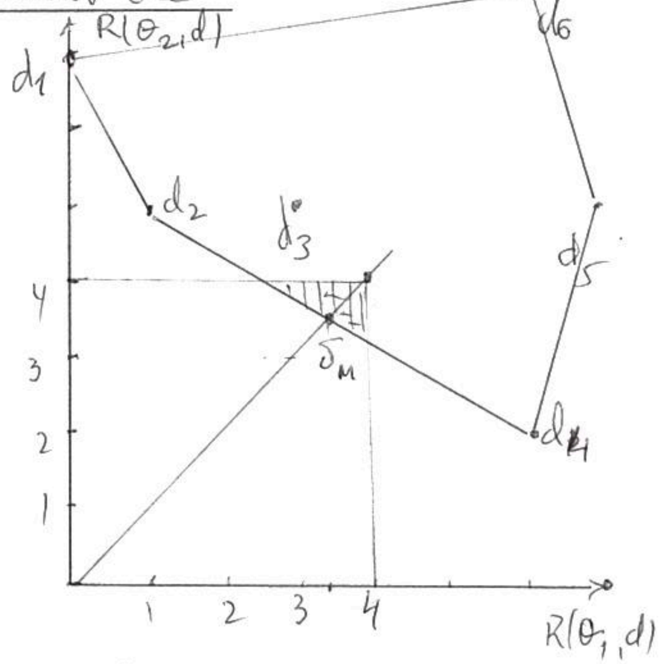
Solutions to Assignment one



Q1) a) Taking max for each of the 6 desion rules' risks we get: $(7, 5, 5, 6, 7, 8)$.
Hence both d_2 and d_3 are minimax in the set D.

b) See graph $\to$ the convex hull

c) Need to find the intersection point of

$$\left.\begin{array}{l} y = x \\ y - 2 = \dfrac{5-2}{1-6}(x-6) \end{array}\right\} \to y = x = 3.5$$

Hence the risk point δ_M of the minimax rule in $\mathcal{D}$ is $(4.5, 4.5)$

d) Looking for α which is such that

$$\begin{array}{l} \alpha * 1 + (1-\alpha)6 = 3.5 \\ \alpha * 5 + (1-\alpha)2 = 3.5 \end{array} \to \alpha = \frac{1}{2}$$

i.e. $\delta_M =$ $\nearrow$ choose d_2 w.p. $\frac{1}{2}$
$\searrow$ choose d_4 w.p. $\frac{1}{2}$

e) If the prior is $(p, 1-p)$ this leads to a line with a normal vector $(p, 1-p)$, i.e., a slope $-\frac{p}{1-p}$ and this slope should coincide with the slope of $\overline{d_2 d_4}$, i.e. $-\frac{p}{1-p} = \frac{5-2}{1-6} = -\frac{3}{5}$ should hold, i.e. $p = \frac{3}{8}$ and the least favorable prior, w.r. to which δ_M is Bayes, is $\left(\frac{3}{8}, \frac{5}{8}\right)$ on (θ_1, θ_2).

f) The line with a normal vector $\left(\frac{5}{6}, \frac{1}{6}\right)$ has a slope of -5. When moving such a line south-west as much as possible by retaining an intersection with the risk set, we end up with d_1 which is the corresponding Bayes rule. Its Bayes risk is $\frac{5}{6} * 0 + \frac{1}{6} * 7 = \underline{7/6.}$

g) See the sheded area.

i) $f(\underline{X} \mid \theta) = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^n$

$f(\underline{X} \mid \theta)\, \pi(\theta) = 3\theta^{\sum_{i=1}^{n} x_i + 2} (1-\theta)^n$. This implies that

$\pi(\theta) f(\theta \mid \underline{X}) \propto \theta^{\sum_{i=1}^{n} x_i + 2} (1-\theta)^n$ which means that the posterior of θ given the sample is $\text{Beta}\left(\sum_{i=1}^{n} x_i + 3, n+1\right)$

Then $\hat{\theta}_{Bayes}$ = mean of this distribution = $\dfrac{\sum_{i=1}^{n} x_i + 3}{\sum_{i=1}^{n} x_i + 4 + n}$

ii) Now we have $\hat{\theta} = \frac{20}{26}$ which is in the vicinity of .75 so need the effect of the prior to see if H_0 or H_1 is more relevant. The posterior is Beta(20, 6)

$$P(H_0 \mid \text{sample}) = \frac{1}{B(20,6)} \int_0^{.75} x^{19} (1-x)^5 \, dx \approx .3783$$

Hence H_0 should be rejected.
(The value of the integral has been calculated using your favorite program for numerical integration)

Q3|

$X_1, X_2, \dots, X_n$ are i.i.d. uniform in $(0, \theta)$

$\Rightarrow \quad f(x_i|\theta) = \frac{1}{\theta} I_{(x_i, \infty)}(\theta)$

Hence $\prod_{i=1}^{n} f(x_i|\theta) = \frac{1}{\theta^n} I_{(x_{(n)}, \infty)}(\theta)$

The prior $\pi(\theta) = \begin{cases} \beta\alpha^\beta \theta^{-(\beta+1)}, & \theta > \alpha > 0 \\ 0 \end{cases}$ can also be written

via indicators on "one line" as $\beta\alpha^\beta \theta^{-(\beta+1)} I_{(\alpha, \infty)}(\theta)$

The joint is a product; the product of indicators is an indicator

so we end up with

$$\prod_{i=1}^{n} f(x_i|\theta)\pi(\theta) = \frac{\beta\alpha^\beta \theta^{-(\beta+1)}}{\theta^n} I_{(\max(x_{(n)}, \alpha), \infty)}(\theta) =$$

$$= \beta\alpha^\beta \theta^{-(n+\beta+1)} I_{(\max(x_{(n)}, \alpha), \infty)}(\theta)$$

Hence the Bayes estimator w.r. quadratic loss is

$$E(\theta|\underline{X}) = \frac{\cancel{\beta\alpha^\beta} \int_{\max(x_{(n)}, \alpha)}^{\infty} \theta^{-(n+\beta)} d\theta}{\cancel{\beta\alpha^\beta} \int_{\max(x_{(n)}, \alpha)}^{\infty} \theta^{-(n+\beta+1)} d\theta} =$$

$$= \frac{(n+\beta)}{(n+\beta-1)} \max(x_{(n)}, \alpha)$$

Q4 The observation scheme: we have $n=1$ observation only from a geometric distribution with

$$f(x|\theta) = (1-\theta)^{x-1}\theta$$

where $\theta \in (0,1)$ is the probability of success in a single trial (since our data expresses the total number of trials ~~until~~ until the first success).

The two priors are $\pi_1(\theta) = 6\theta(1-\theta)$ of the aviation minister and $\pi_2(\theta) = 4\theta^3$ of the prime minister.

The two corresponding posteriors are:

$h_1(\theta/x) \propto \theta^2(1-\theta)^x$ and $h_2(\theta/x) \propto \theta^4(1-\theta)^{x-1}$

These can easily be identified as

$h_1(\theta/x) \sim \text{Beta}(3, x+1)$

$h_2(\theta/x) \sim \text{Beta}(5, x)$

We have 2 actions available: $a_0 \equiv$ continue

$a_1 \equiv$ abandon

The losses related to these actions are:

$$L(\theta, a_0) = \begin{cases} \frac{1}{2} - \theta & \text{if } \theta < \frac{1}{2} \\ 0 & \text{if } \theta \geq \frac{1}{2} \end{cases}$$ and

$$L(\theta, a_1) = \begin{cases} 0 & \text{if } \theta < \frac{1}{2} \\ \theta - \frac{1}{2} & \text{if } \theta \geq \frac{1}{2} \end{cases}$$

⑤

For an optimal Bayes decision we need to compare: $Q(x, a_0) = \int_0^{\frac{1}{2}} (\frac{1}{2} - \theta) h(\theta) d\theta = \frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta - \int_0^{1/2} \theta h(\theta|x) d\theta$ with

$$Q(x, a_1) = \int_{\frac{1}{2}}^{1} (\theta - \frac{1}{2}) h(\theta|x) d\theta = \int_{\frac{1}{2}}^{1} \theta h(\theta|x) dx - \frac{1}{2} \int_{1/2}^{1} h(\theta|x) d\theta.$$

Then a_0 would be preferred to a_1 if $Q(x, a_0) < Q(x, a_1)$ (alternatively if $Q(x, a_0) > Q(x, a_1)$ then a_1 would be preferred (and there is hesitation if $Q(x, a_0) = Q(x, a_1)$)

From the inequality

$$\frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta - \int_0^{1/2} \theta h(\theta|x) dx < \int_{1/2}^{1} \theta h(\theta|x) d\theta - \frac{1}{2} \int_{1/2}^{1} h(\theta|x) d\theta$$

we see that adding $\pm \frac{1}{2} \int_0^{\frac{1}{2}} h(\theta|x) d\theta$, noting that $\frac{1}{2} \int_0^1 h(\theta|x) d\theta = \frac{1}{2}$ and re-arranging we get

$$\cancel{\frac{1}{2} \int_0^{1/2} h(\theta|x) d\theta} - \int_0^{\frac{1}{2}} \theta h(\theta|x) d\theta < \int_{\frac{1}{2}}^{1} \theta h(\theta|x) d\theta - \frac{1}{2} + \cancel{\frac{1}{2} \int_0^{1/2} h(\theta|x) d\theta}$$

Hence $\frac{1}{2} < \int_0^1 \theta h(\theta|x) d\theta = E(\theta|x)$

In other words, we choose a_0 = continue if $E(\theta|x) > \frac{1}{2}$ (and, of course, this decision is also intuitively appealing).

Now, for a Beta (α, β) distribution, the expected value is $\frac{\alpha}{\alpha+\beta}$ which implies in our case:

$E(\theta|x) = 3/(x+4)$ for aviation minister

$E(\theta|x) = 5/(x+5)$ for prime minister

Hence the aviation minister wants the project to continue when $x = 1$, hesitates when $x = 2$ and wants to stop when $x = 3, 4, 5, \ldots$. The prime minister wants to continue when $x = 1, 2, 3, 4$; hesitates when $x = 5$ and wants to stop when $x = 6, 7, 8, \ldots$. Obviously, for $x = 3$ and $x = 4$ we have the most serious disagreement