MATH5905 - Assignment 1

Robert Clifford (z5058692)

16 August 2016

I declare that all work on this assignment is my own, unless acknolwedged. I have read and understand the University Rules in respect to Student Academic Misconduct.

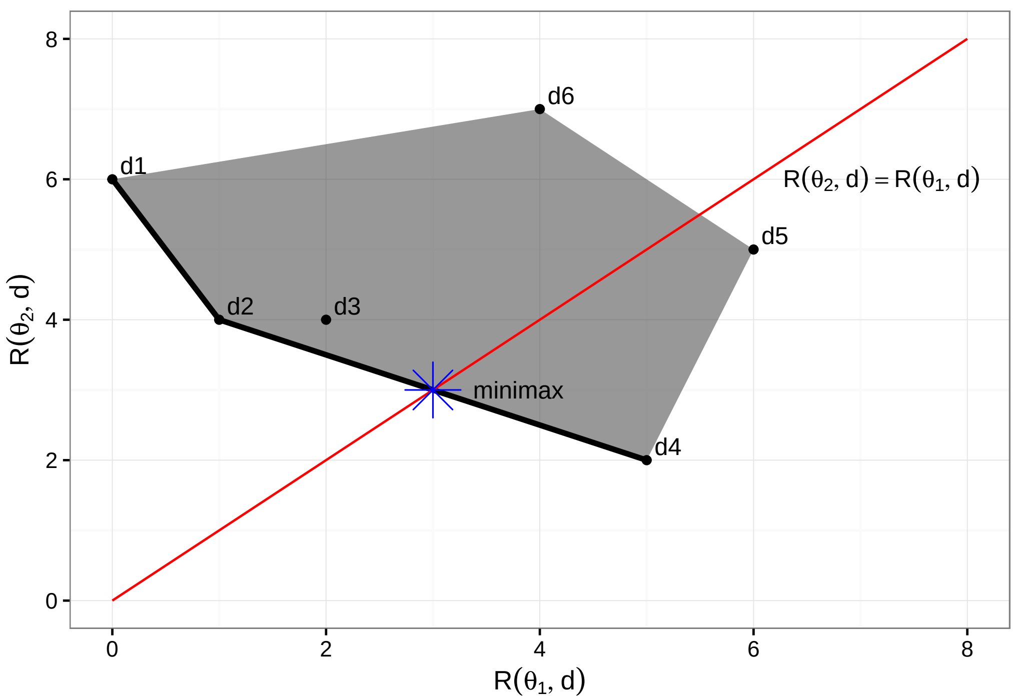
1. Consider a decision problem with parameters space $\Theta = \{\theta_1, \theta_2\}$ and a set of non randomised decisions $D = \{d_i, 1 \leq i \leq 6\}$ with risk points $\{R(\theta_1, d_i), R(\theta_2, d_i)\}$ as follows:

i	1	2	3	4	5	6
$R(\theta_1, d_i)$	0	1	2	5	6	4
$R(\theta_2, d_i)$	6	4	4	2	5	7

a) Find the minimax rule(s) amongst the **nonrandomized** rules in D;

The minimax rules are the set $\{d_2, d_3\}$ since the minimal between the 6 maximum values is $\{6, 4, 4, 5, 6, 7\}$.

b) Plot the risk set of all **randomised** rules $\mathcal{D}$ generated by the set of rules in D.



The risk set of all randomised rules generated by the rules in D is given by the shaded area generated via the convex, S. The points of the decision rules show the non-randomised decision rules, while the admissable decision rules are given along the shaded lower left boundary of points $\{d_1, d_2, d_4\}$.

c) Find the risk point of the minimax rule in $\mathcal{D}$ and determine its minimax risk.

Using points $(1, 4), (5, 2)$, the slope of the curve is $m = \frac{4-2}{1-5} = -\frac{1}{2}$. We can then use $(1, 4)$ to find the intercept, $R_2 = -0.5R_1 + c \rightarrow c = 4.5$. Therefore we have to solve for the equations $R_1 = R_2$

and $R_2 = -0.5R_1 + 4.5$ to find the intersect.

$$R_1 = -0.5R_1 + 4.5 \rightarrow R_1 = 3$$

Therefore the minimax risk is equal to 3.

d) Define the minimax rule in the set $\mathcal{D}$ in terms of rules in D.

The minimax rule is a randomised rule of the form $d^* = \alpha d_2 + (1-\alpha)d_4$, such that $R(\theta_1, d^*) = R(\theta_2, d^*)$. This point is marked in S, and represents the randomised decision rule.

We can now find alpha, to express the minimax rule in terms of the rules in D:

$$\begin{aligned} \alpha d_2 + (1-\alpha)d_4 = 3 \\ \text{Solving for } (1,4) \\ \alpha \cdot 1 + (1-\alpha) \cdot 4 = 3 \\ \alpha = \frac{1}{3} \end{aligned}$$

So we choose $d_2$with probablity$\frac{1}{3}$ And we choose $d_4$with probablity$\frac{2}{3}$

e) For which prior on $\{\theta_1, \theta_2\}$ is the minimax rule a Bayes rule?

The minimax rule occurs along the line $R_2 = -\frac{1}{2}R_1 + c$. Given the Bayes rule can be found along $\alpha R_1 + (1-\alpha)R_2 = c$, we need to set α such that:

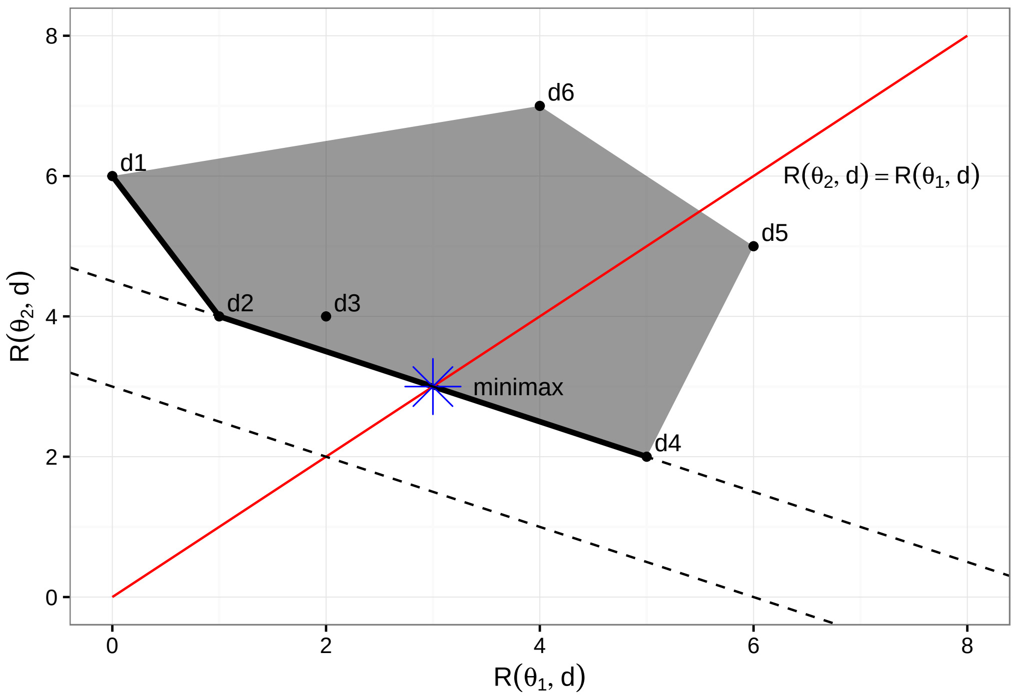
$$\frac{\alpha}{1-\alpha} = \frac{1}{2} \rightarrow \alpha = \frac{1}{3}$$

.

Then the prior is $\{\theta_1, \theta_2\} = \{\frac{1}{3}, \frac{2}{3}\}$.

$$\begin{aligned} \frac{1}{3}R_1 + \frac{2}{3}R_2 = c \\ R_2 = \frac{1}{2}R_1 + \frac{3}{2}c \end{aligned}$$

With these priors and $c = 4.5$ the Bayes rule will include the minimax rule.



f) Determine the Bayes rule and the Bayes risk for the prior $\left(\frac{4}{5}, \frac{1}{5}\right)$ on $\{\theta_1, \theta_2\}$.

The Bayes rule is determined by the straight line $\pi_1 R_1 + \pi_2 R_2 = c$ representing a class of decision rules with the same Bayes Risk. By varying c, we get the family of straight lines, choosing the one that intersects S. In this case, $c = 6$

$$\frac{4}{5}R_1 + \frac{1}{5}R_2 = c$$

Solving for the point (0,6)

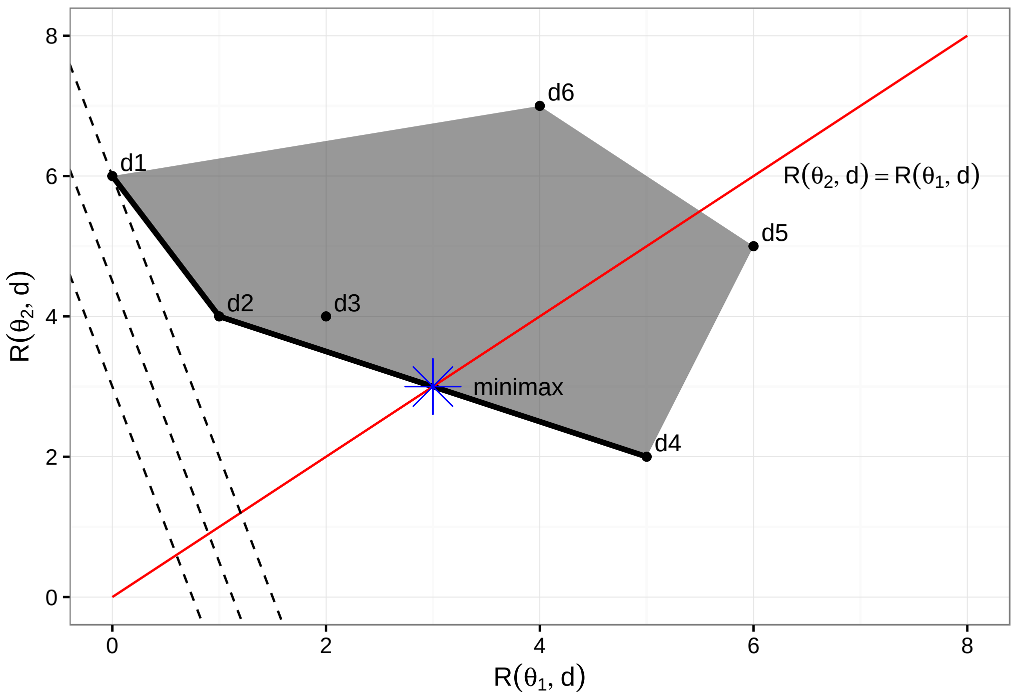
$$\frac{1}{5} \cdot 6 = c \rightarrow c = \frac{6}{5}$$

So the optimal line is:

$$\frac{4}{5}R_1 + \frac{1}{5}R_2 = \frac{6}{5}$$
$$R_2 = -4R_2 + 6$$

This intersects the admissable rules with d_1, so the bayes risk given the prior is:

$$\frac{4}{5}R(\theta_1, d_1) + \frac{1}{5}R(\theta_2, d_1) = 1.2$$



g) For $\epsilon = \frac{4}{5}$, illustrate on the risk set the risk points of all rules which are ϵ-minimax.

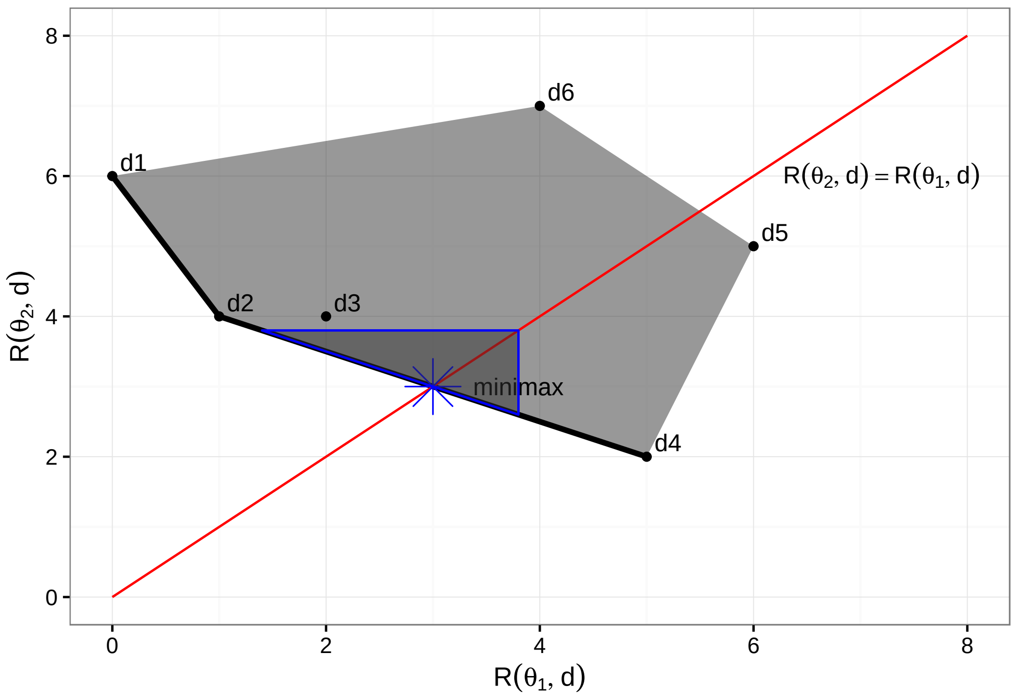
The ϵ-minimax decision rule is equal to

$$\sup_{\theta \in \Theta} R(\theta, \delta_\epsilon) \leq \inf_{\delta \in \mathcal{D}} \sup_{\theta \in \Theta} R(\theta, \delta) + \epsilon$$

The δ_ϵ rules are those those that are:

$$\sup_{\theta \in \Theta} R(\theta, \delta_\epsilon) \leq 3 + \frac{4}{5}$$

These include $\{d_2, d_3\}$



2. In a sequence of consecutive years $1, 2, ..., T$ an annual number of high-risk events is recorded by a bank. The random number N_t of high-risk events in a given year is modelled via Poisson(λ) distribution. This gives a sequence of independent counts $n_1, n_2, ..., n_T$. The prior on λ is Gamma(a,b) with known $a > 0, b > 0$:

$$\tau(\lambda) = \frac{\lambda^{a-1}e^{-\lambda/b}}{\Gamma(a)b^a}, \lambda > 0.$$

   a) Determine the Bayesian estimator of the intensity λ with respect to quadratic loss.

   The distribution of the T events is given by:

$$\begin{aligned} f(x_1, ..., x_n|\lambda) &= \frac{\lambda^{\Sigma_{i=1}^n x_i} e^{-\lambda n}}{\prod_{i=1}^n x_i!} \\ f(x_1, ..., x_n, \lambda) &= f(X|\lambda)\tau(\lambda) \\ &= \frac{\lambda^{\Sigma_{i=1}^n x_i + a - 1} e^{-\lambda(n+1/b)}}{\prod_{i=1}^n x_i!\Gamma(a)b^a} \\ h(x_1, ..., x_n) &= \int_0^\infty f(x_1, ..., x_n, \lambda)\partial\lambda \\ &= \int_0^\infty \frac{\lambda^{\Sigma_{i=1}^n x_i + a - 1} e^{-\lambda(n+1/b)}}{\prod_{i=1}^n x_i!\Gamma(a)b^a}\partial\lambda \\ &= \frac{1}{\prod_{i=1}^n x_i!\Gamma(a)b^a} \int_0^\infty \lambda^{\Sigma_{i=1}^n x_i + a - 1} e^{-\lambda(n+1/b)}\partial\lambda \end{aligned}$$

   Noticing that the integral is a gamma distribution, we can multiple by a constant and, and integrating over the entire distribution is equal to 1.

$$h(x_1, ..., x_n) = \frac{\Gamma(\Sigma_{i=1}^n x_i + a)}{\prod_{i=1}^n x_i!\Gamma(a)b^a(n+1/b)^{\Sigma_{i=1}^n x_i + a}}$$

   The conditional prior distribtuion is therefore:

$$\begin{aligned}\tau(\lambda|x_1,...,x_n) &= \frac{f(x_1,...,x_n,\lambda)}{h(x_1,...,x_n)} \\ &= \frac{\lambda^{\Sigma_{i=1}^n x_i+a-1}e^{-\lambda(n+1/b)}}{\prod_{i=1}^n x_i!\Gamma(a)b^a} \Big/ \frac{\Gamma(\Sigma_{i=1}^n x_i + a)}{\prod_{i=1}^n x_i!\Gamma(a)b^a(n+1/b)^{\Sigma_{i=1}^n x_i+a}} \\ &= \frac{(n+1/b)^{\Sigma_{i=1}^n x_i+a}}{\Gamma(\Sigma_{i=1}^n x_i + a)}\lambda^{\Sigma_{i=1}^n x_i+a-1}e^{-\lambda(n+1/b)} \sim \text{Gamma}(a+\Sigma x_i, n+1/b)\end{aligned}$$

The Bayes estimator with respect to quadratic loss is equal to the expected value of the conditional prior distribtion $E(\lambda) = \int_\Lambda \lambda\tau(\lambda|X)\partial\lambda$. For a Gamma distribution, the mean is given by $\frac{\alpha}{\beta}$. Therefore;

$$\hat{\lambda}_{\text{Bayes}} = \frac{a+\Sigma x_i}{n+1/b}$$

b) Assume $a = 3, b = 2$. Within the last seven years counts were $2, 4, 7, 3, 4, 4, 5$. Find the Baye's estimate of the intensity λ using quadratic loss and the above data.

$$\hat{\lambda}_{\text{Bayes}} = \frac{3+29}{7+1/2} = \frac{64}{15} \approx 4.267$$

c) The bank claims that the yearly intensity λ is less than 4. Test the bank's claim via Bayesian testing with a zero-one loss.

The null hypothesis is:

$$H_0 : \lambda < 4$$
$$H_1 : \lambda > 4$$

Testing the hypothesis:

$$\begin{aligned}\tau(\lambda \in \Lambda_0|X) &= \int_0^4 \frac{(n+1/b)^{\Sigma_{i=1}^n x_i+a}}{\Gamma(\Sigma_{i=1}^n x_i + a)}\lambda^{\Sigma_{i=1}^n x_i+a-1}e^{-\lambda(n+1/b)}\partial\lambda \\ &= \int_0^4 \frac{(7+1/2)^{29+3}}{\Gamma(29+3)}\lambda^{29+3-1}e^{-\lambda(7+1/2)}\partial\lambda\end{aligned}$$

```r
fn <- function(x) ((7.5)^32*x^(31)*exp(-x*(7.5)))/(gamma(32));
integrate(fn,0,4)
```

```
## 0.381357 with absolute error < 2.6e-06
```

The test is rejected if $H_0 < \frac{1}{2}$, so we reject the null hypothesis that the intensity λ is less than 4 and accept the alternative that $\lambda > 4$.

3. Let $X_1, X_2, ..., X_n$ be i.i.d. uniform in $(0, \theta)$ and let the prior on θ be the Pareto prior given by $\tau(\theta) = \beta\alpha^\beta\theta^{-(\beta+1)}, \theta > \alpha$. (Here $\alpha > 0$ and $\beta > 0$ are assumed to be known). Show that the Bayes estimator with respect to quadratic loss is given by $\hat{\theta} = \max(\alpha, x_{(n)})\frac{n+\beta}{n+\beta-1}$. Justify all steps in derivation.

$$f(x_1,...,x_n|\theta) = \theta^{-n}, \theta > x_{(n)}$$
$$\tau(\theta) = \beta\alpha^\beta\theta^{-(\beta+1)}, \theta > \alpha$$

We can calculate the conditional prior distribution:

$$\begin{aligned}
f(x_1,...,x_n,\theta) &= \beta\alpha^\beta\theta^{-(\beta+n+1)}, \theta > x_{(n)}, \alpha \\
h(x_1,...,x_n) &= \int_{x_{(n)}}^{\infty} \beta\alpha^\beta\theta^{-(\beta+n+1)}\partial\theta \\
&= \begin{cases} \frac{\beta\alpha^\beta}{(n+\beta)\alpha^{n+\beta}} = \frac{\beta}{(n+\beta)\alpha^n}, & \text{if } \alpha > x_{(n)} \\ \frac{\beta\alpha^\beta}{(n+\beta)x_{(n)}^{n+\beta}}, & \text{if } \alpha < x_{(n)} \end{cases} \\
\tau(\theta|x_1,...,x_n) &= \begin{cases} \frac{\beta\alpha^\beta\theta^{-(\beta+n+1)}}{\frac{\beta}{(n+\beta)\alpha^n}} = \alpha^{n+\beta}(n+\beta)\theta^{-(\beta+n+1)}, & \text{if } \alpha > x_{(n)} \\ \frac{\beta\alpha^\beta\theta^{-(\beta+n+1)}}{\frac{\beta\alpha^\beta}{(n+\beta)x_{(n)}^{n+\beta}}} = x_{(n)}^{n+\beta}(n+\beta)\theta^{-(\beta+n+1)}, & \text{if } \alpha > x_{(n)} \end{cases} \\
&= \max(\alpha, x_{(n)})^{n+\beta}(n+\beta)\theta^{-(\beta+n+1)} \sim \text{Pareto}(\max(\alpha, x_{(n)}), n+\beta)
\end{aligned}$$

The Bayes estimator with respect to quadratic loss is equal to the expected value of the conditional prior distribtion $E(\theta) = \int_\Theta \theta\tau(\theta|X)\partial\theta$. For a Pareto distribution, the mean is given by $\frac{\alpha\beta}{\alpha-1}$. Therefore;

$$\hat{\theta} = \max(\alpha, x_{(n)})\frac{n+\beta}{n+\beta-1}$$

4. At a critical stage in the development of a new aeroplane in the UK (jointly contracted with the French), a decision must be taken to continue or to abandon the project. The financial viability of the project can be measured by a parameter $\theta \in (0,1)$, the project being profitable if $\theta > \frac{1}{2}$, the cost to the taxpayer of continuing the project is $(\frac{1}{2} - \theta)$(in units of \$ billion) whereas if $\theta > \frac{1}{2}$, it is zero (since the project will be privatised if profitable). Data x provides information about θ : the prototype aeroplane is subjected to trials, each independently having probabliity θ of success, and the data x consists of the total number of trials requried for the first successful result to be obtained (in other wrods, we observe a single realisation of a geometrically distributed random variable). If $\theta > \frac{1}{2}$ the cost of abandoning the project is $(\theta - \frac{1}{2})$(due to contractual arrangements for purchasing the aeroplane from the French), whereas if $\theta < \frac{1}{2}$ it is zero. Two actions are on the table: a_0 : continue the project and a_1 : abandon it.

   i) Derive the Bayesian decision rule, for a given prior on θ. In particular, show that the rule is equivalent to comparing the posterior mean of θ given x with a threshold constant.

   There are two actions: a_0: continue the project a_1: abandon the project.

   The cost of continueing: $L(\theta, a_0) = \begin{cases} 0, & \text{if } \theta > \frac{1}{2} \\ \frac{1}{2} - \theta, & \text{if } \theta < \frac{1}{2} \end{cases}$

   The cost of abandoning: $L(\theta, a_1) = \begin{cases} \theta - \frac{1}{2}, & \text{if } \theta > \frac{1}{2} \\ 0, & \text{if } \theta < \frac{1}{2} \end{cases}$

   Project will be continued if the cost is less than abandoning.

$$E_\theta\left(L(\theta, a_0)\right) > E_\theta\left(L(\theta, a_1)\right)$$
$$\int_0^{\frac{1}{2}} \left(\frac{1}{2} - \theta\right)\tau\left(\theta|x\right)\partial\theta > \int_{\frac{1}{2}}^1 \left(\theta - \frac{1}{2}\right)\tau\left(\theta|x\right)\partial\theta$$

$$0 > \int_0^1 \left(\theta - \frac{1}{2} - \frac{1}{2} + \theta\right) \tau\left(\theta|x\right) \partial\theta$$
$$0 > \int_0^1 \left(\theta - \frac{1}{2}\right) \tau\left(\theta|x\right) \partial\theta$$
$$0 > \int_0^1 \left(\theta\right) \tau\left(\theta|x\right) \partial\theta - \frac{1}{2}$$
$$\frac{1}{2} > \int_0^1 \left(\theta\right) \tau\left(\theta|x\right) \partial\theta = \mu\left(\theta|x\right)$$

ii) The Minister of Aviation has prior density $6\theta(1-\theta)$ for θ. The prime minister has prior density $4\theta^3$. For which values of x will there be most serious ministerial disagreement?

There will be disagreement when one Minister wants to abandon the project, while the other wants to continue it.

The conditional distribution on x is

$$f\left(x;\theta\right) = \theta\left(1-\theta\right)^{x-1}, x = 1, 2, ...$$

Both the Minister for Aviation and the Prime Minster have beta prior distributions.
For the Minister for Aviation (MA):

$$\tau_{MA}\left(\theta\right) = 6\theta(1-\theta) \sim Beta(2,2)$$

So calculating the conditonal prior distribution:

$$\begin{aligned} f(x,\theta) &= 6\theta^2(1-\theta)^x \\ h(x) &= \int_0^1 6\theta^2(1-\theta)^x \partial\theta \\ &= 6 \cdot \mathrm{beta}(3, x+1) \\ \tau(\theta|x) &= \frac{6\theta^2(1-\theta)^x}{6 \cdot \mathrm{beta}(3, x+1)} \\ &= \frac{1}{\mathrm{beta}(3, x+1)}\theta^2(1-\theta)^x \sim \mathrm{Beta}(3, x+1) \end{aligned}$$

The prior distribtuion for the Prime Minister (PM):

$$\tau_{PM}\left(\theta\right) = 4\theta^3 \sim Beta(5,1)$$

So calculating the conditional prior distribution:

$$\begin{aligned} f(x,\theta) &= 4\theta^4(1-\theta)^{x-1} \\ h(x) &= \int_0^1 4\theta^4(1-\theta)^{x-1} \cdot \partial\theta \\ &= 4 \cdot \mathrm{beta}(5, x) \\ \tau(\theta|x) &= \frac{4\theta^4(1-\theta)^{x-1}}{4 \cdot \mathrm{beta}(5, x)} \\ &= \frac{1}{\mathrm{beta}(5, x)}\theta^4(1-\theta)^{x-1} \sim \mathrm{Beta}(5, x) \end{aligned}$$

If $x \sim \mathrm{Beta}(a, b)$ then $E(x) = \frac{a}{a+b}$.

The Minister of Aviation will want to abandon if $\frac{3}{4+x} < \frac{1}{2}$, ie $x > 2$ with $x = 2$ being on the edge. The Prime Minster will want to abandon if $\frac{5}{5+x} < \frac{1}{2}$, ie $x > 5$ with $x = 5$ being on the edge. Therefore there will be the greatest amount of disagreement when $x \in \{3, 4\}$.