인하대학교
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Electromagnetics 1 (ICE2003)
-- Ch. 5. Conductors and Dielectrics --

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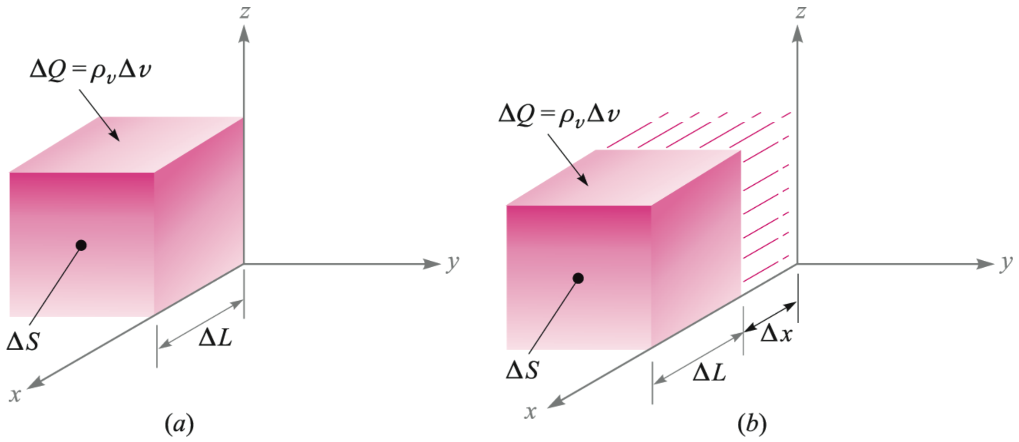
Chapter Outline

- Current and Current Density
- Resistance
- Conductors
  - Electrostatic properties
  - Boundary condition
  - Method of images
- Dielectrics
  - Polarization field
  - Electric susceptibility and dielectric constant
  - Boundary condition

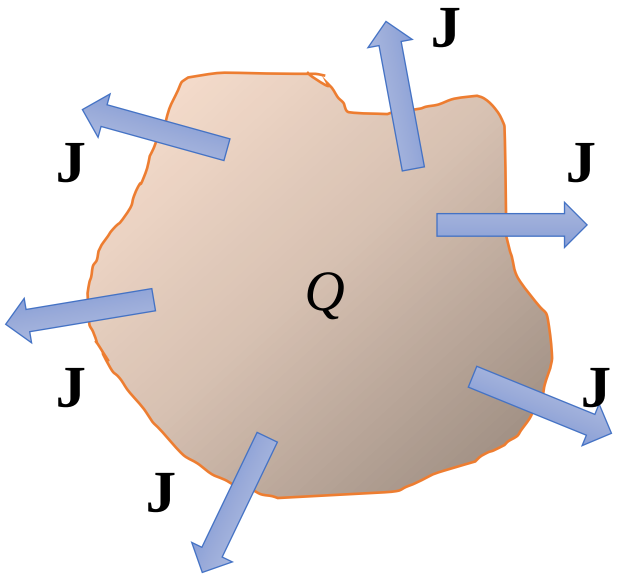
Current and Current Density

- Current(I): The amount of charge that flows through a certain area in a unit time
- Current density(**J**): Current in a unit area = Charge density x Charge velocity



$$\boxed{I = \frac{dQ}{dt} = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_S \rho_v \mathbf{v} \cdot d\mathbf{s}}$$

Continuity of Current



$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} = -\frac{d}{dt}\int_{\text{vol}} \rho_v \, dv$$

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}}$$

The total amount of current flowing outward from an arbitrary closed surface equals to the decreasing rate of the charge inside the closed surface

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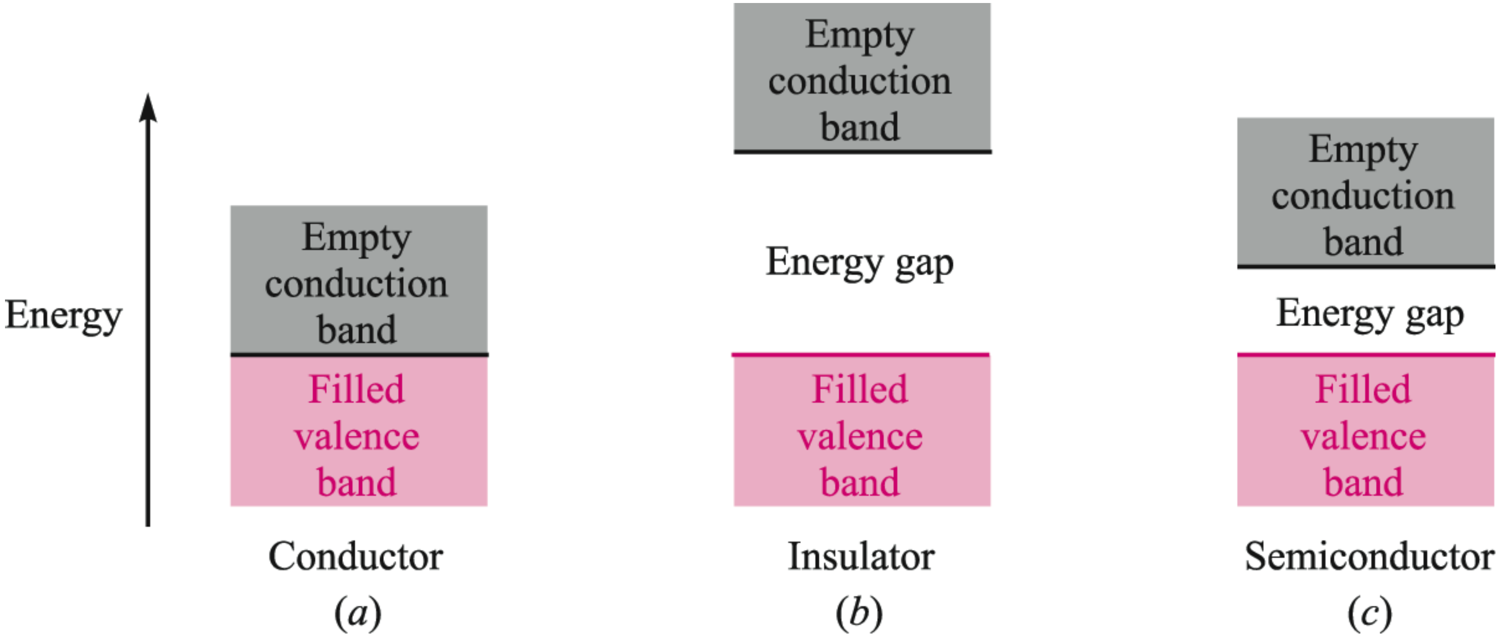
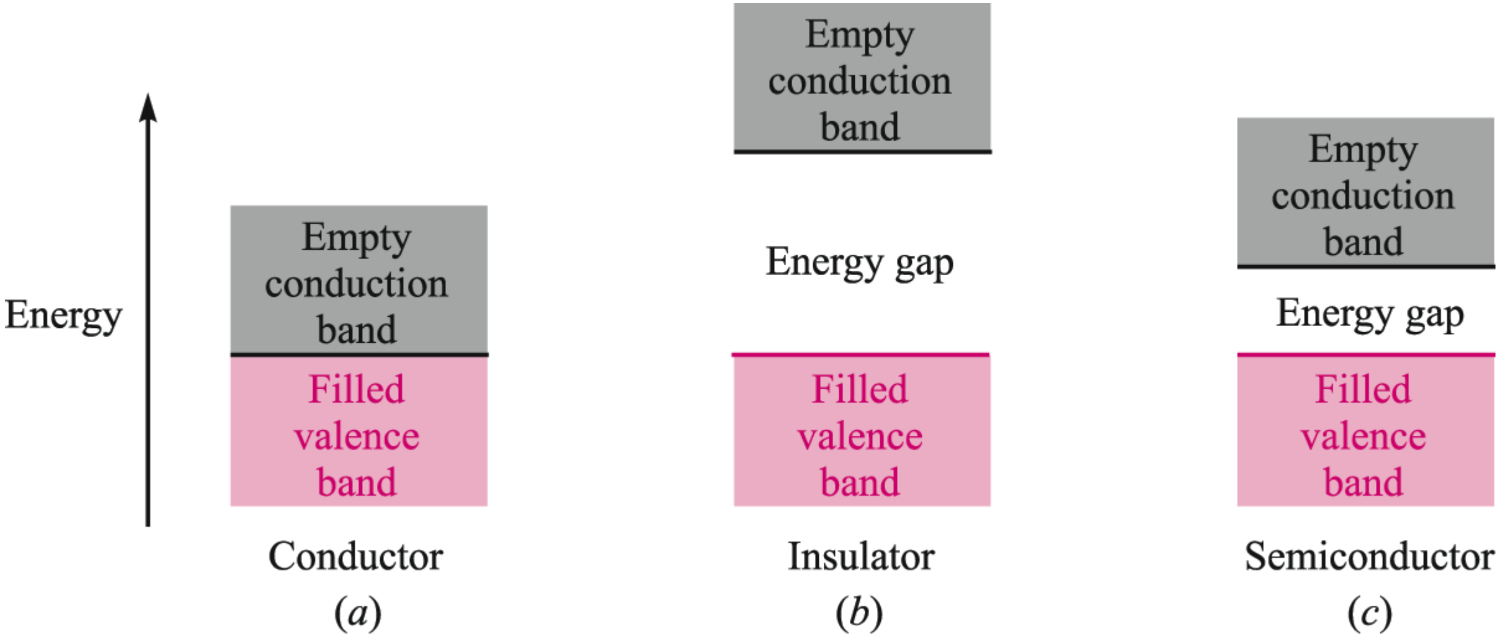
Ex) $\mathbf{J} = \frac{1}{r} e^{-t} \mathbf{a_r}$ $\qquad I, \rho_v, \mathbf{v}$

Energy band structure in three material types



a) Conductors exhibit no energy gap between valence and conduction bands so electrons move freely
b) Insulators show large energy gaps, requiring large amounts of energy to lift electrons into the conduction band. When this occurs, the dielectric breaks down.
c) Semiconductors have a relatively small energy gap, so modest amounts of energy (applied through heat, light, or an electric field) may lift electrons from valence to conduction bands.

Electron flow in conductors



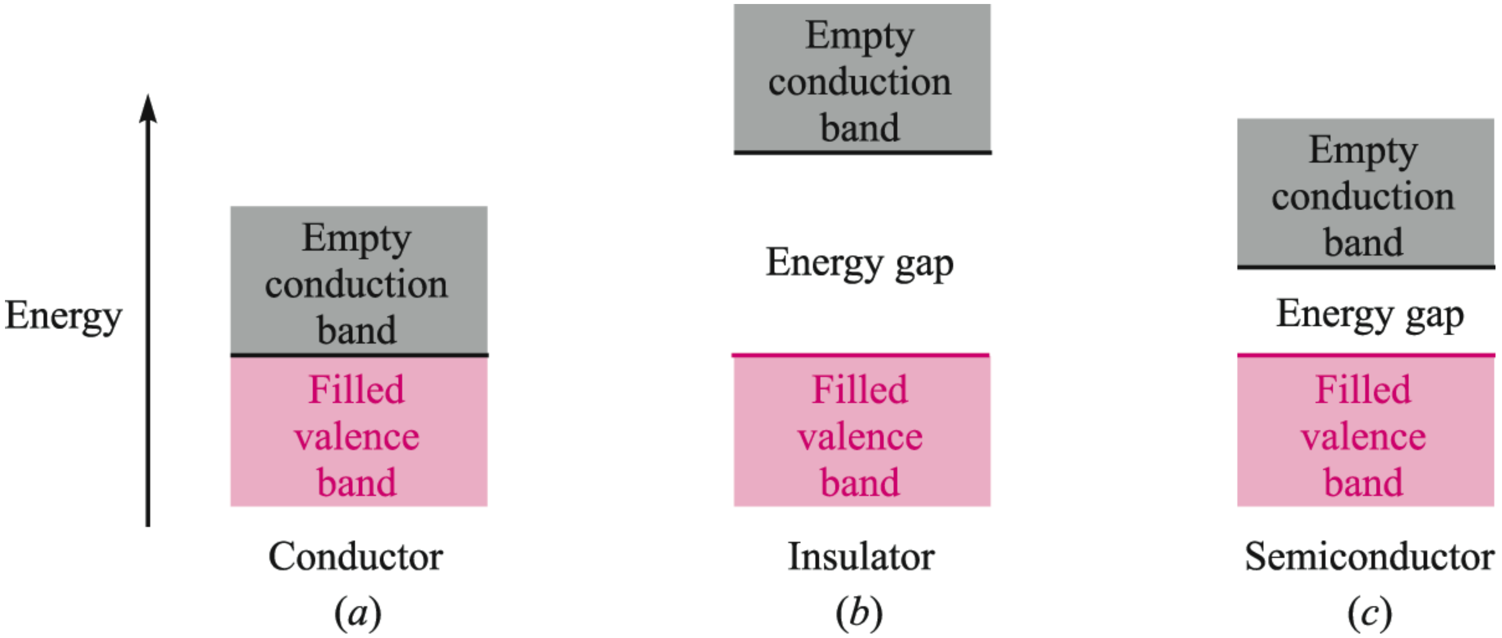
$$\boxed{\mathbf{J} = \rho_v \mathbf{v} = \rho_e \mathbf{v_d} = -\rho_e \mu_e \mathbf{E} = \sigma \mathbf{E}}$$

ρ_e : Charge density of free electrons (negative)

μ_e : Electron mobility

σ : Conductivity

Current in semiconductors



$$\boxed{\mathbf{J} = \rho_v \mathbf{v} = (-\rho_e \mu_e + \rho_h \mu_h)\mathbf{E} = \sigma \mathbf{E}}$$

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$

ρ_e : Charge density of free electrons (negative)

ρ_h : Charge density of free holes (positive)

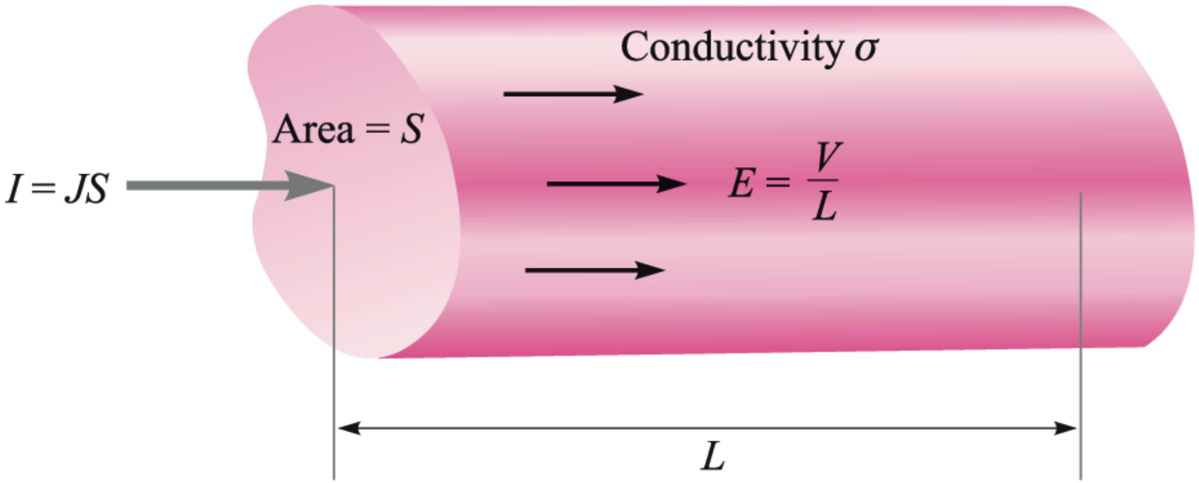
μ_e : Electron mobility

μ_h : Hole mobility

σ : Conductivity

Resistance

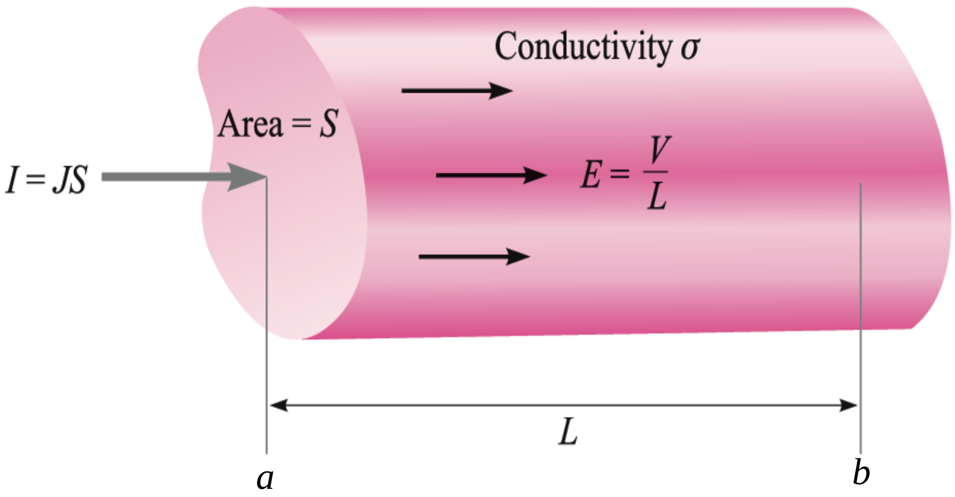
$$\mathbf{J} = \sigma \mathbf{E}$$



$$\mathbf{J} = \sigma \mathbf{E} \quad \Rightarrow \quad \frac{I}{S} = \sigma \frac{V}{L} \quad \Rightarrow \quad V = I \frac{L}{\sigma S} = IR$$

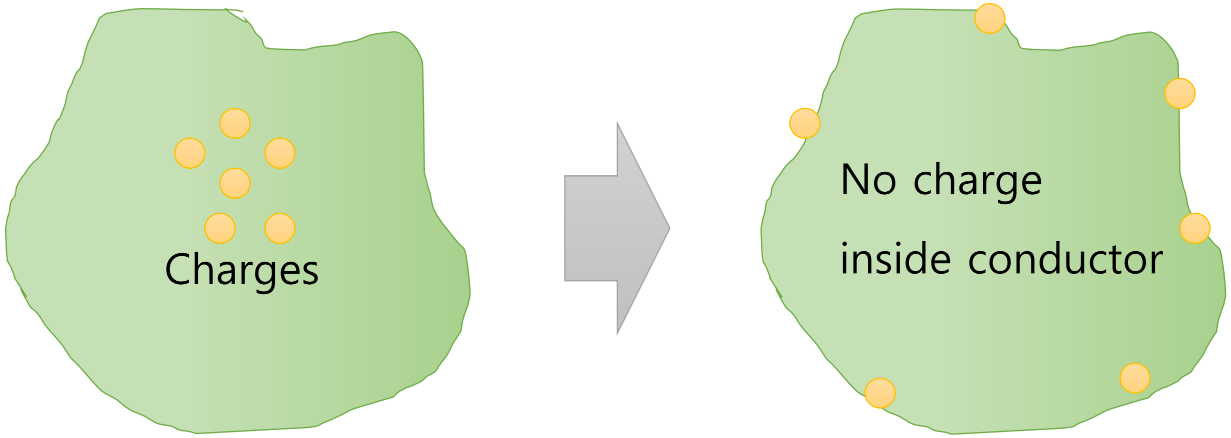
General expression for resistance

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}}$$



Electrostatic properties of Conductors

1. Charges inside Conductor — Immediately redistributed to conductor boundary



2. Ideal Conductor Conductivity is infinite

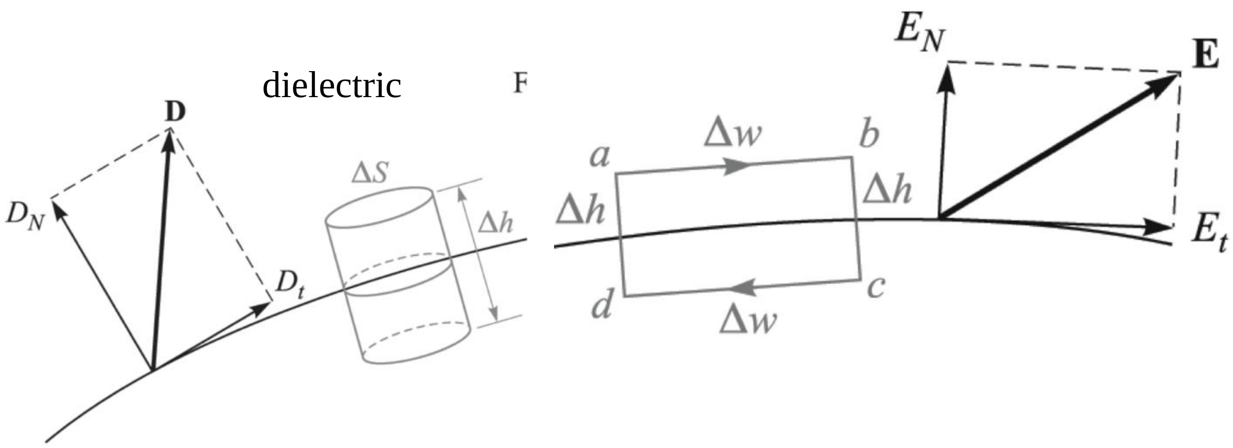
$$\mathbf{J} = \underbrace{\sigma}_{\text{infinite}}\mathbf{E} \quad \Rightarrow \quad \mathbf{E} = 0$$

$\therefore$ Inside conductor

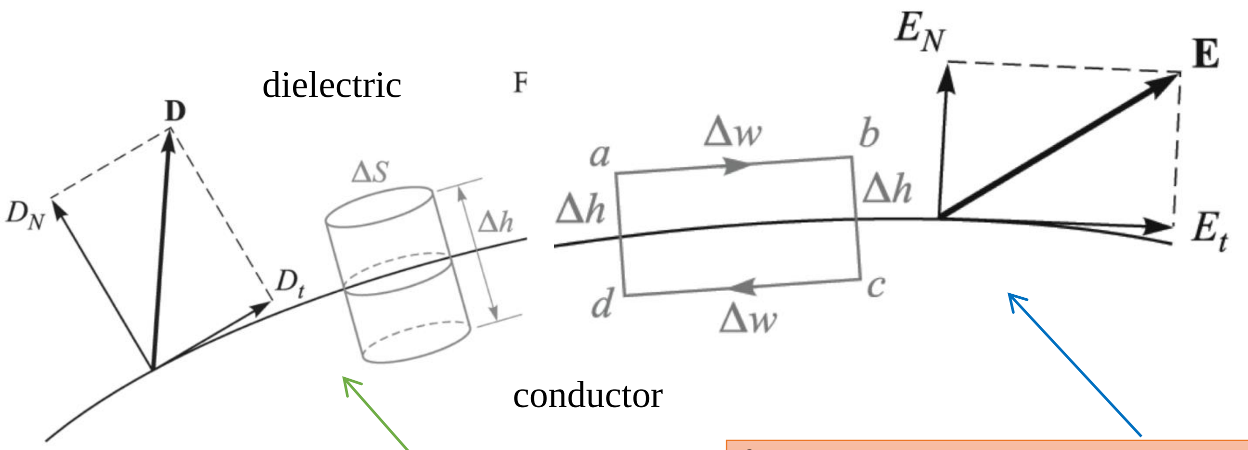
$$\mathbf{E} = \mathbf{D} = 0$$

$$\rho_v = 0$$

Boundary condition



Boundary condition



$$\oint \mathbf{E}\cdot d\mathbf{L} = E_{t,air}\Delta w - E_{t,conductor}\Delta w = E_{t,air}\Delta w = 0$$

$$\oint \mathbf{D}\cdot d\mathbf{S} = D_{N,air}\Delta S - D_{N,conductor}\Delta S = D_{N,air}\Delta S = \rho_s \Delta S$$

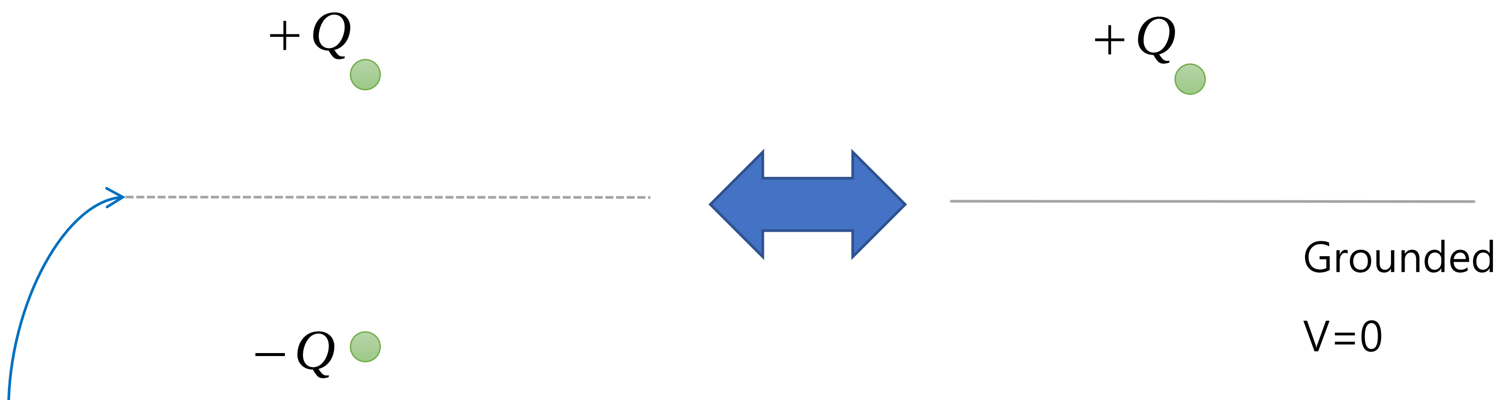
$\therefore$ Conductor boundary
$$D_t = E_t = 0$$
$$D_N = \varepsilon_o E_N = \rho_s$$

D, E outward from the conductor boundary

Example

전위가 $V=100(x^2 - y^2)$로 주어지고 도체와 자유공간과의 경계면상에 한 점 P(2,-1,3)가 있을 때, P에서의 V, **E**, **D** 및 ρ_s를 구하고 또 도체 표면의 방정식을 구하자.

Method of Images

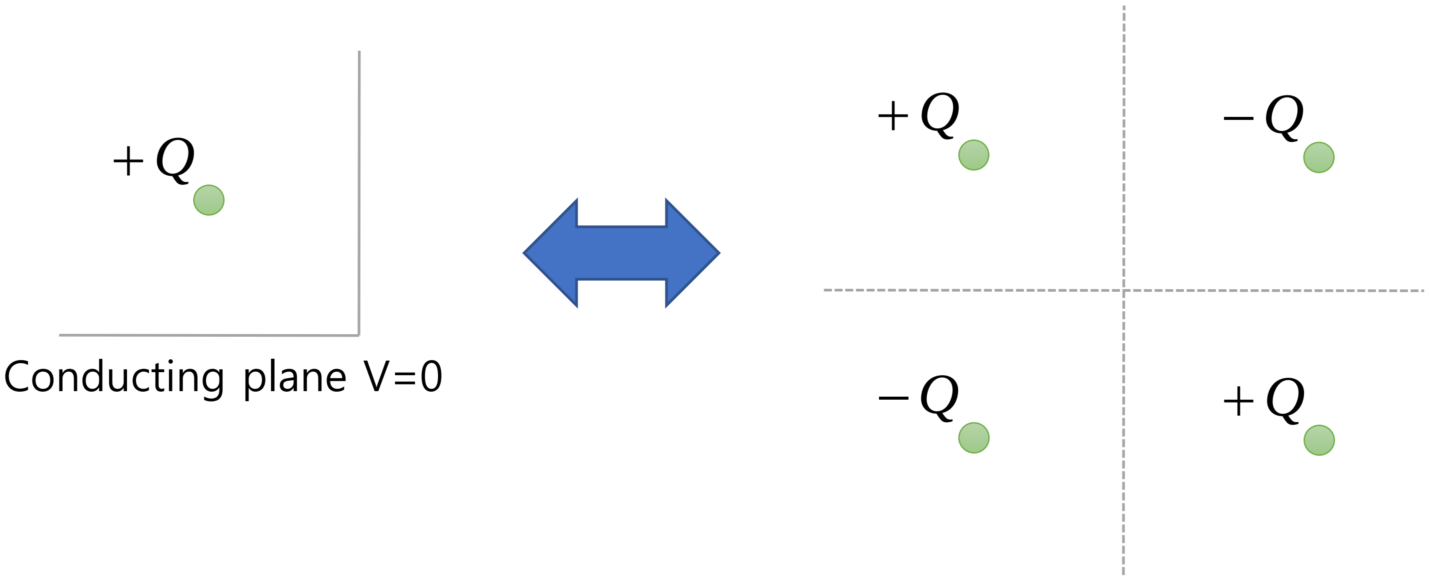


Grounded Conducting plane

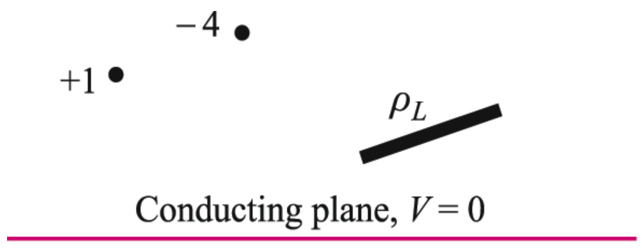
V=0

At Conducting plane $$V = \frac{+Q}{4\pi\varepsilon_o r_+} + \frac{-Q}{4\pi\varepsilon_o r_-} = 0$$

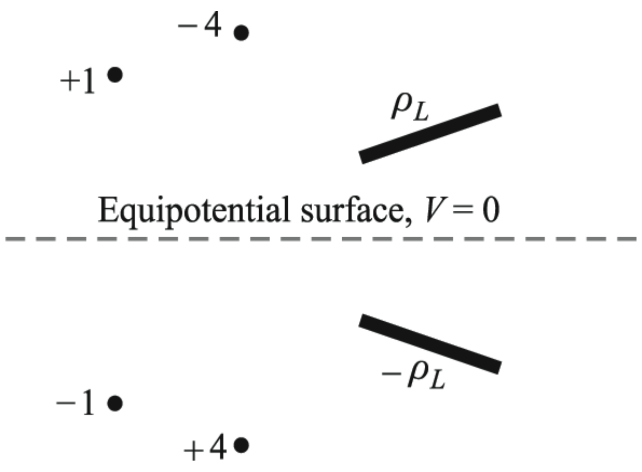
Example



Example



−4
+1
ρ_L
Conducting plane, $V = 0$

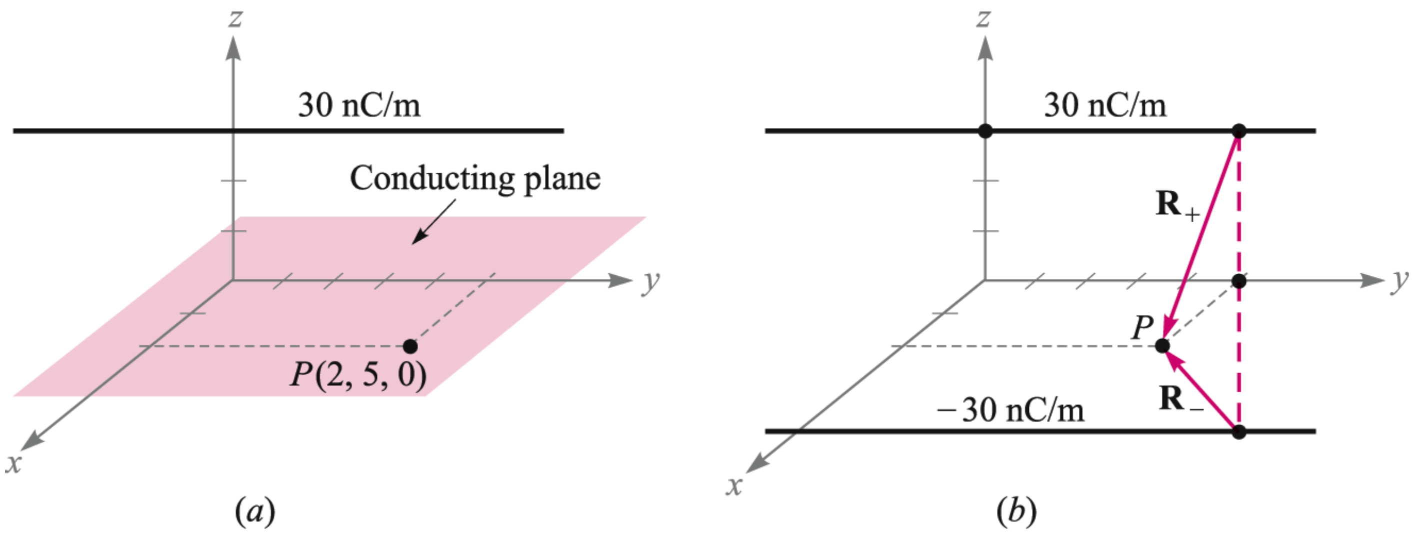


−4
+1
ρ_L
Equipotential surface, $V = 0$
$-\rho_L$
−1
+4

Example

In this case, we are to find the surface charge density on the conducting plane at the point (2,5,0). A 30-nC line charge lies parallel to the *y* axis at $z = 3$.

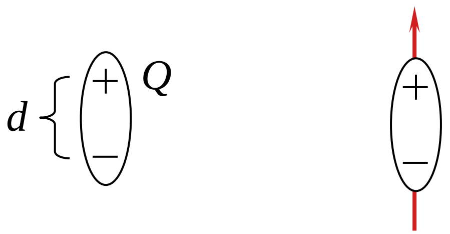
The first step is to replace the conducting plane by a line charge of -30 nC at $z = -3$.



Electric dipole and dipole moment

In dielectric, charges are held in position (bound), and ideally there are no free charges that can move and form a current. Atoms and molecules may be polar (having separated positive and negative charges), or may be polarized by the application of an electric field.

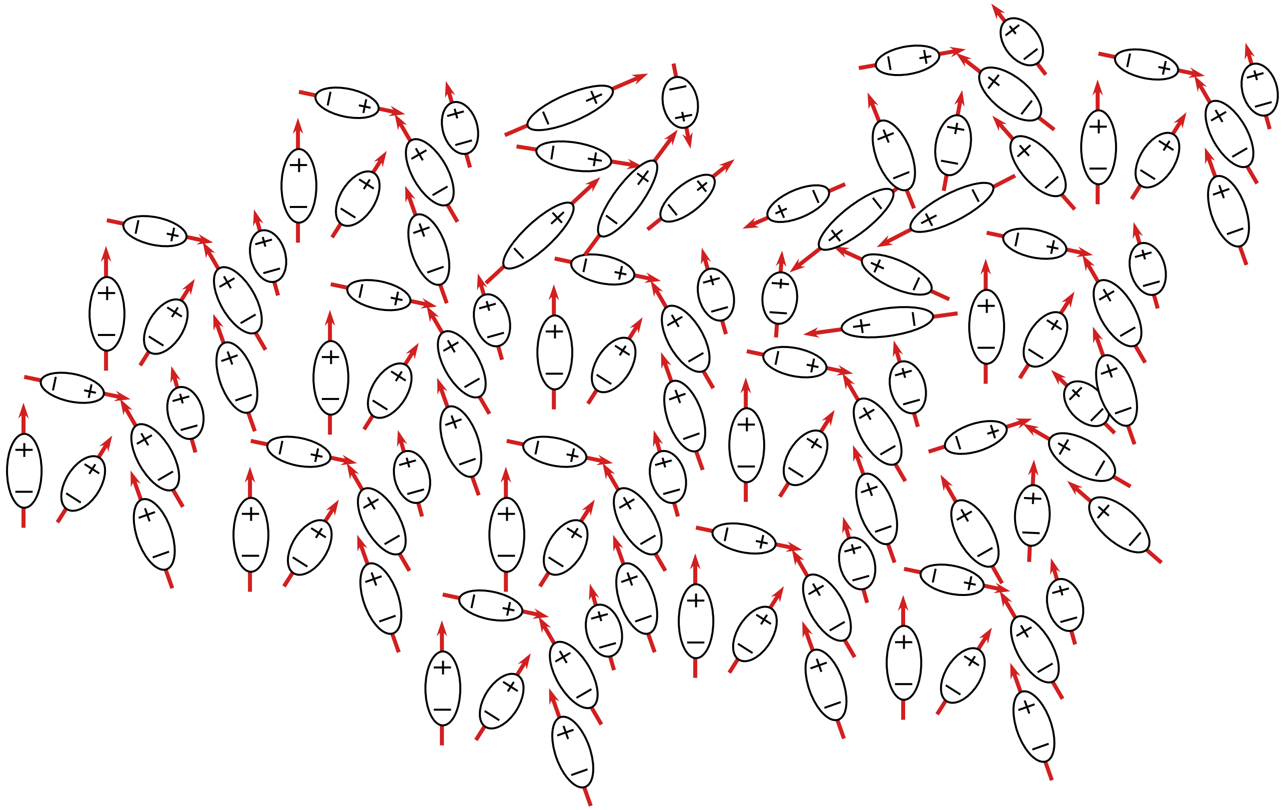
Consider such a polarized atom or molecule, which possesses a *dipole moment*, **p**, defined as the charge magnitude present, Q, times the positive and negative charge separation, d. Dipole moment is a vector that points from the negative to the positive charge.



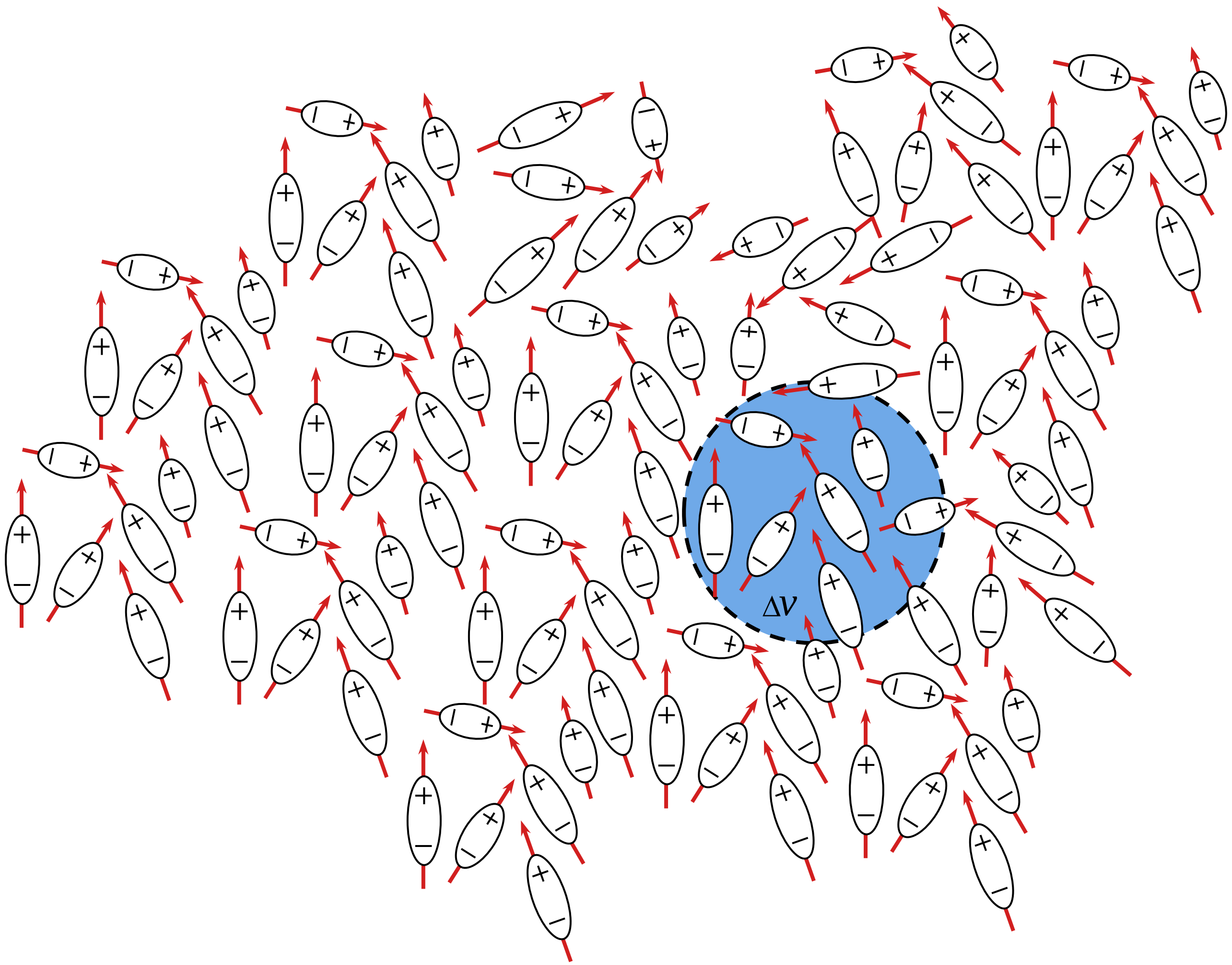
$$\mathbf{p} = Qd\,\mathbf{a}_x$$

Model of a Dielectric

A dielectric can be modeled as an ensemble of bound charges in free space, associated with the atoms and molecules that make up the material. Some of these may have intrinsic dipole moments, others not. In some materials (such as liquids), dipole moments are in random directions.



Polarization Field



The number of dipoles is expressed as a density, *n* dipoles per unit volume.

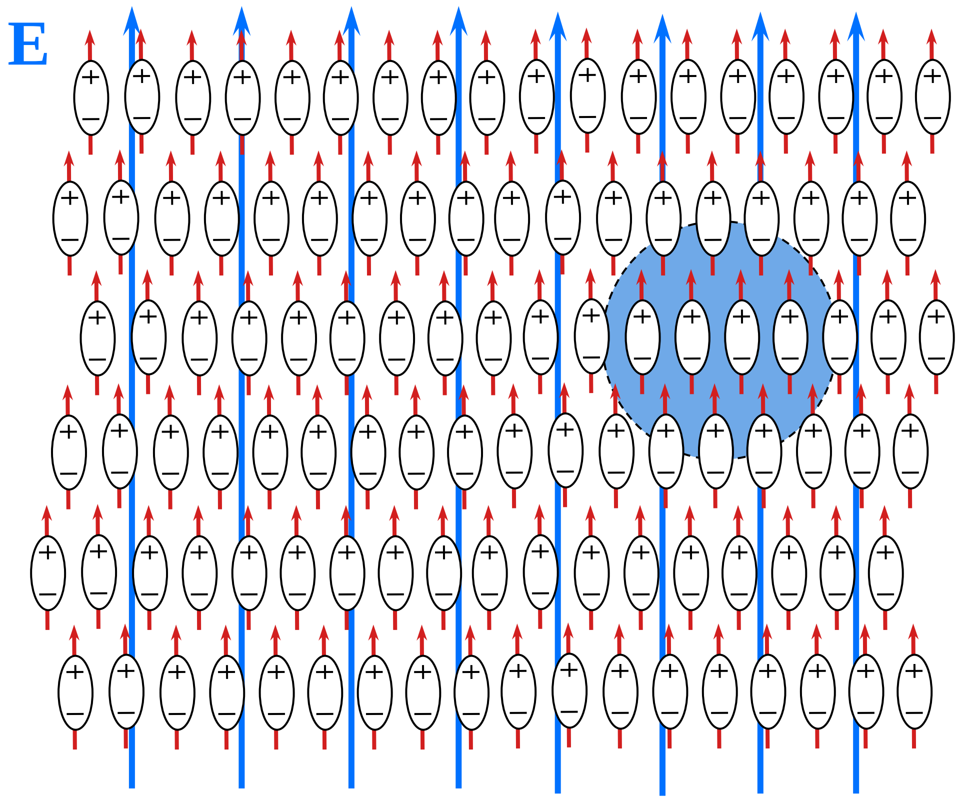
The *Polarization Field* of the medium is defined as:

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

[dipole moment/vol]
or
[C/m^2]

Polarization Field (with Electric Field Applied)

Introducing an electric field may increase the charge separation in each dipole, and possibly re-orient dipoles so that there is some aggregate alignment, as shown here. The effect is small, and is greatly exaggerated here!



The effect is to increase **P**.

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

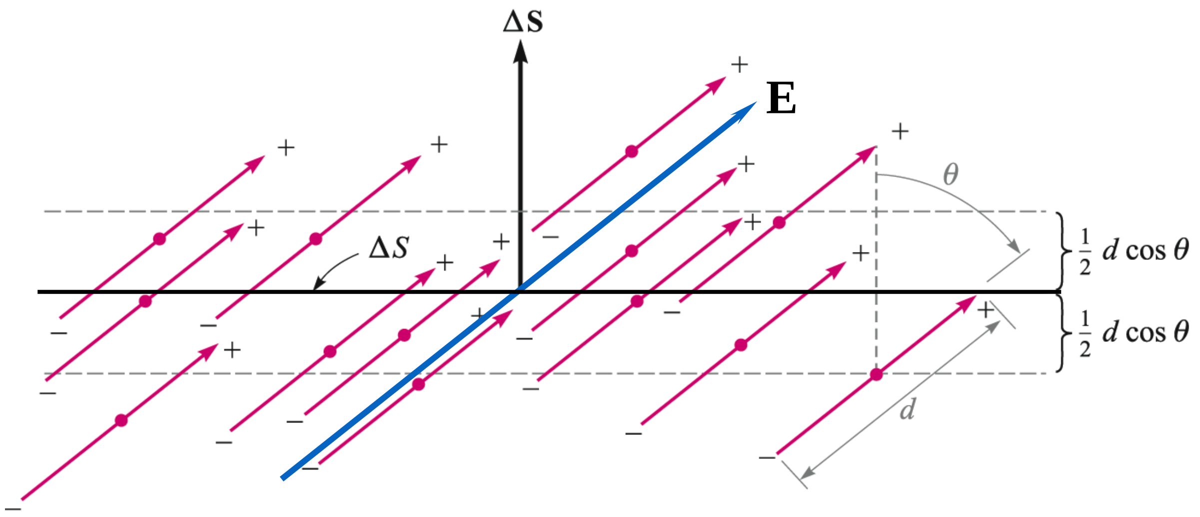
$$= n\mathbf{p}$$

if all dipoles are identical

Migration of Bound Charge

Consider an electric field applied at an angle θ to a surface normal as shown. The resulting separation of bound charges (or re-orientation) leads to positive bound charge crossing upward through surface of area ΔS, while negative bound charge crosses downward through the surface.

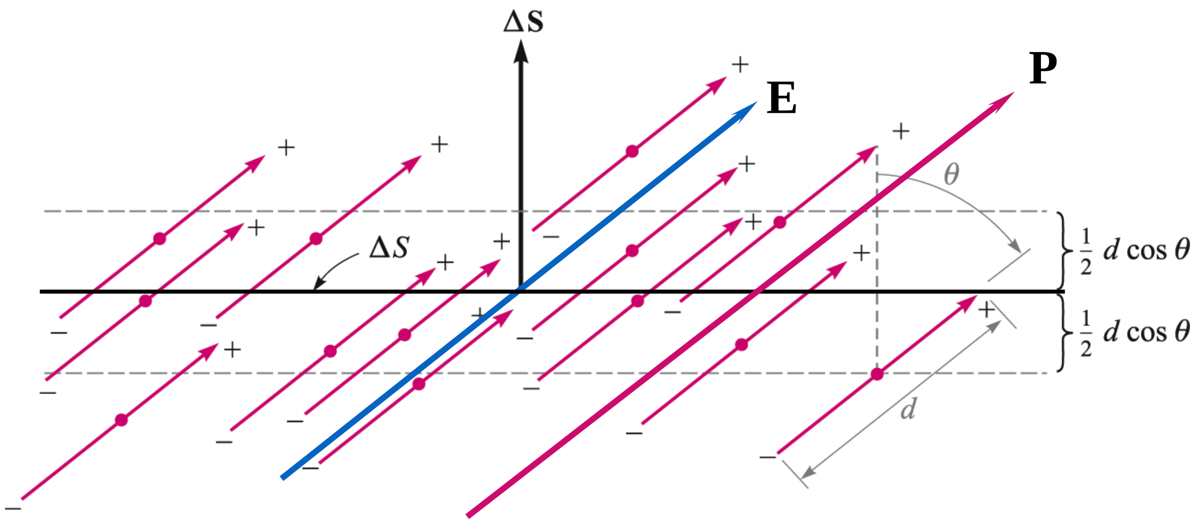
Dipole centers (red dots) that lie within the range (1/2) $d \cos\theta$ above or below the surface will transfer charge across the surface.



Bound Charge Motion as a Polarization Flux

The total bound charge that crosses the surface is given by:

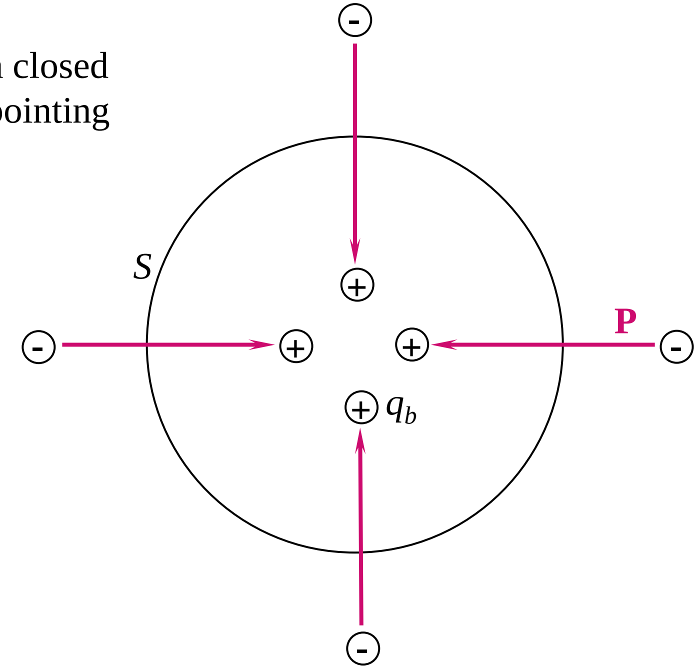
$$\Delta Q_b = nQ\underbrace{d\cos\theta\Delta S}_{\text{volume}} = nQ\mathbf{d}\cdot\Delta\mathbf{S} = \underline{\mathbf{P}\cdot\Delta\mathbf{S}}$$



Polarization Flux Through a Closed Surface

The accumulation of positive bound charge within a closed surface means that the polarization vector must be pointing *inward*. Therefore:

$$Q_b = -\oint_S \mathbf{P} \cdot d\mathbf{S}$$



$$Q_T = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

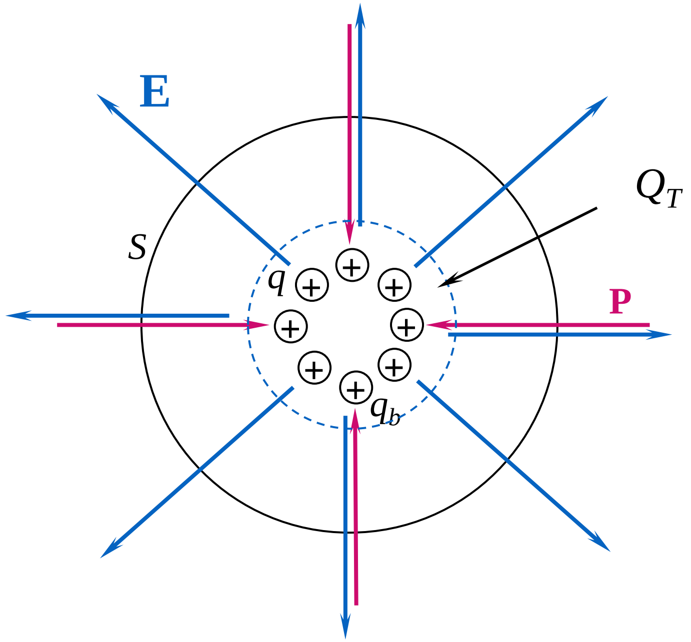
Bound and Free Charge

Now consider the charge within the closed surface consisting of bound charges, q_b, *and* free charges, q. The total charge will be the sum of all bound and free charges. We write Gauss' Law in terms of the total charge, Q_T as:

$$Q_T = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

where

$$Q_T = \underbrace{Q_b}_{\text{bound charge}} + \overbrace{Q}^{\text{free charge}}$$



Gauss Law for Free Charge

$$Q_b = -\oint_S \mathbf{P} \cdot d\mathbf{S} \quad \text{and} \quad Q_T = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

where $Q_T = Q_b + Q$

$$\boxed{Q = Q_T - Q_b = \oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S}}$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}} \qquad Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

Charge Densities

Taking the previous results and using the divergence theorem, we find the point form expressions:

Bound Charge: $Q_b = \int_v \rho_b \, dv = -\oint_S \mathbf{P} \cdot d\mathbf{S} \longrightarrow \boxed{\nabla \cdot \mathbf{P} = -\rho_b}$

Total Charge: $Q_T = \int_v \rho_T \, dv = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} \longrightarrow \boxed{\nabla \cdot \epsilon_0 \mathbf{E} = \rho_T}$

Free Charge: $Q = \int_v \rho_v \, dv = \oint_S \mathbf{D} \cdot d\mathbf{S} \longrightarrow \boxed{\nabla \cdot \mathbf{D} = \rho_v}$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}}$$

Electric Susceptibility and the Dielectric Constant

A stronger electric field results in a larger polarization in the medium. In a *linear* medium, the relation between **P** and **E** is linear, and is given by:

$$\boxed{\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}}$$

where χ_e is the electric *susceptibility* of the medium.

We may now write: $$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$$

where the *dielectric constant*, or *relative permittivity* is defined as:

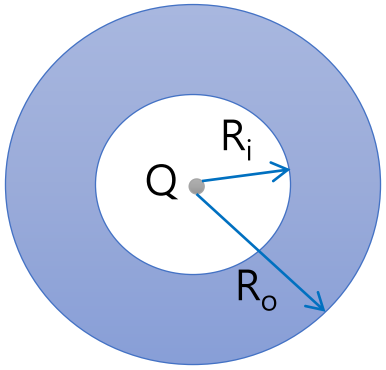
$$\boxed{\epsilon_r = \chi_e + 1}$$

Leading to the overall permittivity of the medium: $\boxed{\epsilon = \epsilon_r \epsilon_0}$ where $\boxed{\mathbf{D} = \epsilon \mathbf{E}}$

Example

Spherical dielectric(ε) shell of inner radius R_i and outer radius R_o enclose a point charge Q.

Find **E**, **D**, **P** at all regions

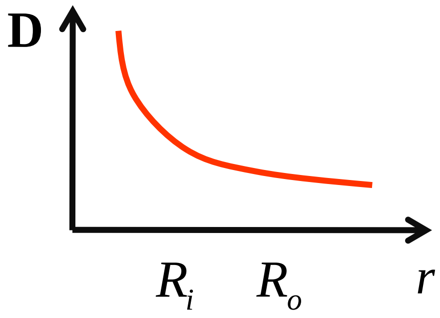


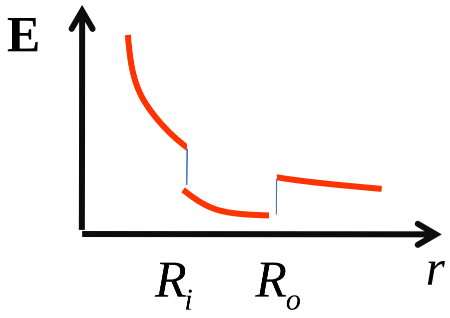
$0 < r < R_i$, $R_o < r$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a_r} \qquad \mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a_r} \qquad \mathbf{P} = 0$$

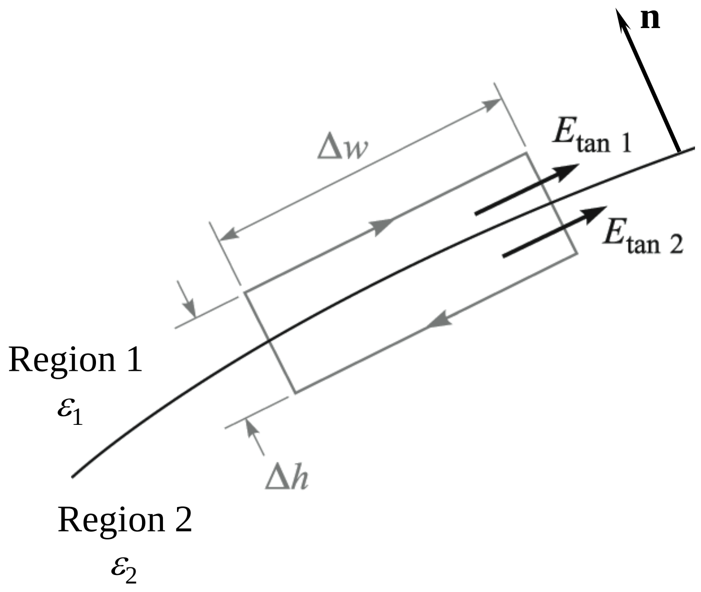
$R_i < r < R_o$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\mathbf{a_r} \qquad \mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a_r} \qquad \mathbf{P} = \frac{Q}{4\pi r^2}(1 - \frac{\varepsilon_o}{\varepsilon})\mathbf{a_r}$$





Boundary Condition for Tangential Electric Field



$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

$$\boxed{\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}}$$

We use the fact that **E** is conservative:

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

So therefore:

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$

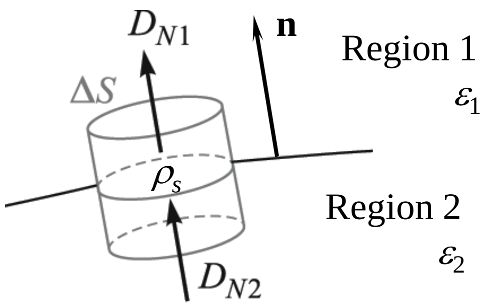
Leading to: $\boxed{E_{\tan 1} = E_{\tan 2}}$

More formally:

$$\boxed{(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{n} = 0}$$

Boundary Condition for Normal Electric Flux Density

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$



The electric flux enters and exits only through the bottom and top surfaces, respectively.

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

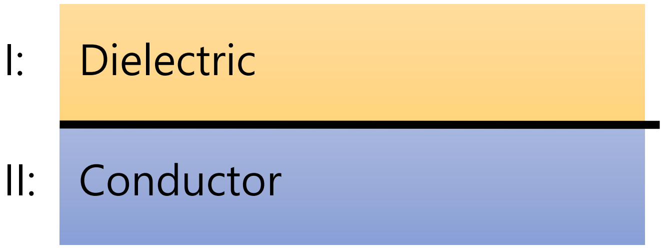
From which: $\boxed{D_{N1} - D_{N2} = \rho_S}$

and if the charge density is zero: $\boxed{D_{N1} = D_{N2}}$

More formally:

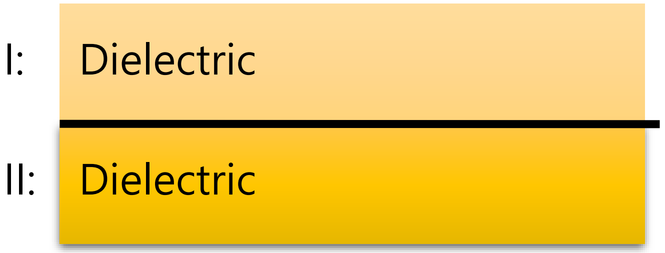
$$\boxed{(\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{n} = \rho_s}$$

Continued



$$E_{t,1} = E_{t,2} = 0$$

$$D_{N,1} = \rho_s, \quad D_{N,2} = 0$$



$$E_{t,1} = E_{t,2}$$

$$D_{N,1} - D_{N,2} = \rho_s \quad (=0)$$

Free surface charge

(unless inserted deliberately, 0)

Example

We wish to find the relation between the angles θ_1 and θ_2, assuming no charge density on the surface.

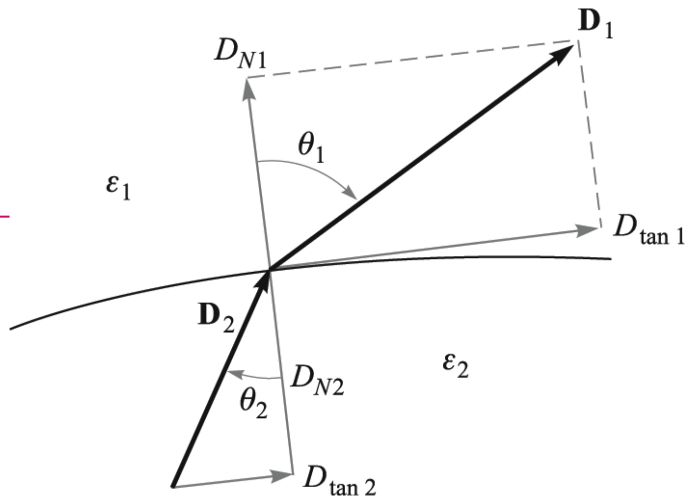
The normal components of **D** will be continuous across the boundary, so that:

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2}$$

Then, with tangential **E** continuous across the boundary, it follows that:

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

or… $\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2$



Now, taking the ratio of the two underlined equations, we finally obtain:

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}}$$

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