The University of New South Wales
Department of Statistics
Session 2, 2018
MATH5905 - Statistical Inference
Assignment 1

This assignment must be submitted no later than at the beginning of the lecture at 5pm on Friday, 24th August 2018. Please, declare on the first page that the assignment is your own work, except where acknowledged. State also that you have read and understood the University Rules in respect to Student Academic Misconduct.

The assignment is to be handed in as a **hard copy** (no e-mails!)

Maximal number of pages: 8

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1. Consider a decision problem with parameter space $\Theta = \{\theta_1, \theta_2\}$ and a set of non randomized decisions $D = \{d_i, 1 \le i \le 6\}$ with risk points $\{R(\theta_1, d_i), R(\theta_2, d_i)\}$ as follows:

i	1	2	3	4	5	6
$R(\theta_1, d_i)$	0	1	3	6	7	6
$R(\theta_2, d_i)$	7	5	5	2	5	8

a) Find the minimax rule(s) amongst the **nonrandomized** rules in D;

b) Plot the risk set of all **randomized** rules $\mathcal{D}$ generated by the set of rules in D.

c) Find the risk point of the minimax rule in $\mathcal{D}$ and determine its minimax risk.

d) Define the minimax rule in the set $\mathcal{D}$ in terms of rules in D.

e) For which prior on $\{\theta_1, \theta_2\}$ is the minimax rule in the set $\mathcal{D}$ also a Bayes rule?

f) Determine the Bayes rule and the Bayes risk for the prior $\left(\frac{5}{6}, \frac{1}{6}\right)$ on $\{\theta_1, \theta_2\}$.

g) For a small positive $\epsilon = \frac{1}{2}$, illustrate on the risk set the risk points of all rules which are ϵ-minimax.

2. In a Bayesian estimation problem, we sample n i.i.d. observations $\mathbf{X} = (X_1, X_2, \dots, X_n)$ from a population with conditional distribution of each single observation being the geometric distribution

$$f_{X_1|\Theta}(x|\theta) = \theta^x(1-\theta), x = 0, 1, 2, \dots; 0 < \theta < 1.$$

The parameter θ is considered as random in the interval $\Theta = (0, 1)$.

i) If the prior on Θ is given by $\tau(\theta) = 3\theta^2, 0 < \theta < 1$, show that the posterior distribution $h(\theta|\mathbf{X} = (x_1, x_2, \dots, x_n))$ is in the Beta family. Hence determine the Bayes estimator of θ with respect to quadratic loss.

Hint: For $\alpha > 0$ and $\beta > 0$ the beta function $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ satisfies $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ where $\Gamma(\alpha) = \int_0^\infty exp(-x)x^{\alpha-1}dx$. A Beta (α, β) distributed random variable X has a density $f(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1$, with $E(X) = \alpha/(\alpha+\beta)$.

ii) Five observations form this distribution were observed: 2, 4, 7, 1, 3. Using zero-one loss, what is your decision when testing $H_0 : \theta \leq 0.75$ against $H_1 : \theta > 0.75$. (You may use the `integrate` function in `R` or another numerical integration routine from your favourite programming package to answer the question.)

3. Let $X_1, X_2, \ldots, X_n$ be i.i.d. uniform in $(0, \theta)$ and let the prior on θ be the Pareto prior given by $\tau(\theta) = \beta\alpha^\beta\theta^{-(\beta+1)}, \theta > \alpha$. (Here $\alpha > 0$ and $\beta > 0$ are assumed to be known constants). Show that the Bayes estimator with respect to quadratic loss is given by $\hat{\theta}_{Bayes} = \max(\alpha, x_{(n)})\frac{n+\beta}{n+\beta-1}$. Justify all steps in the derivation.

4. At a critical stage in the development of a new aeroplane in the UK, a decision must be taken to continue or to abandon the project. The financial viability of the project can be measured by a parameter $\theta \in (0, 1)$, the project being profitable if $\theta > \frac{1}{2}$. Data x provide information about θ. If $\theta < 1/2$, the cost to the taxpayer of continuing the project is $(\frac{1}{2} - \theta)$(in units of \$ billion) wheres if $\theta > 1/2$ it is zero (since the project will be privatised if profitable). If $\theta > \frac{1}{2}$ the cost of abandoning the project is $(\theta - \frac{1}{2})$ (due to contractual arrangements for purchasing the aeroplane from the French), whereas if $\theta < \frac{1}{2}$ it is zero. Two actions are on the table: a_0 : continue the project and a_1 : abandon it.

Derive the Bayesian decision rule in terms of the posterior mean of θ given x. The Minister of Aviation has prior density $6\theta(1-\theta)$ for θ. The prime minister has prior density $4\theta^3$. The prototype aeroplane is subjected to trials, each independently having probability θ of success, and the data x consists of the total number of trials required for the first successful result to be obtained (i.e., one realization from a geometric distribution). For which values of x will there be most serious ministerial disagreement?