Question 1

(a) According to Service demand law:

$$D(j) = \frac{U(j)}{X(0)}, \quad U(j) = \frac{B(j)}{T}, \quad X(0) = \frac{C(0)}{T}$$

$\cancel{B(j)} =$ $B(\text{Disk}) = 2765s$, $B(\text{CPU}) = 2929s$

$T = 60 \text{ mins} = 60 \times 60 = 3600s$, $C(0) = 1267$

$$\because \frac{U(j)}{X(0)} = \frac{B(j)}{T} \times \frac{T}{C(0)} = \frac{B(j)}{C(0)}$$

$$\therefore D(\text{disk}) = \frac{2765}{1267} \approx 2182 \text{ ms}$$

$$D(\text{cpu}) = \frac{2929}{1267} \approx 2312 \text{ ms}$$

(b) Since the bottleneck analysis is

$$X(0) \le \min\left[\frac{1}{\max D_i}, \frac{N}{\sum_{i=1}^{K} D_i}\right]$$

if there is thinking time, then bottleneck analysis is:

$$X(0) \le \min\left[\frac{1}{\max D_i}, \frac{N}{\sum_{i=1}^{K} D_i + \text{thinking time}}\right]$$

$\because D(\text{disk}) = 2182 \text{ ms}$, $D(\text{cpu}) = 2312 \text{ ms}$

$$\therefore \frac{1}{\max D_i} = \frac{1}{2.312(s)} = 0.4326 \text{ (jobs/s)}$$

$\because$ there are 20 active terminals and thinking time per job is 14 second

$$\therefore \frac{N}{\sum_{i=1}^{K} D_i + \text{thinking time}} = \frac{20}{2.182 + 2.312 + 14} = 1.0814 \text{ (jobs/s)}$$

$\therefore$ so the asymptotic bound should be 0.4326 (jobs/s).

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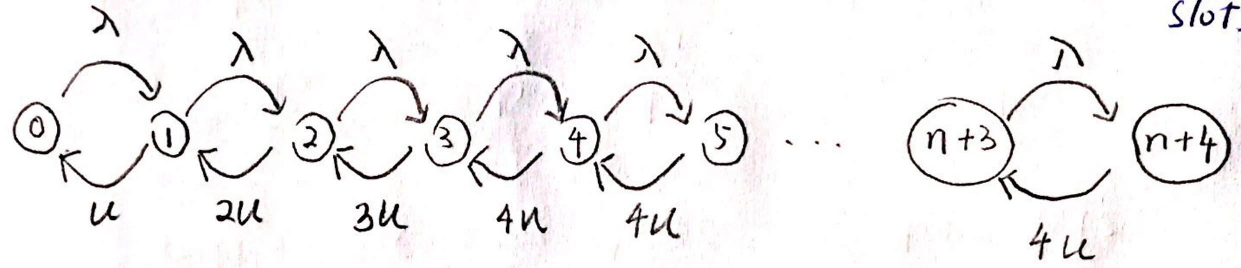
Question 2

(a) Formulate Markov chain for 4 operators and n waiting slots.



State (0) means all operators are idle and there are no calls in the center.

State (1) means one operator is busy (no calls in the queue)

State (2) means two operators are busy (no calls in the queue)

State (3) means three operators are busy (no calls in the queue)

State (4) means four operators are busy (no calls in the queue)

State (5) means all operators are busy (one call in the queue)

....

State (n+4) means all operators are busy (n calls in the queue)

(b) $p(0)$ means the probability that no calls in the queue and all operators are idle.

$p(K)$ means the probability that K calls in the center,

$\lambda P(0) = \mu P(1) \Rightarrow P(1) = \frac{\lambda P(0)}{u} = \frac{\lambda}{u} P(0)$

$\lambda P(1) = 2u P(2) \Rightarrow P(2) = \frac{\lambda P(1)}{2u} = (\frac{\lambda}{u})^2 P(0) \cdot \frac{1}{2}$

$\lambda P(2) = 3u P(3) \Rightarrow P(3) = \frac{\lambda P(2)}{3u} = (\frac{\lambda}{u})^3 P(0) \cdot \frac{1}{6}$

$\lambda P(3) = 4u P(4) \Rightarrow P(4) = \frac{\lambda P(3)}{4u} = (\frac{\lambda}{u})^4 P(0) \cdot \frac{1}{24}$

$\cancel{\lambda P(4) = 4u P(5)}$

$\lambda P(4) = 4u P(5) \Rightarrow P(5) = \frac{\lambda P(4)}{4u} = (\frac{\lambda}{u})^5 \cdot P(0) \cdot \frac{1}{96}$

$\lambda P(n+2) = 4u P(n+3) \Rightarrow P(n+3) = \frac{\lambda P(n+2)}{4u}$

$= (\frac{\lambda}{u})^{n+3} \cdot P(0) \cdot \frac{1}{24} \cdot (\frac{1}{4})^{n-1}$

$\lambda p(n+3) = 4u\, p(n+4) \Rightarrow p(n+4) = \dfrac{\lambda p(n+3)}{4u}$

$= (\frac{\lambda}{u})^{n+4} \cdot p(0) \cdot \frac{1}{24} \cdot (\frac{1}{4})^{n}$

Since all states' probability is 1

$p(0) + p(1) + p(2) + \dots p(n+4) = 1$

$\because \rho = \frac{\lambda}{u}$.

$\therefore (1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{6}\rho^3 + \frac{1}{24}\rho^4 + \dots + (\frac{1}{4})^{n-1}\frac{1}{24}\rho^{n+3} + (\frac{1}{4})^{n}\frac{1}{24}\rho^{n+4}) \cdot p(0)$
$= 1$

$$p(0) = \frac{1}{1+\rho + \frac{1}{2}\rho^2 + \frac{1}{6}\rho^3 + \frac{1}{24}\rho^4 + \sum_{i=5}^{n+4} (\frac{1}{4})^{K-4} \cdot \frac{1}{24}\rho^K}$$

State (K) means there are K jobs in the center.

it can be simplified as ~~below~~ below:

$$p(0) = \frac{1}{\sum_{K=0}^{4} \frac{1}{K!}\rho^K + \sum_{i=5}^{n+4} \frac{1}{24} \cdot (\frac{1}{4})^{K-4}\rho^K}$$

(c) if state (K) $K \le 4$ then

$p(K) = p(0) \frac{1}{K!}\rho^K \quad (\rho = \frac{\lambda}{u})$

if state (K) $K > 4$ then

$p(K) = p(0) \frac{1}{24} \cdot (\frac{1}{4})^{K-4}\rho^K \quad (\rho = \frac{\lambda}{u})$

According to the answer of (b)

$$p(0) = \frac{1}{\sum_{K=0}^{4} \frac{1}{K!}\rho^K + \sum_{i=5}^{n+4} \frac{1}{24} \cdot (\frac{1}{4})^{K-4}\rho^K} \quad (\rho = \frac{\lambda}{u})$$

(d) For the current configuration $n=2$. means there are 2 ~~slot~~ waiting slots.

~~手~~ suppose there are K jobs in the center

if K < operators + waiting slots, no calls can be rejected. This means the probability of rejection will equal to p (#operators + #waiting slots).

(i) the probability that an arring query will be rejected

$$p(6) = p(0) \cdot \frac{1}{24} \left(\frac{1}{4}\right)^2 \left(\frac{\lambda}{\mu}\right)^6$$

$\because$ the voice-over-IP record shows that the center is getting on average 15 queries per hour. The arrivals can be modelled by using poisson distribution

$\therefore \quad \lambda = 15$ (calls / hour)

$\because$ the record also shows that each support staff can complete on average 3 queries per hour. The amount of time required by each query is exponentially distributed

$\therefore \quad \mu = 3$ (calls / hour)

$$\therefore \quad \rho = \frac{\lambda}{\mu} = \frac{15}{3} = 5$$

$$p(0) = \frac{1}{\sum_{K=0}^{4} \frac{1}{K!} \rho^K + \sum_{K=5}^{n+4} \frac{1}{24} \cdot \left(\frac{1}{4}\right)^{K-4} \rho^K}$$

$K = 6, \quad n = 2 \quad \rho = 5.$

$$\sum_{K=0}^{4} \frac{1}{K!} \rho^K = \frac{1}{0!} 5^0 + \frac{1}{1!} 5^1 + \frac{1}{2!} 5^2 + \frac{1}{3!} 5^3 + \frac{1}{4!} 5^4$$

$$= 1 + 5 + \frac{25}{2} + \frac{125}{6} + \frac{625}{24}$$

$$= \frac{24 + 120 + 300 + 500 + 625}{24} = \frac{1569}{24}$$

$$= 65.375$$

$$\cancel{\sum_{i=5}^{6} \frac{1}{24} \left(\frac{1}{4}\right)^2 5}$$

$$\sum_{K=5}^{n+4} \frac{1}{24} \cdot \left(\frac{1}{4}\right)^{K-4} 5^K = \left(\frac{1}{4}\right)^1 \times \frac{1}{24} \times 5^5 + \left(\frac{1}{4}\right)^2 \times \frac{1}{24} \times 5^6$$

$$= 73.2422$$

$P(0) = (65.375 + 73.1422)^{-1} = 0.007214$

$P(6) = P(0) \cdot \frac{1}{24} (\frac{1}{4})^2 \times 5^6$

$= 0.2935$

$\therefore X = 0.2935$ the probability that an arriving query will be rejected is 0.2935

(ii) According to the little's law

mean number of customers = throught × response time

$N = X \cdot R$

$\because$ waiting slots are 2, operators are 4,

$\therefore$ the sum of calls in the center is

$N = \sum_{K=0}^{6} K \cdot P(K) = 0 \cdot P(0) + 1 \times P(1) + 2 \times P(2) + 3P(3) + 4P(4) + 5P(5) + 6P(6)$

$P(0) = \cancel{0.2935}\ 0.007214$

$P(1) = \frac{1}{1!} 5^1 P(0) = 0.03607$

$P(2) = \frac{1}{2!} 5^2 P(0) = 0.090176$

$P(3) = \frac{1}{3!} 5^3 P(0) = 0.150294$

$P(4) = \frac{1}{4!} 5^4 P(0) = 0.187867$

$P(5) = (\frac{1}{4}) \frac{1}{24} 5^5 P(0) = 0.234834$

$P(6) = (\frac{1}{4})^2 \frac{1}{24} \times 5^6 P(0) = 0.2935$

$$P(K) = \begin{cases} P(0) \frac{1}{K!} \rho^K & K \leq 4 \\ P(0) (\frac{1}{4})^{K-4} \frac{1}{24} \rho^K & K > 4 \end{cases}$$

service time $= \frac{1}{u} = \frac{1}{3}(h)$

$\frac{1}{3}(h) = 1200s$

$\therefore N \approx 4.3542$

$\because X(0) = (1 - P(\text{reject})) \cdot \lambda = (1 - 0.2935) \times 15 \approx 10.5968$

$\therefore R = \frac{N}{X(0)} = \frac{4.3542}{10.5968} = 0.410896h = 1479.23s$

Waiting time = R − service time $= 1479.23 - 1200 = 279.23s$

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∴ the mean waiting time of an accepted query in the queue is 279.23 seconds.

(e) $p(0) = \left[\sum_{K=0}^{4} \frac{1}{K!} \rho^K + \sum_{K=5}^{n+4} \frac{1}{24} \left(\frac{1}{4}\right)^{K-4} \rho^K \right]^{-1}$

$$p(K) = \begin{cases} p(0) \frac{1}{K!} \rho^K & (K \leq 4) \\ p(0) \left(\frac{1}{4}\right)^{K-4} \frac{1}{24} \rho^K & (K > 4) \end{cases}$$

there are 4 operators and 2 holding slots in the center.

After adding 5 waiting slots (please refer to q2e.py in supp.zip)

$K = 4+2+5 = 11$

$p(0) = 0.0017984$

$p(11) = 0.22332$

After adding 10 waiting slots (About the calculation, please refer to q2e.py in supp.zip)

$K = 4+2+10 = 16$

$p(0) = 0.000546$

$p(16) = 0.207085$

After adding 15 waiting slots

$K = 4+2+15 = 21$

$p(0) = 0.0001749$

$p(21) = 0.20227$

After adding 20 waiting slots

$K = 4+2+20 = 26$

$p(0) = 5.688 \times 10^{-5}$

$p(26) = 0.20074$

(f) After adding 10 waiting slots, the value of $p(0)$ gradually approaches 0.

Since the $p(K)$ depends on the value of $p(0)$

So it results in a small change for $p(K)$

In order to reduce the blocking probability,

we should add more operators. ~~to~~

Also we should ~~increase~~ decrease the ~~value~~ value of ρ $(\frac{\lambda}{\mu})$

One method is to reduce the arrival rate

Another method is to increase the completion rate for each staff.

Question 3

(a) there are 12 possible states : in a list of 3-tuple states.

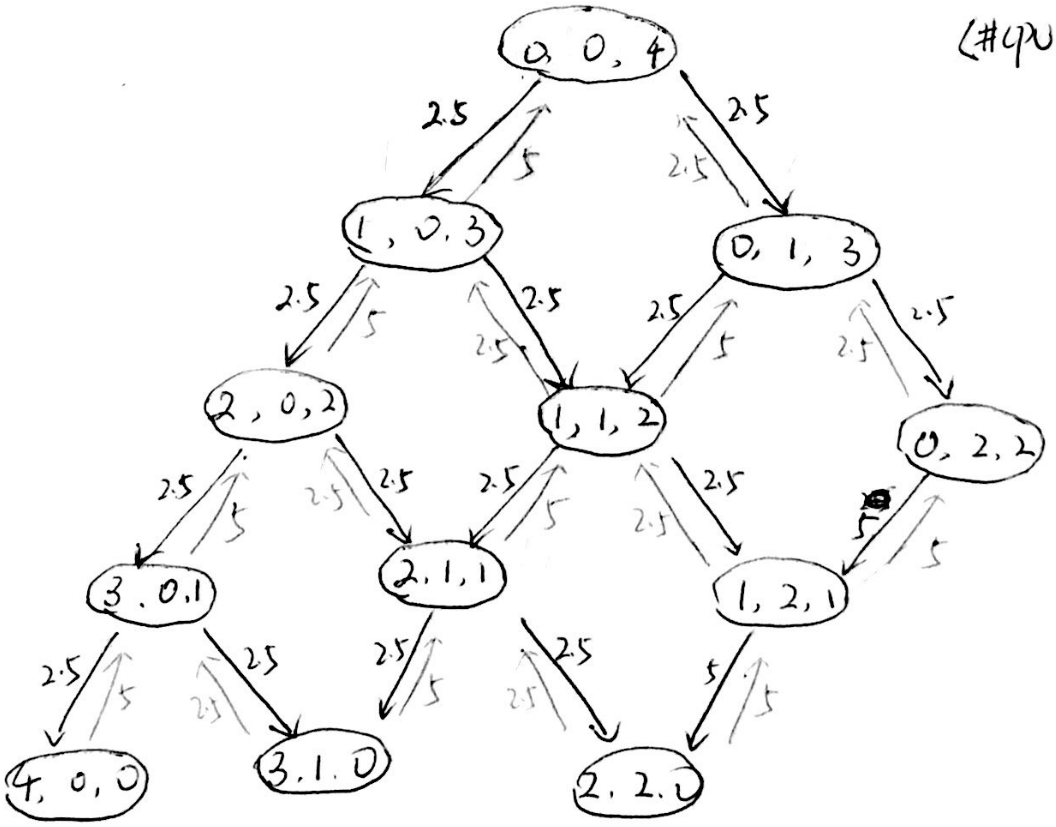
(number of users in the CPU1, number of users in the CPU2, number of users in th disk)

(# CPU1 , # CPU2, # Disk)

(0, 0, 4)	(3, 0, 1)
(1, 0, 3)	(2, 1, 1)
(0, 1, 3)	(1, 2, 1)
(2, 0, 2)	(4, 0, 0)
(1, 1, 2)	(3, 1, 0)
(0, 2, 2)	(2, 2, 0)

Then the transition rates between the states

(#CPU1, #CPU2, #Disk)



$\mu(CPU1) = 1/0.2 = 5$

$\mu(CPU2) = 1/0.5 = 2.5$

$\mu(Disk) = 1/0.2 = 5$

(b) ① $5P_{(0,0,4)} - 5P_{(1,0,3)} - 2.5P_{(0,1,3)} + 0P_{(2,0,2)} + 0P_{(1,1,2)} + 0P_{(0,2,2)}$
$+ 0P_{(3,0,1)} + 0P_{(2,1,1)} + 0P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} + 0P_{(2,2,0)} = 0$

②
$-2.5P_{(0,0,4)} + 10P_{(1,0,3)} + 0P_{(0,1,3)} - 5P_{(2,0,2)} - 2.5P_{(1,1,2)} + 0P_{(0,2,2)}$
$+ 0P_{(3,0,1)} + 0P_{(2,1,1)} + 0P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} + 0P_{(2,2,0)} = 0$

③
$-2.5P_{(0,0,4)} + 0P_{(1,0,3)} + 7.5P_{(0,1,3)} + 0P_{(2,0,2)} - 5P_{(1,1,2)} - 2.5P_{(0,2,2)}$
$+ 0P_{(3,0,1)} + 0P_{(2,1,1)} + 0P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} + 0P_{(2,2,0)} = 0$

④
$0P_{(0,0,4)} - 2.5P_{(1,0,3)} + 0P_{(0,1,3)} + 10P_{(2,0,2)} + 0P_{(1,1,2)} + 0P_{(0,2,2)}$
$- 5P_{(3,0,1)} - 2.5P_{(2,1,1)} + 0P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} + 0P_{(2,2,0)} = 0$

⑤
$0P_{(0,0,4)} - 2.5P_{(1,0,3)} - 2.5P_{(0,1,3)} + 0P_{(2,0,2)} + 12.5P_{(1,1,2)} + 0P_{(0,2,2)}$
$+ 0P_{(3,0,1)} - 5P_{(2,1,1)} - 2.5P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} + 0P_{(2,2,0)} = 0$

⑥
$0P_{(0,0,4)} + 0P_{(1,0,3)} - 2.5P_{(0,1,3)} + 0P_{(2,0,2)} + 0P_{(1,1,2)} + 7.5P_{(0,2,2)}$
$+ 0P_{(3,0,1)} + 0P_{(2,1,1)} - 5P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} + 0P_{(2,2,0)} = 0$

⑦
$0P_{(0,0,4)} + 0P_{(1,0,3)} + 0P_{(0,1,3)} - 2.5P_{(2,0,2)} + 0P_{(1,1,2)} + 0P_{(0,2,2)}$
$+ 10P_{(3,0,1)} + 0P_{(2,1,1)} + 0P_{(1,2,1)} - 5P_{(4,0,0)} - 2.5P_{(3,1,0)} + 0P_{(2,2,0)} = 0$

⑧
$0P_{(0,0,4)} + 0P_{(1,0,3)} + 0P_{(0,1,3)} - 2.5P_{(2,0,2)} - 2.5P_{(1,1,2)} + 0P_{(0,2,2)}$
$+ 0P_{(3,0,1)} + 12.5P_{(2,1,1)} + 0P_{(1,2,1)} + 0P_{(4,0,0)} - 5P_{(3,1,0)} - 2.5P_{(2,2,0)} = 0$

(9)

$$0P_{(0,0,4)} + 0P_{(1,0,3)} + 0P_{(0,1,3)} + 0P_{(2,0,2)} - 2.5P_{(1,1,2)} - 5P_{(0,2,2)} + 0P_{(3,0,1)} + 0P_{(2,1,1)} + 12.5P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} - 5P_{(2,2,0)} = 0$$

(10)

$$0P_{(0,0,4)} + 0P_{(1,0,3)} + 0P_{(0,1,3)} + 0P_{(2,0,2)} + 0P_{(1,1,2)} + 0P_{(0,2,2)} - 2.5P_{(3,0,1)} + 0P_{(2,1,1)} + 0P_{(1,2,1)} + 5P_{(4,0,0)} + 0P_{(3,1,0)} + 0P_{(2,2,0)} = 0$$

(11)

$$0P_{(0,0,4)} + 0P_{(1,0,3)} + 0P_{(0,1,3)} + 0P_{(2,0,2)} + 0P_{(1,1,2)} + 0P_{(0,2,2)} - 2.5P_{(3,0,1)} - 2.5P_{(2,1,1)} + 0P_{(1,2,1)} + 0P_{(4,0,0)} + 7.5P_{(3,1,0)} + 0P_{(2,2,0)} = 0$$

(12)

$$0P_{(0,0,4)} + 0P_{(1,0,3)} + 0P_{(0,1,3)} + 0P_{(2,0,2)} + 0P_{(1,1,2)} + 0P_{(0,2,2)} + 0P_{(3,0,1)} - 2.5P_{(2,1,1)} - 5P_{(1,2,1)} + 0P_{(4,0,0)} + 0P_{(3,1,0)} + 7.5P_{(2,2,0)} = 0$$

(13)

$$P_{(0,0,4)} + P_{(1,0,3)} + P_{(0,1,3)} + P_{(2,0,2)} + P_{(1,1,2)} + P_{(0,2,2)} + P_{(3,0,1)} + P_{(2,1,1)} + P_{(1,2,1)} + P_{(4,0,0)} + P_{(3,1,0)} + P_{(2,2,0)} = 1$$

(c)

$P_{(0,0,4)} = 0.1711$	$P_{(3,0,1)} = 0.0259$
$P_{(1,0,3)} = 0.0912$	$P_{(2,1,1)} = 0.0572$
$P_{(0,1,3)} = 0.1598$	$P_{(1,2,1)} = 0.1021$
$P_{(2,0,2)} = 0.0501$	$P_{(4,0,0)} = 0.0130$
$P_{(1,1,2)} = 0.0935$	$P_{(3,1,0)} = 0.0277$
$P_{(0,2,2)} = 0.1213$	$P_{(2,2,0)} = 0.0871$

About the calculation, please refere to 93c.m in supp.zip

(d) the throughput of the system is exactly the output of the disk, so we should calculate the utilisation of the disk.

We find the disk state ~~whose state is~~ which is idle.

$P(4,0,0)$ $P(3,1,0)$ $P(2,2,0)$

$$U(disk) = 1 - P(4,0,0) - P(3,1,0) - P(2,2,0)$$
$$= 1 - 0.0130 - 0.0277 - 0.0871$$
$$= 0.8722$$

According to the ~~little's law~~ Utilisation law

$$X = \frac{U}{S} \quad \because U = \frac{B}{T} \quad S = \frac{B}{C} \quad X = \frac{C}{T}$$

$$\therefore U = SX \quad \therefore X = \frac{U}{S}$$

$$\therefore X(disk) = \frac{U(disk)}{S(disk)} = \frac{0.8722}{0.2} = 4.361$$

$$\therefore X(0) = X(disk) = 4.361 \text{ jobs/s}$$

(e) According to the little's Law:

$$R = N/X \Rightarrow N = XR$$

$$N(cpu_1) = \sum_{k=0}^{K} n(k) \cdot P(cpu_1)$$

$$\therefore N(cpu_1) = P(1,0,3) + 2P(2,0,2) + P(1,1,2) + 3P(3,0,1) + 2P(2,1,1) + P(1,2,1) + 4P(4,0,0) + 3P(3,1,0) + 2P(2,2,0)$$

$$= \cancel{0.1598}\ 0.0912 + 2\times 0.0501 + 0.0935 + 3\times 0.0259 + 2\times 0.0572 + 0.1021 + 4\times 0.0130 + 3\times 0.0277 + 2\times 0.0871 = \cancel{0.957}\ 0.8884$$

$\therefore$ the mean number of ~~cpu~~ jobs in cpu1 is ~~0.957~~ 0.8884

(f) According to the Utilisation law

$U = \frac{B}{T}$ $S = \frac{B}{C}$, $X = \frac{C}{T}$ $\Rightarrow U = SX \Rightarrow X = \frac{U}{S}$

$\therefore$ throughput of cpu1 is

$$X(cpu1) = \frac{U(cpu1)}{S(cpu1)} =$$

$$U(cpu1) = P(1,0,3) + P(2,0,2) + P(1,1,2) + P(3,0,1) + P(2,1,1)$$
$$+ P(1,2,1) + P(4,0,0) + P(3,1,0) + P(2,2,0)$$
$$= 0.0912 + 0.0501 + 0.0935 + 0.0259 + 0.0572$$
$$+ 0.1021 + 0.0130 + 0.0277 + 0.0871$$
$$= 0.5478$$

$S(cpu1) = 0.2$ (s)

$$X(cpu1) = \frac{U(cpu1)}{S(cpu1)} = \frac{0.5478}{0.2} = 2.739 \text{ jobs/s}$$

According to the little's law

$$R = \frac{N}{X} = \frac{N(cpu1)}{X(cpu1)} = \frac{0.8884}{2.739} = 0.3243 \text{ (s)}$$

$\therefore$ The response time of cpu1 is 0.3243 seconds.