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Question 1 We let $\mathbf{X} = (X_1, X_2, ..., X_n)$ be a sample of n observations each with a uniform density in $[0, \theta)$

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

Where $\theta > 0$ is an unknown parameter. The CDF of this function is

$$F(x,\theta) = \begin{cases} 0 & \text{if } x < 0 \\ \int_0^x \frac{1}{\theta} dx = \frac{x}{\theta} & \text{if } 0 < x < \theta \\ 1 & x > \theta \end{cases}$$

(a) Denoting the joint density as $L(X, \theta)$, we can show that the family $\{L(X,\theta)\}, \theta > 0$ has a monotone likelihood ratio in $X_{(n)}$. First, rewriting the density as

$$f(x,\theta) = \frac{1}{\theta} I_{[0,\theta]}(x)$$

Then the likelihood function is

$$L(X,\theta) = \prod_{i=1}^{n} f(x_i,\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I_{[0,\theta]}(x_i) = \frac{1}{\theta^n} \prod_{i=1}^{n} I_{[0,\theta]}(x_i)$$

Which is true if and only if $\theta > max[x_1, ..., x_n]$, denoted as $X_{(n)}$, the value of the largest order statistic. The likelihood function is,

$$L(X,\theta) = \frac{1}{\theta^n} I_{[0,\theta]}(X_{(n)})$$

For $\theta_1 < \theta_2$,

$$\frac{L(X,\theta_2)}{L(X,\theta_1)} = \frac{\frac{1}{\theta_2^n} I_{[0,\theta_2]}(X_{(n)})}{\frac{1}{\theta_1^n} I_{[0,\theta_1]}(X_{(n)})} = (\frac{\theta_1}{\theta_2})^n \frac{I_{[0,\theta_2]}(X_{(n)})}{I_{[0,\theta_1]}(X_{(n)})}$$

Therefore,

$$\frac{L(X,\theta_2)}{L(X,\theta_1)} = (\frac{\theta_1}{\theta_2})^n \times \begin{cases} 1 & 0 \le X_{(n)} \le \theta_1 \\ \infty & \theta_1 \le X_{(n)} \le \theta_2 \end{cases}$$

Which is a monotone non-decreasing function of $T(X) = X_{(n)}$. ✓

(b) Given the density function of $\mathbf{X} = (X_1, X_2, ..., X_n)$ is in a family with monotone likelihood ratio in $T(X) = X_{(n)}$, then for testing $H_0 : \theta \le 2$ versus $H_1 : \theta > 2$, the UMP level α-test is given by

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 & \text{if } X_{(n)} > k \\ 0 & \text{if } X_{(n)} \le k \end{cases}$$

To find k, we need the distribution of $x_{(n)}$. Using the fact that by independence of the observations,

$$P_\theta(X_{(n)} > k) = (P_\theta(X_1 > k))^n$$

Also, $P_\theta(X_1 > k) = 1 - P(X_1 \le k) = 1 - F(k,\theta)$, then using the CDF function from part a,

$$P_\theta(X_1 > k) = \begin{cases} 1 - 0 = 1 & \text{if } k < 0 \\ 1 - \frac{k}{\theta} & \text{if } 0 < k < \theta \\ 1 - 1 = 0 & \text{if } k > \theta \end{cases}$$

Hence

$$E_\theta \varphi^* = P_\theta(X_{(n)} > k) = (P_\theta(X_1 > k))^n = \begin{cases} 1 & \text{if } k < 0 \\ 1 - (\frac{k}{\theta})^n & \text{if } 0 < k < \theta \\ 0 & \text{if } k > \theta \end{cases}$$

Therefore under the null hypothesis, $\theta = \theta_0 = 2$, so to solve for k we solve the equation $E_{\theta_0}\varphi^* = 1-(\frac{k}{\theta})^n = \alpha$,

$$\alpha = 1 - (\frac{k}{2})^n$$

$$1 - \alpha = (\frac{k}{2})^n$$

$$[2^n(1-\alpha)]^{\frac{1}{n}} = [k^n]^{\frac{1}{n}}$$

$$2(1-\alpha)^{\frac{1}{n}} = k$$

Hence, the UMP α-size test is

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 & \text{if } X_{(n)} > 2(1-\alpha)^{\frac{1}{n}} \\ 0 & \text{if } X_{(n)} \leq 2(1-\alpha)^{\frac{1}{n}} \end{cases}$$

(c) The power function is defined as part b with

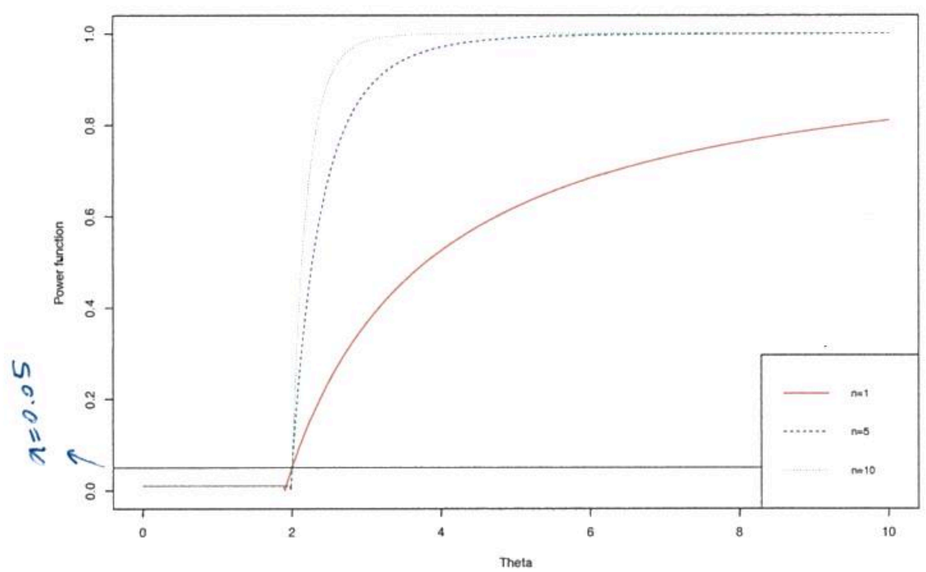
$$E_\theta\varphi^* = 1 - (\frac{k}{\theta})^n$$

Substituting in $k = 2(1-\alpha)^{\frac{1}{n}}$,

$$E_\theta\varphi^* = 1 - \frac{[2(1-\alpha)^{\frac{1}{n}}]^n}{\theta^n} = 1 - (\frac{2}{\theta})^n(1-\alpha)I_{(k,\infty)}(\theta)$$

With $E_\theta\varphi^* = \alpha$, if $\theta = 2$. Plotting this, the power function increases as θ increases, approaching 1 as θ approaches ∞. As n gets larger, the function approaches one very quickly.

Figure 1: Power Function - Uniform



(d) First, from part c, $f_{X_{(n)}}(x) = \frac{nx^{n-1}}{\theta^n}, 0 < x < \theta$, with

$$F_{X_{(n)}}(x) = \begin{cases} 0 & \text{if } x < 0 \\ (\frac{x}{\theta})^n & \text{if } 0 \le x < \theta \\ 1 & \text{if } x \ge \theta \end{cases}$$

Defining random variable $Y_n = n(1 - \frac{X_{(n)}}{\theta})$, then the distribution of Y_n is

$$F_{Y_n}(y) = P[Y_n \le y] = P[n(1 - \frac{X_{(n)}}{\theta}) \le y]$$

$$= P[1 - \frac{X_{(n)}}{\theta} \le \frac{y}{n}]$$

$$= P[X_{(n)} \ge \theta(1 - \frac{y}{n})]$$

$$= 1 - P[X_{(n)} < \theta(1 - \frac{y}{n})]$$

$$= 1 - P[X_1 \le \theta(1 - \frac{y}{n})] \times P[X_2 \le \theta(1 - \frac{y}{n})] \times ... \times P[X_n \le \theta(1 - \frac{y}{n})]$$

By mutual independence of X_i, then

$$1 - F_{X_1}(\theta(1 - \frac{y}{n})) \times F_{X_2}(\theta(1 - \frac{y}{n})) \times ... \times F_{X_n}(\theta(1 - \frac{y}{n}))$$

$$= 1 - [F_{X_n}(\theta(1 - \frac{y}{n}))]^n$$

Therefore,

$$F_{Y_n}(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - \frac{(\theta(1-\frac{y}{n}))}{\theta}^n = 1 - (1 - \frac{y}{n})^n & \text{if } 0 \le y < n \\ 1 & \text{if } y \ge n \end{cases}$$

Since,

$$lim_{n \to \infty}(1 - \frac{y}{n})^n = e^{-y}$$

Then,

$$lim_{n \to \infty} F_{Y_n}(y) = F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-y} & \text{if } y \ge 0 \end{cases}$$

Where $F_Y(y)$ is the distribution function of an exponential random variable with mean of one. Therefore Y_n converges in distribution to an exponential random variable with mean of one as $n \to \infty$. We can evaluate $P(|X_{(n)} - \theta| < \epsilon)$ directly by noting that $X_{(n)}$ cannot possible be larger than θ, so

$$P(|X_{(n)} - \theta| < \epsilon) = P(X_{(n)} > \theta - \epsilon) = 1 - P(X_{(n)} \le \theta - \epsilon)$$

The maximum $X_{(n)}$ is less than some constant if and only if each of the random variables $X_1, ..., X_n$ is less than that constant. Therefore, since the X_i are i.i.d.,

$$P(X_{(n)} \le \theta - \epsilon) = [P(X_1 \le \theta - \epsilon)]^n = \begin{cases} [1 - (\epsilon/\theta)]^n & \text{if } 0 < \epsilon < \theta \\ 0 & \text{if } \epsilon \ge \theta \end{cases}$$

Since $1 - (\epsilon/\theta)$ is strictly less than 1, we conclude that no matter what positive value ϵ takes,

$$P(X_{(n)} \le \theta - \epsilon) \to 0$$

as desired.

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Question 2 We let $\mathbf{X} = (X_1, X_2, ..., X_n)$ be i.i.d. variables with density

$$f(x,\theta) = \begin{cases} \frac{2}{\theta} x e^{\frac{-x^2}{\theta}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Where $\theta > 0$ is an unknown parameter.

(a) We can find the information matrix using the following formula

$$I(\theta) = -E(\frac{\partial^2 \log L(X,\theta)}{\partial \theta^2}) \quad (1)$$

First, finding the likelihood function for one observation (i.e. X_1)

$$L(X,\theta) = \frac{2X}{\theta} e^{\frac{-X^2}{\theta}}$$

With a log-likelihood function as

$$\log L(X,\theta) = \log\{\frac{2X}{\theta} e^{\frac{-X^2}{\theta}}\}$$

$$= \log 2 - \log \theta + \log e^{-\frac{1}{\theta}X^2} + \log X$$

$$= \log 2 - \log \theta - \frac{X^2}{\theta} + \log X$$

Taking the first derivative w.r.t to θ,

$$\frac{\partial \log L(X,\theta)}{\partial \theta} = -\frac{1}{\theta} + \frac{X^2}{\theta^2}$$

Then the second derivative w.r.t to θ is

$$\frac{\partial^2 \log L(X,\theta)}{\partial \theta^2} = \frac{1}{\theta^2} - \frac{2X^2}{\theta^3}$$

Substituting this into (1), yields

$$I(\theta) = -E(\frac{1}{\theta^2} - \frac{2X^2}{\theta^3}) = -\frac{1}{\theta^2} + \frac{2}{\theta^3} E[X^2]$$

To find $E[X^2]$, we can prove the square of a Rayleigh(θ) random variable is an Exponential(θ) random variable. Considering the transformation $Y = g(X) = X^2$ is a 1-1 transformation from $X = \{x|x > 0\}$ to $Y = \{y|y > 0\}$ with inverse $X = g^{-1}(Y) = \sqrt{y}$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{2} Y^{-\frac{1}{2}} = \frac{1}{2\sqrt{Y}}$$

Therefore by the transformation technique, the probability density function of Y is

$$f_Y(y) = f_X(g^{-1}(y))|\frac{dx}{dy}|$$

$$= \frac{2\sqrt{y}}{\theta} e^{-\sqrt{y}^2/\theta} |\frac{1}{2\sqrt{y}}|$$

$$= \frac{1}{\theta} e^{-y/\theta}, y > 0$$

Which is the density function of the exponential distribution with expectation of θ, i.e. $X^2 \approx Exp(\theta)$ and $E[X^2] = \theta$. Therefore,

$$I(\theta) = -\frac{1}{\theta^2} + \frac{2\theta}{\theta^3} = -\frac{1}{\theta^2} + \frac{2}{\theta^2} = \frac{1}{\theta^2}$$

For a sample of n i.i.d. observations,

$$I_n(\theta) = nI(\theta) = \frac{n}{\theta^2}$$ ✓

(b) To find the MLE, we find the likelihood function for the sample as

$$L(X,\theta) = \prod_{i=1}^{n} \frac{2x_i}{\theta} e^{\frac{-x_i^2}{\theta}} = \frac{2^n}{\theta^n} e^{\frac{-\sum_{i=1}^{n} x_i^2}{\theta}} \sum_{i=1}^{n} x_i$$

With a log-likelihood function as

$$\log L(X,\theta) = \log\{\frac{2^n}{\theta^n} e^{\frac{-\sum_{i=1}^{n} x_i^2}{\theta}} \sum_{i=1}^{n} x_i\}$$

$$= \log 2^n - \log\theta^n + \log e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i^2} + \log\sum_{i=1}^{n} x_i$$

$$= n\log 2 - n\log\theta - \frac{\sum_{i=1}^{n} x_i^2}{\theta} + \sum_{i=1}^{n}\log x_i$$

Taking the first derivative w.r.t to θ,

$$\frac{\partial \log L(X,\theta)}{\partial\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^2}$$

Setting to 0 to solve for the MLE, yields

$$-\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^2} = 0$$

$$\frac{\sum_{i=1}^{n} x_i^2}{\theta^2} = \frac{n}{\theta}$$

$$\theta\sum_{i=1}^{n} x_i^2 = n\theta^2$$

$$\sum_{i=1}^{n} x_i^2 = n\theta$$

Therefore the MLE of θ is,

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i^2}{n}$$ ✓

See part (a), we know the square of a rayleigh-distributed random variable follows an exponential distribution with parameter θ, then

$$E[\hat{\theta}] = E[\frac{\sum_{i=1}^{n} x_i^2}{n}] = \frac{\sum_{i=1}^{n} E[x_i^2]}{n} = \frac{n\theta}{n} = \theta$$

Therefore, the MLE of θ is unbiased. ✓ To obtain the variance of the MLE,

$$V[\hat{\theta}] = V[\frac{1}{n}\sum_{i=1}^{n} x_i^2] = \frac{1}{n^2}\sum_{i=1}^{n} V[x_i^2] = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}$$

Since the variance of an exponential distribution with parameter θ is θ^2. The Cramer-Rao bound is

$$\frac{1}{nI(\theta)} = \frac{1}{n\frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

Therefore, the MLE of $\hat{\theta}$ attains the Cramer-Rao lower bound. ✓

(c) The asymptotic distribution of $\hat{\theta}$ is

$$\sqrt{n}(\hat{\theta}-\theta) \to N(0, \frac{1}{I(\theta)})$$

Where $\frac{1}{I(\theta)} = \theta^2$. From lecture notes, page 43, $\hat{\theta} \approx N(\theta, \frac{1}{nI(\theta)})$. Using the results from part a and b,

$$\hat{\theta} \approx N(\theta, \frac{\theta^2}{n})$$

(d) First, rewriting the density function as follows,

$$f(x,\theta) = \frac{2}{\theta}xe^{-\frac{x^2}{\theta}} = \frac{2}{\theta}xe^{\frac{1}{\theta}-x^2}$$

This is in the form of a one-parameter exponential family, $a(\theta)b(x)e^{[c(\theta)d(x)]}$, with

$$a(\theta) = \frac{2}{\theta}$$

$$b(x) = x$$

$$c(\theta) = -\frac{1}{\theta}$$

$$d(x) = x^2$$

Clearly this family has MLR in $T(X) = \sum_{i=1}^n d(X_i) = \sum_{i=1}^n x_i^2$ because $c(\theta)$ is increasing in θ. This also means T(X) is minimal sufficient.

(e) Given $T(X) = \sum_{i=1}^n x_i^2$ is a sufficient statistic for θ and the family is an MLR family, then testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ exists and the UMP level α-size test is given by rejecting H_0 if and only if $T(X) > t_0$ where $\alpha = P[T(X) > t_0|\theta_0]$. Hence, the UMP α-size test is

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n X_i^2 \geq k \\ 0 & \text{if} \sum_{i=1}^n X_i^2 < k \end{cases}$$

This can be evaluated as an exact test using the Gamma distribution, that is, using the fact that $\sum_{i=1}^n x_i^2 \approx Gamma(n,\theta)$. This becomes difficult to integrate, so we can use the fact that $\hat{\theta} = \frac{\sum_{i=1}^n x_i^2}{n} \approx N(\theta, \frac{\theta^2}{n})$, from part c,

$$P[\sum_{i=1}^n x_i^2 > k] = \alpha$$

$$= P[\frac{\sum_{i=1}^n x_i^2}{n} > \frac{k}{n}] = \alpha$$

Under the null hypothesis, $\theta = \theta_0$,

$$P[\frac{\frac{\sum_{i=1}^n x_i^2}{n} - \theta_0}{\theta_0/\sqrt{n}} \geq \frac{\frac{k}{n} - \theta_0}{\theta_0/\sqrt{n}}] = \alpha$$

So that $Z = \frac{\frac{\sum_{i=1}^n x_i^2}{n} - \theta_0}{\theta_0/\sqrt{n}}$ is approximately $N(0,1)$. When H_0 is true,

$$\Phi = [\frac{\frac{\sum_{i=1}^n x_i^2}{n} - \theta_0}{\theta_0/\sqrt{n}}] = 1 - \alpha \to (\frac{\frac{\sum_{i=1}^n x_i^2}{n} - \theta_0}{\theta_0/\sqrt{n}}) = z_\alpha$$

Now solving for k,

$$z_\alpha = \frac{\frac{k}{n} - \theta_0}{\theta_0/\sqrt{n}}$$

$$\frac{k}{n} = \theta_0 + \frac{z_\alpha \theta_0}{\sqrt{n}}$$

$$k = n\theta_0 + \sqrt{n} z_\alpha \theta_0 = \theta_0(n + \sqrt{n} z_\alpha)$$

Therefore the UMP α-size test is

$$\varphi^*(\mathbf{X}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n X_i^2 \geq n\theta_0 + \sqrt{n} z_\alpha \theta_0 \\ 0 & \text{if} \sum_{i=1}^n X_i^2 < n\theta_0 + \sqrt{n} z_\alpha \theta_0 \end{cases}$$

(f) From part e, the power function can be obtained using the normal approximation as

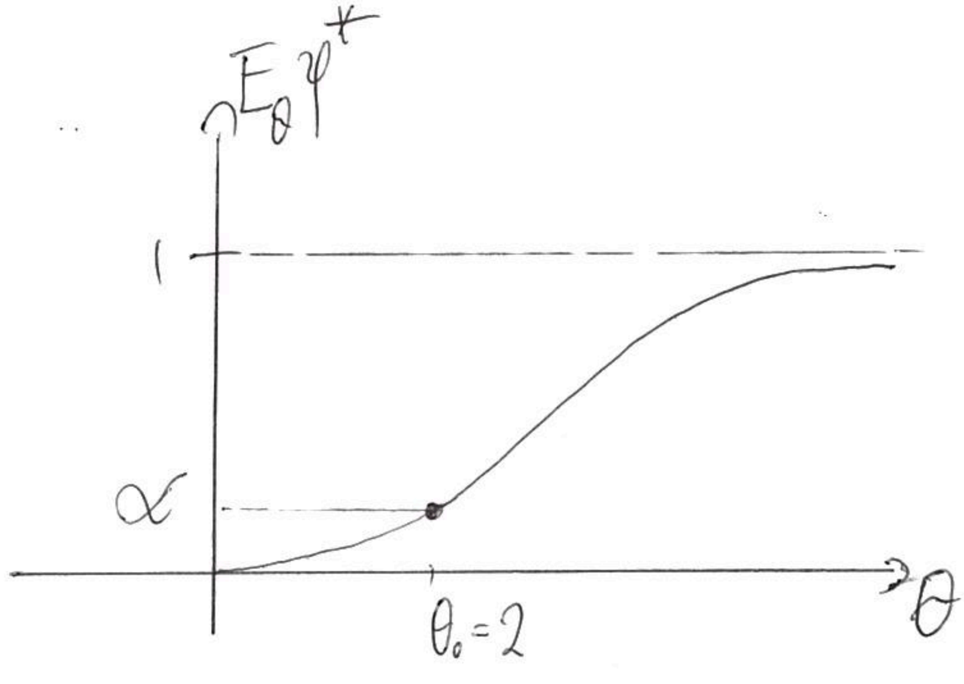
$$E_\theta \varphi^* \approx P[\frac{\sum_{i=1}^n X_i^2}{n} > \theta_0 + \frac{z_\alpha \theta_0}{\sqrt{n}})]$$

$$= P[\frac{\frac{\sum_{i=1}^n X_i^2}{n} - \theta}{\theta/\sqrt{n}} > \frac{\theta_0 + \frac{z_\alpha \theta_0}{\sqrt{n}} - \theta}{\theta/\sqrt{n}}]$$

$$= P[Z > \frac{\theta_0 + \frac{z_\alpha \theta_0}{\sqrt{n}} - \theta}{\theta/\sqrt{n}}] = 1 - P[Z \leq \frac{\theta_0 + \frac{z_\alpha \theta_0}{\sqrt{n}} - \theta}{\theta/\sqrt{n}}]$$

$$= 1 - \Phi(\frac{\frac{k}{n} - \theta}{\theta/\sqrt{n}})$$

As an example, setting $\theta_0 = 2$, and setting $\alpha = 0.05$ so that $z_{1-0.05/2} = 1.96$, then the power function can be plotted as follows. The power function increases as θ increases, approaching 1 as θ approaches ∞. As n gets larger, the function approaches one very quickly.



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Question 3 Please see MathStatica output attached for answers to Question 3, parts a, b and c as marked on the output.

Question 4

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(a) The order statistics, $X_{(1)} < X_{(2)} < X_{(3)}$, are based on a random sample size with $n = 3$, from the standard exponential family distribution with density function

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The CDF of this function is

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \int_0^x e^{-x}dx = 1 - e^{-x} & x > 0 \end{cases}$$

(a) Using Theorem 7.3, given the order statistics are from a continuous population with cdf $F(X)$ and pdf $f(x)$ as defined, then the pdf of $X_{(j)}$ is

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)[F(x)]^{j-1}[1 - F(x)]^{n-j}$$

The pdf of $X_{(2)}$, is

$$\begin{aligned} f_{X_{(2)}}(x) &= \frac{3!}{(2-1)!(3-2)!}(e^{-x})[1 - e^{-x}]^{2-1}[1 - (1 - e^{-x})]^{3-2}, x > 0 \\ &= \frac{3!}{1!^2}(e^{-x})[1 - e^{-x}](e^{-x}) \\ &= \frac{3 \times 2 \times 1}{1 \times 1}(e^{-2x})[1 - e^{-x}] \\ &= 6[e^{-2x} - e^{-3x}] \quad \checkmark \end{aligned}$$

With,

$$f_{X_{(2)}}(x) = \begin{cases} 6[e^{-2x} - e^{-3x}] & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

We can now find the expectation of the second order statistic,

$$E[X_{(2)}] = \int_0^\infty 6x[e^{-2x} - e^{-3x}]dx$$

Because of linearity we can evaluate the following integral terms separately

$$= \int_0^\infty 6xe^{-2x}dx - \int_0^\infty 6xe^{-3x}dx, \ (1)$$

First, solving $\int_0^\infty 6xe^{-2x}dx$ by integration by parts ($\int uv' = uv - \int u'v$),

$$\mathbf{u} = x, \mathbf{u'} = \frac{d}{dx}x = 1, \mathbf{v'} = e^{-2x}, \mathbf{v} = \int_0^\infty e^{-2x}dx = -\frac{e^{-2x}}{2}$$

Then this integral becomes,

$$\begin{aligned} &= -\frac{xe^{-2x}}{2} - \int_0^\infty -\frac{e^{-2x}}{2}dx \\ &= -\frac{xe^{-2x}}{2} + \frac{1}{2}\int_0^\infty e^{-2x}dx \end{aligned}$$

$$= -\frac{xe^{-2x}}{2} + \frac{1}{2} \times -\frac{e^{-2x}}{2}$$
$$= -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}$$

Then, solving $\int_0^\infty 6xe^{-3x}dx$, in a similar manner

$$\mathbf{u} = x, \mathbf{u'} = \frac{d}{dx}x = 1, \mathbf{v'} = e^{-3x}, \mathbf{v} = \int_0^\infty e^{-3x}dx = -\frac{e^{-3x}}{3}$$

Then this integral becomes,

$$= -\frac{xe^{-3x}}{3} - \int_0^\infty -\frac{e^{-3x}}{3}dx$$
$$= -\frac{xe^{-3x}}{3} + \frac{1}{3}\int_0^\infty e^{-3x}dx$$
$$= -\frac{xe^{-3x}}{3} + \frac{1}{3} \times -\frac{e^{-3x}}{3}$$
$$= -\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9}$$

Plugging these integrals back into (1) and evaluating

$$= 6[-\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} - (-\frac{xe^{-3x}}{3} - \frac{e^{-3x}}{9})]_0^\infty$$
$$= [-3xe^{-2x} - \frac{3}{2}e^{-2x} + 2xe^{-3x} + \frac{2}{3}e^{-3x}]_0^\infty$$
$$= -(-\frac{3}{2}e^0 + \frac{2}{3}e^0)$$
$$= \frac{3}{2} - \frac{2}{3}$$
$$= \frac{5}{6}$$

✓

(b) First, we can find the joint density of $X_{(1)}$ and $X_{(n)}$ using Theorem 7.3,

$$f_{X_{(i)},X_{(j)}}(u,v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!}f_x(u)f_x(v)[F_x(u)]^{i-1}[F_x(v)-F_x(u)]^{j-1-i}[1-F_x(v)]^{n-j}, 0 < u < v < \infty$$
$$= \frac{3!}{0!1!0!}f_x(x_{(1)})f_x(x_{(3)})[F_x(x_{(3)}) - F_x(x_{(1)})]$$
$$= 6(e^{-x_{(1)}})(e^{-x_{(3)}})[1 - e^{-x_{(3)}} - (1 - e^{-x_{(1)}})]$$
$$= 6e^{-x_{(1)}-x_{(3)}}[e^{-x_{(1)}} - e^{-x_{(3)}}]$$
$$f_{X_{(1)},X_{(n)}}(x_{(1)}, x_{(3)}) = 6[e^{-2x_{(1)}-x_{(3)}} - e^{-x_{(1)}-2x_{(3)}}], 0 < x_{(1)} < x_{(3)} < \infty$$

✓

We are given the transformation,

$$B = \frac{X_{(1)} + X_{(n)}}{2} = \frac{X_{(1)} + X_{(3)}}{2}$$

and defining

$$W = X_{(1)}$$

then solving for $X_{(1)}$ and $X_{(3)}$ yield

$$X_{(1)} = W$$
$$X_{(3)} = 2B - X_{(1)} = 2B - W$$

The value of the Jacobian is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial X_{(1)}}{\partial B} = 0 & \frac{\partial X_{(1)}}{\partial W} = 1 \\ \frac{\partial X_{(3)}}{\partial B} = 2 & \frac{\partial X_{(3)}}{\partial W} = -1 \end{bmatrix}$$

The determinant of this matrix is

$$\frac{\partial X_{(1)}}{\partial B} \times \frac{\partial X_{(3)}}{\partial W} - \frac{\partial X_{(1)}}{\partial W} \times \frac{\partial X_{(3)}}{\partial B} = |(0 \times 1) - (2 \times 1)| = |-2|$$

Therefore, the joint density of B and W becomes

$$f_{B,W}(b,w) = 6[e^{-2W-(2B-W)} - e^{-W-2(2B-W)}]|2|$$

$$f_{B,W}(b,w) = 12[e^{-2B-W} - e^{-4B+W}]$$

The relationship $0 < x_{(1)} < x_{(3)} < \infty$ transfers into $0 < W < 2B - W < \infty$, which is equivalent to the domain of $0 < W < B$ for W. Hence the density of $B = \frac{X_{(1)}+X_{(3)}}{2}$ is

$$f_B(b) = \int_0^B 12[e^{-2B-W} - e^{-4B+W}]dW$$

Applying the linearity rule:

$$= 12e^{-2B}\int_0^B e^{-W}dW - 12e^{-4B}\int_0^B e^{W}dW$$

$$= 12e^{-2B}(-e^{-B}+1) - 12e^{-4B}(e^{B}-1)$$

$$= 12e^{-2B} - 12e^{-3B} + 12e^{-4B} - 12e^{-3B}$$

$$= 12e^{-4B} + 12e^{-2B} - 24e^{-3B}$$

Now, evaluating $P[B > 2]$,

$$P[B > 2] = \int_2^\infty 12e^{-4B} + 12e^{-2B} - 24e^{-3B}dB$$

$$= [-\frac{1}{4}12e^{-4B} - \frac{1}{2}12e^{-2B} + \frac{1}{3}24e^{-3B}]_2^\infty$$

$$= -(-3e^{-4(2)} - 6e^{-2(2)} + 8e^{-3(2)})$$

$$= 3e^{-8} + 6e^{-4} - 8e^{-6} \quad \checkmark$$

Therefore,

$$P[B > 2] = 0.09107 \quad \checkmark$$

Question 3 (a)

```
In[12]:= << mathStatica.m
f = PDF[NormalDistribution[], x]; domain[f] = {x, -∞, ∞};
g = OrderStat[r, f, 3]
domain[g] = OrderStatDomain[r, f, 3]
g /. r → {1, 2, 3} // Simplify
Prob[y, g /. r → {1, 2, 3}]
PlotDensity[g /. r → {1, 2, 3}]
Expect[x, g /. r → {1, 2, 3}]
```

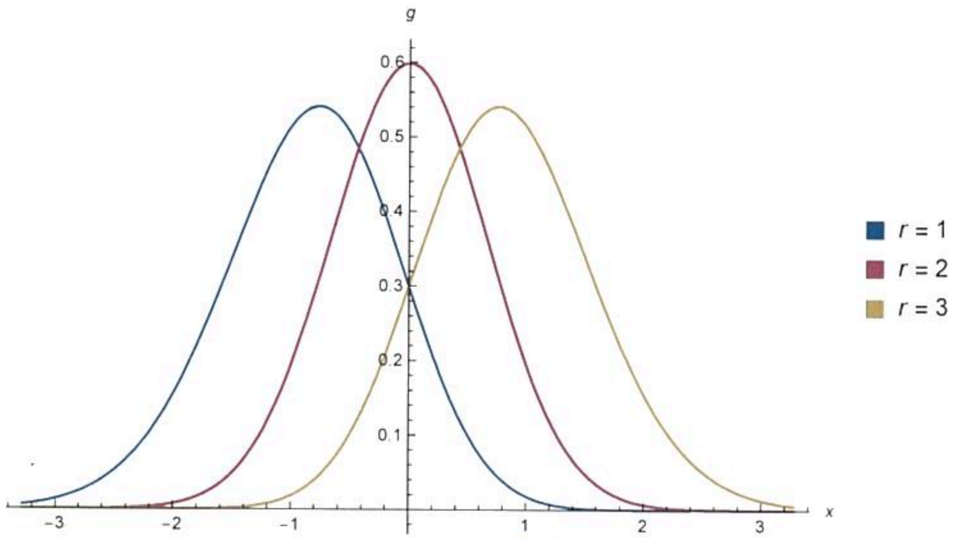
Out[14]= $$\frac{3\, e^{-\frac{x^2}{2}} \left(1 - \mathrm{Erf}\left[\frac{x}{\sqrt{2}}\right]\right)^{3-r} \left(1 + \mathrm{Erf}\left[\frac{x}{\sqrt{2}}\right]\right)^{-1+r}}{2\sqrt{2\pi}\,(3-r)!\,(-1+r)!}$$

Out[15]= $\{x, -\infty, \infty\}$ && $\{r \in \mathbb{Z}, 1 \le r \le 3\}$

Out[16]= $$\left\{\frac{3\, e^{-\frac{x^2}{2}} \left(-1 + \mathrm{Erf}\left[\frac{x}{\sqrt{2}}\right]\right)^{2}}{4\sqrt{2\pi}}, -\frac{3\, e^{-\frac{x^2}{2}} \left(-1 + \mathrm{Erf}\left[\frac{x}{\sqrt{2}}\right]^{2}\right)}{2\sqrt{2\pi}}, \frac{3\, e^{-\frac{x^2}{2}} \left(1 + \mathrm{Erf}\left[\frac{x}{\sqrt{2}}\right]\right)^{2}}{4\sqrt{2\pi}}\right\}$$ PART i.

Out[17]= $$\left\{1 + \frac{1}{8}\left(-1 + \mathrm{Erf}\left[\frac{y}{\sqrt{2}}\right]\right)^{3}, -\frac{1}{4}\left(-2 + \mathrm{Erf}\left[\frac{y}{\sqrt{2}}\right]\right)\left(1 + \mathrm{Erf}\left[\frac{y}{\sqrt{2}}\right]\right)^{2}, \frac{1}{8}\left(1 + \mathrm{Erf}\left[\frac{y}{\sqrt{2}}\right]\right)^{3}\right\}$$

Out[18]=



PART ii.

Out[19]= $$\left\{-\frac{3}{2\sqrt{\pi}}, 0, \frac{3}{2\sqrt{\pi}}\right\}$$ PART iii.

✓

Question 3 (b)

```
In[1]:= << mathStatica.m
f = PDF[LaplaceDistribution[], x]; domain[f] = {x, -∞, ∞};
g = OrderStat[r, f, 3]
domain[g] = OrderStatDomain[r, f, 3]
g /. r → {1, 2, 3} // Simplify
Prob[y, g /. r → {1, 2, 3}]
PlotDensity[g /. r → {1, 2, 3}]
Expect[x, g /. r → {1, 2, 3}]
```

Out[3]=
$$\begin{cases} \dfrac{3\, e^{r x} \left(2-e^{x}\right)^{3-r}}{4\, (3-r)!\, (-1+r)!} & x \le 0 \\[2ex] \dfrac{3\, e^{(-3+r) x} \left(2-e^{-x}\right)^{r}}{4 \left(-1+2 e^{x}\right) (3-r)!\, (-1+r)!} & \text{True} \end{cases}$$

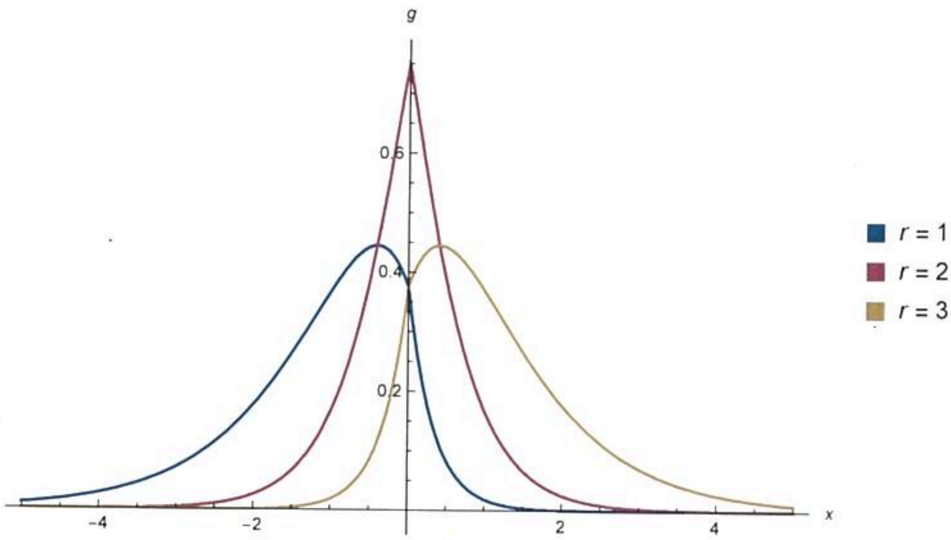
Out[4]= $\{x, -\infty, \infty\}\ \&\&\ \{r \in \mathbb{Z}, 1 \le r \le 3\}$

Out[5]=
$$\begin{cases} \left\{\frac{3}{8} e^{x} \left(-2+e^{x}\right)^{2}, -\frac{3}{4} e^{2x} \left(-2+e^{x}\right), \frac{3 e^{3x}}{8}\right\} & x \le 0 \\[1ex] \left\{\frac{3 e^{-3x}}{8}, \frac{3}{4} e^{-3x} \left(-1+2 e^{x}\right), \frac{3}{8} e^{-3x} \left(1-2 e^{x}\right)^{2}\right\} & \text{True} \end{cases}$$

] ⇒ $\boxed{\text{PART i}}$

Out[6]=
$$\begin{cases} \begin{pmatrix} \frac{1}{8} e^{y} \left(12+e^{y}\left(-6+e^{y}\right)\right) & -\frac{1}{4} e^{2y}\left(-3+e^{y}\right) & \frac{e^{3y}}{8} \\ \frac{1}{8} e^{y} \left(12+e^{y}\left(-6+e^{y}\right)\right) & -\frac{1}{4} e^{2y}\left(-3+e^{y}\right) & \frac{e^{3y}}{8} \\ \frac{1}{8} e^{y} \left(12+e^{y}\left(-6+e^{y}\right)\right) & -\frac{1}{4} e^{2y}\left(-3+e^{y}\right) & \frac{e^{3y}}{8} \end{pmatrix} & y \le 0 \\[3ex] \begin{pmatrix} 1-\frac{e^{-3y}}{8} & \frac{1}{4}\left(4+e^{-3y}-3 e^{-2y}\right) & \frac{1}{8} e^{-3y}\left(-1+2 e^{y}\right)^{3} \\ 1-\frac{e^{-3y}}{8} & \frac{1}{4}\left(4+e^{-3y}-3 e^{-2y}\right) & \frac{1}{8} e^{-3y}\left(-1+2 e^{y}\right)^{3} \\ 1-\frac{e^{-3y}}{8} & \frac{1}{4}\left(4+e^{-3y}-3 e^{-2y}\right) & \frac{1}{8} e^{-3y}\left(-1+2 e^{y}\right)^{3} \end{pmatrix} & \text{True} \end{cases}$$

Out[7]=



] ⇒ $\boxed{\text{PART ii}}$

Out[8]= $\left\{-\frac{9}{8}, 0, \frac{9}{8}\right\}$] ⇒ $\boxed{\text{PART iii}}$

✓