인하대학교
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Electromagnetics 1 (ICE2003)
-- Ch. 6. Capacitance --

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Spring, 2021

Chapter Outline

- Capacitance definition
- Capacitance calculation
  - Parallel plate, Coaxial transmission line, Spherical capacitor
  - Isolated sphere with a dielectric coating, Parallel plate with two-layer dielectric
  - Two-wire line, one-wire line and a plane
- Energy stored in a capacitor
- Poisson / Laplace equation
  - Laplacian operator
  - Parallel plate, Coaxial transmission line, Angled plate, Concentric sphere, Cone above a conducting plane
  - Product solution

Capacitance Definition

A simple capacitor consists of two oppositely charged conductors surrounded by a uniform dielectric.

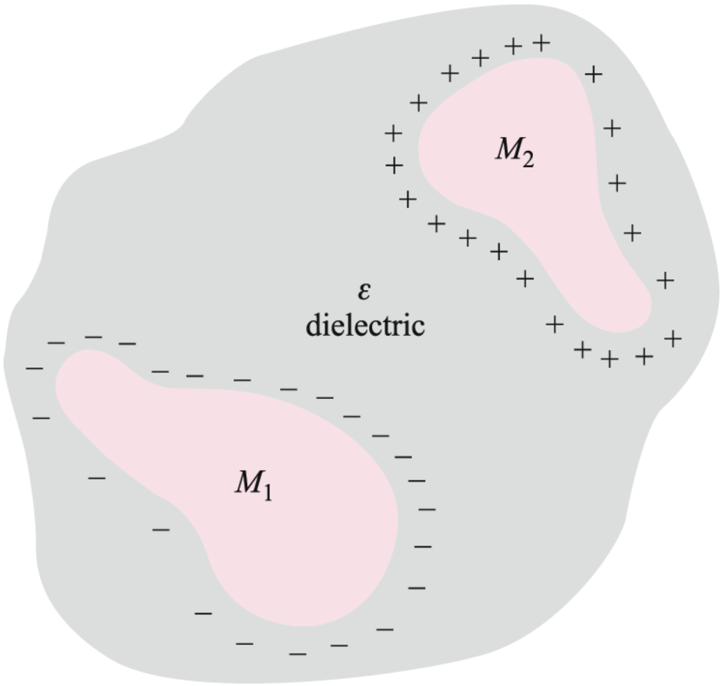
An increase in Q by some factor results in an increase in E (and in D) by the same factor.

$$\text{where} \quad Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

Consequently, the potential difference between conductors:

$$V_0 = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

will also increase by the same factor -- so the ratio of Q to V_0 is a constant. We define the *capacitance* of the structure as the ratio of the stored charge to the applied voltage, or

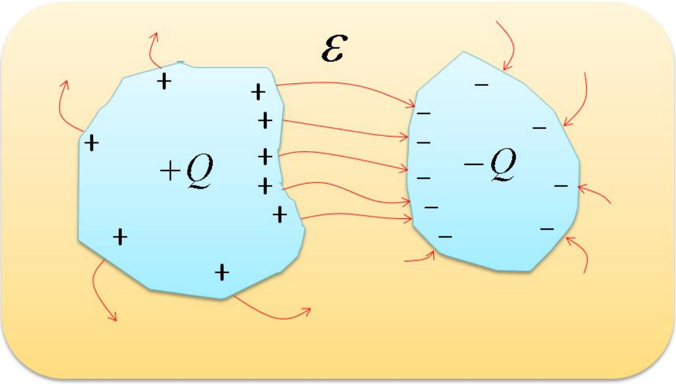


$$\boxed{C = \frac{Q}{V_0}}$$

Units are Coul/V or *Farads*

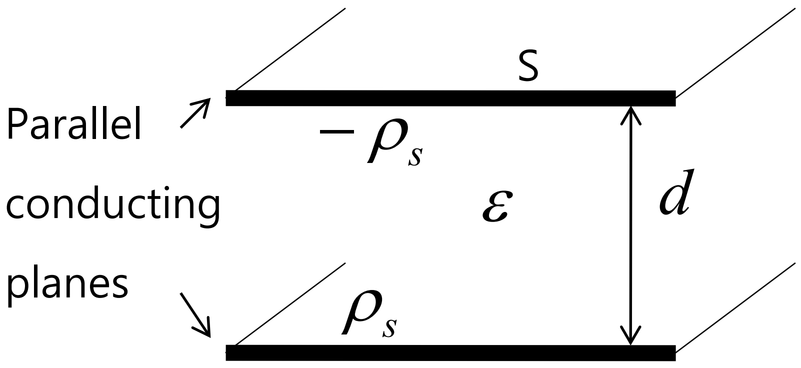
- Determined by Capacitor geometry and material
- Independent from V and Q

Calculating Capacitance



1. Set Coordinate system
2. Assume +Q, -Q to two conductors
3. Calculate E
4. Calculate V by integrating E
5. C=Q/V

Parallel Plate Capacitor



$$Q = \rho_s S$$

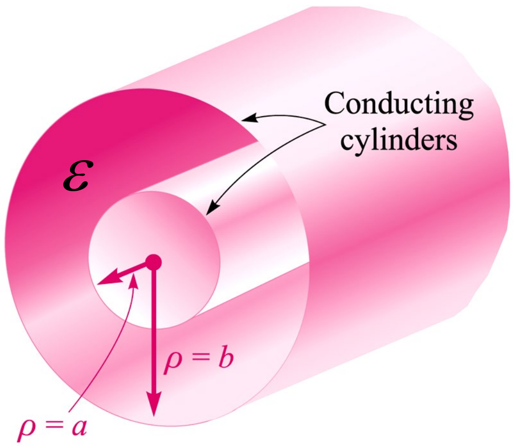
$$\mathbf{D} = \rho_s \mathbf{a_z}$$

$$\mathbf{E} = \frac{\rho_s}{\varepsilon}\mathbf{a_z}$$

$$V = -\int_{init}^{final} \mathbf{E}\cdot d\mathbf{L} = -\int_{d}^{0} \mathbf{E}\cdot \mathbf{a_z}\,dz = -\int_{d}^{0} \frac{\rho_s}{\varepsilon}\,dz = \frac{\rho_s d}{\varepsilon}$$

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon S}{d}}$$

Coaxial Transmission Line



$$Q = 2\pi a \rho_s L$$

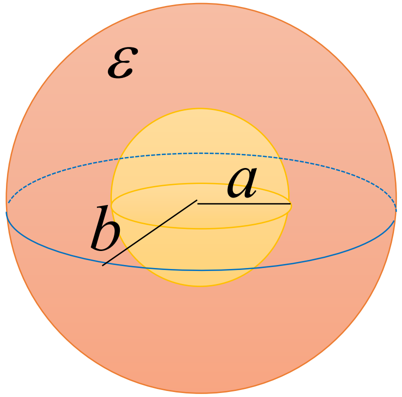
$$\mathbf{D} = \frac{a\rho_s}{\rho}\mathbf{a}_\rho$$

$$\mathbf{E} = \frac{a\rho_s}{\varepsilon\rho}\mathbf{a}_\rho$$

$$V = -\int_{init}^{final} \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \mathbf{E}\cdot \mathbf{a}_\rho d\rho = -\int_b^a \frac{a\rho_s}{\varepsilon\rho} d\rho = \frac{a\rho_s}{\varepsilon}\ln\frac{b}{a}$$

$$\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}}$$

Spherical Capacitor



$$Q = 4\pi a^2 \rho_s$$

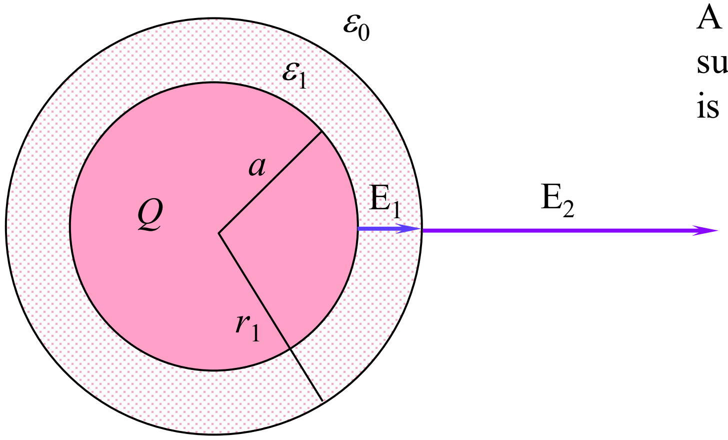
$$\mathbf{D} = \frac{a^2 \rho_s}{r^2}\mathbf{a_r}$$

$$\mathbf{E} = \frac{a^2 \rho_s}{\varepsilon r^2}\mathbf{a_r}$$

$$V = -\int_{init}^{final} \mathbf{E} \cdot d\mathbf{L} = -\int_b^a \mathbf{E} \cdot \mathbf{a_r} dr = -\int_b^a \frac{a^2 \rho_s}{\varepsilon r^2} dr = \frac{a^2 \rho_s}{\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}}$$

Isolated Sphere with a Dielectric Coating



A conducting sphere of radius a carries charge Q.
A dielectric layer of thickness r_1 - a and of permittivity ε_1 surrounds the conductor. Electric field in the two regions is found from Gauss' Law to be:

$$E_r = \frac{Q}{4\pi\epsilon_1 r^2} \qquad (a < r < r_1)$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r_1 < r)$$

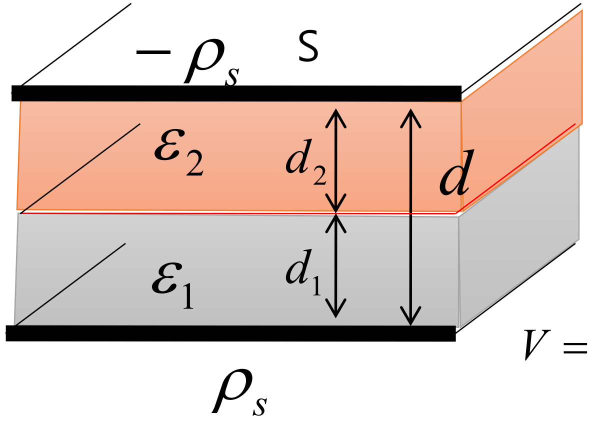
The potential at the sphere surface is (with zero reference at infinity):

$$V_a - V_\infty = -\int_{r_1}^{a} \frac{Q\,dr}{4\pi\epsilon_1 r^2} - \int_{\infty}^{r_1} \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi}\left[\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}\right] = V_0$$

and the capacitance is:

$$\boxed{C = \frac{4\pi}{\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}}}$$

Capacitor with a Two-Layer Dielectric



$$Q = S\rho_s$$

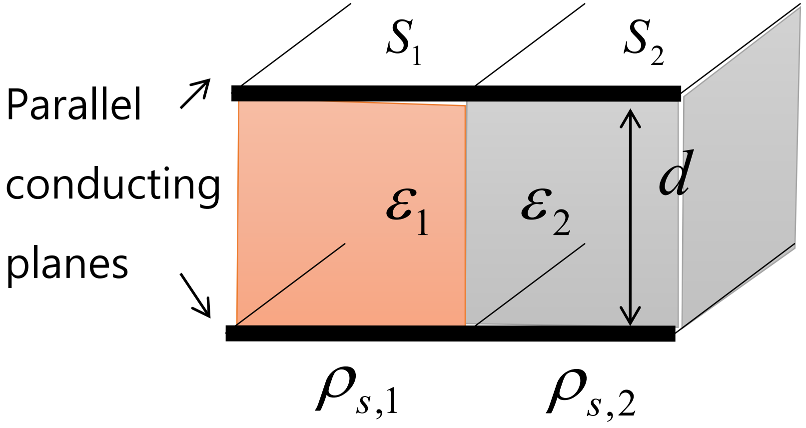
$$\mathbf{D} = \rho_s \mathbf{a_z}$$

$$\mathbf{E}_1 = \frac{\rho_s}{\varepsilon_1}\mathbf{a_z} \qquad \mathbf{E}_2 = \frac{\rho_s}{\varepsilon_2}\mathbf{a_z}$$

$$V = -\int_{init}^{final} \mathbf{E}\cdot d\mathbf{L} = -\int_{d_1+d_2}^{d_1} \mathbf{E}_2 \cdot \mathbf{a_z} dz - \int_{d_1}^{0} \mathbf{E}_1 \cdot \mathbf{a_z} dz = \frac{\rho_s d_1}{\varepsilon_1} + \frac{\rho_s d_2}{\varepsilon_2}$$

$$\boxed{C = \frac{Q}{V} = \frac{1}{\dfrac{d_1}{\varepsilon_1 S} + \dfrac{d_2}{\varepsilon_2 S}}}$$

The other case



$$Q = S_1\rho_{s,1} + S_2\rho_{s,2}$$

$$\mathbf{D}_1 = \rho_{s,1}\mathbf{a_z} \qquad \mathbf{D}_2 = \rho_{s,2}\mathbf{a_z}$$

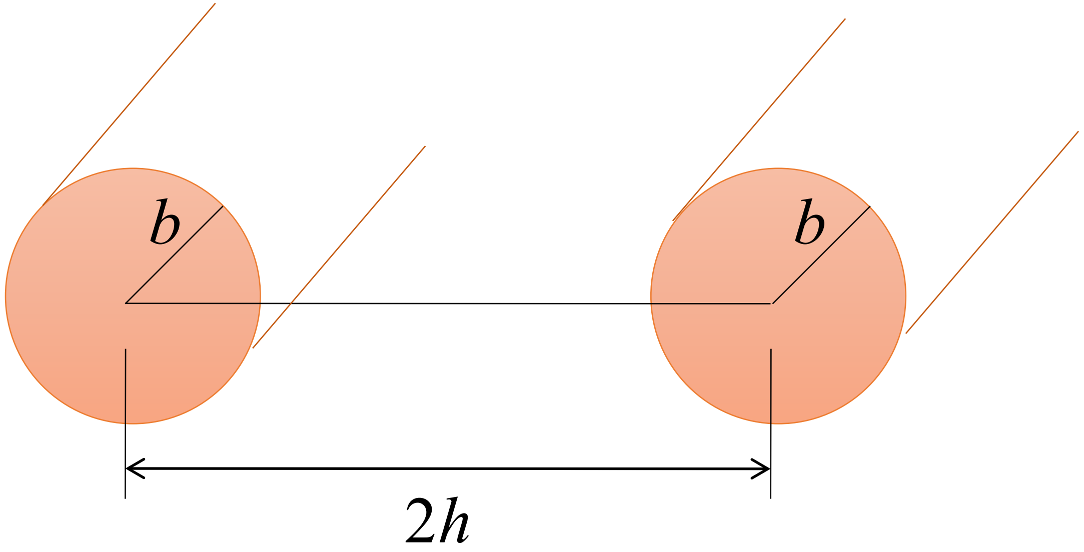
$$\mathbf{E}_1 = \frac{\rho_{s,1}}{\varepsilon_1}\mathbf{a_z} \qquad \mathbf{E}_2 = \frac{\rho_{s,2}}{\varepsilon_2}\mathbf{a_z}$$

$$V = -\int_d^0 \mathbf{E}_1 \cdot \mathbf{a_z} dz = \frac{\rho_{s,1} d}{\varepsilon_1} = -\int_{d_1}^0 \mathbf{E}_2 \cdot \mathbf{a_z} dz = \frac{\rho_{s,2} d}{\varepsilon_2}$$

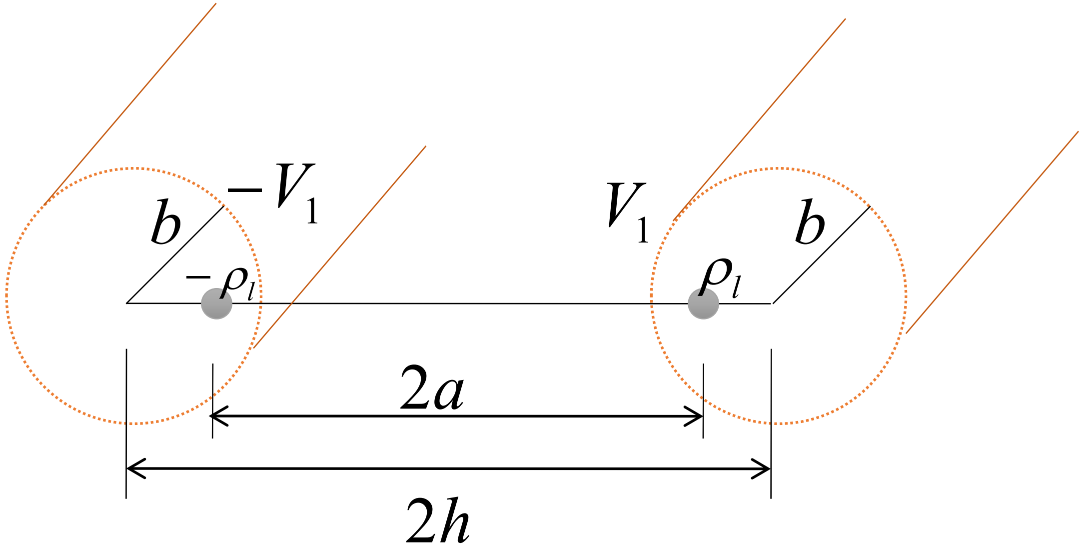
$$\therefore \frac{\rho_{s,1}}{\varepsilon_1} = \frac{\rho_{s,2}}{\varepsilon_2}$$

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon_1 S_1}{d} + \frac{\varepsilon_2 S_2}{d}}$$

Capacitance of two-wire line



Capacitance of two-wire line



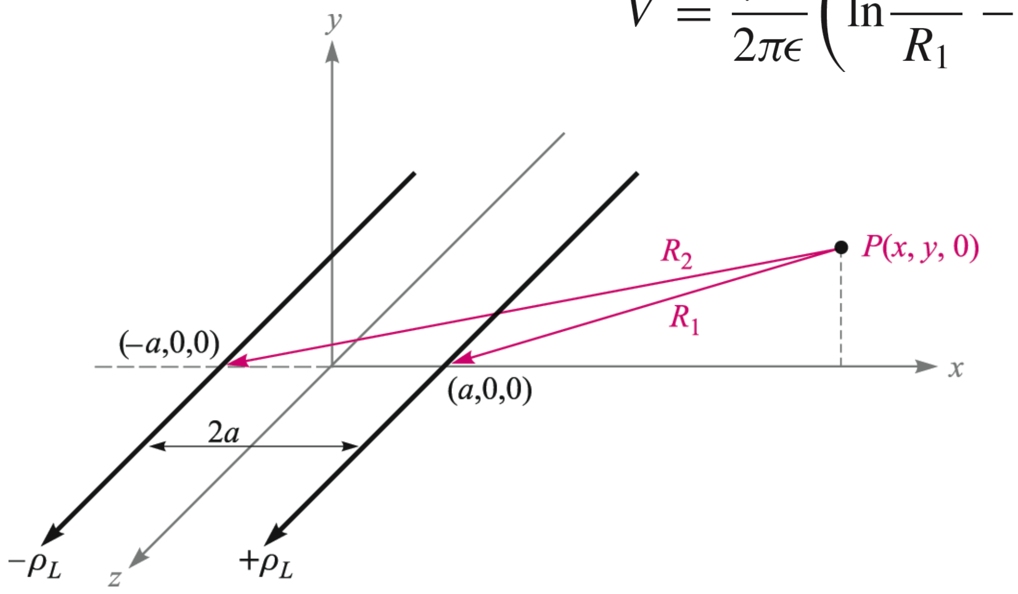
Capacitance of two-wire line

We begin with the potential field of a single line charge on the z axis, with a zero reference at $\rho = R_0$

$$V = \frac{\rho_L}{2\pi\epsilon} \ln\frac{R_0}{R}$$

We can use this result to write the potential at point P from two line charges of opposite sign, postioned as shown:

$$V = \frac{\rho_L}{2\pi\epsilon}\left(\ln\frac{R_{10}}{R_1} - \ln\frac{R_{20}}{R_2}\right) = \frac{\rho_L}{2\pi\epsilon}\ln\frac{R_{10}R_2}{R_{20}R_1}$$

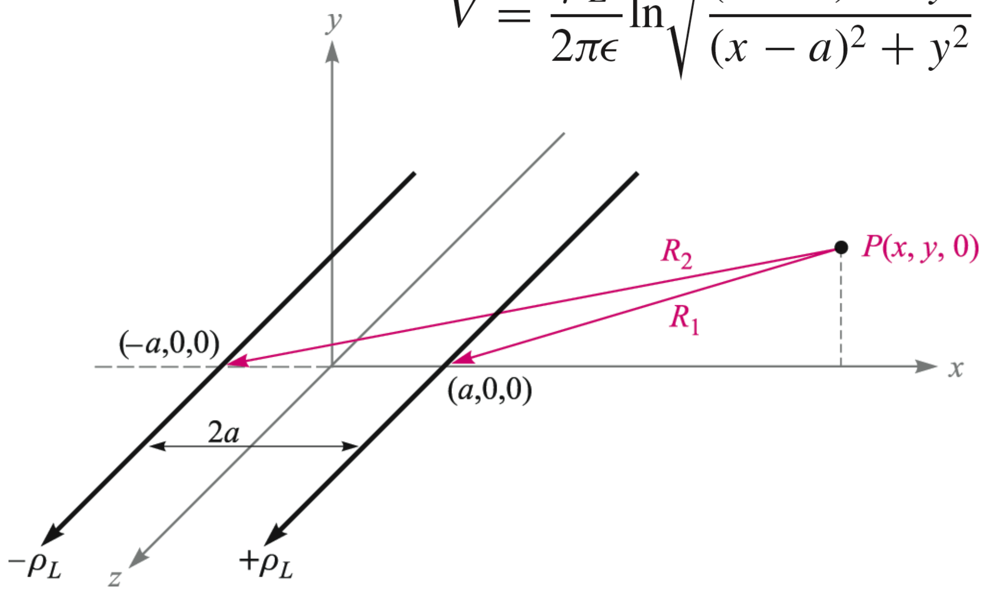


Capacitance of two-wire line

We now have: $$V = \frac{\rho_L}{2\pi\epsilon}\left(\ln\frac{R_{10}}{R_1} - \ln\frac{R_{20}}{R_2}\right) = \frac{\rho_L}{2\pi\epsilon}\ln\frac{R_{10}R_2}{R_{20}R_1}$$

We choose $R_{10} = R_{20}$, thus placing the zero reference at equal distances from each line. This surface is the $x = 0$ plane. Expressing R_1 and R_2 in terms of x and y,

$$V = \frac{\rho_L}{2\pi\epsilon}\ln\sqrt{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}} = \frac{\rho_L}{4\pi\epsilon}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$



Capacitance of two-wire line

We now have: $$V = \frac{\rho_L}{2\pi\epsilon} \ln \sqrt{\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}} = \frac{\rho_L}{4\pi\epsilon} \ln \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

Choose an equipotential surface on which $V = V_1$, and define the dimensionless parameter: $K_1 = e^{4\pi\epsilon V_1/\rho_L}$

from which we identify: $$K_1 = \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

This is the equation of the equipotential surface on which the voltage is V_1

To better identify the surface, expand the squares, and collect terms:

$$x^2 - 2ax\frac{K_1 + 1}{K_1 - 1} + y^2 + a^2 = 0 \Rightarrow \boxed{\left(x - a\frac{K_1 + 1}{K_1 - 1}\right)^2 + y^2 = \left(\frac{2a\sqrt{K_1}}{K_1 - 1}\right)^2}$$

This is the equation of a circle (actually a cylinder), displaced along the x axis, and of radius b, where

$$\boxed{b = \frac{2a\sqrt{K_1}}{K_1 - 1}}$$

It is all independent of z, as you might expect!

Capacitance of two-wire line

The equation of an equipotential surface is:

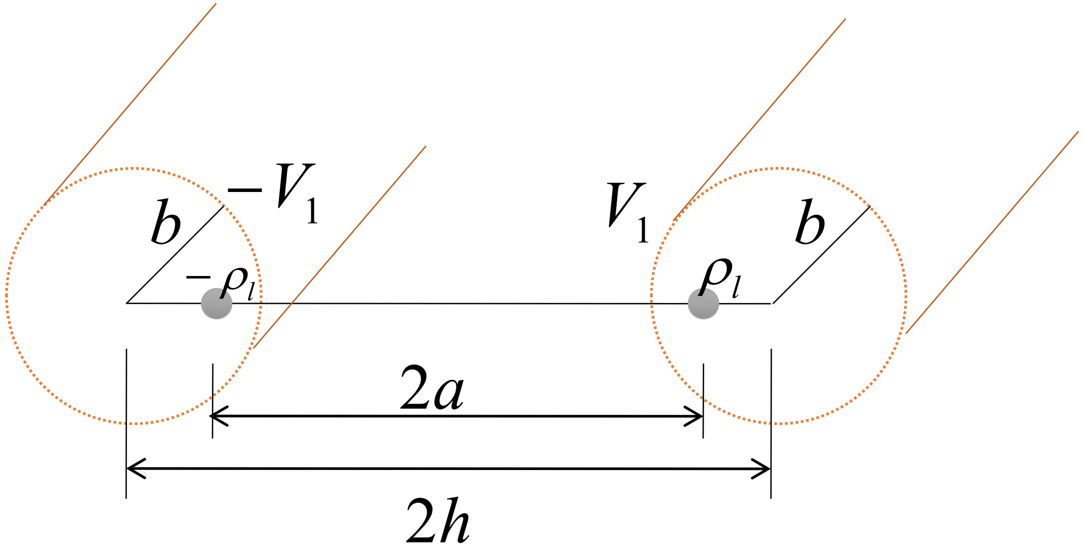
$$\left(x - a\frac{K_1 + 1}{K_1 - 1}\right)^2 + y^2 = \left(\frac{2a\sqrt{K_1}}{K_1 - 1}\right)^2$$

This is the equation of a circle (actually a cylinder), displaced along the x axis by distance h, and having radius b, where

$$\boxed{h = a\frac{K_1 + 1}{K_1 - 1}} \quad \text{and} \quad \boxed{b = \frac{2a\sqrt{K_1}}{K_1 - 1}}$$

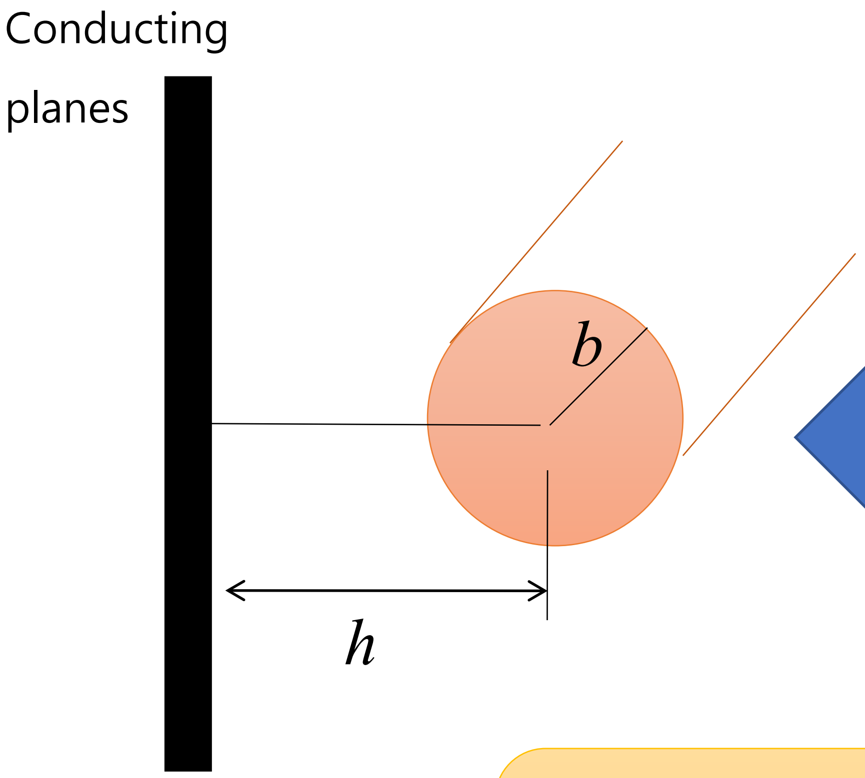
It is all independent of z, as you might expect!

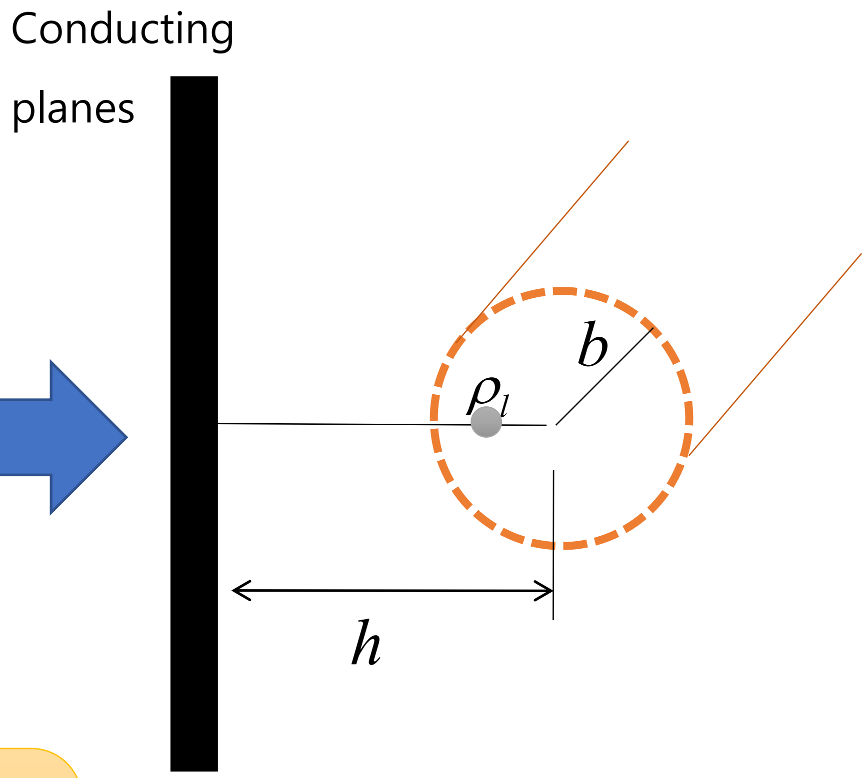
Capacitance of two-wire line



$$C = \frac{\pi \varepsilon L}{\ln\left[\frac{\left(h+\sqrt{h^2-b^2}\right)}{b}\right]}$$

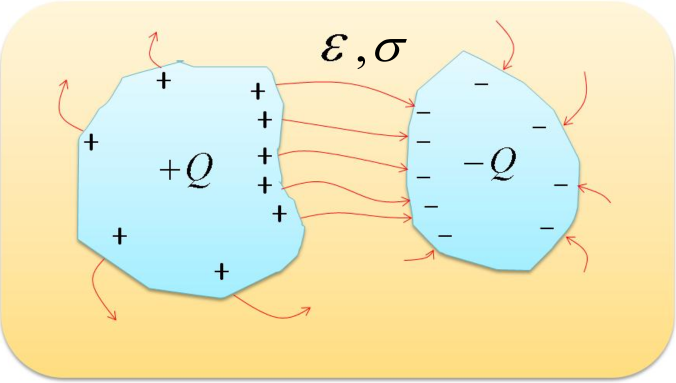
One-wire line and a conducting plane





$$C = \frac{2\pi\varepsilon L}{\ln\left[\dfrac{\left(h+\sqrt{h^2-b^2}\right)}{b}\right]}$$

Current Analogy



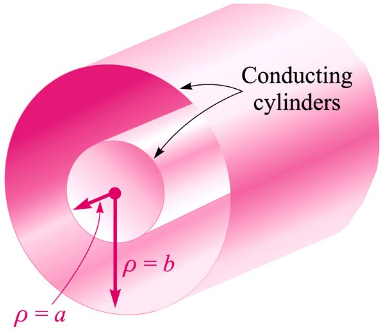
$$C = \frac{Q}{V} = \frac{\int_v \rho_v dv}{-\int \mathbf{E}\cdot d\mathbf{L}} = \frac{\int_v \nabla\cdot\mathbf{D} dv}{-\int \mathbf{E}\cdot d\mathbf{L}} = \frac{\oint_S \mathbf{D}\cdot d\mathbf{s}}{-\int \mathbf{E}\cdot d\mathbf{L}} = \frac{\oint_S \varepsilon\mathbf{E}\cdot d\mathbf{s}}{-\int \mathbf{E}\cdot d\mathbf{L}}$$

$$R = \frac{V}{I} = \frac{-\int \mathbf{E}\cdot d\mathbf{L}}{\oint_S \mathbf{J}\cdot d\mathbf{s}} = \frac{-\int \mathbf{E}\cdot d\mathbf{L}}{\oint_S \sigma\mathbf{E}\cdot d\mathbf{s}}$$

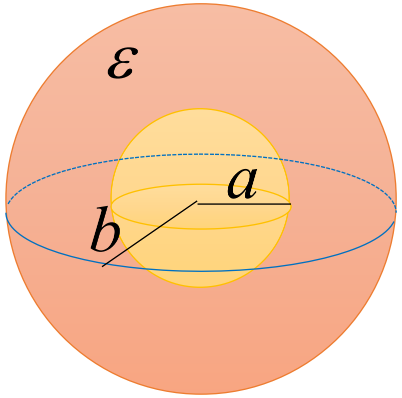
$$\boxed{RC = \frac{\varepsilon}{\sigma}}$$

Linear, homogeneous, isotropic material only

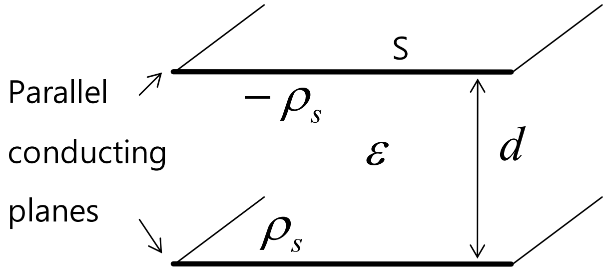
Examples



$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

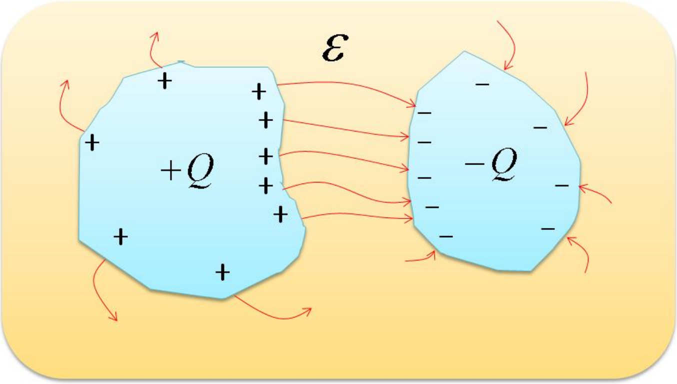


$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$



$$C = \frac{Q}{V} = \frac{\varepsilon S}{d}$$

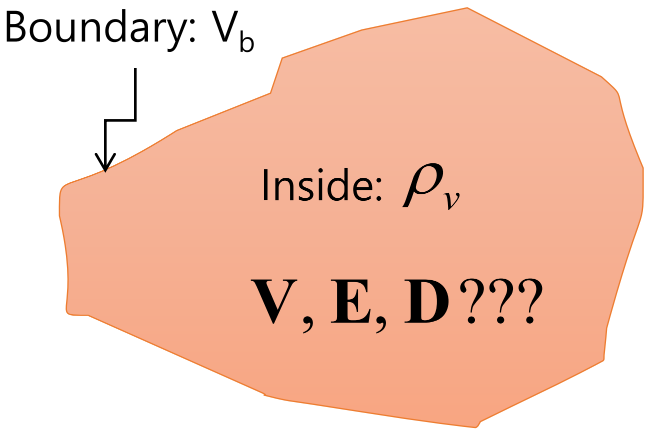
Energy stored in a capacitor



$$W_E = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_{vol} \mathbf{D}\cdot\mathbf{E} dv = \frac{1}{2}\int_{vol} \varepsilon E^2 dv$$

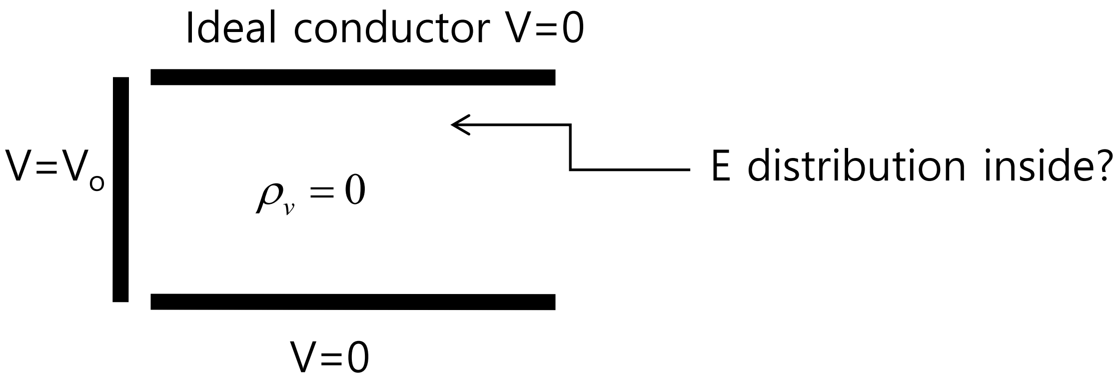
$$= \boxed{\frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}}$$

Poisson's / Laplace's Equation

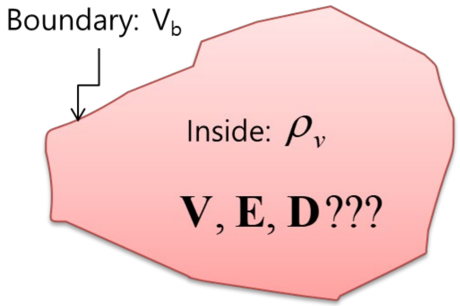


- When (1) internal charge distribution and (2) boundary condition are known,
- Calculate V, E, and D inside the boundary

Example



Poisson's / Laplace's Equation



$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\nabla \cdot \varepsilon \mathbf{E} = \rho_v$$

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon}$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$ ➡ Poisson's Equation

When no charge, $$\boxed{\nabla^2 V = 0}$$ ➡ Laplace's Equation

- E can be obtained by solving Poisson's or Laplace equation with boundary conditions

Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$ ➡ Rectangular coordinates

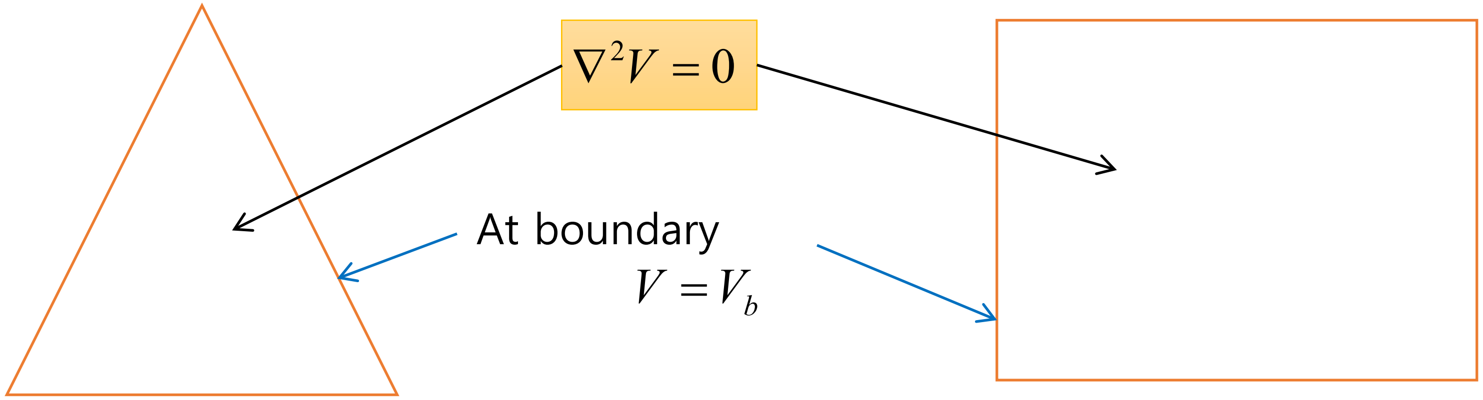
$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$ ➡ Cylindrical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$ ➡ Spherical coordinates

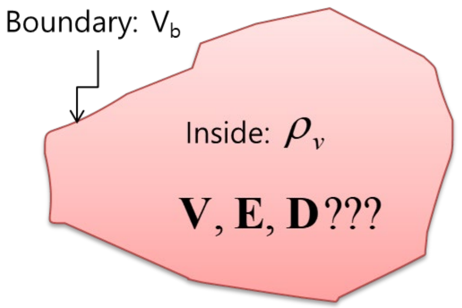
Uniqueness Theorem

Is the solution of Poisson’s or Laplace’s equation unique for given boundary condition?

Yes!!



Using Poisson's / Laplace's Equation



$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}}$$

1. Set coordinate system
2. Obtain V by solving Poisson's / Laplace equation with given boundary condition
3. Obtain E,D,Q using V

$$\mathbf{E} = -\nabla V$$

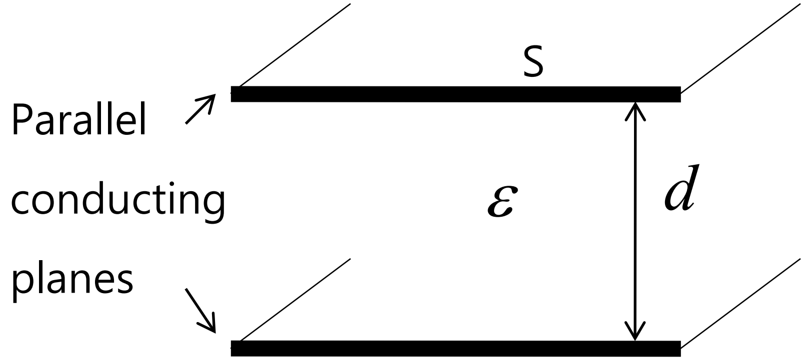
$$\mathbf{D} = \varepsilon \mathbf{E}$$

Surface charge $\rho_s = |\mathbf{D}|$

Capacitance calculation using Laplace's Equation

Laplace equation $\nabla^2 V = 0$

BC: $V=V_o$ (at $z=0$), $V=0$ (at $z=d$)



$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

From symmetry

$$\frac{d^2 V}{dz^2} = 0$$

$$V = Az + B$$

BC

$$V = -\frac{V_o}{d} z + V_o$$

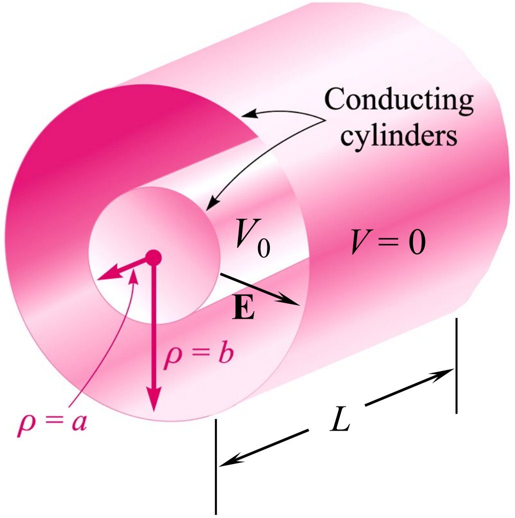
$$\mathbf{E} = -\nabla V = \frac{V_o}{d}\mathbf{a_z}$$

$$\mathbf{D} = \frac{\varepsilon V_o}{d}\mathbf{a_z}$$

$$\rho_s = \frac{\varepsilon V_o}{d}$$

$$\boxed{C = \frac{Q}{V} = \frac{\frac{\varepsilon V_o}{d} S}{V_o} = \frac{\varepsilon S}{d}}$$

Capacitance calculation using Laplace's Equation



Laplace equation $\nabla^2 V = 0$

BC: $V=V_o$ (at $\rho=a$), $V=0$ (at $\rho=b$)

$$\nabla^2 V = 0$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

From symmetry

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

$$V = A \ln \rho + B$$

BC

$$V = V_o \frac{\ln(b/\rho)}{\ln(b/a)}$$

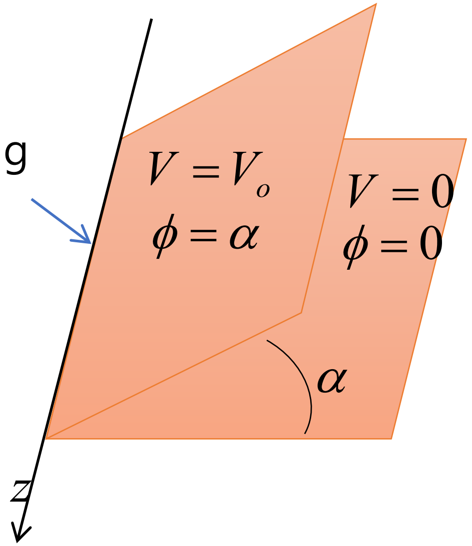
$$\mathbf{E} = -\nabla V = \frac{V_o}{\rho}\frac{1}{\ln(b/a)}\mathbf{a}_\rho$$

$$\mathbf{D} = \frac{\varepsilon V_o}{\rho}\frac{1}{\ln(b/a)}\mathbf{a}_\rho$$

$$\rho_s = \frac{\varepsilon V_o}{a}\frac{1}{\ln(b/a)}$$

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

Capacitance calculation using Laplace's Equation



Laplace equation $\nabla^2 V = 0$

BC: $V=V_o$ (at $\phi=\alpha$), $V=0$ (at $\phi=0$)

$$\nabla^2 V = 0$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

From symmetry

$$\frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) = 0$$

$$V = A\phi + B$$

BC

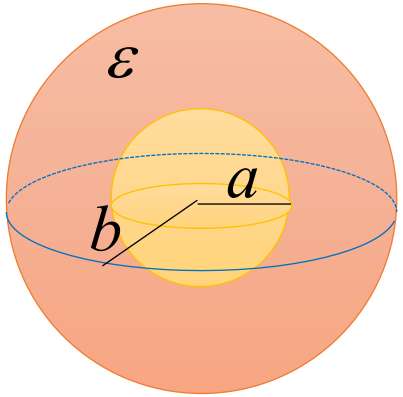
$$V = V_o \frac{\phi}{\alpha}$$

$$\mathbf{E} = -\nabla V = -\frac{V_o}{\alpha \rho}\mathbf{a}_\phi$$

$$\mathbf{D} = -\frac{\varepsilon V_o}{\alpha \rho}\mathbf{a}_\phi$$

$$\rho_s = \frac{\varepsilon V_o}{\alpha \rho}$$

Capacitance calculation using Laplace's Equation



Laplace equation $\nabla^2 V = 0$

경계 조건: V=V$_o$ (at r=a), V=0 (at r=b)

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

From symmetry

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=0$$

$$V=\frac{A}{r}+B$$

BC

$$V=V_o\frac{\frac{1}{r}-\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$$

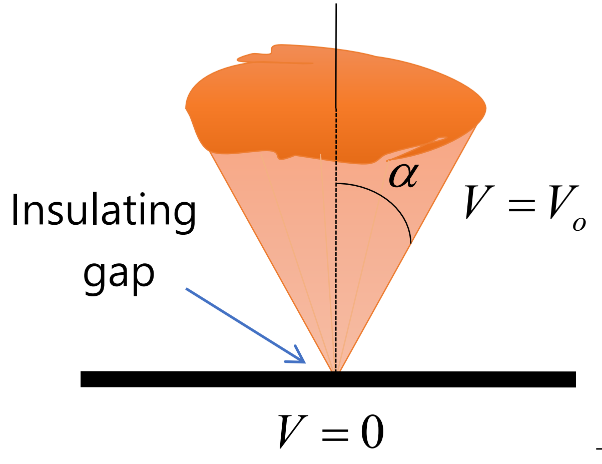
$$\mathbf{E}=-\nabla V=\frac{1}{r^2}\frac{V_o}{\frac{1}{a}-\frac{1}{b}}\mathbf{a}_\mathbf{r}$$

$$\mathbf{D}=\frac{\varepsilon}{r^2}\frac{V_o}{\frac{1}{a}-\frac{1}{b}}\mathbf{a}_\mathbf{r}$$

$$\rho_s=\frac{\varepsilon}{a^2}\frac{V_o}{\frac{1}{a}-\frac{1}{b}}$$

$$C=\frac{Q}{V}=\frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}}$$

Capacitance calculation using Laplace's Equation



Laplace equation $\nabla^2 V = 0$

BC: $V=V_o$ (at $\theta=\alpha$), $V=0$ (at $\theta=\pi/2$)

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

From symmetry

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

$$V = A\ln\left(\cot\frac{\theta}{2}\right)+B$$

BC

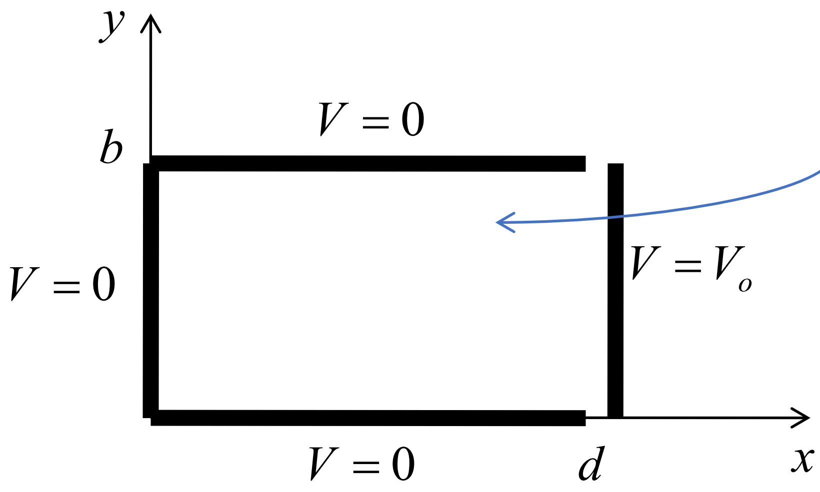
$$V = V_o\frac{\ln\left(\cot\frac{\theta}{2}\right)}{\ln\left(\cot\frac{\alpha}{2}\right)}$$

$$\mathbf{E} = -\nabla V = \frac{V_o}{r\sin\theta\ln\left(\cot\frac{\alpha}{2}\right)}\mathbf{a}_\theta$$

$$\mathbf{D} = \frac{\varepsilon V_o}{r\sin\theta\ln\left(\cot\frac{\alpha}{2}\right)}\mathbf{a}_\theta$$

$$\rho_s = \frac{\varepsilon V_o}{r\sin\alpha\ln\left(\cot\frac{\alpha}{2}\right)}$$

Product Solution of Laplace's Equation



$$\nabla^2 V = 0$$

V is dependent on x, y

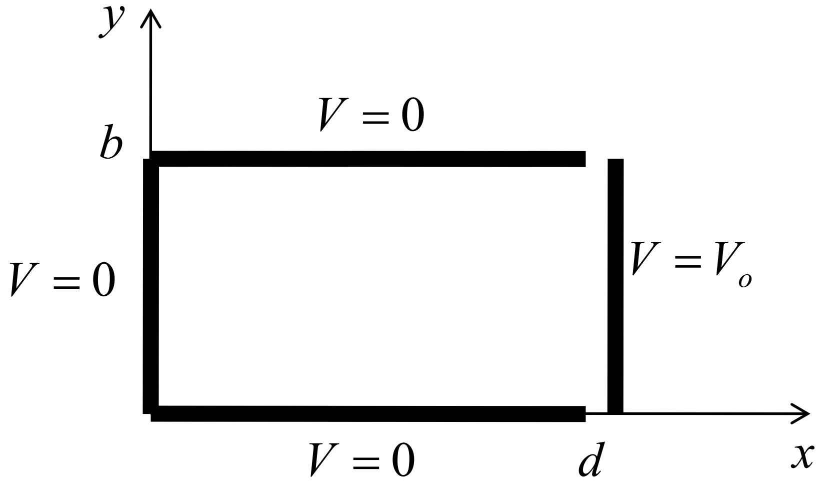
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Assume product solution

(V=X(x)Y(y))

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

Product Solution of Laplace's Equation



$$Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0$$

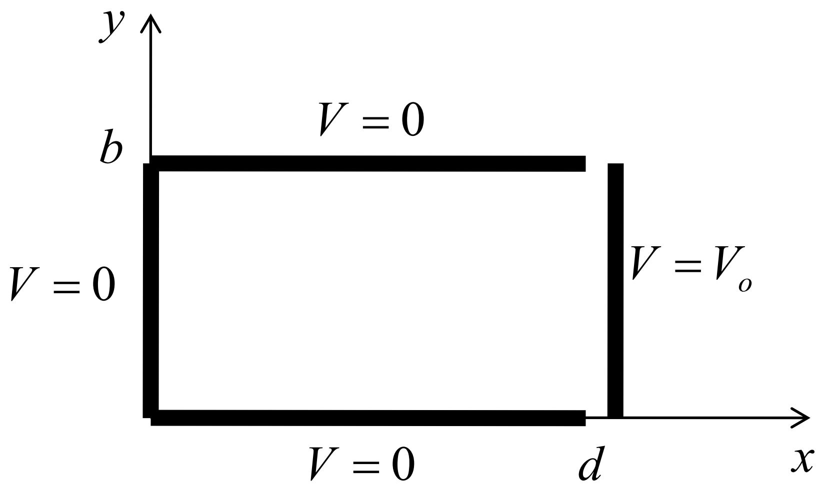
$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=k^2$$

$$\frac{\partial^2 X}{\partial x^2}=k^2 X \qquad \frac{\partial^2 Y}{\partial y^2}=-k^2 Y$$

$$\begin{cases} X = A\cosh kx + B\sinh kx = A'e^{kx}+B'e^{-kx} \\ Y = C\cos ky + D\sin ky = C'e^{iky}+D'e^{-iky} \end{cases}$$

A, B, C, D, k obtained from BC

Product Solution of Laplace's Equation



$$\begin{cases} X = A\cosh kx + B\sinh kx = A'e^{kx} + B'e^{-kx} \\ Y = C\cos ky + D\sin ky = C'e^{iky} + D'e^{-iky} \end{cases}$$

BC

$$\begin{cases} V = V_o \quad at \quad x = d \\ V = 0 \quad at \quad y = 0,\ y = d,\ x = 0 \end{cases}$$

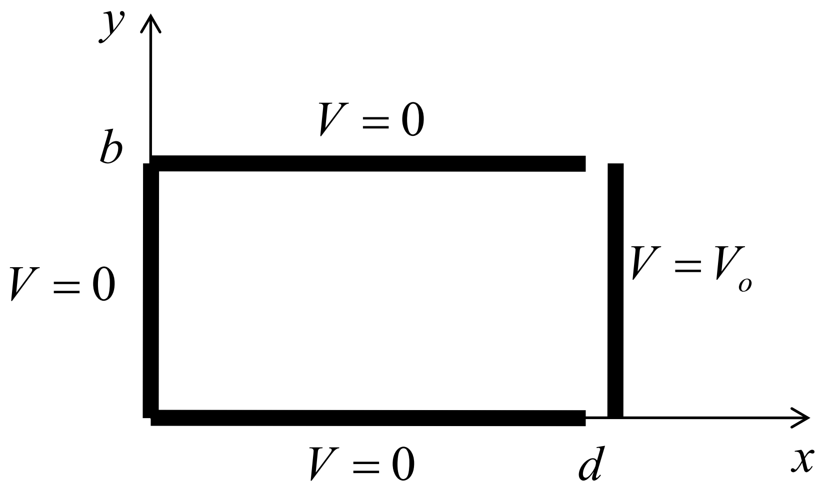
1

$V = 0 \quad at \quad x = 0$

$$V = (A\cosh kx + B\sinh kx)(C\cos ky + D\sin ky) = A(C\cos ky + D\sin ky) = 0$$

$$\therefore A = 0$$

Product Solution of Laplace's Equation



$$\begin{cases} X = A\cosh kx + B\sinh kx = A'e^{kx} + B'e^{-kx} \\ Y = C\cos ky + D\sin ky = C'e^{iky} + D'e^{-iky} \end{cases}$$

BC

$$\begin{cases} V = V_o \quad at \quad x = d \\ V = 0 \quad at \quad y = 0,\ y = d,\ x = 0 \end{cases}$$

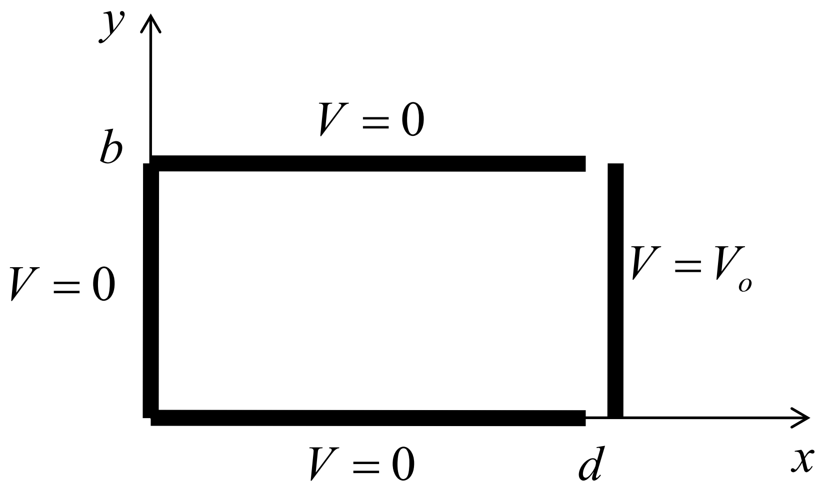
2

$V = 0 \quad at \quad y = 0$

$$V = B\sinh kx(C\cos ky + D\sin ky) = BC\sinh kx = 0$$

$$\therefore C = 0$$

Product Solution of Laplace's Equation



$$\begin{cases} X = A\cosh kx + B\sinh kx = A'e^{kx} + B'e^{-kx} \\ Y = C\cos ky + D\sin ky = C'e^{iky} + D'e^{-iky} \end{cases}$$

BC

$$\begin{cases} V = V_o \quad at \quad x = d \\ V = 0 \quad at \quad y = 0,\ y = d,\ x = 0 \end{cases}$$

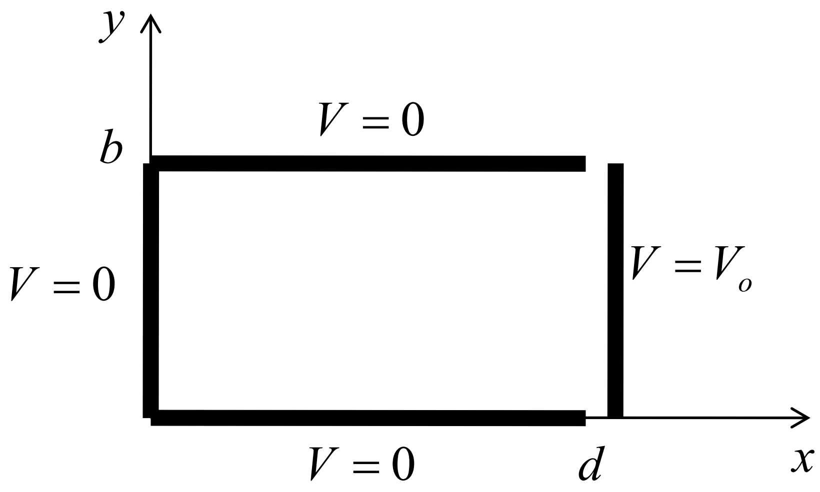
3

$V = 0 \quad at \quad y = b$

$$V = BD\sinh kx \sin ky = BD\sinh kx \sin kb = 0$$

$$\therefore k = \frac{m\pi}{b}, \quad m = 0,1,\cdots$$

Product Solution of Laplace's Equation



$$\begin{cases} X = A\cosh kx + B\sinh kx = A'e^{kx} + B'e^{-kx} \\ Y = C\cos ky + D\sin ky = C'e^{iky} + D'e^{-iky} \end{cases}$$

BC

$$\begin{cases} V = V_o \quad at \quad x = d \\ V = 0 \quad at \quad y = 0,\ y = d,\ x = 0 \end{cases}$$

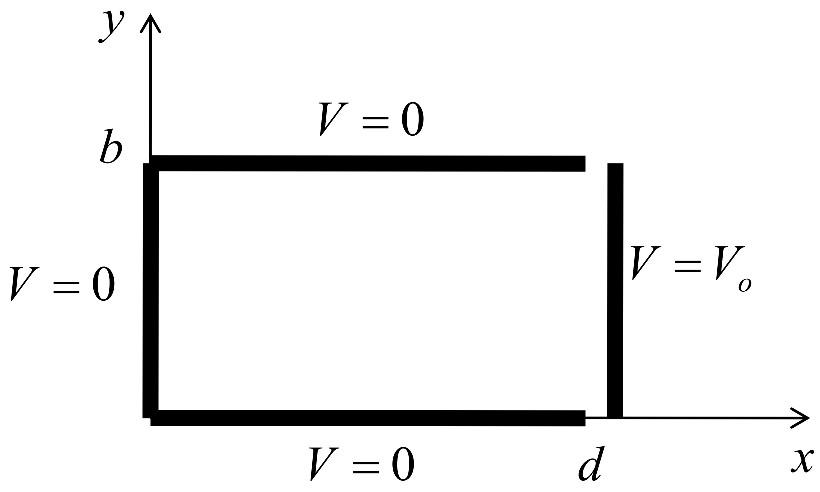
4

$$V = V_o \ at \quad x = d$$

$$V = \sum_{m=0}^{\infty} V_m \sinh\frac{m\pi x}{b}\sin\frac{m\pi y}{b} = \sum_{m=0}^{\infty} V_m \sinh\frac{m\pi d}{b}\sin\frac{m\pi y}{b} = \sum_{m=0}^{\infty} c_m \sin\frac{m\pi y}{b} = V_o$$

$$c_m = \frac{1}{b}\left[\int_0^b V_o \sin\frac{m\pi y}{b}dy + \int_b^{2b} -V_o \sin\frac{m\pi y}{b}dy\right] = \begin{cases} \dfrac{4V_o}{m\pi} & (m\ odd) \\ 0 & (m\ even) \end{cases}$$

Product Solution of Laplace's Equation



$$\begin{cases} X = A\cosh kx + B\sinh kx = A'e^{kx} + B'e^{-kx} \\ Y = C\cos ky + D\sin ky = C'e^{iky} + D'e^{-iky} \end{cases}$$

BC

$$\begin{cases} V = V_o \quad at \quad x = d \\ V = 0 \quad at \quad y = 0,\ y = d,\ x = 0 \end{cases}$$

$$V = \frac{4V_o}{\pi}\sum_{1,odd}^{\infty}\frac{1}{m}\frac{\sinh(m\pi x/b)}{\sinh(m\pi d/b)}\sin\frac{m\pi y}{b}$$

Chapter Summary

- Capacitance definition
- Capacitance calculation
  - Parallel plate, Coaxial transmission line, Spherical capacitor
  - Isolated sphere with a dielectric coating, Parallel plate with two-layer dielectric
  - Two-wire line, one-wire line and a plane
- Energy stored in a capacitor
- Poisson / Laplace equation
  - Laplacian operator
  - Parallel plate, Coaxial transmission line, Angled plate, Concentric sphere, Cone above a conducting plane
  - Product solution