ECON526: Quantitative Economics with Data Science Applications

Probability and Uncertainty

Jesse Perla

jesse.perla@ubc.ca

University of British Columbia

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Overview

Summary

- Will provide background on probability, simulation of randomness, independence, and expectations
- See the following for extra material - some of which were used in these notes
  - → QuantEcon Probability
  - → QuantEcon Distributions and Probabilities
  - → QuantEcon LLN and CLT
- Using the following packages and definitions

```python
import matplotlib.pyplot as plt
import pandas as pd
import numpy as np
import scipy.stats
import seaborn as sns
from matplotlib.animation import FuncAnimation
import IPython.display
```

Probability

Definitions

To formalize probability always be careful to separate

1. **Events** i.e., probability space.
2. **Probability** an events occurs. i.e., probability measure
3. **Value** or implications of an event. i.e., random variables

Probability Space

Probability space is a $(\Omega, \mathcal{A})$:

- Set Ω of possible **outcomes** and $\omega \in \Omega$ is a particular outcome
  - $\rightarrow$ e.g. $\Omega = \{U, E, R, D\}$ for unemployed, employed, retired, or dead
- Subsets $A \subseteq \Omega$ are **events**
  - $\rightarrow$ e.g. $A = \{U, E\}$ is the event of being employed or unemployed
  - $\rightarrow$ $\Omega \setminus A = \{R, D\}$ (the `\setminus`) is event of not being either
- The collection of all possible events is $\mathcal{A}$ where $A \in \mathcal{A}$
  - $\rightarrow$ $\Omega \in \mathcal{A}$, i.e. we can consider the event of any outcome occurring
  - $\rightarrow$ $\emptyset \in \mathcal{A}$, i.e. we can consider the event of nothing occurring

Probability Measure

Probability Measure is a function which assigns a numerical value on the likelihood of an event

- For us, $\mathbb{P} : \mathcal{A} \to [0, 1]$
  - $\rightarrow$ e.g. $\mathbb{P}(\{U, E\}) = 0.7$ is probability either U or E
  - $\rightarrow$ $\mathbb{P}(\Omega \setminus \{U, E\}) = 0.3$
- Will see denoted as a function, $\mu(A)$ for integrals in advanced uses
  - $\rightarrow$ Overkill for probability spaces with a finite, discrete number of elements
  - $\rightarrow$ Important for probability spaces with a continuous number of elements
  - $\rightarrow$ Essential for stochastic processes (e.g., flipping a coin until heads)

Random Variables

Random Variable: $X(\omega)$ assigns a numerical value to a particular outcome

- $X : \Omega \to \mathbb{R}$, but could be vector or matrix valued
  - $\rightarrow$ e.g. $X(\omega = E) = 1$ if employed, $X(\omega = U) = 0$ if unemployed. Useful for doing counts
- Or $X(\omega = E) = \$40,000$ if employed, $X(\omega = E) = \$15,000$ if unemployed. Useful for finding average incomes
- Random variables defined on Ω, and inherit the probability measure
  - $\rightarrow$ So can query values like $\mathbb{P}(X = \$40,000)$

Axioms of Probability

Probability measure $\mathbb{P}$ on probability space $(\Omega, \mathcal{A})$ must satisfy axioms:

- **Non-negativity**: $\mathbb{P}(A) \geq 0$
- **Normalization**: $\mathbb{P}(\Omega) = 1$
- **Additivity**: If $A \cap B = \emptyset$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

These imply other results such as:

- $\mathbb{P}(\emptyset) = 0$
- $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- $\mathbb{P}(\Omega \setminus A) = 1 - \mathbb{P}(A)$

Discrete Distributions

Discrete Distributions

- A discrete probability spaces have finite (or countable) number of outcomes
- When convenient, we can number the outcomes arbitrarily as $n = 1, \ldots N$ (or ∞) and then work with $\Omega = \{1, \ldots N\}$ and $\omega \in \Omega$
- Axioms especially simple because we use $\mathbb{P}(\omega = n) = p_n$,
  - → **Non-negativity**: $p_n \geq 0$
  - → **Normalization**: $\sum_{n=1}^{N} p_n = 1$
  - → **Additivity**: $\mathbb{P}(A) = \sum_{n \in A} p_n$

Random Variables

- Notation can become a little confusing because we will sometimes use the same index number for the event and for the random value, but they are separate!
- Frequently we will assign the random variable as just that index
  - → $X(\omega = n) = n$ and then denote $\mathbb{P}(X = n) = p_n$
- Other times we may want to associate a value with each outcome
  - → $X(\omega = n) = x_n$ and then denote $\mathbb{P}(X = x_n) = p_n$

PDF and CDF

- **Probability Mass Function (PMF)** is the probability of a single outcome for random variable X. Will assume X itself has discrete values

$$p_n \equiv \mathbb{P}(X = n)$$

- **Cumulative Distribution Function (CDF)** is the probability of all outcomes less than or equal to a particular outcome.

$$\mathbb{P}(X \leq n) = \sum_{i=1}^{n} p_i$$

Expectation

- Expectation of a random variable is the sum of the values weighted by the probabilities. Continuous Ω uses integrals, or measure theory if “weird”
- Especially easy to compute for discrete random variables

$$\mathbb{E}[X] = \sum_{n=1}^{N} x_n \mathbb{P}(X = x_n)$$

- Generalized to functions of a random variables

$$\mathbb{E}[f(X)] = \sum_{n=1}^{N} f(x_n) \mathbb{P}(X = x_n)$$

Expectations and Linear Algebra

Vectors can help with the accounting and notation of expectations. Let

- $x \equiv \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^\top$ be the list of values for the random variable X
- $p \equiv \begin{bmatrix} p_1 & p_2 & \dots & p_N \end{bmatrix}^\top$ be the list of probabilities
- Then the expectation is (broadcasting $f(\cdot)$ across x as required)

$$\mathbb{E}[X] = \sum_{n=1}^{N} x_n \mathbb{P}(X = x_n) = p \cdot x = p^\top x$$

$$\mathbb{E}[f(X)] = \sum_{n=1}^{N} f(x_n) \mathbb{P}(X = x_n) = p \cdot f(x) = p^\top f(x)$$

Example with a Discrete Distribution

- $\Omega = \{U, E, R\}$
- $\mathbb{P}(U) = 0.1, \mathbb{P}(E) = 0.8, \mathbb{P}(R) = 0.1$
- $X(U) = 15000, X(E) = 40000, X(R) = 10000$
- $\mathbb{E}[X]$ and $\mathbb{E}[\sqrt{X}]$

```python
p = np.array([0.1, 0.8, 0.1])
x = np.array([15000, 40000, 10000])
def f(x):
    return np.sqrt(x)
print(f"E(X) = {p @ x}")
print(f"E(f(X)) = {p @ f(x)}")
print(f"CDF(X) = {np.cumsum(p)}")
```

```
E(X) = 34500.0
E(f(X)) = 182.2474487139159
CDF(X) = [0.1 0.9 1. ]
```

Note that the CDF was easy to calculate as cumulative sums. Interpretable?

Using the `discrete_rv`

- `scipy.stats` has a `discrete_rv` type with built-in functions
- Useful for working with discrete random variables

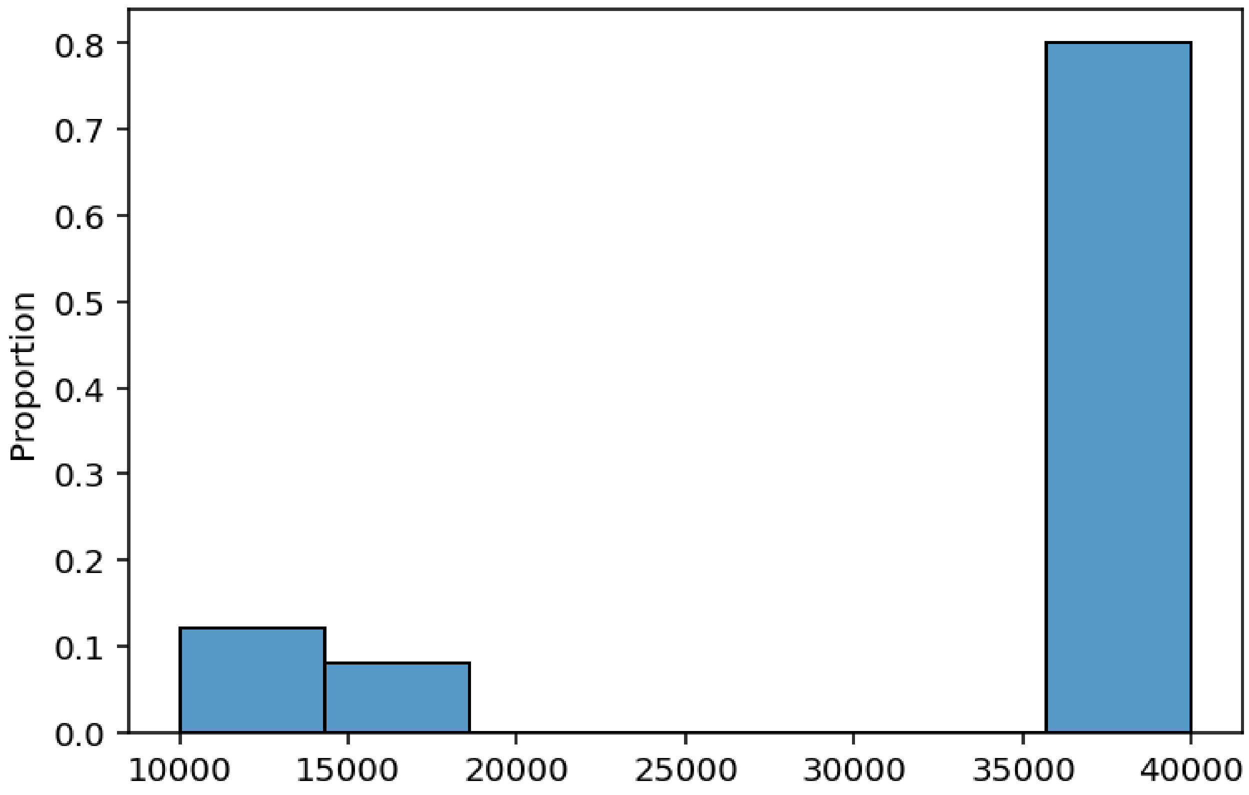
```python
p = np.array([0.1, 0.8, 0.1])
x = np.array([15000, 40000, 10000])
u = scipy.stats.rv_discrete(
  values=(x, p))
samples = u.rvs(size=5)
print(f"E(X) = {u.mean()}")
print(f"E(f(X)) = {u.expect(f)}")
print(f"CDF(X) = {u.cdf(x)}")
print(f"Samples of X = {samples}")
```

```
E(X) = 34500.0
E(f(X)) = 182.2474487139159
CDF(X) = [0.2 1.  0.1]
Samples of X = [15000 40000 40000 40000 40000]
```

Histogram $N = 50$

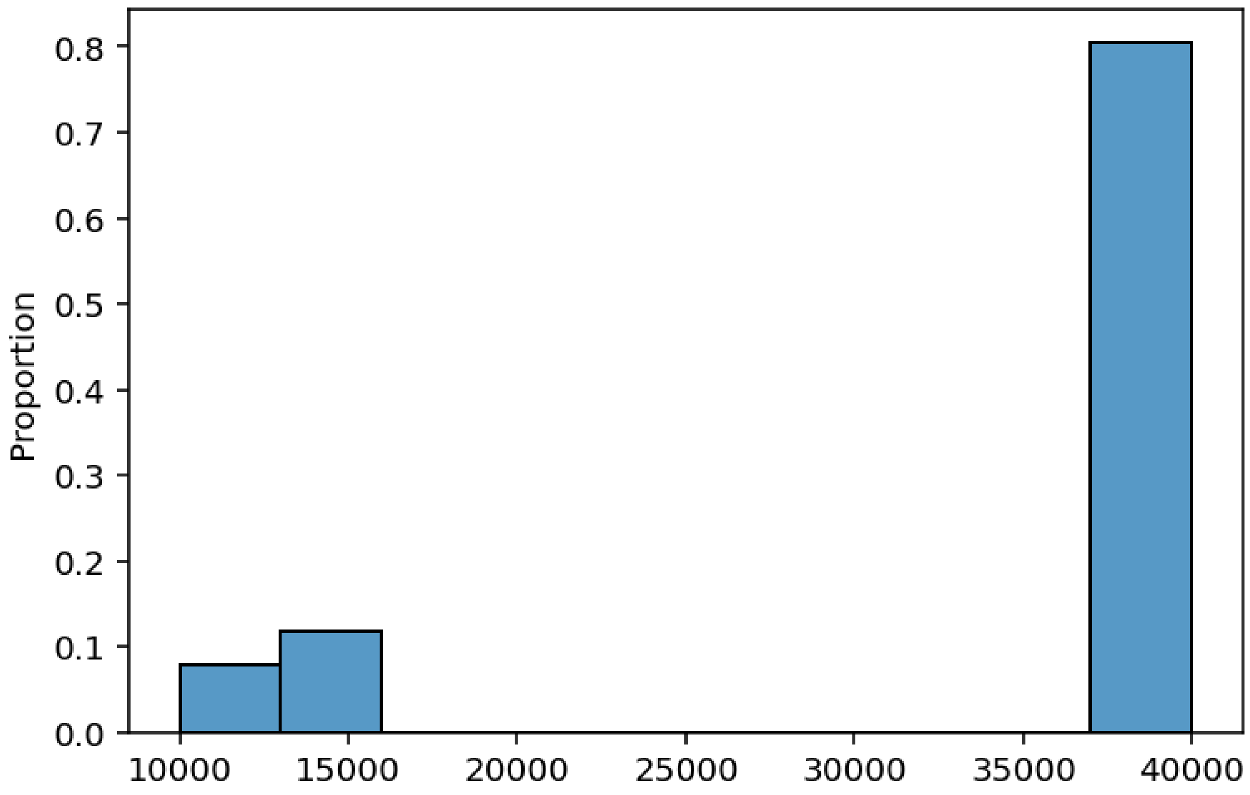
```python
N = 50
samples = u.rvs(size=N)
ax = sns.histplot(samples,
  stat="proportion")

# Alternative
# Density doesn't add up
# plt.hist(samples_1,
#   density=True)
# Or must build barchart
```



Histogram $N = 500$

```python
N = 500
samples = u.rvs(size=N)
ax = sns.histplot(samples,
  stat="proportion")
```



The Binomial Distribution

For $n = 1 \ldots N$, the **binomial distribution** is defined by the PMF

$$\mathbb{P}(X = n) = \binom{N}{n} \theta^n (1 - \theta)^{N-n}$$

$$\mathbb{E}(X) = \sum_{n=0}^{N} n \binom{N}{n} \theta^n (1 - \theta)^{N-n} = N\theta$$

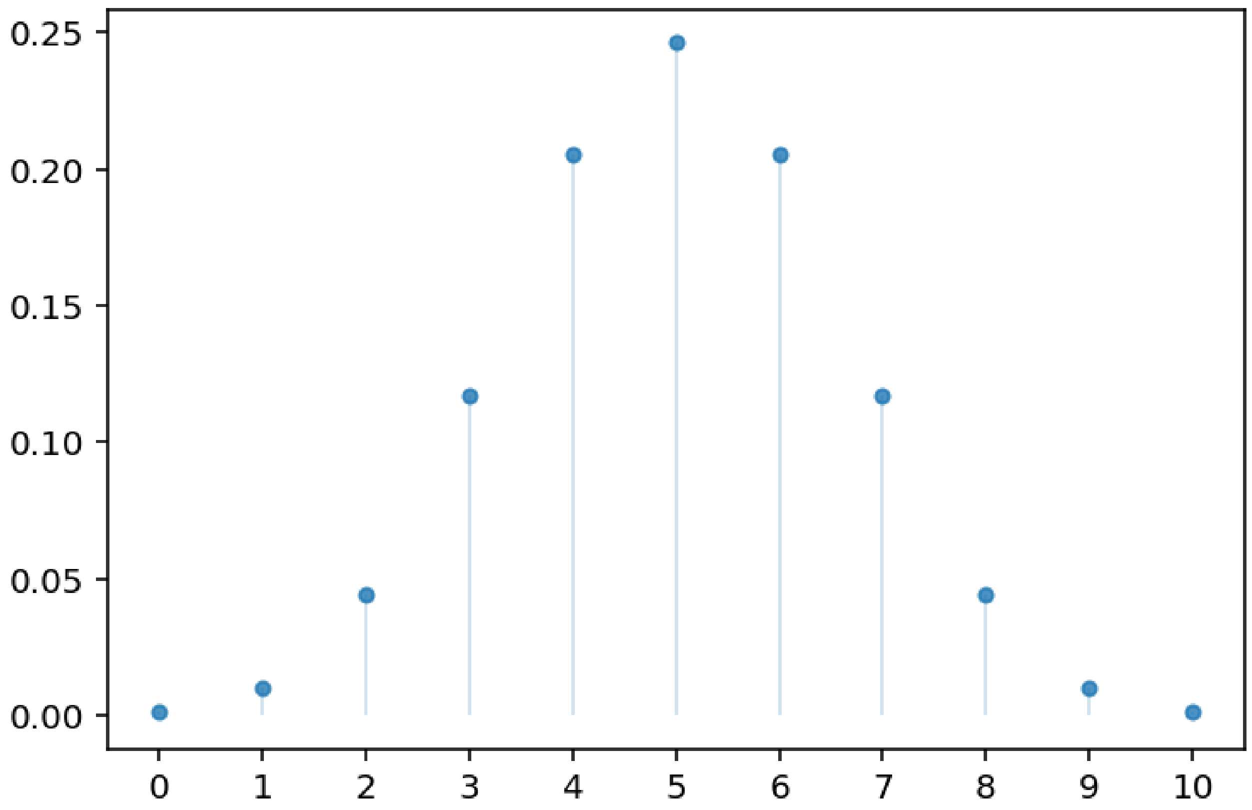
```python
N = 10
θ = 0.5
u = scipy.stats.binom(N, θ)
print(f"Mean: {u.mean():.2f}")
print(f"Variance: {u.var():.2f}")
print(f"Draws of u: {u.rvs(5)}")
```

```
Mean: 5.00
Variance: 2.50
Draws of u: [6 3 5 4 8]
```

The Binomial Probability Mass Function

```python
grid = np.arange(N+1)
u_pmf = u.pmf(grid)

fig, ax = plt.subplots()
ax.plot(grid, u_pmf,
  linestyle='',
  marker='o',
  alpha=0.8, ms=4)
ax.vlines(grid, 0,
  u_pmf,
  lw=0.2)
ax.set_xticks(grid)
plt.show()
```

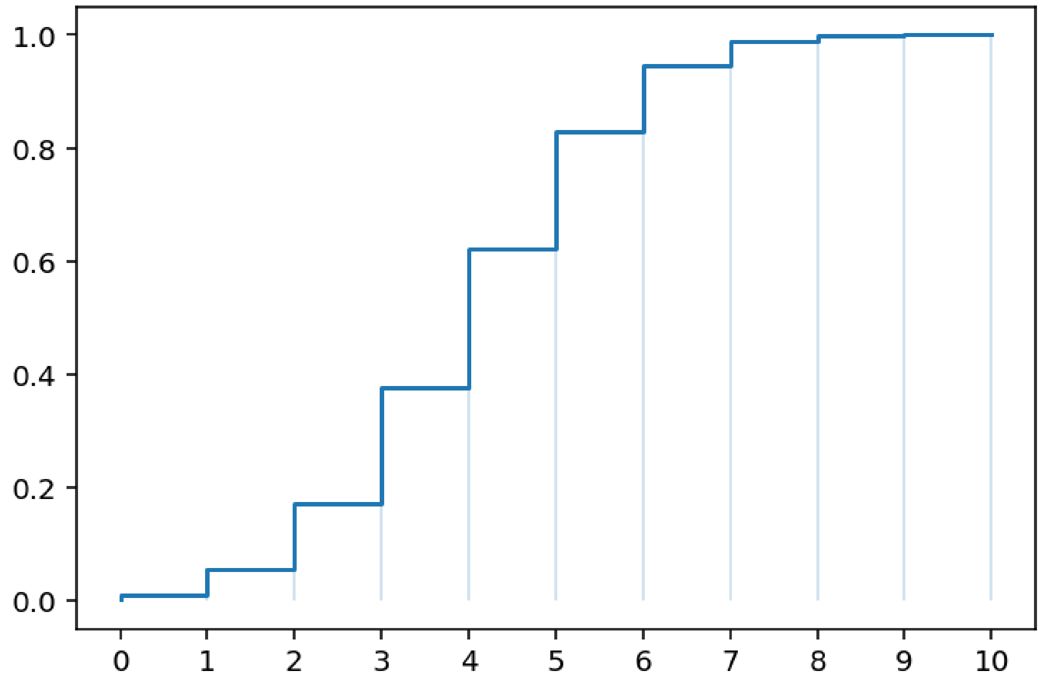


The Binomial Cumulative Distribution Function

$$\mathbb{P}(X \leq n) = \sum_{i=0}^{n} \binom{N}{i} \theta^i (1-\theta)^{N-i}$$

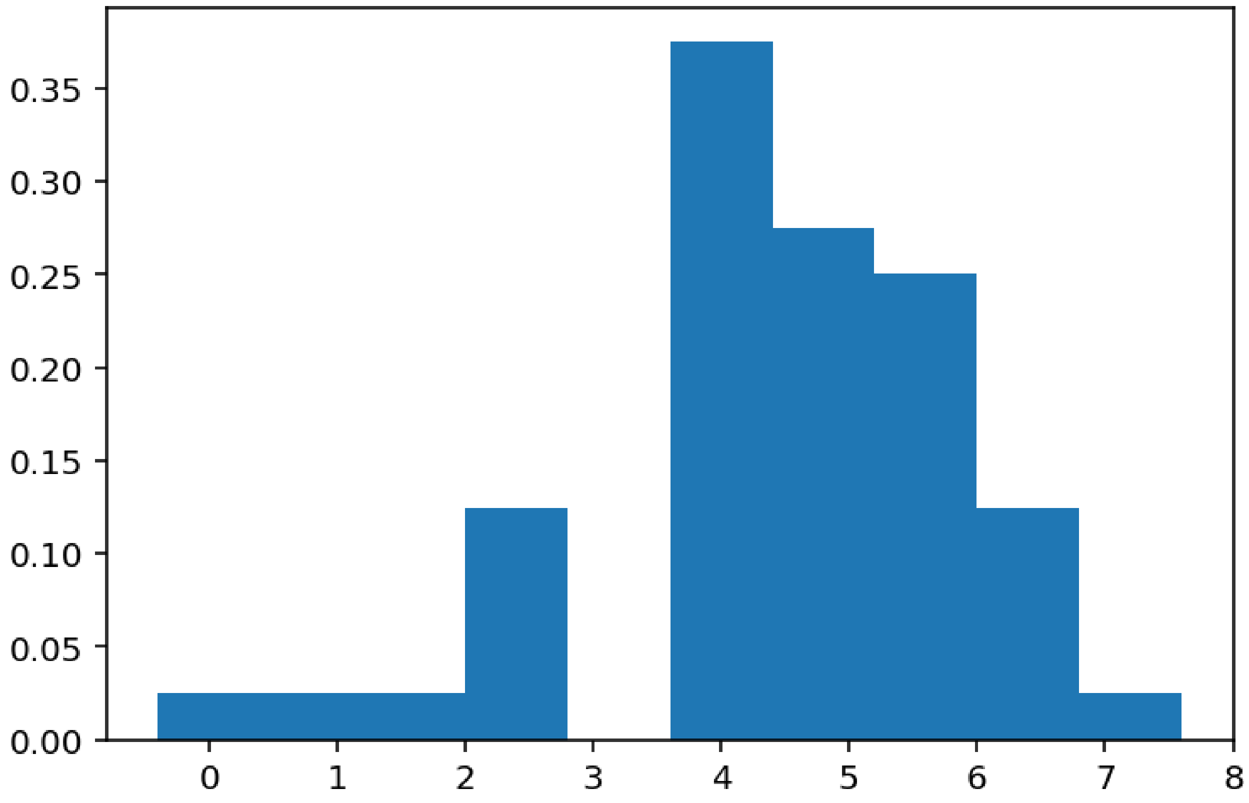
```
grid = np.arange(N+1)
u_cdf = u.cdf(grid)

fig, ax = plt.subplots()
ax.step(grid, u_cdf)
ax.vlines(grid, 0, u_cdf,
  lw=0.2)
ax.set_xticks(grid)
plt.show()
```



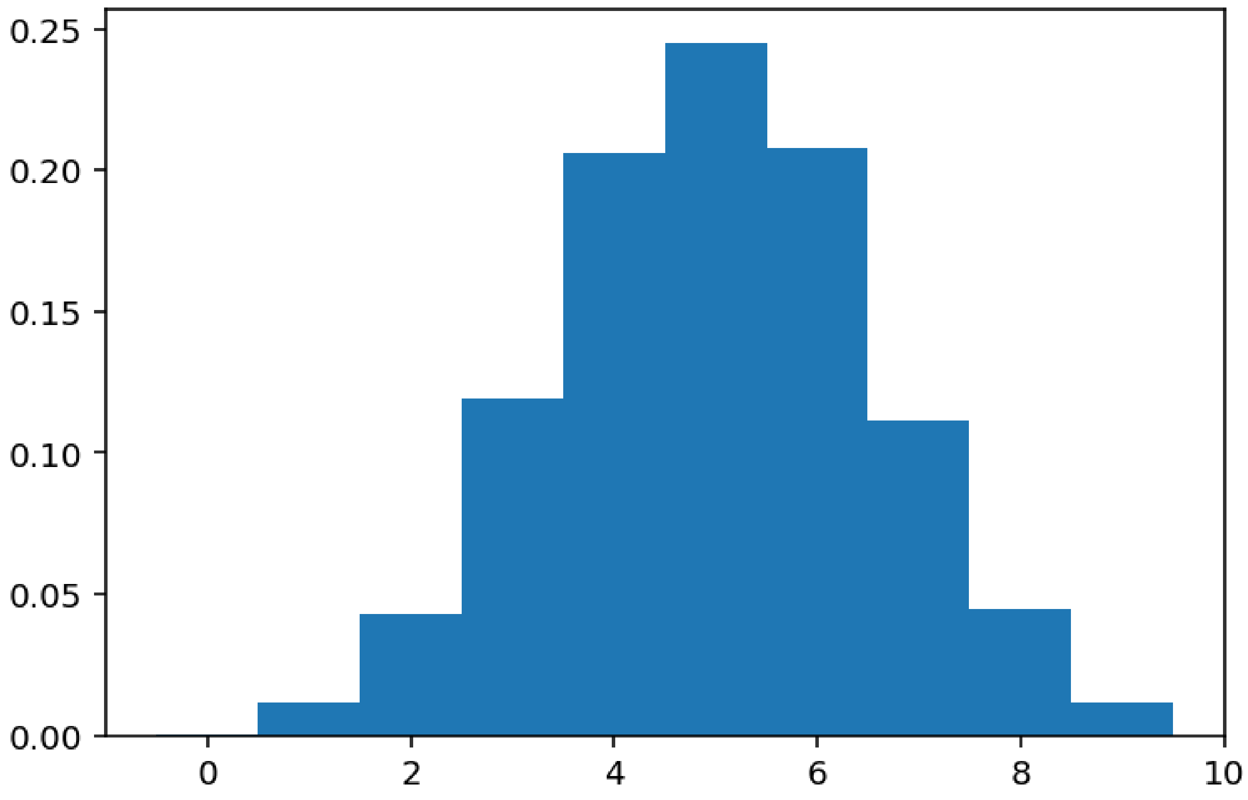
Histogram $N = 50$

```python
N = 50
u = scipy.stats.binom(10, 0.5)
plt.hist(u.rvs(size=N),
  density=True,
  align='left')
plt.show()
```



Histogram $N = 5000$

```
N = 5000
u = scipy.stats.binom(10, 0.5)
plt.hist(u.rvs(size=N),
  density=True,
  align='left')
plt.show()
```



LLN and CLT

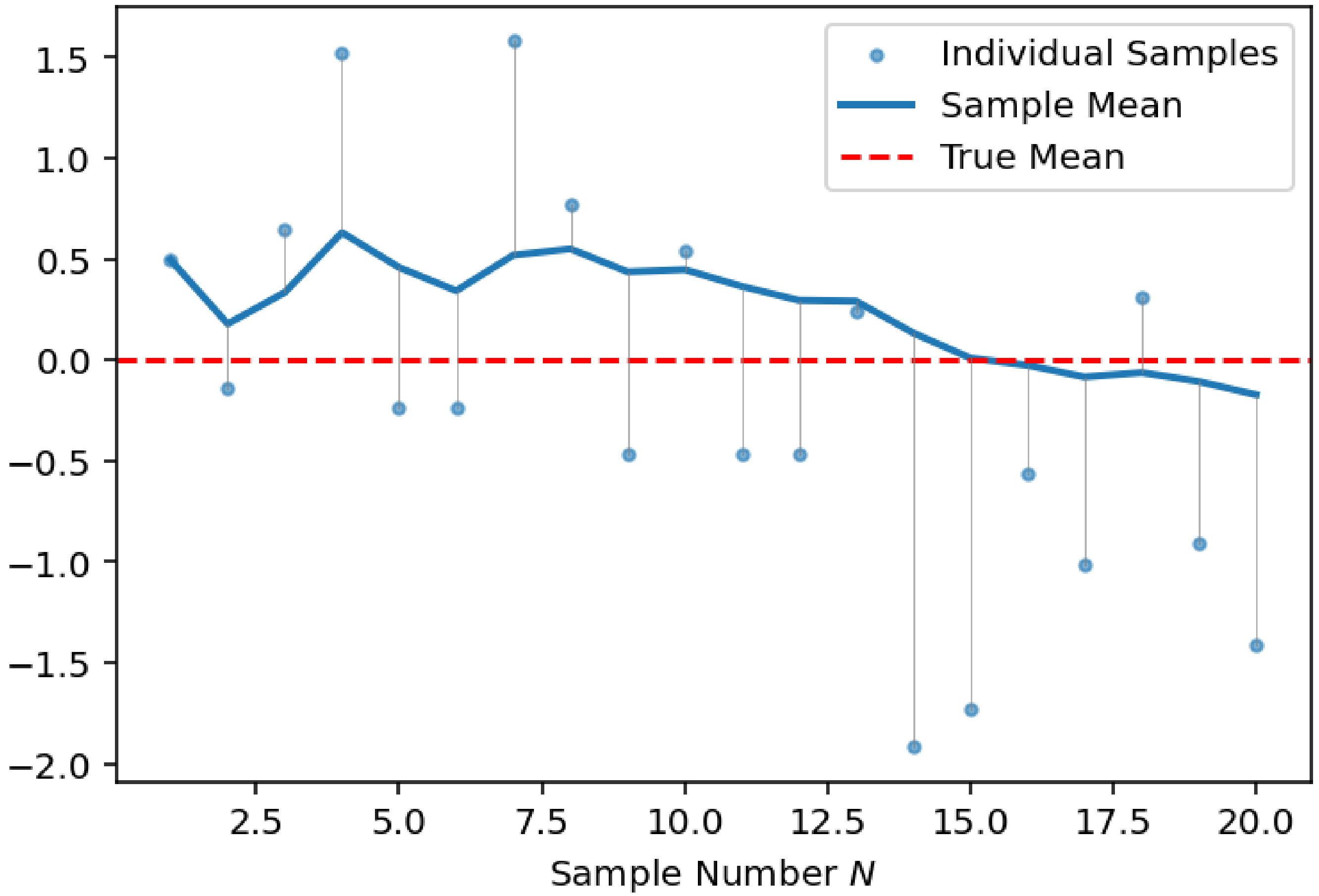
Law of Large Numbers (LLN)

- A classic LLN is the Strong Law of Large Numbers
- Take a sequence of independent, identically distributed random variables $X_1, X_2, \ldots$ with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2 < \infty$. Then,
  - → If X_n is a random variable then $\bar{X}_N \equiv \frac{1}{N}\sum_{i=1}^{N} X_n$ is also an RV
  - → The law says for any $\epsilon > 0$, $\lim_{N\to\infty} \mathbb{P}(|\bar{X}_N - \mu| > \epsilon) \to 0$
  - → Sometimes denoted $\bar{X}_N \xrightarrow{p} \mu$ for "convergence in probability"
- Powerful and frequently used, but remember assumptions!

Visualizing the LLN with Gaussian RVs $N = 20$

```python
N = 20  # Number of samples
mu, sigma = 0, 1
np.random.seed(42)
samples = np.random.normal(mu, sigma, N)
sample_means = np.cumsum(samples) / np.arange(1, N + 1)
plt.scatter(range(1, N + 1), samples, label='Individual Samples', alpha=0.6, s=10)
plt.plot(range(1, N + 1), sample_means, label='Sample Mean', linewidth=2)
plt.axhline(mu, color='r', linestyle='--', label='True Mean')
for n in range(N): # add lines to samples from sample mean
    plt.plot([n + 1, n + 1], [sample_means[n], samples[n]], color='gray', linewidth=0.5, alpha=0.6)
plt.xlabel('Sample Number $N$')
plt.legend()
plt.show()
```

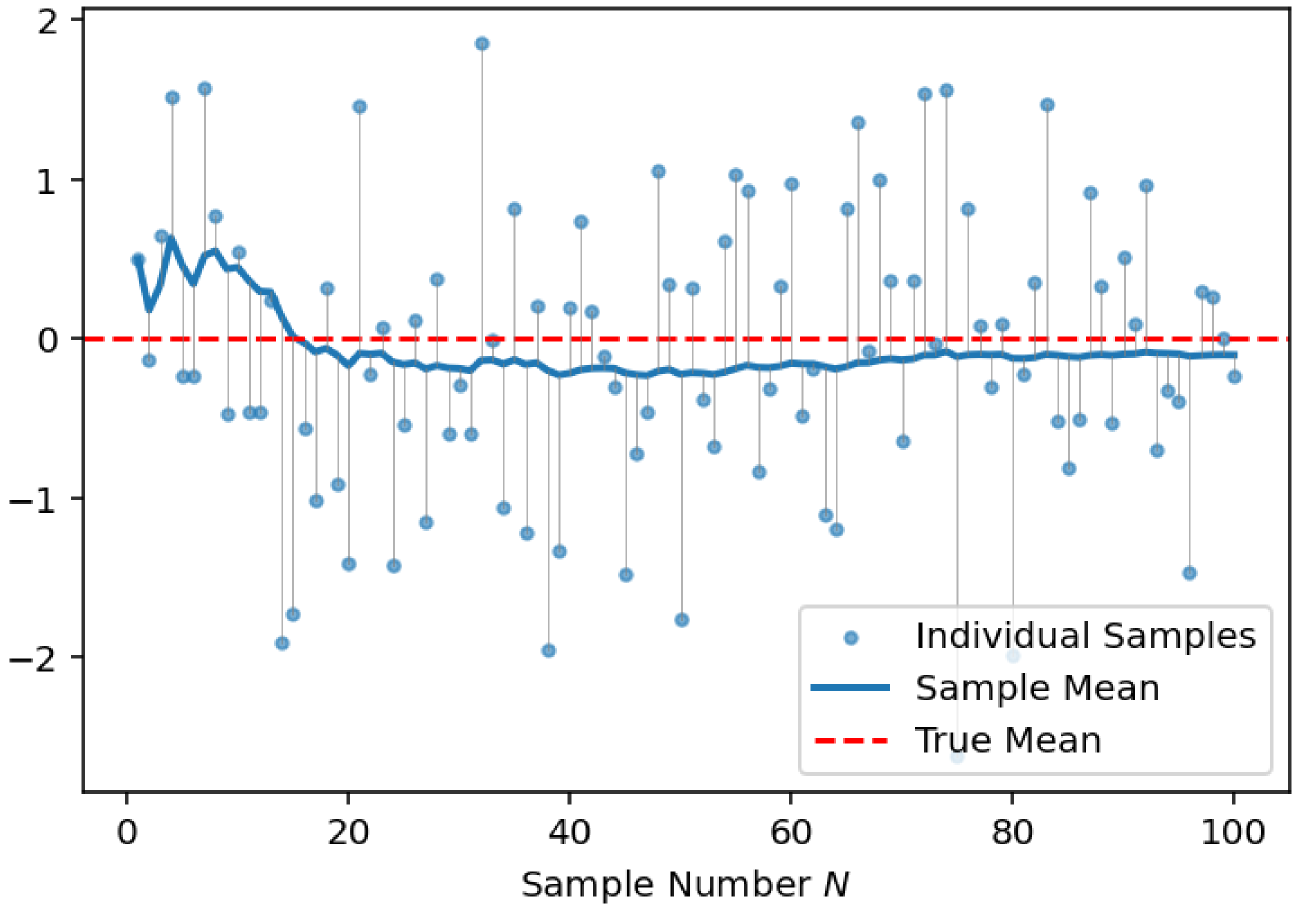
Visualizing the LLN with Gaussian RVs $N = 20$



Visualizing the LLN with Gaussian RVs $N = 100$

```python
N = 100  # Number of samples
mu, sigma = 0, 1
np.random.seed(42)
samples = np.random.normal(mu, sigma, N)
sample_means = np.cumsum(samples) / np.arange(1, N + 1)
plt.scatter(range(1, N + 1), samples, label='Individual Samples', alpha=0.6, s=10)
plt.plot(range(1, N + 1), sample_means, label='Sample Mean', linewidth=2)
plt.axhline(mu, color='r', linestyle='--', label='True Mean')
for n in range(N): # add lines to samples from sample mean
    plt.plot([n + 1, n + 1], [sample_means[n], samples[n]], color='gray', linewidth=0.5, alpha=0.6)
plt.xlabel('Sample Number $N$')
plt.legend()
plt.show()
```

Visualizing the LLN with Gaussian RVs $N = 100$



Pareto Distributions

- Pareto distributions are a family of distributions with a power-law tail
- Parameterized by (x_m, α) with the PDF

$$p(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$$

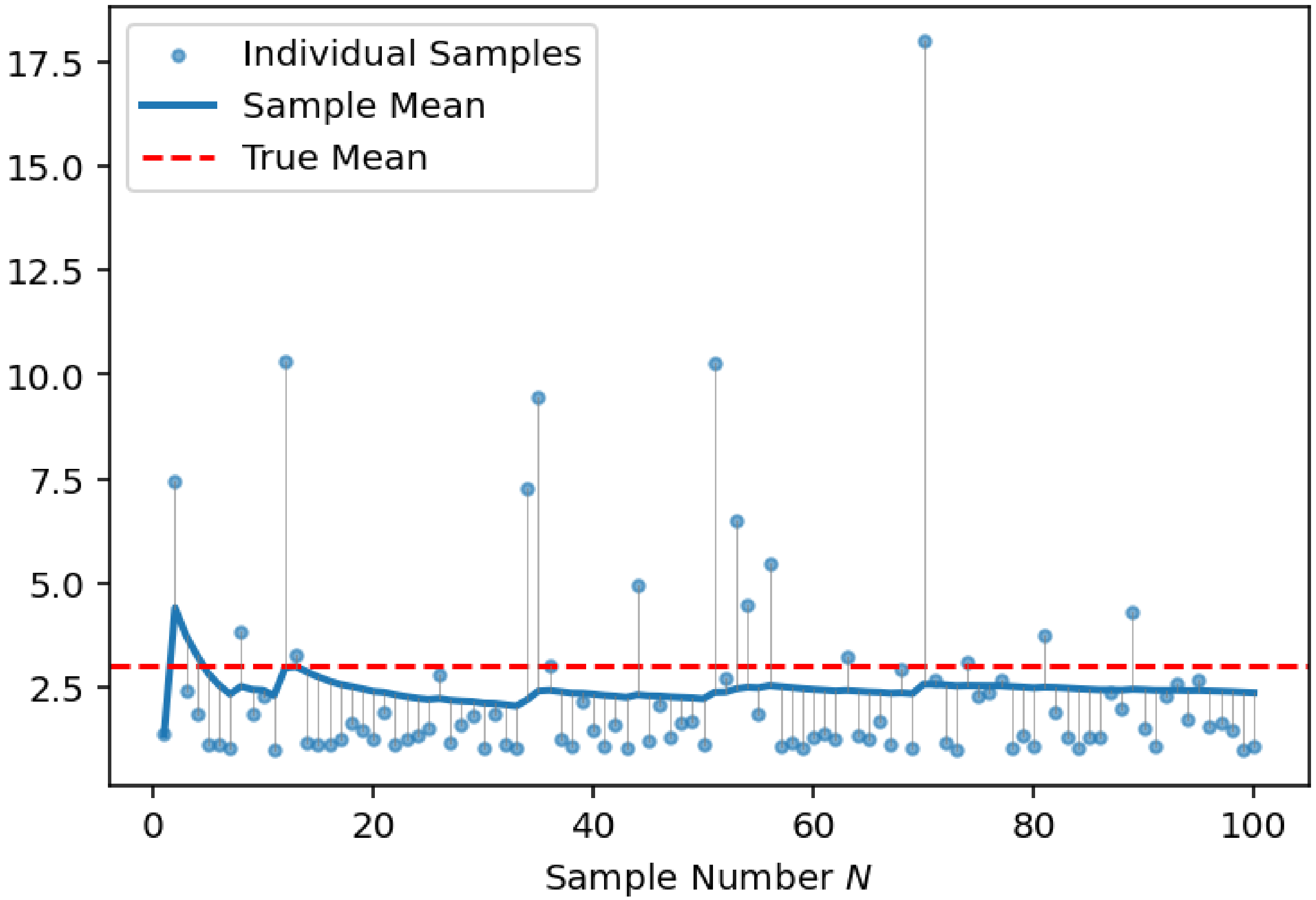
- The mean is $\mathbb{E}[X] = \frac{\alpha x_m}{\alpha - 1}$ for $\alpha > 1$
- The variance is $\mathbb{V}[X] = \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$

A distribution with pdf $p(x)$ is power-law if $p(x) \propto x^{-\alpha}$ for some $\alpha > 0$ as $x \to \infty$. More formally, if $\lim_{x\to\infty} \frac{\log p(x)}{\log x} = -\alpha$

Visualizing the Sample Means for a Pareto Distribution

```python
N = 100 # Number of samples
alpha = 1.5
np.random.seed(42)
dist = scipy.stats.pareto(alpha)
samples = dist.rvs(size=N)
sample_means = np.cumsum(samples) / np.arange(1, N + 1)
plt.scatter(range(1, N + 1), samples, label='Individual Samples', alpha=0.6, s=10)
plt.plot(range(1, N + 1), sample_means, label='Sample Mean', linewidth=2)
plt.axhline(dist.mean(), color='r', linestyle='--', label='True Mean')
for n in range(N): # add lines to samples from sample mean
    plt.plot([n + 1, n + 1], [sample_means[n], samples[n]], color='gray', linewidth=0.5, alpha=0.6)
plt.xlabel('Sample Number $N$')
plt.legend()
plt.show()
```

Visualizing the Sample Means for a Pareto Distribution



Central Limit Theorem (CLT)

- The Central Limit Theorem (CLT) is a classic result in statistics
- Again, lets assume we have IID observations with $\mathbb{E}[X_i] = \mu$ and $\mathbb{V}[X_i] = \sigma^2 < \infty$
- Define the sample mean $\bar{X}_N \equiv \frac{1}{N} \sum_{i=1}^{N} X_i$
- Then the CLT is

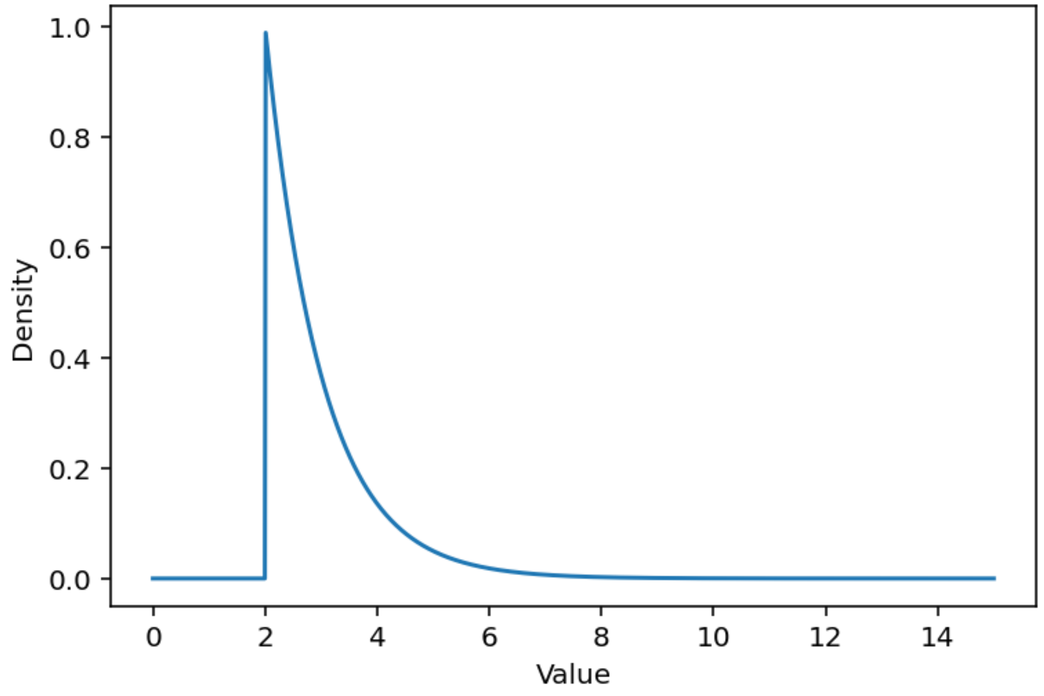
$$\sqrt{n}\left(\bar{X}_n - \mu\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

→ That notation means converges in distribution, which roughly means that as $n \to \infty$ the CDF are getting closer to each other

Visualizing the CLT with Exponential Distributions

- See QuantEcon CLT lecture for the source.
- Exponential distributions $p(x) = \lambda e^{-\lambda x}$ for $\lambda = 0.5$

```python
distribution = scipy.stats.expon(1/0.5)
mu, s = distribution.mean(), distribution.std()
x = np.linspace(0, 15.0, 1000)
y = distribution.pdf(x)
plt.plot(x, y)
plt.xlabel('Value')
plt.ylabel('Density')
plt.show()
```



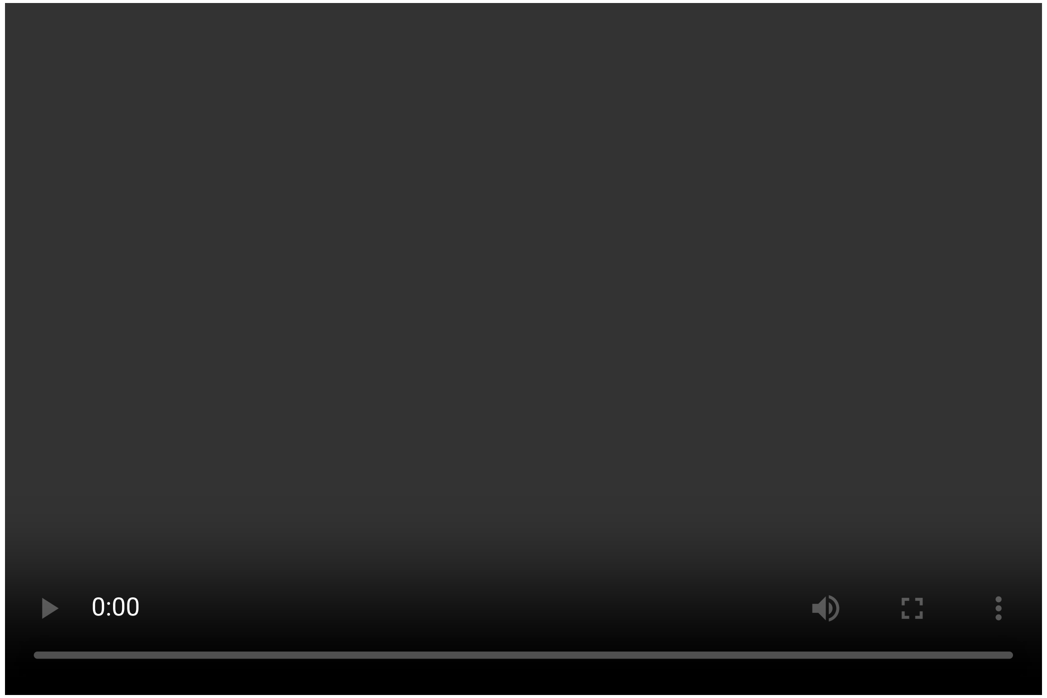
CLT of Exponential to $N = 100$

```
fig, ax = plt.subplots()
def update(n):
    ax.clear()
    data = distribution.rvs((5000, n))
    sample_means = data.mean(axis=1)
    Y = np.sqrt(n) * (sample_means - mu)
    ax.set_xlim(-3 * s, 3 * s)
    ax.set_ylim(0, 0.5)
    ax.hist(Y, bins=60, alpha=0.5,
      density=True)
    ax.set_title(f"CLT for $N = {n}$")
ani = FuncAnimation(fig,update,
  frames=range(1, 100, 5),
  interval=500,blit=False, repeat=False)
plt.close()
IPython.display.HTML(ani.to_html5_video())
```

CLT of Exponential to $N = 2000$

```
fig, ax = plt.subplots()
def update(n):
    ax.clear()
    data = distribution.rvs((10000, n))
    sample_means = data.mean(axis=1)
    Y = np.sqrt(n) * (sample_means - mu)
    ax.set_xlim(-3 * s, 3 * s)
    ax.set_ylim(0, 0.5)
    ax.hist(Y, bins=60, alpha=0.5, density=True)
    ax.set_title(f"CLT for $N = {n}$")
    x = np.linspace(-3 * s, 3 * s, 100)
    ax.plot(x, scipy.stats.norm.pdf(x, 0, s), 'r-', lw=2, alpha=0.7, label='N(0, 1)')
ani = FuncAnimation(fig,update,frames=range(1, 2000, 250), interval=500,blit=False, repeat=False)
plt.close()
IPython.display.HTML(ani.to_html5_video())
```

CLT of Exponential to $N = 2000$



Joint Distributions

Joint Probability Distributions

- Key concepts are **marginal distributions**, **conditional distributions**, **independence**, and **conditional expectations**
- Will demonstrate with bivariate discretely valued distributions
  - → Similar for multivariate distributions, except we replace sums with sums over multiple indices
  - → Similar for continuous or mixed discrete-continuous distributions, except we replace sums with integrals
- Interpretation of the joint distribution of X and Y is the probability of each pair of outcomes occurs
  - → e.g., prob you get a cash transfer and are unemployed, don't get a cash transfer and are unemployed, get a cash transfer and are employed, etc.

Bivariate Probability Distributions

- Let X, Y be two discrete random variables that take values:

$$X \in \{1, \ldots, I\}, \quad Y \in \{1, \ldots, J\}$$

- Then their **joint distribution** is described by a matrix

$$P \equiv [\mathbb{P}(X = i, Y = j)]_{i=1\ldots I, j=1,\ldots J} \in \mathbb{R}^{I \times J}$$

- Which fulfills the key axioms of probability

$$p_{ij} \equiv \mathbb{P}(X = i, Y = j) \geq 0$$
$$\sum_{i=1}^{I} \sum_{j=1}^{J} p_{ij} = 1$$

Marginal Probability Distributions

- The joint distribution induces marginal distributions

$$\mathbb{P}(X = i) = \sum_{j=1}^{J} p_{ij} = \mu_i, \quad i = 1, \ldots, I$$

$$\mathbb{P}(Y = j) = \sum_{i=1}^{I} p_{ij} = \nu_j, \quad j = 1, \ldots, J$$

- The marginal distributions are also probability distributions
  - $\rightarrow$ i.e., $\mu_i \geq 0$ and $\sum_{i=1}^{I} \mu_i = 1$
  - $\rightarrow$ e.g. the probability you were given a conditional cash transfer regardless of your employment status

Conditional Probability

- Conditional probabilities are defined according to

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

- $A \cap B$ is the event that both A and B occur, i.e., the intersection
  - $\longrightarrow$ e.g. you were given a conditional cash transfer AND you were unemployed
- The conditional probability is the probability of A given B has occurred
  - $\longrightarrow$ e.g. the probability you were given a conditional cash transfer given you were unemployed

Conditional Distributions

For a pair of discrete random variables, we have the **conditional distribution**

$$\mathbb{P}(X = i|Y = j) = \frac{p_{ij}}{\sum_{i=1}^{I} p_{ij}} = \frac{\mathbb{P}(X = i, Y = j)}{\mathbb{P}(Y = j)}$$

- Fix $Y = j$, then the conditional distribution of $X \mid Y = j$ is a probability distribution. Trivially positive since $p_ij \geq 0$. Verify it sums to 1

$$\sum_{i=1}^{I} \mathbb{P}(X = i \mid Y = j) = \frac{\sum_{i=1}^{I} p_{ij}}{\sum_{i=1}^{I} p_{ij}} = 1$$

Law of Total Probability

- **Law of Total Probability** is a useful identity for conditional probabilities
  - → Let $A_1, \ldots, A_N$ be a partition of Ω
  - → i.e., $\Omega = \cup_{i=1}^{N} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$
- Then for any event B,

$$\mathbb{P}(B) = \sum_{i=1}^{N} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{N} \mathbb{P}(B \mid A_i)\mathbb{P}(A_i)$$

→ e.g. the probability of being unemployed is the probability of being unemployed and getting a cash transfer plus the probability of being unemployed and not getting a cash transfer

Statistical Independence

- Random variables $X \sim p$ and $Y \sim g$ are statistically **independent** if

$$\mathbb{P}(X = i, Y = j) = p_i g_j, \text{ for all } i, j$$

- i.e., the joint distribution is the product of the marginal distributions
- e.g., the probability you were given a conditional cash transfer AND you were unemployed is probability you were given a conditional cash transfer $\times$ the probability you were unemployed

Conditional Distributions and Independence

- When X and Y are independent, use the definitions of conditional and marginal distributions

$$\mathbb{P}(X = i \,|\, Y = j) = \frac{\mathbb{P}(X = i, Y = j)}{\mathbb{P}(Y = j)} = \frac{p_i g_j}{\sum_{i=1}^{I} p_i g_j} = \frac{p_i g_j}{g_j} = p_i$$

$$\mathbb{P}(Y = j \,|\, X = i) = \frac{\mathbb{P}(X = i, Y = j)}{\mathbb{P}(X = i)} = \frac{p_i g_j}{\sum_{j=1}^{J} p_i g_j} = \frac{p_i g_j}{p_i} = g_j$$

- i.e, independent X and Y implies the conditional distributions are the marginals

- $\mathbb{P}(X = i \,|\, Y = j) = \mathbb{P}(X = i)$ and $\mathbb{P}(Y = j \,|\, X = i) = \mathbb{P}(Y = j)$

Notation for (Conditional) Independence

- Let X, Y, Z be random variables
- Common notation for independence is

$$X \perp Y$$
$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y)$$

- Common notation for conditional independence

$$X \perp\!\!\!\perp Y \mid Z$$
$$\mathbb{P}(X = x, Y = y|Z = z) = \mathbb{P}(X = x|Z = z)\mathbb{P}(Y = y|Z = z)$$

- Central to causal inference and treatment effects

Classic Example of Conditional Independence

- Let X be the number of cigarettes smoked per day
- Let Y be the number of years of life remaining
- Let Z be the number of years of smoking
- Then $X \perp\!\!\!\perp Y \mid Z$

  $\rightarrow$ i.e., the number of cigarettes smoked per day is independent of the number of years of life remaining given the number of years of smoking

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Simpson's "Paradox"

- Simpson's paradox is a warning on composition effects
- Recall the law of total probability
  $\mathbb{P}(X = x|Y = y) = \sum_z \mathbb{P}(X = x|Y = y, Z = z)\mathbb{P}(Z = z|Y = y)$
- Lets say you see $\mathbb{P}(X|Y = y_1) > \mathbb{P}(X|Y = y_2)$

  $\rightarrow$ Might suggests positive relationship on X and Y?
- If $\mathbb{P}(X|Y = y_1, Z = z) < \mathbb{P}(X|Y = y_2, Z = z)$ for many z then

  $\rightarrow$ X and Y may have a negative relationship after conditioning on Z?

UC Berkeley Gender Bias: Overall Data

Total Applicants	Admitted	Men Applicants	Men Admitted	Women Applicants	Women Admitted
12,763	41%	8,442	44%	4,321	35%

- Classic example is the Berkeley Gender Bias is a classic example of Simpson's paradox
- But if you look at individual departments the results are different
- Seemed to show that 4 out of 85 departments had significant bias against women and 6 significant bias against men
- But the biggest difference was in which departments women applied to
- The following shows the top 6 departments to get a sense of heterogeneity

Conditional Probabilities for 6 Largest Departments

Dept	All Applicants	Admitted	Men Applicants	Men Admitted	Women Applicants	Women Admitted
A	933	64%	**825**	62%	108	82%
B	585	63%	**560**	63%	25	68%
C	918	35%	325	37%	**593**	34%
D	792	34%	417	33%	375	35%
E	584	25%	191	28%	**393**	24%
F	714	6%	373	6%	341	7%

greater number of applicants than other gender and less number of applicants than other gender **bold** the two "most applied for" departments for each gender

Explanation Using Conditional Probabilities

Overall, $\mathbb{P}(\textbf{Admitted} \mid \textbf{Men}) = \mathbf{0.44}$ and $\mathbb{P}(\textbf{Admitted} \mid \textbf{Women}) = \mathbf{0.35}$

But this is different when conditioning on departments!

- $\mathbb{P}(\textbf{Admitted} \mid \textbf{Men, A}) = \mathbf{0.62}$, $\mathbb{P}(\textbf{Admitted} \mid \textbf{Women, A}) = \mathbf{0.82}$
- $\mathbb{P}(\textbf{Admitted} \mid \textbf{Men, B}) = \mathbf{0.63}$, $\mathbb{P}(\textbf{Admitted} \mid \textbf{Women, B}) = \mathbf{0.68}$
- “Paradox” because women tend to apply to more competitive departments

Does this Old Data Imply There was No Bias?

- All data requires assumptions to interpret! Most assumptions are implicit, so you need to reflect on what assumptions you may have made
- This simply corrected for the mechanical composition effect
- Interpreting bias better requires reflecting on your “model” and assumptions
  - $\longrightarrow$ Is average quality is identical conditional on department and gender? Especially in 1973 when there was enormous selection bias?
  - $\longrightarrow$ What if bias leads women to apply to the more competitive departments?

Bayes' Law

Conditional probability is used for **Bayes' Law**:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Sometimes:

- $\mathbb{P}(B \mid A)$ is called the "likelihood"
- $\mathbb{P}(A)$ is called the "prior"
- $\mathbb{P}(A \mid B)$ is called the "posterior"
- $\mathbb{P}(B)$ is called the "marginal likelihood", which normalizes the expression

Example with Bayes' Law

A is the event of being unemployed, B is the event of getting a cash transfer

- $\mathbb{P}(B \mid A)$ is the probability of being given a cash transfer given you were unemployed
- $\mathbb{P}(A)$ is the probability of being unemployed within the whole distribution
- $\mathbb{P}(A \mid B)$ is the probability of being unemployed given you were given a cash transfer
- $\mathbb{P}(B)$ is the probability of being given a cash transfer within the whole distribution
- **Bayes' law**: probability of being unemployed given you were given a cash transfer $\propto$ probability of being given cash transfer given you were unemployed $\times$ probability of being unemployed

Bayes Law with Bivariate Random Variables

- For discrete bi-variate random variables, we can write Bayes' Law as

$$\mathbb{P}(X = i \,|\, Y = j) = \frac{\mathbb{P}(X = i, Y = j)}{\mathbb{P}(Y = j)} = \frac{\mathbb{P}(Y = j \,|\, X = i)\mathbb{P}(X = i)}{\mathbb{P}(Y = j)}$$

- If X and Y are independent
  - $\rightarrow$ $\mathbb{P}(Y = j \,|\, X = i) = \mathbb{P}(Y = j)$
  - $\rightarrow$ Bayes' Law simplifies to just the marginal distribution

$$\mathbb{P}(X = i \,|\, Y = j) = \mathbb{P}(X = i)$$

Calculating Marginal Distributions

- Lets create a bivariate $\mathbb{P}(X = i, Y = j)$ with $I = 5$ and $J = 4$
- Use matrix P to calculate $\mathbb{P}(X = i)$ and $\mathbb{P}(Y = j)$

```python
np.set_printoptions(precision=3)
P = np.array([[0.05, 0.07, 0.02, 0.01],
              [0.04, 0.1,  0.06, 0.03],
              [0.08, 0.09, 0.07, 0.04],
              [0.02, 0.03, 0.02, 0.01],
              [0.09, 0.08, 0.04, 0.05]])

print(f"sum = 1? {np.isclose(P.sum(), 1.0)}")
print(f"p_ij >= 0? {np.all(P >= 0)}")
margin_x = P.sum(axis=1) # sum over j
margin_y = P.sum(axis=0) # sum over i
print(f"P(X=i) = {margin_x}")
print(f"Sum_i P(X=i) = {margin_x.sum()}")
print(f"P(Y=j) = {margin_y}")
print(f"Sum_j P(Y=j) = {margin_y.sum()}")
```

```
sum = 1? True
p_ij >= 0? True
P(X=i) = [0.15 0.23 0.28 0.08 0.26]
Sum_i P(X=i) = 0.9999999999999999
P(Y=j) = [0.28 0.37 0.21 0.14]
Sum_j P(Y=j) = 1.0
```

Calculating Conditional Distributions

- Now use P to calculate $\mathbb{P}(X = i \mid Y = j)$, etc.

```python
print(f"P(X=i|Y=1)=\n{P[:,0] / margin_y[0]}\,")
cond_x_y = np.row_stack(
  [P[:,i] / margin_y[i] for i in range(4)])
# or (P / margin_y[np.newaxis, :]).T
cond_y_x = np.row_stack(
  [P[j,:] / margin_x[j] for j in range(5)])
# or (P.T / margin_x[np.newaxis, :]).T
print(f"P(X=i|Y=2)=\n{cond_x_y[:, 1]}")
print(f"sum_i P(X=i|Y=2)=\
{cond_x_y[1,:].sum():.2f}")
print(f"P(Y=j|X=1)=\n{cond_y_x[:, 0]}")
```

```
P(X=i|Y=1)=
[0.179 0.143 0.286 0.071 0.321]\,
P(X=i|Y=2)=
[0.143 0.27  0.286 0.214]
sum_i P(X=i|Y=2)=1.00
P(Y=j|X=1)=
[0.333 0.174 0.286 0.25  0.346]
```

Check Bayes' Law

$$\mathbb{P}(X=1 \mid Y=2) = \frac{\mathbb{P}(Y=2 \mid X=1)\mathbb{P}(X=1)}{\mathbb{P}(Y=2)}$$

```python
x = 1
y = 2
p_y_x = cond_y_x[x-1, y-1]
p_x = margin_x[x-1]
p_y = margin_y[y-1]
p_x_y = cond_x_y[y-1, x-1]
p_bayes = p_y_x * p_x / p_y
print(f"P(Y={y}|X={x}) = {p_y_x:.2g}")
print(f"P(X={x}) = {p_x:.2g}")
print(f"P(Y={y}) = {p_y:.2g}")
print(f"P(X={x}|Y={y})={p_x_y:.2g}")
print(f"P(Y={y}|X={x})P(X={x})\
/P(Y={y})={p_bayes:.2g}")
```

```
P(Y=2|X=1) = 0.47
P(X=1) = 0.15
P(Y=2) = 0.37
P(X=1|Y=2)=0.19
P(Y=2|X=1)P(X=1)/P(Y=2)=0.19
```

Conditional Expectations

Conditional Expectation

- Recall: $\mathbb{P}(X = i \mid Y = j)$ is itself a probability distribution if we vary j
- A **conditional expectation** is an expectation using the conditional probability distribution. For a discrete random variable X and Y,

$$\mathbb{E}[X \mid Y = j] = \sum_{i=1}^{I} i\, \mathbb{P}(X = i \mid Y = j)$$

- If X and Y are independent then
  - → Recall that $\mathbb{P}(X = i \mid Y = j) = \mathbb{P}(X = i)$
  - → Which implies $\mathbb{E}[X \mid Y = j] = \mathbb{E}[X]$
  - → That the expected value of X is the same regardless of the value of Y

Key Properties of Expectations

- Let A and B be scalar/vector/matrix constants, and X and Y are scalar/vector/matrix random variables
- Expectations are **linear operators**, which gives us some useful properties
  - $\rightarrow$ $\mathbb{E}[AX + BY] = A\mathbb{E}[X] + B\mathbb{E}[Y]$
- $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$ in general
  - $\rightarrow$ But if X and Y are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- $\mathbb{E}[f(X)] \neq f(\mathbb{E}[X])$ in geneal
  - $\rightarrow$ Unless $f(\cdot)$ is linear or if X is degenerate (i.e., a constant)
- **Jensen's Inequality**: If $f(\cdot)$ is a convex function, then $\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$

Law of Total Expectations

- Let $\{A_1, \ldots, A_N\}$ be a partition of Ω. For any random variable X,
- **Law of Total Expectations**

$$\mathbb{E}[X] = \sum_{i=1}^{N} \mathbb{E}[X \mid A_i]\mathbb{P}(A_i)$$

- e.g. the expected value of income is the expected value of income given you were unemployed times the probability of being unemployed plus the expected value of income given you were employed times the probability of being employed

Related decomposition is the Law of Total Variances: $\mathbb{V}[X] = \mathbb{E}[\mathbb{V}[Y \mid X]] + \mathbb{V}[\mathbb{E}[Y \mid X]]$

Conditional Expectations and Iterated Expectations

- Same properties all hold e.g. $\mathbb{E}[AX + BY \mid Z] = A\mathbb{E}[X \mid Z] + B\mathbb{E}[Y \mid Z]$
- Conditional expectations are themselves random variables if the conditional is. e.g. $\mathbb{E}[X \mid Y]$ is a random variable in Y
- **Law of Iterated Expectations**

$$\mathbb{E}\left[\mathbb{E}[X \mid Y]\right] = \mathbb{E}[X]$$

$\longrightarrow$ The expected value of X is the average of the conditional expectations of X given Y over the distribution of Y

$\longrightarrow$ Similarly for conditionals: $\mathbb{E}\left[\mathbb{E}[X \mid Y, Z] \mid Z\right] = \mathbb{E}[X \mid Z]$

Calculating Conditional Expectations

Assign an RV to each state then find $\mathbb{E}[X \mid Y = 1]$

```
P = np.array([[0.05, 0.07, 0.02, 0.01],
              [0.04, 0.1,  0.06, 0.03],
              [0.08, 0.09, 0.07, 0.04],
              [0.02, 0.03, 0.02, 0.01],
              [0.09, 0.08, 0.04, 0.05]])
margin_x = P.sum(axis=1)
margin_y = P.sum(axis=0)
cond_x_y = (P / margin_y[np.newaxis, :]).T
cond_y_x = (P.T / margin_x[np.newaxis, :]).T
# Give RV values to states
vals_x = np.arange(P.shape[0]) + 1
vals_y = np.arange(P.shape[1]) + 1

print("E(X | Y = 1) =",
  np.sum([vals_x[i]*cond_x_y[0,i]
          for i in range(0,5)]))
```

```
E(X | Y = 1) = 3.2142857142857144
```

Conditional Expectations and the Law of Iterated Expectations

```python
E_x_y = np.array([
  np.sum([vals_x[i]*cond_x_y[j,i]
          for i in range(0,5)])
  for j in range(0,4)])
E_y_x = np.array([
  np.sum([vals_y[j]*cond_y_x[i,j]
          for j in range(0,4)])
  for i in range(0,5)])
# Or use np broadcasting with *
E_x_y = np.sum(vals_x * cond_x_y, axis=1)
E_y_x = np.sum(vals_y * cond_y_x, axis=1)
print("E(X | Y = j) =", E_x_y)
print("E(Y | X = i) =", E_y_x)
print(f"E(X) = {vals_x @ margin_x:.3g},\
  E(E(X|Y)) = {E_x_y @ margin_y}")
```

```
E(X | Y = j) = [3.214 2.865 3.    3.429]
E(Y | X = i) = [1.933 2.348 2.25  2.25  2.192]
E(X) = 3.07,  E(E(X|Y)) = 3.07
```

Tips for using Numpy Broadcasting

1. **Don't**
   - Loops, list comprehensions (e.g., `[x[i, i+1] for i in range(5)]`), or a combination are usually clearer and often fast enough
2. Write the slow version first
3. Ask Github Copilot or ChatGPT to do a numpy broadcasting version
4. Test it for a few values! Easy to make mistakes
5. For more advanced usage you may be working in a ML library. If so, then packages such as jax.vmap of torch.vmap help