Chapter 10

Channel Coding

10.1 Problems

To accommodate the two common data rates specified in the problems below, we use the convention that the rate *bits per sond* is written as ”bps” and the rate *bytes per sond* as ”Bps”.

Problem 10.1 (Error correction by repetition) *Design a bit-level code that corrects up to 2 errors per bit repetition. What is the data increase factor for your code?*

(ans: To correct up to 2 errors per bit, repeat each bit five times, as

$$0 \to 00000 \qquad 1 \to 11111$$

Use majority count to correct a maximum of two errors per code word. For example,

$$\overbrace{\textit{received:}}^{\textit{corrected:}} \quad \overbrace{00101}^{00000}\ \overbrace{11010}^{11111}\ \overbrace{11101}^{11111}\ \overbrace{00001}^{00000}$$

$$DIF = \frac{5}{1} = 5$$

)

Problem 10.2 (Data-packet error correction) *A data packet contains the 5-digit number 24680. Each code word contains a single digit encoded using binary-coded-decimal (BCD) codes that form quadbit (4-bit) data plus an even-parity bit. Generate a data packet that includes a LRC code word. What is the data increase factor for your data packet?*

(ans:

digit					*ep*
2 :	0	0	1	0	1
4 :	0	1	0	0	1
6 :	0	1	1	0	0
8 :	1	0	0	0	1
0 :	0	0	0	0	0
LRC (ep) :	1	0	0	0	1

$$DIF = \frac{30}{20} = 1.5$$

)

Problem 10.3 (Data rate of texting) *You enter text on a smartphone keypad at a rate of ten characters per sond and an 8-bit ASCII code word is generated for each character. What is your data rate in bps and Bps?*

(ans:

$$10 \textit{ char/s} \times 8 \textit{ bits/char} = 80 \textit{ bps}$$

$$10 \textit{ char/s} \times 1 \textit{ byte/char} = 10 \textit{ Bps}$$

)

Problem 10.4 (Data rate of a video source) *A digital video is a sequence of digital frames, each forming* 480×640 *pixel image with each pixel encoded with 24-bit color. The video displays 15 frames/s. What is the data rate produced by this video source?*

(ans:

$$480 \times 640 \textit{ pixels/frame} \times 24 \textit{ bits/pixel} \times 15 \textit{ frames/s} = 110,592,000 \textit{ bps}$$

)

Problem 10.5 (Digital camera source) *My 10 Megapixel camera has three (R, G, and B) sensors in each pixel and sensor outputs 256 levels of intensity. The camera allows me to take multiple pictures at a rate of four per sond when I keep the shutter button depressed.*

1. *What is the maximum data rate from the camera into the camera memory card when I keep the shutter button depressed?*
2. *If I want to store 300 images, what size memory card should I purchase for my new camera? (Specify size as an integer number of gigabytes, e.g., X Gb memory.)*

(ans:

1.

$$10 \times 2^{20} \textit{ pixels/image} \times 3 \textit{sensors/pixel} \times 8 \textit{ bits/sensor} = 251,658,240 \textit{ bits/image}$$

$$251,658,240 \textit{ bits/image} \times 4 \textit{ images/s} = 1,006,632,960 \textit{ bps}$$

2.

$$251,658,240 \textit{ bits/image} \times 300 \textit{ images} = 75,497,472,000 \textit{ bits}$$

$$\frac{75,497,472,000 \textit{ bits}}{8 \textit{ bits/bytes} \times 2^{30} \textit{ bytes/gigabyte}} = 8.79 \rightarrow 9 \textit{ Gbyte}$$

)

Problem 10.6 (Data rate in the telephone system) *If the audio waveform sampling period in the smartphone is T_s = 0.1 ms and each sample is quantized to 256 levels, what is the data rate produced by your smartphone?*

(ans:

$$8 \text{ bits/sample} \times \frac{1 \text{ sample}}{10^{-4} \text{ s}} = 8 \times 10^4 \text{ bits/s} = 80 \text{ kbps}$$

)

Problem 10.7 (Time to transfer 1 TB disk data) *A external hard disk drive has a 1 terabyte (TB) capacity. How long does it take to transfer 1 TB through a USB 2.0 port having a 480 Mbps transmission rate? How long does it take if you upgrade to a USB 3.0 port having a 5 Gbps transmission rate?*

(ans: When computing mega- (2^{20} or 10^6) and giga- (2^{30} or 10^9), either powers of two or ten can be used, as long as they are used consistently.

USB 2.0 *Using powers of two gives*

$$\frac{8 \times 2^{30} \text{ bits}}{480 \times 2^{20} \text{ bits/s}} = 17.07 \text{ s}$$

Using powers of ten gives

$$\frac{8 \times 10^{9} \text{ bits}}{480 \times 10^{6} \text{ bits/s}} = 16.67 \text{ s}$$

USB 3.0 *Using powers of two gives*

$$\frac{8 \times 2^{30} \text{ bits}}{5 \times 2^{30} \text{ bits/s}} = 1.6 \text{ s}$$

Using powers of ten gives

$$\frac{8 \times 10^{9} \text{ bits}}{5 \times 10^{9} \text{ bits/s}} = 0.16 \text{ s}$$

)

Problem 10.8 (Video download to your smartphone) *A 10 MB video file requires a data rate D_V = 6 Mbps for continuous playback. The file is transmitted over a channel having transmission data rate equal to 1 Mbps. At what point in the data transfer should your smartphone begin to play the video to produce continuous playback and with minimal delay?*

(ans: The time to transmit n_f = 10 MB over a $\mathcal{C}$ = 1 Mbps channel is computed as

$$T_D = \frac{n_f}{\mathcal{C}} = \frac{8 \times 10 \times 2^{20} \text{ bits}}{2^{20} \text{ bits/s}} = 80 \text{ s}$$

The video duration is computed as

$$T_V = \frac{n_f}{D_V} = \frac{8 \times 10 \times 2^{20} \text{ bits}}{6 \times 2^{20} \text{ bits/s}} = 13.3 \text{ s}$$

The video can start at time after beginning of data transmission

$$T_{VS} = T_D - T_V = 80 - 13.3 = 66.7\ s$$

During this time, the amount of data transferred is

$$\mathcal{C}\ T_{VS} = 2^{20}\ bit/s \times 66.7\ s = 66.7\ Mb$$

During playback, the amount of data transferred is

$$\mathcal{C}\ T_V = 2^{20}\ bit/s \times 13.3\ s = 13.3\ Mb$$

)

Problem 10.9 (Data transmission between your smartphone and antenna tower) *The more-powerful antenna tower transmits data* down *to your smartphone at a rate equal to 300 Mbps, while your less-powerful smartphone transmits data back* up *to the tower at 75 Mbps. Assume the bandwidths and noise levels are the same in each transmission direction. How are the signal energies at the tower and cellphone related? Evaluate for $\mathcal{B}$ = 1 MHz and 10 MHz. (Hint: Use the channel capacity equation and ignore the +1 in the $\log_2$ term.)*

(ans: From antenna to smartphone,

$$\mathcal{C}_{a2s} = \mathcal{B}\log_2(SNR_{a2s}) = \mathcal{B}\log_2\left(\frac{\mathcal{E}_a}{\sigma_N^2}\right)$$

Solving for $\mathcal{E}_a$ gives

$$\mathcal{E}_a = 2^{\frac{\mathcal{C}_{a2s}}{\mathcal{B}}}\sigma_N^2$$

From smartphone to antenna,

$$\mathcal{C}_{s2a} = \mathcal{B}\log_2(SNR_{s2a})\mathcal{B}\log_2\left(\frac{\mathcal{E}_s}{\sigma_N^2}\right)$$

Solving for $\mathcal{E}_s$ gives

$$\mathcal{E}_s = 2^{\frac{\mathcal{C}_{s2a}}{\mathcal{B}}}\sigma_N^2$$

After cancelling σ_N^2, the ratio $\mathcal{E}_a/\mathcal{E}_s$ gives

$$\frac{\mathcal{E}_a}{\mathcal{E}_s} = \frac{2^{\frac{\mathcal{C}_{a2s}}{\mathcal{B}}}}{2^{\frac{\mathcal{C}_{s2a}}{\mathcal{B}}}}$$

When $\mathcal{B}$ = 1 MHz,

$$\frac{\mathcal{C}_{a2s}}{\mathcal{B}} = 300 \quad \textit{and} \quad \frac{\mathcal{C}_{a2s}}{\mathcal{B}} = 75$$

which gives

$$\frac{\mathcal{E}_a}{\mathcal{E}_s} = \frac{2^{300}}{2^{75}} = 2^{225}$$

When $\mathcal{B}$ = 10 MHz,

$$\frac{\mathcal{C}_{a2s}}{\mathcal{B}} = 30 \quad \textit{and} \quad \frac{\mathcal{C}_{a2s}}{\mathcal{B}} = 7.5$$

which gives

$$\frac{\mathcal{E}_a}{\mathcal{E}_s} = \frac{2^{30}}{2^{7.5}} = 2^{22.5}$$

)

Problem 10.10 (Minimum signal-to-noise ratio) *The data rate produced by your smartphone is $\mathcal{D} = 2 \times 10^5 bps$. Your smartphone transmits data over the frequency band from 100 kHz to 200 kHz. What is the minimum SNR required to transmit your data in real time?*

(ans: $\mathcal{B} = 100$ kHz ($= 10^5$ Hz). For real-time operation $\mathcal{C} = \mathcal{D}$, which gives

$$2 \times 10^5 \; bps = 10^5 \log_2(1 + SNR_{min})$$

Solving for SNR gives

$$1 + SNR_{min} = 2^2 = 4 \quad \rightarrow \quad SNR_{min} = 3$$

)

Problem 10.11 (Channel capacity reduction with range) *The signal energy detected by your smartphone at range r_o =0.5 km equals $\mathcal{E}_s(r_o) = 10^6\sigma_N^2$. The bandwidth $\mathcal{B}$ =100 kHz and is constant with range.*

1. *What is the value of $\mathcal{C}(r_o)$?*
2. *As the smartphone distance from the antenna increases, at what range does $\mathcal{E}_s(\alpha r_o)$ become less than $100\sigma_N^2$?*
3. *What is the channel capacity at that range?*

(ans:

1.

$$\mathcal{C}(r_o) = 10^5 \log_2\left(\frac{10^6\sigma_N^2}{\sigma_N^2} + 1\right) = 10^5 \overbrace{\log_2(10^6)}^{=19.93} = 2 \times 10^6 \; bps = 2 \; Mbps$$

2.

$$\mathcal{E}(\alpha_x r_o) = \frac{\mathcal{E}(r_o)}{\alpha_x^2} = 100\sigma_N^2$$

Solving for α_x gives

$$\alpha_x = \sqrt{\frac{\mathcal{E}(r_o)}{100\sigma_N^2}} = \sqrt{\frac{10^6\sigma_N^2}{100\sigma_N^2}} = \sqrt{10^4} = 100$$

The range at which this occurs is given by

$$r_x = \alpha_x r_o = 100 \times 0.5 \; km = 50 \; km$$

3.

$$\mathcal{C}(50\ km) = 10^5 \log_2\left(\frac{10^2\sigma_N^2}{\sigma_N^2}\right) = 10^5 \overbrace{\log_2(10^2)}^{=6.64} = 6.64 \times 10^5\ bps = 0.64\ Mbps$$

)

Problem 10.12 (Channel capacity of space probe) *A space probe transmitter operates with $\mathcal{B} = 10^5$ Hz and $\mathcal{E}_s(r_o)/\sigma_N^2 = 10^6$ at r_o = 1 km. The probe transmits images that each contain 1 Mb of data. If an image takes approximately 1 hour to transmit, what is the range of the probe from the antenna on Earth?*

(ans: The channel capacity at the probe range $r = \alpha r_o$ is computed as

$$\mathcal{C}_{min}(\alpha r_o) = \frac{2^{20}\ bits}{1\ hr \times 3600\ s/hr} = 291\ bps$$

The capacity at $r = \alpha r_o$ equals

$$\mathcal{C}(\alpha r_o) = 10^5 \log_2\left(\frac{\mathcal{E}_s(r_o)}{\alpha^2\sigma_N^2} + 1\right) = 10^5 \log_2\left(\frac{10^6}{\alpha^2} + 1\right) = 291\ bps$$

Solving for α simplifies by using powers of two to give

$$\log_2\left(\frac{2^{20}}{\alpha^2} + 1\right) = 0.00291$$

or

$$\frac{2^{20}}{\alpha^2} + 1 = 2^{0.00291} = 1.002019 \quad \rightarrow \quad \alpha = \sqrt{\frac{2^{20}}{0.002019}} = 22,789$$

The probe range is

$$\alpha r_o = 22,789\ km$$

)

10.2 Excel Projects

Project 10.1 (Bit-level error correction) *In work sheet* Repeated bits, *form a truth table that has input section formed by logic values DA_r, DB_r and DC_r, and write an Excel formula that determines D_r the correct value under the single error assumption.*

(ans:

	A	B	C	D	E	F
1	row	DA_r	DB_r	DC_r		D_r
2	0	0	0	0		0
3	1	0	0	1		0
4	2	0	1	0		0
5	3	0	1	1		1
6	4	1	0	0		0
7	5	1	0	1		1
8	6	1	1	0		1
9	7	1	1	1		1

	A	B	C	D	E	F
1	row	DA_r	DB_r	DC_r		D_r
2	0	=MOD(FLOOR(A2/4,1),2)	=MOD(FLOOR(A2/2,1),2)	=MOD(A2,2)		=IF(SUM(B2:D2)>1, 1, 0)
3	=1+A2	=MOD(FLOOR(A3/4,1),2)	=MOD(FLOOR(A3/2,1),2)	=MOD(A3,2)		=IF(SUM(B3:D3)>1, 1, 0)
4	=1+A3	=MOD(FLOOR(A4/4,1),2)	=MOD(FLOOR(A4/2,1),2)	=MOD(A4,2)		=IF(SUM(B4:D4)>1, 1, 0)
5	=1+A4	=MOD(FLOOR(A5/4,1),2)	=MOD(FLOOR(A5/2,1),2)	=MOD(A5,2)		=IF(SUM(B5:D5)>1, 1, 0)
6	=1+A5	=MOD(FLOOR(A6/4,1),2)	=MOD(FLOOR(A6/2,1),2)	=MOD(A6,2)		=IF(SUM(B6:D6)>1, 1, 0)
7	=1+A6	=MOD(FLOOR(A7/4,1),2)	=MOD(FLOOR(A7/2,1),2)	=MOD(A7,2)		=IF(SUM(B7:D7)>1, 1, 0)
8	=1+A7	=MOD(FLOOR(A8/4,1),2)	=MOD(FLOOR(A8/2,1),2)	=MOD(A8,2)		=IF(SUM(B8:D8)>1, 1, 0)
9	=1+A8	=MOD(FLOOR(A9/4,1),2)	=MOD(FLOOR(A9/2,1),2)	=MOD(A9,2)		=IF(SUM(B9:D9)>1, 1, 0)

)

Project 10.2 (Data-packet error correction) *Following Example 13.53 generate a data packet with random binary data having 5 rows and 5 columns. Introduce a single-bit error to show that the erroneous bit can be located. Introduce two errors to illustrate the inability to correct them using parity-bits and LRC.*

(ans: First, generate a 5×5 block of random data:

	A	B	C	D	E
1	0	1	1	1	1
2	1	1	1	1	0
3	0	0	0	1	1
4	0	0	1	1	0
5	1	1	1	1	0

	A	B	C	D	E
1	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)
2	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)
3	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)
4	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)
5	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)	=IF(RAND()<0.5,0,1)

The random bits from above are copied and paste/special/values into the transmitted data *worksheet that computes the parity bits of the columns (PC) and the rows (PR) to form the transmitted data packet.*

The even-parity formula (shown here) is

```
=IF(MOD(SUM(B2:F2)+1,2),0,1)
```

and the odd-parity formula reverse the bits as

```
=IF(MOD(SUM(B2:F2)+1,2),1,0)
```

	A	B	C	D	E	F	G
1		Data					PR
2		0	1	0	1	1	1
3		0	1	1	1	0	1
4		1	1	1	1	1	1
5		1	1	1	1	0	0
6		1	1	0	0	0	0
7	PC	1	1	1	0	0	1

	A	B	C	D	E	F	G
1		Data					PR
2		0	1	0	1	1	=IF(MOD(SUM(B2:F2)+1,2),0,1)
3		0	1	1	1	0	=IF(MOD(SUM(B3:F3)+1,2),0,1)
4		1	1	1	1	1	=IF(MOD(SUM(B4:F4)+1,2),0,1)
5		1	1	1	1	0	=IF(MOD(SUM(B5:F5)+1,2),0,1)
6		1	1	0	0	0	=IF(MOD(SUM(B6:F6)+1,2),0,1)
7	PC	=IF(MOD(SUM(B2:B6)+1,2),0,1)	=IF(MOD(SUM(C2:C6)+1,2),0,1)	=IF(MOD(SUM(D2:D6)+1,2),0,1)	=IF(MOD(SUM(E2:E6)+1,2),0,1)	=IF(MOD(SUM(F2:F6)+1,2),0,1)	=IF(MOD(SUM(B7:F7)+1,2),0,1)

The transmitted data packet is copied and paste/special/values into the received data *worksheet that computes the parity bits of the columns (PC') and the rows (PR') (of the data only to form receiver parity bits. The worksheet with received data containing no error follows. The indicator formulas are appear in row 9 and column I.*

	A	B	C	D	E	F	G	H
1		Data					PC	PC'
2		0	1	0	1	1	1	1
3		0	1	1	1	0	1	1
4		1	1	1	1	1	1	1
5		1	1	1	1	0	0	0
6		1	1	0	0	0	0	0
7	PR	1	1	1	0	0	1	
8	PR'	1	1	1	0	0	1	

	A	B	C	D	E	F
1		Data				
2		0	1	0	1	1
3		0	1	1	1	0
4		1	1	1	1	1
5		1	1	1	1	0
6		1	1	0	0	0
7	PR	=IF(MOD(SUM(B2:B6)+1,2),0,1)	=IF(MOD(SUM(C2:C6)+1,2),0,1)	=IF(MOD(SUM(D2:D6)+1,2),0,1)	=IF(MOD(SUM(E2:E6)+1,2),0,1)	=IF(MOD(SUM(F2:F6)+1,2),0,1)
8	PR'	=IF(MOD(SUM(B2:B6)+1,2),0,1)	=IF(MOD(SUM(C2:C6)+1,2),0,1)	=IF(MOD(SUM(D2:D6)+1,2),0,1)	=IF(MOD(SUM(E2:E6)+1,2),0,1)	=IF(MOD(SUM(F2:F6)+1,2),0,1)
9		=IF(B7=B8," ","^")	=IF(C7=C8," ","^")	=IF(D7=D8," ","^")	=IF(E7=E8," ","^")	

	G	H	I
1	PC	PC'	
2	=IF(MOD(SUM(B2:F2)+1,2),0,1)	=IF(MOD(SUM(B2:F2)+1,2),0,1)	=IF(G2=H2," ","<")
3	=IF(MOD(SUM(B3:F3)+1,2),0,1)	=IF(MOD(SUM(B3:F3)+1,2),0,1)	=IF(G3=H3," ","<")
4	=IF(MOD(SUM(B4:F4)+1,2),0,1)	=IF(MOD(SUM(B4:F4)+1,2),0,1)	=IF(G4=H4," ","<")
5	=IF(MOD(SUM(B5:F5)+1,2),0,1)	=IF(MOD(SUM(B5:F5)+1,2),0,1)	=IF(G5=H5," ","<")
6	=IF(MOD(SUM(B6:F6)+1,2),0,1)	=IF(MOD(SUM(B6:F6)+1,2),0,1)	
7	=IF(MOD(SUM(B7:F7)+1,2),0,1)		
8	=IF(MOD(SUM(G2:G6)+1,2),0,1)		
9	=IF(G7=G8," ","^")		

The received data packet containing one transmission error follows. The difference between the received and computed parity bits indicate the erroneous bit in `D4`*.*

	A	B	C	D	E	F	G	H	I
1		Data					PC	PC'	
2		0	1	0	1	1	1	1	
3		0	1	1	1	0	1	1	
4		1	1	0	1	1	1	0	<
5		1	1	1	1	0	0	0	
6		1	1	0	0	0	0	0	
7	PR	1	1	1	0	0	1		
8	PR'	1	1	0	0	0	1		
9				^					

The received data packet containing two transmission errors follows. The difference between the received and computed parity bits indicate the erroneous bit in `D4` *and in* `E2`*. However the errors could have also occurred in* `D2` *and* `E4`*. Hence, while error detection still occurs, error correction fails.*

	A	B	C	D	E	F	G	H	I
1		Data					PC	PC'	
2		0	1	0	0	1	1	0	<
3		0	1	1	1	0	1	1	
4		1	1	0	1	1	1	0	<
5		1	1	1	1	0	0	0	
6		1	1	0	0	0	0	0	
7	PR	1	1	1	0	0	1		
8	PR'	1	1	0	1	0	1		
9				^	^				

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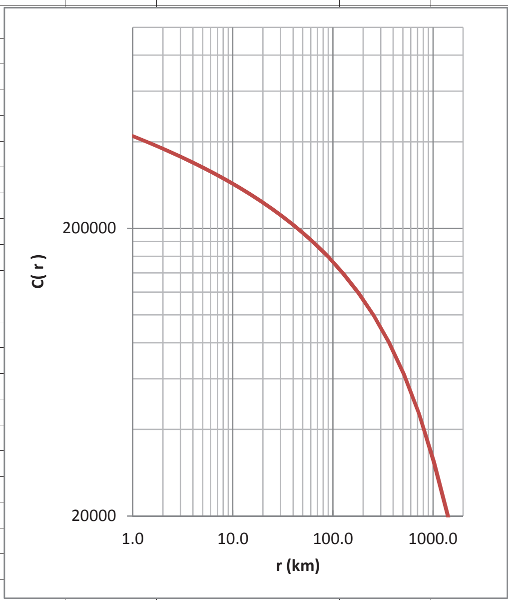
Project 10.3 (Channel capacity) *Using Example 13.54 as a guide, compute the channel capacity as a function of range when $\mathcal{E}_s/\sigma_N^2 = 2 \times 10^6$ at range $r_o = 1km$ and $B = 20kHz$.*

1. *At what range does $\mathcal{E}_s < 10\sigma_N^2$?*

2. *What is the channel capacity at that range?*

(ans:

	A	B	C	D	E	F	G	H	I
1	E_s/σ_N^2=	2.00E+06		r_0 (km)=	1				
2	B=	2.00E+04							
3	**r (km)**	**E_s(r)/σ_N^2**	**C(r)**						
4	1.0	2000000.0	4.19E+05						
5	1.4	1000000.0	3.99E+05						
6	2.0	500000.0	3.79E+05						
7	2.8	250000.0	3.59E+05						
8	4.0	125000.0	3.39E+05						
9	5.7	62500.0	3.19E+05						
10	8.0	31250.0	2.99E+05						
11	11.3	15625.0	2.79E+05						
12	16.0	7812.5	2.59E+05						
13	22.6	3906.3	2.39E+05						
14	32.0	1953.1	2.19E+05						
15	45.3	976.6	1.99E+05						
16	64.0	488.3	1.79E+05						
17	90.5	244.1	1.59E+05						
18	128.0	122.1	1.39E+05						
19	181.0	61.0	1.19E+05						
20	256.0	30.5	9.96E+04						
21	362.0	15.3	8.05E+04						
22	512.0	7.6	6.22E+04						
23	724.1	3.8	4.53E+04						
24	1024.0	1.9	3.08E+04						
25	1448.2	1.0	1.93E+04						
26									
27									
28									
29									
30									



E_s(r)/σ_N^2<10 for r > 512 km and C(512) = 62,200 bps

	A	B	C
1	E_s/σ_N^2=	2000000	
2	B=	20000	
3	**r (km)**	**E_s(r)/σ_N^2**	**C(r)**
4	=E1	=$B$1/(A4/$E$1)^2	=$B$2*LOG(1+B4,2)
5	=SQRT(2)*A4	=$B$1/(A5/$E$1)^2	=$B$2*LOG(1+B5,2)
6	=SQRT(2)*A5	=$B$1/(A6/$E$1)^2	=$B$2*LOG(1+B6,2)
7	=SQRT(2)*A6	=$B$1/(A7/$E$1)^2	=$B$2*LOG(1+B7,2)
8	=SQRT(2)*A7	=$B$1/(A8/$E$1)^2	=$B$2*LOG(1+B8,2)
9	=SQRT(2)*A8	=$B$1/(A9/$E$1)^2	=$B$2*LOG(1+B9,2)
10	=SQRT(2)*A9	=$B$1/(A10/$E$1)^2	=$B$2*LOG(1+B10,2)

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