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Exemplary Solutions – Sheet 10

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Solution to Exercise 26

Let $f\colon \mathbb{N} \to \mathbb{N}$ be a monotonically increasing function such that $f(n) \geq n$ for all $n \in \mathbb{N}$.

(a) Let $L_1, L_2 \in \text{NTIME}(f)$. We seek to prove that $L_1 \cup L_2 \in \text{NTIME}(f)$. By definition of NTIME$(f)$, there exist two nondeterministic multitape Turing machines M_1 with k_1 tapes and M_2 with k_2 tapes for $k_1, k_2 \geq 1$, such that $L_1 = L(M_1)$, $L_2 = L(M_2)$, and $\text{Time}_{M_1}(w_1), \text{Time}_{M_2}(w_2) \in O(f(n))$ for all inputs $w_1 \in L_1, w_2 \in L_2$ of length n. From them, we construct a nondeterministic (k_1+k_2+1)-tape TM M for $L = L_1 \cup L_2$ as follows.

The machine M first copies its input onto the first working tape and then brings the heads on the tapes back to the beginning. This is clearly feasible in time $O(n)$ and thus also in $O(f(n))$, since $f(n) \geq n$, for all $n \in \mathbb{N}$.

Now M simulates the work of M_1 on the input tape and the working tapes 2 through $k_1 + 1$ and the work of M_2 on the working tape 1 as the input tape and the working tapes $k_1 + 2$ through $k_1 + k_2 + 1$ in parallel. The machine M accepts if and only if at least one of the two simulations of M_1 and M_2 accepts. Clearly, there exists an accepting computation of M on the word w if and only if there exists an accepting computation of M_1 or M_2 on w. Furthermore, the length of the shortest accepting computation of M on w does not exceed the time to copy the input plus the length of the shortest accepting computation of M_1 or M_2 on w. Hence, $\text{Time}_M(w) \in O(n) + \min\{\text{Time}_{M_1}(w), \text{Time}_{M_2}(w)\} \subseteq O(f(n))$ for all $w \in L_1 \cup L_2$. It follows that $L_1 \cup L_2 \in \text{NTIME}(f)$.

(b) Let $L \in \text{NTIME}(f)$ and $L' \in \text{TIME}(f)$. Then there exist a nondeterministic k_1-tape Turing machine M_1 for L and a deterministic k_2-tape Turing machine M_2 for L' such that $\text{Time}_{M_1}(n), \text{Time}_{M_2}(n) \in O(f(n))$. From them, we construct a nondeterministic $(k_1 + k_2)$-tape TM M for $L - L'$ with a time complexity in $O(f(n))$ as follows.

The machine M first simulates the work of M_2 on the input w of length n on the working tapes k_1+1 through k_1+k_2. Because $\text{Time}_{M_2}(n) \in O(f(n))$, this simulation is guaranteed to terminate on every input of length n in time $O(f(n))$. If M_2 reaches the accepting state, then $w \in L'$, hence, $w \notin L - L'$, and M thus rejects its input. If M_2 reaches the rejecting state, then $w \notin L'$. In this case, M brings the reading head on the input tape back to the beginning and starts a simulation of M_1 on w on the first k_1 working tapes. If M_1 accepts the word w, then M accepts as well and vice versa.

The time complexity of M can be bounded as follows. The simulation of M_2 clearly requires $O(f(n))$ steps, bringing back the reading head then requires again at most $O(n) \subseteq O(f(n))$ steps. If the word w is accepted by M_1, then, by definition of nondeterministic time complexity, there exists a computation of M_1 that can be simulated in $O(f(n))$ time. Hence, $\text{Time}_M(n) \in O(f(n))$.

Solution to Exercise 27

(a) Let M be a nondeterministic MTM with $\text{Time}_M(n) \in O(n^2)$ that in addition uses at most $O(n)$ space during every computation. To prove that $L(M) \in \text{SPACE}(n \log n)$, we construct a deterministic MTM A such that $L(A) = L(M)$, that simulates the computation of M. For the construction, we follow the proof of Savitch's theorem from the textbook. This means we again assume that M has a unique accepting configuration for all words $w \in L(M)$, so that a deterministic simulation of the work of M, performed by A, only needs to determine if the accepting configuration $C_{\text{accept}}(w)$ is reachable from the initial configuration $C_{\text{start}}(w)$. Due to the assumption on the time complexity of M, the length of the shortest accepting computation of M on w is at most $d \cdot |w|^2$, for a suitable constant d. To determine whether $C_{\text{accept}}(w)$ is reachable from $C_{\text{start}}(w)$ in $d \cdot |w|^2$ steps, we use the same procedure REACHABLE as in the proof of Savitch's theorem. Because the function $\log_2(d \cdot n^2) = 2\log_2 n + \log_2 d$ is space constructible, A can compute and save the value $d \cdot |w|^2$ for an arbitrary word using $2\log_2 |w| + \log_2 d$ space. Every internal configuration of a computation of M on w can be represented in $c \cdot |w|$ space, because $O(n)$ space is sufficient for every computation by assumption. To execute REACHABLE, at most $O(\log_2 |w|)$ configurations have to be saved at the same time, because the recursion depth is logarithmic in the time complexity of M. Hence, we derive a space requirement in $O(|w| \cdot \log |w|)$ for A by the same argument as in the proof of Savitch's theorem.

(b) An analogous construction as in part (a) or in the proof of Savitch's theorem cannot be applied here. For every language L in $\text{NSPACE}(f(n)) \cap \text{NTIME}(f(n)^k)$, $L \in \text{NSPACE}(f(n))$ and $L \in \text{NTIME}(f(n)^k)$ holds as well. Hence, there exists a nondeterministic MTM M_1 such that $L(M_1) = L$ and $\text{Space}_{M_1}(n) \in O(f(n))$ and a nondeterministic MTM M_2 such that $L(M_2) = L$ and $\text{Time}_{M_2}(n) \in O((f(n))^k)$. However, it is still possible that $\text{Space}_{M_2}(n) \notin O(f(n))$ and $\text{Time}_{M_1}(n) \notin O((f(n))^k)$. In this case, we would only have, on the one hand, an MTM that decides L with sufficiently low space complexity, but too high time complexity and, on the other hand, another MTM that decides L with sufficently low time complexity, but too high time complexity. To make an analogous argument as in part (a), we need a *single* nondeterministic MTM that satisfies both bounds for space and time.

(please turn over)

Solution to Exercise 28

Let $L \in \mathrm{VP}$ and let A be a polynomial-time verifier for L. Assume that, for every word $w \in L$, there exists a witness x such that $|x| \leq \log_2 |w|$ and A accepts the input (w, x).

The following multitape Turing machine M decides L: For all potential witnesses $x \in \Sigma_{\text{bool}}^*$ such that $|x| \leq \log_2 |w|$, M simulates successively the work of A on (w, x). If A accepts a pair (w, x), then M accepts as well. If A rejects each pair (w, x), then M rejects.

We first prove that $L(M) = L$. We start with $L(M) \supseteq L$. If $w \in L$, then there exists a witness $x \in \Sigma_{\text{bool}}^*$ such that $|x| \leq \log_2 |w|$ by assumption, so that (w, x) is accepted by A. This witness will be considered by M in one of its simulations, M thus accepts. Hence, $w \in L(M)$.

Now we prove that $L(M) \subseteq L$. If $w \in L(M)$, then M accepts its input in one of its simulations. But this can only happen if the input (w, x) is accepted by the verifier A for the witness x used in that particular simulation. Hence, the condition for $w \in L$ is satisfied.

It remains to show that M works in polynomial time. As A is a polynomial-time verifier, there exists a polynomial p such that $\mathrm{Time}_A(w, x) \leq p(|w|)$ for all $x \in \Sigma_{\text{bool}}^*$. The running time of one simulation performed by M is thus bounded by $p(|w|) + |w|$ for a constant c. The additional overhead $|w|$ comes from the fact that M has to bring the head on the input tape back to the beginning of the respective tapes after each simulation and must produce the next potential witness.

The number of potential witnesses $x \in \Sigma_{\text{bool}}^*$ such that $|x| \leq \log_2 |w|$ is

$$\sum_{i=0}^{\lfloor \log_2 |w| \rfloor} 2^i = 2^{\lfloor \log_2 |w| \rfloor + 1} - 1 < 2^{\lfloor \log_2 |w| \rfloor + 1} \leq 2|w| \, .$$

This is an upper bound on the number of simulations performed by M, it follows that, for a constant c,

$$\mathrm{Time}_M(w) \leq 2|w| \cdot (p(|w|) + |w|) + c \in O(|w|^2 p(|w|)) \, .$$

Hence, M is a deterministic polynomial-time MTM deciding L, i.e., $L \in \mathrm{P}$.