ON THE DEVELOPMENT OF ORDER RELATIONS IN THE XIXº CENTURY

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1. Introduction

This paper concerns a story from the history of set theory's role in real analysis. A controversy created by an unsolved problem in analysis, initially posed by Paul du Bois-Reymond, drove many to develop a more robust theory of ordered sets in the late 1800s and early 1900s. Du Bois-Reymond introduced the idea of a boundary between convergence and divergence based on the rate of growth of a function, which he sought to derive from some total ordering on the growth of any number of functions as they went to infinity. He evaluated growth rate in the sense of whether one sum attains a larger infinity than another, and in so doing, claimed to have ordered the set of all functions, including series as functions returning partial sums. His approach later attracted some criticism, including that of Felix Hausdorff, who showed that some functions are incomparable at infinity, and so the ordering that du Bois-Reymond constructed could never truly exist. Hausdorff proceeded to use his prior work on gaps in ordered sets to evaluate the structure of a slightly different ordering, and showed a boundary could never exist in a countably infinite set. Later results show that this cannot occur in general, not even in the uncountable case. The two primary

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references are [Plo05], J M Plotkin's 2005 book *Hausdorff on Ordered Sets*, and [Fis81], G Fisher's 1981 article *The Infinite and Infinitesimal Quantities of du Bois-Reymond and their Reception*. Translations are the author's.

2. A question about convergence from du Bois-Reymond

Analysis, broadly characterised, is the study of functions; real analysis, then, is the study of real-valued functions. Separate from this, an ordering or order relation is an anti-symmetric binary relation modelled on $<$ and $>$, perhaps first suggested in Euclid's fifth book, regarding magnitudes and ratios of magnitudes.

An interesting bit of historiography is suggested by the unidirectional nature of the so-called 'limit test' for the convergence of a function $f : \mathbb{N} \to \mathbb{R}_{>0}$, found in analysis.[1] Take an infinite series of the form

$$\sum_{n=0}^{\infty} f(n) = \mathscr{L}.$$

We have the following observation: $\lim_{n\to\infty} f(n) > 0 \implies \mathscr{L} = \infty$, or, if the function defining the summand increases at all, the series certainly diverges. This is easy to ideate, in that the Cauchy criterion for the bounding of partial sums will obviously be violated if the elements being added to each sum themselves increase. There is also the following counterexample to the converse of this observation: in the infinite limit of $f(n) = \frac{1}{n}$, we have zero, and yet, the harmonic series

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

still diverges. This is since $\frac{1}{n}$ does not decrease *fast enough* for the sequence of partial sums to remain bounded. This criteria, a boundary between divergence and convergence based on a sufficient speed of decrease for the function to sum to a finite number, is the formulation (in somewhat modern language) of a problem first posed by Paul du Bois-Reymond, which took decades and a series of new results on ordered sets to solve conclusively.

In an 1871 paper [dBR71] by du Bois-Reymond entitled « Sur la Grandeur Relative des Infinis des Fonctions » [EN: On the Relative Size of the Infinities of Various Functions], he defines an "algorithm" that orders any set of functions according to how large their infinities are, a kind of ordering according to the speed with which they grow. Taking two functions f and g, he denoted by the binary relation $f \succ g$ that f should be regarded as larger than g if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty,$$

indicating that f grows more quickly (attains a larger infinity) than g. Likewise, $f \sim g$ if the above algorithm returns a finite, non-zero number, and $g \succ f$ if this algorithm

[1]Here we go against the Bourbaki convention for the natural numbers by excluding zero. On the other hand, every sequence we take is strictly positive in the Bourbaki sense, in that it is not negative and excludes zero.

returns zero. Du Bois-Reymond refers to this ordering as his infinitary rank ordering, and the $\succ$ relation as an infinitary inequality. There are some fatal problems with this first iteration of the infinitary rank ordering, however—many of them related in someway to the fact that this set is only partially orderable, as there are large equivalence classes of functions with the same limiting behaviour. A specific problem du Bois-Reymond acknowledges is that the limit is forgetful of any terms in a function not contributing to the 'largest' infinity of that function: he uses the example of a sequence of functions $f \succ f_1 \succ f_2 \ldots \succ f_N$ such that each f_i has a 'smaller infinity' as its individual limiting behaviour. Clearly, the following two ratios,

$$\lim_{n\to\infty} \frac{f(n) + f_1(n) + f_2(n) + \ldots + f_N(n)}{g(n)}$$

and

$$\lim_{n\to\infty} \frac{f(n)}{g(n)},$$

are equivalent for any finite N, since in the first ratio, the leading infinity is that of f, not the finite series of smaller infinities in the first one. As a result, f dominates in the limit. This is at least one instance of multiple different functions occupying the same point on du Bois-Reymond's line, and indeed, there were other difficulties, though many went unrecognised for a time after his publication. He suggests a way to patch over the problem by also considering the difference between two functions, but does not seem to elaborate much, beyond suggesting that beside some technical difficulties that can easily be resolved, a well-defined criteria for ordering a set of functions by the size they attain surely exists and can be implemented.

In a paper [dBR77] following that one, „Über die Paradoxen des Infinitärcalcül" [EN: Regarding Some Paradoxes in the Calculus of Infinitary Quantities], we see the first mention of the boundary between convergence and divergence, as encoded in the infinitary rank ordering. This article seeks to investigate what he calls paradoxes, which are features of his infinitary rank ordering that are not present in the usual real line. He is first interested in what we would now call the density of his line, and with its properties as a kind of continuum.[2] Du Bois-Reymond recognised that, in the continuum of the real numbers, there were particularly special rational and natural numbers embedded amongst innumerable irrational ones. Richard Dedekind had, shortly before, disseminated [Ded72], from which we conceptualise Dedekind cuts—constructing any number as either a rational number (if the cut defines a sequence with smallest element in $\mathbb{Q}$) or an irrational number filling that gap, which we would now say is constructible as the limit of a sequence of rational numbers. Presumably, du Bois-Reymond took inspiration from this, and asked similar questions about the various curves in between two curves that increase at different rates. This idea of cuts in the continuum may indeed have been what led him to seek the existence of a boundary between curves that increase too quickly to converge and curves that increase sufficiently slowly to converge. Du Bois-Reymond recognised what Hausdorff would later formalise: due to the density of any set with large cardinality, such as

[2]Crucially, these concepts did not exist properly until the likes of Georg Cantor and Felix Hausdorff formalised them, which we will discuss later.

any continuum-type object, cuts which are ‘clean’ on both sides are not possible—any such cut would be surrounded on one side by an infinite number of smaller and smaller elements. Du Bois-Reymond admits that there is no sense in declaring a gap between the two sides of a cut, because any gap could be filled by some other function which tends to infinity alongside the function at the cut (again, one wonders if the thought was, ‘just like Dedekind did to fill the rationals with irrationals’). Without regard to his own insight, however, du Bois-Reymond persists in saying a boundary must exist—if not by these means, then by some other. In particular, he conjectures that if it is not a function he could write down and put on his line, then perhaps it is some analogue to the irrational numbers embedded amongst the more elementary functions he was concerned with. He suggests that, in analogy to admitting irrational quantities in $\mathbb{R}$, in the infinitary domain there could be kinds of growth which cannot be expressed by the analytic operations he used.

This point is the first building block in what Hausdorff would later formalise about the infinitary rank ordering. Crucially, du Bois-Reymond recognises that there can be no simple object satisfying the property he seeks; but, he also has not yet dismissed the question, suggesting a more complicated positive answer ought to exist.

3. Objections to and affirmations of the infinitary rank ordering

Cantor was a serious critic of du Bois-Reymond’s ideas. He saw the work spelt out above as seriously lacking—imaginative, perhaps, but uninsightful and dreadfully informal. More directly, several mathematicians objected to du Bois-Reymond’s infinitary inequalities, as well as the construction of the infinitary rank ordering based on these inequalities. Dedekind sided immediately with Cantor as regards du Bois-Reymond’s loose description of infinitely large and infinitely small numbers. Felix Klein seemed to agree with the two of them, even writing that allowing such a construction to take hold in mathematics would be irresponsible. Perhaps it is no coincidence that today, Cantor is famous for having gone on to develop the first formal ideas surrounding infinite set theory, as a direct precursor to the Zermelo–Fraenkel axioms. It is curious, then, that Abraham Fraenkel was somewhat supportive of du Bois-Reymond’s ideas in the ensuing controversy over whether his work possessed any philosophical merit, as did Bertrand Russell. It is no exaggeration to say du Bois-Reymond’s work had captured much of the mathematical world of the late 1800s.

An important counterexample to the well-definiteness of the infinitary inequalities is due to Cantor, who established circumstances under which two functions cannot be compared. Take, for instance, any function for whom the limit of the quotient of that function by some other function does not exist. Cantor was also displeased with the infinitary rank ordering, as it violated the Archimedean axiom of the usual reals. It was Otto Stolz who, in 1883, proved that any system with infinitely small quantities was incompatible with the Archimedean axiom [Sto83]. In particular, this includes du Bois-Reymond’s system, where an infinitely small quantity is a function on the inversion of his line. This was published as „Zur Geometrie der Alten, Insbesondere über ein Axiom des Archimede” [EN: On the Geometry of Antiquity, Especially as

Regards an Axiom of Archimedes]. However, in 1885, Stolz proposed a lengthy reaxiomatisation of infinitesimal quantities, which avoided this issue by focussing on limits rather than the explicitly infinitely small. Just as it seemed new doors were opening for his ideas, Paul du Bois-Reymond died, four years later in 1889. The set-theoretic side of the issue appears to have gone dormant at this time.

The story does continue on the analytic side with Alfred Pringsheim, who became involved in search of this boundary. It is Pringsheim's 1890 paper [Pri90], „Allgemeine Theorie der Divergenz und Convergenz von Reihen mit Positiven Gliedern" [EN: A More General Theory of Divergence and Convergence of Sequences with Positive Terms] wherein appears the first proof that the limit test is necessary, but not sufficient, for convergence. Pringsheim publishes a series of papers after this, arguing against the idea that a boundary exists, and suggesting that at the very least, du Bois-Reymond's infinitary rank ordering has nothing technical to say about the problem.

Du Bois-Reymond had his own supporters in terms of the idea of organising functions by growth. In [Bor98], his 1898 « Leçons sur la Théorie des Fonctions » [EN: Lectures on the Theory of Functions], Émile Borel took an interest in the problem, revising du Bois-Reymond's approach with more technical details. He first takes only those functions in that are comparable, and then asks if there is a system analogous to the theory of measures on this set. Let a denumerable set be a set in bijection with the infinite set of integers, and let $\varphi_k(x)$ be the k-fold iteration of φ, i.e.,

$$\underbrace{(\varphi \circ \varphi \circ \ldots \circ \varphi)}_{k}(x).$$

For a concrete example, consider the exponential function and $\varphi_2 = \exp\{\exp\{x\}\}$. Borel used du Bois-Reymond's construction to prove that, for any increasing, denumerable sequence of functions φ_k, there is always a function φ_ω with a 'larger infinity' than any sequence φ_k for any k. In direct analogy with defining the transfinite ordinal numbers, we can now take the iteration $\varphi_{k\omega}$, and the iteration of the function φ_{ω^2} surpassing that sequence, $\varphi_{k\omega^2}$, and so forth. Borel called this transfinite iteration, and argued it had a proper place in any inductive scenario exceeding the small cardinals of ZFC set theory. Having more formally established the idea of transfinitely large infinite numbers, Borel was regarded as having redeemed du Bois-Reymond's idea of larger and smaller infinities. Today, we have exactly this situation in formal approaches to ∞-category theory, whose models require the existence of large, inaccessible cardinals,[3] and a semi-formal calculus of infinite numbers with varying sizes. This idea is itself built out of the von Neumann universe of sets, which requires transfinite procedures to construct.

It is interesting, then, that Borel was led to the same question as du Bois-Reymond, suddenly becoming interested in the boundary between convergence and divergence in an 1899 memoir. He did not complete his work on this idea until [Bor10], the 1910 « Leçons sur la Théorie de la Croissance » [EN: Lectures on the Theory of Growth], wherein he presented some detailed results on 'order types.' Hardy also worked on

[3]This is loosely used to specify when a category is not locally small or small, and exactly how non-small it is.

the idea around 1910. Both made progress towards resolving the problem, but never could a concrete solution be given; it is known that Borel continued to speak about this problem as late as 1946, and it appears that Hardy moved swiftly on from his contributions, to tackle other problems.

4. Hausdorff and a more robust theory of ordered sets

In 1906, with Borel having steered the issue partially back towards set-theoretic foundations, Felix Hausdorff became interested in this problem.

Hausdorff had been interested in the set-theoretic structure of ordered continua independently of this problem, as since Hilbert's address at the Second International Congress of Mathematicians in 1900, he had been interested in proving Cantor's continuum hypothesis about the possible cardinalities of infinity between the reals and the naturals (Hilbert's first problem). Du Bois-Reymond's work, having incidentally defined a prototype of a *pantachie*, had likely been of interest to Hausdorff for this reason.[4] As a result, it is somewhat misleading to say Hausdorff took an interest in the problem; rather, through his independent interest in ordered sets, he (much like both du Bois-Reymond and Borel) eventually became captured by the problem.

From 1901 to 1904, Hausdorff does quite a lot of foundational work on ordered sets, defining their properties in various ways. In [Hau06] and [Hau07], respectively, he publishes two particularly interesting papers for us—these are „Untersuchungen über Ornungstypen I, II, III" and „Untersuchungen über Ornungstypen IV, V" [EN: Investigations into Order Types I, II, and III; Investigations into Order Types IV and V]. As Hausdorff writes, this work was new at the time, in that it dealt with infinite ordered sets in their totality, rather than merely "subsets of the linear continuum" or the easily understood well-ordered set. Order types define some types of infinitely large ordered sets, classified by various features akin to topological invariants or characteristic classes. In this way, they classify totally ordered sets, in the same way that cardinal numbers classify sets by their number of elements. An early conceptualisation of order types was known prior to Hausdorff's work, mostly due to Cantor's work on ordinal numbers. For example—the order type of the natural numbers, and thus any totally ordered countable collection of sets by isomorphism with $\mathbb{N}$, is the first ordinal number, ω. Indeed, Hausdorff's papers do not define order types, *per ser*. Rather, he constructs new ones and investigates their properties. A brief account of this work is as follows.

4.1. Order type I.

Hausdorff's order type I, the order type $\mu(\alpha)$ of powers of order types, regards that of sets which can be formed from cardinal exponentiation.

If $\#A$ is the cardinal number measuring the size of the set A, then it is now known that

$$\#A^{\#B} = \#\mathrm{Hom}(B, A).$$

Hausdorff seeks to describe how the order type of the resulting set depends on the order type of the base and exponent sets, in the most generality possible. To do so,

[4]This model for a pantachie being an everywhere-dense subset of a totally ordered set, mentioned by du Bois-Reymond in later work on his infinitary rank ordering. Hausdorff was likely interested in the idea of density as it related to the cardinality of continuum objects.

he seems to introduce some concepts that would serve as precursors to his work in topology—namely, the idea of a covering set. If μ and α are arbitrary order types corresponding to sets M and A with respective cardinalities $\mathfrak{m}$ and $\mathfrak{a}$, then the cardinal exponentiation $\#M^{\#A}$ has the same order type as the covering of A by M, such that subsets of M are indexed by elements of A. In other words, each element $a \in A$ is assigned a set M_a, of order type μ_a, such that we can take a particular element $x_a \in M_a$ covering the element a, in the sense of occupying a point in a topological space. The set of all such x_a is the covering set of A by M_a, and the set of all x (the cover of A by M) now has cardinality $\mathfrak{m}^{\mathfrak{a}}$. He is then able to say various things about the order type $\mu(\alpha)$ corresponding to the cardinal exponentiation $\mathfrak{m}^{\mathfrak{a}}$ by looking at the ordering of this covering set, such as whether $\mu(\alpha)$ is finite, and if not, what transfinite ordinal bounds it. Hausdorff asserts the cardinal derived from the covering set is a much easier object to manipulate than the cardinal of the power set, and thus, it is better to construct $\mu(\alpha)$ from this.

4.2. **Order type II.** Order type II concerns the order type of subsets of ordered sets with arbitrary order type μ. Whilst initially an unimpressive statement, we discover that Hausdorff wishes to speak in complete generality about what sort of order types we can expect to be contained in another order type.

Here, cofinality and coinitiality are defined. A subset B of A is said to be coinitial if for every $a \in A$, there exists some $b \in B$ such that $b \leq a$. Likewise, a cofinal subset is a set for which there exists a $b \geq a$ for every $a \in A$. The coinitiality and cofinality of a set A is the number equivalent to the cardinality of the smallest cofinal subset of A. For example, the cofinality of a partially ordered set with a greatest element is one, as the set containing that greatest element is cofinal, and is the smallest cofinal subset of that set. It follows that the cofinality of an ordinal number α is the least cardinality of a cofinal subset of α. Since ordinals correspond to order types, we can say the cofinality of an order type α is the least ordinal β which occurs as the order type of some cofinal subset of α. Intuitively, this is the order type of the smallest fully cofinal set. When we speak of 'greatest elements' in an ordinal sense, we must take limits to treat the concept formally, so that the cofinality of some ordinal α is the ordinal β such that there exists a β-indexed sequence with α as its limit. This can be intuited as the length of the shortest possible sequence leading up to it from below. As an example, take ω^2, the order type of the countably infinite set of countably infinite sets.[5] Its cofinality is ω, since the limit of the ω-sequence of ω's, ωn with n ranging over all of $\mathbb{N}$, is ω^2. By definition of a limit ordinal, the cofinality of any countable limit ordinal is ω. Another example is of the first uncountable ordinal (Hausdorff denotes this Ω), the first ordinal number not in bijection with $\mathbb{N}$. The cofinality of Ω is itself Ω, by the same limiting argument—Ω is the collection of *all* countable ordinals. Reversing the order of an order type is given by the 'adjoint' order type α^*, such that the definitions for coinitial ordinals follow as limits from below of reversed sequences. By defining cofinality, Hausdorff accomplishes what he sets out to do. In analogy with ordinal numbers $[n] = \{1, \ldots, n\}$, the cofinal element of an ordered set

[5]Think of a copy of $\mathbb{N}$ at every place in $\mathbb{N}$, or the product $\mathbb{N} \times \mathbb{N}$. The resulting pairs (i, j) range across $\mathbb{N}$ in both arguments.

is its order type. Hence, the order type of a subset of a set cannot possibly be larger than the cofinality of its parent set.

In this way, he is able to say something about the kinds of subsets associated to a set with a given order type, and in particular, he is able to say when a set is *dense-in-itself*. A set which is dense-in-itself is a continuum: it contains no isolated points. It seems a tautology to say that a set is only dense if it is 'big enough' (or contains sufficiently many elements) for all of its subsets to also be dense; Hausdorff formalises this by showing that only particular order types have a cofinal element large enough to admit infinitely many subsets of infinite order type (i.e., particular ω-sequences), such that there is no isolated element in the set.

He goes on to relate subset order types to the order type $\mu(\alpha)$, offering a theorem that if either the argument α is a limit ordinal or the base μ is a dense-in-itself type, then $\mu(\alpha)$ is a dense-in-itself type.

4.3. **Order type III.** In discussing order type III, the idea of "Dedekind continuity" is also defined, where a set is of continuous type if any cut in the set produces at most one bounded subset. In other words, for a set of order type μ, any decomposition is given by $\mu = \alpha \cup \beta$, where α and β are generic order types of the resulting subsets. If α has a last element, or β a first one—but not both—then the set is of continuous type. Now, if a set of type μ is coinitial with a set of type β^* and cofinal with α, where α and β are ordinals, it clearly means that every element of μ is a limit of an α-sequence and a β^*-sequence. Hausdorff mentions the particular case where we have $\Omega\omega^*$-elements. These elements in general are of crucial importance to us, because coinitiality and cofinality determine whether there is some limitingly smallest or largest element in an infinite set—and so here, we have encountered a primordial Hausdorff gap.[6]

5. Order types IV and V, their gaps, and final behaviour

The first relevant result to us is Hausdorff's exploitation of incomparability, in the opening of the second half of the referenced 1907 paper. Like Borel (who encountered this issue at one point in his work) and Cantor, Hausdorff noted there are pairs of functions f and g that are infinitarily incomparable when the limit of their quotient does not exist. He denoted such pairs by $f \parallel g$, alongside du Bois-Reymond's $f \succ g$ and $f \sim g$. Incomparability is a crucial flaw in du Bois-Reymond's infinitary rank ordering, according to Hausdorff. He says, since it "has no analogy in the domain of ordinary numbers," that "the relation of the infinitary rank-order to the simple order of quantities [the real line] is completely destroyed." As a result, "all attempts to produce a simply (linearly) ordered set of elements in which infinite has its definite place, must

[6]Such gaps were of crucial importance to Hausdorff as well. This was positioned as a way of proving the continuum hypothesis—Hausdorff intended a proof along the lines of $\mathbb{R}$ contains an $\Omega\Omega^*$-gap, and thus must be large enough to contain such a gap, and thus must be larger than Ω. With the benefit of hindsight we know that the existence of such large cardinals is independent of ZFC, and indeed, so is the continuum hypothesis. The approach Hausdorff desired to take shows strong parallels to Cohen's forcing, and had he the model-theoretic information available in the 1960's, he may just have been successful in generating the immense insight he wished to.

on these grounds fail: the infinitary pantachie in the sense of du Bois-Reymond does not exist."

Hausdorff carries on with a closely related variant of du Bois-Reymond's pantachie, intending to treat it with his own newfound insights into the structure of ordered sets. Using the Hausdorff gap $\omega\omega^*$, indicating the presence of a sequence or set of values of type ω and a sequence of type ω^* such that between these two sets there is no or only finitely-many quantities—in other words, a clean or finitely unclean cut, on both sides of the set containing each sequence as a subset—Hausdorff proved there was no $\omega\omega^*$-gap in his variant of the infinitary rank ordering. We will briefly describe the construction his pantachie and the proof of its gapless-ness.

5.1. **On pantachies.** Under Hausdorff's demonstration of incomparibility, the set which du Bois-Reymond intended is at best partially ordered. Hausdorff takes what we would now call a *maximal chain* in this partially ordered set; it is Hausdorff's maximal principle, an early version of which appears in this 1907 paper on order types IV and V, that provides the existence of such a chain. Here, a pantachie is formally defined as a totally linearly ordered set not contained in any other such set. Again, this is precisely what we today call a maximal chain. Hausdorff also changes the ordering from the quotient of two functions to the 'final behaviour' of the function. This condition is rather like one function producing a greater cofinality than another. Hausdorff formalises this as: a function f is *finally greater than* g if there is an $f(n)$ greater than all $g(n)$. The author imagines an equivalent cofinality condition in the following sense: let $\{x_n\}_f$ denote the entire sequence of values $x_n = f(n)$, or the function evaluated on all $n \in N$. We have $f \succ g$ if there exists an x_n in $\{x_n\}_f$ greater than all of the elements of $\{x_n\}_g$. If that is the case, then the greatest such element in $\{x_n\}_f$ must be the cofinal element in $\{x_n\}_f$, since any x_n satisfying this is either the greatest element in $\{x_n\}_f$, or a lower bound for a greater also satisfactory element in $\{x_n\}_f$. Thus, by definition of a cofinal element, the cofinality of $\{x_n\}_f$ is strictly greater than the cofinality of $\{x_n\}_g$. In addition to finally greater, the other relations (finally less than, equal to, and incomparable to) follow as expected.

Hausdorff then defines his fifth order type: that of pantachies. He calls this the H-type. This is itself informed by his investigations into order type IV, occurring in the section preceding this one in [Hau07], which focusses on unbounded sets with no initial or final element and everywhere density.[7] It is at this point in [Hau07] that he retrospectively describes a kind of maximal principle, formally proving the existence of pantachies. He provides a proof that such a maximal set must exist, even if it is trivially the original set, or else every ordered set would have greater cardinality than the continuum by virtue of being contained in a still larger ordered set. This is clearly a direct precursor to Hausdorff's maximal principle, proven in 1914 using Zorn's lemma. He then gives a series of results about H-types, especially as they relate to features of the final rank ordering constructed earlier in the paper. Ultimately, he provides the following characterisation: the H-type is everywhere dense

[7]Aside from its relevance to dealing with continuum type objects, this order type does not affect our discussion much, as the key results in this paper relate the existence of certain gaps to the cardinality of continuum objects, in pursuit of the continuum hypothesis.

and unbounded, neither cofinal with ω nor coinitial with ω^*, and contains no $\omega\omega^*$-gaps. Intuitively, density is equivalent to being gapless, since any everywhere dense set will have an infinite number of smaller quantities on one side of any cut defining a supposed boundary. Take, for instance, the cut at 1 defining the set of real numbers containing the natural numbers as embedded items. What is the largest number in the set not containing any naturals? Is it $0.9999\ldots9$? How many nines are included in this ideal boundary between the reals containing and not containing any natural numbers? Without taking limits, this question is poorly defined: there is *no* such number. We can always construct a slightly larger number which is still not big enough to be in the set of real numbers containing a natural number. In $\mathbb{R}$, we can take limits, so our problem is solved. There is no sense in which this pantachie is complete, so we are stuck. In a non-standard analytic setting where infintesimal quantities are formalised through ultrafilters, there would be slightly more to say about this, but as it stands, there is no gap. In fact, Hausdorff proves a stronger condition about mixture in decompositions, as described.

At this point, using the definition of a pantachie and the derived properties of H-types, he applies his new insights to the final rank ordering as a case study in pantachies. In particular, he tidies up some results due to Pringsheim, who suggested that a more slowly growing divergent series, or more quickly growing convergent sequence, can always be constructed. Using arguments about the cofinality of subsets, and the unboundedness and density of H-types, he proves that

(1) If $\{U_i\}$ is a countable set of convergent sequences[8] ordered by their final behaviour, and P is an arbitrary sequence finally greater such that $\{U_i\} \prec P$, there always exists a convergent sequence X between them such that $\{U_i\} \prec X \prec P$,

(2) If $\{V_i\}$ is a similar set of divergent sequences, and A a finally greater sequence $\{V_i\} \prec V$, there is always a divergent sequence Y between them such that $A \prec Y \prec \{V_i\}$, and,

(3) If every sequence in $\{U_i\}$ is less than every sequence in $\{V_i\}$, then there are infinitely many convergent sequences X *and* divergent sequences Y between $\{U_i\}$ and $\{V_i\}$ such that $\{U_i\} \prec X, Y \prec \{V_i\}$.

The lattermost result is of particular interest, as it establishes that for any 'gap' we can always take a further 'gap', *ad infinitum*. Statement 3 clearly follows from Statements 1 and 2, so we look at those. To prove Statement 1, note the following: we can construct a converging sequence $\{U_i\} \prec X$ for any set of sequences by general facts of real analysis, at this point known due to Pringsheim. This means that, for a countable set of convergent sequences, there exists a convergent ω-sequence[9] X finally greater than all in $\{U_i\}$. Now, iterating the construction of X, we construct a set P

[8]This ought to be isomorphic to the ordering of $\mathbb{R}$-valued functions, given the canonical identification $U_i = \{x_n\}_f \mapsto f(n)$. Indeed, Hausdorff is implicitly using the fact that functions produce sequences in some sense. Moreover, this applies just as easily to series, regarded as sequences of partial sums S_n.

[9]Recall that an ω-sequence is a sequence of values x_n indexed by all of $\mathbb{N}$—hence, this is just the result of a function $f : \mathbb{N} \to \mathbb{R}_{>0}$.

finally greater than X, such that $\{U_i\} \prec X \prec P$. In fact, inducting on the process reveals there are infinitely many such X's. To prove Statement 2, we repeat the process, changing directions of bounds as necessary.

This is certainly an indictment of du Bois-Reymond's boundary—no gap can be realised in Hausdorff's countable set. Hausdorff states in a remark under this proof that "in particular, between an ω-sequence of convergent elements and a following ω^*-sequence of divergent elements there always exists further convergent and divergent sequences; so that it seems totally wrong to want to fill in any such [postulated] $\omega\omega^*$-gap, in which by right there belong infinitely many real elements, by a single 'ideal' element."

However, this is somehow unsatisfactory to him, as he returns to this issue in a paper following his investigation into order types IV and V.

5.2. **Final behaviour.** In 1909, Hausdorff revisits the question in [Hau09], his paper „Die Graduireung nach dem Endverlauf" [EN: Graduation by Final Behaviour], proving a slightly better result. Perhaps he was unsatisfied with the previous results he had, which rely on having countable sets of sequences, whilst du Bois-Reymond had already known that this was not where the answer to his question would lie—see, for instance, his remark about an 'irrational' function.

After briefly reviewing the results he gave in his investigation into order type V, Hausdorff establishes some results about gaps in the uncountable case. He comments explicitly that this is dependent on the continuum hypothesis, as the existence of certain types of gaps in a pantachie ought to imply that a continuum object must have a certain cardinality. He first develops a more complicated ordering, introducing intermediate relationships based on how many elements of a sequence are finally greater than all the elements of another. Let $\mathfrak{A} = \{A_\alpha\}$ be a transfinite set of sequences, and $\mathfrak{A} \prec B$. One intermediary possibility is that B *non-uniformly surpasses* the set $\mathfrak{A}$. Recall a set B is finally greater than a set A_α if B contains at least one element that is greater than every element in A_α, and that this can be expressed by

$$a_n < b_n \quad \forall n > \nu_\alpha,$$

where ν_α is a particular lower bound for how quickly B becomes finally greater than A_α—i.e., the smaller ν_α is, the faster B grows compared to A_α. We say B non-uniformly surpasses $\mathfrak{A}$ if (in Hausdorff's words) "no infinite set of ν_α remains below a fixed number"—clearly, this is the condition that there is no upper bound on the sequence $\{\nu_\alpha\}$. As a simple example, consider that the set $B = \{1, 2, 3, 4\}$ has $\nu_1 = 1$ for $\{1\}$, $\nu_2 = 2$ for $\{1, 2\}$, and $\nu_3 = 3$ for $\{1, 2, 3\}$. It is certainly true that

$$\mathfrak{A} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\} \prec B,$$

since B is finally greater than every $A_\alpha \in \mathfrak{A}$. Due to the variability in the 'critical numbers' ν_α, we would say that B non-uniformly surpasses $\mathfrak{A}$.

Hausdorff now does the following: he constructs two Ω-sequences of ω-sequences, $\mathfrak{A}$ and $\mathfrak{B}$, where each A_α is non-uniformly surpassed by some B_α, for whom there is an $\Omega\Omega^*$-gap lying between, where no numerical sequence is realised. He then tries to relate this to the final rank ordering, quoting du Bois-Reymond on the irrational boundary found in a decomposition of an uncountable set. Indeed, he is able to prove that such a pantachie does not necessarily prohibit a gap—but evidently was unable to construct one, as he leaves the problem open.

In some greater detail, Hausdorff is able to give the following result. Let $\mathfrak{B}_c$ be the Ω-sequence of all convergent numerical sequences, and $\mathfrak{B}_d$ divergent. If $\mathfrak{X}$ is any interval of a pantachie consisting of real sequences, and $\mathfrak{X} = \mathfrak{X}' + \mathfrak{X}''$ is a cut of this interval, then there is an interval $\mathfrak{B} = \mathfrak{B}_c + \mathfrak{B}_d$ similar to $\mathfrak{X}$ such that $\mathfrak{B}_c$ and $\mathfrak{B}_d$ are similar to $\mathfrak{X}'$ and $\mathfrak{X}''$, respectively. In particular, if $\mathfrak{X}$ contains a gap, then so too could $\mathfrak{B}$. The best he is able to actually construct is that $\mathfrak{B}_c$ could have a final element, finally greater than all others, interpreted in this sense as a fastest growing convergent sequence; independently, $\mathfrak{B}_d$ could have a first element. Hausdorff is evidently unable to realise both simultaneously in the same pantachie. Again, he leaves open the question of whether a patch exists, suggesting an ambient domain may realise both simultaneously.

All things considered, it seems that, despite the controversy du Bois-Reymond created, there is something like his infinitary rank ordering, given by Hausdorff using his maximal principle, and this could exhibit something like a gap, also investigated by him. Hausdorff comments explicitly that the construction of such a gap inside an ambient domain would link a continuum with other results following from the various order types, including cardinality, suggesting a solution to Cantor's continuum hypothesis. This is Hausdorff's last remark on the matter, as he moved on to do what would become foundational work in topology.[10]

6. Final reflections on du Bois-Reymond's boundary

To the author's knowledge, a conclusive proof of the non-existence of a single function constituting the boundary between convergence and divergence does not appear in print until 44 years later. Walter Rudin describes in [Rud53], his 1953 book "Principles of Mathematical Analysis" a proof that, given any divergent series, a more slowly growing divergent series can be constructed; likewise, given any convergent series, a more quickly growing convergent series can be constructed. He refers here to [Kno22], Konrad Knopp's 1922 book "Theory and Application of Infinite Series," where a diagonalisation-type argument due originally to Pringsheim is found. Many years after this, in [Fol99], Gerald Folland proves in chapter five of his 1999 textbook "Real Analysis: Modern Techniques and Their Applications" that there is no possible worst convergent series. This proof is of particular interest to the author, so it is described below.

Take two sequences, $\{a_n\}$ and $\{c_n\}$, perhaps the sequence of values given by evaluating a function at every natural number, like $a(n)$ or $c(n)$. Indeed, take individual terms in the sequence as a_n and c_n. Recall that a series $\sum a_n|c_n|$ converges absolutely if $\sum a_n$ converges absolutely and c_n is bounded. Suppose there exists an a_n such that if *and only if* c_n is bounded,

$$\sum_{n=0}^{\infty} a_n|c_n| < \infty.$$

[10]It was shortly after this that he defined the various notions of a Hausdorff space, with which most everyone is familiar.

That is to say, we claim there exists a single sufficiently slow rate of increase of the partial sums of a series to guarantee the convergence of a_n, and thus of any $a_n|c_n|$, and then take the sequence with that rate for a_n and bound c_n arbitrarily. Assume the claim is true, and such a sequence exists. If $\mathbb{B}(\mathbb{N})$ is the space of bounded sequences on N and ℓ^1 is the space of functions with $\sum_{n=0}^{\infty} x_n < \infty$, then the claimed result induces a map $T : \mathbb{B}(\mathbb{N}) \to \ell^1$ given by $a_n \mapsto a_n|c(n)|$ for any $c(n)$. Since T is both injective and surjective, T restricts to a bijection. Moreover, by Hölder's inequality, the restriction is bounded, and so using the open mapping theorem, T^{-1} is bounded too. However, there exist sets which are not dense in $\mathbb{B}(\mathbb{N})$ whilst being dense in ℓ^1, making such a bijection impossible. The counterexample given by Folland is the set of functions for which $c(n) = 0$ for all but finitely many n, e.g., element-wise indicator functions like the Kronecker delta $\{\delta_{kn}\}$, returning zero for $k \neq n$ and one for $k = n$. This contradiction disproves the claim.

So, finally, we are assured there can be no single worst convergent series. We could consider this satisfactory, in that it implies there are infinitely-many boundary functions, so that there is an infinitely large class of sequences giving this boundary. Nonetheless, with this result, we seem to close the book on du Bois-Reymond's final hope of some unknown 'irrational function' providing the boundary he sought.

7. Conclusion

It is remarkable that a large part of the historical motivation for ordered set theory is due to, essentially, a side-quest in the analysis of real-valued functions. It was surely a difficult question—an interesting question that took many years and many minds to solve—and this is often the most fertile ground for growing new mathematical ideas.

My own motivation for studying this question was along the same lines as many of the characters in this story. When I was quite young, I fashioned for myself a course in logic, set theory, and real analysis. At the time, the existence of this boundary seemed like a natural question to ask. I went on to formulate my own proof—in fact, my first proof, ever—of its non-existence. The proof I gave was implied by Hausdorff's restatement of Pringhseim's results, but I didn't realise whilst writing it. Accordingly, imagine my surprise when I then consulted the literature and learned that there was a wealth of discussion about it already, and that some truly great mathematicians had been ensnared by the very same question. To write my own proof, I had unintentionally gone through the same set of steps as they all had—notice some functions decrease 'too slowly' for their infinite series to converge, ask what 'too slowly' could possibly mean, use set theory to try and order everything by growth rate, and find that this is impossible to formalise. Moreover, this experience was my first real exposure to research mathematics. Although it is seemingly unknown to the rest of the world, it has become a story of great personal importance for this reason.

With the benefit of hindsight, we now know how much we *all* owe to this controversy in the history of modern mathematics. In particular, the work by Hausdorff that was partially inspired by this problem went to the very heart of set theory, and du Bois-Reymond's misadventures with infinities sharpened our ideas of sets, real analysis, and even the more philosophical side of logic and mathematics. Perhaps it would not be hyperbole to say that his curiosity, and the work it inspired, has led us directly to modern mathematics.

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