[과제 36] 원 $C : |z|=2.$ $(r=2)$

$\sqrt{x^2+y^2}=2.$ $x^2+y^2=4.$ $\int_C |dz| = 2\pi r = 4\pi = L$

(1) $|e^z| = |e^x|$ $\quad |z+1| = \sqrt{(x+1)^2+y^2}$

$\left|\frac{e^z}{z+1}\right| = \frac{|e^x|}{\sqrt{2x+5}}$ 의 大 $= \sqrt{2x+5}$

$M=\infty.$

$ML=\infty$

(3). $\int_C \frac{2z+1}{5+z^2}\,dz$ $\quad L=4\pi$

$\left|\frac{2z+1}{5+z^2}\right| = \left|\frac{(2x+1)+i(2y)}{(x^2-y^2+5)+2xy\,i}\right|$

$= \sqrt{\frac{4x^2+4y^2+4x+1}{x^4+y^4+25+2x^2y^2+10x^2-10y^2}}$

$= \sqrt{\frac{4\cdot 4+4x+1}{4^2+25+20(x^2-2)}} = \sqrt{\frac{4x+17}{20x^2+1}}$ 의 大

$M=\sqrt{17}.$
$(x=0)$

$\left|\int_C \frac{dz}{3+5z^2}\right| \le \sqrt{17}\cdot 4\pi = 4\sqrt{17}\,\pi$

(2). $\int_C \frac{dz}{3+5z^2}$ $\quad L=4\pi.$

$\left|\int_C \frac{dz}{3+5z^2}\right| \le \int_C |f(z)|\,|dz| \le ML$

$\left|\frac{1}{3+5z^2}\right| = \frac{1}{|3+5z^2|}$

$|3+5z^2| = |5(x^2-y^2)+3+10ixy|$

$= \sqrt{\{5(x^2-y^2)+3\}^2+(10xy)^2}$

$= \sqrt{25x^4+25y^4+9-50x^2y^2+30x^2-30y^2+100x^2y^2}$

$= \sqrt{25(x^4+2x^2y^2+y^4)+30(x^2-y^2)+9}$

$= \sqrt{25(x^2+y^2)^2+30(2x^2-4)+9}$

$= \sqrt{25\cdot 16+9+60(x^2-2)}$ 의 小

$\Rightarrow x=0$일 때. $\sqrt{25\cdot 16+9-120} = 17$

$\frac{1}{|3+5z^2|} \le \frac{1}{17} = M$

$\therefore \left|\int_C \frac{dz}{3+5z^2}\right| \le \frac{1}{17}\cdot 4\pi = \frac{4}{17}\pi$

[과제 37]

(1) $f(z) = \dfrac{1+z}{z}$

$C: z(t) = e^{it} \quad (-\frac{\pi}{2} \le t \le \frac{\pi}{2})$

$\dot{z}(t) = ie^{it}$

$\int_C f(z)\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dfrac{1+z(t)}{z(t)} \cdot \dot{z}(t)\,dt$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dfrac{1+e^{it}}{e^{it}} \cdot ie^{it}\,dt$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i(1+e^{it})\,dt$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i(1+\cos t + i\sin t)\,dt$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{-\sin t}_{기} + i(\underbrace{1+\cos t}_{우})\,dt$

$= i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+\cos t\,dt = 2i\int_0^{\frac{\pi}{2}} 1+\cos t\,dt$

$= 2i\left[t+\sin t\right]_0^{\frac{\pi}{2}} = 2i\left(\frac{\pi}{2}+1\right) = i(\pi+2)$

(2) $f(z) = \mathrm{Im}(z)$

$C: z(t) = r\cdot e^{it} \quad (0 \le t \le 2\pi)$

$\dot{z}(t) = rie^{it} = ri(\cos t + i\sin t)$

$\mathrm{Im}(z) = \mathrm{Im}(r\cdot e^{it}) = \mathrm{Im}(r(\cos t + i\sin t))$

$= r\sin t.$

$\int_C f(z)\,dz = \int_0^{2\pi} r\sin t \cdot \dot{z}(t)\,dt$

$= \int_0^{2\pi} ir^2\sin t(\cos t + i\sin t)\,dt$

$= r^2 i\int_0^{2\pi} \sin t\cos t + i\sin^2 t\,dt$

$= r^2 i\int_0^{2\pi} \frac{1}{2}\sin 2t + i\dfrac{1-\cos 2t}{2}\,dt$

$= r^2 i\int_0^{2\pi} \frac{1}{2} i\,dt = -\frac{r^2}{2}\int_0^{2\pi} 1\,dt = -\frac{r^2}{2}\cdot 2\pi$

$= -r^2\pi$

(3) $f(z) = |z|^2.$

$z(t) = t^2 + \frac{i}{t} \quad (1 \le t \le 2)$

$\dot{z}(t) = 2t - \frac{i}{t^2}$

$f(z) = \left|t^2 + \frac{i}{t}\right|^2 = (t^2)^2 + \left(\frac{1}{t}\right)^2 = t^4 + \frac{1}{t^2}$

$\int_C f(z)\,dz = \int_1^2 \left(t^4 + \frac{1}{t^2}\right)\cdot\left(2t - \frac{i}{t^2}\right)dt$

$= \int_1^2 \left(2t^5 + \frac{2}{t} - i\left(t^2 + \frac{1}{t^4}\right)\right)dt$

$= \left[2\cdot\frac{1}{6}t^6 - 2\ln|t| - i\left(\frac{1}{3}t^3 - \frac{1}{3}t^{-3}\right)\right]_1^2$

$= \frac{1}{3}(32-1) - 2\ln 2 - i\left(\frac{1}{3}(8-1) - \frac{1}{3}\left(\frac{1}{8}-1\right)\right)$

$= \frac{31}{3} - 2\ln 2 - i\left(\frac{7}{3} + \frac{7}{24}\right)$

$= \frac{31}{3} - 2\ln 2 - i\cdot\frac{21}{8}$

(4) $f(z) = \bar{z}.$

$C: z(t) = 2e^{it} + 1 \quad (0 \le t \le \pi)$

$\dot{z}(t) = 2ie^{it} = 2i(\cos t + i\sin t)$

$= 2(i\cos t - \sin t)$

$z(t) = 2(\cos t + i\sin t) + 1 = 2\cos t + 1 + i2\sin t$

$\bar{z} = 2\cos t + 1 - 2i\sin t.$

$\int_C f(z)\,dz = \int_0^{\pi} 2(2\cos t + 1 - 2i\sin t)(i\cos t - \sin t)\,dt$

$= \int_0^{\pi} 2(2i + i\cos t - \sin t)\,dt$

$= 2\left[2it + i\sin t + \cos t\right]_0^{\pi} = 2(2\pi i - 2)$

$= 4(\pi i - 1)$

[과제 38]

곡선의 길이 : $\int_C |dz| = \int_a^b |\dot{z}(t)|\, dt$

(1) $z(t) = 3e^{2it} + 2 \quad (0 \le t \le 2\pi)$

$\dot{z}(t) = 6ie^{2it} = 6i(\cos(2t) + i\sin(2t))$

$= -6\sin(2t) + 6i\cos(2t)$

$|\dot{z}(t)| = \sqrt{(6\sin(2t))^2 + (6\cos(2t))^2}$

$= 6.$

$\int_0^{2\pi} 6\, dt = 6\int_0^{2\pi} 1\, dt = 6 \cdot 2\pi = 12\pi$

(2) $z(t) = e^t\cos t + ie^t\sin t \quad (0 \le t \le 2\pi)$

$\dot{z}(t) = e^t\cos t + ie^t\sin t - e^t\sin t + ie^t\cos t$

$= (e^t\cos t - e^t\sin t) + i(e^t\sin t + e^t\cos t)$

$|\dot{z}(t)| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t\sin t + e^t\cos t)^2}$

$= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}$

$= e^t\sqrt{2(\cos^2 t + \sin^2 t)}$

$= \sqrt{2}e^t.$

$\int_0^{2\pi} \sqrt{2}e^t\, dt = \sqrt{2}\int_0^{2\pi} e^t\, dt$

$= \sqrt{2}\left[e^t\right]_0^{2\pi} = \sqrt{2}(e^{2\pi} - 1)$

[과제 39]

(1) $\int_0^{\pi+i} e^{iz}\,dz$ $\quad$ e^{iz}는 해석적.

$= \left[\frac{1}{i}e^{iz}\right]_0^{\pi+i}$

$= \frac{1}{i}\left(e^{i(\pi+i)}-1\right) = \frac{1}{i}\left(e^{i\pi-1}-1\right)$

$= \frac{1}{i}\left(e^{-1}\cdot e^{i\pi}-1\right) = \frac{1}{i}\left(-e^{-1}-1\right) = -\frac{1}{i}\left(1+\frac{1}{e}\right)$

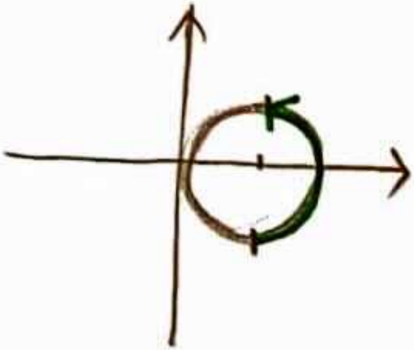
(2) $\int_{-1}^{1} z\cosh z^2\,dz$

$= \left[\frac{1}{2}\sinh z^2\right]_{-1}^{1} = \frac{1}{2}\left(\sinh(1)-\sinh(1)\right) = 0$

(3) $\int_{-i}^{i} \sin z\,dz$

$= \left[-\cos z\right]_{-i}^{i} = -\left(\cos(i)-\cos(-i)\right) = 0$

[과제40]

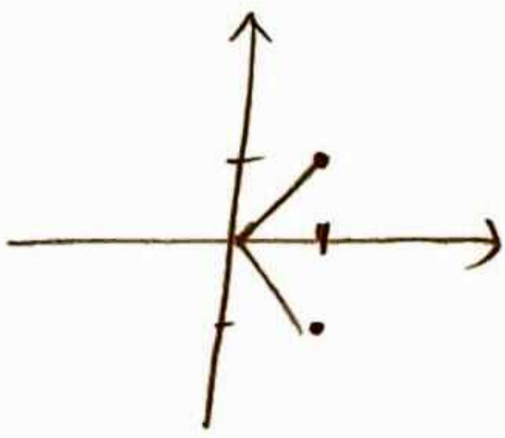


$\int_C \frac{1}{z} dz$. 0과 음수실수를 제외한 범위에서

$\frac{1}{z}$은 해석적이다. 즉.

$$\int_C \frac{1}{z} dz = \Big[Ln\ z \Big]_{1-i}^{1+i} = Ln(1+i) - Ln(1-i)$$

$$= \left(\ln|1+i| + \frac{\pi}{4} i \right) - \left(\ln|1-i| - \frac{\pi}{4} i \right)$$



$$= \frac{\pi}{2} i$$