[과제 31]

(1) $f(z)=Re(z)$

$z=x+iy$

$f(z)=x$ $u=x$ $v=0$

$u_x=1$ $v_y=0$ Cauchy-Riemann equation을 만족X → 해석적X

(2) $f(z)=Re(z^2)$

$z^2=(x+iy)^2=(x^2-y^2)+i2xy$

$Re(z^2)=x^2-y^2$

$u=x^2-y^2$ $v=0$

$u_x=2x$ $v_y=0$

$u_y=-2y$ $v_x=0$

$u_x\neq v_y$. $u_y\neq -v_x$ → 해석적X.

(3) $f(z)=y+ix$

$u=y$ $v=x$

$u_x=0$ $v_y=0$

$u_y=1$ $v_x=1 \rightarrow -v_x=-1$.

$u_y\neq -v_x$ 이므로 해석적 X.

(4) $f(z)=z-\bar{z}$

$z=x+iy$, $\bar{z}=x-iy$

$z-\bar{z}=2iy$

$u=0$ $v=2y$

$u_x=0$ $v_y=2$

$u_y=0$ $v_x=0$

$u_x\neq v_y$ 이므로 해석적X.

(5). $f(z)=\bar{z}^2$

$\bar{z}^2=(x-iy)^2=(x^2-y^2)-i2xy$

$u=x^2-y^2$ $v=-2xy$

$u_x=2x$ $v_y=-2x$

$u_y=-2y$ $v_x=-2y$

$u_x\neq v_y$, $u_y\neq -v_x$ 이므로 해석적X

(6) $f(z)=x^2+y^2$

$u=x^2+y^2$ $v=0$

$u_x=2x$ $v_y=0$

$u_y=2y$ $v_x=0$

$u_x\neq v_y$, $u_y\neq -v_x$ 이므로 해석적X

[과제 32]

(1) $f(z)=z^2+z$

$z=x+iy$

$z^2=(x^2-y^2)+2xyi$

$f(z)=(x^2+x-y^2)+(2xy+y)i$

$u=x^2+x-y^2$, $v=2xy+y$

$u_x=2x+1$ $v_y=2x+1$

$u_y=-2y$ $-v_x=-2y$

$u_x=v_y$, $u_y=-v_x$ 이므로 해석적이다.

(2) $f(z)=z^3$.

$z^3=(x+iy)^3=x^3+3x^2(iy)+3x(iy)^2+(iy)^3$

$=x^3+3x^2yi-3xy^2-y^3i$

$=(x^3-3xy^2)+(3x^2y-y^3)i=f(z)$

$u_x=3x^2-3y^2$ $v_y=3x^2-3y^2$

$u_y=-6xy$ $-v_x=-6xy$

$u_x=v_y$, $u_y=-v_x$ 이므로 해석적이다.

[과제32]

(3) $f(z)=z+\frac{1}{z}$

$z=x+iy$

$$f(z)=x+iy+\frac{1}{x+iy}=x+iy+\frac{x-iy}{x^2+y^2}$$

$$u=x+\frac{x}{x^2+y^2} \qquad v=y-\frac{y}{x^2+y^2}$$

$$u_x=1+\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=1+\frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$v_y=1-\frac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}=1+\frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$u_y=\frac{-x\cdot 2y}{(x^2+y^2)^2}=-\frac{2xy}{(x^2+y^2)^2}$$

$$-v_x=+\frac{-y\cdot 2x}{(x^2+y^2)^2}=\frac{-2xy}{(x^2+y^2)^2}$$

$u_x=v_y$, $u_y=-v_x$이므로 해석적이다.

(4) $f(z)=\frac{1}{1-z}$

$z=x+iy$

$$f(z)=\frac{1}{(1-x)-iy}=\frac{(1-x)+iy}{\{(1-x)-iy\}\{(1-x)+iy\}}$$

$$=\frac{(1-x)+iy}{(1-x)^2-(iy)^2}=\frac{(1-x)+iy}{(1-x)^2+y^2}$$

$$u=\frac{1-x}{x^2-2x+y^2+1} \qquad v=\frac{y}{x^2-2x+y^2+1}$$

$$u_x=\frac{-(x^2-2x+y^2+1)-(1-x)(2x-2)}{(x^2-2x+y^2+1)^2}=\frac{x^2-2x-y^2+1}{(x^2-2x+y^2+1)^2}$$

$$u_y=\frac{-(1-x)(2y)}{(x^2-2x+y^2+1)^2}=\frac{2xy-2y}{(x^2-2x+y^2+1)^2}$$

$$v_y=\frac{(x^2-2x+y^2+1)-y(2y)}{(x^2-2x+y^2+1)^2}=\frac{x^2-2x-y^2+1}{(x^2-2x+y^2+1)^2}$$

$$-v_x=-\frac{-y(2x-2)}{(x^2-2x+y^2+1)^2}=\frac{2xy-2y}{(x^2-2x+y^2+1)^2}$$

$u_x=v_y$, $u_y=-v_x$ 이므로 해석적.

(5) $f(z)=\frac{x-1}{(x-1)^2+y^2}-i\frac{y}{(x-1)^2+y^2}$

$$u=\frac{x-1}{(x-1)^2+y^2} \qquad v=-\frac{y}{(x-1)^2+y^2}$$

$$u_x=\frac{-x^2+2x+y^2-1}{(x^2-2x+y^2+1)^2}$$

$$u_y=\frac{-2xy+2y}{(x^2-2x+y^2+1)^2}$$

$$v_y=\frac{-x^2+2x+y^2-1}{(x^2-2x+y^2+1)^2}$$

$$-v_x=\frac{-2xy+2y}{(x^2-2x+y^2+1)^2}$$

$u_x=v_y$, $u_y=-v_x$ 이므로 해석적.

[과제 33]

(1) $u(x,y)=x^2-y^2-x$

$u_x=2x-1$ $\qquad$ $u_y=-2y$

$u_x=v_y=2x-1$ $\qquad$ $u_y=-v_x=-2y$

$v=2xy-y+h(x)$ $\qquad$ $v_x=2y$

$\downarrow$

$v_x=2y+h'(x)$

$h'(x)=0$ $\quad$ $h(x)=c$

$\therefore v=2xy-y+c$

(2) $u(x,y)=ax+by$

$u_x=a$ $\qquad$ $u_y=b$

$v_y=u_x=a$ $\qquad$ $v_x=-u_y=-b$

$v=ay+h(x)$

$\downarrow$

$v_x=h'(x)=-b$

$h(x)=-bx+c$

$\therefore v=ay-bx+c$

(3) $u(x,y)=x^3-3xy^2$

$u_x=3x^2-3y^2$ $\qquad$ $u_y=-6xy$

$v_y=u_x=3x^2-3y^2$ $\qquad$ $v_x=-u_y=6xy$

$v=3x^2y-y^3+h(x)$

$\downarrow$

$v_x=6xy+h'(x)=6xy$

$h'(x)=0 \rightarrow h(x)=c$

$\therefore v=3x^2y-y^3+c$

[과제33]

(4) $u(x,y) = \dfrac{y}{x^2+y^2}$ $(x^2+y^2 \neq 0)$

$$u_x = \frac{-y \cdot 2x}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$v_y = u_x = \frac{-2xy}{(x^2+y^2)^2}, \quad v_x = -u_y = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v = \frac{x}{x^2+y^2} + h(x)$$

$$\downarrow$$

$$v_x = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + h'(x)$$

$h'(x) = 0$. $h(x) = c$

$$\therefore v = \frac{x}{x^2+y^2} + c$$

(5) $u(x,y) = \ln(x^2+y^2)$

$$u_x = \frac{2x}{x^2+y^2} \qquad u_y = \frac{2y}{x^2+y^2}$$

$$v_y = u_x = \frac{2x}{x^2+y^2} \qquad v_x = -u_y = \frac{-2y}{x^2+y^2}$$

$$v = 2\arctan\left(\frac{y}{x}\right) + h(x)$$

$$v_x = 2 \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{d}{dx}\left(\frac{y}{x}\right) + h'(x)$$

$$= 2 \cdot \frac{y}{-x^2\left(1+\left(\frac{y}{x}\right)^2\right)} + h'(x) = -\frac{2y}{x^2+y^2} + h'(x)$$

$h'(x) = 0$. $h(x) = c$.

$$\therefore v = 2\arctan\left(\frac{y}{x}\right) + c$$

[과제34] $f(z) = u + iv$

해석적이므로 $u_x = v_y$, $u_y = -v_x$ 이다.

$Re(f(z)) = u =$ 상수

$u_x = 0$, $u_y = 0$

$v_y = u_x = 0$, $v_x = -u_y = 0$

즉, v도 상수함수이다.

따라서 u, v 모두 상수함수이므로

$f(z)$도 상수함수이다.

[과제35]

$x = r\cos\theta,\quad y = r\sin\theta$

$$\left.\begin{aligned}
&- u_r = u_x x_r + u_y y_r = u_x \cdot \cos\theta + u_y \cdot \sin\theta \\
&- u_\theta = u_x x_\theta + u_y y_\theta = -u_x \cdot r\sin\theta + u_y \cdot r\cos\theta \\
&- v_r = v_x x_r + v_y y_r = v_x \cos\theta + v_y \sin\theta. \\
&- v_\theta = v_x x_\theta + v_y y_\theta = -v_x r\cos\theta + v_y \cdot r\cos\theta.
\end{aligned}\right\} \text{연대}$$

$f(z)$의 u와 v가 코시-리만 방정식은 만족한다고 하면, $u_x = v_y$, $u_y = -v_x$ 이다.

$$v_r = -u_y \cos\theta + u_x \sin\theta = -\frac{1}{r} u_\theta.$$

$$v_\theta = u_y \cdot r\sin\theta + u_x r\cos\theta = r u_r.$$

즉 $u_r = \frac{1}{r} v_\theta$, $v_r = -\frac{1}{r} u_\theta$를 만족한다.