[과제1]

$\cos 2x \rightarrow \frac{2\pi}{2} = \pi$

$\sin \pi x \rightarrow \frac{2\pi}{\pi} = 2$

$\cos \frac{2\pi n x}{T} \rightarrow \frac{2\pi}{\frac{2\pi n}{T}} = \frac{T}{n}$

$\sin 2k\pi x \rightarrow \frac{2\pi}{2k\pi} = \frac{1}{k}$

[과제2]

$f(x)$와 $g(x)$가 $[-1,1]$에서 서로 직교임을 보이려면

$[-1,1]$에서 $(f(x), g(x))$가 0임을 보이면 된다.

$(f(x), g(x)) = \int_{-1}^{1} f(x)\cdot g(x)\, dx = \int_{-1}^{1} x\cdot x^2\, dx = \int_{-1}^{1} x^3\, dx.$

$= 0$ ($\because$ 기함수)

즉, f와 g는 $[-1,1]$에서 서로 직교이다.

f와 h, g와 h가 직교이기 위해선

① $(f, h) = \int_{-1}^{1} f(x)\cdot h(x)\, dx = 0$

② $(g, h) = \int_{-1}^{1} g(x)\cdot h(x)\, dx = 0$

이어야 한다.

따라서 ① $\int_{-1}^{1} x(x+ax^2+bx^3)\, dx = \int_{-1}^{1} x^2+ax^3+bx^4\, dx$

$= 2\int_{0}^{1} x^2+bx^4\, dx = 2\left[\frac{1}{3}x^3+\frac{b}{5}x^5\right]_0^1 = 2\left(\frac{1}{3}+\frac{b}{5}\right) = 0$

즉, $\frac{1}{3}+\frac{b}{5} = 0$ 이므로 $b = -\frac{5}{3}$

② $\int_{-1}^{1} x^2(x+ax^2+bx^3)\, dx = \int_{-1}^{1} x^3+ax^4+bx^5\, dx$

$= 2\int_{0}^{1} ax^4\, dx = 2\left[\frac{a}{5}x^5\right]_0^1 = \frac{2}{5}a = 0$ 이므로 $a=0$.

$\therefore$ $a=0$, $b=-\frac{5}{3}$

[과제3] 제시된 집합이 $[a, a+2p]$에서 직교임을 보이기 위해

$\int_a^{a+2p} 1 \cdot \cos\frac{n\pi}{p}x\, dx$ ① $\int_a^{a+2p} \sin\frac{n\pi}{p}x \cdot \sin\frac{m\pi}{p}x\, dx$ ④

$\int_a^{a+2p} 1 \cdot \sin\frac{m\pi}{p}x\, dx$ ② $\int_a^{a+2p} \cos\frac{n\pi}{p}x \cdot \cos\frac{m\pi}{p}x\, dx$ ⑤

$\int_a^{a+2p} \cos\frac{n\pi}{p}x \cdot \sin\frac{m\pi}{p}x$ ③ 가 모두 0임을 보이자.

① $\int_a^{a+2p} \cos\frac{n\pi}{p}x\, dx$

$$= \frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_a^{a+2p} = \frac{p}{n\pi}\left(\sin\frac{n\pi}{p}(a+2p) - \sin\frac{n\pi}{p}a\right)$$

$$= \frac{p}{n\pi}\left(\sin\left(2n\pi + \frac{an}{p}\pi\right) - \sin\frac{an}{p}\pi\right)$$

$$= \frac{p}{n\pi}\left(\sin\frac{an}{p}\pi - \sin\frac{an}{p}\pi\right) = 0$$

② $\int_a^{a+2p} \sin\frac{m\pi}{p}x\, dx$

$$= \frac{-p}{m\pi}\cos\frac{m\pi}{p}x\Big|_a^{a+2p} = -\frac{p}{m\pi}\left(\cos\frac{m\pi}{p}(a+2p) - \cos\frac{m\pi}{p}a\right)$$

$$= -\frac{p}{m\pi}\left(\cos\left(2m\pi + \frac{am}{p}\pi\right) - \cos\frac{am}{p}\pi\right)$$

$$= -\frac{p}{m\pi}\left(\cos\frac{am}{p}\pi - \cos\frac{am}{p}\pi\right) = 0$$

③ $\int_a^{a+2p} \cos\frac{n\pi}{p}x \cdot \sin\frac{m\pi}{p}x\, dx$

$$= \int_a^{a+2p} \frac{1}{2}\left(\sin\frac{\pi}{p}(m+n)x + \sin\frac{\pi}{p}(m-n)x\right) dx$$

↳ 삼각함수 덧셈공식 이용

$$= \frac{1}{2}\left[\frac{-p}{(m+n)\pi}\cos\frac{\pi}{p}(m+n)x - \frac{p}{(m-n)\pi}\cos\frac{\pi}{p}(m-n)x\right]_a^{a+2p}$$

$$= -\frac{p}{2\pi}\left(\frac{1}{m+n}\cos\frac{\pi}{p}(m+n)(a+2p) + \frac{1}{m-n}\cos\frac{\pi}{p}(m-n)(a+2p)\right.$$
$$\left. - \frac{1}{m+n}\cos\frac{\pi}{p}(m+n)a - \frac{1}{m-n}\cos\frac{\pi}{p}(m-n)a\right)$$

$$= -\frac{p}{2\pi}\left\{\frac{1}{m+n}\left(\cos\left(2(m+n)\pi + \frac{\pi}{p}a(m+n)\right) - \cos\frac{\pi}{p}a(m+n)\right)\right.$$
$$\left. + \frac{1}{m-n}\left(\cos\left(2(m-n)\pi + \frac{\pi}{p}a(m-n)\right) - \cos\frac{\pi}{p}a(m-n)\right)\right\}$$

$$= 0$$

④ $\int_a^{a+2p} \sin\frac{n\pi}{p}x \cdot \sin\frac{m\pi}{p}x \, dx$

$= \int_a^{a+2p} \frac{1}{2}\left(\cos\frac{\pi}{p}(n-m)x - \cos\frac{\pi}{p}(n+m)x \right) dx$

$= \frac{1}{2}\int_a^{a+2p} \cos\frac{\pi}{p}(n-m)x - \cos\frac{\pi}{p}(n+m)x \, dx$

$= \frac{1}{2}\left[\frac{p}{(n-m)\pi}\sin\frac{\pi}{p}(n-m)x - \frac{p}{(n+m)\pi}\sin\frac{\pi}{p}(n+m)x \right]_a^{a+2p}$

$= \frac{1}{2}\left\{ \frac{p}{(n-m)\pi}\left(\sin\frac{\pi}{p}(n-m)(a+2p) - \sin\frac{\pi}{p}(n-m)a \right) \right.$

$\left. - \frac{p}{(n+m)\pi}\left(\sin\frac{\pi}{p}(n+m)(a+2p) - \sin\frac{\pi}{p}(n-m)a \right) \right\}$

$= 0$

⑤ $\int_a^{a+2p} \cos\frac{n\pi}{p}x \cdot \cos\frac{m\pi}{p}x \, dx$

$= \int_a^{a+2p} \frac{1}{2}\left(\cos\frac{\pi}{p}(n-m)x + \cos\frac{\pi}{p}(n+m)x \right) dx$

$= \frac{1}{2}\left[\frac{p}{(n-m)\pi}\sin\frac{\pi}{p}(n-m)x + \frac{p}{(n-m)\pi}\sin\frac{\pi}{p}(n+m)x \right]_a^{a+2p}$

$= \frac{1}{2}\left\{ \frac{p}{(n-m)\pi}\left(\sin\frac{\pi}{p}(n-m)(a+2p) - \sin\frac{\pi}{p}(n-m)a \right) \right.$

$\left. + \frac{p}{(n+m)\pi}\left(\sin\frac{\pi}{p}(n+m)(a+2p) - \sin\frac{\pi}{p}(n+m)a \right) \right\}$

$= 0$

따라서 제시된 함수는 직교 집합이다.

- 1의 norm

$$\|1\| = \sqrt{\int_a^{a+2p} 1^2\,dx} = \sqrt{2p}$$

- $\cos\frac{n\pi}{p}x$의 norm

$$\left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{\int_a^{a+2p} \cos^2\frac{n\pi}{p}x\,dx}$$

$$\int_a^{a+2p} \cos^2\frac{n\pi}{p}x\,dx = \frac{1}{2}\int_a^{a+2p} 1+\cos\frac{2n\pi}{p}x\,dx$$

$$= \frac{1}{2}\int_a^{a+2p} 1\,dx = \frac{1}{2}\cdot 2p = p. = \left\|\cos\frac{n\pi}{p}x\right\|^2$$

$$\therefore \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{p}.$$

- $\sin\frac{n\pi}{p}x$의 norm

$$\left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{\int_a^{a+2p} \sin^2\frac{n\pi}{p}x\,dx}$$

$$\int_a^{a+2p} \sin^2\frac{n\pi}{p}x\,dx = \frac{1}{2}\int_a^{a+2p} 1-\cos\frac{2n\pi}{p}x\,dx.$$

$$= \frac{1}{2}\int_a^{a+2p} 1\,dx = \frac{1}{2}\cdot 2p = p = \left\|\sin\frac{n\pi}{p}x\right\|^2$$

$$\therefore \left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{p}$$