ETH zürich
Departement Informatik

Theoretische Informatik

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Exercises – Sheet 11

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Exercise 29

The rook in chess can move any number of squares horizontally along a row (rank) or vertically along a column (file). We say that a rook attacks a piece if the rook can get at the square of the piece in a single move.

Describe a CNF formula whose truth assignments represent all possible placings of up to n^2 rooks on an $(n \times n)$ chess board and which is satisfied if and only if exactly n rooks are placed that do not attack each other. **10 points**

Exercise 30

(a) Let THREEFOLD-SAT be the set of all CNF formulas that have at least *three* satisfying truth assignments. Prove that SAT $\leq_p$ THREEFOLD-SAT.

(b) Let E3SAT be the set of all satisfiable CNF formulas that have exactly three literals of pairwise distinct variables per clause. Prove that 3SAT $\leq_p$ E3SAT.

10 points

(please turn over)

Exercise 31

We seek to prove that the vertex cover problem is NP-hard even on graphs with maximum vertex degree 3. To this end, let

$$\begin{aligned}\text{VC-deg-3} = \{(G,k) \mid\ & G \text{ is an undirected graph with maximum vertex degree 3} \\ & \text{that contains a vertex cover of size at most } k\}.\end{aligned}$$

We seek to show E3SAT $\leq_p$ VC-deg-3. To this end, for every E3-CNF formula ϕ, we construct a graph G_ϕ with maximum vertex degree 3 and a number k_ϕ, such that ϕ is satisfiable if and only if G_ϕ contains a vertex cover of size at most k_ϕ.

Let $\phi = C_1 \wedge \ldots \wedge C_q$ be a E3-CNF formula over the variables $x_1, \ldots, x_n$, such that $C_j = (l_{j,1} \vee l_{j,2} \vee l_{j,3})$ for $1 \leq j \leq q$. For every variable x_i, we enumerate its occurrences in the clauses (as a positive or negative literal) arbitrarily as $x_{i,1}, \ldots, x_{i,m(i)}$.

Then we define the graph G_ϕ as follows: For every variable x_i, G_ϕ contains a cycle of length $2 \cdot m(i)$, consisting of the vertices

$$T_{i,1}, F_{i,1}, T_{i,2}, F_{i,2}, \ldots, T_{i,m(i)}, F_{i,m(i)}$$

in this order. For every clause C_j, G_ϕ contains a triangle consisting of the vertices $V_{j,1}$, $V_{j,2}$, $V_{j,3}$. These subgraphs are connected via the following two sets of edges:

$$\begin{aligned}E_1 = \{\{T_{i,j}, V_{s,t}\} \mid\ & \text{the } j\text{-th occurrence } x_{i,j} \text{ of the variable } x_i \\ & \text{is the positive literal } l_{s,t} \text{ in the clause } C_s\} \quad \text{and} \\ E_2 = \{\{F_{i,j}, V_{s,t}\} \mid\ & \text{the } j\text{-th occurrence } x_{i,j} \text{ of the variable } x_i \\ & \text{is the negative literal } l_{s,t} \text{ in the clause } C_s\}.\end{aligned}$$

Prove that G_ϕ contains a vertex cover of size at most $k_\phi = 5q$ if and only if ϕ is satisfiable.

10 points

Submission: Friday, December 4, by 11:15 at the latest, as a clearly legible PDF via e-mail directly to the respective teaching assistant.