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Exemplary Solutions – Sheet 5

Zürich, October 23, 2020

Solution to Exercise 13

(a) We use the pumping lemma to show that the language

$$L_1 = \{wbw^{\mathsf{R}} \mid w \in \{a,b\}^*\}$$

is not regular. Assume that L_1 is regular. According to the pumping lemma (Lemma 3.14), there exists a constant $n_0 \in \mathbb{N}$ such that every word $w \in \{a,b\}^*$ with $|w| \geq n_0$ can be expressed as $w = yxz$, where

(i) $|yx| \leq n_0$,

(ii) $|x| \geq 1$, and

(iii) either $\{yx^kz \mid k \in \mathbb{N}\} \subseteq L_1$ or $\{yx^kz \mid k \in \mathbb{N}\} \cap L_1 = \emptyset$.

We choose the word $w = a^{n_0}ba^{n_0}$. It clearly holds that $|w| \geq n_0$. Hence, there must exist a decomposition $w = yxz$ of w such that the conditions (i), (ii), and (iii) are satisfied. Due to (i), $|yx| \leq n_0$ holds, hence, $y = a^l$ and $x = a^m$ for $l, m \in \mathbb{N}$ with $l + m \leq n_0$, in particular $m \leq n_0$. Due to (ii), it further holds that $m > 0$. Since $w \in L_1$, according to (iii), it also holds that

$$\{yx^kz \mid k \in \mathbb{N}\} = \{a^{n_0+(k-1)m}ba^{n_0} \mid k \in \mathbb{N}\} \subseteq L_1\,.$$

However, this leads to a contradiction, since $yx^2z = a^{n_0+m}ba^{n_0} \notin L_1$ due to $m > 0$. Hence, the assumption is false and L_1 is not regular.

(b) We use the pumping lemma to show that the language

$$L_2 = \{w \in \{0,1\}^* \mid |w|_0 \neq |w|_1\}$$

is not regular. Assume that L_2 is regular. According to the pumping lemma (Lemma 3.14), there exists a constant $n_0 \in \mathbb{N}$ such that every word $w \in \{0,1\}^*$ with $|w| \geq n_0$ can be expressed as $w = yxz$, where

(i) $|yx| \leq n_0$,

(ii) $|x| \geq 1$, and

(iii) either $\{yx^kz \mid k \in \mathbb{N}\} \subseteq L_2$ or $\{yx^kz \mid k \in \mathbb{N}\} \cap L_2 = \emptyset$.

We choose the word $w = 0^{n_0}1^{n_0}$. It clearly holds that $|w| \geq n_0$. Hence, there must exist a decomposition $w = yxz$ of w such that the conditions (i), (ii), and (iii) are satisfied. Due to (i), $|yx| \leq n_0$ holds, hence, $y = 0^l$ and $x = 0^m$ for $l, m \in \mathbb{N}$ with $l + m \leq n_0$, in particular $m \leq n_0$. Due to (ii), it further holds that $m > 0$. Since $w \notin L_2$, according to (iii), it also holds that

$$\{yx^kz \mid k \in \mathbb{N}\} = \{0^{n_0+(k-1)m}1^{n_0} \mid k \in \mathbb{N}\} \cap L_2 = \emptyset\,.$$

However, this leads to a contradiction, since $yx^2z = 0^{n_0+m}1^{n_0} \in L_2$ due to $m > 0$. Hence, the assumption is false and L_2 is not regular.

Solution to Exercise 14

(a) We use the Kolmogorov complexity argument to show that the language

$$L_1 = \{0^{\binom{2n}{n}} \mid n \in \mathbb{N}\}$$

is not regular. Assume that $L = L_1$ is regular. For every $n \geq 1$, it holds that

$$\Delta_n := \binom{2(n+1)}{n+1} - \binom{2n}{n} = \binom{2n}{n} \cdot \left(\frac{(2n+1)(2n+2)}{(n+1)^2} - 1 \right).$$

For every $m \in \mathbb{N}$, $0^{\Delta_m - 1}$ is thus the first word in the language

$$L_{0^{\binom{2m}{m}+1}} = \{y \mid 0^{\binom{2m}{m}+1}y \in L\}\,.$$

According to Theorem 3.19 from the textbook, there exists a constant c, independent of m, such that

$$K(0^{\Delta_m - 1}) \leq \lceil \log_2(1+1) \rceil + c = 1 + c\,.$$

As there are only finitely many programs of a constant length at most $1 + c$, but infinitely many words of the form $0^{\Delta_m - 1}$, we derive a contradiction. Hence, the assumption is false and L_1 is not regular.

(b) We use the Kolmogorov complexity argument to show that the language

$$L_2 = \{ww \mid w \in \{0,1\}^*\}$$

is not regular. Assume that $L = L_2$ is regular. For every $m \in \mathbb{N}$, 0^m1 is the first word in the language

$$L_{0^m1} = \{y \mid 0^m1y \in L\}.$$

According to Theorem 3.19 from the textbook, there exists a constant c, independent of m, such that

$$K(0^m1) \leq \lceil \log_2(1+1) \rceil + c = 1 + c\,.$$

As there are only finitely many programs of a constant length at most $1 + c$, but infinitely many words of the form 0^m1, we derive a contradiction. Hence, the assumption is false and L_2 is not regular.

Solution to Bonus Exercise 2

(a) We show that the language

$$L = \{w \in \{0,1\}^* \mid |w|_0 \neq |w|_1\},$$

known to be nonregular from Exercise 13 (b), satisfies the assumptions of the weaker pumping lemma with the conditions (i′), (ii), and (iii′). Let $n_0 = 3$. For every word $w \in L$ with $|w| \geq 3$, we have to provide a decomposition $w = yxz$ such that the conditions (i′), (ii), and (iii′) are satisfied. To this end, we distinguish three cases:

1. Let $w = 0^m$ for some $m \geq 3$. Then we choose $y = \lambda$ and $x = 0$. This choice clearly satisfies the conditions (i′) and (ii). We show that (iii′) is satisfied, too: all words of the form yx^kz contain a positive number of zeros, but no one. Hence, $\{yx^kz \mid k \in \mathbb{N}\} \subseteq L$.
2. Let $w = 1^m$ for some $m \geq 3$. This case is completely analogous.
3. Let $|w|_0, |w|_1 \geq 1$, and $|w| \geq 3$. Let $w = a_1a_2 \ldots a_m$ with $a_i \in \{0,1\}$ for all i. Then there exists a position i such that $a_i \neq a_{i+1}$, this means that 01 or 10 is a subword of w. We choose $y = a_1 \ldots a_{i-1}$ and $x = a_ia_{i+1}$. This choice clearly satisfies the conditions (i′) and (ii). (Note that the condition (i) from the original pumping lemma does not need to be satisfied, since we do not know if the subword 01 or 10 occurs among the first n_0 symbols of w.) Insertion or deletion of 01 (or 10, respectively) from or to, respectively, a word with a different number of zeros and ones yields again a word with a different number of zeros and ones. Hence, $\{yx^kz \mid k \in \mathbb{N}\} \subseteq L$.

(b) We consider the language

$$L = \{1\}^+\{0^{n^2} \mid n \in \mathbb{N}\} \cup \{0\}^*.$$

We first show that L is nonregular. Assume that L is regular. Then there exists an automaton $A = (Q, \{0,1\}, \delta, q_0, F)$ such that $L(A) = L$. Let $m = |Q|$. We consider the words

$$10^{0^2+1}, 10^{1^2+1}, 10^{2^2+1}, \ldots, 10^{m^2+1}.$$

Since these are $m+1$ words, i.e., more words than the number of A's states, there exist $i, j \in \{0, \ldots, m\}$ such that $i < j$ and

$$\hat{\delta}(q_0, 10^{i^2+1}) = \hat{\delta}(q_0, 10^{j^2+1}).$$

By Lemma 3.12, for every $z \in \{0,1\}^*$, we have

$$10^{i^2+1}z \in L \iff 10^{j^2+1}z \in L.$$

However, the choice of $z = 0^{2i}$ leads to a contradiction, since

$$10^{i^2+1}z = 10^{i^2+1}0^{2i} = 10^{(i+1)^2} \in L$$

and $10^{j^2+1}z = 10^{j^2+1}0^{2i} \notin L$ hold. The latter holds, because $j^2 + 1 + 2i$ is strictly smaller than $(j+1)^2$ and there exists no perfect square between j^2 and $(j+1)^2$. The assumption is thus false and L is nonregular.

Now we show that the language L satisfies the conditions (i), (ii), and (iii′) of the less weakened pumping lemma. Let $n_0 = 1$. For every word $w \in L$ with $|w| \geq 1$, we have to provide a decomposition $w = yxz$ such that the conditions (i), (ii), and (iii′) are satisfied. To this end, we distinguish two cases:

1. If the word w starts with 1, then it is of the form $w = 1^m 0^{n^2}$ for $m > 0$. We choose $y = \lambda$ and $x = 1$. This choice clearly satisfies the conditions (i) and (ii). If $m = 1$, then $yz \in \{0\}^*$, and thus $yz \in L$. If $m > 1$, then $yz = 1^{m-1}0^{n^2}$ is also in L. For all $k \geq 1$, yx^kz is clearly in L, because only the number of ones changes due to pumping.
2. If the word w starts with 0, then we choose $y = \lambda$ and $x = 0$. The conditions (i) and (ii) are again satisfied and, for all $k \in \mathbb{N}$, it holds that $yx^kz \in \{0\}^*$ and thus $yx^kz \in L$.

Remark: Although being more complicated, it is also possible to find a language L that is nonregular, but satisfies the conditions (i), (ii), and (iii) of the pumping lemma from the lecture.