1.4 17, 21, 23, 32, 45, 47, 48, 50

In Exercises 15–18, use the given information to find A.

V **17.** $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

In Exercises 21–22, compute $p(A)$ for the given matrix A and the following polynomials.

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$

(c) $p(x) = x^3 - 2x + 1$

V **21.** $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

In Exercises 23–24, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

V **23.** Find all values of a, b, c, and d (if any) for which the matrices A and B commute.

32. The numerical equation $a^2 = 1$ has exactly two solutions. Find at least eight solutions of the matrix equation $A^2 = I_3$. [*Hint:* Look for solutions in which all entries off the main diagonal are zero.]

45. (a) Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

(b) What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

47. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$$

48. Show that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfies the equation

$$A^2 - (a + d)A + (ad - bc)I = 0$$

50. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$ABC^TDBA^TC = AB^T$$

1.5 8, 19(b), 20, 21, 22, 30

In Exercises 7–8, use the following matrices and find an elementary matrix E that satisfies the stated equation.

ıt satisfy the

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

✓ **8.** (a) $EB = D$ (b) $ED = B$

(c) $EB = F$ (d) $EF = B$

In Exercises 19–20, find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, and k are all nonzero.

19. (a) $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$ ✓ (b) $\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

✓ **20.** (a) $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$ ✓ (b) $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

In Exercises 21–22, find all values of c, if any, for which the given matrix is invertible.

21. $\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$

22. $\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$

30. Show that

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

1.6 15, 18, 21, 22

In Exercises 13–17, determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

15. $$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

18. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$ and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.

(b) Solve $A\mathbf{x} = 4\mathbf{x}$.

Working with Proofs

21. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Prove that if k is any positive integer, then the system $A^k\mathbf{x} = \mathbf{0}$ also has only the trivial solution.

22. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Prove that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

1.7 13, 17, 19, 31, 34, 45, 40

In Exercises 13–14, compute the indicated quantity.

13. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39}$

In Exercises 17–18, create a symmetric matrix by substituting appropriate numbers for the $\times$'s.

17. (a) $\begin{bmatrix} 2 & -1 \\ \times & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & \times & \times & \times \\ 3 & 1 & \times & \times \\ 7 & -8 & 0 & \times \\ 2 & -3 & 9 & 0 \end{bmatrix}$

In Exercises 19–22, determine by inspection whether the matrix is invertible.

19. $\begin{bmatrix} 0 & 6 & -1 \\ 0 & 7 & -4 \\ 0 & 0 & -2 \end{bmatrix}$

In Exercises 31–32, find a diagonal matrix A that satisfies the given condition.

✓

31. $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

✓

34. Let A be an $n \times n$ symmetric matrix.

(a) Show that A^2 is symmetric.

(b) Show that $2A^2 - 3A + I$ is symmetric.

45. Prove the following facts about skew-symmetric matrices.

V(a) If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

(b) If A and B are skew-symmetric matrices, then so are A^T, $A + B$, $A - B$, and kA for any scalar k.

40. If the $n \times n$ matrix A can be expressed as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ can be expressed as $LU\mathbf{x} = \mathbf{b}$ and can be solved in two steps:

Step 1. Let $U\mathbf{x} = \mathbf{y}$, so that $LU\mathbf{x} = \mathbf{b}$ can be expressed as $L\mathbf{y} = \mathbf{b}$. Solve this system.

Step 2. Solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

In each part, use this two-step method to solve the given system.

(a) $$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$$

Supplemental exercises: 9, 12, 14, 15, 17, 19, 20, 24

9. Let

$$\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix}$$

be the augmented matrix for a linear system. Find for what values of a and b the system has

(a) a unique solution.

(b) a one-parameter solution.

(c) a two-parameter solution. (d) no solution.

12. How should the coefficients a, b, and c be chosen so that the system

$$\begin{aligned} ax + by - 3z &= -3 \\ -2x - by + cz &= -1 \\ ax + 3y - cz &= -3 \end{aligned}$$

has the solution $x = 1$, $y = -1$, and $z = 2$?

14. Let A be a square matrix.

(a) Show that $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$

(b) Show that

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^n$$

if $A^{n+1} = 0$.

15. Find values of a, b, and c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

17. Let J_n be the $n \times n$ matrix each of whose entries is 1. Show that if $n > 1$, then

$$(I - J_n)^{-1} = I - \frac{1}{n-1} J_n$$

19. Prove: If B is invertible, then $AB^{-1} = B^{-1}A$ if and only if $AB = BA$.

20. Prove: If A is invertible, then $A + B$ and $I + BA^{-1}$ are both invertible or both not invertible.

24. Assuming that the stated inverses exist, prove the following equalities.

(a) $(C^{-1} + D^{-1})^{-1} = C(C + D)^{-1}D$

(b) $(I + CD)^{-1}C = C(I + DC)^{-1}$

(c) $(C + DD^T)^{-1}D = C^{-1}D(I + D^TC^{-1}D)^{-1}$

▶ Partitioned matrices can be multiplied by the row-column rule just as if the matrix entries were numbers provided that the sizes of all matrices are such that the necessary operations can be performed. Thus, for example, if A is partitioned into a 2×2 matrix and B into a 2×1 matrix, then

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} \qquad (*)$$

provided that the sizes are such that AB, the two sums, and the four products are all defined. ◀