Engineering Circuits Analysis (ICE2002)

Chapter 7. Response of First-Order RL and RC Circuit – Part 1/2/3/4

Contents

In Chapter 7

- We will focus on circuits that consists only of source, resistors (R), and either inductors (L) or capacitors (C).

- **Natural response**: we consider the currents and voltage that arises when sored energy in an inductor or capacitor is suddenly released to a resistive network. This happens when the inductor or capacitor is abruptly disconnected from its DC source.

- **Step response**: wen consider when the current and voltage that arises when energy is being acquired by an inductor or capacitor due to the sudden application of a DC voltage or current source.

Natural Response

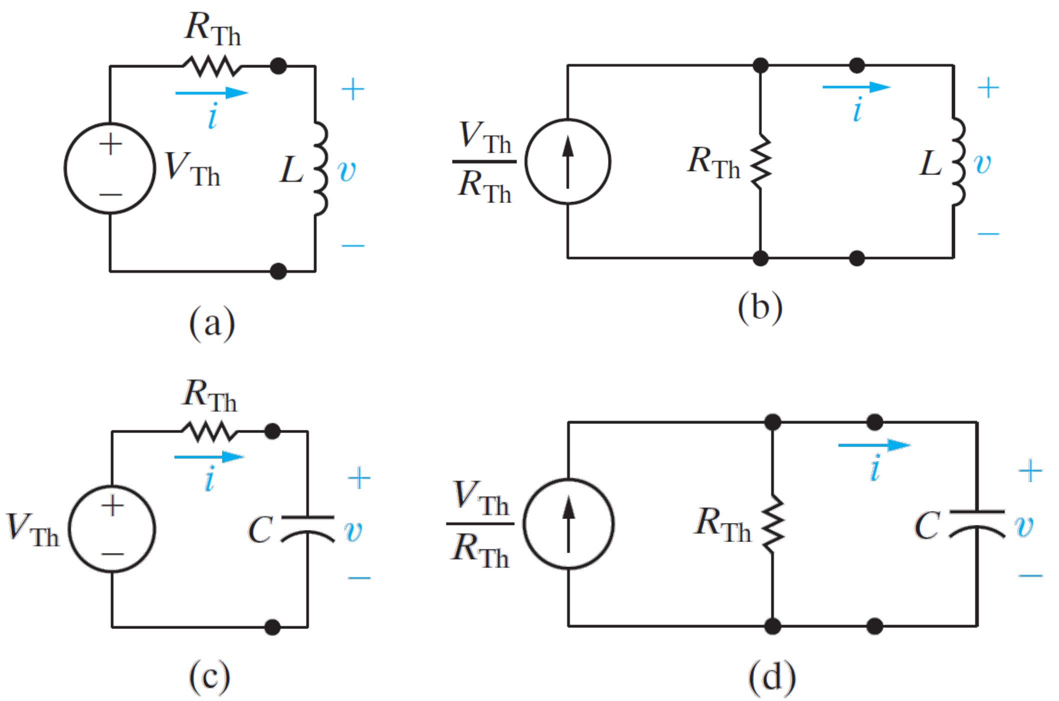
- Two forms of the circuit for natural response: RL and RC Circuits

RL circuit when given initial current, I_0

RC circuit when given initial voltage, V_0

First-Order Circuits

- RL and RC circuits are known as first-order circuits, because their voltage and currents are described by first-order differential equations >> most equations become 1st order differential equations.
- 1st order circuits may be reduced to a Thevenin (or Norton) equivalent



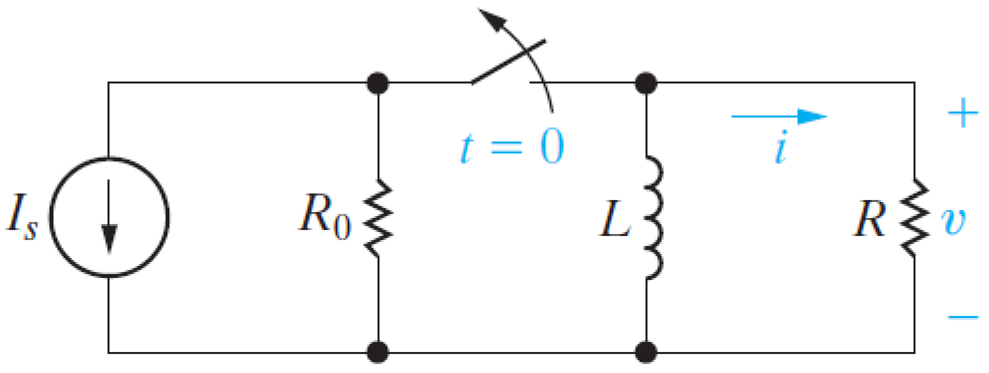
Thevenin equivalent (a and c) to Norton equivalent (b and d) connected with an inductor and a capacitor

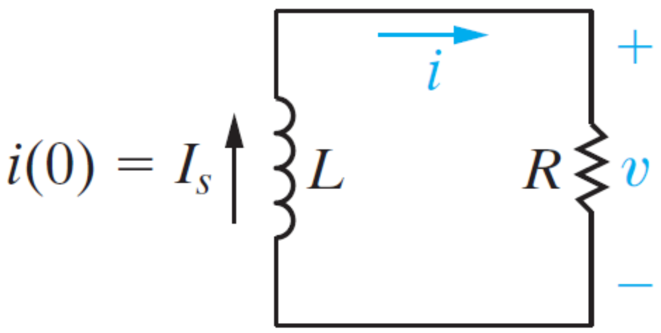
The Natural response of an RL Circuit

- Natural response

  : Response of a circuit due to the initial condition. We will be looking at single time-constant circuit.

  i.e. circuits whose storage elements can be reduced to a single, equivalent storage element and whose resistors can be reduced to a single to a single, equivalent resistance.

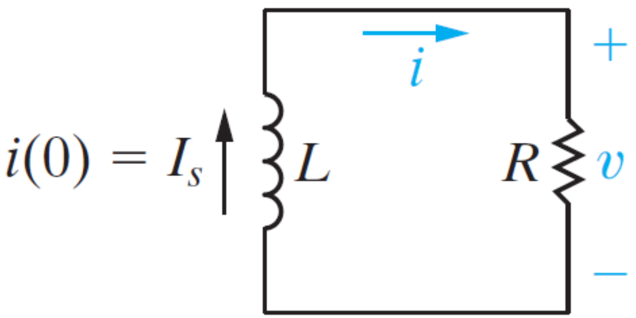




Assuming that the switch is open at t=0

And find i(t) t>0

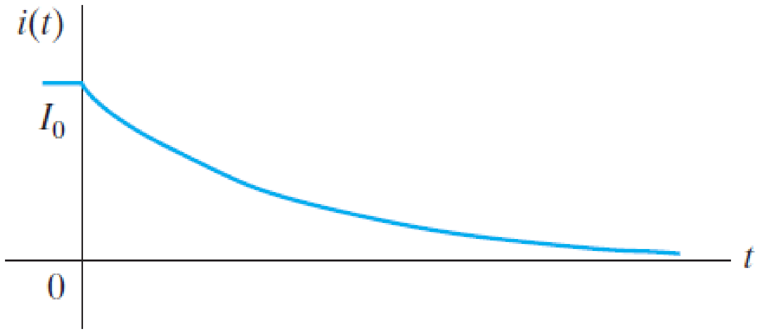
The Natural response of an RL Circuit



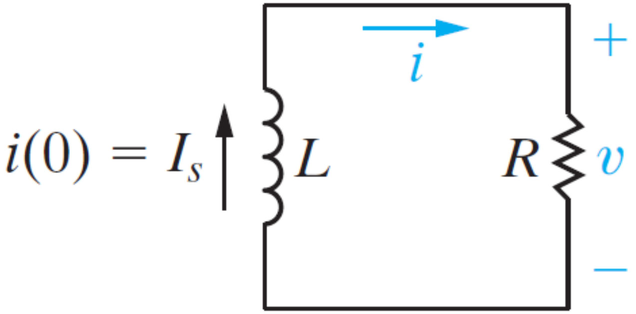
- $L\frac{di}{dt} + Ri = 0$
- $\frac{di}{dt}dt = -\frac{R}{L}idt$
- $\frac{di}{i} = -\frac{R}{L}dt$
- $\int_{i(t_0)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_0}^{t} dy$
- $\ln\frac{i(t)}{i(0)} = -\frac{R}{L}t$

Current i(t) t>0

- $i(t) = i(0)e^{-(R/L)t}$
- $i(0^-) = i(0^+) = I_0$
- $i(t) = I_0 e^{-(R/L)t}, \quad t \geq 0$



The Natural response of an RL Circuit



Voltage

- $\boxed{v = iR = I_0 R e^{-(R/L)t}, \qquad t \geq 0^+}$
- $v(0^-) = 0,\ v(0^+) = I_0 R$

Power and energy

- $p = vi = i^2 R = \dfrac{v^2}{R}$

$$\boxed{= I_0^2 R e^{-2(R/L)t}, \qquad t \geq 0^+}$$

- $w = \int_0^t p dx = \int_0^t I_0^2 R e^{-2(R/L)x}\, dx$

$$= \frac{1}{2(R/L)} I_0^2 R(1 - e^{-2(R/L)t})$$

$$\boxed{= \frac{1}{2} L I_0^2 (1 - e^{-2(R/L)t}), \qquad t \geq 0}$$

The Natural response of an RL Circuit

- Natural response is the current and voltages that exist when stored energy is released to a circuit that contains no independent sources.
- Time constant of an RL circuit equals the equivalent inductance divided by the Thevenin resistance.

$$i(t) = i_0 e^{-(\frac{R}{L})t}, \quad t \geq 0$$

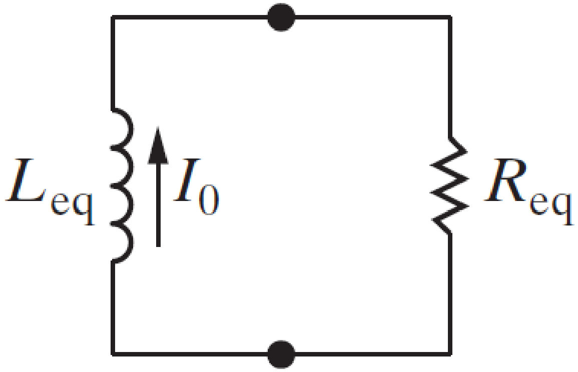
Natural response
of an RL circuit

$$\tau = \frac{L}{R}$$

Time constant
for RL circuit

- $i(t) = I_0 e^{-t/\tau}, \quad t \geq 0$
- $v(t) = I_0 R e^{-t/\tau}, \quad t \geq 0^+$
- $p = I_0^2 R e^{-2t/\tau}, \quad t \geq 0^+$
- $w = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}), \quad t \geq 0$

The Natural response of an RL Circuit

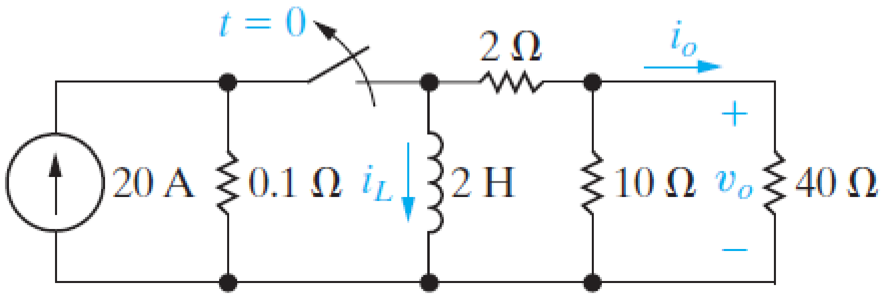


- Calculating the natural response of an *RL* circuit can be summarized as follows:

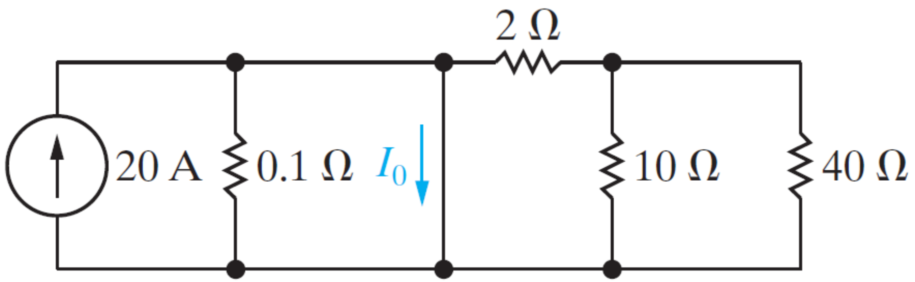
1. Find the initial current, i_0, through the inductor.
2. Find the time constant of the circuit, $\tau = L / R$.
3. Use $i(t) = i_0 e^{-t/\tau}$, $t \geq 0$, to generate $i(t)$ from i_0 and τ.

Example 7.1

Find i_L, i_0, v_0, and the percentage of the total energy stored in 2H inductor that is dissipated in the 10 ohm resistor



t<0



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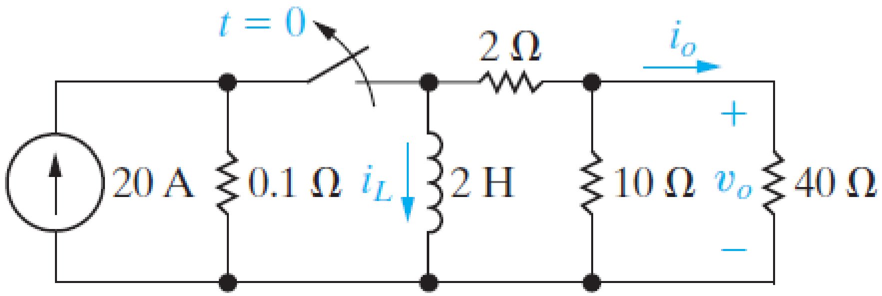
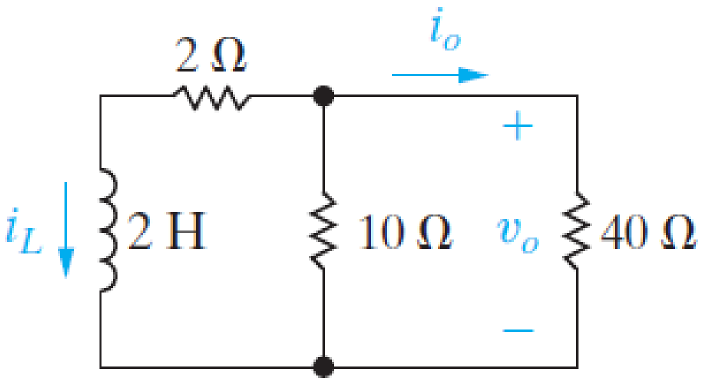


t>0



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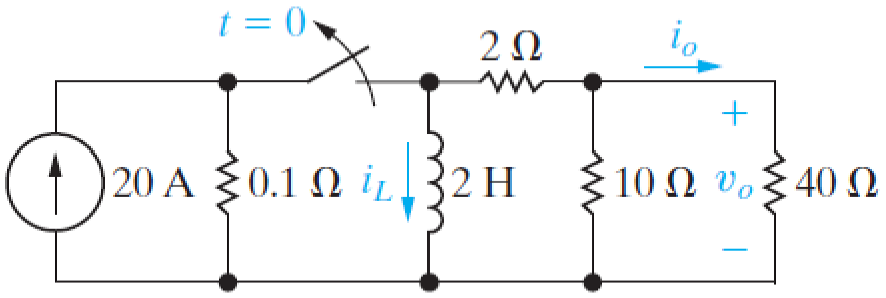
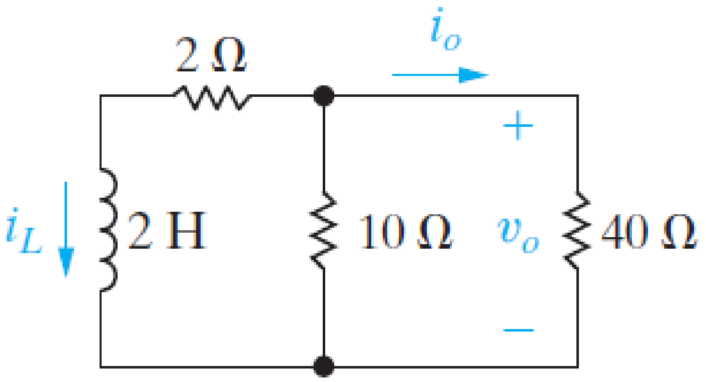


t>0



Example 7.2

Find i_1, i_2, i_3 and
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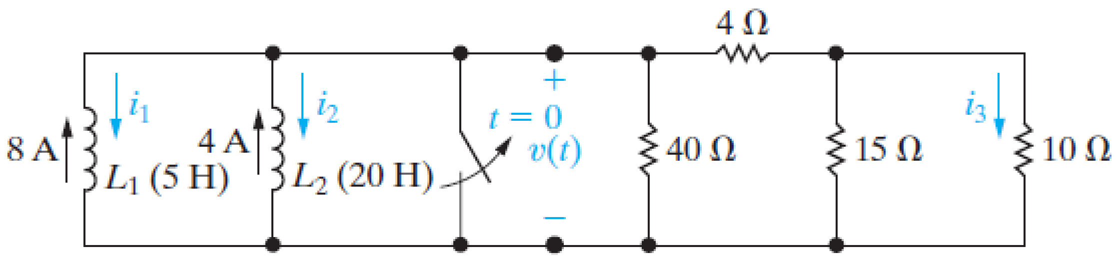

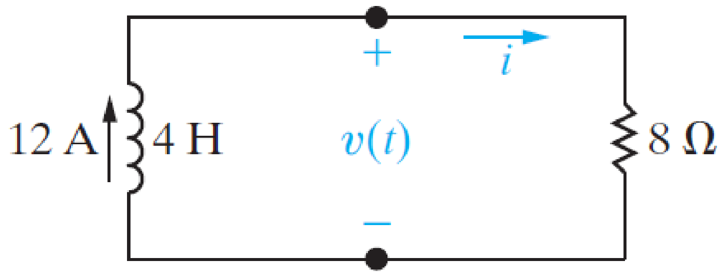

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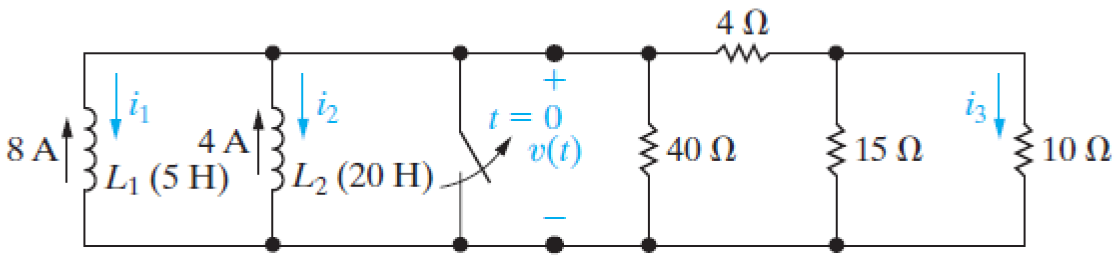

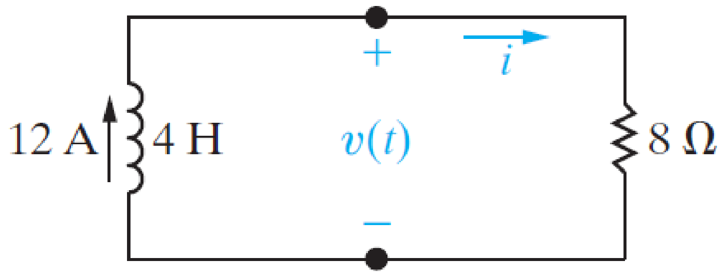

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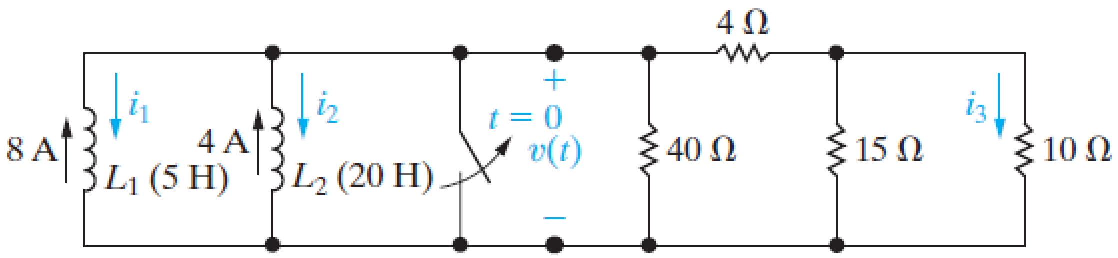

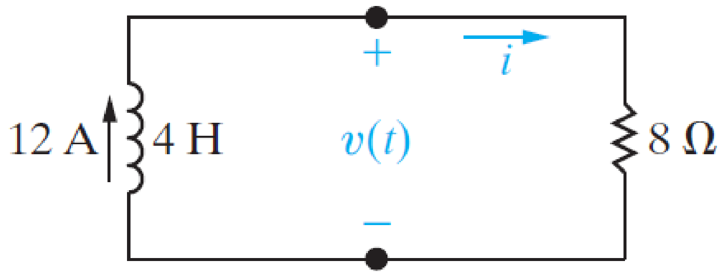

The Natural response of an RC Circuit

- Natural response
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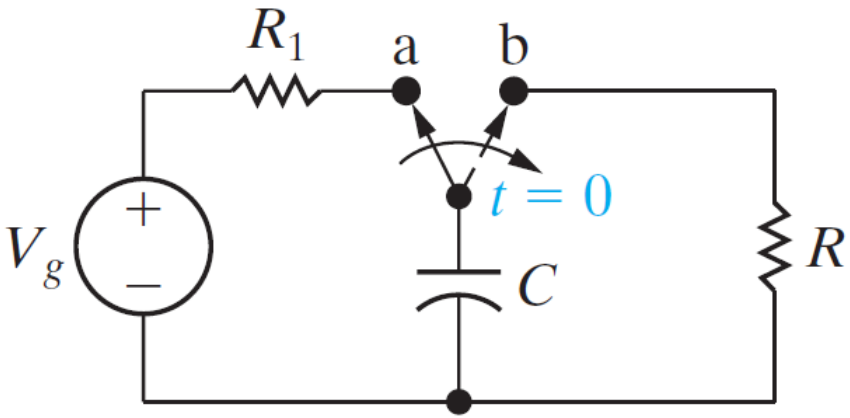
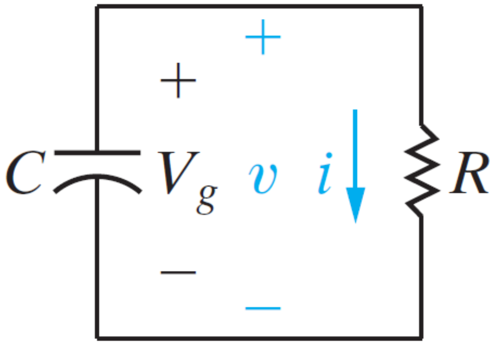
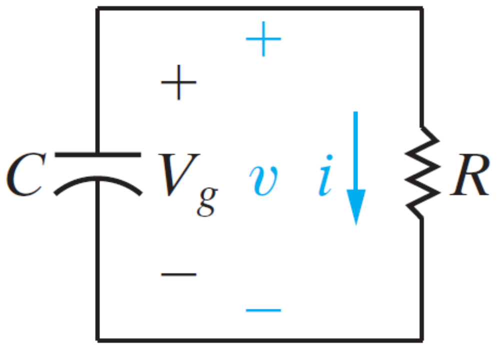


Assuming that the switch is open at t=0



And find v(t) t>0

The Natural response of an RC Circuit

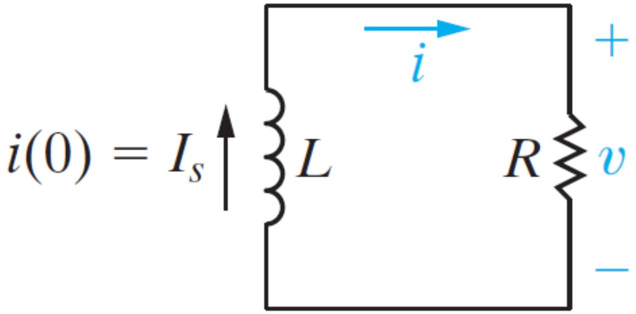


voltage v(t) t>0

- $C\dfrac{dv}{dt} + \dfrac{v}{R} = 0$
- $v(t) = v(0)e^{-t/RC} \quad , t \geq 0$
- $v(0^-) = v(0^+) = V_0$
- $\tau = RC$
- $v(t) = V_0 e^{-t/\tau} \quad , t \geq 0$

Similarity of the RL and RC Circuits

RL Circuit

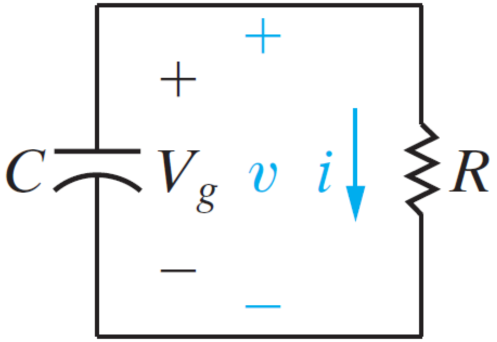


$$\tau = \frac{L}{R}$$

- $L\frac{di}{dt} + Ri = 0$

- $i(0^-) = i(0^+) = I_0$
- $i(t) = I_0 e^{-t/\tau}, \quad t \geq 0$

RC Circuit



$$\tau = \mathrm{RC}$$

- $C\frac{dv}{dt} + \frac{v}{R} = 0$

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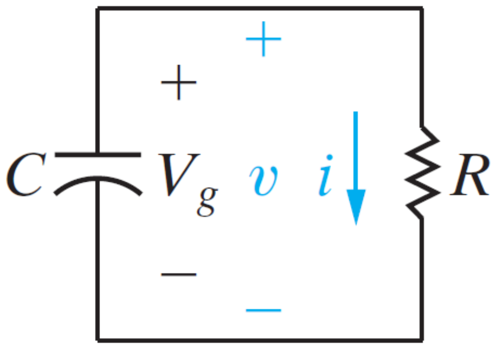
The Natural response of an RC Circuit



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Current

Power and energy

- $i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}, \qquad t \geq 0^+$
- $p = vi = \frac{V_0^2}{R} e^{-2t/\tau}, \qquad t \geq 0^+$
- $w = \int_0^t p dx = \int_0^t \frac{V_0^2}{R} e^{-2x/\tau} dx$

$$= \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad t \geq 0$$

The Natural response of an RC Circuit

- Natural response is the current and voltages that exist when stored energy is released to a circuit that contains no independent sources.
- Time constant of an RL circuit equals the equivalent inductance divided by the Thevenin resistance.

$$v(t) = V_0 e^{-t/\tau},\ t \geq 0$$

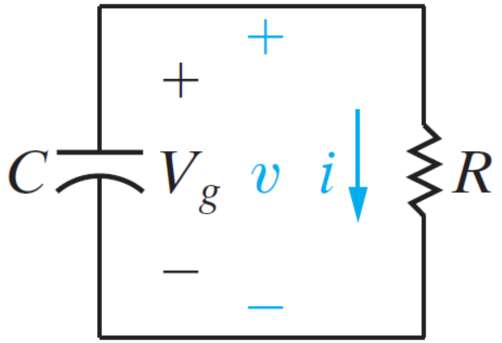
Natural response of an RC circuit

$$\tau = RC$$

Time constant for RC circuit

- $v(t) = V_0 e^{-t/\tau} \quad , t \geq 0$
- $i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}, \qquad t \geq 0^+$
- $p = vi = \frac{V_0^2}{R} e^{-2t/\tau}, \qquad t \geq 0^+$
- $w = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \qquad t \geq 0$

The Natural response of an RC Circuit

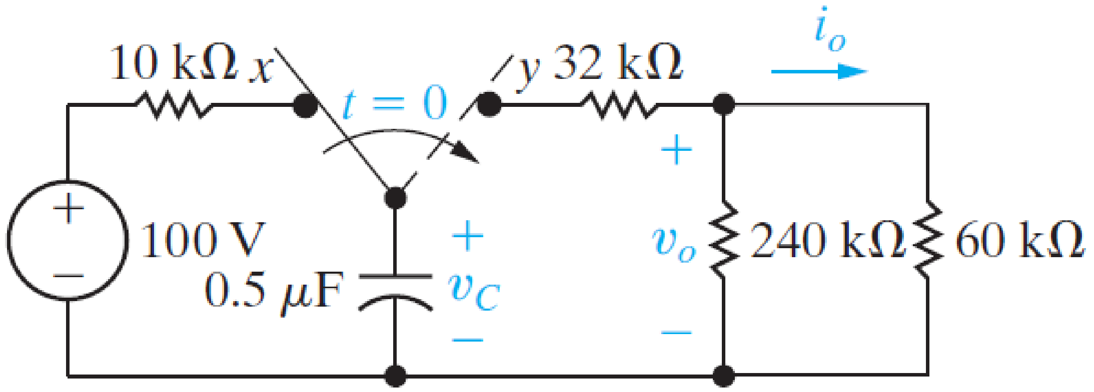


- Calculating the natural response of an *RC* circuit can be summarized as follows:

1. Find the initial voltage, V_0, across the capacitor.
2. Find the time constant of the circuit, $\tau = RC$.
3. Use, $v(t) = V_0 e^{-t/\tau}$, $t \geq 0$ *to* generate $v(t)$ from V_0 and τ.

Example 7.3

Find $v_c(t)$, $v_0(t)$, $i_0(t)$ and the total energy dissipated in the 60 kohm resistor



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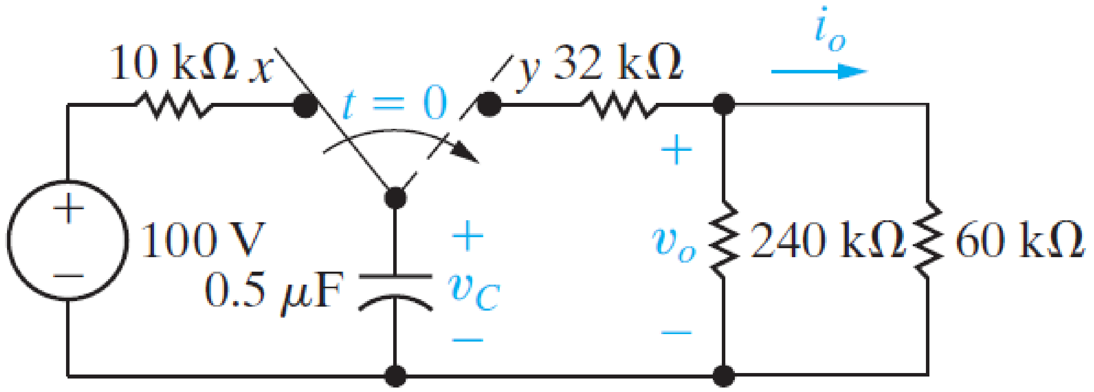

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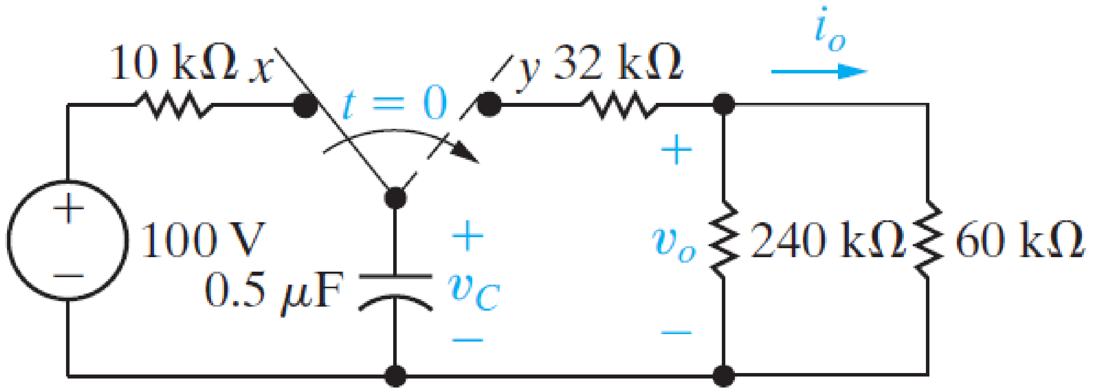
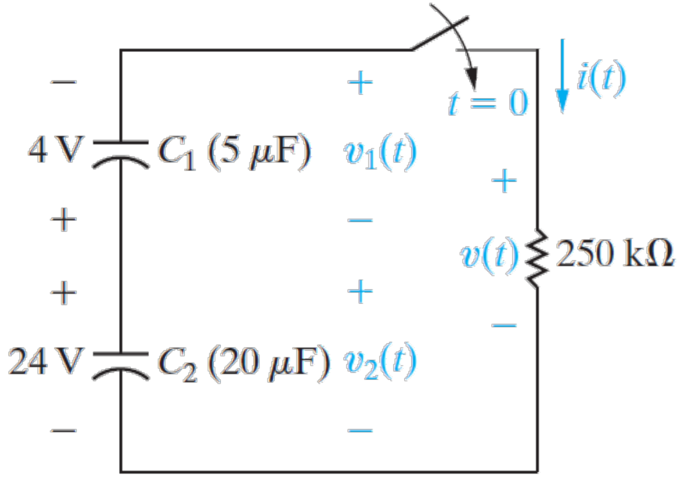
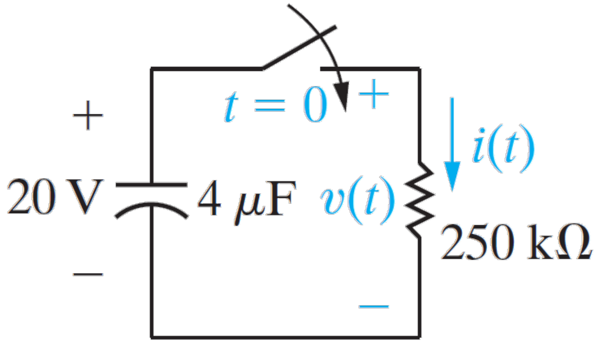


Example 7.4

a) Find $v_1(t)$, $v_2(t)$, and $v(t)$ and $i(t)$.

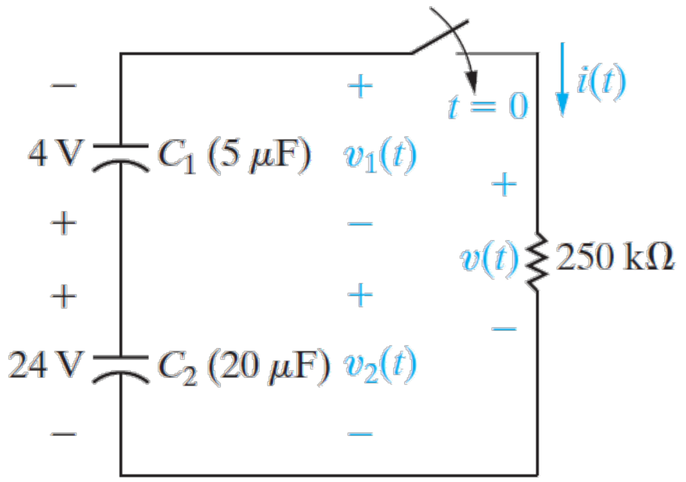




Example 7.4

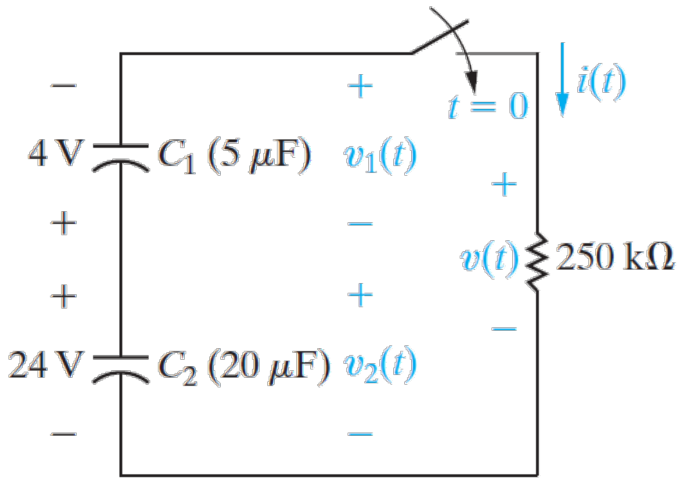
b) Calculate the initial energy stored in the capacitor C1 and C2.

c) How much energy is stored in the capacitor as t goes infinity.



Example 7.4

d) Show that the total energy delivered to the 250 kohm resistor is the difference between the results obtained in (b) - (c)



Summary (Part 1 / 2)

Natural Response of an RL Circuit

- $i(t) = I_0 e^{-t/\tau}, \quad t \geq 0$
- $v(t) = I_0 R e^{-t/\tau}, \quad t \geq 0^+$
- $p = I_0^2 R e^{-2t/\tau}, \quad t \geq 0^+$
- $w = \frac{1}{2} L I_0^2 (1 - e^{-2t/\tau}), \quad t \geq 0$

$$\tau = \frac{L}{R}$$

Natural Response of an RC Circuit

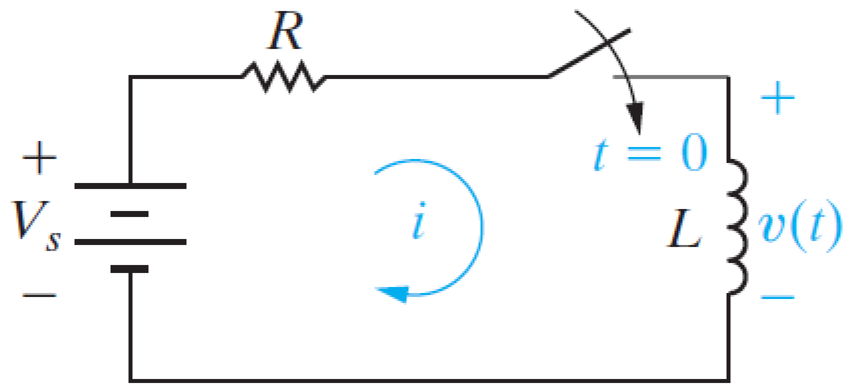
- $v(t) = V_0 e^{-t/\tau} \;, t \geq 0$
- $i(t) = \frac{v(t)}{R} = \frac{V_0}{R} e^{-t/\tau}, \quad t \geq 0^+$
- $p = vi = \frac{V_0^2}{R} e^{-2t/\tau}, \quad t \geq 0^+$
- $w = \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau}), \quad t \geq 0$

$$\tau = \mathrm{RC}$$

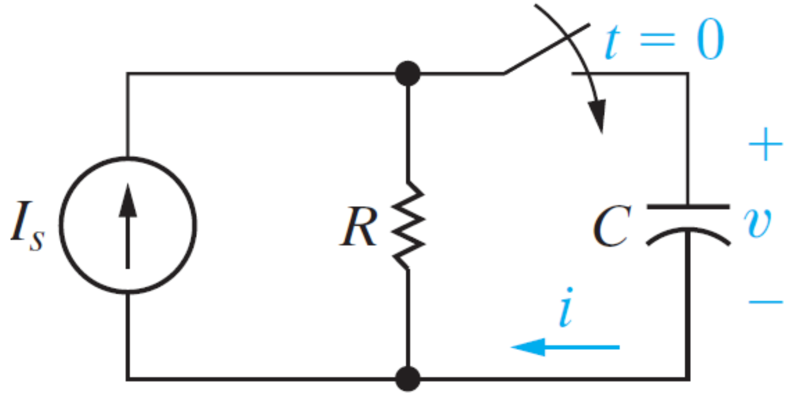
The Step Response of RL and RC Circuits

- **Step Response**

  The response of a circuit to the sudden application of a const voltage or current source is called the step response of the circuit.

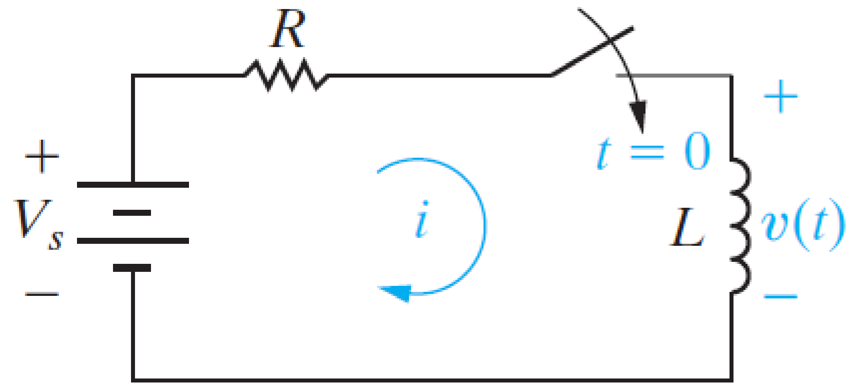


RL circuit



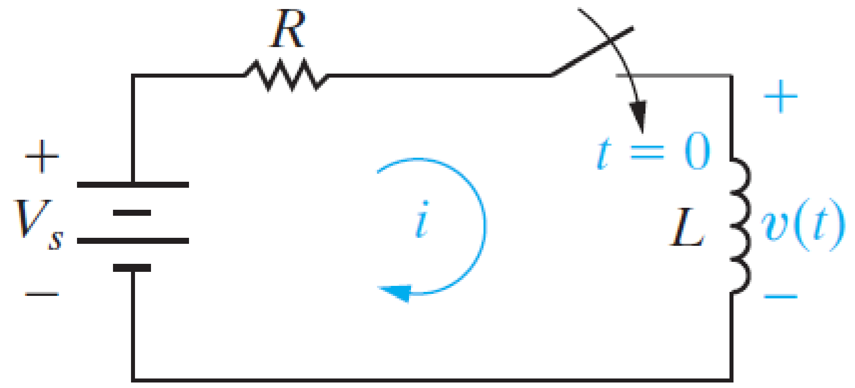
RC circuit

The Step Response of an RL Circuit



- $V_S = Ri + L\frac{di}{dt}$
- $\frac{di}{dt} = \frac{-Ri+V_S}{L} = \frac{-R}{L}\left(i - \frac{V_S}{R}\right)$
- $di = \frac{-R}{L}\left(i - \frac{V_S}{R}\right)dt$
- $\frac{di}{i-(V_S/R)} = \frac{-R}{L}dt$
- $\int_{I_0}^{i(t)} \frac{dx}{x-(V_S/R)} = \frac{-R}{L}\int_0^t dy$
- $\ln\frac{i(t)-(V_S/R)}{I_0-(V_S/R)} = \frac{-R}{L}t$

The Step Response of an RL Circuit

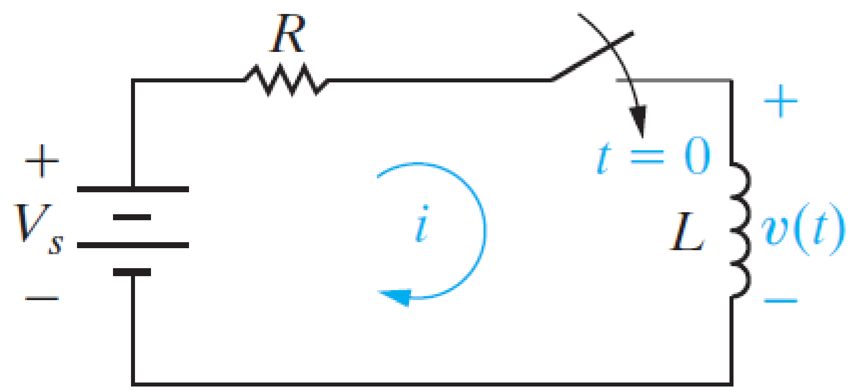


- $\ln \frac{i(t) - (V_S/R)}{I_0 - (V_S/R)} = \frac{-R}{L} t$
- $\frac{i(t) - (V_S/R)}{I_0 - (V_S/R)} = e^{-(R/L)t}$
- $\boxed{i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R}\right) e^{-(R/L)t}}$

when $I_0 = 0$

- $i(t) = \frac{V_S}{R} \quad - \frac{V_S}{R} e^{-(R/L)t}$

The Step Response of an RL Circuit



- $i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R}\right) e^{-(R/L)t}$

$$v = L\frac{di}{dt}$$

- $v = L\left(\frac{-R}{L}\right)\left(I_0 - \frac{V_S}{R}\right) e^{-(R/L)t}$

  $= (V_s - I_0 R)e^{-(R/L)t} \qquad (t \geq 0^+)$

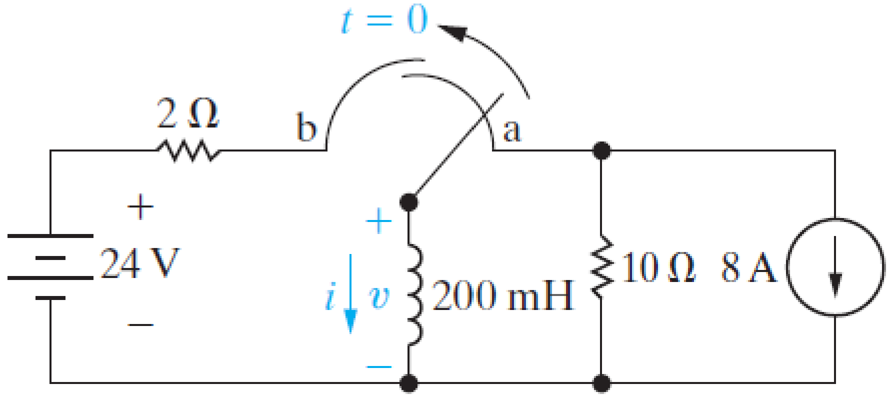
- $v = 0 \qquad (t < 0)$

when $I_0 = 0$

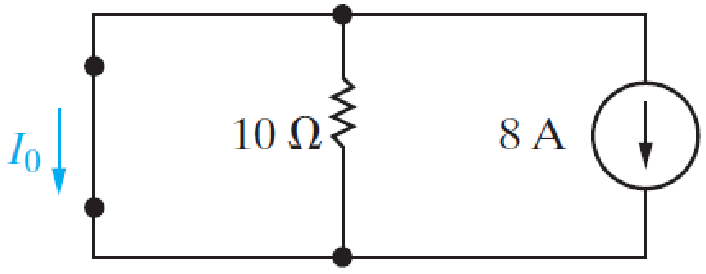
- $v = V_s e^{-(R/L)t}$

Example 7.5

a) i(t) for t>=0 and b) v(t) for t>=0+



t<0



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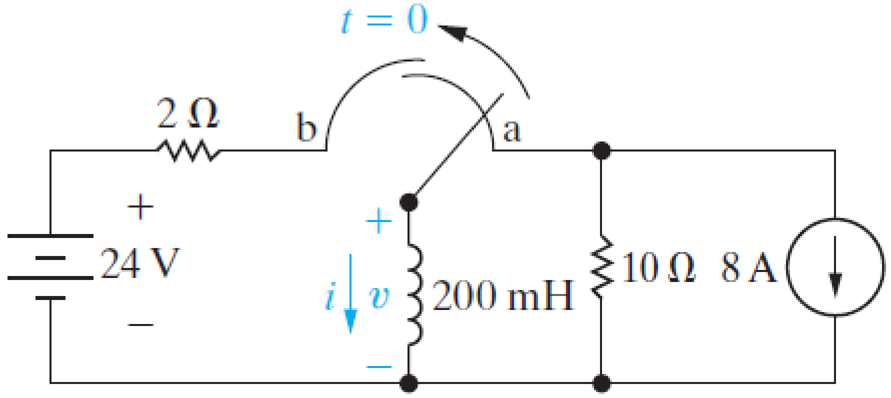
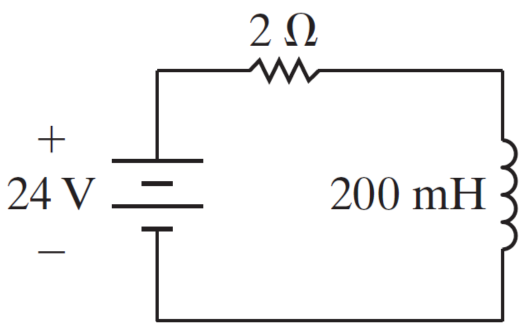
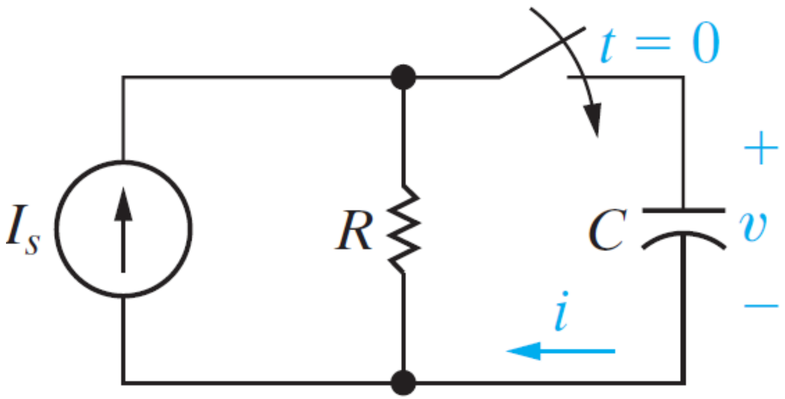


t>=0



The Step Response of an RC Circuit



- $C\frac{dv_C}{dt} + \frac{v_C}{R} = I_s$
- $\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_s}{C}$

RL circuit $\quad \frac{di}{dt} + \frac{R}{L}i = \frac{V_s}{L} \Rightarrow i = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}$

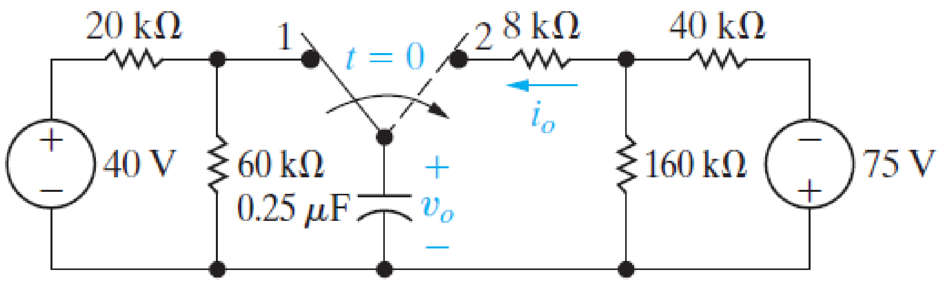
"RC circuit" $\quad \frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_s}{C} \Rightarrow v = I_sR + (V_0 - I_sR)e^{-t/RC}$

RL circuit $\quad v = L\frac{di}{dt} \Rightarrow v = (V_s - I_0R)e^{-(R/L)t}$

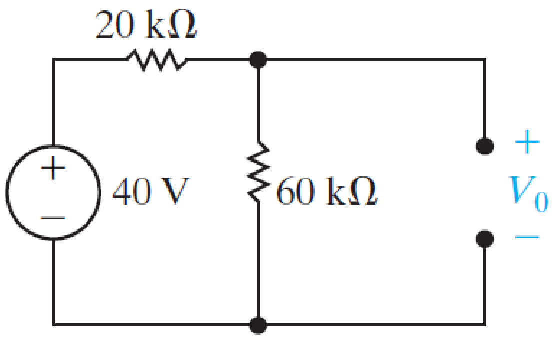
"RC circuit" $\quad i = C\frac{dv}{dt} \Rightarrow i = \left(I_s - \frac{V_0}{R}\right)e^{-t/RC}$

Example 7.6

a) $v_0(t)$ for $t>=0$ and b) $i_0(t)$ for $t>=0+$



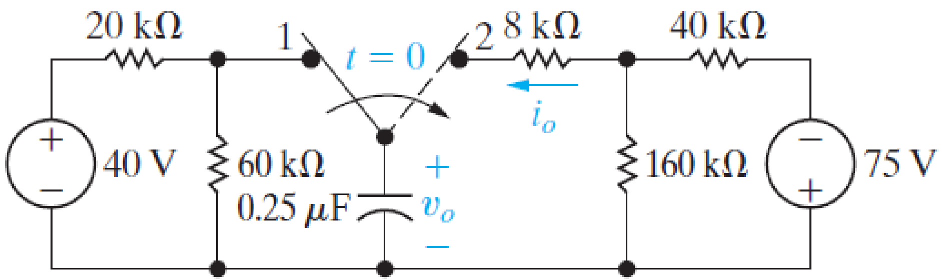
t<0

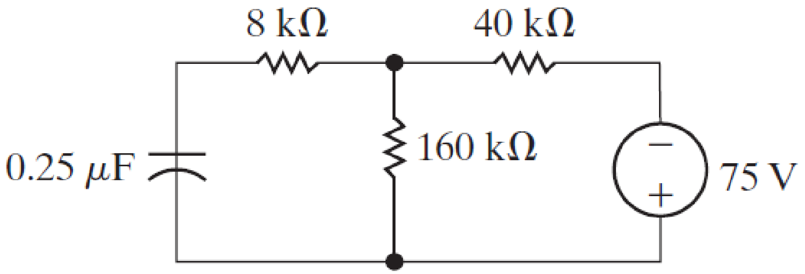


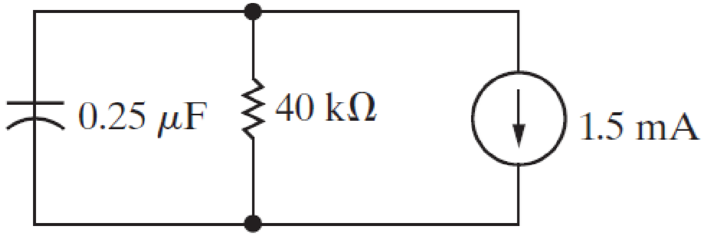
Example 7.6

a) $v_0(t)$ for $t>=0$ and b) $i_0(t)$ for $t>=0+$

t>=0



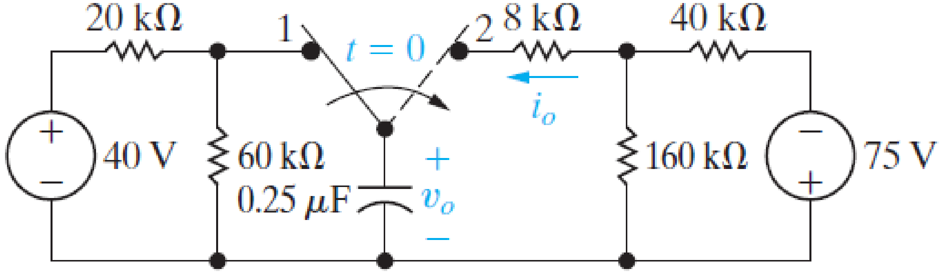


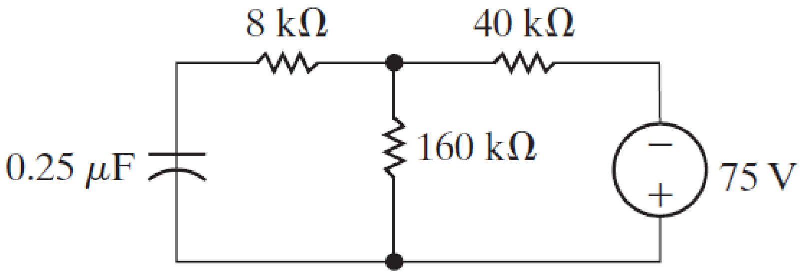


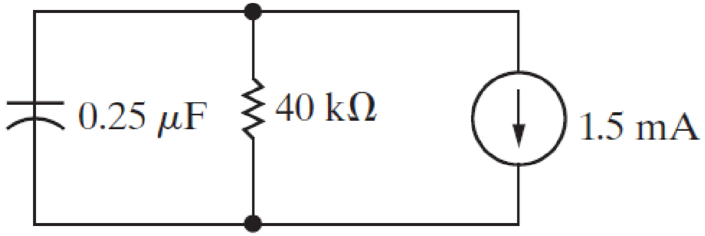
Example 7.6

a) $v_0(t)$ for $t>=0$ and b) $i_0(t)$ for $t>=0+$

t>=0

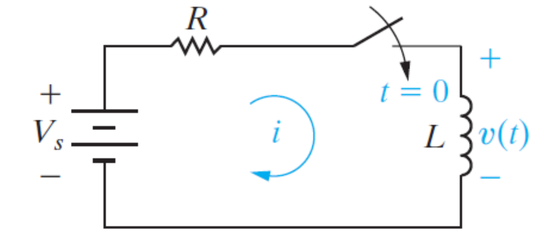






Summary (Part 3)

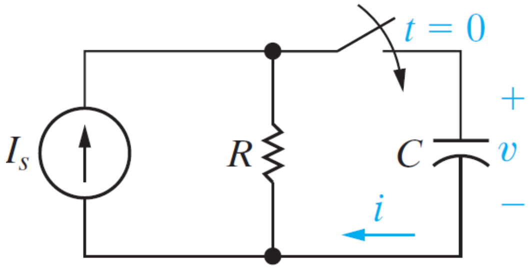
- Natural Response of an RL Circuit
- Natural Response of an RC Circuit
- Step Response of an RL and RC Circuits



RL circuit

RL circuit $\frac{di}{dt} + \frac{R}{L} i = \frac{V_s}{L} \Rightarrow i = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right) e^{-(R/L)t}$

RC circuit $\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_s}{C} \Rightarrow v = I_s R + (V_0 - I_s R) e^{-t/RC}$



RC circuit

RL circuit $v = L\frac{di}{dt} \Rightarrow v = (V_s - I_0 R) e^{-(R/L)t}$

RC circuit $i = C\frac{dv}{dt} \Rightarrow i = \left(I_s - \frac{V_0}{R}\right) e^{-t/RC}$

General Solution for Step and Natural Responses

- General approach to find either the natural response or the step response of the first-order RL and RC circuits.

RL circuit

$$\frac{dv}{dt} + \frac{R}{L}v = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_s}{L}$$

RC circuit

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_s}{C}$$

$$\frac{di}{dt} + \frac{1}{RC}i = 0$$

$$\Rightarrow \quad \frac{dx}{dt} + \frac{x}{\tau} = K$$

General Solution for Step and Natural Responses

$$\frac{dx}{dt} + \frac{x}{\tau} = K$$

$$at \quad t \to \infty \quad \frac{dx}{dt} \to 0, \quad \frac{x_f}{\tau} = K, \quad x_f = \tau \cdot K$$

$$\frac{dx}{dt} = K - \frac{x}{\tau}$$

$$\frac{dx}{dt} = \frac{1}{\tau}(K\tau - x)$$

$$dx = \frac{1}{\tau}(K\tau - x)\, dt$$

$$dx = \frac{1}{\tau}(x_f - x)\, dt$$

$$\frac{dx}{x_f - x} = \frac{1}{\tau} dt$$

$$\int_{x(t_0)}^{x(t)} \frac{dx}{x_f - x} = \int_{t_0}^{t} \frac{1}{\tau} dt$$

$$-\ln(x_f - x)\Big|_{x(t_0)}^{x(t)} = \frac{t}{\tau}\Big|_{t_0}^{t}$$

$$-\ln(x_f - x(t)) + \ln(x_f - x(t_0)) = \frac{t - t_0}{\tau}$$

$$\ln\frac{x_f - x(t)}{x_f - x(t_0)} = \frac{-(t - t_0)}{\tau}$$

$$\frac{x_f - x(t)}{x_f - x(t_0)} = e^{-(t-t_0)/\tau}$$

$$x_f - x(t) = \left[x_f - x(t_0)\right] \cdot e^{-(t-t_0)/\tau}$$

$$-x(t) = -x_f + \left[x_f - x(t_0)\right] \cdot e^{-(t-t_0)/\tau}$$

$$\boxed{x(t) = x_f + \left[x(t_0) - x_f\right] \cdot e^{-(t-t_0)/\tau}}$$

General Solution for Step and Natural Responses

The unknown variables as a function of time

$$x(t) = x_f + [x(t_0) - x_f]e^{-\frac{(t-t_0)}{\tau}}$$

Time constant

The final value of the variable

The initial value of the variable

The final value of the variable

General Solution for Step and Natural Responses

1) Identify the variable of interest for the circuit. For *RC* circuits, it is most convenient to choose the capacitive voltage; for RL circuits, it is best to choose the inductive current.

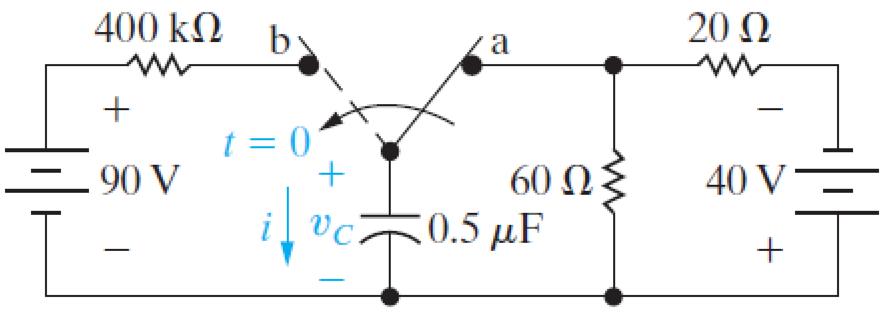
2) Determine the initial value of the variable, which is its value at t_0.

3) Calculate the final value of the variable, which is its value as $t \to \infty$.

4) Calculate the time constant for the circuit.

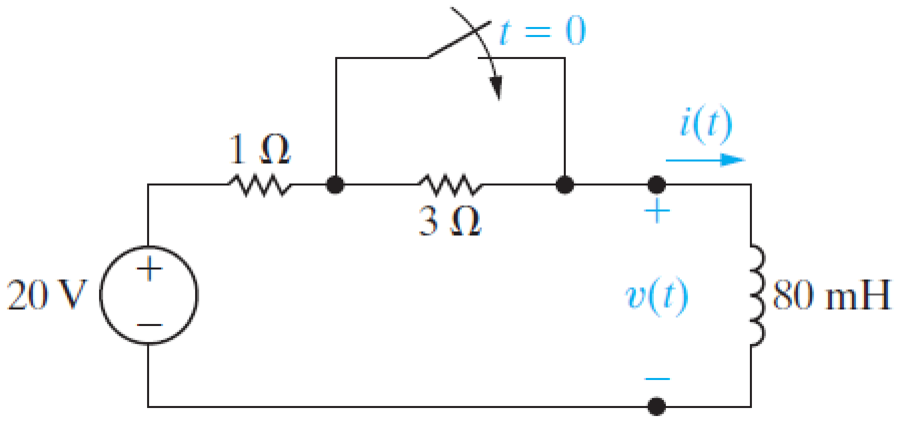
Example 7.8

a) $v_c(t)$ for t>=0 + and b) i(t) for t>=0



Example 7.9

a) i(t) for t>=0 and b) v(t) for t>=0+



Summary (Part 4)

- Natural Response of an RL Circuit
- Natural Response of an RC Circuit
- Step Response of an RL and RC Circuits
- General Solution for Step and Natural Responses

The unknown variables as a function of time

$$x(t) = x_f + [x(t_0) - x_f]e^{-\frac{(t-t_0)}{\tau}}$$

Time constant

The final value of the variable

The initial value of the variable

The final value of the variable