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# Exemplary Solutions – Sheet 11

Zürich, December 4, 2020

## Solution to Exercise 29

We introduce a variable $x_{i,j}$ for each square $(i, j)$ of the $(n \times n)$ chess board. If $x_{i,j}$ is assigned 1, then we interpret this as placing a rook on the square $(i, j)$. Hence, every truth assignment of the $n^2$ variables corresponds to one possible placing of up to $n^2$ rooks on the chess board.

In order for the rooks to pairwise not attack each other, each row (rank) and each column (file) can contain at most one rook.

This can be expressed through the following formula:

- For every row $i$, we use the subformula

  $$Z_i = \bigwedge_{1 \leq j_1 < j_2 \leq n} (\overline{x_{i,j_1}} \vee \overline{x_{i,j_2}}).$$

  This ensures that no two variables at the same row can be simultaneously assigned 1.

- Analogously, we use the following subformula for every column $j$:

  $$S_j = \bigwedge_{1 \leq i_1 < i_2 \leq n} (\overline{x_{i_1,j}} \vee \overline{x_{i_2,j}}).$$

Then the conjunction

$$F_1 = \bigwedge_{1 \leq i \leq n} Z_i \wedge \bigwedge_{1 \leq j \leq n} S_j$$

is a CNF formula which describes the condition that no two rooks may attack each other.

To additionally guarantee that exactly $n$ rooks are placed, we observe that this can only be the case if each row contains exactly one rook. As the above formula already guarantees that each row contains at most one rook, it suffices to add a formula making sure that each row contains at least one rook. This can be expressed for row $i$ through the following formula:

$$\bigvee_{1 \leq j \leq n} x_{i,j} \,.$$

Overall, this yields the desired formula:

$$F = F_1 \wedge \bigwedge_{1 \le i \le n} \bigvee_{1 \le j \le n} x_{i,j}.$$

## Solution to Exercise 30

(a) Let $\phi$ be an input for SAT, i.e., a CNF formula $\phi = C_1 \wedge \ldots \wedge C_m$ with the clauses $C_1, \ldots, C_m$ over the variables from $X = \{x_1, \ldots, x_n\}$. We construct from $\phi$ an input $\psi$ for THREEFOLD-SAT as follows. Let $y_0, y_1 \notin X$ be two new variables that do not occur in $\phi$ and let $C_{m+1} = (y_0 \vee y_1)$. Then we define $\psi = C_1 \wedge \ldots \wedge C_m \wedge C_{m+1}$. The construction of $\psi$ from $\phi$ is clearly feasible in polynomial time.

Now we show that $\phi$ is satisfiable if and only if $\psi$ has at least three satisfying truth assignments.

Assume that $\phi$ is satisfiable. Then there exists a truth assignment $\alpha\colon X \to \{0,1\}$ that satisfies the clauses $C_1, \ldots, C_m$. We can extend $\alpha$ to three truth assignments $\beta_j$, $j \in \{0,1,2\}$, on $X \cup \{y_0, y_1\}$ by setting $\beta_j(x) = \alpha(x)$ for all $x \in X$ and

- $\beta_0(y_0) = 0$ and $\beta_0(y_1) = 1$,
- $\beta_1(y_0) = 1$ and $\beta_1(y_1) = 0$, and
- $\beta_2(y_0) = \beta_2(y_1) = 1$.

All $\beta_j$ clearly satisfy all clauses $C_1, \ldots, C_m$, as $\alpha$ satisfies these clauses. The clause $C_{m+1}$ is also satisfied by all $\beta_j$, since at least one literal in $C_{m+1}$ is set to 1. Hence, the $\beta_j$ are three satisfying truth assignments for $\psi$.

Assume that $\phi$ is not satisfiable. Then there exists no truth assignment $\alpha\colon X \to \{0,1\}$ that satisfies the clauses $C_1, \ldots, C_m$ and thus neither $\beta\colon X \cup \{y_0, y_1\} \to \{0,1\}$, since the new variables $y_0$ and $y_1$ do not occur in $C_1, \ldots, C_m$. As every clause in $\phi$ occurs in $\psi$ as well, there exists no truth assignment satisfying $\psi$.

(b) Let $\phi$ be an input for 3SAT, i.e., a 3-CNF formula $\phi = C_1 \wedge \ldots \wedge C_m$ with the clauses $C_1, \ldots, C_m$ over the variables from $X = \{x_1, \ldots, x_n\}$. We seek to translate $\phi$ into an input $\psi$ for E3SAT. We first modify $\phi$ so that no variable occurs in a clause multiple times. To this end, we drop all repetitions of a literal within a clause. This has no effect on the satisfiability of the formula. If a clause contains a variable $y$ and its negation $\overline{y}$ as a literal, then the clause is satisfied for an arbitrary truth assignment. We can thus drop such clauses from $\phi$ without affecting its satisfiability. Without loss of generality, we can thus assume that every variable occurs at most once in every clause of $\phi$. Then the literals within a clause correspond to pairwise distinct variables.

Now we can construct $\psi$ as follows: all clauses of $\phi$ that already contain three literals stay unchanged. A clause $C_i = (l_{i,1} \vee l_{i,2})$ gets replaced by the two new clauses $C_{i,1} = (l_{i,1} \vee l_{i,2} \vee y_i)$ and $C_{i,2} = (l_{i,1} \vee l_{i,2} \vee \overline{y_i})$, where $y_i$ is a new variable that does not occur anywhere else in $\psi$. A clause $C_i = (l_i)$ gets replaced by the four new clauses $C_{i,1} = (l_i \vee y_{i,1} \vee y_{i,2})$, $C_{i,2} = (l_i \vee y_{i,1} \vee \overline{y_{i,2}})$, $C_{i,3} = (l_i \vee \overline{y_{i,1}} \vee y_{i,2})$, and $C_{i,4} = (l_i \vee \overline{y_{i,1}} \vee \overline{y_{i,2}})$, where $y_{i,1}$ and $y_{i,2}$ are two new variables that do not occur anywhere else in $\psi$. This construction is clearly feasible in polynomial time and the additional conditions of E3SAT with respect to 3SAT are met.

Now we show that $\phi$ is satisfiable if and only if $\psi$ is satisfiable.

Let $\alpha$ be a satisfying truth assignment for $\phi$. Then $\alpha$ sets at least one literal in each of the clauses $C_1, \ldots, C_m$ to 1. Furthermore, an arbitrary extension of $\alpha$ satisfies every clause of $\psi$, since these clauses may only contain additional variables.

Let $\beta$ be a satisfying truth assignment for $\psi$. Then all clauses in $\psi$ are satisfied by $\beta$. We show that the restriction of $\beta$ on the variables from $X$ satisfies $\phi$. Every clause from $\phi$ of length 3 has been left unchanged in $\psi$ and is thus satisfied by $\beta$. For a clause $C_i = (l_{i,1} \vee l_{i,2})$ of length 2 in $\phi$, $\psi$ contains the two clauses $C_{i,1}$ and $C_{i,2}$. The variable $y_i$ occurs positive in one of these two clauses and negative in the other one. We first assume that $\beta(y_i) = 1$. Every clause in $\psi$ is satisfied by $\beta$ by assumption, in particular, $C_{i,2}$ is satisfied although the literal $\overline{y_i}$ is set to 0 here. Hence, $C_i = (l_{i,1} \vee l_{i,2})$ must already be satisfied by $\beta$. The case of $\beta(y_i) = 0$ is analogous.

For a clause $C_i$ of length 1 in $\phi$, there are four clauses in $\psi$, such that, for every truth assignment of the two new variables $y_{i,1}$ and $y_{i,2}$, one of these clauses can only be satisfied, if $C_i$ is satisfied as well. Overall, $\beta$ thus satisfies all clauses in $\phi$.

## Solution to Exercise 31

We consider the reduction from the task statement and seek to prove that the constructed graph $G_\phi$ contains a vertex cover of size at most $5q$ if and only if the E3-CNF formula $\phi$ with $q$ clauses is satisfiable.

Let $\Delta_s$ be the triangle consisting of the vertices $V_{s,1}$, $V_{s,2}$, and $V_{s,3}$ for $1 \leq s \leq q$. Let $H_i$ be the cycle in $G_\phi$ consisting of the vertices $T_{i,1}, F_{i,1}, T_{i,2}, F_{i,2}, \ldots, T_{i,m(i)}, F_{i,m(i)}$ for $i \in \{1, \ldots, n\}$. We first observe that each of the cycles $H_i$ contains exactly two vertex covers of size $m(i)$, i.e., $\{T_{i,1}, T_{i,2}, \ldots, T_{i,m(i)}\}$ and $\{F_{i,1}, F_{i,2}, \ldots, F_{i,m(i)}\}$. Furthermore, there exists no vertex cover of $H_i$ with less than $m(i)$ vertices.

Let $\alpha$ be a satisfying truth assignment for $\phi$. From $\alpha$, we construct a vertex cover $S$ of $G_\phi$ as follows:

- For every variable $x_i$ with $\alpha(x_i) = 1$, we take all the vertices $T_{i,1}, T_{i,2}, \ldots, T_{i,m(i)}$ into the vertex cover $S$,
- for every variable $x_i$ with $\alpha(x_i) = 0$, we take all the vertices $F_{i,1}, F_{i,2}, \ldots, F_{i,m(i)}$ into $S$, and
- for every clause $C_s$, we take exactly two vertices of the triangle $\Delta_s$ into $S$, such that a vertex corresponding to a literal that is set to 1 by $\alpha$ is not taken. There always exists such a vertex, since $\alpha$ is a satisfying truth assignment.

The set $S$ contains exactly $2q$ vertices from the clause triangles $\Delta_s$ and exactly $3q$ vertices from the cycles, as the cycles have a total length of $6q$: every clause contains exactly three occurrences of a variable and for each such occurrence, there are exactly two vertices in the cycles.

The vertices from $S$ cover all edges of the cycles $H_i$ and all edges of the clause triangles $\Delta_s$. It remains to prove that all edges from the sets $E_1$ and $E_2$ are covered as well. Let $\{T_{i,j}, V_{s,t}\} \in E_1$. If $V_{s,t} \in S$, then this edge is covered. Otherwise, by construction of $S$, the vertex $V_{s,t}$ corresponds to a literal that is set to 1 by $\alpha$, i.e., $T_{i,j} \in S$. The argument for edges from $E_2$ is analogous.

Now let $S$ be a vertex cover for $G_\phi$ of size $5q$. At least $2q$ vertices are necessary to cover all clause triangles $\Delta_s$ and at least $3q$ vertices are necessary to cover all cycles $H_i$. Hence, $S$ contains in every clause triangle $\Delta_s$ exactly two vertices and (by the above observation) in every cycle $H_i$ either all the vertices $T_{i,1}, T_{i,2}, \ldots, T_{i,m(i)}$ or all the vertices $F_{i,1}, F_{i,2}, \ldots, F_{i,m(i)}$. We define a truth assignment $\alpha$ as $\alpha(x_i) = 1$, if $T_{i,1} \in S$, and $\alpha(x_i) = 0$, if $F_{i,1} \in S$.

It remains to show that $\alpha$ satisfies the formula $\phi$. Assume that there is a clause $C_s$ in $\phi$ that is not satisfied by $\alpha$. For the sake of simplicity, we assume that $C_s$ only contains positive literals. In all other cases, the argument is analogous. Let $C_s = (x_{i_1} \vee x_{i_2} \vee x_{i_3})$, which is the $j_1$-th, $j_2$-th, and $j_3$-th occurrence of the respective variable. As $C_s$ is not satisfied, we have $\alpha(x_{i_1}) = \alpha(x_{i_2}) = \alpha(x_{i_3}) = 0$. Hence, the vertices $T_{i_1,j_1}$, $T_{i_2,j_2}$, and $T_{i_3,j_3}$ are not contained in $S$ by construction. The three edges $\{T_{i_1,j_1}, V_{s,1}\}$, $\{T_{i_2,j_2}, V_{s,2}\}$, and $\{T_{i_3,j_3}, V_{s,3}\}$ must thus be covered by the three vertices $V_{s,1}$, $V_{s,2}$, and $V_{s,3}$, but only two of them are contained in $S$ by construction. This is a contradiction to the fact that $S$ is a vertex cover. Hence, our assumption was false, $C_s$ is satisfied by $\alpha$, and $\alpha$ is a satisfying truth assignment for $\phi$.