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Theoretische Informatik

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Exemplary Solutions – Sheet 4

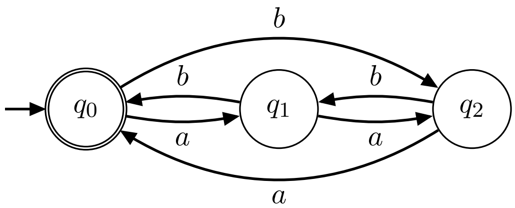
Zürich, October 16, 2020

Solution to Exercise 10

The language L given in the task statement can be written as $L = L_1 \cup L_2$ where

$$\begin{aligned} L_1 &= \{w \in \{a,b\}^* \mid |w|_a \bmod 3 = |w|_b \bmod 3\} \\ &= \{w \in \{a,b\}^* \mid (|w|_a - |w|_b) \bmod 3 = 0\}\,, \\ L_2 &= \{w \in \{a,b\}^* \mid w \text{ contains the subword } ba \text{ and ends by } a\}\,. \end{aligned}$$

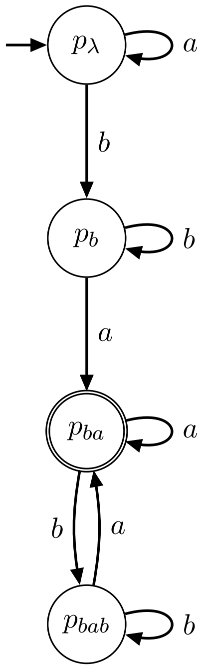
For the language L_1, the following automaton can be constructed:



This automaton counts in its states the number of a's and b's modulo 3 according to the formula from the definition of L_1. For $i \in \{0,1,2\}$, we have

$$\mathrm{Kl}[q_i] = \{w \in \{a,b\}^* \mid (|w|_a - |w|_b) \bmod 3 = i\}\,.$$

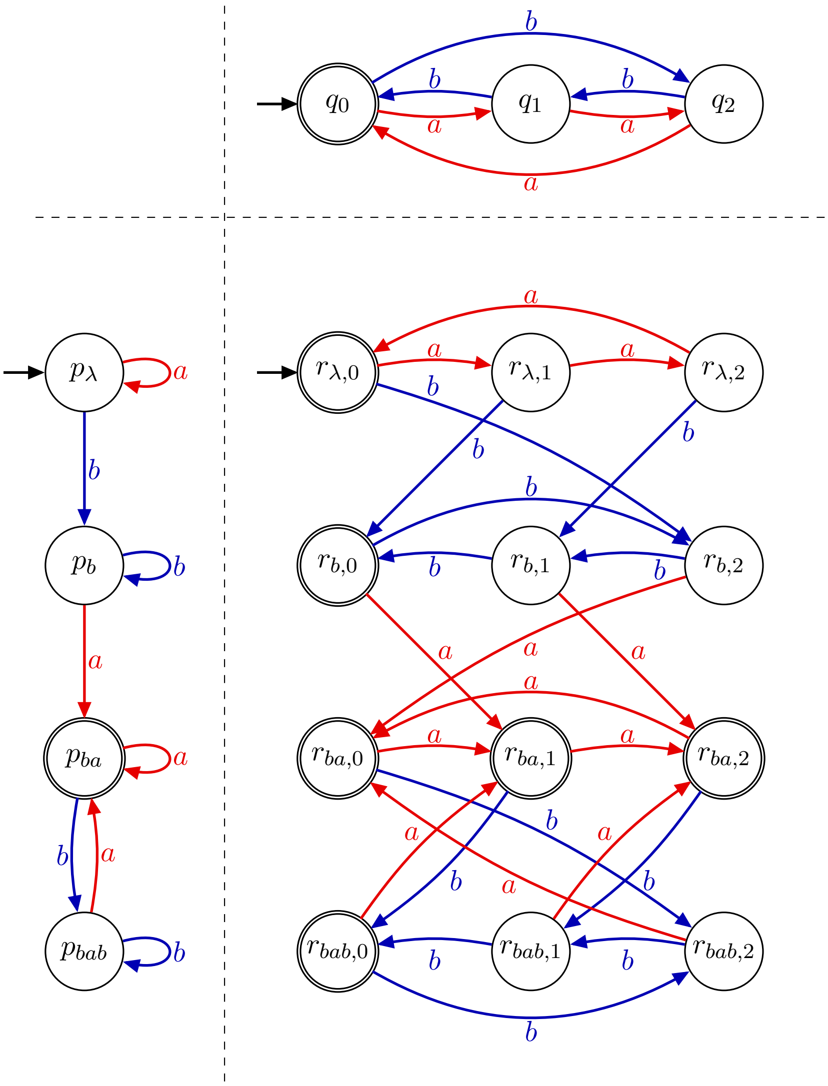
For the language L_2, the following automaton A_2 can be constructed:



The states p_z for $z \in \{\lambda, b, ba\}$ determine the longest prefix of the seeked pattern ba that has been read so far, p_{ba} is the accepting state. The state p_{bab} is reached once the pattern ba has already been found and the currently read prefix of the word ends by b. We thus obtain the following classes:

$$
\begin{aligned}
\mathrm{Kl}[p_\lambda] &= \{a\}^*\,, \\
\mathrm{Kl}[p_b] &= \{xby \mid x \in \{a\}^* \text{ and } y \in \{b\}^*\}\,, \\
\mathrm{Kl}[p_{ba}] &= L_2\,, \\
\mathrm{Kl}[p_{bab}] &= \{a,b\}^* - \bigcup_{z \in \{\lambda, b, ba\}} \mathrm{Kl}[p_z]\,.
\end{aligned}
$$

Using the paradigm of modular design, one can construct the following product automaton from the languages A_1 and A_2 that accepts the language $L = L_1 \cup L_2$. For the sake of readability, we use the notation $r_{x,y} = \langle p_x, q_y \rangle$. Since this product automaton accepts the union of two languages, its accepting states are all states containing an accepting state from one of the two subautomata, i.e., the column corresponding to q_0 and the row corresponding to p_{ba}.



Solution to Exercise 11

(a) Using Lemma 3.12 from the textbook, we show that the language

$$L_1 = \{waww \mid w \in \{a,b\}^+\}$$

is nonregular. Assume that L_1 is regular. Then there exists an automaton $A_1 = (Q, \{a,b\}, \delta, q_0, F)$ such that $L(A_1) = L_1$. Let $m = |Q|$. We consider the words

$$b^k a b^k \quad \text{for } k \in \{1, \dots, m+1\}.$$

Since these are $m+1$ words, i.e., more words than the number of A_1's states, there exist $i, j \in \{1, \dots, m+1\}$ such that $i < j$ and

$$\hat{\delta}(q_0, b^i a b^i) = \hat{\delta}(q_0, b^j a b^j).$$

By Lemma 3.12, for every $z \in \{a,b\}^*$, we have

$$b^i a b^i z \in L_1 \iff b^j a b^j z \in L_1.$$

However, the choice of $z = b^j$ leads to a contradiction, since

$$b^i a b^i z = b^i a b^i b^j \notin L_1, \tag{1}$$

$$b^j a b^j z = b^j a b^j b^j \in L_1\,,$$

where (1) is derived from the following observation: There is a single occurrence of a in $b^i a b^i b^j$, hence, $w = b^i$ must hold, but then $b^i b^j \neq ww$ as $j > i$. The assumption is thus false and L_1 is nonregular.

(b) Using Lemma 3.12 from the textbook, we show that the language

$$L_2 = \{a^i b^j \mid i, j \in \mathbb{N} \text{ and there exists some } k \in \mathbb{N} \text{ such that } j = k \cdot i\}$$

is nonregular. Assume that L_2 is regular. Then there exists an automaton $A_2 = (Q, \{a, b\}, \delta, q_0, F)$ such that $L(A_2) = L_2$. Let $m = |Q|$. We consider the words

$$a^l b \quad \text{for } l \in \{1, \ldots, m+1\}\,.$$

Since these are $m + 1$ words, i.e., more words than the number of A_2's states, there exist $i, j \in \{1, \ldots, m+1\}$ such that $i < j$ and

$$\hat{\delta}(q_0, a^i b) = \hat{\delta}(q_0, a^j b)\,.$$

By Lemma 3.12, for every $z \in \{a, b\}^*$, we have

$$a^i b z \in L_2 \iff a^j b z \in L_2.$$

However, the choice of $z = b^{i-1}$ leads to a contradiction, since

$$a^i b z = a^i b b^{i-1} = a^i b^i \in L_2\,,$$
$$a^j b z = a^j b b^{i-1} = a^j b^i \notin L_2\,,$$

where the latter statement follows from the fact that $j > i > 0$, whence i cannot be a multiple of j. The assumption is thus false and L_2 is nonregular.

Solution to Exercise 12

(a) The statement is false: Let $L_1 = \{0^n 1^n \mid n \in \mathbb{N}\}$ and let $L_2 = \{0^i 1^j \mid i, j \in \mathbb{N} \text{ and } i \neq j\}$. We know from the lecture that L_1 is nonregular. Now, using Lemma 3.12, we show that L_2 is nonregular as well: Assume that L_2 is regular. Then there exists an automaton $A_2 = (Q, \{0, 1\}, \delta, q_0, F)$ such that $L(A_2) = L_2$. Let $m = |Q|$. We consider the words

$$0^k 1 \quad \text{for } k \in \{1, \ldots, m+1\}\,.$$

Since these are $m + 1$ words, i.e., more words than the number of A_2's states, there exist $i, j \in \{1, \ldots, m+1\}$ such that $i < j$ and

$$\hat{\delta}(q_0, 0^i 1) = \hat{\delta}(q_0, 0^j 1)\,.$$

By Lemma 3.12, for every $z \in \{a, b\}^*$, we have

$$0^i 1 z \in L_2 \iff 0^j 1 z \in L_2.$$

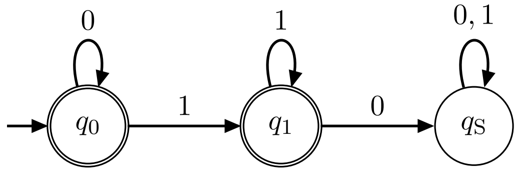
However, the choice of $z = 1^{i-1}$ leads to a contradiction, since

$$0^i 1 z = 0^i 1 1^{i-1} = 0^i 1^i \notin L_2\,,$$

$$0^j 1 z = 0^j 1 1^{i-1} = 0^j 1^i \in L_2\,,$$

where the latter statement follows from the fact that $j > i$. The assumption is thus false and L_2 is nonregular.

It holds that $L_1 \cup L_2 = \{0^i 1^j \mid i, j \in \mathbb{N}\} \in \mathcal{L}_{\mathrm{EA}}$ as one can easily construct a finite automaton with three states for this language:



(b) The statement is false: Let $L_1 = \{0^n 1^n \mid n \in \mathbb{N}\}$ and $L_2 = \{1^n 0^n \mid n \in \mathbb{N}\}$. We know from the lecture that L_1 is nonregular. Hence, the same clearly holds for L_2 as well. Furthermore, $L_1 \cap L_2 = \{\lambda\}$ and $\{\lambda\}$ is regular, just like any finite language.

(c) The statement is false: Let $L_1 = L_2$ be an arbitrary nonregular language. Then $L_1 - L_2 = \emptyset \in \mathcal{L}_{\mathrm{EA}}$.