The University of New South Wales

Department of Statistics

Session 2, 2018

MATH5905 - Statistical Inference

Assignment 2

The assignment must be submitted by 5PM on Friday, 12th October 2018 at the latest (i.e., at the beginning of the lecture in week 11). Please, declare on the first page that the assignment is your own work, except where acknowledged. State also that you have read and understood the University Rules in respect to Academic Misconduct.

1) Let $X = (X_1, X_2, \dots, X_n)$ be a sample of n observations each with a uniform in $[0, \theta)$ density

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} \text{ if } 0 < x < \theta \\ 0 \text{ else} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Denote the joint density by $L(X, \theta)$.

a) Show that the family $\{L(X, \theta)\},\ \theta > 0$ has a monotone likelihood ratio in $X_{(n)}$.

b) Show that the uniformly most powerful α-size test of $H_0 : \theta \leq 2$ versus $H_1 : \theta > 2$ is given by

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 \text{ if } X_{(n)} > 2(1-\alpha)^{\frac{1}{n}} \\ 0 \text{ if } X_{(n)} \leq 2(1-\alpha)^{\frac{1}{n}} \end{cases}$$

c) Find the power function of the test and sketch the graph of $E_\theta \varphi^*$ as accurately as possible.

d) Show that the random variable $Y_n = n(1 - \frac{X_{(n)}}{\theta})$ converges in distribution to the exponential distribution with mean 1 as $n \to \infty$. Hence justify that $X_{(n)}$ is a consistent estimator of θ.

2) Let $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_n})$ be i.i.d. random variables, each with a density

$$f(x, \theta) = \begin{cases} \frac{2}{\theta} x e^{-\frac{x^2}{\theta}}, x > 0 \\ 0 \text{ elsewhere} \end{cases}$$

where $\theta > 0$ is a parameter. (This is called the Raleigh-distribution.)

a) Find the Fisher information about θ in one observation and in the sample of n observations.

b) Find the MLE of θ. Is it unbiased? If YES, does its variance attain the Cramer Rao bound?

c) What is the asymptotic distribution of the MLE of θ?

d) Does that the family $L(\mathbf{X}, \theta)$ has a monotone likelihood ratio? If YES, in which statistic?

e) Does a UMP α test of $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ exist? If YES, outline its structure. Also, using asymptotic arguments (e.g., from c)), find the threshold constant in the definition of the test.

f) Calculate (asymptotic approximation to) the power function $E_\theta \varphi^*$ and sketch a graph.

3. (Use `mathStatica` **only** to solve parts a) and b) of this problem.) During the lab in week 7, capabilities of `mathStatica` to deal with distributions of order statistics have been demonstrated. Even more demonstrations can be found in Section 9.4 of the on-line `MathStatica` textbook (accessible within MATHEMATICA). Examining the files `5905demo_2018.nb` and `intrograpshort.nb`) in the computing subfolder on moodle and possibly examining the material of Section 9.4, solve the following problems and attach the `mathStatica` printout to your assignment:

a) Let $X_{(1)}, X_{(2)}, X_{(3)}$ denote the order statistics of a random sample of size $n = 3$ from $X \sim N(0, 1)$.

i) Obtain the densities of each of these three order statistics. (Note: $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$)

ii) Plot on a *single* diagram the densities of each order statistic (use the interval (-3.5,3.5)).

iii) Determine $E[X_{(r)}]$ for $r = 1, 2, 3$. In particular, you will check in this way the answer to tutorial question 5b) from the Part four exercise sheet.

b) Repeat the steps i), ii), iii) in a) for the case of order statistic of size $n = 3$ from the standard Laplace distribution with density

$$f(x) = \frac{1}{2} \exp(-|x|), x \in (-\infty, \infty).$$

To get a nicer expression about the density of the order statistic, you might need to use the option `FullSimplify`. When plotting on a single diagram, use the interval (-5,5).

4. (For this problem, you are asked to present your analytic derivations. However, if you present a `mathStatica` solution instead, you will get the marks allocated.)

Suppose $X_{(1)} < X_{(2)} < X_{(3)}$ are the order statistics based on a random sample of size 3 from the standard exponential density $f(x) = e^{-x}, x > 0$.

i) Find $E(X_{(2)})$.

ii) Find the density of the midrange $B = \frac{1}{2}(X_{(1)} + X_{(3)})$. Using this result (or otherwise), find $P(B > 2)$.