[과제 19]

(1) $u_x = x+y$.

$\frac{du}{dx} = x+y$. $\xrightarrow{\text{변수분리}}$ $du = (x+y)\,dx$

$\int du = \int x+y\ dx$

$\therefore u = \frac{1}{2}x^2 + yx + A(y)$ ($A(y)$는 임의의 함수)

(2) $u_{xy} = 2x-y$

$u_x = P$. $u_{xy} = P_y$

$P_y = \frac{dP}{dy} = 2x-y$

$dP = (2x-y)\,dy$. $P = 2xy - \frac{1}{2}y^2 + A(x)$

$u_x = 2xy - \frac{1}{2}y^2 + A(x)$

$\frac{du}{dx} = 2xy - \frac{1}{2}y^2 + A(x)$. $u = x^2y - \frac{1}{2}xy^2 + B(x) + C(y)$

$\therefore u = x^2y - \frac{1}{2}xy^2 + B(x) + C(y)$ ($B(x)$, $C(y)$는 임의의 함수)

(3) $u_{xx} = 0$

$\frac{du_x}{dx} = 0$. $du_x = 0$. $u_x = f(y)$

$\frac{du}{dx} = f(y)$ $du = f(y)\cdot dx$

$u = f(y)x + g(y)$

$\therefore u = f(y)x + g(y)$

(4) $u_{yy} = 0$

$\frac{du_y}{dy} = \frac{d^2u}{dy^2} = 0$

$\frac{du}{dy} = f(x)$, $du = f(x)\,dy$

$\therefore u = f(x)y + g(x)$

(5) $u_{xx} = 1$.

$\frac{d^2u}{dx^2} = 1$. $\frac{du}{dx} = x + f(y)$

$du = (x + f(y))\,dx$.

$\therefore u = \frac{1}{2}x^2 + f(y)x + g(y)$

(6) $u_y - u = y$.

y에 관한 도함수가 없으므로 상미분방정식

$u' - u = y$의 해를 구한다.

$u' - u = y$ *비제차 선형

$p(y) = -1$. $y' + p(x)y = r(x)$

$r(y) = y$. $\Rightarrow y = e^{-\int p\,dx}\left(C + \int r(x)\cdot e^{\int p\,dx}\,dx\right)$

$u = e^{-\int -1\,dy}\left(C + \int y\cdot e^{\int -1\,dy}\,dy\right)$

$= e^{y}\left(C + \int y\cdot e^{-y}\,dy\right) = e^{y}\left(C + y\cdot(-1)e^{-y} - \int(-1)e^{-y}\,dy\right)$

$= e^{y}\left(C - ye^{-y} - e^{-y}\right) = C\cdot e^{y} - y - 1$

$\therefore u(x,y) = C(x)\cdot e^{y} - y - 1$

[과제 20]

(1) $2z_x + 3z_y + 8z = 0$. $z(x,0) = \sin(x)$

$2z_x + 3z_y = -8z$. $a(x,y) = 2$. $b(x,y) = 3$.

$g(x,y,z) = -8z$.

$$\frac{dx}{2} = \frac{dy}{3} = \frac{dz}{-8z}$$

$dy = \frac{3}{2}dx$, $y = \frac{3}{2}x + C_1 \rightarrow C_1 = y - \frac{3}{2}x$

$\frac{1}{z}dz = -4dx$. $\ln|z| = -4x + C_2$.

$z = e^{-4x+C_2} = C_3 \cdot e^{-4x}$

$y=0$. $C_1 = -\frac{3}{2}x \rightarrow x = -\frac{2}{3}C_1$.

$z = \sin(x) = C_3 e^{-4x}$.

$C_3 = \sin(x) \cdot e^{4x} = \sin(-\frac{2}{3}C_1) \cdot e^{-\frac{8}{3}C_1}$

$z = C_3 \cdot e^{-4x} = \sin(-\frac{2}{3}C_1) \cdot e^{-4x-\frac{8}{3}C_1}$

$= \sin(x - \frac{2}{3}y) \cdot e^{-4x-\frac{8}{3}y+4x}$

$= \sin(x - \frac{2}{3}y) \cdot e^{-\frac{8}{3}y}$

$\therefore z(x,y) = \sin(x - \frac{2}{3}y) \cdot e^{-\frac{8}{3}y}$

(2) $3z_x - 4z_y + 2z = 7$. $z(x,0) = e^x$

$a(x,y) = 3$. $b(x,y) = -4$. $g(x,y,z) = 7 - 2z$.

$$\frac{dx}{3} = \frac{dy}{-4} = \frac{dz}{7-2z}$$

$dx = -\frac{3}{4}dy$, $x = -\frac{3}{4}y + C_1 \rightarrow C_1 = x + \frac{3}{4}y$

$-\frac{1}{4}dy = \frac{1}{7-2z}dz$, $-\frac{1}{4}y + C_2 = -\frac{1}{2}\ln|7-2z|$

$\ln|7-2z| = \frac{1}{2}y$. $7 - 2z = e^{\frac{1}{2}y} \cdot C_3$

$2z = 7 - C_3 \cdot e^{\frac{1}{2}y}$

$z = \frac{7}{2} - \frac{C_3}{2} \cdot e^{\frac{1}{2}y}$

$C_1 = x + \frac{3}{4}y \xrightarrow{y=0} C_1 = x$

$z = \frac{7}{2} - \frac{C_3}{2} \cdot e^{\frac{1}{2}y} \xrightarrow{y=0} \frac{7}{2} - \frac{C_3}{2} = e^x$.

$7 - C_3 = 2e^x$.

$C_3 = 7 - 2e^x = 7 - 2e^{C_1}$.

$z = \frac{7}{2} - \frac{1}{2}(7 - 2e^{C_1}) \cdot e^{\frac{1}{2}y}$

$= \frac{7}{2} - \frac{1}{2}(7 - 2e^{x+\frac{3}{4}y})e^{\frac{1}{2}y}$

$= \frac{7}{2} - \frac{7}{2}e^{\frac{1}{2}y} + e^{x+\frac{5}{4}y}$.

[과제 20]

(3) $z_x + z_y - z = e^x, \quad z(x,0) = 0.$

보조방정식 $dx = dy = \dfrac{dz}{e^x + z}$

$dx = dy \rightarrow x = y + c_1, \quad c_1 = x - y$

$dx = \dfrac{dz}{e^x + z}, \quad \dfrac{dz}{dx} = e^x + z. \quad \underline{z_x - z = e^x}$

비제차 선형

$y' + P(x)y = r(x)$의 해

$\Rightarrow y = e^{-\int P dx}\left(C + \int r(x) \cdot e^{\int P dx} dx\right)$

$z = e^{-\int -1 dx}\left(C_2 + \int e^x \cdot e^{\int -1 dx} dx\right)$

$= e^x (C_2 + x)$

$C_1 = x - y \xrightarrow{y=0} C_1 = x$

$z = e^x (x + C_2) \xrightarrow{y=0} e^x (x + C_2) = 0 \Rightarrow x = -C_2 = C_1.$

$C_2 = -C_1.$

$z = e^x (x - C_1) = e^x (x - x + y)$

$= y e^x$

[과제 21]

(1) $xz_x + z_y = 1$. $z(1, y) = e^{-y}$.

보조방정식 $\frac{dx}{x} = \frac{dy}{1} = \frac{dz}{1}$

$\frac{1}{x}dx = dy$. $\ln|x| = y + C_1$.

$x = e^{y} \cdot C_2 \rightarrow C_2 = xe^{-y}$.

$dy = dz$, $z = y + C_3$.

$C_2 = xe^{-y} \xrightarrow{x=1} C_2 = e^{-y}$, $y = -\ln|C_2|$

$z = y + C_3 \xrightarrow{x=1} z = y + C_3 = e^{-y}$.

$C_3 = e^{-y} - y = e^{\ln|C_2|} + \ln|C_2|$

$= C_2 + \ln|C_2|$

$z = y + C_3 = y + C_2 + \ln|C_2| = y + xe^{-y} + \ln|xe^{-y}|$

$= y + xe^{-y} + \ln|x| + \underbrace{\ln|e^{-y}|}_{= -y}$

$= xe^{-y} + \ln|x|$

(2) $z_x + yz_y = z$, $z(x, 1) = xe^{-x}$

$dx = \frac{1}{y}dy = \frac{1}{z}dz$.

$dx = \frac{1}{y}dy$, $C_1 + x = \ln|y| \rightarrow y = C_2 e^{x} \Rightarrow C_2 = y \cdot e^{-x}$

$dx = \frac{1}{z}dz \rightarrow z = C_3 e^{x}$.

$C_2 = y \cdot e^{-x} \xrightarrow{y=1} C_2 = e^{-x}$, $x = -\ln|C_2|$

$z = C_3 \cdot e^{x} \xrightarrow{y=1} C_3 \cdot e^{x} = xe^{-x}$.

$C_3 = xe^{-2x}$ 대입

$= -\ln|C_2| \cdot C_2^{2}$

$z = C_3 \cdot e^{x} = -\ln|C_2| \cdot C_2^{2} \cdot e^{x}$.

$= -\ln(y \cdot e^{-x}) \cdot y^2 \cdot e^{-2x} \cdot e^{x}$

$= (-\ln|y| + x) \cdot y^2 \cdot e^{-x}$

[과제2]

(3) $x^2 z_x + y^2 z_y = z^2$.

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2},$$

$$\frac{dx}{x^2} = \frac{dy}{y^2}, \quad \frac{1}{x} = \frac{1}{y} + A, \quad A = \frac{-x+y}{xy} = \frac{1}{x} - \frac{1}{y}$$

$$\frac{dz}{z^2} = \frac{dy}{y^2}, \quad \frac{1}{z} = \frac{1}{y} + B, \quad z = \frac{1}{\frac{1}{y}+B} = \frac{y}{1+By}$$

$$\xrightarrow{xy = x+y} \quad 1 = \frac{y}{1+By}, \quad 1+By = y. \quad B = \frac{y-1}{y}$$

$$xy - x = x(y-1) = y$$

대입

$$B = \frac{1}{x}, \quad A = \frac{-x+y}{xy} = \frac{-x+y}{x+y} \rightarrow (x+y)A = -x+y$$

$$(A+1)x = (1-A)y, \quad x = \frac{1-A}{A+1} y$$

$B = \frac{A+1}{(1-A)y} \rightarrow$ z식에 대입.

$$z = \frac{y}{1 + \frac{A+1}{(1-A)\cancel{y}} \cdot \cancel{y}} = \frac{(1-A)y}{(1-A)+A+1} = \frac{(1-A)y}{2}$$

$$= \frac{(1+\frac{1}{y}-\frac{1}{x})y}{2} = \frac{y+1-\frac{y}{x}}{2}$$

$$= \frac{1}{2}y + \frac{1}{2} - \frac{y}{2x}$$

(4). $(x-y)y^2 z_x + (y-x)x^2 z_y = (x^2+y^2)z$.

보조방정식 $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{x^2+y^2}$

[과제22]

$u_{tt}=c^2u_{xx}\ (0<x<L)$

$u(0,t)=0,\quad u(L,0)=0\ (t>0)$

$u(x,0)=\frac{1}{4}x(L-x)\ =f(x)$

$u_t(x,0)=0\ (0<x<L)\ =g(x)\ \rightarrow\ B_n^*=0$

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\cdot\sin\frac{n\pi}{L}x$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$=\frac{2}{L}\int_0^L\frac{1}{4}x(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$=\frac{1}{2L}\int_0^L x(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow\int_0^L x(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$=\underbrace{\cancel{(Lx-x^2)\cos\left(\frac{n\pi x}{L}\right)\cdot\left(-\frac{L}{n\pi}\right)\Big|_0^L}}_{=0}+\frac{L}{n\pi}\int_0^L(L-2x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{L}{n\pi}\left\{\underbrace{\cancel{(L-2x)\sin\left(\frac{n\pi x}{L}\right)\cdot\frac{L}{n\pi}\Big|_0^L}}_{=0}-\int_0^L(-2)\sin\left(\frac{n\pi x}{L}\right)\cdot\frac{L}{n\pi}\,dx\right\}$$

$$=2\left(\frac{L}{n\pi}\right)^2\int_0^L\sin\left(\frac{n\pi x}{L}\right)dx=-2\left(\frac{L}{n\pi}\right)^3\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$=-2\left(\frac{L}{n\pi}\right)^3(\cos n\pi-1)=2\left(\frac{L}{n\pi}\right)^3(1-(-1)^n)$$

$$B_n=\frac{1}{2L}\cdot2\left(\frac{L}{n\pi}\right)^3(1-(-1)^n)=\frac{L^2}{n^3\pi^3}(1-(-1)^n)$$

$$=\begin{cases}0 & (n\text{이 짝})\\ \dfrac{2L^2}{n^3\pi^3} & (n\text{이 홀})\end{cases}$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2L^2}{(2n)^3\pi^3}\cdot\cos\lambda_{2n}t\cdot\sin\frac{2n\pi}{L}x.$$

$$=\frac{L^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\cos\lambda_{2n}t\cdot\sin\frac{2n\pi}{L}x$$

$$=\frac{L^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\cos\frac{2cn\pi}{L}t\ \sin\frac{2n\pi}{L}x$$

[과제23]

$u_{tt}=9u_{xx}$. , $c^2=9$

$u(0,t)=0$, $u(5,0)=0$ $(t\ge 0)$, $L=5$

$u(x,0)=4\sin(\pi x)-3\sin(5\pi x)$, $=f(x)$

$u_t(x,0)=0$ $(0\le x\le 5)$ $=g(x)\to B_n^*=0$

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\lambda_n t\sin\frac{n\pi}{L}x$$

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$=\frac{2}{L}\int_0^L\{4\sin(\pi x)-3\sin(5\pi x)\}\cdot\sin\frac{n\pi x}{L}dx$$

$$=\frac{2}{L}\int_0^L 4\cdot\sin(\pi x)\cdot\sin\frac{n\pi x}{L}-3\sin(5\pi x)\cdot\sin\frac{n\pi x}{L}dx$$

$$=\frac{2}{L}\int_0^L-\frac{4}{2}\{\cos(\pi x+\frac{n\pi x}{L})-\cos(\pi x-\frac{n\pi x}{L})\}$$
$$+\frac{3}{2}\{\cos(5\pi x+\frac{n\pi x}{L})-\cos(5\pi x-\frac{n\pi x}{L})\}dx$$

$$=\frac{2}{L}\int_0^L-2\{\cos(1+\frac{n}{L})\pi x-\cos(1-\frac{n}{L})\pi x\}$$
$$+\frac{3}{2}\{\cos(5+\frac{n}{L})\pi x-\cos(5-\frac{n}{L})\pi x\}dx$$

$$=\frac{2}{L}\Big[\frac{-2\sin(1+\frac{n}{L})\pi x}{(1+\frac{n}{L})\pi}+\frac{2\sin(1-\frac{n}{L})\pi x}{(1-\frac{n}{L})\pi}$$
$$+\frac{3\sin(5+\frac{n}{L})\pi x}{2(5+\frac{n}{L})\pi}-\frac{3\sin(5-\frac{n}{L})\pi x}{2(5-\frac{n}{L})\pi}\Big]_0^L=0.$$

$B_n=0$. $B_n^*=0\Rightarrow u(x,t)=0$.

[과제24]

$u_{tt}=u_{xx}$. $c^2=1$ $\lambda_n=\frac{cn\pi}{L}=n\pi$

$u(0,t)=0$, $u(1,0)=0$ $(t\ge 0)$

$u(x,0)=x(1-x)$, $=f(x)$

$u_t(x,0)=x(1-x)$ $(0\le x\le 1)$ $=g(x)$ $L=1$.

$$B_n=2\int_0^1 f(x)\sin n\pi x\,dx$$

$$=2\int_0^1 x(1-x)\sin n\pi x\,dx$$

$$B_n^*=\frac{2}{n\pi}\int_0^1 x(1-x)\sin n\pi x\,dx$$

$$\int_0^1 x(1-x)\sin n\pi x\,dx=\int_0^1(x-x^2)\sin n\pi x\,dx$$

$$=\overset{=0}{\cancel{(x-x^2)(-\frac{1}{n\pi})\cos n\pi x}}\Big|_0^1+\frac{1}{n\pi}\int_0^1(1-2x)\cdot\cos n\pi x\,dx$$

$$=\frac{1}{n\pi}\Big\{(1-2x)\cdot\frac{1}{n\pi}\overset{=0}{\cancel{\sin n\pi x}}\Big|_0^1-\int_0^1(-2)\cdot\frac{1}{n\pi}\sin n\pi x\,dx\Big\}$$

$$=(\frac{1}{n\pi})^2\cdot 2\int_0^1\sin n\pi x\,dx=-2(\frac{1}{n\pi})^3[\cos n\pi x]_0^1$$

$$=2(\frac{1}{n\pi})^3\cdot(1-(-1)^n)$$

$$B_n=4(\frac{1}{n\pi})^3(1-(-1)^n)$$

$$B_n^*=4(\frac{1}{n\pi})^4(1-(-1)^n)$$

$$u(x,t)=\sum_{n=1}^{\infty}(B_n\cos\lambda_n t+B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x$$

$$=\sum_{n=1}^{\infty}(B_n\cos n\pi t+B_n^*\overset{=0}{\cancel{\sin n\pi t}})\sin n\pi x.$$

$$=\sum_{n=1}^{\infty}4(\frac{1}{2n\pi})^3\cos 2n\pi t\cdot\sin 2n\pi x.$$

$$=\frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\cos 2n\pi t\cdot\sin 2n\pi x$$

[과제 25]

(1) $A=y^2$, $B=-xy$. $C=x^2$

$AC-B^2=x^2y^2-x^2y^2=0$.

$A(y')^2-2By'+C=y^2(y')^2+2xy\cdot y'+x^2=0$

$=(y\cdot y'+x)^2=0$

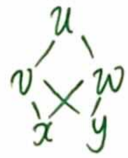
$y'=-\frac{x}{y}$, $y\cdot y'=-x$. $y\cdot dy=-x\,dx$

$\to \frac{1}{2}y^2=-\frac{1}{2}x^2$.

$v_x=1$. $v_{xx}=0$ $w_x=x$. $w_{xx}=1$. $v_{xy}=0$. $w_{xy}=0$

$v_y=0$. $v_{yy}=0$ $w_y=y$. $w_{yy}=1$,

$v=x$. $w=\frac{1}{2}(x^2+y^2)$



$u_x=u_v\underset{1}{v_x}+u_w\underset{x}{w_x}=\underline{u_v+x u_w}$

$u_{xy}=u_{vy}\underset{1}{v_x}+u_v\underset{0}{v_{xy}}+u_{wy}\underset{x}{w_x}+u_w\underset{0}{w_{xy}}=u_{vy}+u_{wy}\cdot x$

$=(u_{vv}\underset{0}{v_y}+u_{vw}\underset{y}{w_y})+(u_{wv}\underset{0}{v_y}+u_{ww}\underset{y}{w_y})\cdot x=\underline{y\cdot u_{vw}+xy\cdot u_{ww}}$

$u_{xx}=u_{vx}\underset{1}{v_x}+u_v\underset{0}{v_{xx}}+u_{wx}\underset{x}{w_x}+u_w\underset{1}{w_{xx}}=u_{vv}\underset{1}{v_x}+u_{vw}\underset{x}{w_x}+(u_{wv}\underset{1}{v_x}+u_{ww}\underset{x}{w_x})\cdot x+u_w\cdot\underset{1}{w_{xx}}$

$=\underline{u_{vv}+x u_{vw}+x u_{wv}+x^2u_{ww}+u_w}$

$u_y=u_v\underset{0}{v_y}+u_w\underset{y}{w_y}=\underline{y\cdot u_w}$

$u_{yy}=u_{vy}\underset{0}{v_y}+u_v\underset{0}{v_{yy}}+u_{wy}\underset{y}{w_y}+u_w\cdot\underset{1}{w_{yy}}=(u_{wv}\underset{0}{v_y}+u_{ww}\underset{y}{w_y})\cdot y+u_w$

$=\underline{y^2u_{ww}+u_w}$

$y^2u_{xx}-2xyu_{xy}+x^2u_{yy}=y^2\cdot u_{vv}+\cancel{xy^2u_{vw}}+\cancel{xy^2u_{wv}}+\cancel{x^2y^2u_{ww}}+\cancel{y^2u_w}-\cancel{2xy^2\cdot u_{vw}}-\cancel{2x^2y^2\cdot u_{ww}}+\cancel{x^2y^2u_{ww}}+\cancel{x^2u_w}$

①

$\frac{1}{xy}(y^3u_x+x^3u_y)=\frac{1}{xy}(y^3\cdot u_v+xy^3u_w+x^3yu_w)=\frac{y^2}{x}\cdot u_v+\cancel{y^2u_w}+\cancel{x^2u_w}$

$\Rightarrow y^2\cdot u_{vv}=\frac{y^2}{x}\cdot u_v$, $\underbrace{x}_{=v}u_{vv}=u_v$., $P=u_v$라 치환. $u_{vv}=P_v=P'$, $vP'=P$.

$\frac{1}{P}dP=\frac{1}{v}dv$, $\ln|P|=\ln|v|+C_1(w)=\ln|C_2(w)\cdot v|$, $P=C_2(w)\cdot v=u_v$. $\Rightarrow u=\frac{1}{2}C_2(w)v^2+C_3(w)$

$\therefore u=\frac{1}{2}A(\frac{1}{2}x^2+\frac{1}{2}y^2)\cdot x^2+B(\frac{1}{2}x^2+\frac{1}{2}y^2)$

[과제25]

(2) $xu_{xx} - yu_{xy} = 0$

$A(x,y) = x.\quad 2B(x,y) = -y.\quad C(x,y) = 0$

$AC - B^2 = -\frac{y^2}{4} < 0$: hyperbolic.

특성방정식 : $x(y')^2 + y(y') = 0$

$y'(xy' + y) = 0$

$y' = 0$ or $y' = -\frac{y}{x}$

I) $y' = 0.\quad y = C_1.$

II) $\frac{1}{y}y' = -\frac{1}{x}$, $\quad \ln|y| = -\ln|x| + C_2.$

$= \ln\left|\frac{C_2}{x}\right|.\quad y = \frac{C_2}{x},\ xy = C_2.$

$v = y,\ w = xy.\quad v_x = 0 \quad v_{xx} = 0 \quad v_y = 1. \quad v_{xy} = 0.$

$w_x = y \quad w_{xx} = 0 \quad w_y = x \quad w_{xy} = 1$

$u_x = u_v v_x + u_w w_x$

$u_{xx} = u_{vx}v_x + u_v v_{xx} + u_{wx}w_x + u_w w_{xx}$

$= \cancel{(u_{vv}\underset{0}{v_x} + u_{vw}\underset{y}{w_x})}\,\underset{0}{v_x} + \cancel{u_v \underset{0}{v_{xx}}}$

$+(\cancel{u_{wv}\underset{0}{v_x}} + u_{ww}\underset{y}{w_x})\underset{y}{w_x} + \cancel{u_w \underset{0}{w_{xx}}} = y^2 u_{ww}$

$u_{xy} = u_{vy}v_x + u_v v_{xy} + u_{wy}w_x + u_w w_{xy}$

$= \cancel{(u_{vv}v_y + u_{vw}w_y)}\cdot \underset{0}{v_x} + \cancel{u_v \underset{0}{v_{xy}}}$

$+ (u_{wv}\underset{1}{v_y} + u_{ww}\underset{x}{w_y})\cdot \underset{y}{w_x} + u_w \underset{1}{w_{xy}}$

$= y u_{wv} + xy u_{ww} + u_w$

$xu_{xx} - yu_{xy} = xy^2\cancel{u_{ww}} - y^2 u_{wv} - \cancel{xy^2 u_{ww}} - y u_w = 0$

$y^2 u_{wv} + y u_w = 0.\qquad y\cdot u_{wv} + u_w = v\cdot u_{wv} + u_w = 0$

$p = u_w.\quad p' = u_{wv}.$

$p + vp' = 0. \Rightarrow \frac{1}{p}p' = -\frac{1}{v},\quad \frac{1}{p}dp = -\frac{1}{v}dv$

$vp' = -p$

이어서 →

$\ln|p| = \ln\left|\frac{1}{v}\right| + A(w)$

$= \ln\left|\frac{B(w)}{v}\right|$

$p = \frac{B(w)}{v} = u_w.$

$u_w = \frac{du}{dw} = \frac{B(w)}{v},\quad du = \frac{B(w)}{v}\cdot dw \Rightarrow u = \frac{1}{v}C(w) + D(v)$

$u = \frac{1}{v}f(w) + g(v)$

$= \frac{1}{y}f(xy) + g(y)$

[과제 26]

$u_t = \frac{1}{4} u_{xx} \quad (t>0,\ 0<x<1) \qquad c^2 = \frac{1}{4},\ L=1.$

$u_x(0,t)=0,\ u_x(1,t)=0 \quad (t>0)$

$u(x,0) = 100x(1-x) \quad (0<x<1) \quad = f(x)$

$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} e^{-\lambda_n^2 t}$

$\lambda_n^2 = \left(\frac{cn\pi}{L}\right)^2 = \frac{c^2 n^2 \pi^2}{L^2} = \frac{\frac{1}{4} \cdot n^2\pi^2}{1} = \frac{n^2\pi^2}{4} = \left(\frac{n\pi}{2}\right)^2$

$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 100x(1-x)\,dx$

$= 100\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 100 \cdot \frac{1}{6} = \underline{\frac{50}{3}}$

$A_n = \frac{2}{L}\int_0^L f(x) \cdot \cos\frac{n\pi x}{L}\,dx$

$= 2\int_0^1 f(x) \cdot \cos n\pi x\,dx$

$= 200\int_0^1 x(1-x)\cos n\pi x\,dx$

$\Rightarrow \int_0^1 x(1-x)\cos n\pi x\,dx$

$= \frac{x(1-x)}{n\pi} \cdot \cancel{\sin n\pi x}^{=0} \Big|_0^1 - \int_0^1 \frac{(1-2x)}{n\pi}\sin n\pi x\,dx$

$= -\left\{ -\frac{(1-2x)}{(n\pi)^2}\cos n\pi x \Big|_0^1 - \int_0^1 -\frac{-2}{(n\pi)^2} \cdot \cos n\pi x\,dx \right\}$

$= \frac{(1-2x)}{(n\pi)^2}\cos n\pi x \Big|_0^1 - \int_0^1 \frac{2}{(n\pi)^2}\cos n\pi x\,dx$

$= -\frac{1}{(n\pi)^2}\cos n\pi - \frac{1}{(n\pi)^2} - \left(\frac{2}{(n\pi)^3} \cdot \cancel{\sin n\pi x}^{=0} \Big|_0^1 \right)$

$= -\frac{1}{(n\pi)^2}\left(\cos n\pi + 1\right) = -\frac{1}{(n\pi)^2}\left((-1)^n + 1\right)$

$A_n = -\frac{200}{(n\pi)^2}\left((-1)^n + 1\right)$

$= \begin{cases} 0 & (n\text{이 홀수}) \\ -\frac{400}{(n\pi)^2} & (n\text{이 짝수}) \end{cases}$

$u(x,t) = \frac{50}{3} + \sum_{n=0}^{\infty} A_{2n} \cos\frac{2n\pi x}{L} e^{-\left(\frac{2n\pi}{2}\right)^2 t}$

$= \frac{50}{3} + \sum_{n=0}^{\infty} -\frac{400}{(2n\pi)^2} \cdot \cos 2n\pi x\, e^{-(n\pi)^2 t}$

$= \frac{50}{3} + \sum_{n=1}^{\infty} -\frac{100}{n^2\pi^2} \cdot \cos 2n\pi x \cdot e^{-n^2\pi^2 t}$

[과제1 2기]

$u_t = 4u_{xx}$. $\quad c^2 = 4 \qquad L = 5$.

$u(0,t) = 5, \quad u(5,t) = 20$

$u(x,0) = 30 - 2x \ (0 < x < 5) = f(x)$

$L(x,t) = ax + b$.

$L(0,t) = b = 5$,

$L(5,t) = 5a + 5 = 20$. $\quad a = 3$.

$L(x,t) = 3x + 5$.

$$w(x,t) = u(x,t) - L(x,t)$$

$$= 30 - 2x - (3x+5) = -5x + 25$$

$$B_n = \frac{2}{5}\int_0^5 (-5x+25)\sin\frac{n\pi}{5}x\,dx$$

$$= \frac{2}{5}\left\{ -\frac{(-5x+25)\cdot\cos\frac{n\pi}{5}x}{\frac{n\pi}{5}}\Bigg|_0^5 - \overset{=0}{\cancel{\int_0^5 \frac{-(-5)}{\frac{n\pi}{5}}\cdot\cos\frac{n\pi}{5}x\,dx}} \right\}$$

$$= \frac{2}{5}\cdot\frac{5}{n\pi}\Big[(5x-25)\cdot\cos\frac{n\pi}{5}x\Big]_0^5$$

$$= \frac{2}{n\pi}(25) = \frac{50}{n\pi}$$

$$w(x,t) = \sum_{n=1}^{\infty}\frac{50}{n\pi}\sin\frac{n\pi}{5}x\cdot e^{-\lambda_n^2 t}$$

$$\lambda_n^2 = \left(\frac{2n\pi}{5}\right)^2 = \frac{4n^2\pi^2}{25}$$

$$w(x,t) = \sum_{n=1}^{\infty}\frac{50}{n\pi}\sin\frac{n\pi}{5}x\cdot e^{-\frac{4n^2\pi^2}{25}t}$$

$$u(x,t) = w(x,t) + L(x,t)$$

$$= 3x + 5 + \sum_{n=1}^{\infty}\frac{50}{n\pi}\sin\frac{n\pi}{5}x\cdot e^{-\frac{4n^2\pi^2}{25}t}$$

[과제 26]

$$u_{xx}+u_{yy}=0$$
$$u_x(0,y)=0,\quad u_x(1,y)=0$$
$$u(x,0)=100,\quad u(x,1)=200$$

$$u=u(x,y)=F(x)G(y)$$

$$F''G=-FG''$$

$$\frac{F''}{F}=-\frac{G''}{G}=k \Rightarrow \begin{cases} F''-kF=0 \\ G''+kG=0 \end{cases}$$

$$u_x=F'G$$

$$u_x(0,y)=F'(0)G(y)=0.\quad u_x(1,y)=F'(1)G(y)=0$$

$G(y)=0 \to u=0$ (no interest)

즉. $F'(1)=0,\ F'(0)=0.$

I) $k=0.\quad F''=0.\quad F(x)=Ax+B.$

$F'(x)=A \to F'(0)=A=0.\quad \underline{A=0}$

$\Rightarrow F(x)=B$

$G''=0.\quad G(y)=Cy+D.$

$$u_0(x,y)=F_0(x)G_0(y)=A_0y+B_0$$

II) $k>0$

$$F''-kF=0 \Rightarrow F(x)=Ae^{\sqrt{k}x}+Be^{-\sqrt{k}x}.$$

$$F'(x)=A\sqrt{k}e^{\sqrt{k}x}-B\sqrt{k}e^{-\sqrt{k}x}$$

$$F'(0)=A\sqrt{k}-B\sqrt{k}=0.\quad A=B$$

$$F'(1)=A\sqrt{k}(\underbrace{e^{\sqrt{k}}-e^{-\sqrt{k}}}_{\neq 0})=0$$

$\therefore A=0 \to B=0. \to F(x)=0.$

$u(x,t)=0$ (no interest)

III) $k<0.\quad k=-p^2.$

$$\begin{cases} F''+p^2F=0 \to F=A\cos px+B\sin px \\ G''-p^2G=0 \to G=Ce^{py}+De^{-py} \end{cases}$$

$$F(x)=A\cos px+B\sin px$$

$$F'(x)=-Ap\sin px+Bp\cos px$$

$$F'(0)=Bp\cos px=0. \to B=0$$

$$F'(1)=-Ap\sin p=0$$

(A가 0이면 $F=0 \to u=0$ 즉 no interest!)

$\sin p=0 \to p=n\pi\ (n=1,2,3,\cdots)$

$$F_n(x)=A_n\cos n\pi x.$$
$$G_n(y)=C_ne^{n\pi y}+D_ne^{n\pi y}$$

$$u_n(x,y)=F_n(x)G_n(y)$$
$$=(C_ne^{n\pi y}+D_ne^{-n\pi y})\cos n\pi x$$

$$u(x,y)=u_0(x,y)+\sum_{n=1}^{\infty}(A_ne^{n\pi y}+B_ne^{-n\pi y})\cos n\pi x$$

$$u(x,0)=B_0+\sum_{n=1}^{\infty}(A_n+B_n)\cdot\cos n\pi x=100$$

100에 대한 급수식

$$\hookrightarrow A_n+B_n=\frac{2}{1}\int_0^1 100\cos n\pi x\,dx$$

$$=200\left[\frac{1}{n\pi}\sin n\pi x\right]_0^1=0$$

$\therefore B_0=100$

$$u(x,1)=u_0(x,1)+\sum_{n=1}^{\infty}(A_ne^{n\pi}+B_ne^{-n\pi})\cos n\pi x$$

$$=A_0+B_0+\sum_{n=1}^{\infty}(A_ne^{n\pi}+B_ne^{-n\pi})\cos n\pi x$$

$$A_ne^{n\pi}+B_ne^{-n\pi}=\frac{2}{1}\int_0^1 200\cdot\cos n\pi x\,dx=0.$$

$$A_0+B_0=200,\quad A_0=100$$

$$\therefore u(x,y)=u_0(x,y)$$
$$=100y+100$$