MULTIVARIATE VERSION OF CRAMER-RAO's THEOREM

Given: the sample $X=(X_1,\dots,X_n)$; $X_i \sim f(x,\theta), \theta \in R^K$;

$\frac{\partial}{\partial\theta} \ln L(X,\theta) := \left[\frac{\partial}{\partial\theta_1}\ln L(X,\theta), \frac{\partial}{\partial\theta_2}\ln L(X,\theta), \dots, \frac{\partial}{\partial\theta_K}\ln L(X,\theta)\right]$

The matrix $\mathcal{J} \in \mathcal{M}^{>}_{K\times K}$: $\mathcal{J} := E\left[\frac{\partial}{\partial\theta}\ln L(X,\theta)' \cdot \frac{\partial}{\partial\theta}\ln L(X,\theta)\right]$

(or equivalently $\mathcal{J} = -E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln L(X,\theta)\right]_{i,j=1,2,\dots,K}$) is the INFORMATION MATRIX w.r. to the parameter-vector θ.

Let $W(X)$ be an s-dimensional statistic with $E_\theta W(X) = \tau(\theta) \in R^S$ and $\tau(\theta)$ be differentiable w.r. to θ; $\Delta = \frac{\partial\tau(\theta)}{\partial\theta}$ be the $S\times K$ matrix of partial derivatives of the components of τ.

Let $\boldsymbol{\Sigma} = \mathrm{cov}_\theta(W(X)) = E\left\{(W(X)-\tau(\theta))(W(X)-\tau(\theta))'\right\} \in \mathcal{M}^{>}_{S\times S}$

For any statistic $H(X) = \begin{pmatrix} H_1(X) \\ H_2(X) \\ \vdots \\ H_S(X)\end{pmatrix}$ such that

$\int\limits_{R^n}\!\dots\!\int |H_i(X)|\, L(X,\theta)\, dX < \infty$, $i=1,2,\dots,S$, we assume

$$(*)\quad \frac{\partial}{\partial\theta_j}\int\limits_{R^n}\!\dots\!\int H_i(X) L(X,\theta)\,dX = \int\limits_{R^n}\!\dots\!\int H_i(X)\frac{\partial}{\partial\theta_j}L(X,\theta)\,dX$$

holds ; $i=1,2,\dots,S$; $j=1,2,\dots,K$.

Then $\boldsymbol{\Sigma} \geq \Delta\mathcal{J}^{-1}\Delta'$ in the sense of matrices, i.e. $\forall$ vector $Z \in R^S$;

$$Z'\boldsymbol{\Sigma}Z \geq Z'\Delta\mathcal{J}^{-1}\Delta' Z$$

Proof: Under Assumption (*).

$$E\left[W_i \cdot \frac{\partial \ln L}{\partial \theta_j}(X,\theta)\right] = \int_{R^n}\!\!\cdots\!\int W_i \frac{\partial L}{\partial \theta_j}(X,\theta)\,dX = \frac{\partial}{\partial \theta_j} E(W_j) = \frac{\partial}{\partial \theta_j}\tau_i(\theta)$$

$i=1,2,\dots,S;\ j=1,2,\dots,K$

Hence
$$\operatorname{cov}\begin{bmatrix} W(X) \\ \left[\frac{\partial}{\partial\theta}\ln L(X,\theta)\right]' \end{bmatrix} = \begin{pmatrix} \Sigma & \Delta \\ \Delta' & J \end{pmatrix}$$

Both determinants

$$\begin{vmatrix} I_{S\times S} & -\Delta J^{-1} \\ O & J^{-1} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \Sigma & \Delta \\ \Delta' & J \end{vmatrix}$$

are obviously non-negative and, correspondingly, their product

$$\begin{vmatrix} \Sigma - \Delta J^{-1}\Delta' & O \\ J^{-1}\Delta' & I_{K\times K} \end{vmatrix} = |\Sigma - \Delta J^{-1}\Delta'| \geq 0$$

Along the same lines, the non-negativity can be shown when considering just a SUBSET of components of the statistic $W(X) \in R^S$. The conclusion follows now from Silvester's criterion.

In particular (take $Z = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$), we obtain

the (m,n) Element of J^{-1}

$$\Sigma_{ii} = Var(W_i) \geq \sum_{m=1}^{K}\sum_{n=1}^{K} J^{mn} \frac{\partial \tau_i}{\partial \theta_m}\frac{\partial \tau_i}{\partial \theta_n}$$

$i=1,2,\dots,S$ and, in particular, for $\tau_i(\theta) = \theta_i$:

$$\Sigma_{ii} \geq J^{ii}$$