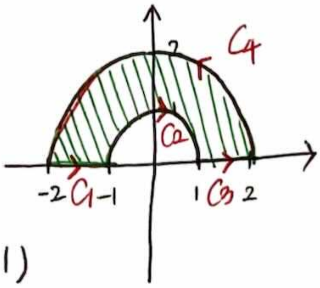
[과제4]

(1) $\oint_C \frac{z}{\bar z}\,dz$.



$C_1 : z(t) = t-2$
$(0 \le t \le 1)$

$C_2' : z(t) = e^{it} \quad (0 \le t \le \pi)$

$C_3 : z(t) = t+1 \quad (0 \le t \le 1)$

$C_4 : z(t) = 2e^{it} \quad (0 \le t \le \pi)$

$$\oint_C \frac{z}{\bar z}\,dz = \int_{C_1} \frac{z}{\bar z}\,dz + \int_{C_2} \frac{z}{\bar z}\,dz + \int_{C_3} \frac{z}{\bar z}\,dz + \int_{C_4} \frac{z}{\bar z}\,dz$$

$$\int_{C_1} \frac{z}{\bar z}\,dz = \int_0^1 \frac{t-2}{t-2}\cdot 1 \cdot dt = 1$$

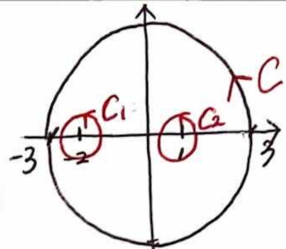
$$\int_{C_2} \frac{z}{\bar z}\,dz = -\int_{C_2'} \frac{z}{\bar z}\,dz = -\int_0^\pi \frac{e^{it}}{e^{-it}}\cdot ie^{it}\,dt$$
$$= -\int_0^\pi ie^{3it}\,dt = -\left[\frac{1}{3}e^{3it}\right]_0^\pi$$
$$= -\frac{1}{3}\left(e^{3i\pi}-1\right) = -\frac{1}{3}(-1-1) = \frac{2}{3}$$

$$\int_{C_3} \frac{z}{\bar z}\,dz = \int_0^1 \frac{t+1}{t+1}\cdot 1\,dt = 1.$$

$$\int_{C_4} \frac{z}{\bar z}\,dz = \int_0^\pi \frac{2e^{it}}{2e^{-it}}\cdot 2ie^{it}\,dt = \int_0^\pi 2ie^{3it}\,dt$$
$$= \left[\frac{2}{3}e^{3it}\right]_0^\pi = \frac{2}{3}\left(e^{3\pi i}-1\right) = -\frac{2}{3}$$

$\therefore\ 1+\frac{2}{3}+1-\frac{2}{3} = 2$

(2) $\oint_C \frac{z^3+3z-1}{(z-1)(z+2)}\,dz$.



다중연결영역에서 코시의 적분정리에 의해 $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$가 성립한다.

$$\oint_C \frac{z^3+3z-1}{(z-1)(z+2)}\,dz$$

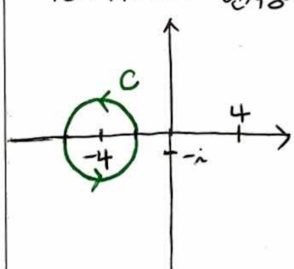
$$= \oint_{C_1} \frac{\overbrace{z^3+3z-1}^{f_1(z)}}{(z-1)(z+2)}\,dz + \oint_{C_2} \frac{\overbrace{z^3+3z-1}^{f_2(z)}}{(z-1)(z+2)}\,dz$$

-2포함 1포함

$$= 2\pi i f(-2) + 2\pi i f(1)$$
$$= 2\pi i\left(\frac{-8-6-1}{-3} + \frac{1+3-1}{3}\right) = 2\pi i\left(\frac{15}{3}+\frac{3}{3}\right)$$
$$= 12\pi i$$

(3) $\oint_C \frac{z}{(16-z^2)(z+i)}\,dz$

$|z+4| = 2$: 반지름이 2, 중심이 (-4, 0)인 원



$$\oint_C \frac{z}{(4+z)(4-z)(z+i)}\,dz$$

-4포함, 4/-i포함 X.

$$\Rightarrow f(z) = \frac{z}{(4-z)(z+i)}$$

$$2\pi i f(-4) = 2\pi i \cdot \frac{-4}{8(-4+i)} = \frac{1}{2(4-i)}\cdot 2\pi i$$

$$= \frac{\pi i}{4-i} = \frac{\pi(4i-1)}{17}$$

(4) $\oint_C \frac{e^{-z}\cos z}{(z-2)^3}\,dz$

도함수에 관한 코시의 적분 공식에 의해

$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}f^{(n)}(z_0)$ 가 성립한다.

$\oint_C \frac{e^{-z}\cos z}{(z-2)^3}\,dz = 2\pi i(e^{-z}\cos z)''\Big|_{z=3}.$

$(e^{-z}\cos z)' = -e^{-z}\cos z - e^{-z}\sin z$

$(e^{-z}\cos z)'' = e^{-z}\cancel{\cos z} + e^{-z}\sin z + e^{-z}\sin z - e^{-z}\cancel{\cos z}$

$= 2e^{-z}\sin z.$

$\therefore 2\pi i(2e^{-z}\sin z)\Big|_{z=3} = 2\pi i(2e^{-3}\sin 3)$

$= 4\pi e^{-3}\sin 3\, i$

(5) $\oint_C (Re(z)+\alpha)\frac{f(z)}{z}\,dz.$

$(Re(z)+\alpha)f(z) = g(z)$라 하자. $g(z)$는 해석적이므로.

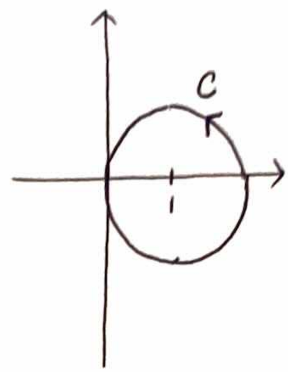
코시의 적분 공식에 의해.

$\oint_C (Re(z)+\alpha)\frac{f(z)}{z}\,dz$

$= 2\pi i\cdot g(z)\Big|_{z=0} = 2\pi i\cdot g(0) = 2\pi i\Big(Re(0)+\alpha\Big)f(0)$

$= 2\pi\alpha f(0) i$

(6) $\oint_C (Im(z))^3\,dz$



$z(t) = 1+e^{it}\ (0\le t\le 2\pi)$

$= 1+\cos t + i\sin t$

$dz = (-\sin t + i\cos t)\,dt$

$\int_0^{2\pi} (Im(z(t)))^3(-\sin t + i\cos t)\,dt$

$= \int_0^{2\pi} \sin^3 t(-\sin t + i\cos t)\,dt$

$= \int_0^{2\pi} -\sin^4 t + i\sin^3 t\cos t\ dt$

$\int_0^{2\pi} \sin^4 t\ dt = \int_0^{2\pi}\left(\frac{1-\cos\frac{t}{2}}{2}\right)^2 dt$

$= \frac{1}{4}\int_0^{2\pi} 1-2\cos\frac{t}{2}+\cos^2\frac{t}{2}\ dt$

$= \frac{1}{4}\int_0^{2\pi} 1-2\cos\frac{t}{2}+\frac{1+\cos\frac{t}{4}}{2}\ dt$

$= \frac{1}{4}\left[t-4\sin\frac{t}{2}+\frac{t}{2}+2\sin\frac{t}{4}\right]_0^{2\pi}$

$= \frac{1}{4}(2\pi+\pi+2) = \frac{1}{4}(3\pi+2)$

$\int_0^{2\pi} i\sin^3 t\cos t\ dt$

$= i\left[\frac{1}{4}\sin^4 t\right]_0^{2\pi} = 0.$

$\Rightarrow \int_0^{2\pi} -\sin^4 t + i\sin^3 t\cos t\ dt$

$= -\frac{1}{4}(3\pi+2)$