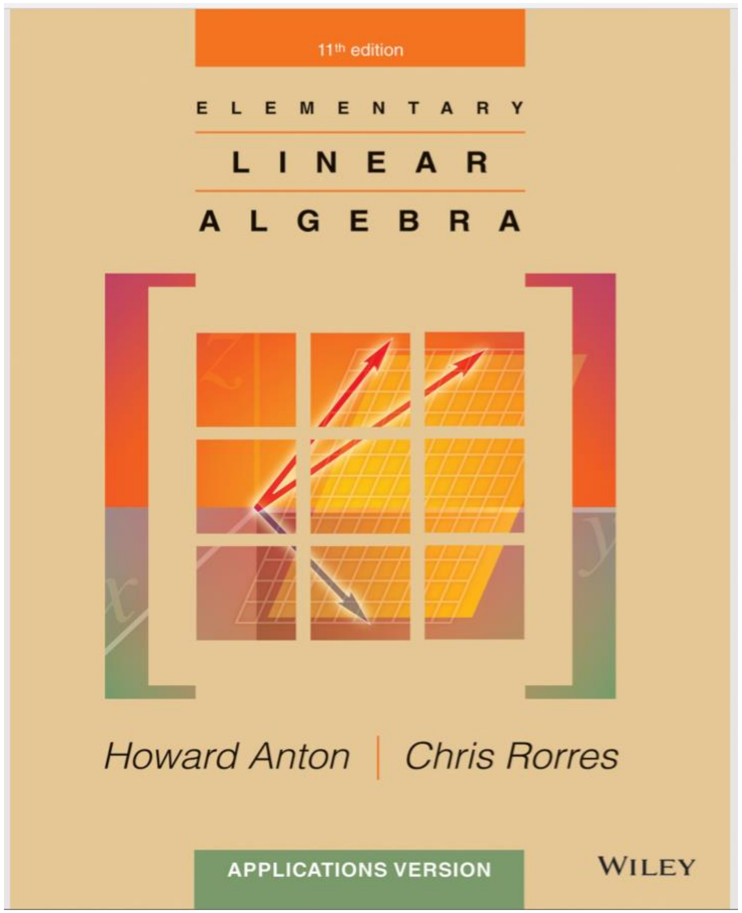
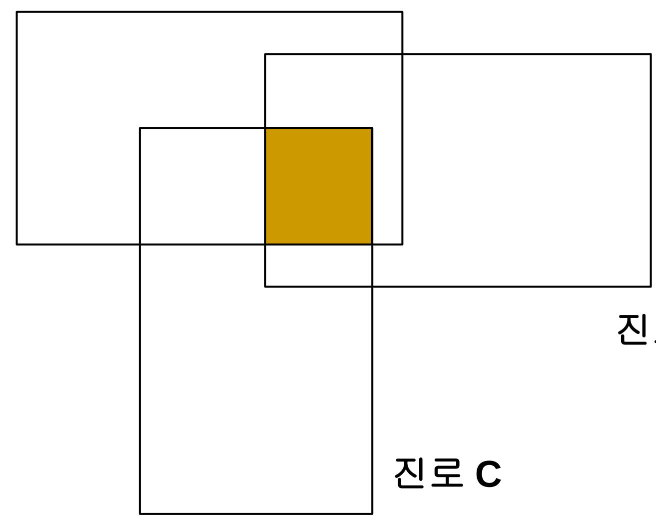
정보통신공학과 선형대수

Introduction (1/4)

- 교과서 11판, 8,278



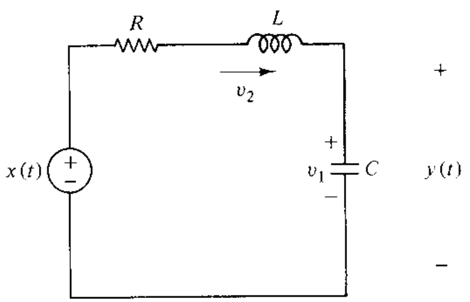
필수과목 이며 수학과목



Introduction (2/4)

- 무엇/어디에 이용되는가?

Example : Circuit analysis



회로이론

$$C\frac{dv_1(t)}{dt} = v_2(t)$$

$$L\frac{dv_2(t)}{dt} = x(t) - Rv_2(t) - v_1(t)$$

$$y(t) = v_1(t)$$

공업수학

주어진 입력에 대한 출력 계산하기

신호 및 시스템, 선형대수

$$\mathbf{v}'(t) = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{-1}{L} & \frac{-R}{L} \end{bmatrix}\mathbf{v}(t) + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}x(t)$$

$$y(t) = [1 \quad 0]\mathbf{v}(t)$$

신호 및 시스템

주어진 입력에 대한 출력 계산하기

Introduction (3/4)

- 무엇/어디에 이용되는가?

교과서의 목차 참조

Introduction (4/4)

- 무엇/어디에 이용되는가?

Q AI에 관심 있는 대학생이나 구직자들에게 조언한다면.

A "지금이라도 늦지 않았다. 제일 쉬운 파이선(python, 프로그래밍언어)부터 배워라. 행렬·확률·통계 기초를 열심히 한 석·박사 인재가 갈수록 간절하다. 문제는 교육이다. 중국은 최근 AI 필수교육을 도입했다. 해당 교재를 보면 대부분 코딩에 관한 내용이다."

◆백상엽 대표

- 1966년생, 서울대 산업공학 학사 · 석사 · 박사
- 미 스탠퍼드대 최고 경영자과정
- (주)LG사장/LG CNS 미래전략사업부장 사장
- 카카오엔터프라이즈 대표(2019년 12월~)

선형대수에서 좋은 성적을 받으려면

- 예습 : 무엇을 모르는지 알고 오기
- 수업 : 예습 때 몰랐던 것 이해하기
- 문제 풀기 : 숙제 스스로 하기

시험은 가급적이면 offline 시험

수업 방법

실시간 강의와 강의 동영상 업로드 병행

실시간 강의로 결석여부 확인

강의 노트 공개 기간

강의 주 포함 3주 + 시험 전 2주

굳이 제한하는 이유 : 학생들이 미루지 않고 그때 그때 공부하기 위함

코딩 프로젝트 예정

4인 1조

본인이 코딩에 문제 있다면 다른 형태로 기여

부작용 : 비대면 형태라도 학생들 사이의 교류 기대

Chapter 1 : Introduction to Systems of Linear Equations

Dept. of Information & Communication. Eng.

Inha University

1.1 Intro. to systems of linear equations

Linear equation (선형식) : 변수의 지수가 1인 식

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1,\ a_2,\ \cdots,\ a_n$: coefficients

$x_1,\ x_2,\ \cdots,\ x_n$: unknowns or variables

Examples

Linear: $x + 3y = 7,\ y = 0.5x + 3z + 1$

Not linear: $x + 3\sqrt{y} = 5,\ 3x + xz = 4,\ y = \sin x$

Linear systems : a set of linear equations

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

m equations, n unknowns

a_{ij}: coefficient for i^{th} equation and variable x_j

선형시스템은 행렬식으로 나타낼 수 있다

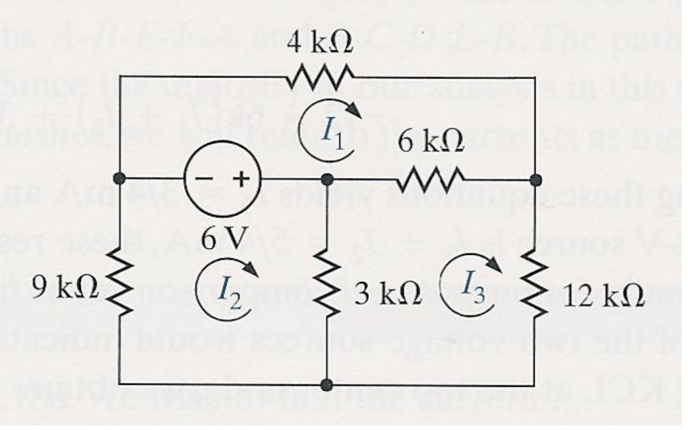
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$$

⇒Ax=b
(mxn)x(nx1)=(mx1)

선형시스템 = 변수가 n개, 식수가 m개인 연립방정식

선형시스템의 해 = 연립방정식의 해 = 연립방정식의 모든 선형식을 만족하는 해 = 행렬식의 해

Example: Electrical Network – 선형대수를 배우는 목적



1. Represent the given circuit using systems of linear equations
2. Find the solution

Applying Kirchhoff's current law, we can obtain

$$4I_1 + 6(I_1 - I_3) + 6 = 0 \Rightarrow 10I_1 - 6I_3 = -6$$
$$9I_2 + 3(I_2 - I_3) - 6 = 0 \Rightarrow 12I_2 - 3I_3 = 6$$
$$6(I_3 - I_1) + 12I_3 + 3(I_3 - I_2) = 0 \Rightarrow -6I_1 - 3I_2 + 21I_3 = 0$$

$$\begin{bmatrix} 10 & 0 & -6 \\ 0 & 12 & -3 \\ -6 & -3 & 21 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{RI} = \boldsymbol{V}$$

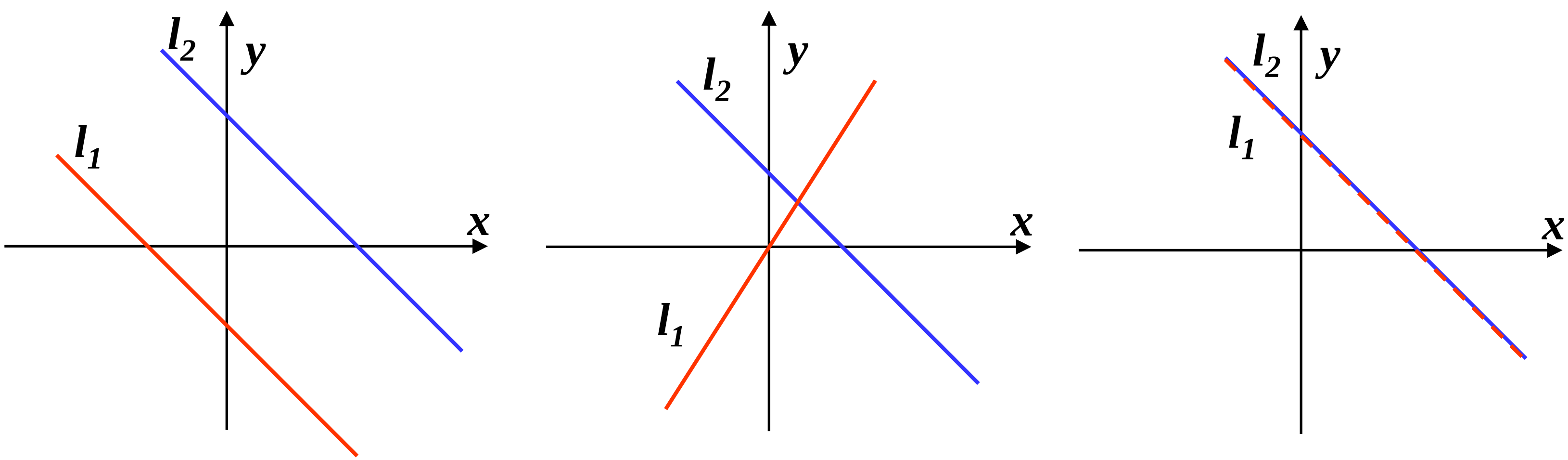
In matrix form

Example

$x + y = 2$	$x + y = 2$	$x + y = 2$
$x + y = 3$	$x - y = 4$	$2x + 2y = 4$

$$\begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

A system of two linear equations



No solution **One solution** **Infinitely many solutions**

- 모든 선형시스템은

  중요!

  - 해가 없거나 (Inconsistent)
  - 유일해가 있거나 (Consistent)
  - 무수히 많은 해가 존재한다 (Consistent)

연립방정식을 손으로 푸는 방법

$2x - y = 4 \qquad (1)$

$-x + 3y = 3 \qquad (2)$

방법 1 : 식(2)의 양변에 2를 곱한다. -> 식(3)
식(1)과 식 (3)을 더한다

무엇을 했는가? : 식(1)과 식(2)를 모두 만족하는 해
= 식(1)과 식(3)를 모두 만족하는 해

=> **한 식에 0이 아닌 상수를 곱해도** 해가 변하지 않는다
=> **한 식에 0이 아닌 상수를 곱하여 다른 식에 더해도** 해가 변하지 않는다
=> 미지수를 소거했다.

$$\begin{aligned} 2x - y &= 4 \quad (1) \\ -x + 3y &= 3 \quad (2) \end{aligned} \quad \Rightarrow \quad \begin{aligned} -x + 3y &= 3 \quad (3) \\ 2x - y &= 4 \quad (4) \end{aligned}$$

무엇을 했는가? : **식(1)과 식(2)의 순서를 바꾸었다.**

Gaussian Elimination 이란?
선형시스템, 연립방정식, 행렬식의 해를 체계적으로 계산하는 방법

STEP 1

***A*x=b 의 해를 계산하는 방법 1**

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

선형시스템, 연립방정식, 행렬식 => Augmented 행렬

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x \quad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x \quad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

STEP 2

Augmented 행렬에 **Elementary row operation**을 적용하여 **(reduced) row echelon form**으로 변환한다.

Elementary row operations

중요!

1. Multiply a row through by a non-zero constant.
2. Interchange two rows.
3. Add a multiple of one row to another row.

(reduced) row echelon form

1. 모두 0이 아닌 row의 첫번째 non-zero number=1이다. .
2. 모두 0인 row는 행렬의 바닥으로 보낸다.
3. 둘 이상의 row가 모두 0이 아닌 경우 위의 row의 leading-1이 아래 row의 leading-1보다 왼쪽에 있다.
4. (reduced의 경우) leading-1을 포함하는 column에서 leading-1이외에는 모두 0이다.

PROJECT 1

Gaussian Elimination을 프로그램 하라.

1. 입력 : 행숫자(식수), 열숫자(미지수 개수)
   Augmented 행렬을 한 줄씩 입력 받는다

<u>2. 출력 : (reduced) row echelon form</u>으로 변환한다.

EX1) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

$$\begin{aligned} x_1 \quad\quad &= 5 \\ x_2 \quad &= -2 \\ x_3 &= 4 \end{aligned}$$

free variable

EX2) $\begin{bmatrix} 1 & 0 & 0 & 4 & -1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 3 & 2 \end{bmatrix}$

$$\begin{aligned} x_1 \quad\quad + 4x_4 &= -1 \\ x_2 \quad + 2x_4 &= 6 \\ x_3 + 3x_4 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 &= -1 - 4x_4 \\ x_2 &= 6 - 2x_4 \\ x_3 &= 2 - 3x_4 \end{aligned}$$

Let $x_4 = t$. Then,

$$x_1 = -1 - 4t, \quad x_2 = 6 - 2t, \quad x_3 = 2 - 3t, \quad x_4 = t$$

Leading variable : leading-1에 해당하는 변수
Free variable : 나머지 변수

매우 중요!

free variable이 존재하면 -> 무수히 많은 해를 갖는다
Leading variable 갯수
　　행의 개수보다 같거나 적다
　　변수 개수보다 같거나 적다
변수 개수 = Leading variable 갯수 + free variable 갯수

Ax=b 에서 A가 nxn 인 linear system에 대해 답하라.

	Reduced row echelon form (nx(n+1))	Leading variable 개수	Free variable 개수
유일해	[I | b']	n	0
무수히 많은 해	모두 0인 행이 존재	<n	0아님
해 없음	식으로 바꾸어주면 0=1 과 같은 식이 만들어 진다,		

문제 1 : $Ax=b$ 에서 A가 mxn (m=n)인 linear system에 대해 답하라.

문제 2 : $Ax=b$ 에서 A가 mxn (m>n)인 linear system에 대해 답하라.

문제 3 : $Ax=b$ 에서 A가 mxn (m<n)인 linear system에 대해 답하라.

문제 4 : $Ax=0$ 에서 A가 mxn (m=n)인 linear system에 대해 답하라.

문제 5 : $Ax=0$ 에서 A가 mxn (m>n)인 linear system에 대해 답하라.

문제 6 : $Ax=0$ 에서 A가 mxn (m<n)인 linear system에 대해 답하라.

Elimination Procedure (1/5)

Reduce the following matrix to reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \qquad \text{(a)}$$

Leftmost nonzero column

Step1. Locate the leftmost column that does not consist entirely of zeros.

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \qquad \text{(b)}$$

The first and second rows in (a) were interchanged

Elimination Procedure (2/5)

Step 3. If the entry that is now at the top of the column found in Step 1 is *a*, multiply the first row by *1/a* in order to introduce a leading 1.

(1) ← (1) x 0.5

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix}$$

Step 4. Add the suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

(3) ← (3) + (1) x -2

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix}$$

Elimination Procedure (3/5)

Step 5. Cover the top row in the matrix and begin again with Step 1 applied to the sub-matrix that remains. Continue in this way until the entire matrix is in row-echelon form.

(2) ← (2) x -0.5

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix}$$

Leftmost nonzero column in the sub-matrix

(3) ← (3) + (2) x -5

(3) ← (3) x 2

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix}$$

Leftmost nonzero column in the sub-matrix

Row-echelon form

: Gaussian elimination completed!

Elimination Procedure (4/5)

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

Should make these zero

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix} \Rightarrow$$

(2) ← (2) + (3) x 7/2

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix} \Rightarrow$$

(1) ← (1) + (3) x -6

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix} \Rightarrow$$

(1) ← (1) + (2) x 5

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix}$$

Reduced row-echelon form

: Gauss-Jordan elimination completed!

Elimination Procedure (5/5)

We converted the augmented matrix to the reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \cdots (1) \\ \cdots (2) \\ \cdots (3) \end{matrix}$$

The system corresponding to this matrix is

$$\begin{aligned} x_1 + 2x_2 \quad + 3x_4 \quad &= 7 \\ x_3 \quad &= 1 \\ x_5 &= 2 \end{aligned}$$

If we assign the free variables x_2 and x_4 arbitrary values *s* and *t*, respectively, the general solution is given by the following:

$$x_1 = 7 - 2s - 3t,\ x_2 = s,\ x_3 = 1,\ x_4 = t,\ x_5 = 2$$

Homogeneous Linear Systems

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

Homogeneous 선형시스템은 선형시스템의 부분집합

Every homogeneous system of linear equations is consistent:

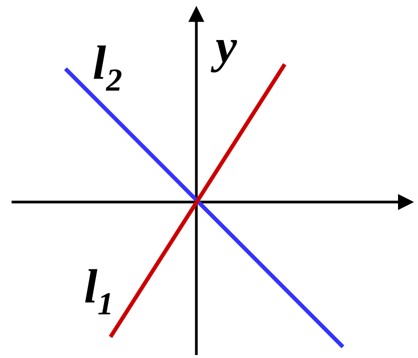
- All such systems have $x_1=0, x_2=0, \ldots, x_n=0$ as a solutions (trivial solution)
- If there are other solutions, they are called nontrivial solutions

참고 : 모든 선형시스템은

- 해가 없거나 (Inconsistent)
- 유일해가 있거나 (Consistent)
- 무수히 많은 해가 존재한다 (Consistent)

Ex) Two equations in two unknowns

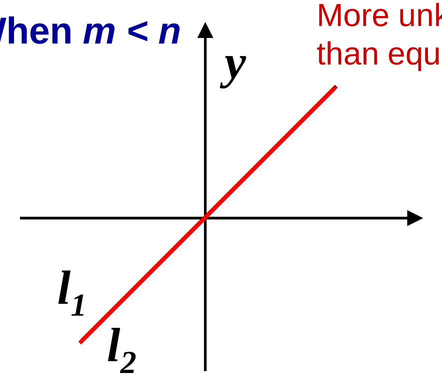
$$a_1x + b_1y = 0$$
$$a_2x + b_2y = 0$$
$$(a_2,\ b_2 \text{ not both zero})$$



Trivial solution

When $m < n$

More unknowns than equations



Infinitely many solution

모두 원점을 지나는 직선

- **Theorem 1.2.1** If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has n – r free variables

r개의 nonzero rows -> r개의 leading variable

-> (n-r)개의 free variable

중요!

- **Theorem 1.2.2** A homogeneous linear system with more unknowns than equations has infinitely many solutions

- m<n   4x7
- Leading 변수 개수 =<4
- 7= leading 변수 개수 + free 변수 개수

중요!

- **Theorem 1.2.2** A homogeneous linear system with more unknowns than equations has infinitely many solutions

식수 m < 변수수 n 이면

Leading variable의 최대 개수=m < n

따라서 free variable 존재 -> 무수히 많은 해

1.3 Matrices and matrix Operations

Vector의 필요성

- 속력과 속도

- 사람의 키와 몸무게를 표현하려면

키=175, 몸무게=75
(키, 몸무게)=(175, 75)

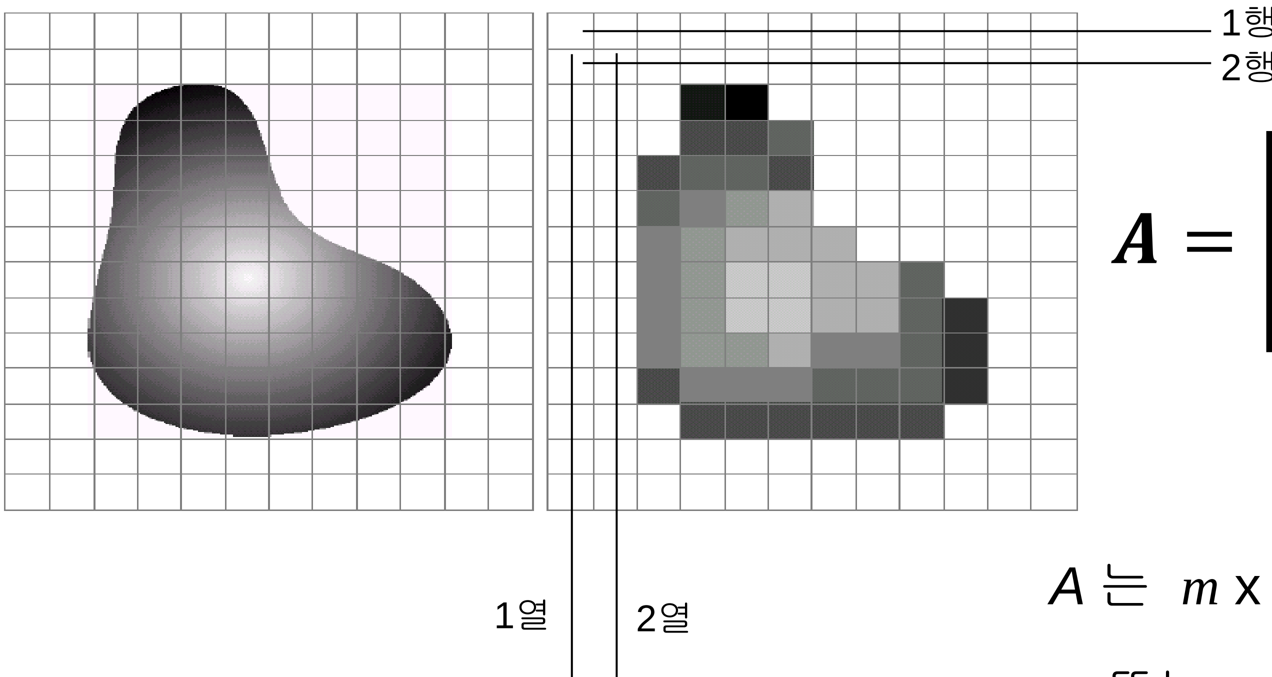
두 사람 각각을 표현하려면 $\mathrm{x} = \begin{bmatrix} 175 \\ 75 \end{bmatrix}, y = \begin{bmatrix} 180 \\ 78 \end{bmatrix}$

두 사람을 한번에 표현하려면 $\begin{bmatrix} 175 & 180 \\ 75 & 78 \end{bmatrix}$

- 사람의 키, 몸무게, 허리둘레를 표현하려면

1.3 Matrices and matrix Operations

행렬의 필요성



$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

A 는 m x n 행렬

m 또는 n 이 1이면 vector

m 과 n 이 모두 1이면 scalar

1.3 Matrices and matrix Operations

Given

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

Equality

$A = B$ if and only if

1) When A and B have the same size and

2) $(\mathrm{A})_{\mathrm{ij}} = (B)_{ij}$, or equivalently, $a_{ij} = b_{ij}$ for all i and j

Addition and Subtraction

1) $(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}$

2) $(A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$

Ex)

$$A+B = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

- Example : AB도 성립하고 BA도 성립하는 경우는??

답1 :정방행렬 -> 정방행렬일 필요는 없다

$A_{3x3}\ B_{3x3}\qquad AB_{3x3}$

$B_{3x3}\ A_{3x3}\qquad BA_{3x3}$

답2 : mxr rxm -> yes

$A_{4x3}\ B_{3x4}\quad AB_{4x4}$

$B_{3x4}\ A_{4x3}\quad BA_{3x3}$

Transpose

A: *mxn* mtarix

A^T: *nxm* mtarix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}$$

Transpose of A

Trace

If A is a square matrix, then the trace of A is defined by\

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ then } tr(A) = a_{11} + a_{22} + a_{33}$$

If A is not a square matrix, the trace of A is undefined

Matrix Multiplication

If *A* is an *mxr* matrix and *B* is an *rxn* matrix, then the product *AB* is defined by

$\mathbf{a}_1$ $\quad$ $\mathbf{b}_1$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

the entry $(AB)_{ij}$ in row i and column j of AB:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

AB can be also represented as

중요!

$$AB = A\left[\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n\right] = \left[A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_n\right]$$

Computed column by column

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

Computed row by row

1.4 Inverses : Algebraic Properties of Matrices

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) $A+B=B+A$ **(Commutative law for addition)**

(b) $A+(B+C)=(A+B)+C$ **(Associative law for addition)**

(c) $A(BC)=(AB)C$ **(Associative law for multiplication)**

(d) $A(B+C)=AB+AC$ **(Left distributive law)**

(e) $(B+C)A=BA+CA$ **(Right distributive law)**

(f) $A(B-C)=AB-AC$

(g) $(B-C)A=BA-CA$

(h) $a(B+C)=aB+aC$

(i) $a(B-C)=aB-aC$

(j) $(a+b)C=aC+bC$

(k) $(a-b)C=aC-bC$

(l) $a(bC)=(ab)C$

(m) $a(BC)=(aB)C=B(aC)$

Properties of Matrix Arithmetic (2/2)

Caution!!

1) For matrices, although $A \neq 0$, we cannot say that

$AB = AC \quad \rightarrow \quad B = C$ **(Cancellation law is not valid)**

2) For matrices,

$AD = 0$, yet $A \neq 0$ and $D \neq 0$ **(It is possible for a product of matrices to be zero without either factor being zero.)**

Ex) Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix},\ C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix},\ D = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

$$1)\ AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix},\ AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

$$\therefore\ AB = AC, \text{ but } B \neq C$$

$$2)\ AD = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ but } A \neq 0 \text{ and } D \neq 0$$

A square matrix with 1's on the main diagonal and zeros elsewhere is called an ***identity matrix***. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

IA=A AI=A

중요!!!

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

THEOREM 1.4.4 B와 C가 정방행렬 A의 역행렬이면 B=C이다. 즉, 역행렬은 유일하다.

가정에 의해 BA=I
양변의 오른쪽에 C를 곱하면 (BA)C=IC
좌변 (BA)C=B(AC)=BI=B
우변 C
=> B=C

역행렬을 계산하는 방법 1

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

중요!!!

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Definition

If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I,\quad A^n = AA\cdots A\ (n > 0)$$

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1}\ (\text{if } A \text{ is invertible})$$

A는 B의 역행렬이다, AB=BA=I

AB는 $B^{-1}A^{-1}$ 의 역행렬이다 $AB\,B^{-1}A^{-1} = AIA^{-1} = AA^{-1}$
$= B^{-1}A^{-1}\,AB = I$

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(a) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(b) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(c) kA *is invertible for any nonzero scalar* k, *and* $(kA)^{-1} = k^{-1}A^{-1}$.

Theorem 1.4.8 If the sizes of the matrices are such that the stated operations can be performed, then

(a) $((A)^T)^T = A$

(b) $(A+B)^T = A^T + B^T$ and $(A-B)^T = A^T - B^T$

(c) $(kA)^T = kA^T$, where k is any scalar

(d) $(AB)^T = B^T A^T$ **The transpose of a product of any number of matrices = the product of their transposes in the reverse order**

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

$A^T (A^{-1})^T = (A^{-1})^T A^T = I$

1.5 Elementary Matrices and a method for finding A^{-1}

역행렬을 계산하는 방법 2

Elementary matrix : E

A matrix obtained from the I_n by performing a single elementary row operation

Examples of elementary matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3x3 identity matrix

I_3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Multiply the 3rd row of I_3 by -3

(3) ← (3) x -3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Interchange the 2nd and 3rd rows of I_3

(2) ↔ (3)

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Add three times the 3rd row of I_3 to the 1st row

(1) ← (1) + (3) x 3

THEOREM 1.5.1

행렬 A에 E란 elementary 행렬을 곱한 결과는 즉, EA는 A에 E을 얻을때 사용한 ERO를 적용한것과 일치한다.

$$I \xrightarrow{\text{ERO}} E$$

$$A \xrightarrow{\text{ERO}} B = EA$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1*3+r_3} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} \xrightarrow{r_1*3+r_3} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix} \boxed{=} EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

Inverse Row Operations

If an elementary row operation is applied to *I* to produce *E*, then there is a second row operation that, when applied to *E*, reproduces *I*

Row operation **Inverse row operation**

$$\underset{I_2}{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}} \xrightarrow[\text{Multiply the } 2^{nd} \text{ row by } 7]{} \underset{E}{\begin{bmatrix}1 & 0\\0 & 7\end{bmatrix}} \xrightarrow[\text{Multiply the } 2^{nd} \text{ row by } 1/7]{} \underset{I_2}{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}$$

Every elementary matrix is invertible, and the inverse is also an elementary matrix

Row operation on *I* that produces *E*	Row operation on *E* that reproduces *I*
Multiply row i by $c \neq 0$	Multiply row i by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add $-c$ times row i to row j

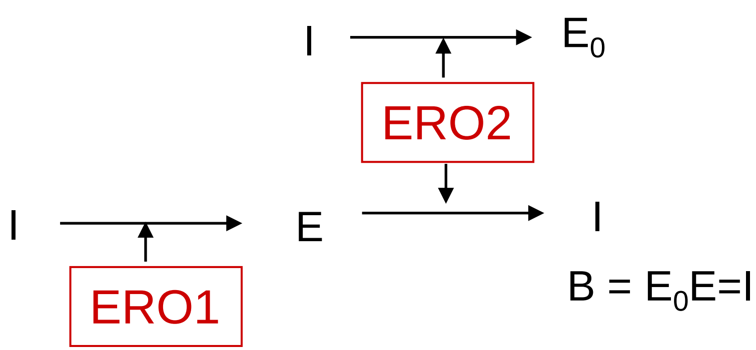
THEOREM 1.5.2

(1) 모든 elementary 행렬 (E)은 invertible이며
(2) elementary 행렬 (E) 의 역행렬도 elementary 행렬 (E0) 이다.

THEOREM 1.5.2

(1) 모든 elementary 행렬 (E)은 invertible이며
(2) elementary 행렬 (E) 의 역행렬도 elementary 행렬 (E0) 이다.

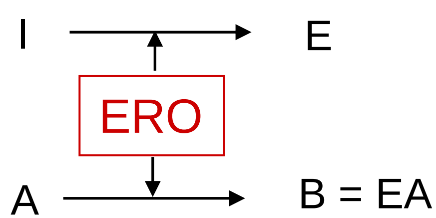
Proof $E_0E=I$ 라는 사실과 E0가 elementary 행렬 **임을 보인다**



근거 2: 정리 1.5.1

근거 1: 앞페이지

참고 : 정리 1.5.1



Theorem 1.5.3 If *A* is an *nxn* matrix, then the following statements are equivalent:

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

중요!!!

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

Proof (a)->(b)

Assume A is invertible and let $\mathbf{x_0}$ be any solution of $A\mathbf{x} = \mathbf{0}$.

Thus, $A\mathbf{x_0} = \mathbf{0}$ (1)

Multiplying both sides of (1) by A^{-1} gives

$$A^{-1}A\mathbf{x_0} = A^{-1}\mathbf{0}$$

$\therefore \mathbf{x_0} = \mathbf{0}$ ($A\mathbf{x} = \mathbf{0}$ has only trivial solution)

Proof (b)->(c)

Consider *A**x**=**0*** :

Augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

*A**x**=**0*** has only the trivial solution.

Augmented matrix

$$\begin{matrix} x_1 & & & = 0 \\ & x_2 & & = 0 \\ & & \ddots & \\ & & & x_n = 0 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Proof (c)->(d)

$$E_k \cdots E_2 E_1 A = I_n \ \text{(where } E_1, \cdots, E_k \text{ are elementary matrices)}$$

A method for inverting matrices : 역행렬을 계산하는 방법 2

We know that

$$E_k \cdots E_2 E_1 A = I_n \quad (1),$$

Multiplying (1) on the right by A^{-1} yields

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

To find A^{-1}, we must find a sequence of elementary row operations that reduces A to I_n and then perform this same sequence of operations on I_n to obtain A^{-1}.

$$[A \mid I]$$ **Adjoin the identity matrix to the right side of A**

$$\Downarrow$$

$$[I \mid A^{-1}]$$ **Apply row operations to this matrix until the left side is reduces to *I*; These operations will convert the right side to A^{-1}**

Ex) Find the inverse of

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$$

Solution

1) Construct $\left[A\,|\,I\right]$

2) Apply row operations to make $\left[A\,|\,I\right]\rightarrow\left[I\,|\,A^{-1}\right]$

$$\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 1 & -3 & -2 & 1 & 0\\ 0 & -2 & 5 & -1 & 0 & 1\end{array}\right]$$

1.6 More on Linear Systems and Invertible Matrices

Theorem 1.6.1 모든 선형시스템은
- 해가 없거나 (Inconsistent)
- 유일해가 있거나 (Consistent)
- 무수히 많은 해가 존재한다 (Consistent)

Proof

해가 두개 이상일 경우 무수히 많은 해가 존재함을 보인다.

<table>
<tr><td rowspan="2">해 없음</td><td>유일해</td></tr>
<tr><td>두개 이상</td></tr>
</table>

x_1 과 x_2 가 서로 다른 두 개의 해 일때 $(x_1 + kx_0)$도 해이다.

$$x_0 = x_1 - x_2 \neq 0$$

$$(x_1 + kx_0) \neq x_1 \qquad (x_1 + kx_0) \neq x_2$$

k 가 임의의 상수 $\rightarrow (x_1 + kx_0)$의 갯수는 무한개

Proof If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, exactly one of the following is true: (a) the system has no solutions, (b) the system has exactly one solution, or (c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case (c).

Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, the matrix $\mathbf{x}_0$ is nonzero; moreover,

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

If we now let k be any scalar, then

$$\begin{aligned} A(\mathbf{x}_1 + k\mathbf{x}_0) &= A\mathbf{x}_1 + A(k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) \\ &= \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b} \end{aligned}$$

But this says that $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{x}_0$ is nonzero and there are infinitely many choices for k, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. ◀

Theorem1.6.2 If A is an invertible *nxn* matrix, then for each *nx1* matrix **b**, the system of equations $A\mathbf{x}=\mathbf{b}$ has exactly one solution, namely, $\mathbf{x}=A^{-1}\mathbf{b}$

Proof

중요!!!

Ax=b 의 해를 계산하는 방법 2, 단 A는 nxn

$\boldsymbol{POINT}$: $\boldsymbol{A^{-1}b}$가 해이다. $\boldsymbol{x_0}$ 가 임의의 해 라 가정하면 $\boldsymbol{x_0 = A^{-1}b}$

Proof Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. To show that this is the only solution, we will assume that $\mathbf{x}_0$ is an arbitrary solution and then show that $\mathbf{x}_0$ must be the solution $A^{-1}\mathbf{b}$.

If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x}_0 = \mathbf{b}$. Multiplying both sides of this equation by A^{-1}, we obtain $\mathbf{x}_0 = A^{-1}\mathbf{b}$. ◀

EXAMPLE 1 Solution of a Linear System Using A^{-1}

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1,\ x_2 = -1,\ x_3 = 2$. ◀

Diagonal Matrices

Diagonal matrices

A square matrix in which all the entries off the main diagonal are zero

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverses of diagonal matrices

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}, \; D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}, \; D^k = \begin{bmatrix} d^k & 0 & \cdots & 0 \\ 0 & d^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d^k \end{bmatrix}$$

Symmetric Matrices

Symmetric matrices

A square matrix A is called symmetric if $A=A^T$

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$$

Properties of symmetric matrices

Theorem) If A and B are symmetric matrices with the same size, and if k is any scalar, then:

(a) A^T is symmetric
(b) $A+B$ and $A-B$ are symmetric
(c) kA is symmetric

Theorem) If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Assume that $A = A^T$. Then, we have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}:\ A^{-1} \text{ is symmetric}$$

1.7 Matrix transformations

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We call the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ the ***standard basis vectors*** for R^n. For example, the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are the standard basis vectors for R^3.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

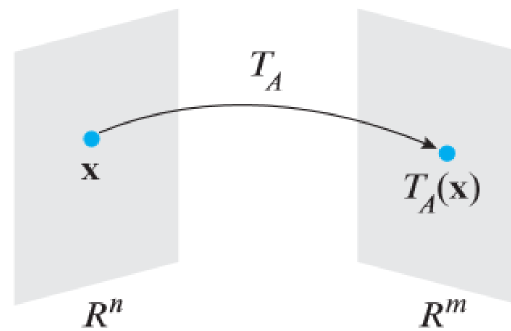
$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

DEFINITION 1 If f is a function with domain R^n and codomain R^m, then we say that f is a ***transformation*** from R^n to R^m or that f ***maps*** from R^n to R^m, which we denote by writing

$$f: R^n \to R^m$$

In the special case where $m = n$, a transformation is sometimes called an ***operator*** on R^n.

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



$$\mathbf{w} = A\mathbf{x}$$

$$\mathbf{w} = T_A(\mathbf{x})$$

$$T_A: R^n \to R^m$$

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

Read Example 1 in Section 1.7

THEOREM 1.8.1 *For every matrix A the matrix transformation $T_A: R^n \to R^m$ has the following properties for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *and for every scalar k:*

(a) $T_A(\mathbf{0}) = \mathbf{0}$

(b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ **[Homogeneity property]**

(c) $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ **[Additivity property]**

(d) $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

Proof All four parts are restatements of the following properties of matrix arithmetic given in Theorem 1.4.1:

$$A\mathbf{0} = \mathbf{0}, \quad A(k\mathbf{u}) = k(A\mathbf{u}), \quad A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(\mathbf{u}-\mathbf{v}) = A\mathbf{u} - A\mathbf{v} \quad ◀$$

THEOREM 1.8.2 *$T : R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors* **u** *and* **v** *in R^n and for every scalar k:*

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ **[Homogeneity property]**

Proof If T is a matrix transformation, then properties (i) and (ii) follow respectively from parts (c) and (b) of Theorem 1.8.1.

Conversely, assume that properties (i) and (ii) hold. We must show that there exists an $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for every vector $\mathbf{x}$ in R^n. Recall that the derivation of Formula (10) used only the additivity and homogeneity properties of T_A. Since we are assuming that T has those properties, it must be true that

$$T(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r) = k_1T(\mathbf{u}_1) + k_2T(\mathbf{u}_2) + \cdots + k_rT(\mathbf{u}_r) \tag{12}$$

for all scalars $k_1, k_2, \ldots, k_r$ and all vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ in R^n. Let A be the matrix

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)] \tag{13}$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n. It follows from Theorem 1.3.1 that $A\mathbf{x}$ is a linear combination of the columns of A in which the successive coefficients are the entries $x_1, x_2, \ldots, x_n$ of $\mathbf{x}$. That is,

$$A\mathbf{x} = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n)$$

Using Formula (10) we can rewrite this as

$$A\mathbf{x} = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = T(\mathbf{x})$$

which completes the proof. ◀

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)] \tag{15}$$

This suggests the following procedure for finding standard matrices.

Finding the Standard Matrix for a Matrix Transformation

Step 1. Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n.

Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

▶ **EXAMPLE 4 Finding a Standard Matrix**

Find the standard matrix A for the linear transformation $T: R^2 \to R^3$ defined by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix} \tag{16}$$

Solution We leave it for you to verify that

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Thus, it follows from Formulas (15) and (16) that the standard matrix is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$