인하대학교
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Electromagnetics 1 (ICE2003)

-- Ch. 3. Electric Flux Density, Gauss' Law, and Divergence --

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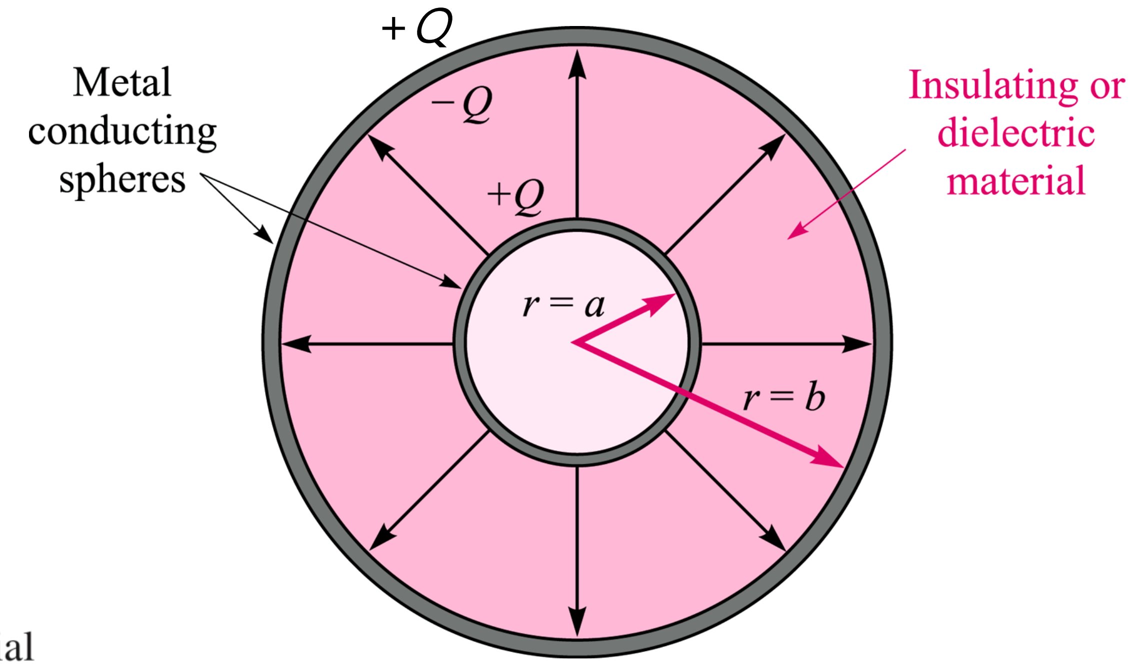
Chapter Outline

- Faraday Experiment
- Electric Flux Density (**D**) 개념 이해
- Gauss's Law
- Gauss's Law를 이용하여 몇 가지 단순한 전하 분포에서의 **E, D** 계산 연습
  - 점전하
  - 선전하
  - 면전하
  - 부피전하
- Divergence Theorem

Faraday's experiment

He started with a pair of metal spheres of different sizes;
the larger one consisted of two hemispheres that could be assembled around the smaller sphere

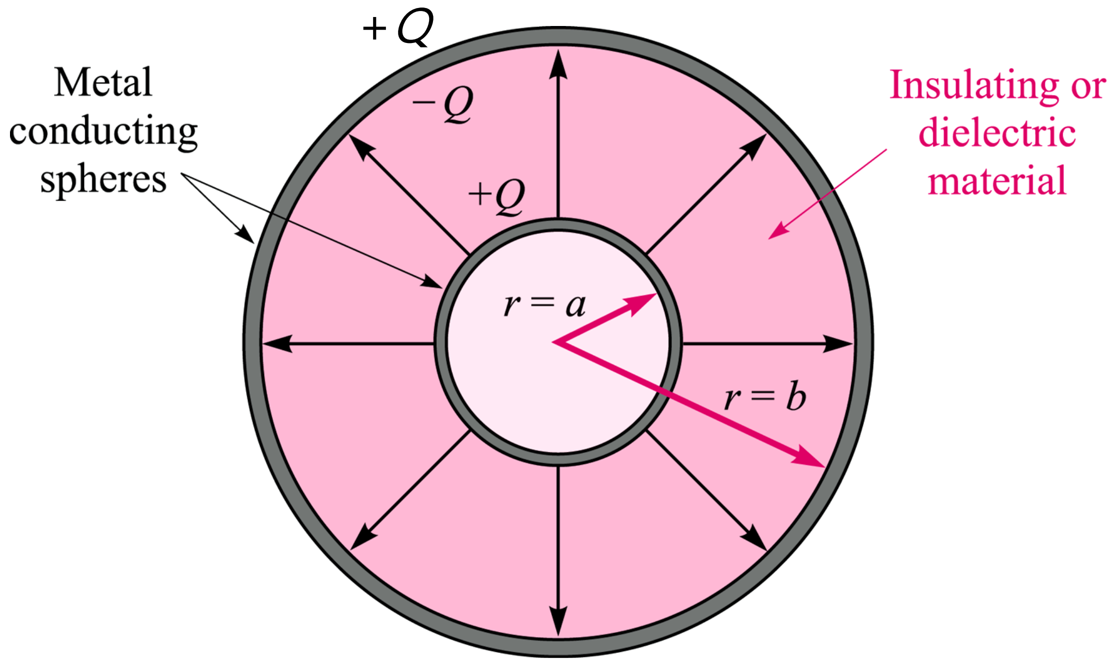


1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.

Faraday's experiment

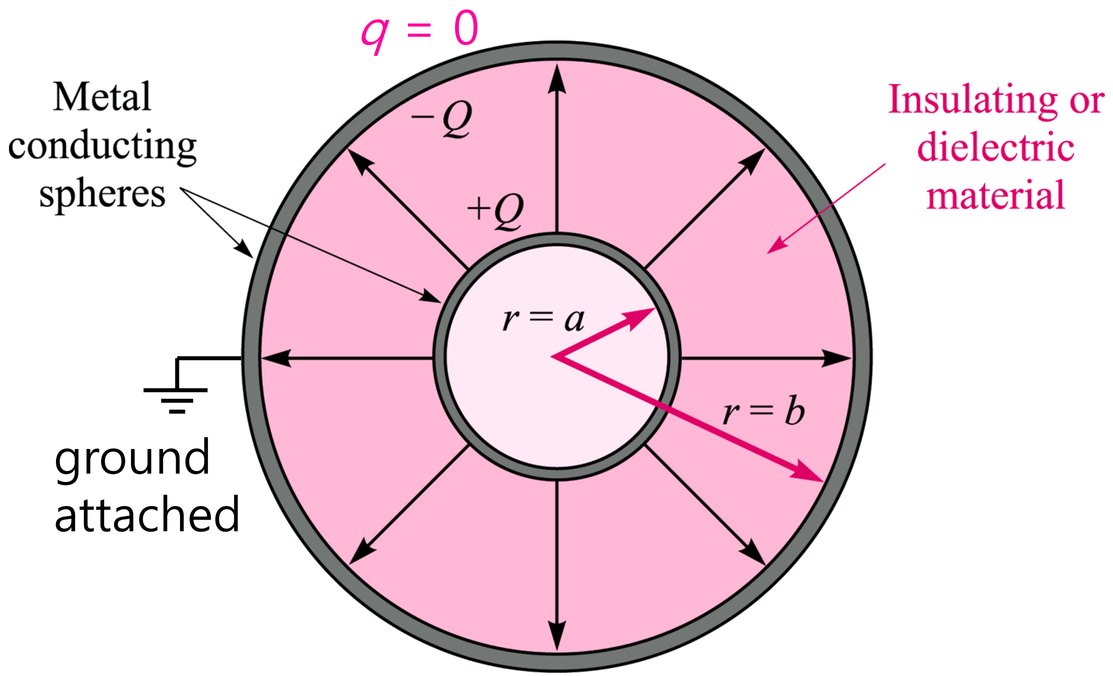
The inner charge, *Q* induces an equal and opposite charge, -*Q* on the inside surface of the outer sphere, by attracting free electrons in the outer material toward the positive charge.

This means that before the outer sphere is grounded, charge +*Q* resides on the *outside* surface of the outer conductor.



Faraday's experiment

Attaching the ground connects the outer surface to an unlimited supply of free electrons, which then neutralize the positive charge layer. The net charge on the outer sphere is then the charge on the inner layer, or -*Q*.



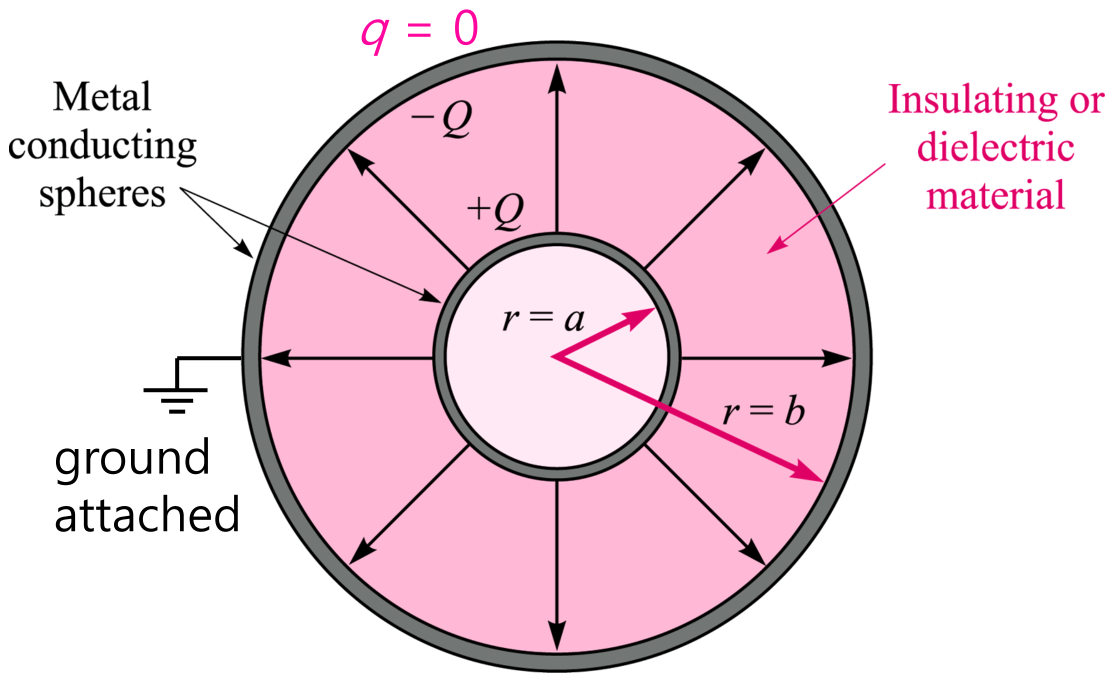
Faraday's experiment

Faraday concluded that there occurred a charge "displacement" from the inner sphere to the outer sphere.

Displacement involves a *flow* or *flux,* Ψ, existing within the dielectric, and whose magnitude is equivalent to the amount of "displaced" charge.

Specifically:

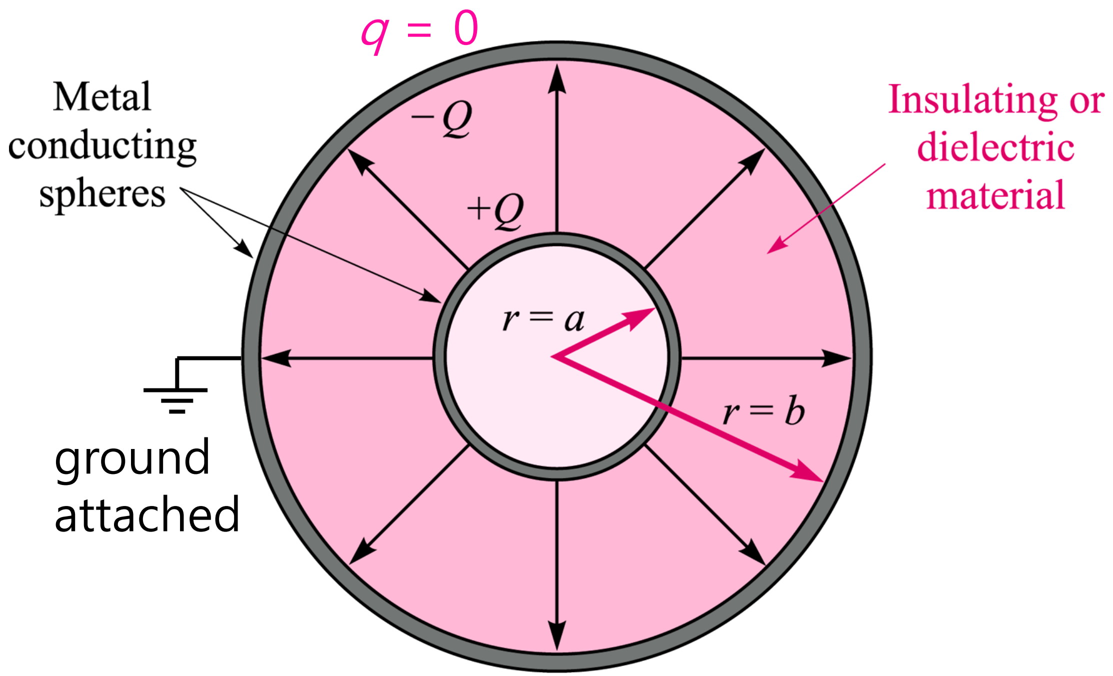
$$\boxed{\Psi = Q}$$



Electric Flux Density

The density of flux at the inner sphere surface is equivalent to the density of charge there (in Coul/m^2)

$$\boxed{D(r=a) = \frac{\Psi}{4\pi a^2} = \frac{Q}{4\pi a^2}}$$

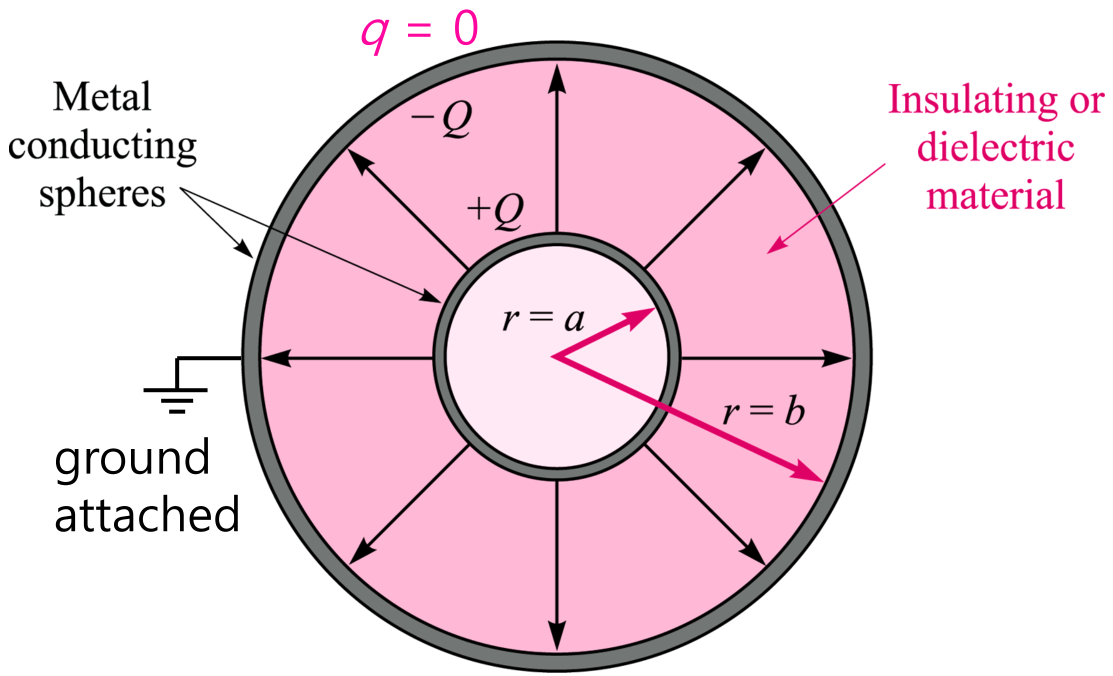


Electric Flux Density

A vector field is established which points in the direction of the "flow" or displacement. In this case, the direction is the outward radial direction in spherical coordinates. At each surface, we would have:

$$\mathbf{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\mathbf{a}_r \qquad \text{(inner sphere)}$$

$$\mathbf{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\mathbf{a}_r \qquad \text{(outer sphere)}$$

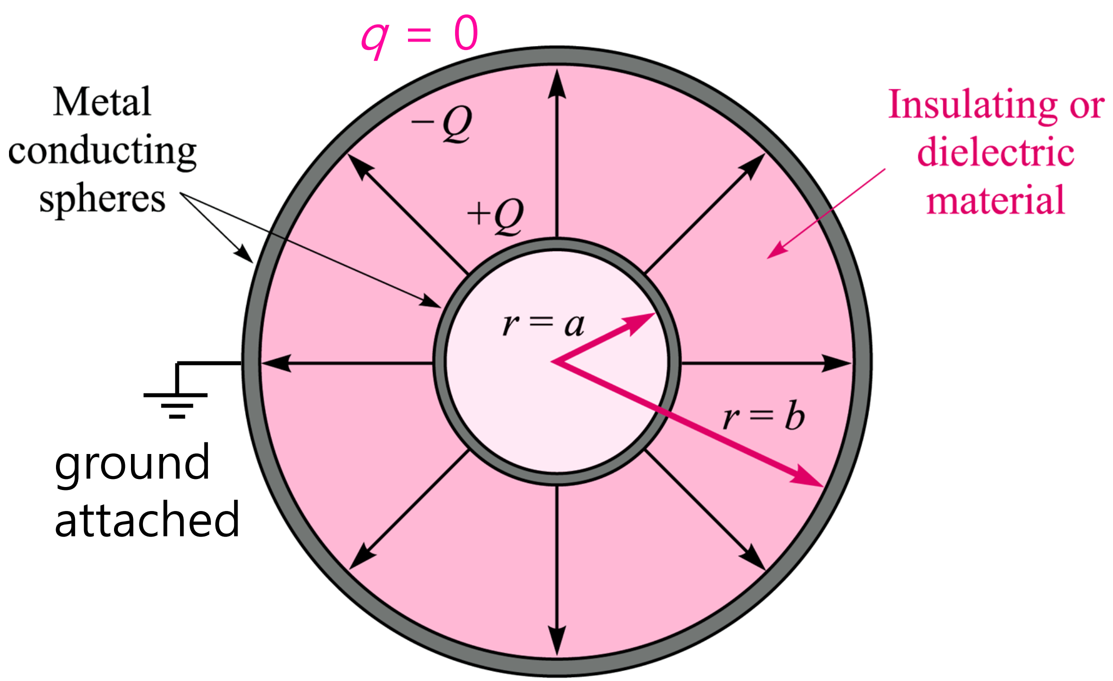


Electric Flux Density

At a general radius r between spheres, we would have:

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

Expressed in units of Coulombs/m^2, and defined over the range ($a \leq r \leq b$)



Point Charge Fields

If we now let the inner sphere radius reduce to a point, while maintaining the same charge, and let the outer sphere radius approach infinity, we have a point charge. The electric flux density is unchanged, but is defined over all space:

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r} \quad \text{C/m}^2 \qquad (0 < r < \infty)$$

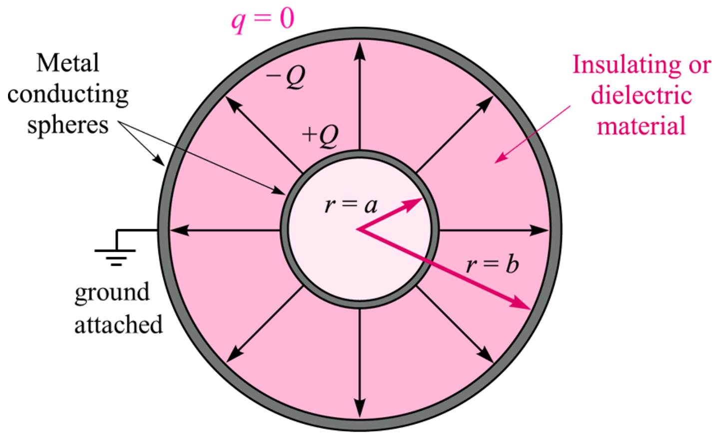
We compare this to the electric field intensity in free space:

$$\boxed{\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\mathbf{a}_r} \quad \text{V/m} \qquad (0 < r < \infty)$$

..and we see that: $\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}}$ (free space only)

Electric Flux Density

Faraday's Experiment



- Electric Flux defined $\Psi = Q$
- **D**: Electric Flux Density
- In this case,

Flux = Charge

Flux density

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a_r} \quad (\mathrm{C/m^2})$$

Spherical surface

- And $$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a_r}$$
- Therefore $$\boxed{\mathbf{D} = \varepsilon_o \mathbf{E}}$$

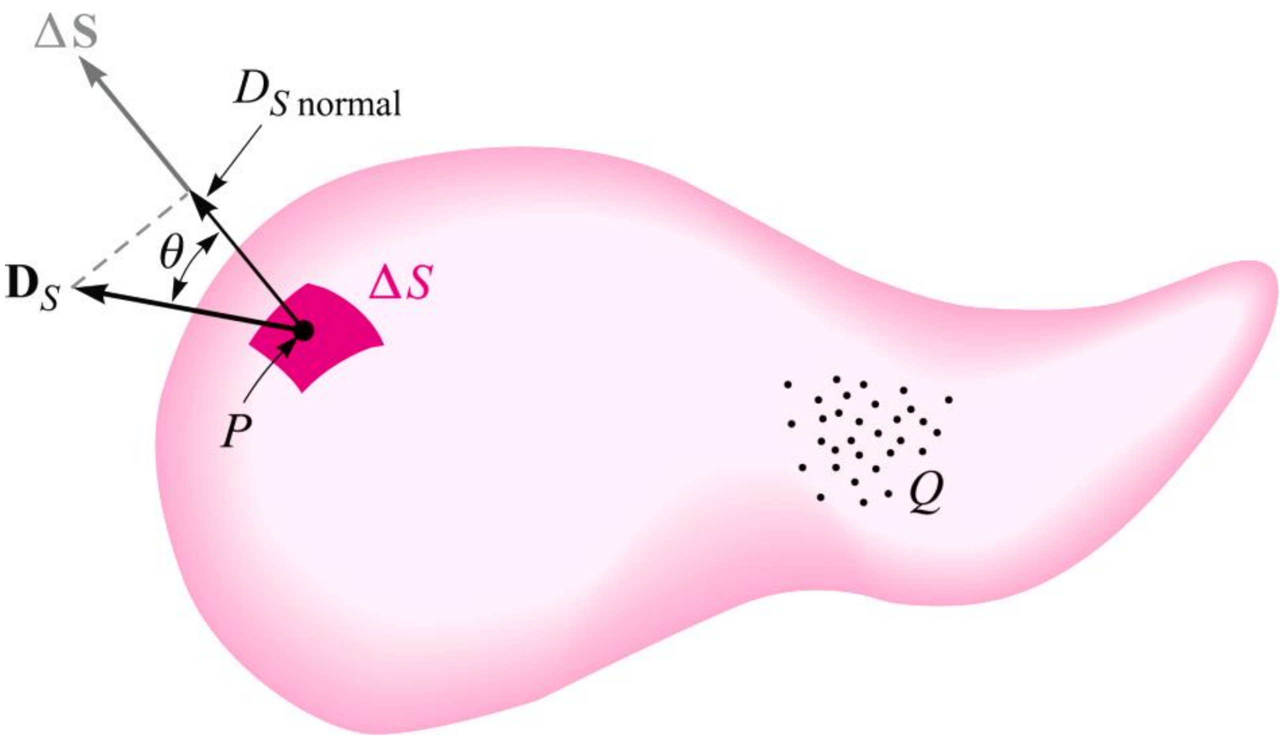
Electric Flux Density

$$\mathbf{D} = \varepsilon_o \mathbf{E}$$

- Electric Flux Density
- The # of flux lines per unit area
- Dielectric medium independent (in simple case)
- (Determined by Q, Geometry)
- Usually linear to **E**

- Electric Field Intensity (Force)
- Force exerted to a Test charge
- Dielectric medium dependent

Gauss's Law

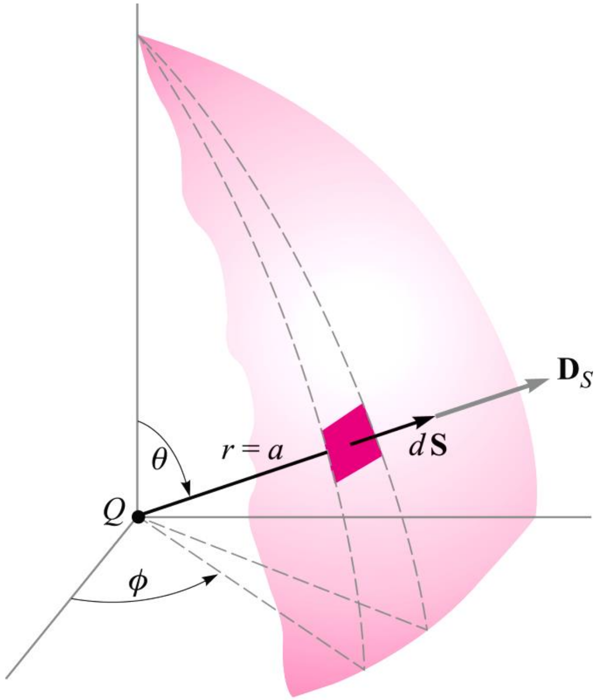


$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q$$

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\mathrm{vol}} \rho_v \, dv$$

Total outward electric flux flowing through an arbitrary closed surface equals to the total amount of charge within the closed surface.

Point charge field

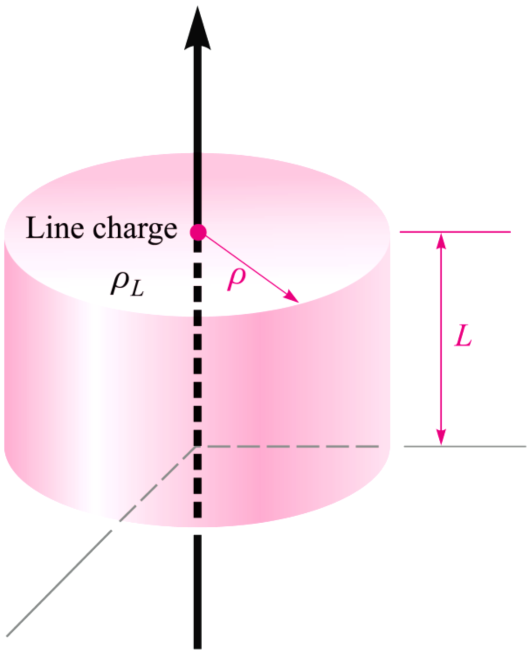


$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a_r}$$

$$\mathbf{D} = \varepsilon_o \mathbf{E} = \frac{Q}{4\pi r^2}\mathbf{a_r}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S \frac{Q}{4\pi r^2}\mathbf{a_r} \cdot d\mathbf{s} = Q$$

Line charge field



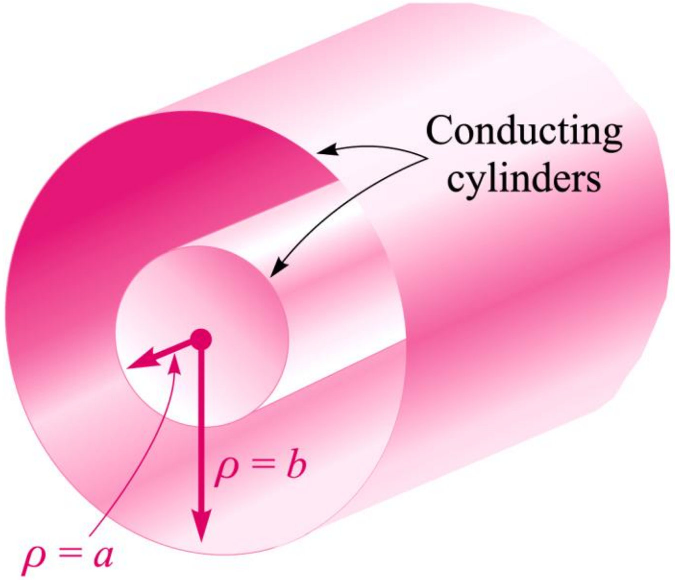
$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_0^L \int_0^{2\pi} D_\rho \mathbf{a}_\rho \cdot \mathbf{a}_\rho \rho \, d\phi \, dz = \rho_L L$$

$$\mathbf{D} = \frac{\rho_L}{2\pi\rho} \mathbf{a}_\rho$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o \rho} \mathbf{a}_\rho$$

Coaxial Transmission Line



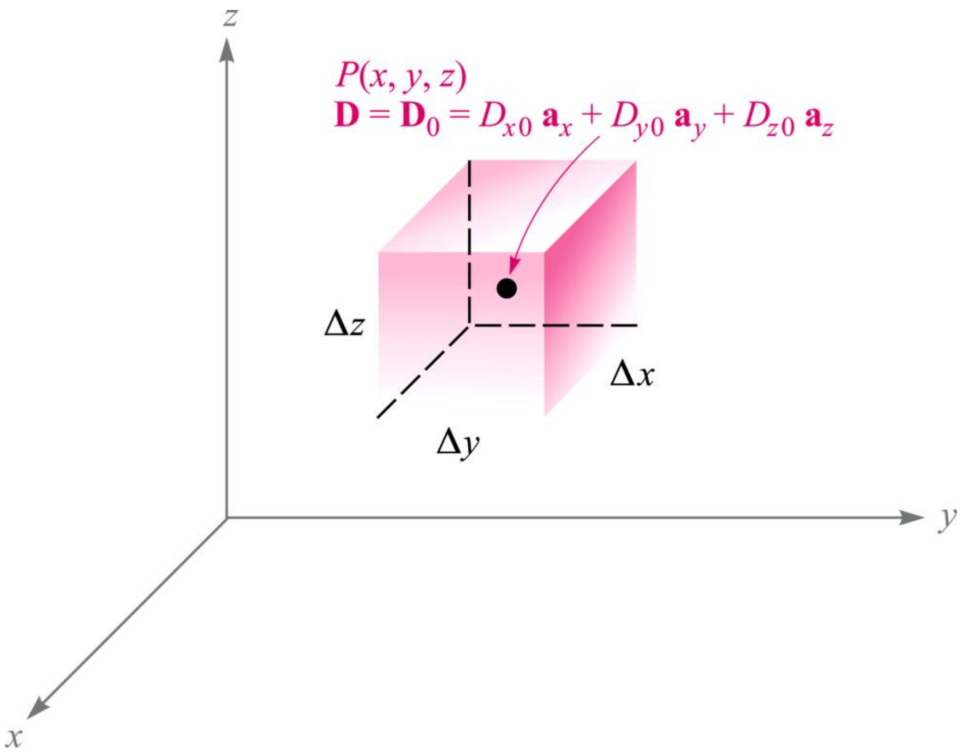
$$\mathbf{D} = D_\rho \mathbf{a}_\rho$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_0^L \int_0^{2\pi} D_\rho \mathbf{a}_\rho \cdot \mathbf{a}_\rho \rho d\phi dz = 2\pi a \rho_S L$$

$$\mathbf{D} = \frac{a\rho_s}{\rho} \mathbf{a}_\rho$$

$$\mathbf{E} = \frac{a\rho_s}{\varepsilon_o \rho} \mathbf{a}_\rho$$

Divergence

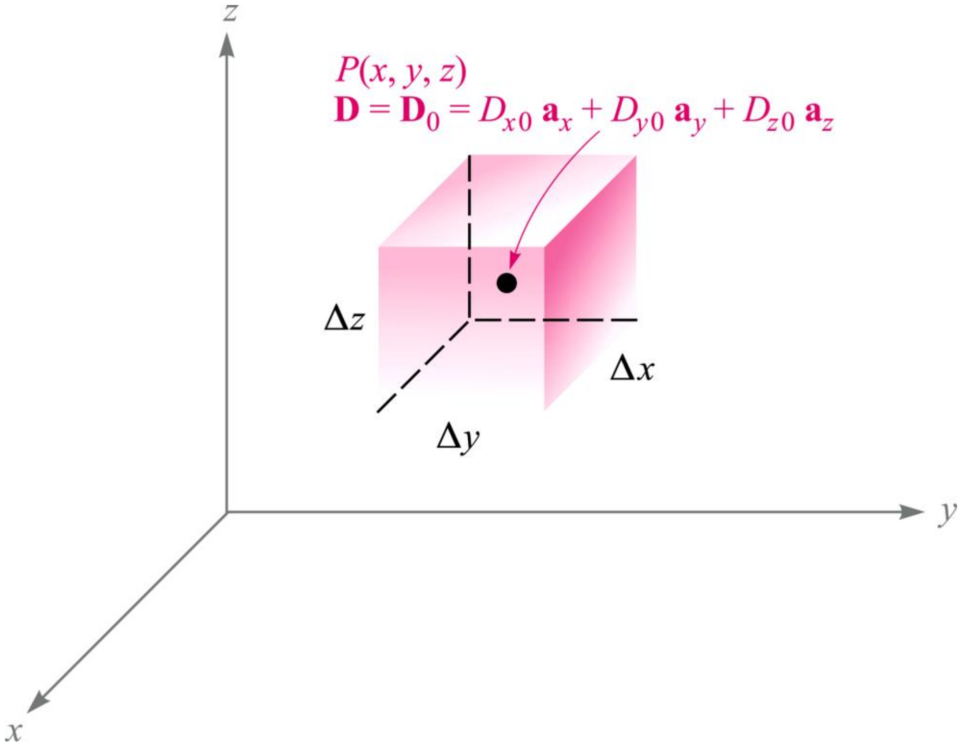


$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

$$\lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v}$$

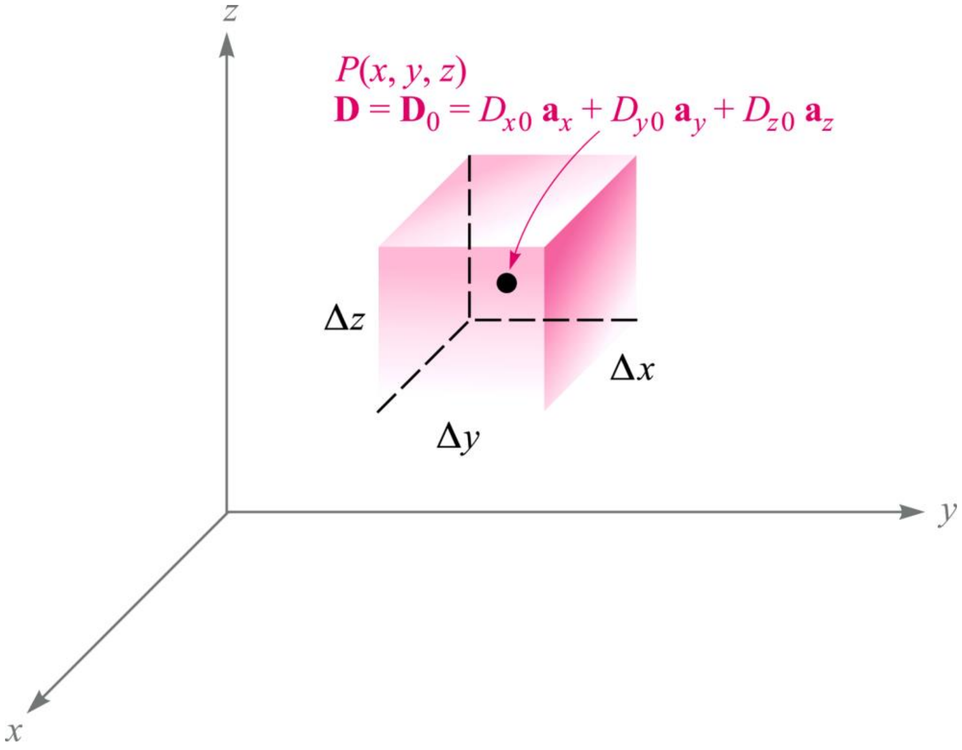
$$\operatorname{div} \mathbf{D} = \rho_v$$

Divergence



$$\boxed{\operatorname{div}\mathbf{D} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D}\cdot d\mathbf{s}}{\Delta v}}$$

Divergence



$$\boxed{\operatorname{div}\mathbf{D} \equiv \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D}\cdot d\mathbf{s}}{\Delta v}}$$

Rectangular

$$\operatorname{div}\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Cylindrical

$$\operatorname{div}\mathbf{D} = \frac{1}{\rho}\frac{\partial(\rho D_x)}{\partial \rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Spherical

$$\operatorname{div}\mathbf{D} = \frac{1}{r^2}\frac{\partial(r^2 D_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta D_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Example

Find div D at the origin if $\mathbf{D} = e^{-x} \sin y \mathbf{a_x} - e^{-x} \sin y \mathbf{a_y} + 2z \mathbf{a_z}$

Maxwell's First Equation

Gauss's Law

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

Point form of Guass's Law

$$\lim_{\Delta v -> 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{\Delta v} = \lim_{\Delta v -> 0} \frac{Q}{\Delta v}$$

Maxwell's First Equation

$$\boxed{div\mathbf{D} = \rho_v}$$

$$\operatorname{div} \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\operatorname{div} \mathbf{D} = \frac{1}{\rho} \frac{\partial (\rho D_x)}{\partial \rho} + \frac{1}{\rho} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\operatorname{div} \mathbf{D} = \frac{1}{r^2} \frac{\partial (r^2 D_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta D_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi}$$

Example

Point charge $$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a_r}$$

$$\mathrm{div}\,\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\mathrm{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial(\rho D_x)}{\partial \rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\mathrm{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial(r^2 D_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta D_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

$$div\,\mathbf{D} = \rho_v = 0$$ For everywhere except r=0

Vector operator

$$div\mathbf{D} = \rho_v$$

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z}$$

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}\right) \cdot \left(D_x\mathbf{a_x} + D_y\mathbf{a_y} + D_z\mathbf{a_z}\right) = \frac{\partial D_x}{\partial x}\mathbf{a_x} + \frac{\partial D_y}{\partial y}\mathbf{a_y} + \frac{\partial D_z}{\partial z}\mathbf{a_z}$$

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial(\rho D_x)}{\partial \rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial(r^2 D_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta D_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

Also applied to scalar functions

$$\nabla u = \frac{\partial u}{\partial x}\mathbf{a_x} + \frac{\partial u}{\partial y}\mathbf{a_y} + \frac{\partial u}{\partial z}\mathbf{a_z}$$

$$\mathrm{div}\,\mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\mathrm{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial(\rho D_x)}{\partial \rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\mathrm{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial(r^2 D_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta D_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

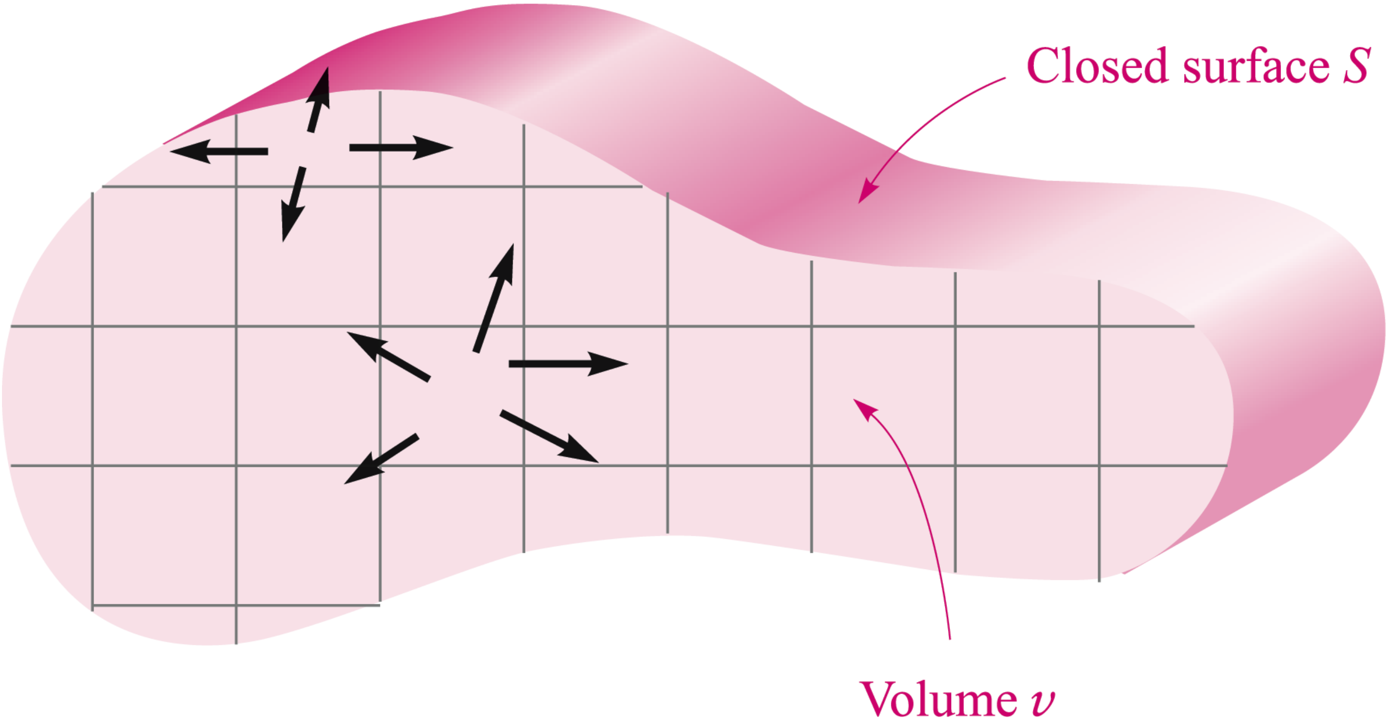
Divergence Theorem

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

$$Q = \int_{vol} \rho_v dv$$

$$\rho_v = \nabla \cdot \mathbf{D}$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv}$$



Example

$$\mathbf{D} = 2xy\mathbf{a}_{\mathbf{x}} + x^2\mathbf{a}_{\mathbf{y}} \qquad x = 0{,}1 \quad y = 0{,}2 \quad z = 0{,}3$$

Chapter Summary

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