[과제 41] Integrate counterclockwise around the simple closed path C.

(1) $\displaystyle\oint_C (z/\bar{z})dz$, C the upper semicircle $|z| = 2$ from 2 to -2, the line segment $[-2,-1]$, the upper semicircle $|z| = 1$ from -1 to 1 and the line segment $[1,2]$.

(답)

$$\oint_C (z/\bar{z})dz = \int_0^\pi \frac{2e^{it}}{2e^{-it}} 2ie^{it}dt + \int_{-2}^{-1} 1dt - \int_0^\pi \frac{e^{it}}{e^{-it}} ie^{it}dt + \int_1^2 1dt$$

$$= 2i\int_0^\pi e^{3it}dt + 1 - i\int_0^\pi e^{3it}dt + 1$$

$$= i\int_0^\pi e^{3it}dt + 2 = i\left[\frac{1}{3i}e^{3it}\right]_0^\pi + 2 = \frac{1}{3}\left(e^{i3\pi} - 1\right) + 2 = -\frac{2}{3} + 2 = \frac{4}{3}$$

[과제 41] Integrate counterclockwise around the simple closed path C.

(2) $\displaystyle\oint_C \frac{z^3+3z-1}{(z-1)(z+2)}dz,\ C$ the circle $|z|=3$.

(답) 미분이 불가능한 점 $z=1, z=-2$는 모두 $C: |z|=3$ 내부에 있으므로 그 점들을 제거하기 위해, 각각 중심이 $z=1, z=-2$, 반지름이 r_1, r_2이고 겹치지 않고 반시계방향인 원 C_1과 C_2를 경계곡선으로 갖는 구멍을 내면, **코시 적분 공식**에 의해

$$\oint_{C:|z|=3}\frac{z^3+3z-1}{(z-1)(z+2)}dz = \oint_{C_1:|z-1|=r_1}\frac{z^3+3z-1}{(z-1)(z+2)}dz + \oint_{C_2:|z+2|=r_2}\frac{z^3+3z-1}{(z-1)(z+2)}dz$$

$$= \oint_{C_1:|z-1|=r_1}\frac{(z^3+3z-1)/(z+2)}{(z-1)}dz + \oint_{C_2:|z+2|=r_2}\frac{(z^3+3z-1)/(z-1)}{(z+2)}dz$$

$$= 2\pi i\left[\frac{z^3+3z-1}{z+2}\right]_{z=1} + 2\pi i\left[\frac{z^3+3z-1}{z-1}\right]_{z=-2}$$

$$= 2\pi i\left(\frac{1+3-1}{3}\right) + 2\pi i\left(\frac{-8-6-1}{-3}\right) = 2\pi i + 10\pi i = 12\pi i$$

[과제 41] Integrate counterclockwise around the simple closed path C.

(3) $\displaystyle\oint_C \frac{z}{(16-z^2)(z+i)}dz,\quad C$ the circle $|z+4|=2$.

(답) 미분이 불가능한 점 $z=\pm 4, z=-i$ 중 $z=-4$가 $|z+4|=2$ 안에 포함되므로 **코시 적분 공식**에 의해

$$\oint_C \frac{z}{(16-z^2)(z+i)}dz = \oint_C \frac{-z/(z-4)(z+i)}{z+4}dz = 2\pi i\left[\frac{-z}{(z-4)(z+i)}\right]_{z=-4}$$

$$= 2\pi i\left(\frac{4}{(-8)(-4+i)}\right) = \frac{\pi i}{4-i} = \frac{-\pi+4\pi i}{17}$$

[과제 41] Integrate counterclockwise around the simple closed path C.

(4) $\displaystyle\oint_C \frac{e^{-z}\cos z}{(z-2)^3}dz,$ C the circle $|z| = 3$.

(답) 미분이 불가능한 점 $z = 2$가 $|z| = 3$ 안에 포함되므로 **코시 적분 공식**에 의해

$$\oint_{C:|z|=3} \frac{e^{-z}\cos z}{(z-2)^3}dz = \frac{2\pi i}{2!}f''(2) = \pi i f''(2)$$

$$f(z) = e^{-z}\cos z,\ \ f'(z) = -e^{-z}\cos z - e^{-z}\sin z$$

$$f''(z) = e^{-z}\cos z + e^{-z}\sin z + e^{-z}\sin z - e^{-z}\cos z = 2e^{-z}\sin z$$

$$f''(2) = 2e^{-2}\sin 2$$

따라서 구하는 답은 $2\pi i e^{-2}\sin 2$

[과제 41] Integrate counterclockwise around the simple closed path C.

(5) $\displaystyle\oint_C (Re(z)+\alpha)\frac{f(z)}{z}dz,\quad C$ the circle $|z|=1$ and $\alpha \in \mathbb{C}$.

(답) $\displaystyle Re(z) = \frac{z+\bar{z}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$ $\qquad z\bar{z} = |z|^2 = 1$

$$\oint_C (Re(z)+\alpha)\frac{f(z)}{z}dz = \oint_C \left(\frac{z^2+1}{2z}+\alpha\right)\frac{f(z)}{z}dz = \oint_C \frac{\frac{1}{2}(z^2+1+2\alpha z)f(z)}{z^2}dz = 2\pi i g'(0)$$

$$g(z) = \frac{1}{2}(z^2+1+2\alpha z)f(z)$$

$$g'(z) = \frac{1}{2}(2z+2\alpha)f(z) + \frac{1}{2}(z^2+1+2\alpha z)f'(z)$$

$$g'(0) = \frac{1}{2}(2\alpha)f(0) + \frac{1}{2}(1)f'(0) = \alpha f(0) + \frac{1}{2}f'(0)$$

따라서 구하는 값은 $\pi i(2\alpha f(0) + f'(0))$

[과제 41] Integrate counterclockwise around the simple closed path C.

(6) $\displaystyle\oint_C (Im(z))^3\, dz, \quad C$ the circle $|z-1| = 1$.

(답1) $z - 1 = e^{it},\ z = 1 + e^{it}\ (0 \le t \le 2\pi),\ dz = ie^{it}dt,\ Im(z) = \sin t$

$$\oint_C (Im(z))^3\, dz = \int_0^{2\pi} \sin^3 t(ie^{it})dt = i\int_0^{2\pi} \sin^3 t(\cos t + i\sin t)dt$$

$$= i\int_0^{2\pi} (\sin^3 t\cos t + i\sin^4 t)dt = i\int_0^{2\pi}\left(\sin^3 t\cos t + i\left(\frac{1-\cos 2t}{2}\right)^2\right)dt$$

$$= i\left[\frac{1}{4}\sin^4 t\right]_0^{2\pi} - \int_0^{2\pi} \frac{1 - 2\cos 2t + \cos^2 2t}{4}dt$$

$$= -\frac{1}{4}\int_0^{2\pi}\left(1 - 2\cos 2t + \frac{1+\cos 4t}{2}\right)dt$$

$$= -\frac{1}{4}\left[\frac{3}{2}t - \sin 2t + \frac{1}{8}\sin 4t\right]_0^{2\pi} = -\frac{3\pi}{4}$$

[과제 41] Integrate counterclockwise around the simple closed path C.

(6) $\oint_C (Im(z))^3\, dz$, C the circle $|z-1| = 1$.

$(z-1)(\overline{z-1}) = |z-1|^2 = 1$

$e^{-it} = \overline{z-1} = \frac{1}{z-1}$

(답2) $$z - 1 = e^{it},\ z = 1 + e^{it},\ \bar{z} = 1 + e^{-it} = 1 + \frac{1}{z-1} = \frac{z}{z-1}$$

$$Im(z-1) = Im(z) = \frac{z - \bar{z}}{2i} = \frac{z - \frac{z}{z-1}}{2i} = \frac{1}{2i}\frac{z^2 - 2z}{z-1}$$

$$\oint_{C:|z-1|=1} (Im(z))^3\, dz = \oint_{C:|z-1|=1} \left(\frac{1}{2i}\frac{z^2-2z}{z-1}\right)^3 dz = \frac{1}{-8i}\oint_{C:|z-1|=1} \frac{(z^2-2z)^3}{(z-1)^3} dz \ = \frac{i}{8}\frac{2\pi i}{2!} f''(1)$$

$$f(z) = (z^2 - 2z)^3,\ f'(z) = 3(z^2-2z)^2(2z-2)$$

$$f''(z) = 6(z^2-2z)(2z-2)^2 + 3(z^2-2z)^2(2),\ f''(1) = 3(-1)^2(2) = 6$$

따라서 구하는 답은 $\frac{i}{8}\frac{2\pi i}{2!} f''(1) = \frac{i}{8}\frac{2\pi i}{2!} 6 = -\frac{6\pi}{8} = -\frac{3\pi}{4}$