SOME VOCABULARY AND MOTIVATION FOR ∞-CATEGORY THEORY

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These are notes for the 2021-2022 algebraic topology reading group on ∞-category theory. These notes provide a brief overview of important material in chapter one of our reference text, Land's *Introduction to ∞-Category Theory*. Section 3 includes some further remarks. The notation and concepts are (mostly) consistent with Land. This is not a comprehensive discussion of these topics, and so I also recommend Emily Riehl's stellar set of notes "A Leisurely Introduction to Simplicial Sets," and the material found in Kerodon. I can further recommend Friedman's survey on simplicial sets [arXiv:0809.4221] and Groth's survey on ∞-categories [arXiv:1007.2925]. Chapter 6 (6.1.2 in particular) of Lurie's *Higher Topos Theory* contains more information on classifying spaces.

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1. Simplicial objects

Here are some brief remarks about simplicial sets, to motivate what we will discuss. We have a host of similar, but not identical, objects, that will be distinguished in the interest of formality. Loosely, these objects have different types, and are certainly realised in different contexts—e.g., unrealised, realised in sets, realised in spaces, and so forth. Throughout, points are zero-simplices, and so we (somewhat accidentally) choose the convention that anytime an n is seen, it is typically in fact an $n+1$ in some sense. This is made explicit where needed.

A totally ordered set isomorphic to some sequence of natural numbers is an ordinal number $[n]$, also written $\underline{n}$. The category Δ is the abstract category of ordinal numbers $[n] = \{0, \ldots, n\}$ for all $n \in \mathbb{N}$. This is sometimes misleadingly called the simplex category, but this should be avoided, as Δ does not contain

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simplices. Every $[n]$ contains the ordinal $[n-1]$, which itself contains the ordinal $[n-2]$, and so forth. Effectively, each $[n]$ contains n terminal ordinals $[0] = 0$, labelled by their index in $[n]$. Any sequence of n ordinals can be mapped to a sequence of any m ordinals by inserting or skipping sufficiently many elements—in particular, adding $m - n$ elements.[1] In any case, the data encoded by Δ is purely combinatorial, but has a useful recursive character.

A priori, an n-simplex is a particular, isolated object with some shape. A zero-simplex is a point, a one-simplex is an edge filling the space between points, a two-simplex is a face filling the space between edges, and so forth. Clearly, an n-simplex admits n types of sub-simplex, as yet unspecified. The way we capture this recursion is as follows: given an initial set of interest, $[n]$, the standard n-simplex Δ^n is $\mathrm{Hom}_\Delta(-, [n])$. This works by taking $[n]$ and giving us all the maps between all the subsets of and supersets containing $[n]$; this records every simplex for every i up to n and every j greater than $[n]$. Indeed, any i-simplex in Δ^n is a monomorphism $[i] \mapsto [n]$, defining a sub-simplex at i.

Define a presheaf $X : \Delta^{\mathrm{op}} \to \mathsf{Set}$ as a simplicial set, also written $X_\bullet$ to indicate we take in all n. This functor maps $[n]$, for every value of n, to the set of n-simplices $X[n]$, also written as X_n. By set of n-simplices, we mean there may be multiple entities at some n, e.g., several points at $n = 0$, a large graph at $n = 1$, and so forth. These points are assumed to be instantiated in some way prior to defining an n-simplex on those points.[2] A simplicial set is simply a contravariant functor C to Set, so there is no need for any standardisation. This is quite a general object, or at least, we have a lot of freedom in terms of how, precisely, it is 'shaped'—it is simply a functor, as far as we have defined it. As a presheaf, we can use the Yoneda embedding to say that this functor is related somehow to $\mathrm{Hom}_\Delta([m], [n])$, collecting the set of all maps between every pair $[m]$ and $[n]$ for different values of m and n. Since a simplex is defined by exactly these maps, X gives us the set of simplices associated to any set of vertices, and it appears like Δ^n is the thing that assigns each $X[n]$ its n-simplices (and implicitly its sub-simplices). We can make this relationship precise, as follows below.

Given a locally small category $\mathcal{C}$, presheaf that is naturally isomorphic to the contravariant functor $\mathrm{Hom}(-, A)$ for some $A \in \mathrm{Obj}(\mathcal{C})$ is called a representable presheaf. Since we can define a simplicial set as a presheaf, the density theorem says that any simplicial set ought to be the colimit of some representable presheaf,

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[1] The epimorphism s_i (a degeneracy map) mapping $[n] \mapsto [n-1]$ by degenerating a map to the i-th element of $[n]$, and the monomorphism d_i (a face map) mapping $[n] \mapsto [n+1]$ by inserting an element at i, accomplish this. The degeneracy map is of particular interest, since there are several context-sensitive ways of creating such a degeneracy. In Δ, this is given by a surjective map with two arrows from i and $i + 1$ in $[n]$, the domain of s_i, both onto i in the codomain $[m] = [n-1]$. In other words, $s_i(i) = s_i(i + 1) = i$.

[2] The standard topological n-simplex is 'standard' because there are $n+1$ points aligned with the $n + 1$ standard basis vectors of $\mathbb{R}^{n+1}$—so in other words, the minimal set of points needed to define an n-simplex, if each i-simplex for $i \in [n]$ is stored in a space of dimension no greater nor less than i. This will be apparent as soon as we discuss realisation.

such that every simplicial set has a factorisation related to a more 'fundamental' object. The representable presheaf is thus the object satisfying the following isomorphism:

$$\mathrm{Nat}(h^n, X) \cong X[n]$$

where $h^n = \mathrm{Hom}(-, [n])$ by the Yoneda embedding. That object is what we previously defined as the standard n-simplex. In this way, any simplicial set can be related to the more familiar notion of maps in Delta. From this, we apply the density theorem to get

$$X \cong \operatorname*{colim}_{n \in \Delta/X} \Delta^n.$$

2. Realisation and singularisation

In general, but especially in ∞-category theory, we conceptualise space and quantity as being dual.[3] This is modelled on the fact that we can identify simplicial sets as volumes of spaces, and pass from one to the other. Here, we speak of realisation as constructing an object out of a simplicial set, and singularisation as constructing a simplicial set out of an object. These things are typically spaces, but in the particular case of the nerve of a category, a category is singularised by a simplicial set. Duality is encoded in the adjoint-ness of this pair of operations.

Using the preceding definition of Δ^n, we can construct Δ^n_{top}, or the standard topological n-simplex, as the convex hull of the set of points around the origin of $\mathbb{R}^{n+1}$ such that the vertices of the n-simplex are aligned with the standard basis $e_0, \ldots, e_n$. In fact, the standard topological n-simplex is contained entirely in $[0,1]^n$. Suppose this is the simplex on which we model simplicial complexes. Like the freedom we have in simplicial sets as opposed to their standard representable objects, we have simplicial complexes covering a space, defined more freely, but still built out of standard (topological) n-simplices. The simpler case of defining Δ^n_{top} corresponds to a covariant functor $\Delta_{\mathrm{top}} : \Delta \to \mathsf{Top}$, sending $[n]$ to Δ^n_{top}. In turn, this induces a more general geometric realisation functor

$$|-| : \mathsf{sSet} \to \mathsf{Top},$$

inspired by declaring Δ^n to be the convex hull of some points in some topological space. In general, identifying a simplicial set as the volume of some space, and passing to that space, is the realisation of that set (as a geometric space). Geometric realisation takes a simplicial set X, and fashions a topological space $|X|$ using each element in X_n—all the n-simplices in X between every relevant vertex—as being identical to a copy of Δ^n_{top}, just like how a simplicial set is built. These are glued together to make a larger topological space, using the information encoded

[3]Spaces are generalised by objects on which we model quantities, and by quantity, we mean some algebraic structure with values in the collection of objects of interest. This terminology is inspired by the evaluation of a homomorphism on a space, giving a particular space as its image, which is the quantity returned by the map. There is a philosophical sense in which this underlies the ethos of algebraic topology, in that it is the study of algebraic qualities (such as invariants) defining a space.

in the face and degeneracy maps of X regarding what shares a sub-simplex with what. This equivalence relation defines the relevant quotient space. Indeed, we have

$$|X| = \operatorname*{colim}_{\Delta^n \in \Delta/X} \Delta^n_{\text{top}},$$

or identically, the coproduct of $X_n \times \Delta^n_{\text{top}}$ modulo the equivalence relation of shared sub-simplices, measured by the content of the images of the face and degeneracy maps. To return to complexes, we have the following interesting fact: the realisation of a simplicial set is a CW complex. Also, the realisation of Δ^n is Δ^n_{top}, in that the object corresponding to an n-simplex is an n-dimensional object in $\mathbb{R}^{n+1}$.

We could, in theory, go about this adjointly—define a functor such that, rather than realising a simplicial set as a space by building its faces and edges into a topological space, we probe an existing space with a simplicial set, built by a set of continuous functions $\sigma : \Delta^n_{\text{top}} \to Y$ defining a set of topological n-simplices in Y. A single such map is the singular n-simplex of Y. In analogy to the above, this identifies the volume of some space with a simplex—for some simplicial set, we view a volume of the space Y as an instance of X. This in turn is called the singular simplicial complex (or singular simplex for brevity) of Y, $\mathbf{Sing}(Y)$. It should be clear that the singular simplex of a space sends that space to a simplicial set, such that we induce another functor

$$\mathbf{Sing} : \mathsf{Top} \to \mathsf{sSet},$$

called singularisation. This functor can be constructed as

$$\mathbf{Sing}(Y) : Y \mapsto \mathrm{Hom}_{\mathsf{Top}}(\Delta^\bullet_{\text{top}}, Y),$$

such that we build a simplicial set out of the data in the space, taking all the maps describing every n-simplex in the space that the singular simplex could contain. Technically, we represent ordinals $[n]$ by the set of topological n-simplices in Y, $\mathrm{Hom}_{\mathsf{Top}}(\Delta^n_{\text{top}}, Y)$, like we did with simplicial sets as presheaves. Then, going from Y to $\mathrm{Hom}_{\mathsf{Top}}(\Delta^n_{\text{top}}, Y)$ induces the intended functor for $\mathbf{Sing}(Y)[n]$. This is induced by the Yoneda lemma in the same way as mapping Set to sSet by way of Δ: given that $\Delta^\bullet_{\text{top}}$ is cosimplicial (true by definition), the map $[n] \mapsto \mathrm{Hom}_{\mathsf{Top}}(\Delta^n_{\text{top}}, Y)$ determines a functor from Δ^{op} to Set, such that the result $\mathbf{Sing}^{\Delta_{\text{top}}}_n(Y)$ is in bijection with $\mathrm{Hom}_{\mathsf{Top}}(\Delta^n_{\text{top}}, Y)$. Neglecting to write the informational but perhaps not useful superscript Δ_{top}, we recover our definition of the simplicial set $\mathbf{Sing}(Y)$.

2.1. **Adjointness.** A special note is devoted to the adjointness of these functors. The functors $|-|$ and $\mathbf{Sing}(-)$ are an adjoint pair, since for any space Y which is singularised by some simplicial set $X_\bullet$, we can say Y is exhibited as a realisation of $X_\bullet$. Take the previous example: the realisation of a $\{0, 1, 2\}$-simplex is a {point, edge, face}, whilst {points, edges, faces} are singularised by $\{0, 1, 2\}$-simplices. In that case, singularisation makes an n-simplex out of the vertices, edges, faces, etc, found in the convex hull of $\mathbb{R}^{n+1}$, such that Δ^n_{top} is exhibited as the realisation of Δ^n. The σ defining the representation of $[n]$ as Δ^n_{top}, mapped back to Δ^n_{top} by the singular functor, is $\mathrm{id} : \Delta^n_{\text{top}} \to \Delta^n_{\text{top}}$. In general, the vertices of $\mathbf{Sing}_\bullet(Y)$ can be identified as the points in Y, and the edges as paths $p : I \to Y$.

Formally, singularisation is right-adjoint to realisation. Take the definition of $\mathbf{Sing}(Y)$ as $\mathrm{Hom}_{\mathsf{Top}}(\Delta^n_{\text{top}}, Y)$. The claim is that

$$\mathrm{Hom}_{\mathsf{Top}}(\Delta^n_{\text{top}}, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \mathbf{Sing}(Y)).$$

This seems to follow by definition, and indeed, the composition of maps behaves as one expects it to (see Friedman, Theorem 4.10, for some details). By the density theorem, we can generalise the proof to any $|X|$ (X, respectively), so the more interesting

$$\mathrm{Hom}_{\mathsf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, \mathbf{Sing}(Y))$$

also holds. This statement is arguably even clearer.

A more interesting proof is given by Riehl in "A Leisurely Introduction to Simplicial Sets." Let $R : \mathcal{E} \to \mathsf{sSet}$ be a functor sending $e \in \mathcal{E}$ to a simplicial set by the Yoneda embedding

$$R_n(e) = \mathrm{Hom}_{\mathcal{E}}(F[n], e).$$

For any cosimplicial object $F : \Delta \to \mathcal{E}$, the left Kan extension of F along the Yoneda embedding yields a functor $L : \mathsf{sSet} \to \mathcal{E}$, for whom the the right adjoint is R. This Kan extension formalises the 'it follows by definition' argument above, and the proof itself is far more general than just topological spaces. This generality is useful, as we will see we are not exclusively interested in $\mathcal{E} = \mathsf{Top}$.

3. Nerves, model structures, and Kan complexes

The nerve of a category is the simplicial set that singularises that category. Take a locally small category $\mathcal{C}$. Much like the functor $\Delta_{\text{top}} : \Delta \to \mathsf{Top}$ we modelled on the topological n-simplex, we have a cosimplicial object $\Delta_c : \Delta \to \mathcal{C}$, sending ordinals onto the data of $\mathcal{C}$. In particular, this maps increasing ordinal numbers onto objects, morphisms, and eventually, strings of composable morphisms. Since this forms a simplex out of the data in $\mathcal{C}$, there is a particular singularisation sending $\mathcal{C}$ to Set, given by

$$\mathcal{C} \mapsto \mathrm{Hom}_{\mathcal{C}}(\Delta^\bullet_c, \mathcal{C})$$

with

$$\mathbf{Nerv}(\mathcal{C})[n] = \mathrm{Hom}_{\mathcal{C}}(\Delta_c[n], \mathcal{C}),$$

giving the intended presheaf $\mathbf{Nerv}(\mathcal{C}) : \Delta^{\text{op}} \to \mathsf{Set}$. At this point, this construction is quite familiar, as it really is nothing but a simplicial set (valued like data in $\mathcal{C}$).

Setting nerves aside for a moment, we ought to define a Kan complex if we wish to discuss how particular nerves provide a definition of an ∞-category. And, before that, we ought to define horns and fillers.

The horn of a simplicial set can be ideated as a piece of the boundary of a cell. Let $0 \leq k \leq n$, a natural number in $[n]$. If $\Delta[n]$ is the standard simplicial n-simplex in sSet representing some $[n]$, there exists a sub-simplicial set $\Lambda^k[n] \hookrightarrow \Delta[n]$ defined by taking every face in the boundary $\partial\Delta^n$ with the exception of the k-th one. This is properly referred to as the (n, k)-horn. If k is zero or n, this is called an outer horn; else, an inner horn. A horn filler is a morphism of simplicial sets that has the right lifting property against horn inclusions. All this means is, given two simplicial sets X and Y with $f : X \to Y$, and an (n, k)-horn, there exists a unique morphism f such that the following diagramme commutes:

$$\begin{tikzcd}
\Lambda^k[n] \arrow[r, "s"] \arrow[d, "\iota_k"'] & X \arrow[d, "f"] \\
\Delta[n] \arrow[r, "y"] \arrow[ur, "x"] & Y
\end{tikzcd}$$

If x exists and is unique, then f is said to be right orthogonal to the inclusion ι_k. These sorts of properties are of interest in category theory, as they allow particularly nice factorisations of the system. Moreover, if we take a set of morphisms $C \subset \mathrm{Mor}(\mathcal{C})$ in some category $\mathcal{C}$, then the set of all morphisms left orthogonal to every morphism in C is denoted by $C^{\perp}$ (dually for right orthogonals, ${}^{\perp}C$). These sets contain only isomorphisms.

Fibrations are morphisms with particular factorisation qualities, usually related to homotopy equivalences. As such, orthogonal morphisms are types of fibrations, so the model structure on a category is determined by the lifting properties of its data.[4] For instance, weak equivalences are precisely these fibrations that we are able to treat as isomorphisms. This means that the way we can fill each simplex in X determines the model structure on X; in general, model structures encode precisely these lifting properties. A Quillen model structure arises from a Kan complex, which is a simplicial set with the Kan condition—the property that *every* horn has a filler which is a Kan fibration.[5] Intuitively, this translates to the condition that if a horn exists, we can reconstruct the simplex it came from,

[4]Conversely, anything that is nice enough to have some sort of model structure at some abstract level (i.e. can be realised as a simplicial set) has fibrations. Indeed, homotopy lift and Kan conditions are equivalent through the eyes of model categories, which is what enables simplicial homotopy theory. We'll see why this is of particular interest when we discuss fibrations as the 'quantities' in the duality between space and quantity, and how the structure of simplicial sets allows us to understand the role of fibrations in describing objects by classifying spaces. The generality here leads to Quillen's homotopy hypothesis, and in some sense, to the idea that homotopy theory actually happens in an ambient $(\infty, 1)$-topos.

[5]A Kan fibration is a particular right orthogonal morphism, taken to be $f : X \to \mathrm{pt}$, where pt is the terminal simplicial set.

by extending the inclusion to $\Lambda^k[n] \hookrightarrow \Delta^n \to X$.[6] This can be proven simply by chasing the diagramme. In any case, this brings us back to nerves—clearly, nerves are important, because by singularising a category, we can place a model structure on it and extract some homotopy coherent information.

One other interesting fact, recapitulating a motif throughout this section: the realisation of a Kan fibration is a Serre fibration. So, the singularisation of a topological fibration, already a very general object, leads us directly to model structures. Conversely, this fact follows from the Quillen equivalence of the model structures on Top and sSet. A similarly inspired statement is that a Kan fibration is acyclic precisely if the fibre over each vertex x, $f^{-1}(y) = x$, is contractible.

3.1. **Classifying spaces.** The classifying space of an object is a space constructed for the purpose of probing that object by quantities, i.e, a space such that maps $M \to \mathscr{B}\mathcal{D}$ record data of $\mathcal{D}$ over M, for some generic object M and object of objects $\mathcal{D}$. When M is some category and we want to record categorical data over M, we have the Grothendieck construction inducing a functor $M \to \mathscr{B}\mathcal{D}$ by

$$M^{\mathrm{op}} \to \mathsf{Cat}$$

such that Cat is the classifying space of categories. This is very similar to other ideas like stacks and moduli spaces, but only if we can represent an object of $\mathscr{B}\mathcal{C}$ *inside* M do we have a moduli space. Likewise, fibre bundles, which parameterise copies of a manifold by points in a base manifold, possess classifying spaces when we consider equivalence classes of bundles with similar fibres.

By probing an object with object-valued quantities, it follows that classifying spaces should be related to simplicial objects in some sense. Indeed, this is (in the author's opinion) the most interesting thing which can be formalised by ∞-category theory. Any discrete group $(G, \cdot)$ can be categorified by taking it as a groupoid BG—a category with one element, and with morphisms being automorphisms of the object given by actions of $g \in G$. Since every morphism in a group is composable, n-simplices in $\mathbf{Nerv}(G)$ are just n-tuples $G \times \ldots \times G$, such that the nerve of BG is a particular groupoid $\mathbf{B}G$. We have the following fact given by Land: $|\mathbf{B}G| \cong \mathscr{B}G$, such that the classifying space is the space made of the groupoid that deloops G. The converse can be defined adjointly, relying somewhat on the technology of ∞-categories: the singularisation of $\mathscr{B}G$ is the nerve $\mathbf{B}G$, and indeed, $\mathbf{B}G$ is a Kan complex.[7]

Thus, an example of how we think of nerves is found in a classifying space, in that $\mathbf{Nerv}(\mathcal{C})$ exhibits a classifying space $\mathscr{B}\mathcal{C}$ as its realisation. A nice instance

[6] Since $f \circ s$ now factors through $y \circ \iota_k$, the existence of x simply asks that there is a simplex in X that contains the horn given by s. Note how this map resembles a Serre fibration, when the extension is a homotopy lift property.

[7] There is a different way to approach defining $\mathbf{B}G$, which defines a cosimplicial object in the category of groups. Clearly, singular complexes are to set Set as nerves are to Cat as simplicial groups are to Grp. Due to the particularly nice structure of groups, simplicial groups are automatically ∞-groupoids. This provides an instance of ∞-groupoids being equivalent to spaces.

of this is the above relationship between groupoids and classifying spaces. In line with that, we have the fact that the model structure on Top is the presentation of the $(\infty, 1)$-category of ∞-groupoids as Kan complexes. This is, loosely, where the ubiquity of Serre fibrations comes from—the geometric realisation of Kan fibrations defined by relevant members of this category. This is the aforementioned homotopy hypothesis, that ∞-groupoids localise spaces simplicially (in particular, that every ∞-groupoid is the fundamental ∞-groupoid of some space). Moreover, when $\mathbf{B}G$ is the singularisation of $\mathscr{B}G$, it is a particularly special groupoid called the delooping groupoid. A delooping is defined such that an object of interest is the loop space object of the delooping, encoding some important homotopy information about it.[8] Indeed, deloopings occur in an ∞-category with homotopy pullbacks.

Philosophically, the theory of ∞-categories is useful in that it is the language that formalises a number of things. A particular example is the ubiquity of fibration-type objects, which arise whenever there is a model structure hiding somewhere. Then we can understand how spaces and fibrations interact using ideas modelled on simplicial sets, especially homotopy theory and classifying spaces.

3.2. **Towards ∞-categories.** A model structure on a simplicial set defines an ∞-category. This idea is present throughout Lurie's work, where he uses Kan complexes as a geometric model for ∞-groupoids (and homotopy types). Indeed, Groth states an ∞-category is nothing but a simplicial set that satisfies certain horn extension properties—these are clearly the lifting requirements we discussed previously. In particular, an ∞-category is a simplicial set for which all inner horn inclusions extend.

There are a few other ways of constructing an ∞-categorical object, contingent on some of the relationships recorded by simplicial sets. For example, under the homotopy hypothesis that spaces and ∞-groupoids should be isomorphic, given a space X there is an ∞-groupoid $\pi_\infty(X)$ where the points are objects, one-morphisms paths, two-morphisms homotopies between paths, and so forth. Here, we don't explicitly need Kan complexes—although of course, it is hidden in the groupoid structure of π_∞. The realisation of $\pi_\infty(X)$ as a simplicial set is X.

Likewise, since every singular simplicial complex is a Kan complex, so is every nerve, meaning that the nerve of a category fashions higher categorical data out of a $(1, 1)$-category.[9] A simplicial set is the nerve of such a category if and only if it

[8]In a generalised cohomology theory, the n-th degree cohomology of X is the set of homotopy classes between X and some object $E(n)$ such that the space of based loops in $E(n)$ is $E(n-1)$. The set of spaces $E(n)$ represents $h^n(X)$. Thus, a loop space is a kind of homotopy coherent abelian group. A loop space object also satisfies particular lifting properties, which is nice, and relates to its capturing of homotopical data. See `https://math.ucr.edu/home/baez/calgary/BG.html` for a brief overview.

[9]Recall that an (n, r)-category has n 'meaningful' types of morphism (anything higher is trivial) and $n - r$ distinct reversible (undirected) morphisms, with anything greater than r trivially reversible. This is correspondingly called an r-directed homotopy n-type.

satisfies the Segal condition. In turn, this means any nerve is a Kan complex. This is constistent with the construction of $\mathbf{B}G$ out of the monoid BG, for example.

There is a lot more to mention about loop spaces, classifying spaces, and the homotopical and topological information encoded in ∞-categories, which I presume will be discussed sometime quite soon.

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