbb{Z})$.

A by-product of the above theorem is that the set $\{\varphi_n\}_{n\in\mathbb{Z}}$ of functions in the Hilbert space $L^2(\mathbb{T})$ forms a complete orthonormal system (and hence an unconditional basis) with respect to the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)} \, dt.$$

It is harder to establish whether or not $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a basis for $L^p(\mathbb{T})$ when $1 \leq p < \infty$ and $p \neq 2$. Standard results about Fejér's kernel (see Section 3.1 and [19, I.2.6]) give the following facts. First recall that a *trigonometric polynomial* is a function on $\mathbb{T}$ of the form $\sum_{n=-N}^{N} a_n \varphi_n$ where $N \in \mathbb{N}$ and $a_n \in \mathbb{C}$.

Theorem 3.1.11. *Fix $1 \leq p < \infty$. Then*

(i) the trigonometric polynomials are dense in $L^p(\mathbb{T})$;

(ii) if $f, g \in L^p(\mathbb{T})$ and $\widehat{f}(n) = \widehat{g}(n)$ for all $n \in \mathbb{Z}$, then $f = g$; and

(iii) if the Fourier series of a function $f \in L^p(\mathbb{T})$ does converge in L^p-norm, then it converges to f.

So if the Fourier series does converge for each function in $L^p(\mathbb{T})$, $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a basis for $L^p(\mathbb{T})$. Otherwise, one would like to know whether or not $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a basic sequence in $L^p(\mathbb{T})$ for $p \neq 2$, and whether this sequence is unconditional. One way to study such problems is through multiplier transforms.

Definition 3.1.12. *Let G be an LCA group and let $\phi : \widehat{G} \to \mathbb{C}$ be bounded and measurable on $\widehat{G}$. By Plancherel's Theorem, define S_ϕ to be the continuous linear mapping of $L^2(G)$ into itself for which $(S_\phi f)\widehat{\ } = \phi \widehat{f}$ for all $f \in L^2(G)$. Then for $1 \leq p \leq \infty$, ϕ is said to be an $L^p(G)$* multiplier *if and only if there is a constant $C > 0$ such that*

$$\|S_\phi f\|_p \leq C_p \|f\|_p \tag{3.1.1}$$

for all $f \in L^2 \cap L^p(\mathbb{T})$. In this case S_ϕ is called the multiplier transform corresponding to ϕ on $L^p(G)$ *and ϕ may also be referred to as a* Fourier multiplier *or a* multiplier function *for $L^p(G)$.*

The space of Fourier multipliers for $L^p(G)$ will be denoted by $M_p(\widehat{G})$. We turn $M_p(\widehat{G})$ into a Banach space by equipping it with the following norm: for $\phi \in M_p(\widehat{G})$ define $\|\phi\|_{M_p(\widehat{G})}$ to be the usual operator norm on $L^p(G)$ for the operator S_ϕ. Thus $\|\phi\|_{M_p(\widehat{G})}$ is the smallest possible $C \geq 0$ for which (3.1.1) holds. It is trivial to check from the definitions that if $\phi_1, \phi_2 \in M_p(\widehat{G})$, then the product $\phi_1\phi_2$, defined by pointwise operations, is in $M_p(\widehat{G})$ and

$$\|\phi_1\phi_2\|_{M_p(\widehat{G})} \leq \|\phi_1\|_{M_p(\widehat{G})} \|\phi_2\|_{M_p(\widehat{G})}.$$

In fact, we have the following result.

Proposition 3.1.13. *[3, §3] Let G be an LCA group. Then for $1 \leq p < \infty$, $M_p(\widehat{G})$ is a Banach algebra under pointwise operations, and the mapping $\phi \mapsto S_\phi$ is an identity-preserving algebra homomorphism of $M_p(\widehat{G})$ into $\mathfrak{B}(L^p(G))$.*

The next example illustrates some of the concepts that have been introduced so far.

Example 3.1.14. *Let $G = \mathbb{T}$ and let $\varphi_n : t \mapsto e^{int}$ for all $t \in \mathbb{T}$ and $n \in \mathbb{Z}$. Fix an $m \in \mathbb{Z}$ and consider the characteristic function $\chi_{\{m\}}$ on $\mathbb{Z}$. Then for any $f \in L^2(\mathbb{T})$ we have, by Definition 3.1.3.1.12,*

$$\big(S_{\chi_{\{m\}}} f\big)^\frown(n) = \chi_{\{m\}}(n)\widehat{f}(n) = \chi_{\{m\}}(n)\widehat{f}(m) = \widehat{\varphi}_m(n)\widehat{f} = \big(\widehat{f}(m)\varphi_m\big)^\frown(n)$$

for all $n \in \mathbb{Z}$. Thus $S_{\chi_{\{m\}}} f = \widehat{f}(m)\varphi_m$, and $S_{\chi_{\{m\}}}$ projects each f onto the mth term of its Fourier series.

Now we consider the convolution product of f with φ_m. For almost all $t \in \mathbb{T}$ we have

$$\begin{aligned}(\varphi_m * f)(t) &= \frac{1}{2\pi}\int_0^{2\pi} \varphi_m(t-s)f(s)ds \\ &= e^{imt}\frac{1}{2\pi}\int_0^{2\pi} e^{-ims}f(s)ds \\ &= e^{imt}\widehat{f}(m).\end{aligned}$$

Thus $S_{\chi_{\{m\}}}$ and the operator $f \mapsto \varphi_m * f$ coincide on $L^2(\mathbb{T})$. It is easy to see that $\|S_{\chi_{\{m\}}}\| = 1$ and hence $\|\chi_{\{m\}}\|_{M_p(\widehat{G})} = 1$.

We now give one reason why multiplier transforms are of interest to us. Let $\varphi_n(t) = e^{int}$ for each $n \in \mathbb{Z}$ and $t \in \mathbb{T}$. One might hope that $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a basis for $L^p(\mathbb{T})$. Assume for the moment that this is the case. The expansion of a function $f \in L^p(\mathbb{T})$ with respect to this basis is then given by $\sum_{n\in\mathbb{Z}} \widehat{f}(n)\varphi_n$. If one wanted to prove that the basis was unconditional, it would suffice to find (by Proposition 2.3.2.3.4) a constant C_p depending only on p such that

$$\Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n \widehat{f}(n)\varphi_n\Big\|_p \leq C_p \|f\|_p$$

for all $f \in L^p(\mathbb{T})$ and all choices $\varepsilon_n = \pm 1$. This would be equivalent to finding a constant C_p such that

$$\|S_\varepsilon f\|_p \leq C_p \|f\|_p$$

for all $f \in L^2 \cap L^p(\mathbb{T})$ and all functions $\varepsilon : \mathbb{Z} \to \mathbb{C}$ taking values in $\{-1, 1\}$. Motivated by this, we develop the general theory of multipliers a little further in the next section.

Chapter 4

Conclusion

In mathematics, certain kinds of mistaken proof are often exhibited, and sometimes collected, as illustrations of a concept of mathematical fallacy. There is a distinction between a simple mistake and a mathematical fallacy in a proof: a mistake in a proof leads to an invalid proof just in the same way, but in the best-known examples of mathematical fallacies, there is some concealment in the presentation of the proof. For example, the reason validity fails may be a division by zero that is hidden by algebraic notation. There is a striking quality of the mathematical fallacy: as typically presented, it leads not only to an absurd result, but does so in a crafty or clever way. Therefore these fallacies, for pedagogic reasons, usually take the form of spurious proofs of obvious contradictions. Although the proofs are flawed, the errors, usually by design, are comparatively subtle, or designed to show that certain steps are conditional, and should not be applied in the cases that are the exceptions to the rules.

The traditional way of presenting a mathematical fallacy is to give an invalid step of deduction mixed in with valid steps, so that the meaning of fallacy is here slightly different from the logical fallacy. The latter applies normally to a form of argument that is not a genuine rule of logic, where the problematic mathematical step is typically a correct rule applied with a tacit wrong assumption. Beyond pedagogy, the resolution of a fallacy can lead to deeper insights into a subject (such as the introduction of Pasch's axiom of Euclidean geometry and the five color theorem of graph theory). Pseudaria, an ancient lost book of false proofs, is attributed to Euclid.

Mathematical fallacies exist in many branches of mathematics. In elementary algebra, typical examples may involve a step where division by zero is performed, where a root is incorrectly extracted or, more generally, where different values of a multiple valued function are equated. Well-known fallacies also exist in elementary Euclidean geometry and calculus.

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