전자기학 I 중간고사 모범답안

1

(1) $\mathbf{A} = x\,\mathbf{a}_x = A_\rho \mathbf{a}_\rho + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z$

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho = x\,\mathbf{a}_x\cdot\mathbf{a}_\rho = x\cos\phi = \rho\cos^2\phi$$

$$A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi = x\,\mathbf{a}_x\cdot\mathbf{a}_\phi = x\cos(\phi+\tfrac{\pi}{2}) = -x\sin\phi = -\rho\sin\phi\cos\phi$$

$$A_z = \mathbf{A}\cdot\mathbf{a}_z = x\,\mathbf{a}_x\cdot\mathbf{a}_z = 0$$

$$\therefore \mathbf{A} = \rho\cos^2\phi\,\mathbf{a}_\rho - \rho\sin\phi\cos\phi\,\mathbf{a}_\phi$$

$$= \rho\cos\phi\,\{\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi\} \;//$$

(2) $\mathbf{B} = 2\,\mathbf{a}_r = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$

$$B_x = \mathbf{B}\cdot\mathbf{a}_x = 2\,\mathbf{a}_r\cdot\mathbf{a}_x = 2\sin\theta\cos\phi$$

$$= 2\cdot\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\frac{x}{\sqrt{x^2+y^2}} = \frac{2x}{\sqrt{x^2+y^2+z^2}}$$

$$B_y = \mathbf{B}\cdot\mathbf{a}_y = 2\,\mathbf{a}_r\cdot\mathbf{a}_y = 2\sin\theta\sin\phi$$

$$= 2\cdot\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\frac{y}{\sqrt{x^2+y^2}} = \frac{2y}{\sqrt{x^2+y^2+z^2}}$$

$$B_z = \mathbf{B}\cdot\mathbf{a}_z = 2\,\mathbf{a}_r\cdot\mathbf{a}_z = 2\cos\theta = 2\,\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \mathbf{B} = \frac{2}{\sqrt{x^2+y^2+z^2}}\,(x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z) \;//$$

$\boxed{2}$

(1)

$$\mathbb{E} = \int \frac{\rho_s\, ds' (\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r}-\mathbf{r}'|^3} \qquad \begin{cases} \mathbf{r} = z\, \mathbf{a}_z, & ds' = \rho' d\rho' d\phi' \\ \mathbf{r}' = \rho' \mathbf{a}_\rho & \\ |\mathbf{r}-\mathbf{r}'| = \sqrt{z^2+\rho'^2} & \end{cases}$$

$$= \int_0^{2\pi}\int_0^b \frac{\rho_s \rho' (z\mathbf{a}_z - \rho'\mathbf{a}_\rho)}{4\pi\epsilon_0 (z^2+\rho'^2)^{3/2}} d\rho' d\phi'$$

$$= \int_0^b \frac{\rho_s\, \rho' z\, \mathbf{a}_z}{2\epsilon_0 (z^2+\rho'^2)^{3/2}} d\rho' = \frac{\rho_s z \mathbf{a}_z}{2\epsilon_0} \int_0^b \frac{\rho'}{(z^2+\rho'^2)^{3/2}} d\rho'$$

(let $z^2+\rho'^2 = t$, $2\rho' d\rho' = dt$)

$$= \frac{\rho_s z \mathbf{a}_z}{2\epsilon_0} \int_{z^2}^{z^2+b^2} \frac{1}{2t^{3/2}} dt = \frac{\rho_s z\, \mathbf{a}_z}{2\epsilon_0} \left(\frac{1}{z} - \frac{1}{\sqrt{z^2+b^2}} \right)$$

$$= \frac{\rho_s \mathbf{a}_z}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2+b^2}} \right)$$

(2)

binomial expansion

$$\mathbb{E} = \frac{\rho_s \mathbf{a}_z}{2\epsilon_0} \left(1 - \frac{1}{\sqrt{1+\frac{b^2}{z^2}}} \right) \overset{z \gg b}{=} \frac{\rho_s \mathbf{a}_z}{2\epsilon_0} \left(1 - \left(1 - \frac{b^2}{2z^2}\right)\right)$$

$$= \frac{\rho_s \mathbf{a}_z}{2\epsilon_0} \frac{b^2}{2z^2} = \frac{\rho_s b^2}{4\epsilon_0 z^2} \mathbf{a}_z = \frac{\rho_s b^2 \pi}{4\pi\epsilon_0 z^2} \mathbf{a}_z = \frac{Q}{4\pi\epsilon_0 z^2} \mathbf{a}_z$$

3.

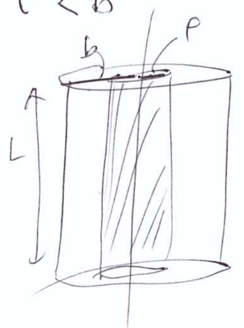
(1) Gauss' law

$$\oint \mathbf{D} \cdot d\mathbf{s} = Q$$

let $\mathbf{D} = D_\rho(\rho)\,\mathbf{a}_\rho$.

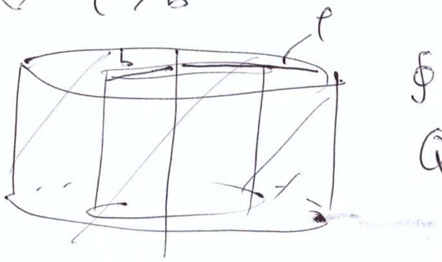
반지름 ρ인 원통면을 Gauss 면으로 하여 적용

① $\rho < b$



$$\left.\begin{aligned} \oint \mathbf{D} \cdot d\mathbf{s} &= D_\rho\, 2\pi\rho L \\ Q &= \pi\rho^2 L \rho_0 \end{aligned}\right] \rightarrow \quad D_\rho 2\pi\rho \cancel{L} = \pi\rho^2 \cancel{L} \rho_0, \quad D_\rho = \frac{\rho_0 \rho}{2}$$

② $\rho > b$



$$\left.\begin{aligned} \oint \mathbf{D} \cdot d\mathbf{s} &= D_\rho 2\pi\rho L \\ Q &= \pi b^2 L \rho_0 \end{aligned}\right] \rightarrow \quad D_\rho 2\pi\rho \cancel{L} = \pi b^2 \cancel{L} \rho_0, \quad D_\rho = \frac{b^2 \rho_0}{2\rho}$$

$$\therefore \mathbf{D} = \begin{cases} \dfrac{\rho_0 \rho}{2}\,\mathbf{a}_\rho & (\rho < b) \\[2ex] \dfrac{b^2 \rho_0}{2\rho}\,\mathbf{a}_\rho & (\rho > b) \end{cases}$$

$$\mathbf{E} = \begin{cases} \dfrac{\rho_0 \rho}{2\epsilon_0}\,\mathbf{a}_\rho & (\rho < b) \\[2ex] \dfrac{b^2 \rho_0}{2\epsilon_0 \rho}\,\mathbf{a}_\rho & (\rho > b) \end{cases}$$

(2)

① $\rho < b$

$$\nabla \cdot \mathbb{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\rho_0 \rho}{2}\right) = \frac{1}{\rho}\rho_0 \rho = \underline{\underline{\rho_0}}$$

② $\rho > b$

$$\nabla \cdot \mathbb{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{b^2\rho_0}{2\rho}\right) = 0$$

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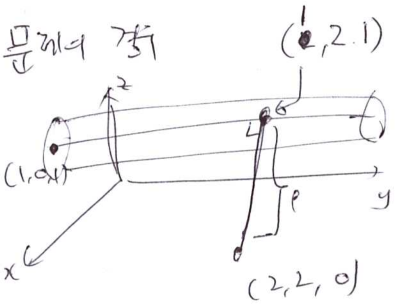
(3) (2,2,0)은 원통 외부. 따라서 (1)의 결과중 $\mathbb{E} = \frac{b^2\rho_0}{2\epsilon_0 \rho} a_\rho$ 활용

이 문제의 경우



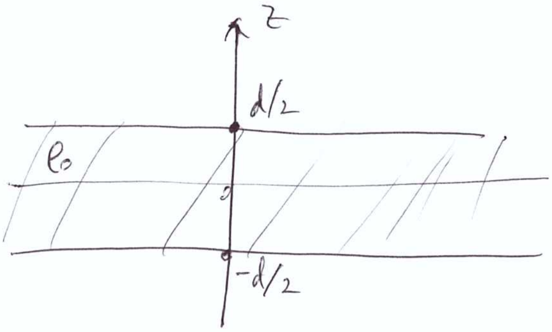
$$\rho \to \sqrt{(2-1)^2+(2-2)^2+(0-1)^2} = \sqrt{2}$$

$$a_\rho \to \frac{(2-1)a_x + (2-2)a_y + (0-1)a_z}{\sqrt{2}}$$

$$= \frac{a_x - a_z}{\sqrt{2}}$$

$$\therefore\ \mathbb{E} = \frac{b^2\rho_0}{2\epsilon_0\sqrt{2}}\frac{a_x - a_z}{\sqrt{2}} = \frac{b^2\rho_0(a_x - a_z)}{4\epsilon_0}$$

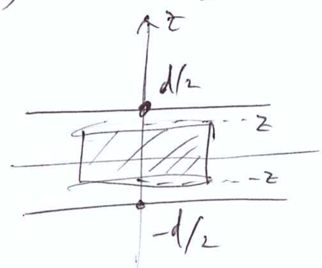
4



넓이 S, 높이 $-z \sim +z$ $(z>0)$ 인 원통면을 Gauss surface로 하여 Gauss 법칙 적용.

문제의 대칭성에서 $E = E_z(z)\, a_z$ $(z>0)$ 이며 $E_z(z) = -E_z(-z)$ 라고 가정 가능.

$\therefore$ ① $0 < z < \frac{d}{2}$ 일 때

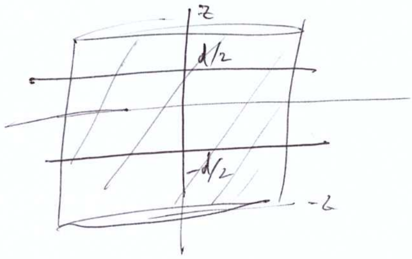


$$\oint D \cdot ds = Q$$

$$2D_z(z)S = \rho_0 \cdot S \cdot 2z$$

$$D_z = \rho_0 z$$

② $\frac{d}{2} < z$



$$\oint D \cdot ds = Q$$

$$2D_z(z)S = \rho_0 S \cdot d$$

$$D_z = \frac{\rho_0 d}{2}$$

$$D = \begin{cases} \frac{\rho_0 d}{2} a_z & (z > \frac{d}{2}) \\ \rho_0 z\, a_z & (0 < z < \frac{d}{2}) \\ \rho_0 z\, a_z & (-\frac{d}{2} < z < 0) \\ -\frac{\rho_0 d}{2} a_z & (z < -\frac{d}{2}) \end{cases}$$

$$\therefore E = \begin{cases} \frac{\rho_0 d}{2\epsilon_0} a_z & (z > \frac{d}{2}) \\ \frac{\rho_0 z}{\epsilon_0} a_z & (0 < z < \frac{d}{2}) \\ \frac{\rho_0 z\, a_z}{\epsilon_0} & (-\frac{d}{2} < z < 0) \\ -\frac{\rho_0 d}{2} a_z & (z < -\frac{d}{2}) \end{cases}$$

5

(1) Gauss law

$$\oint \mathbf{D} \cdot d\mathbf{s} = Q.$$

$$\mathbf{D} = D_r(r)\,\mathbf{a}_r$$

① $r=1$

$$\underbrace{\oint \mathbf{D} \cdot d\mathbf{s}}_{b} = \underbrace{Q}_{a}$$

$$D_r(1)\,4\pi \cdot 1^2 = 0$$

$$D_r(1) = 0$$

② $r=3$

$$\underbrace{\oint \mathbf{D} \cdot d\mathbf{s}}_{b} = \underbrace{Q}_{b}$$

$$D_r(3)\,4\pi \cdot 3^2 = \rho_{s1} \cdot 4\pi r_1^2 = 20 \cdot 4\pi \cdot 2^2 = 320\pi$$

$$D_r(3) = \frac{80}{9}$$

③ $r=5$

$$\underbrace{\oint \mathbf{D} \cdot d\mathbf{s}}_{b} = \underbrace{Q}_{a}$$

$$D_r(3)\,4\pi \cdot 5^2 = \rho_{s1}\,4\pi r_1^2 + \rho_{s2}\,4\pi r_2^2 = 320\pi - 4 \cdot 4\pi \cdot 16$$

$$D_r(3) = \frac{80-64}{25} = \frac{16}{25}$$

$$\therefore \mathbf{D} = \begin{cases} 0 & (r=1) \\ \frac{80}{9}\,\mathbf{a}_r & (r=3) \\ \frac{16}{25}\,\mathbf{a}_r & (r=5) \end{cases}$$

(2)

$$\underbrace{\oint \mathbf{D} \cdot d\mathbf{s}}_{0 \;@ r=9} = \underbrace{Q}_{0}$$

$$Q = \rho_{s1}\,4\pi r_1^2 + \rho_{s2}\,4\pi r_2^2 + \rho_{s3}\,4\pi r_3^2$$

$$= 320\pi - 4 \cdot 4\pi \cdot 16 + \rho_{s3}\,4\pi\,36 = 0$$

$$\therefore \rho_{s3} = \frac{4 \cdot 4\pi \cdot 16 - 320\pi}{4\pi \cdot 36} = \frac{64-80}{36}$$

$$= \frac{-16}{36} = -\frac{4}{9}$$