**Multiple Choice 4.1** True or false? Motivate your answer.

Consider the Cauchy problem

$$\begin{cases} y' = \sqrt{|y|} & \text{for } t > 0, \\ y(0) = 0, \end{cases}$$

You notice that that $y \equiv 0$ solves the problem. Consequently, without any further computation, you can say that there exists a sufficiently small half-interval $I = [0, \varepsilon)$ , $\varepsilon > 0$, where 0 is the only solution.

☐ True ☐ False

**Multiple Choice 4.2** Choose the correct statement. Motivate your answer.

Consider for $n \geq 2$ a function

$$f : (a, b) \to \mathbb{R}^n, \qquad f(t) = (f_1(t), \ldots, f_n(t)).$$

In order for $f$ to be differentiable:

(a) it is *necessary*, but in general not sufficient, that each $f_j$ is differentiable. ☐

(b) it is *sufficient*, but in general not necessary, that each $f_j$ is differentiable. ☐

(c) it is *necessary and sufficient* that each $f_j$'s is differentiable. ☐

**Exercise 4.1** Find the general solution of the ODE:

$$y^{(4)} + 2y'' + y = f(x),$$

when

(a) $f(x) = \sin x$,

(b) $f(x) = \mathrm{e}^{2x}$,

(c) $f(x) = \sin x + \mathrm{e}^{2x}$.

**Exercise 4.2** Solve the following ODE with the method of the variation of constants:

$$y'' + 4y = \frac{1}{\sin(2x)}.$$

**Exercise 4.3** Solve the following ODE/Cauchy problems. If you apply the method of separation of variables, be careful not to divide by zero!

(a) $y' - y = \sin x$,

(b) $\begin{cases} y' = (x+y)^2, & \text{for } x \in (-\pi/2, \pi/2), \\ y(0) = 1. \end{cases}$

(c) $\begin{cases} y' = \sqrt{\dfrac{1-y^2}{1-x^2}}, & \text{for } x \in (-\pi/2, \pi/2), \\ y(0) = 0. \end{cases}$

(d) $yy' - (1+y)x^2 = 0$,

*Note for (d):* you will not be able to write explicitly every solution (this often happens when dealing with nonconstant coefficient ODEs). Is suffices that you find an implicit relation for $y$ that does not involve its derivatives.

**Multiple Choice 10.1** Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and consider its *Jacobian matrix*, namely the $3 \times 3$ matrix of its 1st derivatives:

$$DV = \begin{pmatrix} \partial_{x_1} V_1 & \partial_{x_2} V_1 & \partial_{x_3} V_1 \\ \partial_{x_1} V_2 & \partial_{x_2} V_2 & \partial_{x_3} V_2 \\ \partial_{x_1} V_3 & \partial_{x_2} V_3 & \partial_{x_3} V_3 \end{pmatrix}.$$

Knowing that, in this matrix, there are three *distinct* coinciding pairs of elements, is, in general,

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for $V$ to be conservative.

**Multiple Choice 10.2** Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ and $V$ be as in the previous question.

Knowing that the matrix $DV$ is *symmetric*, is

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for $V$ to be conservative.

**Exercise 10.1** In each of the following, find a parametrization of the curve $\gamma$ and compute the line integral $\int_\gamma F \cdot d\vec{s}$.

(a) $F(x, y) = (x + y, x - y)$ and $\gamma$ runs through the parabola $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ from the point $(-1, 1)$ to the point $(1, 1)$.

(b) $F(x, y) = (0, xy^2)$ and $\gamma$ runs through the half-circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4, y \geq 0\}$ in counter-clockwise direction.

(c) $F(x,y) = (x^2+y^2, x^2-y^2)$ and $\gamma$ runs trough the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ in counter-clockwise direction.

**Exercise 10.2** In each of the following, determine whether the vector field $V\colon \mathbb{R}^3 \to \mathbb{R}^3$ admits a potential and compute the line integral $\int_\gamma V \cdot d\vec{s}$ along the curve $\gamma\colon [0,1] \to \mathbb{R}^3$, $\gamma(t) = (t^3, t^2+t, t)$.

$$\text{(a)} \quad V(x,y,z) = \begin{pmatrix} 2xy^3 \\ 3x^2y^2 + 2yz \\ y^2 \end{pmatrix}, \qquad \text{(b)} \quad V(x,y,z) = \begin{pmatrix} x+z \\ x+y+z \\ x+z \end{pmatrix}.$$

**Exercise 10.3** The following vector field describes, according to the *Biot-Savart law*, the magnetic field generated by an infinitely long, constant-current electric wire displaced along the $z$-axis:

$$B(x,y,z) = \frac{\mu_0 I}{2\pi} \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \text{defined for } (x,y) \neq 0,$$

where $\mu_0$ and $I$ are, respectively, the magnetic constant and $I$ the (also constant) current.

(a) Prove that it is

$$\frac{\partial}{\partial x_i} B_j = \frac{\partial}{\partial x_j} B_i \quad \forall\, i,j \in \{1,2,3\},$$

where we denoted $(x_1, x_2, x_3) = (x,y,z)$.

(b) Consider the curves $\gamma_m : [0, 2\pi m] \to \mathbb{R}^3$, $\gamma_m(t) = (\cos(t), \sin(t), 0)$ for $m \in \mathbb{Z}$, and compute the line integrals $\int_{\gamma_m} B \cdot d\vec{s}$.

(c) Does $B$ admit a potential in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$?

**Multiple Choice 12.1** Choose the correct statement. Motivate your answer.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous functions and let $B_r(0) \subset \mathbb{R}^n$ be the ball of radius $r > 0$ centered ad the origin. The integral

$$\int_{B_r(0)} f(x)\,dx$$

can also be written as

(a) $r^n \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$ ☐

(b) $\frac{1}{r^n} \int_{B_1(0)} f(rx)\,dx$ ☐

(c) $r^n \int_{B_1(0)} f(rx)\,dx$ ☐

(d) $\frac{1}{r^n} \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$ ☐

where we denoted $rx = (rx_1, \dots, rx_n)$ and similarly for $\frac{1}{r}x$.

**Multiple Choice 12.2** Choose the correct statement. Motivate your answer.

Recall from one variable calculus that the improper integral

$$\int_{-1}^{1} \frac{1}{|x|^\alpha} dx$$

is convergent if and only if $\alpha < 1$. The following analogue in 3 dimensions:

$$\int_{B_1(0)} \frac{1}{|x|^\alpha} dx, \quad B_1(0) \subset \mathbb{R}^3,$$

is convergent if and only if

(a) $\alpha < 1$ ☐

(b) $\alpha < 3$ ☐

(c) $\alpha < \frac{3+1}{2} = 2$ ☐

(d) $\alpha < 3!$ ☐

**Exercise 12.1** For each of the following integrals:

– Determine the domain of integration $\Omega \subset \mathbb{R}^2$,

– Rewrite them exchanging the integration order,

– Compute them using this new integration order.

(a) $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx,$ (b) $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx,$

(c) $\displaystyle\int_0^2 \int_x^2 2y^2 \sin(xy)\, dy\, dx,$ (d) $\displaystyle\int_0^{2\sqrt{\log(3)}} \int_{y/2}^{\sqrt{\log(3)}} e^{x^2}\, dx\, dy.$

**Exercise 12.2** Let $K = Z_1 \cap Z_2$ be the intersection of the cylinders

$$Z_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}, \qquad Z_2 = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 \leq 1\}.$$

(a) Draw $K$.

(b) Compute the *volume* of $K$.

(c) If the function $\rho\colon K \to \mathbb{R}$ given by $\rho(x, y, z) = 1 + x^2 + z^2$ represent the *mass density* of $K$, compute its *mass* $m(K) = \int_K \rho\, d\mu$.

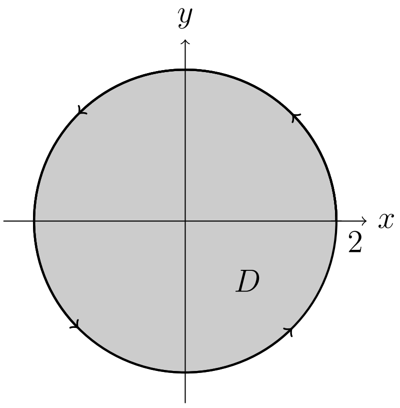
**Exercise 12.3** The "curl" of a vector field in $\mathbb{R}^2$ is, by definition, the function

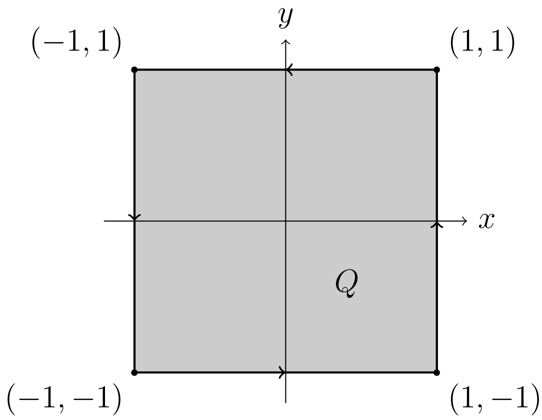
$$\operatorname{curl}(v) = \frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1.$$

Consider the vector field $v(x, y) = (y^2, x)$.

(a) Compute the line integral of $v$ along the circle of radius 2 centered at the origin and along the square of vertices $(\pm 1, \pm 1)$, both oriented counter-clockwise (see the picture).

(b) Now compute the double integral of $\operatorname{curl}(v)$ over the disk $D$ and the square $Q$ enclosed by the curves in (b). What do you notice?



**Multiple Choice 1.1** True or false? Motivate your answers.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^n$, $n \geq 1$, $f(x) = (f_1(x), \ldots, f_n(x))$ be a function. Then:

| | | True | False |
|---|---|---|---|
| (a) | If one of the $f_j$'s is injective, then $f$ is injective. | ☐ | ☐ |
| (b) | If $f$ is injective, at least one of the $f_j$'s is injective. | ☐ | ☐ |
| (c) | If every $f_j$ is surjective, then $f$ is surjective. | ☐ | ☐ |
| (d) | If $f$ is surjective, then every $f_j$ is surjective. | ☐ | ☐ |

**Multiple Choice 1.2** Let $f : [a, b] \to \mathbb{R}^n$, $n \geq 1$ be continuous and differentiable in $(a, b)$. Then there is $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The above statement is

☐ True ☐ False

Motivate your answer.

**Exercise 1.1** For each of the following expressions, determine whether they are ODE, and in such case, whether they are linear/nonlinear, homogeneous/inhomogeneous and their order.

(a) $y'(x) = y(x)(1 - y(x))$.

(b) $a_2\, x^2\, y''(x) + a_1\, x\, y'(x) + a_0\, y(x) = 1$, where $a_0, a_1, a_2 \in \mathbb{R}$.

(c) $y''(x) + y(x) = y(2x)$.

(d) $y(x) = xy'(x) + f(y'(x))$, where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

(e) $y''(x) + 2y'(x) + y(x) = \cos x$.

**Exercise 1.2** Consider the ODE in $y : \mathbb{R} \to \mathbb{R}$:

$$y'' + 2y' + y = 0.$$

(a) Verify that $\mathrm{e}^{-x}$ and $x\mathrm{e}^{-x}$ are solutions of the equation.

(b) Verify that $a\mathrm{e}^{-x} + x\mathrm{e}^{-x}$ is again a solution of the equation for every $a \in \mathbb{R}$.

(c) Can you find all the (twice continuously differentiable) functions $\alpha : \mathbb{R} \to \mathbb{R}$ so that $\alpha(x)\mathrm{e}^{-x} + x\mathrm{e}^{-x}$ is a solution of the equation?

**Exercise 1.3** Find an ODE of the specified order solved by the given function:

(a) $\varphi(t) = \frac{1}{1-t}$, of 1st order,

(b) $\varphi(t) = c_1 \cos t + c_2 \sin t$, of 2nd order, where $c_1, c_2 \in \mathbb{R}$,

(c) $\varphi(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{-t}$, of 2nd order, where $c_1, c_2 \in \mathbb{R}$.

**Multiple Choice 6.1** The following subsets are open, closed, bounded, compact? Motivate your answers.

(a) $S = (0,1) \cup \left(2, 2+\frac{1}{2}\right) \cup \left(3, 3+\frac{1}{3}\right) \cup \left(4, 4+\frac{1}{4}\right) \cup \cdots \subset \mathbb{R}$

☐Open ☐Closed ☐Bounded ☐Compact

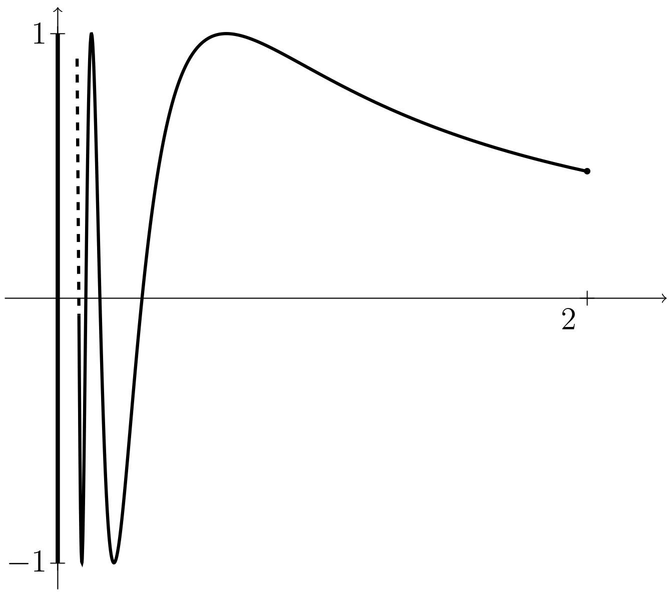
(b) $S = (0,1) \times \{0\} \subset \mathbb{R}^2$

☐Open ☐Closed ☐Bounded ☐Compact

(c) $S = \bigcap_{n=1}^{\infty} B_{1+\frac{1}{n}}(0) \subset \mathbb{R}^n$, where $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$

☐Open ☐Closed ☐Bounded ☐Compact

(d) $S = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x \in (0,2]\right\} \cup \{0\} \times [-1,1] \subset \mathbb{R}^2$




☐Open ☐Closed ☐Bounded ☐Compact

(e) $S = f^{-1}(\{1\}) \subset \mathbb{R}^3$, where $f(x,y,z) = z^2 - x^2 - y^2$

☐Open ☐Closed ☐Bounded ☐Compact

(f) $S = g^{-1}((-2,2]) \subset \mathbb{R}^3$, where $g(x,y,z) = x^2 + y^2 + z^2 + 1$

☐Open ☐Closed ☐Bounded ☐Compact

**Multiple Choice 6.2** True or false? Motivate your answers (For the definition of directional derivative, see Exercise 6.2 below).

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. For $f$ to be continuous at $(0, 0)$:

| | | True | False |
|---|---|---|---|
| (a) | it is sufficient that, along *some* direction $v \neq 0$, the directional derivative $D_v f(0, 0)$ exists. | ☐ | ☐ |
| (b) | it is sufficient that *both* the partial derivatives $\partial_x f(0, 0)$ and $\partial_y f(0, 0)$ exist. | ☐ | ☐ |
| (c) | it is sufficient that the directional derivatives $D_v f(0, 0)$ along *every* direction $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ exists. | ☐ | ☐ |

**Exercise 6.1** Determine the domain and compute the (1st order) partial derivatives of the following functions:

(a) $f(x, y) = \pi x^2$

(b) $f(x, y) = e^{xy}$

(c) $f(x, y) = x^y$

(d) $f(x, y) = \dfrac{x - y}{x^2 + y^2}$

(e) $f(x, y) = x^2 y \sin(xy)$

(f) $f(x, y, z) = xy^2 z^3 + y$.

**Exercise 6.2** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function and $v \in \mathbb{R}^n$ a vector. When it exists, the limit

$$D_v f(x) = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

is called *directional derivative* of $f$ along $v$ at the point $x \in \mathbb{R}^n$. In particular, along the coordinate directions $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ we have $D_{e_i} f = \frac{\partial f}{\partial x_i} = \partial_{x_i} f$.

Compute, using the definition above, the directional derivative of the folloing functions along the direction $v, w \in \mathbb{R}^2$ at the given point:

(a) $f(x, y) = \cos(xy) + x^2, \quad (x, y) = (\pi, 3), \quad v = (1, 1), \quad w = (2, 0)$

(b) $f(x, y) = 2x^2 y + 3xy + y, \quad (x, y) = (2, 1), \quad v = (1, 1), \quad w = (1, 2).$

Compute now the usual partial derivatives at the same point. What do you notice?

**Exercise 6.3** Compute the mixed 2nd partial derivatives i.e.

$$\frac{\partial}{\partial y} \frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial x} \frac{\partial f}{\partial y}$$

of the following functions, and compare them.

(a) $f(x,y) = x^2 e^{y \sin x}$,

(b) $$f(x,y) = \begin{cases} \dfrac{xy}{2}\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

*Note:* For (b), at the point $(0,0)$ use the definition of derivative as limit of difference quotient, or use Exercise 6.2.

**Multiple Choice 11.1** Choose the correct statement. Motivate your answer.

Recall that, for a $C^2$ function $f : \mathbb{R}^3 \to \mathbb{R}$, its gradient $\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$ can be thought as a vector field. The equation

$$\operatorname{curl}(\nabla f) = g$$

(a) has a solution $f$ for *every* given $g$ ☐

(b) has a solution $f$ *only* for $g = 0$ ☐

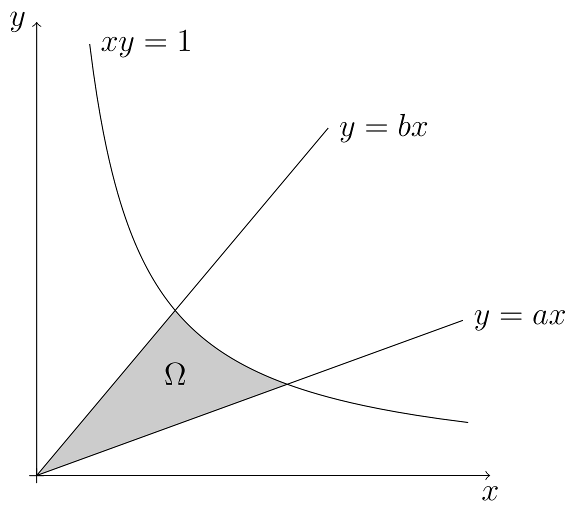
(c) has a solution $f$ *also* for some nonzero $g$'s. ☐

**Multiple Choice 11.2** True or false? Motivate your answer.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected, regular region. Generally speaking, the integral $\int_\Omega d\mu$ represents:

| | | True | False |
|---|---|---|---|
| (a) | the area of $\Omega$ | ☐ | ☐ |
| (b) | the length of the curve bounding $\Omega$ | ☐ | ☐ |
| (c) | the volume of a cylinder with base $\Omega$ and height 1 | ☐ | ☐ |
| (d) | the surface area of some cylinder with base $\Omega$ and height 1. | ☐ | ☐ |

**Exercise 11.1** Let $b > a > 0$. Compute the integral of $f(x, y) = xy$ over the domain $\Omega$ drawn below:



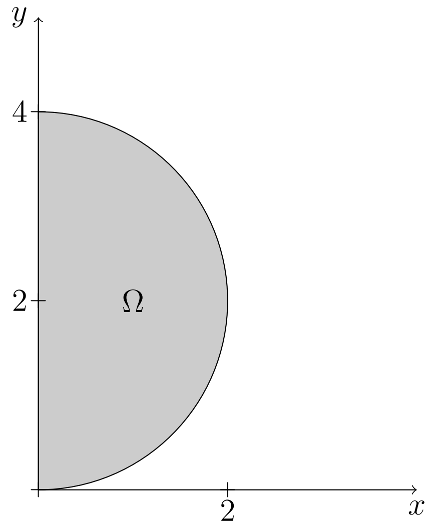
**Exercise 11.2** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function.

(a) Determine and draw (*first* by hand, *then* checking the result with a software of your choice) the domain of integration in $\mathbb{R}^2$ of the following integral:

$$\int_{-1}^{2}\int_{-x}^{2-x^2} f(x,y)\,dy\,dx. \tag{$\int$}$$

(b) Rewrite ($\int$) exchanging the order or integration, that is, rewrite the same integral supposing you want to integrate in $x$ first and then in $y$.

(c) Consider the domain $\Omega$ drawn below. Write $\int_\Omega f\,d\mu$ as double integral, in both orders of integration.




**Exercise 11.3** Compute the following integrals:

(a) $\displaystyle\int_0^1\int_0^x e^{x+y}\;dy\,dx,$ (b) $\displaystyle\int_0^1\int_{\sqrt{y}}^1 x\cos y\;dx\,dy.$

Now describe and draw the domains of integration, and compute the integrals exchanging order of integration. Is the result the same?

**Multiple Choice 3.1** True or false? Motivate your answers.

Consider the ODE $y'(x) = \mathrm{e}^{y(x)}$. Then:

| | | True | False |
|---|---|---|---|
| (a) | Any solution with $y(0) \geq 0$ satisfies $y(t) > 0$ for $t > 0$. | ☐ | ☐ |
| (b) | For any $a, b \in \mathbb{R}$ there always is a solution with $y(0) = a$ and $y(1) = b$. | ☐ | ☐ |
| (c) | Any solution is increasing. | ☐ | ☐ |

**Multiple Choice 3.2** Choose the correct statement(s). Motivate your answers.

The ODE $y^{(4)} + y = 0$ has:

(a) Only periodic solutions. ☐

(b) Some nonzero periodic solutions. ☐

(c) No nonzero periodic solutions. ☐

(d) No nonzero solution $y$ so that $\lim_{t\to+\infty} y(t)$ exists and is finite. ☐

(e) Some nonzero solution $y$ so that $\lim_{t\to+\infty} y(t)$ exists and is finite. ☐

**Exercise 3.1 (A glance at systems)** Let

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad t \in I \subseteq \mathbb{R} \tag{$*$}$$

be a linear, homogeneous ODE of order $n \geq 2$ with constant coefficients.

(a) Show that, by setting $z_1 = y$, $z_2 = y'$, ..., $z_n = y^{(n-1)}$, equation $(*)$ can be seen as a first-order *system* of ODEs:

$$\boldsymbol{z}' = A\boldsymbol{z}, \quad t \in I \tag{$\star$}$$

where $A \in M_{n\times n}(\mathbb{R})$ is a $n \times n$ matrix. Write down explicitly the expression for $A$.

(b) Prove that the characteristic polynomial of the ODE $(*)$ is the characteristic polynomial of the matrix $A$.

(c) Show that

$$\boldsymbol{\zeta}(t) = \mathrm{e}^{\lambda t}\boldsymbol{u}$$

is a solution of the homogeneous problem $(\star)$ if and only if $\lambda$ is an eigenvalue of $A$ and $\boldsymbol{u}$ is a corresponding eigenvector.

(d) *Fact:* Similarly as for ODE of order $n$, one can prove that for a system like $(\star)$, the set of solutions is vector space of dimension $n$.

Assuming that all the eigenvalues of $A$ are distinct, use (c) and the fact above to find an explicit expression for the general solution of homogeneous system.

**Exercise 3.2** Find the general solution of the following ODE and, when specified, the solutions fulfilling the indicated requirements. The solutions must always be expressed in real form.

(a) $y'' - 3y' + 2y = 0$,

(b) $y'' - 4y' = 0$ and the solutions that are always positive,

(c) $y'' - y' + y = 0$ and the solutions satisfying $y(0) = 0$, $y'(0) = 1$,

(d) $y^{(4)} + 1 = 0$ and the solutions so that are even i.e. $y(t) = y(-t)$.

**Exercise 3.3** Solve the following Cauchy problems:

$$\text{(a)} \quad \begin{cases} 2t^2 y' - y = 0, & \text{for } t \geq 1, \\ \quad y(1) = 1. \end{cases}$$

$$\text{(b)} \quad \begin{cases} y' - y = y \log x + 1 + \log x, & \text{for } x > 2, \\ y(2) = 3. \end{cases}$$

Find the values $a \in \mathbb{R}$ so that the problem

$$\text{(c)} \quad \begin{cases} y'' - (a+1)y' + ay = 0 \text{ in } \mathbb{R}, \\ y(t) \text{ is bounded for } t > 0, \end{cases}$$

has nonzero solutions, and write them explicitly.

**Multiple Choice 5.1** True or false? Motivate your answer.

Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a *linear* function. Then if $L$ is continuous at one point $x_0 \in \mathbb{R}^m$, it is continuous at every point in $\mathbb{R}^m$.

☐ True ☐ False

**Multiple Choice 5.2** Choose the correct statement. Motivate your answer.

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0,0) = 0$. For $f$ to be continuous at $(0,0)$, the fact that

$$\lim_{x\to 0} f(x,0) = 0 \quad \text{and} \quad \lim_{y\to 0} f(0,y) = 0$$

(a) is *necessary*, but in general not sufficient. ☐

(b) is *sufficient*, but in general not necessary. ☐

(c) is *necessary and sufficient.* ☐

*Note:* Looking at Exercise 5.1 may be useful!

**Exercise 5.1** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Prove that $f$ is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) Prove that, for every fixed $y \in \mathbb{R}$, there holds

$$\lim_{x\to 0} f(x,y) = 0,$$

and that or every fixed $x \in \mathbb{R}$, there holds

$$\lim_{y\to 0} f(x,y) = 0.$$

(c) Consider now the parabola $P = \{(x, x^2) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Prove that

$$\lim_{\substack{(x,y)\to 0 \\ (x,y)\in P}} f(x,y) = \lim_{x\to 0} f(x,x^2) = \frac{1}{2}.$$

Without any further computation, what can you say about

$$\lim_{(x,y)\to 0} f(x,y)\ldots?$$

Is $f$ continuous at $(0,0)$?

**Exercise 5.2** Compute the limits

(a) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2},$$

(b) $$\lim_{(x,y)\to(1,0)} \frac{y^2 \log x}{(x-1)^2+y^2},$$

(c) $$\lim_{(x,y,z)\to\infty} f(x,y,z),$$
where $f(x,y,z) = x^4 + y^2 + z^2 - x^3 + xyz - x + 4$.

*Note:* similarly as in one variable, "$\lim_{(x,y,z)\to\infty} f(x,y,z) = \alpha$" means that for every $\varepsilon > 0$ there exists $M > 0$, so that if $\|(x,y,z)\| \geq M$, then $|f(x,y,z) - \alpha| \leq \varepsilon$.

**Exercise 5.3** Which of the following subsets of the Euclidean space are compact?

(a) $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 2020\}$;

(b) $B = \{(a,b,c) \in \mathbb{R}^3 \mid a, b, c$ are integers and $a^2 + b^2 + c^2 < 2020\}$;

(c) $C = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in (0,1], f(x) = \sin(\frac{1}{x})\}$;

(d) $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0$ and $y \geq 0\}$;

(e) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 2\}$.

**Multiple Choice 8.1** True or False? Motivate your answers.

The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

at the point $(0,0)$ is:

| | | True | False |
|---|---|---|---|
| (a) | discontinuous | ☐ | ☐ |
| (b) | continuous | ☐ | ☐ |
| (c) | differentiable | ☐ | ☐ |
| (d) | $C^1$ (i.e. continuously differentiable). | ☐ | ☐ |

**Multiple Choice 8.2** Choose the correct statement. Motivate your answer.

Recall that a *critical point* of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is an $x_0 \in \mathbb{R}^n$ so that $df(x_0) = 0$. At such point, the tangent plane to the graph of $f$ is:

(a) not defined ☐

(b) horizontal (looking at $\mathbb{R}^3$ in the usual way with upward-pointing $z$-axis) ☐

(c) vertical (looking at $\mathbb{R}^3$ in the usual way with upward-pointing $z$-axis) ☐

(d) none of the above, in general. ☐

**Exercise 8.1** Let $\mathcal{G} = \{(x,y,f(x,y)) : (x,y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ be the graph of the function

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^{-(x^2+y^2-2x+3y+2)}.$$

(a) Find the equation of the tangent plane $E$ to $\mathcal{G}$ at the point $(0,0,e^{-2})$, both in *Cartesian form*, i.e. with an equation:

$$E = \{(x,y,z) \in \mathbb{R}^3 : \text{“equation in } x,y,z\text{”}\},$$

and in *parametric form* i.e. with a function:

$$\varphi\colon \mathbb{R}^2 \to E \subset \mathbb{R}^3, \quad \varphi(s,t) = \big(x(s,t), y(s,t), z(s,t)\big).$$

(b) Use a plotting software of your choice to verify that $\varphi$ actually plots a plane that is tangent to $\mathcal{G}$ as above.

(c) Find all the points in $\mathcal{G}$ where the tangent plane is parallel to the $x$-$y$ plane $\Pi = \{(x, y, 0) : (x, y) \in \mathbb{R}^2\}$.

**Exercise 8.2** Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = e^x \sin(y).$$

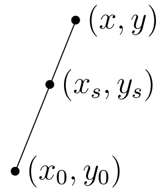
(a) Compute the Taylor polynomials of 1st and 2nd order of $f$ at $(x_0, y_0) = (0, \frac{\pi}{2})$; approximate with each of them the value of $f$ at $(x_1, y_1) = (0, \frac{\pi}{2} + \frac{1}{4})$. Compare the results approximating numerically the value of $f(x_1, y_1)$ with a software of your choice.

(b) Similarly as for the one variable case, one can prove that if a function is $C^2$, one can write

$$f(x) = f(x_0) + df(x_0) \cdot (x - x_0) + R_1 f(x, y),$$

where $R_1 f$ is the *rest*, given by

$$R_1 f(x, y) = \frac{1}{2} \frac{\partial^2 f}{\partial x \partial x}(x_s, y_s) \cdot (x - x_0)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y \partial y}(x_s, y_s) \cdot (y - y_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x_s, y_s) \cdot (x - x_0)(y - y_0)$$




where $(x_s, y_s) = \Big(x_0 + s(x - x_0), y_0 + s(y - y_0)\Big)$ for some $s \in [0, 1]$ (see e.g. Satz 7.5.2 of Struwe's script).

With this information, quantify how precise in the linear approximation in the ball $B_{\frac{1}{4}}(0, \frac{\pi}{2})$ by giving an upper bound for the corresponding error.

**Exercise 8.3** Compute the Taylor polynomials of the following functions at the given point and of the given order.

(a) $f(x, y) = \frac{1}{1 - xy}$, at $(0, 0)$, $2n-$th order with $n \geq 1$.

(b) $f(x, y) = \arctan(x^2 y)$, at $(0, 0)$, 2nd order.

(c) $f(z) = \log(|z|^2 + 1)$ $(z \in \mathbb{C} \simeq \mathbb{R}^2)$, at $z = 0$, $2n-$th order with $n \geq 1$.

(d) $f(x_1, \ldots, x_n) = \prod_{i=1}^n x_i$, at $x_0 = (2, \ldots, 2)$ 2nd order.

**Multiple Choice 13.1** True or false? Motivate your answer.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and consider its measure ("area" if $n = 2$, "volume" if $n = 3$) $|\Omega| = \int_\Omega dx_1 \cdots dx_n$. Then

| | | True | False |
|---|---|---|---|
| (a) | If $\Omega$ is unbounded, then the integral is divergent | ☐ | ☐ |
| (b) | If the integral is divergent, then $\Omega$ is unbounded | ☐ | ☐ |
| (c) | If there exists $\varepsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ so that $B_\varepsilon(x_j) \subset \Omega$ for every $j \in \mathbb{N}$, then the integral is divergent | ☐ | ☐ |
| (d) | If the integral is divergent, there exists $\varepsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ so that $B_\varepsilon(x_j) \subset \Omega$ for every $j \in \mathbb{N}$ | ☐ | ☐ |

**Multiple Choice 13.2** True or false? Motivate your answer.

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a non-negative, continuous function. Then

| | | True | False |
|---|---|---|---|
| (a) | If $\lim_{x\to\infty} f(x) = 0$, the improper integral $\int_{\mathbb{R}^n} f\, dx$ exists and is finite | ☐ | ☐ |
| (b) | If the improper integral $\int_{\mathbb{R}^n} f\, dx$ exists and is finite, then $\lim_{x\to\infty} f(x) = 0$ | ☐ | ☐ |
| (c) | If $\lim_{x\to\infty} f(x)$ does not exist, then the improper integral $\int_{\mathbb{R}^n} f\, dx$ is not finite | ☐ | ☐ |
| (d) | If $\lim_{x\to\infty} f(x)$ exists and is nonzero, the improper integral $\int_{\mathbb{R}^n} f\, dx$ is not finite. | ☐ | ☐ |

**Exercise 13.1** Compute the following integrals over the specified domains:

(a) $\displaystyle\int_D \sqrt{x^2 + y^2}\, dx\, dy, \qquad D = \{(x, y) : x^2 + y^2 - 2x \le 0\},$

(b) $\displaystyle\int_{B_R^+(0)} z\, dx\, dy\, dz, \qquad B_R^+(0) = \{(x, y, z) : x^2 + y^2 + z^2 \le R^2,\ z \ge 0\},$

(c) $\displaystyle\int_{D_2} xy\, dx\, dy, \qquad D_2 = \{(x, y) : 0 \le x \le y \le 1\},$

(d) $\displaystyle\int_{D_3} xyz\, dx\, dy\, dz, \qquad D_3 = \{(x, y, z) : 0 \le x \le y \le z \le 1\},$

**Exercise 13.2** Compute the following line integrals, draw the corresponding domain and compute them first directly, then using Green's theorem (the curves are always oriented counter-clockwise).

(a) $\displaystyle\int_{\partial A} (x^2, y^2) \cdot d\vec{s}, \qquad A = \left\{(x,y) \in \mathbb{R}^2 \mid -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2},\ -1 \le y \le \cos x\right\}$

(b) $\displaystyle\int_{\partial B} (xy, e^x) \cdot d\vec{s}, \qquad B = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ -1 \le y \le 1-x\right\}$

**Exercise 13.3**

(a) For $0 < a < b$ determine whether the improper integral

$$\int_{E(a,b)} x^2 y \mathrm{e}^{-(xy)^2}\, dx\, dy \quad \text{where } E(a,b) = [a,b] \times [0,+\infty)$$

is convergent, and if so compute it.

(b) Use (a) to determine whether the improper integral

$$\int_{E} x^2 y \mathrm{e}^{-(xy)^2}\, dx\, dy \quad \text{where } E = [0,+\infty) \times [0,+\infty)$$

is convergent (no need to compute it).

**Multiple Choice 9.1** True or false? Motivate your answer.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice differentiable function with a critical point $p_0$, whose Hessian matrix at $p_0$ is

$$\mathrm{Hess}_f(p_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then:

| | | True | False |
|---|---|---|---|
| (a) | $p_0$ *cannot* be a local maximum | ☐ | ☐ |
| (b) | $p_0$ *cannot* be a local minimum | ☐ | ☐ |
| (c) | $p_0$ *cannot* be a saddle point | ☐ | ☐ |
| (d) | none of the above. | ☐ | ☐ |

**Multiple Choice 9.2** True or false? Motivate your answers.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function and consider its restriction over the square $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Then:

| | | True | False |
|---|---|---|---|
| (a) | If $f$ has a local max/min/saddle at $x_0$ in $Q$, then $df(x_0) = 0$ | ☐ | ☐ |
| (b) | Let $x_0 \in Q$ be a point such that $df(x_0) = 0$, then $f$ has a local max/min/saddle at $x_0$. | ☐ | ☐ |

**Exercise 9.1** For each of the following functions, determine their critical points and find those for which the 2nd derivative test applies, determining in such case whether they are local maxima, local minima or saddle points.

(a) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^3 + y^3 - 3xy,$

(b) $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = (x^3 - 3x - y^2)z^2 + z^3,$

(c) $f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = xy^2 - \cos(x).$

**Exercise 9.2** Consider a system of $N$ particles in $\mathbb{R}^n$, that is $N$ points $a_1, \ldots, a_N$ with masses $m_1, \ldots, m_N$. Prove that the expression

$$I(x) = \sum_{i=1}^{N} m_i |x - a_i|^2$$

has a unique global minimum, and find it explicitly. Such point $C$ is called the *center of mass* of the system.

**Exercise 9.3** Let $0 \neq a \in \mathbb{R}^n$ be fixed and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{a \cdot x}{|x|^2 + 1}, \quad x \in \mathbb{R}^n,$$

where "$\cdot$" denotes the usual scalar product.

(a) Prove that $f$ attains its global maximum and global minimum.

(b) Compute the global extrema of $f$.

**Multiple Choice 7.1** True or false? Motivate your answer.

Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a *linear* function. Then, without doing any computation, we always know who is its differential $dL(x)$ at every point $x \in \mathbb{R}^m$.

☐ True ☐ False

**Multiple Choice 7.2** True or false? Motivate your answers.

Let $M_{n\times n}(\mathbb{R})$ be the space of $n \times n$ matrices which we identify with the Eulclidean space $\mathbb{R}^{n^2}$. The function "determinant" $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $A \mapsto \det A$, is

| | | True | False |
|---|---|---|---|
| (a) | is continuous on $M_{n\times n}(\mathbb{R})$ | ☐ | ☐ |
| (b) | continuous only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$ | ☐ | ☐ |
| (c) | differentiable on $M_{n\times n}(\mathbb{R})$ | ☐ | ☐ |
| (d) | differentiable only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$ | ☐ | ☐ |

**Exercise 7.1** Compute first the differential $df$ of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, then compute $df(p_0) \cdot v$ at the given $p_0$ and $v$.

(a) $f(x, y) = y, \quad p_0 = (3, \frac{1}{3}), \quad v = (-2, -4),$

(b) $f(x, y) = (x - 2y)^3, \quad p_0 = (6, 2), \quad v = (2, 1),$

(c) $f(x, y) = \sin(2x) + \cos(3y), \quad p_0 = (\pi, \frac{\pi}{2}), \quad v = (6, -7).$

**Exercise 7.2**

(a) Let

- $x(t) = \cos(\pi t)$,
- $y(t)$ be the primitive of the function $t \mapsto e^{-t^2}$, so that $y(1) = 42$,
- $f(x, y) = x^2 + y^2$.

Compute the derivative of the composite function $t \mapsto f(x(t), y(t))$ at $t = 1$.

(b) Compute the differential $df(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = \int_{\cos x + \sin y}^{z} e^{tz}\, dt.$$

(c) Let $f(x,y) = |xy|$. Find the set of points $(x,y) \in \mathbb{R}^2$, where $f$ is differentiable and compute its differential at those points.

*Note for (b):* to compute the derivative in the $z$-direction, one way is to use directly the definition as limit of difference quotient.

**Exercise 7.3 (Gradient and Level Sets)** A *curve in the plane* is a subset $\Gamma \subset \mathbb{R}^2$ so that there is a differentiable function $\gamma : (a,b) \subset \mathbb{R} \to \mathbb{R}^2$ so that $\gamma((a,b)) = \Gamma$ and $\gamma'(t) \neq 0$ for every $t \in (a,b)$. Any such $\gamma$ is called *parametrization* of $\Gamma$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function so that its *level set* at $c$

$$\Gamma = f^{-1}(\{c\}) = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$$

is a curve in the plane, and let $\gamma : I = (a,b) \to \Gamma$ be a parametrization. Prove that:

(a) The gradient to $f$ is orthogonal to $\Gamma$, namely

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0 \quad \text{for every } t \in I.$$

(b) The directional derivative of $f$ along $\Gamma$ vanishes, namely

$$D_{\gamma'(t)} f(\gamma(t)) = 0 \quad \text{for every } t \in I.$$

(c) At a point $(x,y) \in \Gamma$, the direction where $f$ grows the most is orthogonal to $\Gamma$ (this means: among all the vectors $v$ with $|v| = 1$, $D_v f(x,y)$ assumes the maximum value when $v$ is orthogonal to $\Gamma$).

**Multiple Choice 2.1** True or false? Motivate your answers.

Consider the ODE $y''(x) - \sigma^2 y(x) = 0$, where $\sigma > 0$. Then:

| | | True | False |
|---|---|---|---|
| (a) | It has at least one bounded solution. | ☐ | ☐ |
| (b) | If $y$ is a nonzero solution, then $y$ is unbounded. | ☐ | ☐ |
| (c) | For any two values $a, b \in \mathbb{R}$, there is only one solution with $y(0) = a$ and $y(1) = b$. | ☐ | ☐ |
| (d) | There is no solution which is surjective from $\mathbb{R}$ to $\mathbb{R}$. | ☐ | ☐ |

**Multiple Choice 2.2** Choose the correct statement(s). Motivate your answers.

A particular solution of the ODE $y'' + 2y' + y = \mathrm{e}^{-x}$ is:

(a) $k\mathrm{e}^{-x}$ for some constant $k$. ☐

(b) $kx\mathrm{e}^{-x}$ for some constant $k$. ☐

(c) $kx^2\mathrm{e}^{-x}$ for some constant $k$. ☐

(d) $k\mathrm{e}^{-x^2}$ for some constant $k$. ☐

**Exercise 2.1** A piece of mass (of mass $m$) connected to a coil spring that can stretch along its length. If $k > 0$ denotes the spring constant, the equation of motion of such system is given, according to Hooke and Newton's laws, by

$$m\ddot{x}(t) = -kx(t), \tag{\dagger}$$

where $x = x(t)$ denotes the position in time of the piece of mass along the vertical direction. Call $\omega = \sqrt{\frac{k}{m}}$. Find the solution of (†):

(a) with initial position $x(0) = 1$ and initial velocity $\dot{x}(0) = 2\omega$.

(b) with initial position $x(0) = 1$ and final position $x(\frac{\pi}{2\omega}) = 1$.

(c) Is is possible to find a solution so that $x(t) \to -\infty$ as $t \to +\infty$?

**Exercise 2.2** It is observed that the populations of certain species of bacteria grow, when there is plenty of food and space, with a rate proportional to the number of present individuals. So, if $y(t)$ represents the size of the bacteria population with respect to time $t \geq 0$, it satisfies

$$\begin{cases} y'(t) = \kappa y(t) & \text{for } t > 0, \\ y(0) = y_0, \end{cases} \tag{$\circ$}$$

where $y_0$ represents the initial size of the population and $\kappa > 0$ is a constant determined by the biology of the bacteria in consideration.

(a) Find the solution of the problem ($\circ$). Looking at the solution, can you guess what is this kind of growth called?

(b) Suppose you are in a lab where the technology to observe the population delivers one picture per $\varepsilon$ seconds, for some small $\varepsilon > 0$. Explain why ($\circ$) is replaced by

$$\begin{cases} \dfrac{y(t+\varepsilon) - y(t)}{\varepsilon} = \kappa y(t) & \text{for } t = 0, \varepsilon, 2\varepsilon, \ldots \\ y(0) = y_0, \end{cases} \tag{$\diamond$}$$

and find the solution of the problem ($\diamond$).

(c) How does the solution to ($\diamond$) behave as $\varepsilon \to 0$?

**Exercise 2.3** Consider the differential equation

$$xy' = 2y - 3xy^2 \qquad \text{for } x > 0,$$

in the unknown $y = y(x)$.

(a) Rewrite the equation using the change of variable $y = \frac{1}{u}$.

(b) Solve the equation in the new variable $u$ and then write the solution in the original variable $y$.

(c) There is one solution that cannot be obtained with the procedure (a) & (b). What is it, and why?