Electromagnetics 1 (ICE2003)

-- Ch. 4. Energy and Potential --

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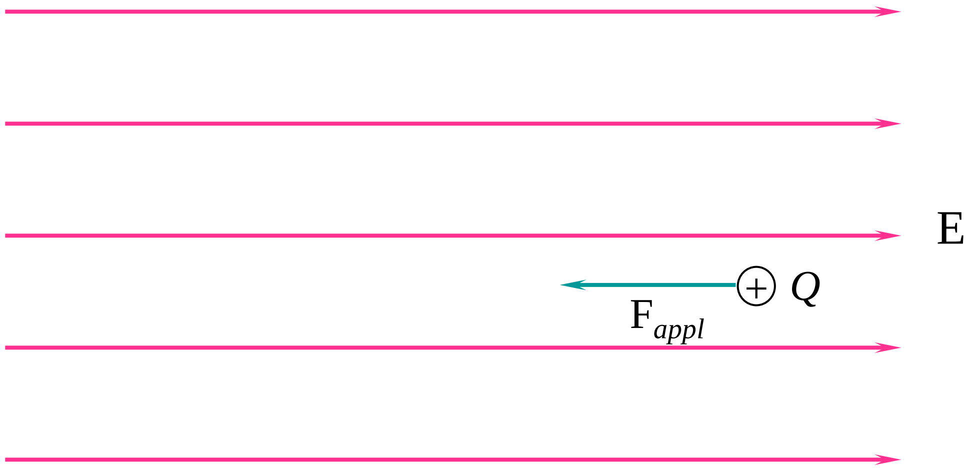
Chapter Outline

- Work done on moving a point charge against an external field
- Potential Difference
- Potential Field
  - 점전하
  - 선전하
  - 면전하
  - 부피전하
- Relation between Potential and Electric Field
- Electric Dipole
- Electric Energy

Point charge in an external field

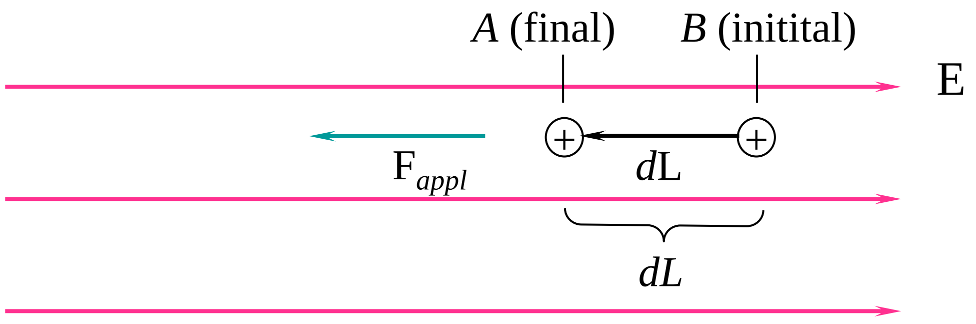


$$\boxed{\mathbf{F}_{appl} = -Q\mathbf{E}}$$

Point charge in an external field

$$dW = \mathbf{F}_{appl} \cdot d\mathbf{L} = -Q\mathbf{E} \cdot d\mathbf{L}\ [J]$$

gives positive result if charge
is forced *against* the electric field

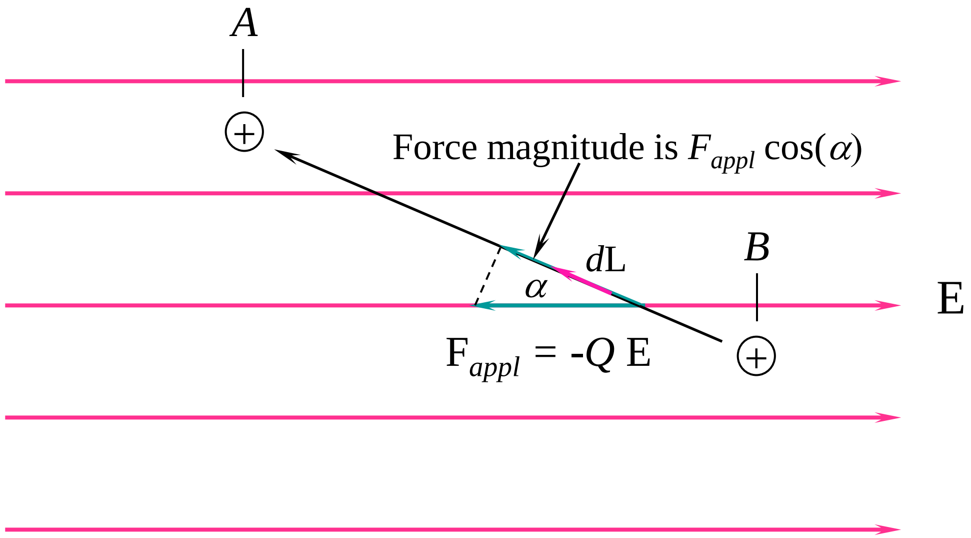


$$\boxed{\mathbf{F}_{appl} = -Q\mathbf{E}}$$

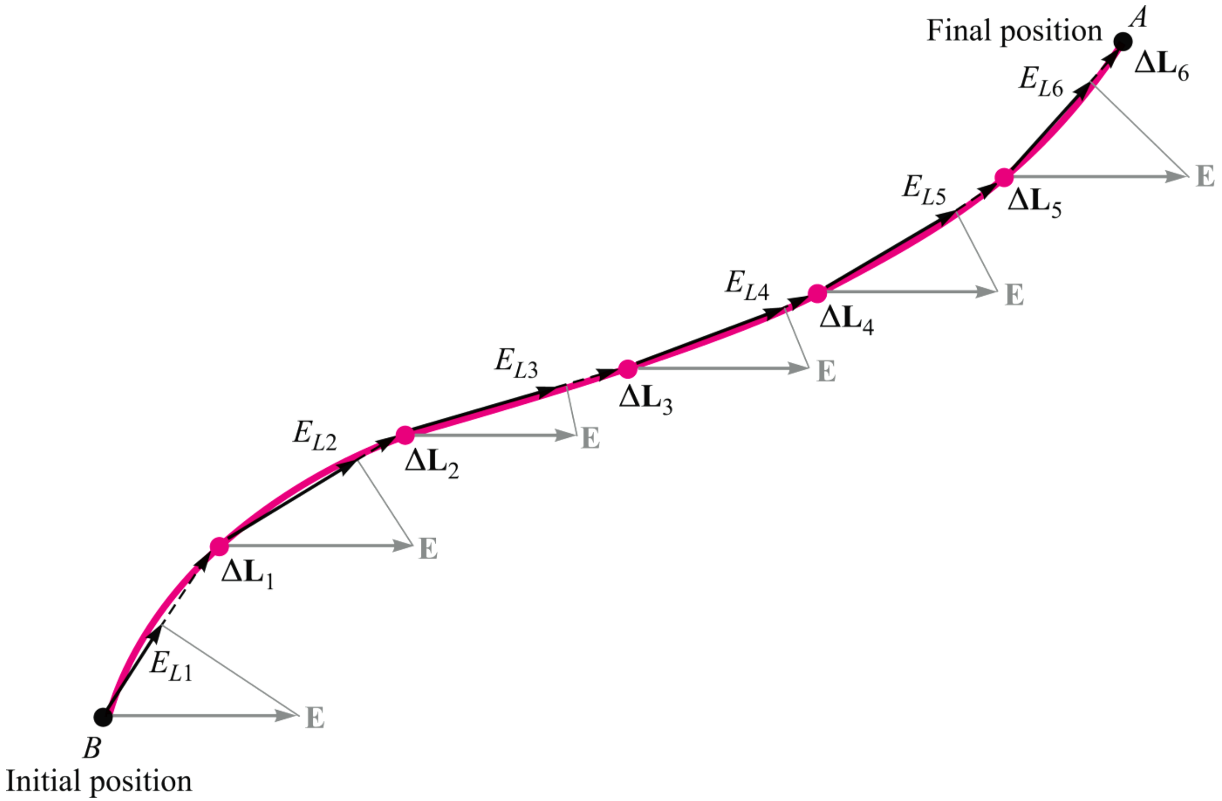
Point charge in an external field

$$dW = \mathbf{F}_{appl} \cdot d\mathbf{L} = -Q\mathbf{E} \cdot d\mathbf{L}\,[J]$$

$$\boxed{W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}}$$



Total work done over an arbitrary path



$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$$

$$d\mathbf{L} = dx\mathbf{a}_\mathbf{x} + dy\mathbf{a}_\mathbf{y} + dz\mathbf{a}_\mathbf{z}$$
$$d\mathbf{L} = d\rho\mathbf{a}_\boldsymbol{\rho} + \rho d\phi\mathbf{a}_\boldsymbol{\varphi} + dz\mathbf{a}_\mathbf{z}$$
$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a}_\boldsymbol{\theta} + r\sin\theta d\phi\mathbf{a}_\boldsymbol{\varphi}$$

Example 1

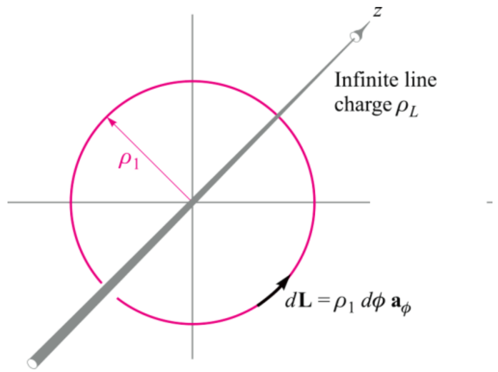
An electric field is given as: $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$

We wish to find the work done in moving a point charge of magnitude $Q = 2$ over the shorter arc of the circle given by $x^2 + y^2 = 1 \quad z = 1$

The initial point is $B(1, 0, 1)$ and the final point is $A(0.8, 0.6, 1)$:

Example 2

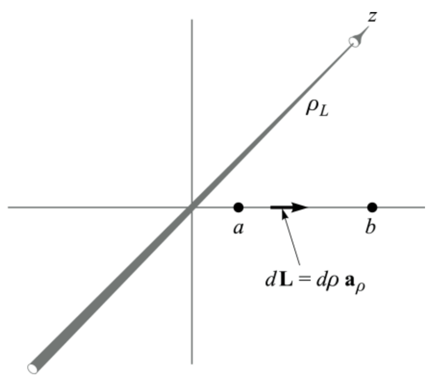
In this example, the work in moving charge Q in a circular path around a line charge is found:



(*a*)

Example 3

Instead, we now move charge Q along a radial line near the same line charge:



(b)

Definition of Potential Difference

$$W = -Q \int_{\mathrm{init}}^{\mathrm{final}} \mathbf{E} \cdot d\mathbf{L}$$

The *potential difference* is defined as the work done (or potential energy gained) *per unit charge*. We express this quantity in units of Joules/Coulomb, or *volts*.

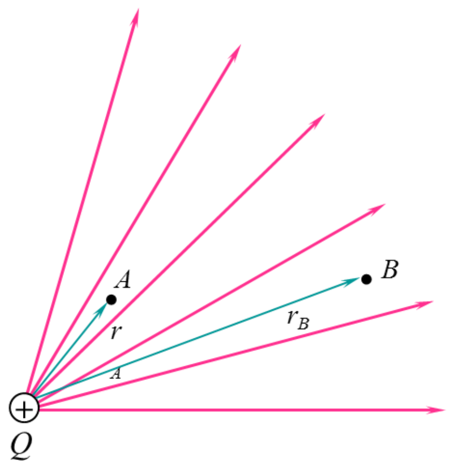
$$potential difference = V = -\int_{init}^{final} \mathbf{E} \cdot d\mathbf{L}$$

$$V_{AB} = -\int_{B}^{A} \mathbf{E} \cdot d\mathbf{L}$$

With reference zero potential

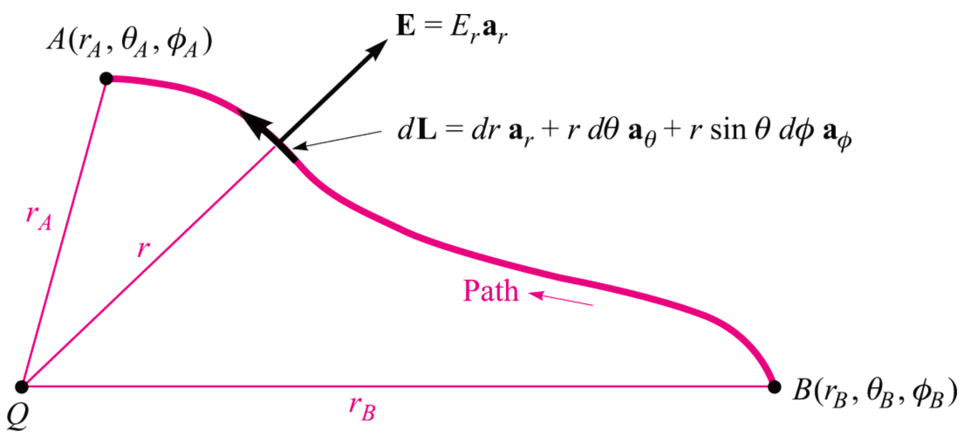
$$V_{AB} = V_A - V_B$$

Potential field of a point charge



$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a_r}$$

$$d\mathbf{L} = dr\mathbf{a}_r + rd\theta\mathbf{a_\theta} + r\sin\theta d\phi\mathbf{a_\varphi}$$



$$V_{AB} = -\int_B^A \mathbf{E}\cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_o r^2}dr$$

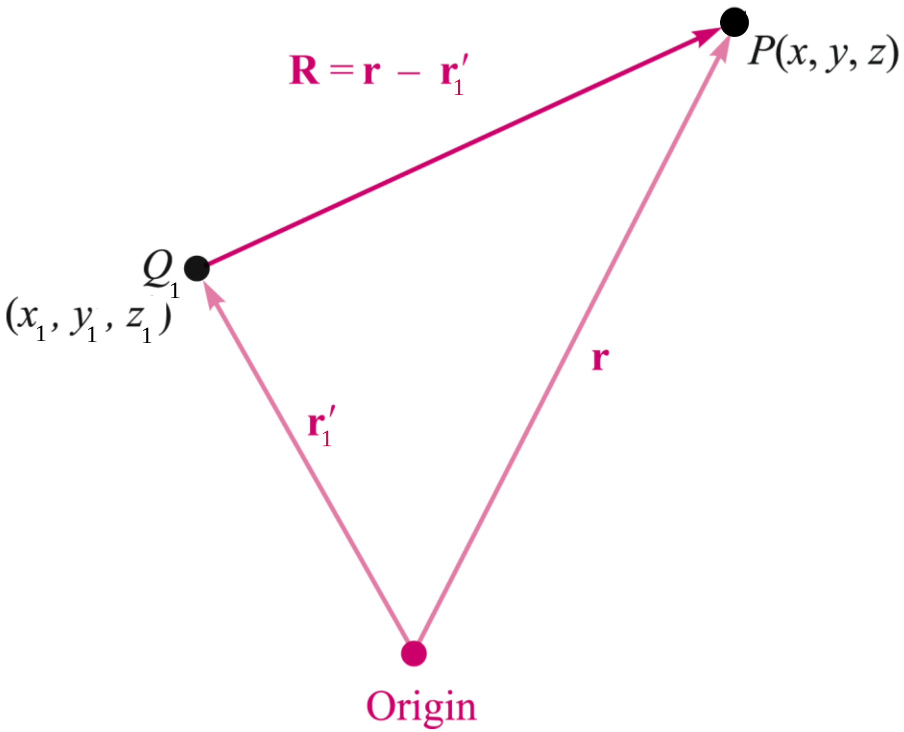
$$= \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

With infinite reference
(r_B goes to infinity)

$$\boxed{V = \frac{Q}{4\pi\varepsilon_o r}}$$

Potential field of a point charge off-origin

$$V_P(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|}$$



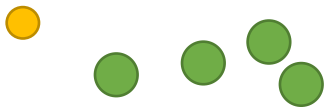
Potential field of a system of charges

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o |\mathbf{r}-\mathbf{r}_1|}$$

Potential field by a point charge Q1 at r_1

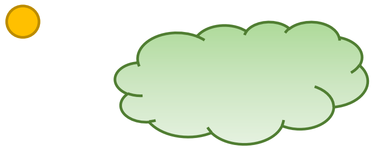
$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o |\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_o |\mathbf{r}-\mathbf{r}_2|}$$

Potential field by two point charges Q_1, Q_2



$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_o |\mathbf{r}-\mathbf{r}_m|}$$

Potential field by N point charges

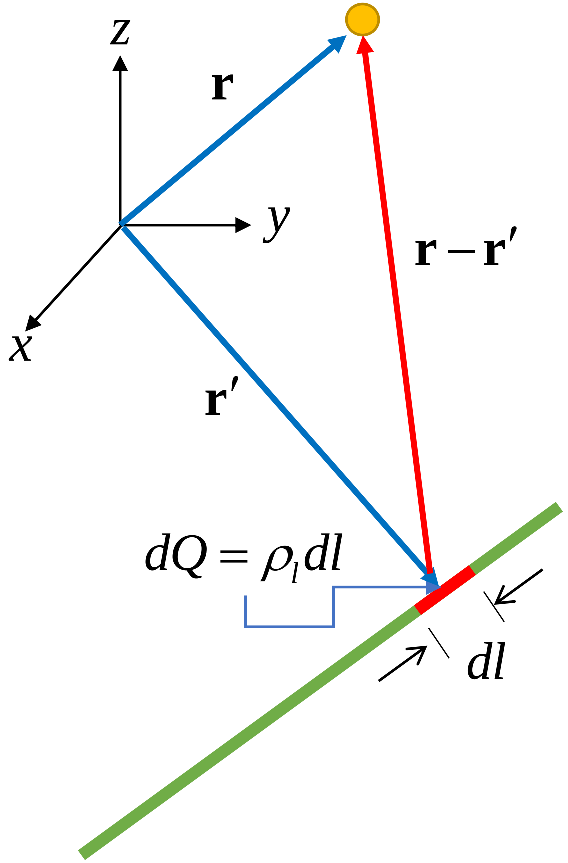


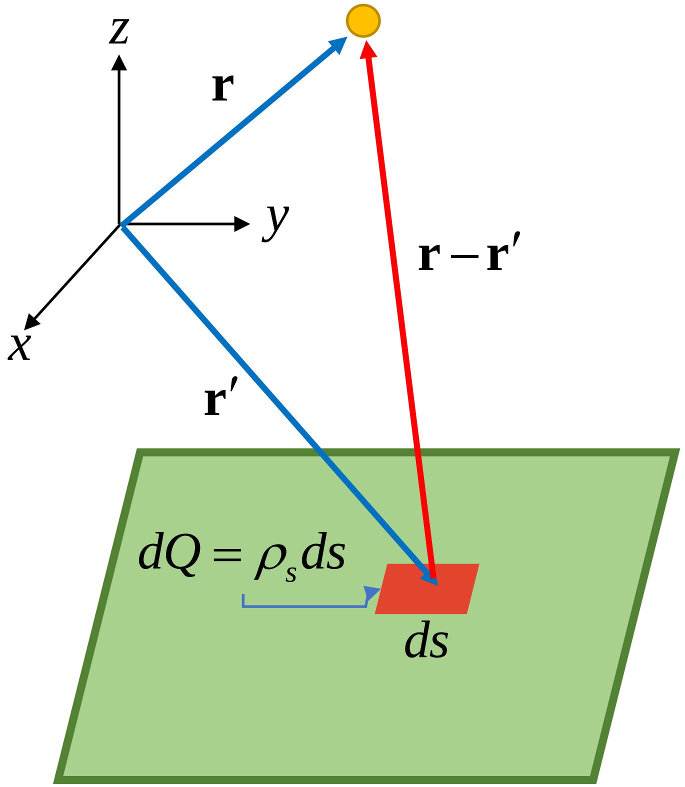
$$V(\mathbf{r}) = \int_{vol} \frac{\rho_v(\mathbf{r}')dv'}{4\pi\varepsilon_o |\mathbf{r}-\mathbf{r}'|}$$

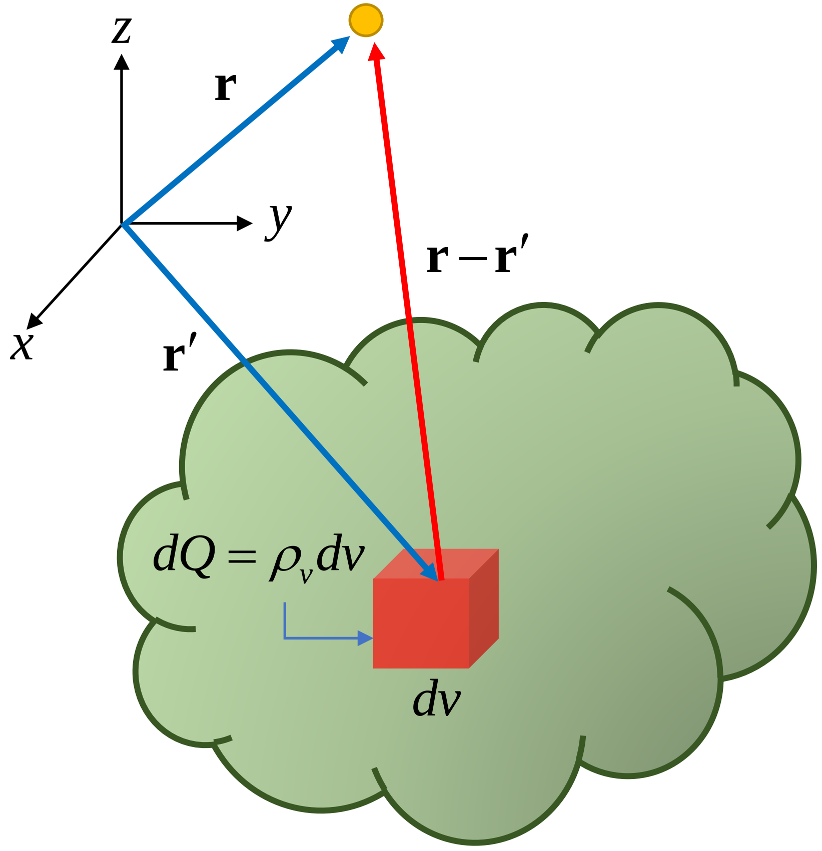
Potential field by continuous volume charge density ρ

$$\mathbf{E}(\mathbf{r}) = \int_{\mathrm{vol}} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

Potential field of a system of charges







$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\,dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

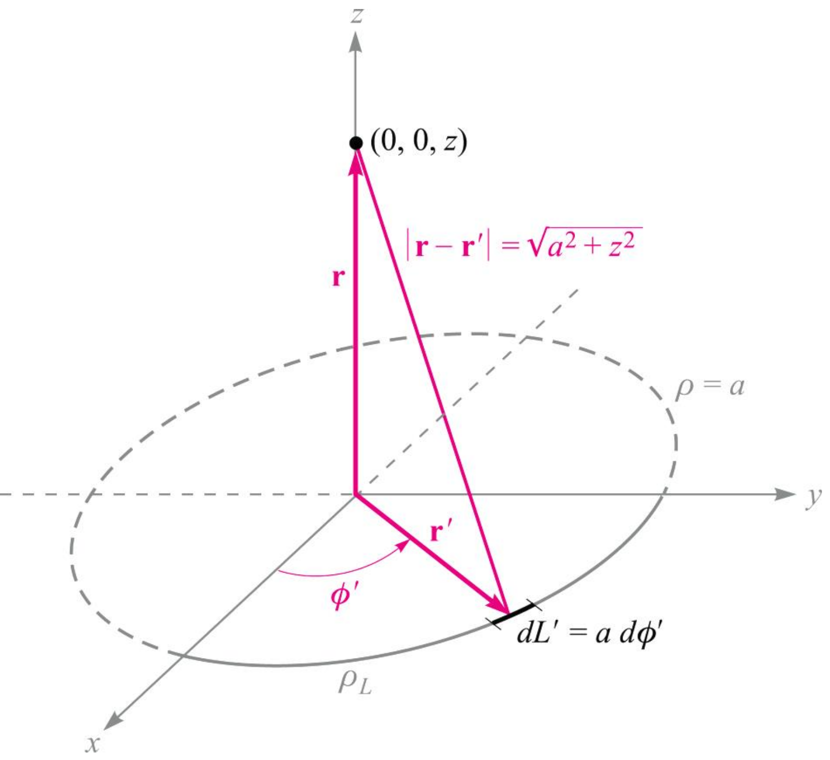
$$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')\,dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

$$V(\mathbf{r}) = \int_{\mathrm{vol}} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Example

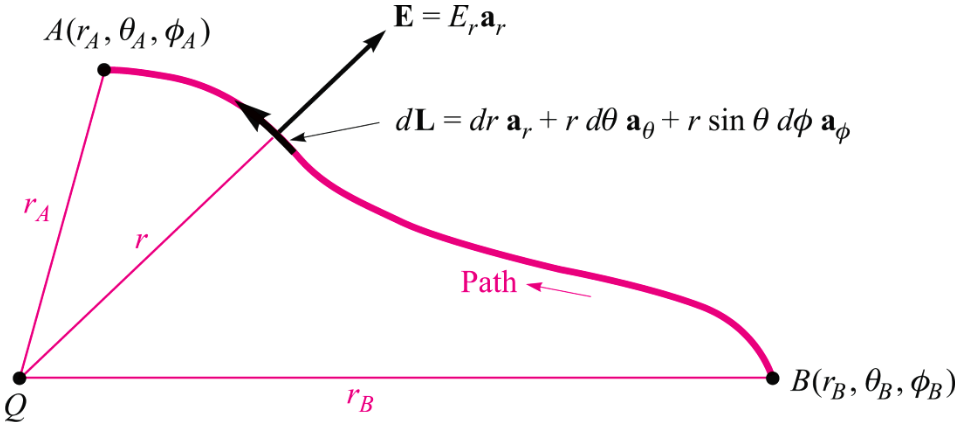
$$V(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_\nu(\mathbf{r}')\, dv'}{4\pi\epsilon_0 |\mathbf{r}-\mathbf{r}'|}$$

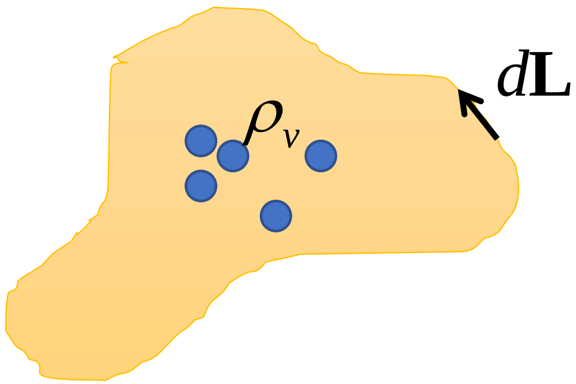
$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_\nu(\mathbf{r}')\, dv'}{4\pi\epsilon_0 |\mathbf{r}-\mathbf{r}'|^2} \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$



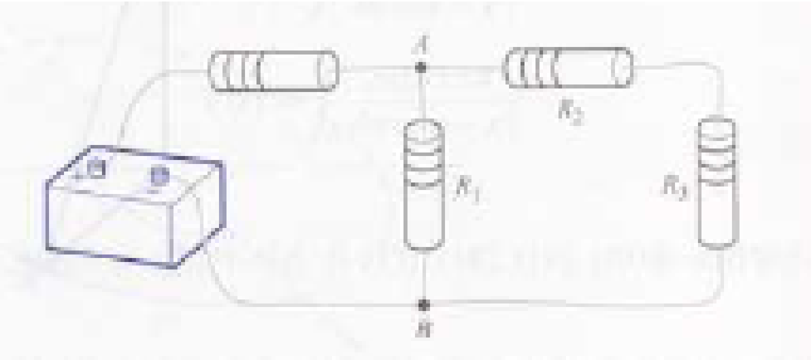
$$V(z) = \int_0^{2\pi} \frac{\rho_L a d\phi'}{4\pi\varepsilon_o \sqrt{a^2+z^2}} = \frac{\rho_L a}{2\varepsilon_o \sqrt{a^2+z^2}}$$

Conservative field





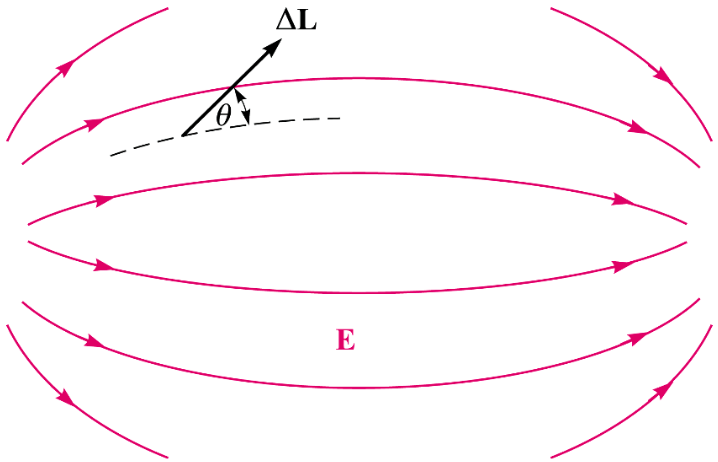
$$\boxed{\oint \mathbf{E} \cdot d\mathbf{L} = 0}$$ holds for static field



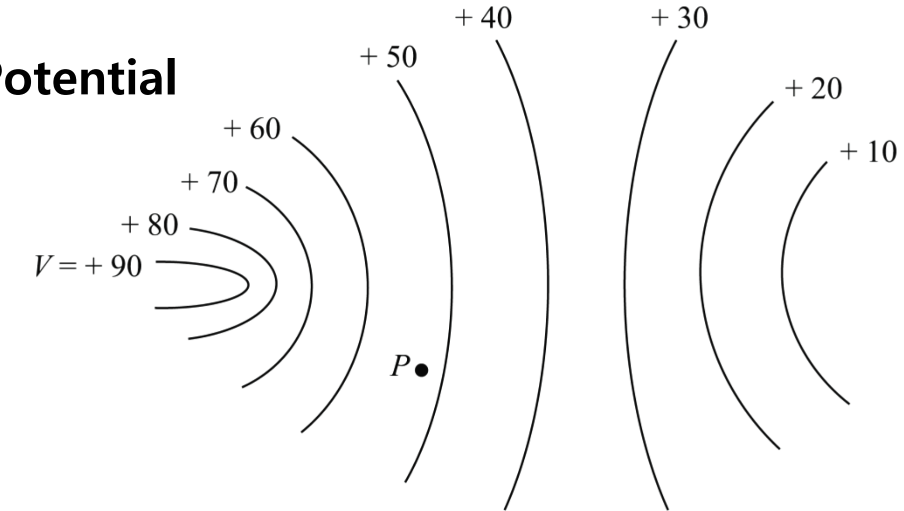
Kirchhoff's voltage law

Potential gradient

Electric field



Potential



$$V = -\int \mathbf{E} \cdot d\mathbf{L}$$

$$\Delta V = -\mathbf{E} \cdot \Delta\mathbf{L}$$

$$\Delta V = -E\Delta L \cos\theta$$

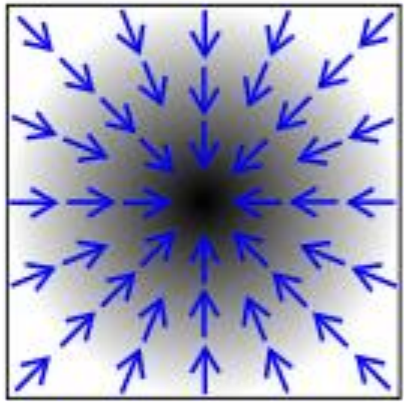
$$\left.\frac{dV}{dL}\right|_{\max} = E$$

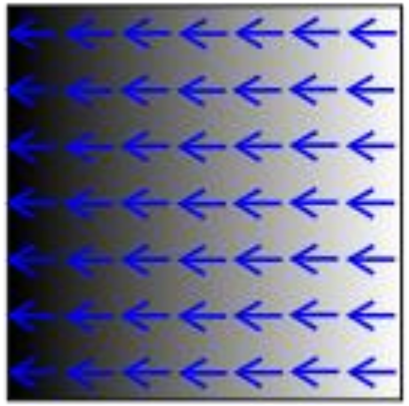
Potential gradient

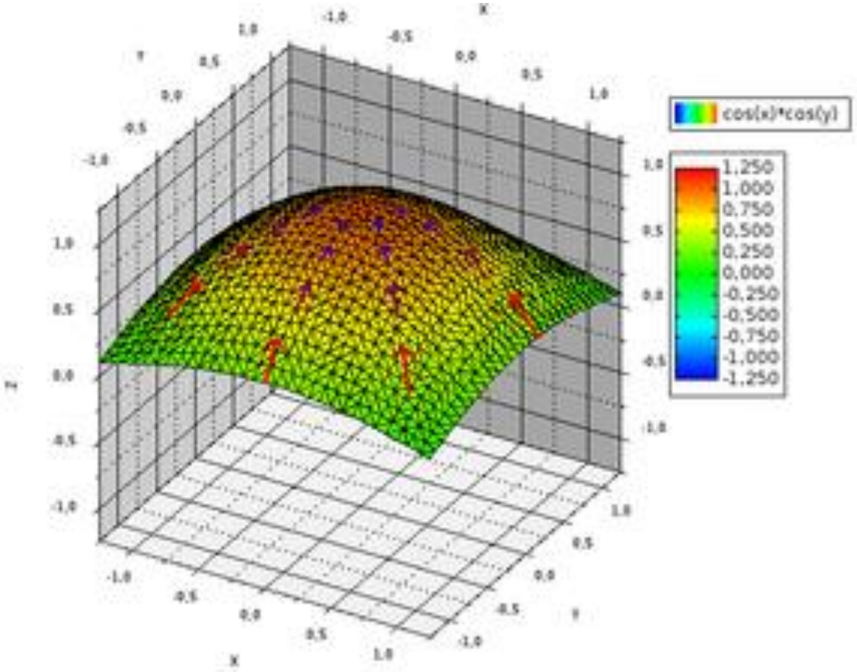
$$\mathbf{E} = \mathbf{a_N}\left(-\frac{dV}{dL}\bigg|_{max}\right) = -\frac{dV}{dN}\mathbf{a_N}$$

$$Gradient of\ T = grad\,T = \frac{dT}{dN}\mathbf{a_N}$$

$$\mathbf{E} = -grad\,V$$







Potential gradient

$$\mathbf{E} = \mathbf{a}_{\mathbf{N}}\left(-\frac{dV}{dL}\bigg|_{\max} \right) = -\frac{dV}{dN}\mathbf{a}_{\mathbf{N}}$$

$$Gradient\, of\ T = grad\, T = \frac{dT}{dN}\mathbf{a}_{\mathbf{N}}$$

$$\mathbf{E} = -grad\, V$$

$$\Delta V = -\mathbf{E} \cdot \Delta\mathbf{L}$$

$$dV = -\mathbf{E} \cdot d\mathbf{L}$$

$$= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_{\mathbf{x}} + \frac{\partial V}{\partial y}\mathbf{a}_{\mathbf{y}} + \frac{\partial V}{\partial z}\mathbf{a}_{\mathbf{z}} \right)$$

$$grad\, V = \frac{\partial V}{\partial x}\mathbf{a}_{\mathbf{x}} + \frac{\partial V}{\partial y}\mathbf{a}_{\mathbf{y}} + \frac{\partial V}{\partial z}\mathbf{a}_{\mathbf{z}}$$

Potential gradient

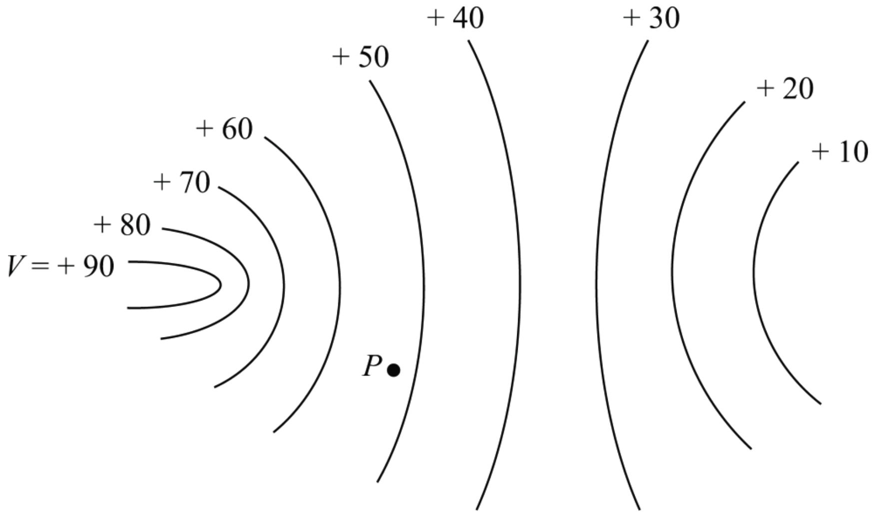
$$V = x + y$$

Direction of the Gradient vector

The gradient of *V* is a directional derivative that represents spatial rate of change. This is a vector which we would assume must be in some fixed direction at a given point. The projection of the gradient along a direction tangent to an equipotential surface *must* give a result of zero, as the potential by definition is constant along that surface: In other words,

$$\nabla V \cdot \mathbf{t} = 0$$

Therefore, must be perpendicular to t, or *normal* to an equipotential surface, and in the direction of *maximum increase* in *V*.



Gradient

$$grad\,V = \frac{\partial V}{\partial x}\mathbf{a_x} + \frac{\partial V}{\partial y}\mathbf{a_y} + \frac{\partial V}{\partial z}\mathbf{a_z}$$

$$\nabla = \frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}$$

$$\nabla V = grad\,V$$

$$\boxed{\mathbf{E} = -\nabla V}$$

Rectangular

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a_x} + \frac{\partial V}{\partial y}\mathbf{a_y} + \frac{\partial V}{\partial z}\mathbf{a_z}$$

Cylindrical

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a_\rho} + \frac{\partial V}{\rho\partial \phi}\mathbf{a_\varphi} + \frac{\partial V}{\partial z}\mathbf{a_z}$$

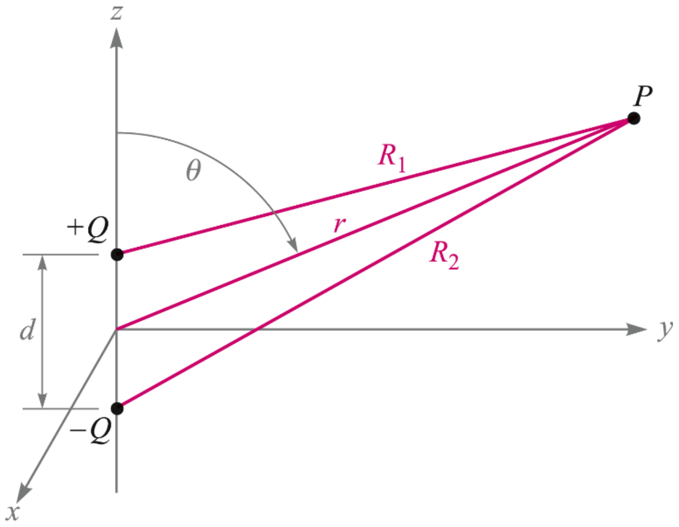
Spherical

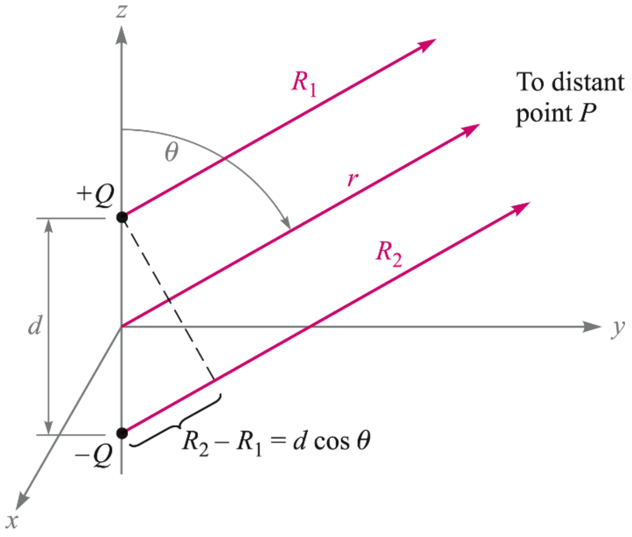
$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a_r} + \frac{\partial V}{r\partial \theta}\mathbf{a_\theta} + \frac{\partial V}{r\sin\theta\partial \phi}\mathbf{a_\varphi}$$

Example

$V = 2x^2 y - 5z$ at P(-4,3,6), obtain V, **E**, **D,** ρ_v ?

Electric dipole





Rectangular

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a_x} + \frac{\partial V}{\partial y}\mathbf{a_y} + \frac{\partial V}{\partial z}\mathbf{a_z}$$

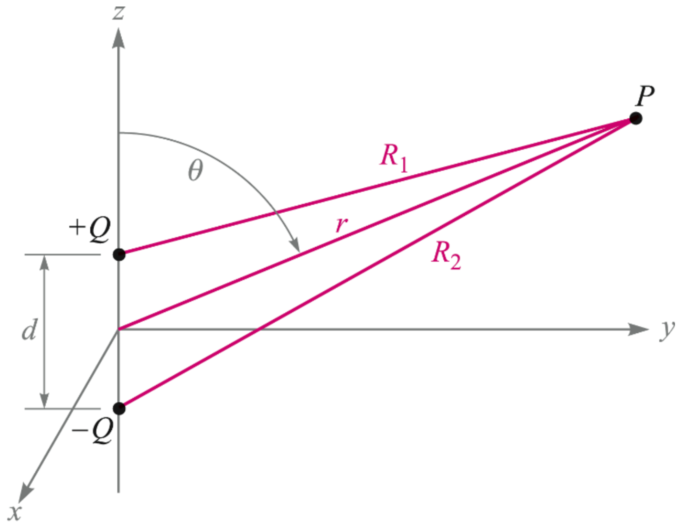
Cylindrical

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a_\rho} + \frac{\partial V}{\rho\partial \phi}\mathbf{a_\varphi} + \frac{\partial V}{\partial z}\mathbf{a_z}$$

Spherical

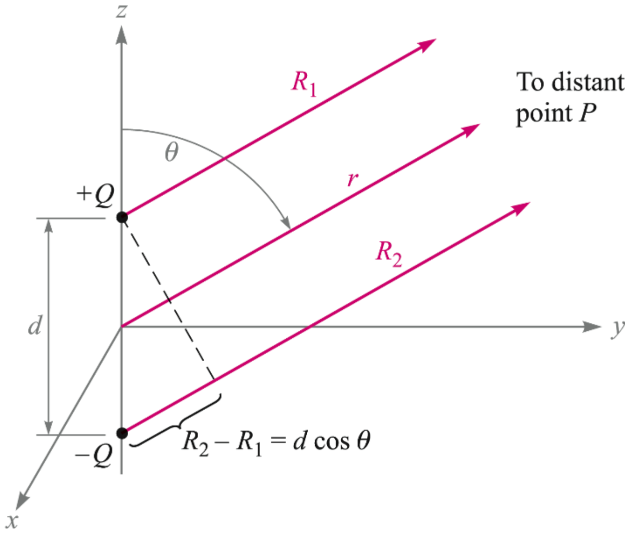
$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a_r} + \frac{\partial V}{r\partial \theta}\mathbf{a_\theta} + \frac{\partial V}{r\sin\theta\partial \phi}\mathbf{a_\varphi}$$

Electric dipole



$$V = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_o}\frac{R_2 - R_1}{R_1 R_2}$$

$$= \frac{Qd\cos\theta}{4\pi\varepsilon_o r^2}$$



$$\mathbf{E} = -\nabla V = \frac{Qd}{4\pi\varepsilon_o r^2}\left(2\cos\theta\mathbf{a_r} + \sin\theta\mathbf{a_\theta}\right)$$

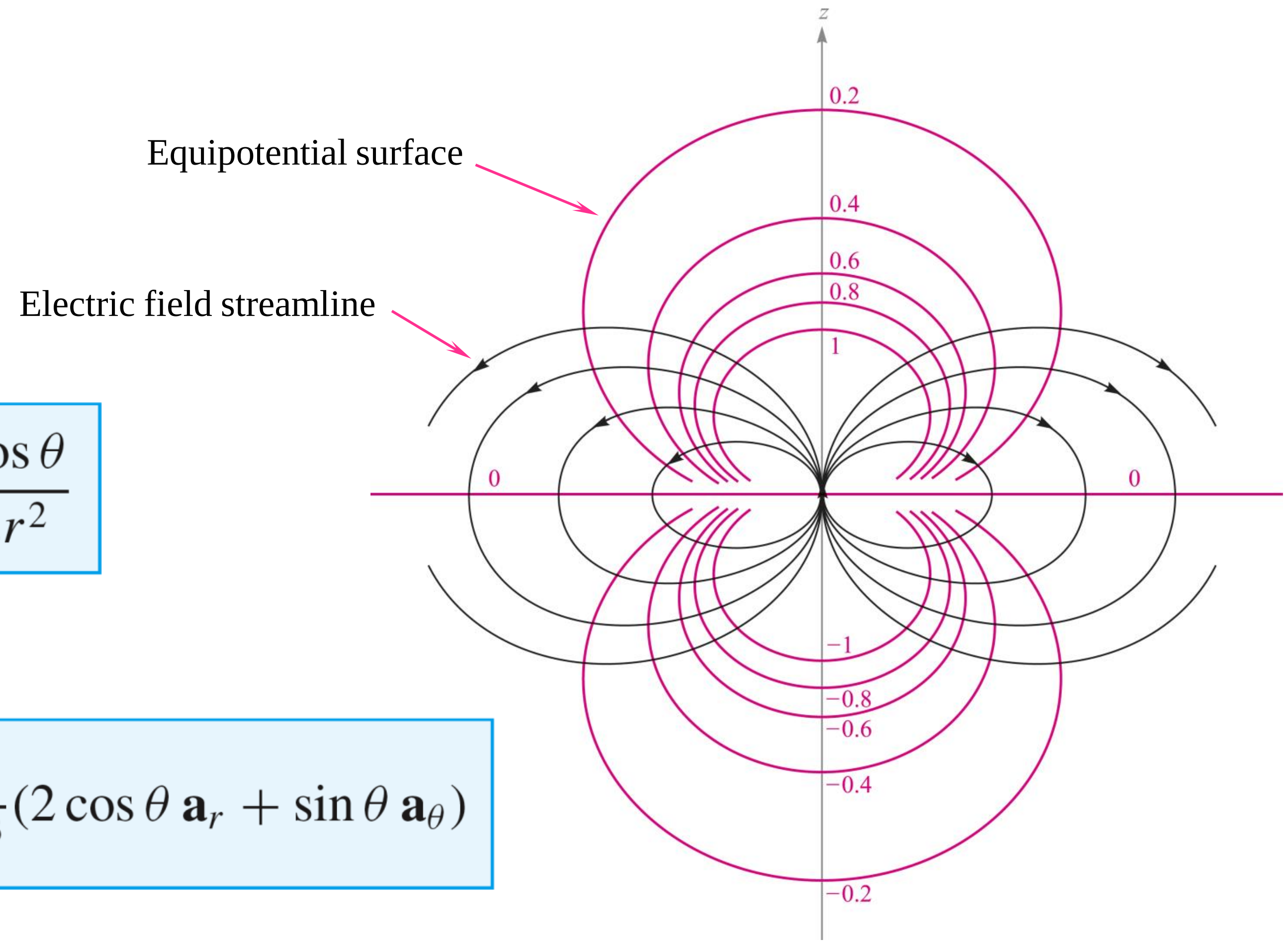
$\mathbf{p} = Q\mathbf{d}$ Dipole moment

$$V = \frac{\mathbf{p}\cdot\mathbf{a_r}}{4\pi\varepsilon_o r^2} = \frac{1}{4\pi\varepsilon_o|\mathbf{r}-\mathbf{r}'|^2}\mathbf{p}\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

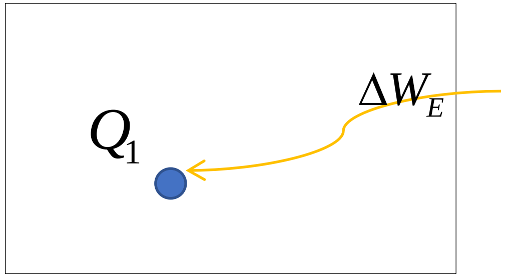
Electric dipole

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

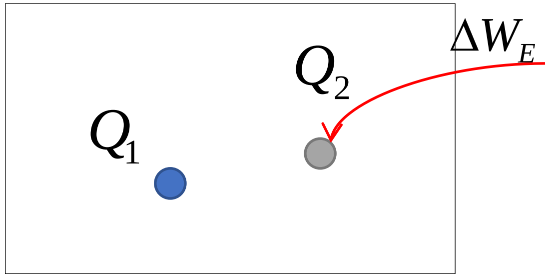


Work required to collect Qs

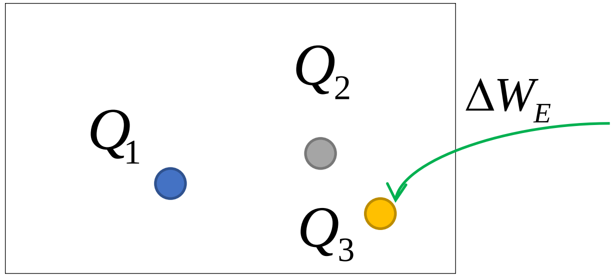


$$W_E = \Delta W_E = 0$$

Potential at Q2 position by
a point charge Q1



$$W_E = 0 + \Delta W_E = Q_2 V_{2,1}$$



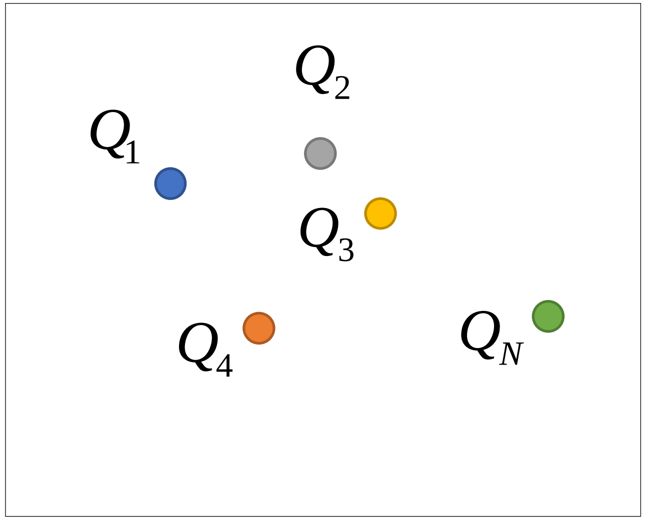
$$W_E = Q_2 V_{2,1} + \Delta W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2}$$

$$\vdots$$

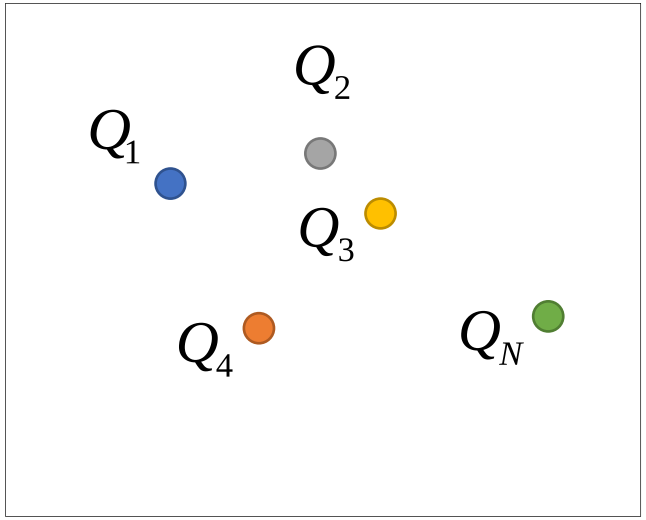
$$W_E = Q_2 V_{2,1} + Q_3 (V_{3,1} + V_{3,2}) + Q_4 (V_{4,1} + V_{4,2} + V_{4,3}) + \cdots + Q_N (V_{N,1} + V_{N,2} + \cdots + V_{N,N-1})$$

Work required to collect Qs

$$W_E = Q_2 V_{2,1} + Q_3(V_{3,1} + V_{3,2}) + Q_4(V_{4,1} + V_{4,2} + V_{4,3}) + \cdots + Q_N(V_{N,1} + V_{N,2} + \cdots + V_{N,N-1})$$



Work required to collect Qs



$$W_E = Q_2V_{2,1} + Q_3(V_{3,1} + V_{3,2}) + Q_4(V_{4,1} + V_{4,2} + V_{4,3}) + \cdots + Q_N(V_{N,1} + V_{N,2} + \cdots + V_{N,N-1}) \quad \dots (1)$$

Using $\quad Q_2V_{2,1} = Q_2\dfrac{Q_1}{4\pi\varepsilon_o R_{2,1}} = Q_1\dfrac{Q_2}{4\pi\varepsilon_o R_{1,2}} = Q_1V_{1,2}$

$$W_E = Q_1(V_{1,2} + V_{1,2} + \cdots + V_{1,N}) + Q_2(V_{2,3} + V_{2,4} + \cdots + V_{2,N}) + \cdots + Q_{N-1}(V_{N-1,N}) \quad \dots (2)$$

From (1) + (2)

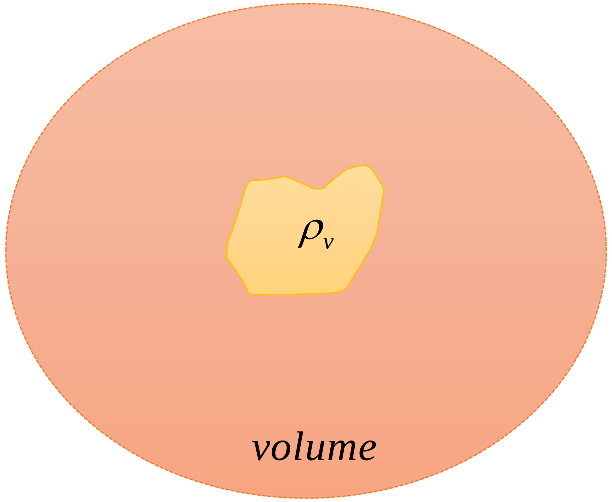
$$2W_E = Q_1\underbrace{(V_{1,2} + V_{1,3} + \cdots + V_{1,N})}_{V_1} + Q_2\underbrace{(V_{2,1} + V_{2,3} + \cdots + V_{2,N})}_{V_2} + \cdots + Q_N\underbrace{(V_{N,1} + V_{N,2} + \cdots + V_{N,N-1})}_{V_N}$$

$$\therefore W_E = \frac{1}{2}\sum_{m=1}^{N} Q_m V_m$$

$$\therefore W_E = \frac{1}{2}\int_{vol} \rho_v V dv$$

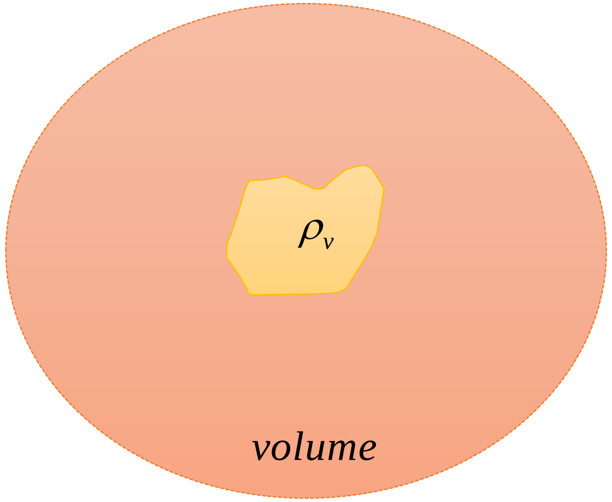
Stored energy in the electric field

$$W_E = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_{vol} \mathbf{D}\cdot\mathbf{E} dv = \frac{1}{2}\int_{vol} \varepsilon_o E^2 dv$$



Stored energy in the electric field

$$\boxed{W_E = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_{vol} \mathbf{D}\cdot\mathbf{E} dv = \frac{1}{2}\int_{vol} \varepsilon_o E^2 dv}$$



$$W_E = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_{vol} (\nabla\cdot\mathbf{D}) V dv = \frac{1}{2}\int_{vol} \nabla\cdot(\mathbf{D}V) - \mathbf{D}\cdot\nabla V dv$$

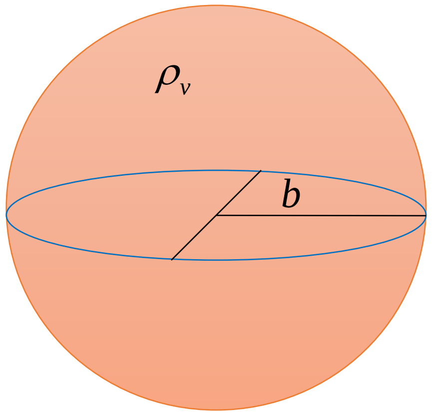
$$\nabla\cdot(\mathbf{D}V) = V(\nabla\cdot\mathbf{D}) + \mathbf{D}\cdot\nabla V$$

$$\oint_S \mathbf{D}V\cdot d\mathbf{S} \to 0 \ as\ R \to \infty$$

$$W_E = \frac{1}{2}\int_{vol} \nabla\cdot(\mathbf{D}V) - \mathbf{D}\cdot\nabla V dv = \overbrace{\frac{1}{2}\oint_S \mathbf{D}V\cdot d\mathbf{S}} - \frac{1}{2}\int_{vol} \mathbf{D}\cdot\nabla V dv = -\frac{1}{2}\int_{vol} \mathbf{D}\cdot\nabla V dv = \frac{1}{2}\int_{vol} \mathbf{D}\cdot\mathbf{E} dv$$

Example – Uniform charge cloud

$$W_E = ?$$



Chapter Summary

- Work done on moving a point charge against an external field
- Potential Difference
- Potential Field
  - 점전하
  - 선전하
  - 면전하
  - 부피전하
- Relation between Potential and Electric Field
- Electric Dipole
- Electric Energy