[**과제 1**] 다음의 각 함수에서 **기본 주기**를 구하여라.

$$\cos 2x, \quad \sin \pi x, \quad \cos \frac{2\pi n x}{T}, \quad \sin 2k\pi x$$

(풀이) $\frac{2\pi}{2} = \pi$

$\frac{2\pi}{\pi} = 2$

$\frac{2\pi}{\frac{2\pi n}{T}} = \frac{T}{n}$

$\frac{2\pi}{2k\pi} = \frac{1}{k}$

[**과제 2**] 함수 $f(x) = x$와 $g(x) = x^2$은 구간 $[-1, 1]$에서 서로 **직교**임을 보이고, 이 두 함수에 대해 $h(x) = x + ax^2 + bx^3$가 직교가 되도록 a, b를 구하여라.

(풀이) $\int_{-1}^{1}(x)(x^2)dx = \int_{-1}^{1} x^3 dx = 0$

$\int_{-1}^{1} x(x + ax^2 + bx^3)dx = 2\int_{0}^{1}(x^2 + bx^4)dx = 2\left(\frac{1}{3} + \frac{b}{5}\right) = 0, \ b = -\frac{5}{3}$

$\int_{-1}^{1} x^2(x + ax^2 + bx^3)dx = 2\int_{0}^{1}(ax^4)dx = 2\left(\frac{a}{5}\right) = 0, \ a = 0$

[**과제 3**] 임의의 실수 a와 양의 실수 p에 대하여 집합 $\left\{1, \cos\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right\}_{n,m=1}^{\infty}$는 구간 $[a, a+2p]$의 **직교 집합**임을 보이고, 각 함수의 **노옴**을 구하여라.

(풀이) $\displaystyle\int_a^{a+2p} \cos\frac{n\pi}{p}x\,dx = \frac{p}{n\pi}\left[\sin\frac{n\pi}{p}x\right]_a^{a+2p} = 0 \qquad \int_a^{a+2p} \sin\frac{m\pi}{p}x\,dx = -\frac{p}{m\pi}\left[\cos\frac{m\pi}{p}x\right]_a^{a+2p} = 0$

$$\int_a^{a+2p} \cos\frac{n\pi}{p}x\cos\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_a^{a+2p}\left(\cos\frac{(m+n)\pi}{p}x + \cos\frac{(m-n)\pi}{p}x\right)dx = 0\ (n \neq m)$$

$$\int_a^{a+2p} \sin\frac{n\pi}{p}x\sin\frac{m\pi}{p}x\,dx = -\frac{1}{2}\int_a^{a+2p}\left(\cos\frac{(m+n)\pi}{p}x - \cos\frac{(m-n)\pi}{p}x\right)dx = 0\ (n \neq m)$$

$$\int_a^{a+2p} \cos\frac{n\pi}{p}x\sin\frac{m\pi}{p}x\,dx = \frac{1}{2}\int_a^{a+2p}\left(\sin\frac{(m+n)\pi}{p}x + \sin\frac{(m-n)\pi}{p}x\right)dx = 0$$

노옴: $\displaystyle\int_a^{a+2p} 1^2\,dx = 2p \ \Rightarrow\ \|1\| = \sqrt{2p}$

$$\int_a^{a+2p} \cos^2\frac{n\pi}{p}x\,dx = \int_a^{a+2p}\frac{1+\cos\frac{2n\pi}{p}x}{2}dx = p \ \Rightarrow\ \|\cos\frac{n\pi}{p}x\| = \sqrt{p}$$

$$\int_a^{a+2p} \sin^2\frac{m\pi}{p}x\,dx = \int_a^{a+2p}\frac{1-\cos\frac{2m\pi}{p}x}{2}dx = p \ \Rightarrow\ \|\sin\frac{m\pi}{p}x\| = \sqrt{p}$$

[**과제 4**] 주기가 2π인 주기함수 $f(x)=x^3-\pi^2 x$에 대하여

(1) $f(x)$의 푸리에 급수를 구하여라.

(2) (1)의 결과를 이용하여 무한급수 $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots$의 합을 구하여라.

(3) Parseval's identity를 이용하여 $\zeta(6)=\sum_{n=1}^{\infty}\frac{1}{n^6}$의 값을 구하여라.

(풀이) $f(x)$는 기함수, $2L=2\pi,\ L=\pi$

$$b_n=\frac{2}{\pi}\int_0^{\pi}(x^3-\pi^2x)\sin nx\,dx=\frac{2}{\pi}\left\{\left[(x^3-\pi^2x)\frac{-1}{n}\cos nx\right]_0^{\pi}+\frac{1}{n}\int_0^{\pi}(3x^2-\pi^2)\cos nx\,dx\right\}=\frac{2}{n\pi}\int_0^{\pi}(3x^2-\pi^2)\cos nx\,dx$$

$$=\frac{2}{n\pi}\left\{\left[(3x^2-\pi^2)\frac{1}{n}\sin nx\right]_0^{\pi}-\frac{1}{n}\int_0^{\pi}(6x)\sin nx\,dx\right\}=-\frac{12}{n^2\pi}\int_0^{\pi}x\sin nx\,dx=-\frac{12}{n^2\pi}\left\{\left[x\frac{-1}{n}\cos nx\right]_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\right\}=\frac{12(-1)^n}{n^3}$$

$$\boxed{f(x)=\sum_{n=1}^{\infty}\frac{12(-1)^n}{n^3}\sin nx}$$

$x=\frac{\pi}{2}$을 대입하면 $\frac{\pi^3}{8}-\frac{\pi^3}{2}=f\left(\frac{\pi}{2}\right)=\sum_{n=1}^{\infty}\frac{12(-1)^n}{n^3}\sin\frac{n\pi}{2}=12\left\{-1+\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{7^3}-\cdots\right\}$ $\quad \therefore\ 1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}$

파르스발 항등식 $\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx=2a_0^2+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$에 의해서

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2\,dx=\frac{2}{\pi}\int_0^{\pi}(x^3-\pi^2x)^2dx=\frac{16}{105}\pi^6,\quad \sum_{n=1}^{\infty}b_n^2=\sum_{n=1}^{\infty}\frac{144}{n^6},\qquad \therefore\ \sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}$$

[과제 5] 주기가 2π인 주기함수 $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$ 에 대하여

(1) $f(x)$의 푸리에 급수를 구하여라.

(2) (1)의 결과를 이용하여 $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$의 값을 구하여라.

(3) Parseval's identity를 이용하여 $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$의 값을 구하여라.

(4) (3)의 결과를 이용하여 $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4}$의 값을 구하여라.

(풀이) $f(x)$는 우함수, $2L = 2\pi,\ L = \pi$ $\quad a_0 = \frac{1}{\pi}\int_0^{\pi}(\pi - x)dx = \frac{\pi}{2}$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi-x)\cos nx dx = \frac{2}{\pi}\left\{\left[(\pi - x)\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi} -\frac{1}{n}\sin nx dx\right\} = \frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{n^2\pi}\left(1 - (-1)^n\right)$$

$$\boxed{f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos nx = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x}$$

$x = 0$을 대입하면 $\quad \pi = f(0) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ $\qquad \therefore \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

파르스발 항등식 $\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$에 의해서

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\{f(x)\}^2 dx = \frac{2}{\pi}\int_0^{\pi}(\pi-x)^2 dx = \frac{2}{3}\pi^2,\ \ 2a_0^2 + \sum_{n=1}^{\infty}a_n^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4},$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{n=1}^{\infty}\frac{1}{(2n)^4} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{16}{15}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{16}{15}\frac{\pi^4}{96} = \frac{\pi^4}{90}$$

[과제 6] $f(x) = |x| \ (-2 < x < 2), \ f(x+4) = f(x)$의 **푸리에 급수**를 구하고,
이를 이용하여 $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n+1)^2} + \cdots$의 합을 구하여라.

(풀이) 우함수, $2L = 4, \ L = 2$

$$a_0 = \frac{1}{2}\int_0^2 x dx = \frac{1}{2}\left[\frac{1}{2}x^2\right]_0^2 = 1 \qquad a_n = \frac{2}{2}\int_0^2 x\cos\frac{n\pi}{2}x dx = \left[x\frac{2}{n\pi}\sin\frac{n\pi}{2}x\right]_0^2 - \int_0^2 \frac{2}{n\pi}\sin\frac{n\pi}{2}x dx$$

$$= -\frac{2}{n\pi}\left[-\frac{2}{n\pi}\cos\frac{n\pi}{2}x\right]_0^2 = -\frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$f(x) = 1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi}{2}x$$

$$= 1 - \frac{8}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi}{2}x + \frac{1}{3^2}\cos\frac{3\pi}{2}x + \frac{1}{5^2}\cos\frac{5\pi}{2}x + \cdots\right\}$$

$x = 0$을 대입

$$0 = f(0) = 1 - \frac{8}{\pi^2}\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} \qquad \therefore \quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + = \frac{\pi^2}{8}$$

[과제 7] 함수 $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi \end{cases}$; $f(x+2\pi) = f(x)$의 **푸리에 급수**를 구하고, 이를 이용하여 무한급수 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$의 값을 구하여라.

(풀이) $2L = 2\pi,\ L = \pi$ $\qquad a_0 = \frac{1}{2\pi}\left\{\int_{-\pi}^{0} \pi^2 dx + \int_0^{\pi} (\pi^2 - x^2) dx\right\} = \frac{1}{2\pi}\left\{\pi^3 + \frac{2}{3}\pi^3\right\} = \frac{5}{6}\pi^2$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} \pi^2 \cos nx dx + \int_0^{\pi} (\pi^2 - x^2) \cos nx dx\right\} = \frac{1}{\pi}\left\{\left[(\pi^2 - x^2)\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi} (-2x)\frac{1}{n}\sin nx dx\right\}$$

$$= \frac{2}{n\pi}\left\{\left[x\frac{-1}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi} \frac{-1}{n}\cos nx dx\right\} = -\frac{2}{n^2}(-1)^n = (-1)^{n+1}\frac{2}{n^2}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} \pi^2 \sin nx dx + \int_0^{\pi} (\pi^2 - x^2) \sin nx dx\right\} = -\frac{\pi}{n}(1-(-1)^n) + \frac{1}{\pi}\left\{\left[(\pi^2 - x^2)\frac{-1}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi} (-2x)\frac{-1}{n}\cos nx dx\right\}$$

$$= -\frac{\pi}{n}(1-(-1)^n) + \frac{\pi}{n} - \frac{2}{n\pi}\left\{\left[x\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi} \frac{1}{n}\sin nx dx\right\} = \frac{\pi}{n}(-1)^n + \frac{2}{n^2\pi}\left[\frac{-1}{n}\cos nx\right]_0^{\pi} = (-1)^n\frac{\pi}{n} + \frac{2}{n^3\pi}(1-(-1)^n)$$

$$f(x) = \frac{5}{6}\pi^2 + \sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{2}{n^2}\cos nx + \left((-1)^n\frac{\pi}{n} + \frac{2}{n^3\pi}(1-(-1)^n)\right)\sin nx\right]$$

$x = 0$을 대입하면 $\qquad \pi^2 = f(0) = \frac{5}{6}\pi^2 + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} \qquad \therefore \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

[과제 8] 함수 $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ 2, & 0 < x \le 2 \end{cases}$; $f(x+4) = f(x)$의 푸리에 급수를 구하고, 이를 이용하여 무한급수 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$의 값을 구하여라.

(풀이) $2L = 4,\ L = 2$ $\qquad a_0 = \dfrac{1}{4}\left\{\int_{-2}^{0}(-1)dx + \int_{0}^{2} 2dx\right\} = \dfrac{1}{4}\{-2+4\} = \dfrac{1}{2}$

$$a_n = \frac{1}{2}\left\{\int_{-2}^{0}(-1)\cos\frac{n\pi}{2}xdx + \int_{0}^{2} 2\cos\frac{n\pi}{2}xdx\right\} = 0$$

$$b_n = \frac{1}{2}\left\{\int_{-2}^{0}(-1)\sin\frac{n\pi}{2}xdx + \int_{0}^{2} 2\sin\frac{n\pi}{2}xdx\right\} = \frac{3}{n\pi}\left(1-(-1)^n\right)$$

$$f(x) = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{3}{n\pi}\left(1-(-1)^n\right)\sin\frac{n\pi}{2}x = \frac{1}{2}+\frac{6}{\pi}\left(\frac{1}{1}\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right)$$

$x=1$을 대입하면 $\qquad 2 = f(1) = \dfrac{1}{2}+\dfrac{6}{\pi}\left(\dfrac{1}{1}\sin\dfrac{\pi}{2} + \dfrac{1}{3}\sin\dfrac{3\pi}{2} + \dfrac{1}{5}\sin\dfrac{5\pi}{2} + \cdots\right) = \dfrac{1}{2}+\dfrac{6}{\pi}\left(1-\dfrac{1}{3}+\dfrac{1}{5}-\cdots\right)$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = 1-\frac{1}{3}+\frac{1}{5}-\cdots = \frac{\pi}{4}$$

[과제 9] 함수 $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$; $f(x+2\pi) = f(x)$의 푸리에 급수를 구하고, 이를 이용하여 무한급수 $\frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \cdots$의 값을 구하여라.

(풀이) $2L = 2\pi,\ L = \pi$ $\qquad a_0 = \dfrac{1}{2\pi}\displaystyle\int_0^\pi \sin x dx = \dfrac{1}{\pi}$

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin x \cos x dx = \frac{1}{2}\left[\sin^2 x\right]_0^\pi = 0$$

$$a_{n\neq 1} = \frac{1}{\pi}\int_0^\pi \sin x \cos nx dx = \frac{1}{\pi}\int_0^\pi \frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right] dx = \frac{1}{2\pi}\left\{\left[\frac{-1}{(1+n)}\cos(1+n)x + \frac{-1}{(1-n)}\cos(1-n)x\right]_0^\pi\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-1}{(1+n)}(-(-1)^n - 1) + \frac{-1}{(1-n)}(-(-1)^n - 1)\right\} = \frac{1}{\pi}\frac{1}{1-n^2}\left(1 + (-1)^n\right)$$

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin x \sin x dx = \frac{1}{\pi}\int_0^\pi \frac{1-\cos 2x}{2} dx = \frac{1}{2}$$

$$b_{n\neq 1} = \frac{1}{\pi}\int_0^\pi \sin x \sin nx dx = \frac{1}{\pi}\int_0^\pi \frac{-1}{2}\left[\cos(1+n)x - \cos(1-n)x\right] dx = 0$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{1+(-1)^n}{n^2-1}\cos nx = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\left(\frac{1}{3}\cos 2x + \frac{1}{15}\cos 4x + \frac{1}{35}\cos 6x + \cdots\right)$$

$x = 0$을 대입하면 $\qquad 0 = f(0) = \dfrac{1}{\pi} - \dfrac{1}{\pi}\displaystyle\sum_{n=2}^{\infty}\dfrac{1+(-1)^n}{n^2-1} = \dfrac{1}{\pi} - \dfrac{2}{\pi}\left(\dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \cdots\right)$ $\qquad \therefore \quad \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \cdots = 1$

[과제 10] 함수 $f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 0 \\ 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$ 의 푸리에 적분을 구하여라.

(풀이)

$f(x) = \int_0^\infty \left(A(w)\cos wx + B(w)\sin wx\right) dw$
$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx dx$
$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx dx$

$$f(x) = \int_0^\infty \left\{\frac{1}{\pi}\frac{\sin w}{w}\cos wx + \frac{3}{\pi}\frac{1-\cos w}{w}\sin wx\right\} dw$$

$$A(w) = \frac{1}{\pi}\int_{-1}^{0}(-1)\cos wx dx + \frac{1}{\pi}\int_0^1 (2)\cos wx dx$$
$$= \frac{1}{\pi}\left\{\left[\frac{-1}{w}\sin wx\right]_{-1}^{0} + 2\left[\frac{1}{w}\sin wx\right]_0^1\right\} = \frac{1}{\pi}\frac{\sin w}{w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{0}(-1)\sin wx dx + \frac{1}{\pi}\int_0^1 (2)\sin wx dx$$
$$= \frac{1}{\pi}\left\{\left[\frac{1}{w}\cos wx\right]_{-1}^{0} + 2\left[-\frac{1}{w}\cos wx\right]_0^1\right\} = \frac{3}{\pi}\frac{1-\cos w}{w}$$

[과제 11] 함수 $f(x) = e^{-|x|}\sin x$의 푸리에 사인 적분 또는 코사인 적분을 구하여라.

(풀이) $f(x)$는 기함수이므로

$$f(x) = \int_0^\infty B(w)\sin wx dw,\ \ B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin wx dx$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{4w\sin wx}{(1+(1+w)^2)(1+(1-w)^2)}dw$$

$$\begin{aligned}
B(w) &= \frac{2}{\pi}\int_0^\infty e^{-x}\sin x\sin wx dx \\
&= \frac{2}{\pi}\int_0^\infty e^{-x}\frac{-\cos(1+w)x+\cos(1-w)x}{2}dx \\
&= -\frac{1}{\pi}\int_0^\infty e^{-x}\cos(1+w)x dx + \frac{1}{\pi}\int_0^\infty e^{-x}\cos(1-w)x dx \\
&= -\frac{1}{\pi}L(\cos(1+w)x)|_{s=1} + \frac{1}{\pi}L(\cos(1-w)x)|_{s=1} \\
&= -\frac{1}{\pi}\frac{1}{1+(1+w)^2} + \frac{1}{\pi}\frac{1}{1+(1-w)^2} = \frac{1}{\pi}\frac{4w}{(1+(1+w)^2)(1+(1-w)^2)}
\end{aligned}$$

[과제 12] 함수 $f(x) = \begin{cases} |x|, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$ 의 푸리에 사인 적분 또는 코사인 적분을 구하여라.

(풀이) $f(x)$는 우함수이므로

$$f(x) = \int_0^\infty A(w)\cos wx dw, \;\; A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos wx dx$$

$$\begin{aligned} A(w) &= \frac{2}{\pi}\int_0^\pi x\cos wx dx \\ &= \frac{2}{\pi}\left\{\left[x\frac{1}{w}\sin wx\right]_0^\pi - \int_0^\pi \frac{1}{w}\sin wx dx\right\} \\ &= \frac{2}{\pi}\left\{\frac{\pi}{w}\sin w\pi - \frac{1}{w}\left[-\frac{1}{w}\cos wx\right]_0^\pi\right\} \\ &= \frac{2}{\pi}\left\{\frac{\pi}{w}\sin w\pi + \frac{1}{w^2}(\cos w\pi - 1)\right\} \\ &= \frac{2}{w}\sin w\pi + \frac{2}{\pi w^2}(\cos w\pi - 1) \end{aligned}$$

$$f(x) = \int_0^\infty \left\{\frac{2}{w}\sin w\pi + \frac{2}{\pi w^2}(\cos w\pi - 1)\right\}\cos wx dw$$

[과제 13] 함수 $f(x) = xe^{-2x}\ (x > 0)$의 푸리에 사인 적분과 코사인 적분을 구하여라.

(풀이) 푸리에 코사인 적분

$$A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos wx dx = \frac{2}{\pi}\int_0^\infty xe^{-2x}\cos wx dx = \frac{2}{\pi}L(x\cos wx)|_{s=2}$$

$$g(x) = x\cos wx,\ \ g'(x) = \cos wx - wx\sin wx,$$

$$g''(x) = -w\sin wx - w\sin wx - w^2 x\cos wx = -2w\sin wx - w^2 x\cos wx$$

$$L(g'') = -2wL(\sin wx) - w^2 L(g) = -2w\tfrac{w}{s^2+w^2} - w^2 L(g)$$

$$L(g'') = s^2 L(g) - sg(0) - g'(0) = s^2 L(g) - 1$$

$$\tfrac{-2w^2}{s^2+w^2} - w^2 L(g) = s^2 L(g) - 1$$

$$L(g) = \left(\tfrac{-2w^2}{s^2+w^2} + 1\right)\tfrac{1}{s^2+w^2} = \tfrac{s^2-w^2}{(s^2+w^2)^2}$$

$$L(g)|_{s=2} = \tfrac{4-w^2}{(4+w^2)^2},\ \ A(w) = \tfrac{2}{\pi}\tfrac{4-w^2}{(4+w^2)^2}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{4-w^2}{(4+w^2)^2}\cos wxdw$$

[과제 13] 함수 $f(x) = xe^{-2x}\ (x > 0)$의 푸리에 사인 적분과 코사인 적분을 구하여라.

(풀이) 푸리에 사인 적분

$$B(w) = \frac{2}{\pi}\int_0^\infty f(x)\sin wx dx = \frac{2}{\pi}\int_0^\infty xe^{-2x}\sin wx dx = \frac{2}{\pi}L(x\sin wx)|_{s=2}$$

$$g(x) = x\sin wx,\ \ g'(x) = \sin wx + wx\cos wx,$$

$$g''(x) = w\cos wx + w\cos wx - w^2 x\sin wx = 2w\cos wx - w^2 x\sin wx$$

$$L(g'') = 2wL(\cos wx) - w^2L(g) = 2w\tfrac{s}{s^2+w^2} - w^2L(g)$$

$$L(g'') = s^2L(g) - sg(0) - g'(0) = s^2L(g)$$

$$\tfrac{2ws}{s^2+w^2} - w^2L(g) = s^2L(g)$$

$$L(g) = \tfrac{2ws}{s^2+w^2}\tfrac{1}{s^2+w^2}$$

$$L(g)|_{s=2} = \tfrac{4w}{(4+w^2)^2},\ \ B(w) = \tfrac{8w}{\pi(4+w^2)^2}$$

$$f(x) = \frac{8}{\pi}\int_0^\infty \frac{w}{(4+w^2)^2}\sin wx dw$$

[과제 14] 적분 방정식 $\int_0^\infty f(x)\cos wx dx = e^{-w}$의 해 f를 구하여라.

(풀이) $f(x)$를 $(-\infty, \infty)$에서 우함수로 생각하면,

$$f(x) = \int_0^\infty A(w)\cos wx dw, \quad A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos wx dx$$

이므로

$$\int_0^\infty f(x)\cos wx dx = \frac{\pi}{2}A(w) = e^{-w}$$

이다. 따라서 $A(w) = \frac{2}{\pi}e^{-w}$이므로 구하는 해 $f(x)$는 다음과 같다.

$$f(x) = \int_0^\infty \frac{2}{\pi}e^{-w}\cos wx dw = \frac{2}{\pi}L(\cos wx)|_{s=1} = \frac{2}{\pi}\frac{1}{1+x^2}$$

[과제 15] $f(x) = \frac{1}{1+x^2}$의 푸리에 변환 $\hat{f}(w)$을 구하여라.

(풀이) $f(x)$가 우함수이므로

$$\hat{f}(w) = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos wx dx \quad = \sqrt{\frac{2}{\pi}} \frac{\pi}{2} e^{-|w|} = \sqrt{\frac{\pi}{2}} e^{-|w|}$$

516페이지 Laplace Integral : $\displaystyle \int_0^\infty \frac{\cos wx}{k^2+w^2} dw = \frac{\pi}{2k} e^{-kx} \quad (k > 0, x > 0)$

[과제 16] $f(x) = \begin{cases} 1-\frac{|x|}{a}, & |x| \le a \\ 0, & |x| > a \end{cases}$ 에 대하여 $f(x)$의 푸리에 변환 $\hat{f}(w)$을 구하고 $\lim_{w\to 0}\hat{f}(w)$의 값을 구하여라.

(풀이) $f(x)$가 우함수이므로

$$\begin{aligned}
\hat{f}(w) = \hat{f}_c(w) &= \sqrt{\frac{2}{\pi}}\int_0^a \left(1-\frac{x}{a}\right)\cos wx dx \\
&= \sqrt{\frac{2}{\pi}}\left\{\left[\left(1-\frac{x}{a}\right)\frac{1}{w}\sin wx\right]_0^a - \int_0^a \frac{-1}{a}\frac{1}{w}\sin wx dx\right\} \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{aw}\left[-\frac{1}{w}\cos wx\right]_0^a \\
&= \sqrt{\frac{2}{\pi}}\frac{1}{aw^2}(1-\cos aw)
\end{aligned}$$

$$\lim_{w\to 0}\hat{f}(w) = \lim_{w\to 0}\sqrt{\frac{2}{\pi}}\frac{1-\cos aw}{aw^2} = \sqrt{\frac{2}{\pi}}\lim_{w\to 0}\frac{a\sin aw}{2aw} = \sqrt{\frac{2}{\pi}}\frac{a}{2} = \frac{a}{\sqrt{2\pi}}$$

[과제 17] 다음 적분방정식의 해 $f(x)$를 구하여라.

$$\int_0^\infty f(x)\cos wx\,dx = \begin{cases} 1-w, & 0 \le w \le 1 \\ 0, & w > 1 \end{cases}$$

(풀이) $f(x)$를 $(-\infty,\infty)$에서 우함수로 생각하면,

$$\sqrt{\frac{\pi}{2}}\hat{f}_c(w) = \int_0^\infty f(x)\cos wx\,dx = \begin{cases} 1-w, & 0 \le w \le 1 \\ 0, & w > 1 \end{cases}$$

$$\hat{f}_c(w) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-w), & 0 \le w \le 1 \\ 0, & w > 1 \end{cases}$$

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\,dw = \sqrt{\frac{2}{\pi}}\int_0^1 \sqrt{\frac{2}{\pi}}(1-w)\cos wx\,dw \\ &= \frac{2}{\pi}\left\{\left[(1-w)\frac{1}{x}\sin wx\right]_0^1 - \int_0^1 (-1)\frac{1}{x}\sin wx\,dw\right\} \\ &= \frac{2}{\pi}\left\{\frac{1}{x}\left[-\frac{1}{x}\cos wx\right]_0^1\right\} \quad = \frac{2(1-\cos x)}{\pi x^2} \end{aligned}$$

[과제 18] $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$ 의 푸리에 코사인 변환 $\hat{f}_c(w)$을 구하여라.

(풀이)

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos wx dx = \sqrt{\frac{2}{\pi}} L(\cos wx)|_{s=1} \quad = \sqrt{\frac{2}{\pi}} \frac{1}{1+w^2}$$