1. $\frac{\sqrt{2}}{2}$

2. $y'(e)=1$, $y''(e)=1$

3. $\frac{1}{2}\ln 3$

4. $\frac{3\pi^2}{16}$

5. $(0,0)$, $\left(\frac{3}{8}, \frac{3\sqrt{3}}{8}\right)$, $\left(-\frac{1}{8}, -\frac{\sqrt{3}}{8}\right)$

6. -72

7. $(-\sqrt{2}, 0)$

8. $\sqrt[3]{2}\left(\cos\frac{1}{18}\pi + i\sin\frac{1}{18}\pi\right)$, $\sqrt[3]{2}\left(\cos\frac{13}{18}\pi + i\sin\frac{13}{18}\pi\right)$, $\sqrt[3]{2}\left(\cos\frac{25}{18}\pi + i\sin\frac{25}{18}\pi\right)$

9. (a) $\lim_{x\to 0}\frac{\cos x - 1}{x} = -\lim_{x\to 0}\sin x = 0$이므로 $a=0$

(b) $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\cos x - 1}{x^2} = \lim_{x\to 0}\frac{-\sin x}{2x} = -\frac{1}{2}$ 이므로 $f'(0) = -\frac{1}{2}$

10. $f(x) = \frac{x}{x+1}$에 대하여 $I = \int_0^{\pi}\theta f(\sin\theta)d\theta$이라 하자.

$u = \pi - \theta$로 치환하면 $I = \int_{\pi}^{0}(\pi - u)f(\sin(\pi - u))(-du) = \pi\int_0^{\pi}f(\sin u)du - I$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi}f(\sin u)du = \frac{\pi}{2}\int_0^{\pi}\frac{\sin u}{\sin u + 1}du = \frac{\pi}{2}\int_0^{\pi}\frac{\sin u(1-\sin u)}{1-\sin^2 u}du$$

$$= \frac{\pi}{2}\int_0^{\pi}\left(\frac{\sin u}{\cos^2 u} - \tan^2 u\right)du = \frac{\pi}{2}\int_0^{\pi}(\sec u\tan u - \sec^2 u + 1)du$$

$$= \frac{\pi}{2}\left[\sec u - \tan u + u\right]_0^{\pi} = \frac{\pi}{2}(-1+\pi-1) = \frac{\pi^2}{2} - \pi$$

11. (1) $\lim_{n\to\infty}\frac{(n+1)^2}{2^{n+1}}\times\frac{2^n}{n^2}=\lim_{n\to\infty}\frac{1}{2}\times\left(1+\frac{1}{n}\right)^2=\frac{1}{2}<1$이므로 비율판정법에 의하여 수렴

(2) $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$이므로

$$\frac{1}{(1-x)^2}=\frac{d}{dx}\frac{1}{1-x}=\sum_{n=1}^{\infty}nx^n,\quad \frac{2}{(1-x)^3}=\frac{d}{dx}\frac{1}{(1-x)^2}=\sum_{n=2}^{\infty}n(n-1)x^{n-2}\text{이다.}$$

따라서 $\sum_{n=1}^{\infty}nx^n=\frac{x}{(1-x)^2}$, $\sum_{n=2}^{\infty}(n^2x^n-nx^n)=\frac{2x^2}{(1-x)^3}$이다.

$$\Rightarrow \sum_{n=1}^{\infty}n^2x^n=x+\sum_{n=2}^{\infty}n^2x^n=x+\frac{2x^2}{(1-x)^3}+\sum_{n=2}^{\infty}nx^n=x+\frac{2x^2}{(1-x)^3}+\sum_{n=1}^{\infty}nx^n-x$$

$$=\frac{2x^2}{(1-x)^3}+\frac{x}{(1-x)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n^2}{2^n}=\frac{2\times\frac{1}{4}}{\frac{1}{8}}+\frac{\frac{1}{2}}{\frac{1}{4}}=4+2=6$$

(다른 풀이)

$S_n=\frac{1}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\cdots+\frac{n^2}{2^n}$이라 하자.

$$\Rightarrow \frac{1}{2}S_n=S_n-\frac{1}{2}S_n=\left(\frac{1}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\cdots+\frac{n^2}{2^n}\right)-\left(\frac{1}{2^2}+\frac{2^2}{2^3}+\frac{3^3}{2^4}+\cdots+\frac{n^2}{2^{n+1}}\right)$$

$$=\frac{1}{2}+\frac{2^2-1}{2^2}+\frac{3^2-2^2}{2^3}+\cdots+\frac{n^2-(n-1)^2}{2^n}-\frac{n^2}{2^{n+1}}$$

$$=\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\frac{7}{2^4}+\cdots+\frac{2n-1}{2^n}-\frac{n^2}{2^{n+1}}$$

$T_n=\frac{3}{2^2}+\frac{5}{2^3}+\frac{7}{2^4}+\cdots+\frac{2n-1}{2^n}$이라 하면

$$\frac{1}{2}T_n=T_n-\frac{1}{2}T_n=\left(\frac{3}{2^2}+\frac{5}{2^3}+\frac{7}{2^4}+\cdots+\frac{2n-1}{2^n}\right)-\left(\frac{3}{2^3}+\frac{5}{2^4}+\frac{7}{2^5}+\cdots+\frac{2n-1}{2^{n+1}}\right)$$

$$=\frac{3}{4}+\frac{2}{2^3}+\frac{2}{2^4}+\frac{2}{2^5}+\cdots+\frac{2}{2^n}-\frac{2n-1}{2^{n+1}}$$

$$=\frac{3}{4}+\frac{1}{4}\left(\frac{1-(1/2)^{n-2}}{1-1/2}\right)-\frac{2n-1}{2^{n+1}}\text{이므로}$$

$T_n=\frac{3}{2}+1-\left(\frac{1}{2}\right)^{n-2}-\frac{2n-1}{2^n}$이다.

따라서 $\frac{1}{2}S_n=\frac{1}{2}-\frac{n^2}{2^{n+1}}+\frac{5}{2}-\left(\frac{1}{2}\right)^{n-2}-\frac{2n-1}{2^n}$ 즉 $S_n=6-\frac{n^2}{2^n}-\left(\frac{1}{2}\right)^{n-1}-\frac{2n-1}{2^{n-1}}$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{n^2}{2^n}=\lim_{n\to\infty}S_n=6$$

12. (1) $a_n = \dfrac{(x+1)^n \ln(n+1)}{2nx^n}$이라 하면

$\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{2n\ln(n+2)}{2(n+1)\ln(n+1)}\left|1+\dfrac{1}{x}\right| = \left|1+\dfrac{1}{x}\right|$이다.

비율판정법에 의해 $\left|1+\dfrac{1}{x}\right| < 1$이면 급수 $\sum_{n=1}^{\infty} a_n$는 수렴한다.

즉, $-1 < 1+\dfrac{1}{x} < 1 \Leftrightarrow -2 < \dfrac{1}{x} < 0 \Leftrightarrow x < -2$이면 급수 $\sum_{n=1}^{\infty} a_n$는 수렴

(2) $x = -2$라 하자.

$\sum_{n=1}^{\infty}\dfrac{(x+1)^n \ln(n+1)}{2nx^n} = \sum_{n=1}^{\infty}\dfrac{\ln(n+1)}{n2^{n+1}}$의 수렴성을 확인하기 위하여 $b_n = \dfrac{\ln(n+1)}{n2^{n+1}}$을 생각한다.

$n > 1$일 때 $\ln(n+1) < n$이므로 $b_n < \left(\dfrac{1}{2}\right)^{n+1}$이고 $\sum_{n=2}^{\infty}\left(\dfrac{1}{2}\right)^{n+1}$은 수렴하므로

비교판정법에 의하여 $\sum_{n=2}^{\infty} b_n$은 수렴한다. 따라서 $\sum_{n=1}^{\infty} b_n$은 수렴한다.

(1)과 (2)에 의하여 수렴구간은 $(-\infty, -2]$이다.