Multiple Choice 7.1 True or false? Motivate your answer.

Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a *linear* function. Then, without doing any computation, we always know who is its differential $dL(x)$ at every point $x \in \mathbb{R}^m$.

☐ True ☐ False

Solution. True. By linearity we have $L(x+y) - L(x) - L(y) = 0$ for every $x, y \in \mathbb{R}^m$, and hence trivially

$$\lim_{y \to 0} \frac{|L(x+y) - L(x) - L(y)|}{|y|} = 0,$$

which means precisely that $dL(x) \equiv L$ for every $x \in \mathbb{R}^m$.

Multiple Choice 7.2 True or false? Motivate your answers.

Let $M_{n\times n}(\mathbb{R})$ be the space of $n \times n$ matrices which we identify with the Eulclidean space $\mathbb{R}^{n^2}$. The function "determinant" $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$, $A \mapsto \det A$, is

		True	False
(a)	is continuous on $M_{n\times n}(\mathbb{R})$	☐	☐
(b)	continuous only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$	☐	☐
(c)	differentiable on $M_{n\times n}(\mathbb{R})$	☐	☐
(d)	differentiable only on a certain subset $U \subsetneq M_{n\times n}(\mathbb{R})$	☐	☐

Solution. The only correct answers are

		True	False
(a)	is continuous on $M_{n\times n}(\mathbb{R})$	☒	☐
(c)	differentiable on $M_{n\times n}(\mathbb{R})$	☒	☐

Indeed if $A = (a_{ij})_{i,j}$ is a matrix, we may compute with respect to the first row

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(\widehat{A}_{1j}),$$

where $\det(\widehat{A}_{1j})$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained by removing the 1st row and the jth column. From this formula and by induction on n one sees that $\det A$ is just a polynomial in the variables a_{ij}, and so differentiable and *a fortiori* continuous.

Exercise 7.1 Compute first the differential df of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, then compute $df(p_0) \cdot v$ at the given p_0 and v.

(a) $f(x,y) = y, \quad p_0 = (3, \frac{1}{3}), \quad v = (-2,-4)$,

(b) $f(x,y) = (x-2y)^3, \quad p_0 = (6,2), \quad v = (2,1)$,

(c) $f(x,y) = \sin(2x) + \cos(3y), \quad p_0 = (\pi, \frac{\pi}{2}), \quad v = (6,-7)$.

Solution. From the general formula $df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$,

(a) We have $df(x,y) \equiv (0,1)$, so for every $p_0 \in \mathbb{R}^2$ it is

$$df(p_0)\begin{pmatrix}-2\\-4\end{pmatrix} = -4.$$

(b) We have $df(x,y) = (3(x-2y)^2, -6(x-2y)^2)$, thus

$$df(6,2)\begin{pmatrix}2\\1\end{pmatrix} = (12,-24)\begin{pmatrix}2\\1\end{pmatrix} = 0.$$

(c) We have $df(x,y) = (2\cos(2x), -3\sin(3y))$, thus

$$df(\pi, \tfrac{\pi}{2})\begin{pmatrix}6\\-7\end{pmatrix} = (2,3)\begin{pmatrix}6\\-7\end{pmatrix} = 2\cdot 6 - 3\cdot 7 = 12 - 21 = -9.$$

Exercise 7.2

(a) Let

- $x(t) = \cos(\pi t)$,
- $y(t)$ be the primitive of the function $t \mapsto e^{-t^2}$, so that $y(1) = 42$,
- $f(x,y) = x^2 + y^2$.

Compute the derivative of the composite function $t \mapsto f(x(t), y(t))$ at $t = 1$.

(b) Compute the differential $df(\frac{\pi}{2}, \frac{\pi}{3}, 0)$ of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x,y,z) = \int_{\cos x + \sin y}^{z} e^{tz}\, dt.$$

(c) Let $f(x,y) = |xy|$. Find the set of points $(x,y) \in \mathbb{R}^2$, where f is differentiable and compute its differential at those points.

Note for (b): to compute the derivative in the z-direction, one way is to use directly the definition as limit of difference quotient.

Solution. (a) The differential of $f(x,y)$ is $df(x,y) = (2x, 2y)$; the chain rule yields

$$\frac{d}{dt} f\Big(x(t), y(t)\Big) = df\Big(x(t), y(t)\Big) \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = 2x(t)\, x'(t) + 2y(t)\, y'(t).$$

The derivative of $x(t)$ is $x'(t) = -\pi \sin(\pi t)$ and since $y(t)$ is a primitive of $t \mapsto e^{-t^2}$, it follows that $y'(t) = e^{-t^2}$. With $y(1) = 42$ we get

$$\begin{aligned}\frac{d}{dt}\Big(f\Big(x(t), y(t)\Big)\Big)\Big|_{t=1} &= 2x(1)\, x'(1) + 2y(1)\, y'(1) \\ &= -2\pi \cos(\pi) \sin(\pi) + 84 e^{-1} = \frac{84}{e}.\end{aligned}$$

(b) By the fundamental theorem of calculus we get

$$\frac{\partial}{\partial s} \int_s^z e^{tz}\, dt = -\frac{\partial}{\partial s} \int_z^s e^{tz}\, dt = -e^{sz}.$$

With the chain rule it follows that

$$\begin{aligned}\frac{\partial}{\partial x} \int_{\cos x + \sin y}^z e^{tz}\, dt &= -e^{(\cos x + \sin y) z} \frac{\partial}{\partial x}(\cos x + \sin y) = e^{(\cos x + \sin y) z} \sin x, \\ \frac{\partial}{\partial y} \int_{\cos x + \sin y}^z e^{tz}\, dt &= -e^{(\cos x + \sin y) z} \frac{\partial}{\partial y}(\cos x + \sin y) = -e^{(\cos x + \sin y) z} \cos y.\end{aligned}$$

By evaluating in $(x, y, z) = (\frac{\pi}{2}, \frac{\pi}{3}, 0)$ we get

$$\frac{\partial f}{\partial x}\Big(\tfrac{\pi}{2}, \tfrac{\pi}{3}, 0\Big) = \sin(\tfrac{\pi}{2}) = 1, \qquad \frac{\partial f}{\partial y}\Big(\tfrac{\pi}{2}, \tfrac{\pi}{3}, 0\Big) = -\cos(\tfrac{\pi}{3}) = -\tfrac{1}{2}.$$

For the derivative in the z direction we use the definition with the difference quotient. We have $\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2} =: a$, so for $h \neq 0$ we get

$$\begin{aligned}f(\tfrac{\pi}{2}, \tfrac{\pi}{3}, h) &= \int_a^h e^{th}\, dt = \Big[\tfrac{1}{h} e^{th}\Big]_a^h = \tfrac{1}{h}(e^{h^2} - e^{ah}), \\ f(\tfrac{\pi}{2}, \tfrac{\pi}{3}, 0) &= \int_a^0 1\, dt = -a.\end{aligned}$$

Recalling from one variable calculus that $e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$, we get

$$\begin{aligned}\frac{f(\frac{\pi}{2},\frac{\pi}{3},h) - f(\frac{\pi}{2},\frac{\pi}{3},0)}{h} &= \frac{e^{h^2} - e^{ah} + ah}{h^2} \\ &= \frac{(1+h^2) - (1 + ah + \frac{1}{2}a^2h^2) + ah + o(h^2)}{h^2} \\ &= \frac{h^2 - \frac{1}{2}a^2h^2 + o(h^2)}{h^2} \xrightarrow{h\to 0} 1 - \frac{1}{2}a^2 = \frac{5}{8},\end{aligned}$$

This gives that $\frac{\partial f}{\partial z}(\frac{\pi}{2},\frac{\pi}{3},0) = \frac{5}{8}$. All in all, it is

$$df(\tfrac{\pi}{2},\tfrac{\pi}{3},0) = \left(1, -\tfrac{1}{2}, \tfrac{5}{8}\right).$$

(c) We have

$$f(x,y) = \begin{cases} xy, & \text{if } xy > 0, \\ 0, & \text{if } xy = 0, \\ -xy, & \text{if } xy < 0. \end{cases}$$

So we see that in $A = \{(x,y) \in \mathbb{R}^2 \mid xy > 0\}$ the partial derivatives exists and are given by

$$\frac{\partial f}{\partial x}(x,y) = y, \qquad \frac{\partial f}{\partial y}(x,y) = x.$$

Since they are continuous, f is differentiable in A with $df|_A = (y,x)$. In $B = \{(x,y) \in \mathbb{R}^2 \mid xy < 0\}$ we have similarly

$$\frac{\partial f}{\partial x}(x,y) = -y, \qquad \frac{\partial f}{\partial y}(x,y) = -x,$$

so f is differentiable in B with $df|_B = (-y,-x)$.

On $\{(x,0) \in \mathbb{R}^2 \mid x \neq 0\}$ and $\{(0,y) \in \mathbb{R}^2 \mid y \neq 0\} f$ is not differentiable, since

$$\frac{f(x,h) - f(x,0)}{h} = \frac{|xh| - 0}{h} = \frac{|h|}{h}\cdot|x|, \qquad \frac{f(h,y) - f(0,y)}{h} = \frac{|h|}{h}\cdot|y|$$

have no limit for $h \to 0$.

It remains to check differentiability at the origin $(0,0)$. Recalling that $|xy| \leq \frac{1}{2}(x^2+y^2)$ we see that

$$\left|\frac{f(x,y) - f(0,0)}{\|(x,y)\|}\right| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\frac{1}{2}(x^2+y^2)}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2} \xrightarrow{(x,y)\to(0,0)} 0.$$

Thus f is differentiable in 0 with $df(0,0) = (0,0)$.

Exercise 7.3 (Gradient and Level Sets) A *curve in the plane* is a subset $\Gamma \subset \mathbb{R}^2$ so that there is a differentiable function $\gamma : (a,b) \subset \mathbb{R} \to \mathbb{R}^2$ so that $\gamma((a,b)) = \Gamma$ and $\gamma'(t) \neq 0$ for every $t \in (a,b)$. Any such γ is called *parametrization* of Γ.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function so that its *level set* at c

$$\Gamma = f^{-1}(\{c\}) = \{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$$

is a curve in the plane, and let $\gamma : I = (a,b) \to \Gamma$ be a parametrization. Prove that:

(a) The gradient to f is orthogonal to Γ, namely

$$\nabla f(\gamma(t)) \cdot \gamma'(t) = 0 \quad \text{for every } t \in I.$$

(b) The directional derivative of f along Γ vanishes, namely

$$D_{\gamma'(t)} f(\gamma(t)) = 0 \quad \text{for every } t \in I.$$

(c) At a point $(x,y) \in \Gamma$, the direction where f grows the most is orthogonal to Γ (this means: among all the vectors v with $|v| = 1$, $D_v f(x,y)$ assumes the maximum value when v is orthogonal to Γ).

Solution. (a) Since $I \ni t \mapsto f \circ \gamma(t)$ is constant, the chain rule yields that for every $t \in I$ there holds

$$0 = \frac{d}{dt} f \circ \gamma(t) = df(\gamma(t))\gamma'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t).$$

(b) By definition of gradient, we have for every $t \in I$

$$D_{\gamma'(t)} f(\gamma(t)) = df(\gamma(t))\gamma'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) = 0.$$

(c) We need to find the unit vector v so that $D_v f(x,y)$ is largest. Let $t \in I$ be so that $\gamma(t) = (x,y)$. Since $\gamma'(t) \neq 0$, we may choose a unit vector normal to $\gamma'(t)$, namely $\mathsf{n} \in \mathbb{R}^2$ with $\|\mathsf{n}\| = 1$ and $\mathsf{n} \cdot \gamma'(t) = 0$, any other vector $v \in \mathbb{R}^2$ may be written uniquely as

$$v = a\gamma'(t) + b\mathsf{n},$$

for some $a, b \in \mathbb{R}$.

From (b) we have that $\nabla f(\gamma(t)) \cdot \gamma'(t) = 0$ and so

$$\begin{aligned} D_v f(\gamma(t)) &= \nabla f(\gamma(t)) \cdot v \\ &= \nabla f(\gamma(t)) \cdot (a\gamma'(t) + b\mathsf{n}) \\ &= b\Big(\nabla f(\gamma(t)) \cdot \mathsf{n}\Big). \end{aligned}$$

Hence, since $|v| = 1$, the maximum of $D_v f(x, y)$ is achieved when either $b = 1$ if $\nabla f(\gamma(t)) \cdot \mathsf{n} \geq 0$ or $b = -1$ if $\nabla f(\gamma(t)) \cdot \mathsf{n} < 0$. In any case it must necessarily be $a = 0$. This means that v is orthogonal to Γ.

Multiple Choice 11.1 Choose the correct statement. Motivate your answer.

Recall that, for a C^2 function $f : \mathbb{R}^3 \to \mathbb{R}$, its gradient $\nabla f : \mathbb{R}^3 \to \mathbb{R}^3$ can be thought as a vector field. The equation

$$\operatorname{curl}(\nabla f) = g$$

(a) has a solution f for *every* given g ☐

(b) has a solution f *only* for $g = 0$ ☐

(c) has a solution f *also* for some nonzero g's. ☐

Solution. It is

(a) has a solution f for *every* given g ☐

(b) has a solution f *only* for $g = 0$ ☒

(c) has a solution f *also* for some nonzero g's. ☐

Indeed an immediate calculation shows that it is always $\operatorname{curl}(\nabla f) = 0$, hence it must be $g = 0$.

Multiple Choice 11.2 True or false? Motivate your answer.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected, regular region. Generally speaking, the integral $\int_\Omega d\mu$ represents:

		True	False
(a)	the area of Ω	☐	☐
(b)	the length of the curve bounding Ω	☐	☐
(c)	the volume of a cylinder with base Ω and height 1	☐	☐
(d)	the surface area of some cylinder with base Ω and height 1.	☐	☐

Solution. It is

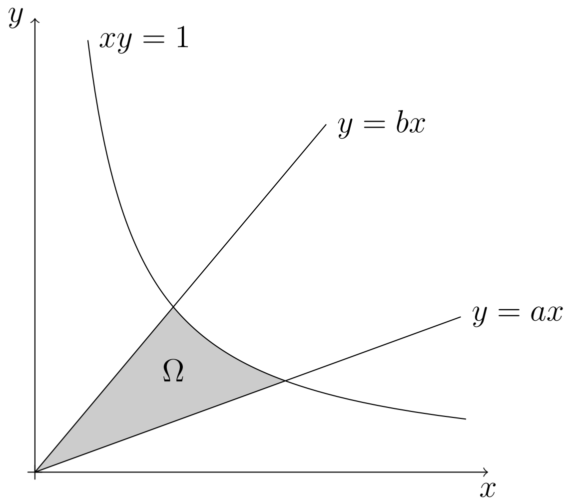
		True	False
(a)	the area of Ω	$\boxtimes$	$\square$
(b)	the length of the curve bounding Ω	$\square$	$\boxtimes$
(c)	the volume of some cylinder with base Ω and height 1	$\boxtimes$	$\square$
(d)	the surface area of some cylinder with base Ω and height 1.	$\square$	$\boxtimes$

Indeed the fact that the integral represents the area is well-known from the lectures. Recalling that the formula for the volume and surface area of a cylinder are, respectively,

$$(\text{Base Area}) \times (\text{height}) \quad \text{and} \quad 2(\text{Base Area}) + (\text{Base profile Length}) \times (\text{height}),$$

we also get that (c) holds. Finally, for instance when Ω is a circle (b) and (d) cannot hold.

Exercise 11.1 Let $b > a > 0$. Compute the integral of $f(x, y) = xy$ over the domain Ω drawn below:

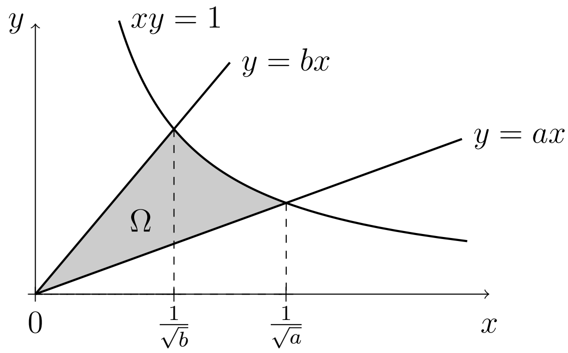


Solution. We determine first the intersection points between the curves:

$$xy = 1 \wedge y = ax \;\Rightarrow\; x = \frac{1}{\sqrt{a}},$$

$$xy = 1 \wedge y = bx \;\Rightarrow\; x = \frac{1}{\sqrt{b}},$$

$$x = bx \wedge x = ax \;\Rightarrow\; x = 0.$$



Consequently, it is

$$\begin{aligned}
\int_\Omega xy\,d\mu &= \int_0^{\frac{1}{\sqrt{b}}}\int_{ax}^{bx} xy\,dy\,dx + \int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\int_{ax}^{\frac{1}{x}} xy\,dy\,dx \\
&= \frac{1}{2}\int_0^{\frac{1}{\sqrt{b}}}\Big(x(bx)^2 - x(ax)^2\Big)\,dx + \frac{1}{2}\int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\Big(x\big(\tfrac{1}{x}\big)^2 - x(ax)^2\Big)\,dx \\
&= \frac{b^2}{2}\int_0^{\frac{1}{\sqrt{b}}} x^3\,dx + \frac{1}{2}\int_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}}\frac{1}{x}\,dx - \frac{a^2}{2}\int_0^{\frac{1}{\sqrt{a}}} x^3\,dx \\
&= \frac{b^2}{2}\left[\frac{x^4}{4}\right]_0^{\frac{1}{\sqrt{b}}} + \frac{1}{2}\Big[\log|x|\Big]_{\frac{1}{\sqrt{b}}}^{\frac{1}{\sqrt{a}}} - \frac{a^2}{2}\left[\frac{x^4}{4}\right]_0^{\frac{1}{\sqrt{a}}} = \frac{1}{2}\log\sqrt{\frac{b}{a}}.
\end{aligned}$$

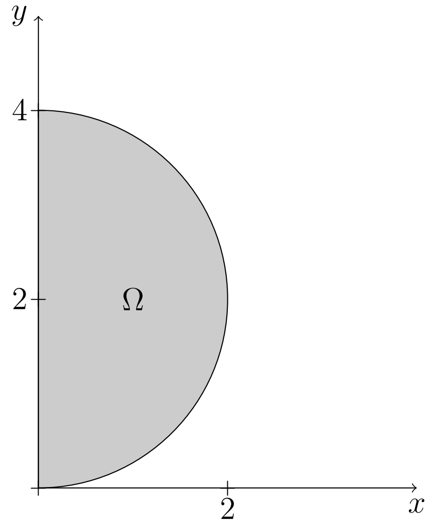
Exercise 11.2 Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function.

(a) Determine and draw (*first* by hand, *then* checking the result with a software of your choice) the domain of integration in $\mathbb{R}^2$ of the following integral:

$$\int_{-1}^{2}\int_{-x}^{2-x^2} f(x,y)\,dy\,dx. \qquad (\textstyle\int)$$

(b) Rewrite $(\int)$ exchanging the order or integration, that is, rewrite the same integral supposing you want to integrate in x first and then in y.

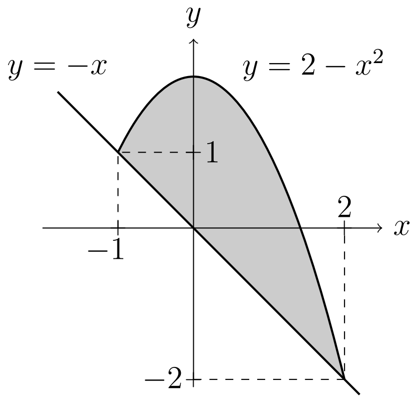
(c) Consider the domain Ω drawn below. Write $\int_\Omega f\,d\mu$ as double integral, in both orders of integration.

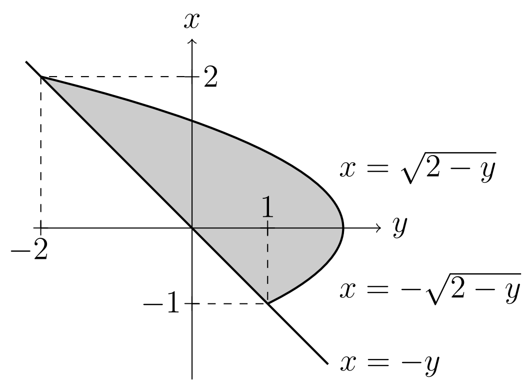


Solution. (a) The domain of integration is the bounded region between the parabola $y = 2 - x^2$ and the line $y = -x$; they intersect in $(-1, 1)$ and $(2, -2)$.

(b) The left branch of the parabola is $x = -\sqrt{2-y}$,and the right one $x = \sqrt{2-y}$. Thus we divide the domain in two and obtain:

$$\int_{-1}^{2}\int_{-x}^{2-x^2} f(x,y)\,dy\,dx = \int_{-2}^{1}\int_{-y}^{\sqrt{2-y}} f(x,y)\,dx\,dy + \int_{1}^{2}\int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)\,dx\,dy.$$



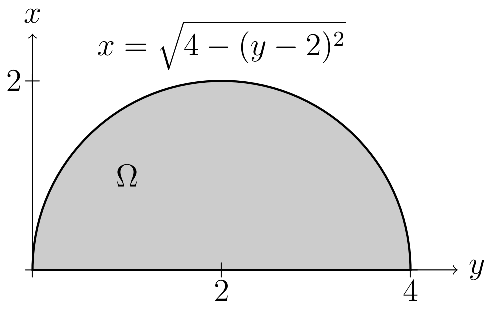


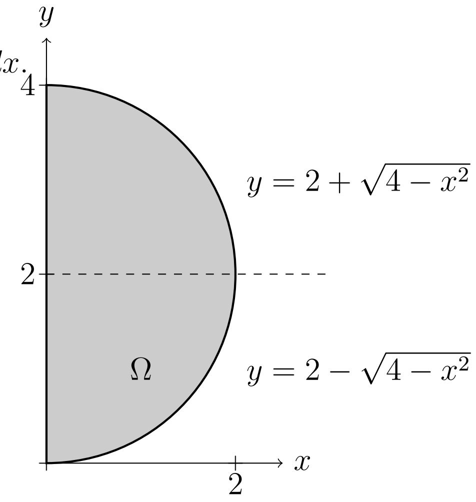
(c) The equation of the circle is $x^2 + (y-2)^2 = 2^2 = 4$, hence the half-cicle has equation $x = \sqrt{4-(y-2)^2}$ and there holds

$$\int_{\Omega} f(x,y)\,d\mu = \int_0^4 \int_0^{\sqrt{4-(y-2)^2}} f(x,y)\,dx\,dy.$$

The lower quarter-of-a-circle has equation $y-2=-\sqrt{4-x^2}$, the uppe one $y-2=\sqrt{4-x^2}$, hence

$$\int_\Omega f(x,y)\,d\mu = \int_0^2 \int_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} f(x,y)\,dy\,dx.$$





Exercise 11.3 Compute the following integrals:

(a) $\displaystyle\int_0^1\int_0^x e^{x+y}\,dy\,dx,$ (b) $\displaystyle\int_0^1\int_{\sqrt{y}}^1 x\cos y\,dx\,dy.$

Now describe and draw the domains of integration, and compute the integrals exchanging order of integration. Is the result the same?

Solution. (a) We have

$$\int_0^1\int_0^x e^{x+y}\,dy\,dx = \int_0^1 e^x\int_0^x e^y\,dy\,dx = \int_0^1 e^x(e^x-1)\,dx = \frac{e^2}{2}-e+\frac{1}{2}.$$

The domain of integration is the triangle the enclosed by the x-axis and the lines $x=1, x=y$. Thus we can exchange the domain of integration as follows:

$$\begin{aligned}\int_0^1\int_y^1 e^{x+y}\,dx\,dy &= \int_0^1 e^y(e-e^y)\,dy = \int_0^1 e^{y+1}-e^{2y}\,dy = [e^{y+1}-\frac{1}{2}e^{2y}]_0^1\\ &= e^2-\frac{e^2}{2}-e+\frac{1}{2} = \frac{e^2}{2}-e+\frac{1}{2}.\end{aligned}$$

The result is the same.

(b) We have

$$\begin{aligned}
\int_0^1 \int_{\sqrt{y}}^1 x \cos y \; dx \, dy &= \int_0^1 \frac{1}{2}(\cos y - y \cos y) \; dy \\
&= \frac{1}{2} \left([\sin y]_0^1 - [y \sin y]_0^1 + \int_0^1 \sin x \; dx \right) \\
&= \frac{1}{2} \left([\sin y]_0^1 - [y \sin y]_0^1 - [\cos x]_0^1 \right) \\
&= \frac{1}{2}(\sin 1 - \sin 1 - \cos 1 + 1) \\
&= \frac{1}{2} - \frac{\cos 1}{2}.
\end{aligned}$$

The domain of integration is the region enclosed by the graphs of $x = \sqrt{y}$ and the line $x = 1$. Exchanging the order of integration yields the region between the x-axis and the parabola $y = x^2$, hence

$$\int_0^1 \int_0^{x^2} x \cos y \; dy \, dx = \int_0^1 x \sin(x^2) \; dx = -\frac{1}{2}[\cos(x^2)]_0^1 = -\frac{1}{2}\cos 1 + \frac{1}{2}.$$

The result is again the same.

Multiple Choice 13.1 True or false? Motivate your answer.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and consider its measure ("area" if $n = 2$, "volume" if $n = 3$) $|\Omega| = \int_\Omega dx_1 \cdots dx_n$. Then

		True	False
(a)	If Ω is unbounded, then the integral is divergent	☐	☐
(b)	If the integral is divergent, then Ω is unbounded	☐	☐
(c)	If there exists $\varepsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ so that $B_\varepsilon(x_j) \subset \Omega$ for every $j \in \mathbb{N}$, then the integral is divergent	☐	☐
(d)	If the integral is divergent, there exists $\varepsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ so that $B_\varepsilon(x_j) \subset \Omega$ for every $j \in \mathbb{N}$	☐	☐

Solution. It is

True False

(a) *If Ω is unbounded, then the integral is divergent* ☐ ☒

Take $\Omega = \{(x,y) : x > 1, 0 < y < \frac{1}{x^2}\} \subseteq \mathbb{R}^2$, i.e. the subgraph of $\frac{1}{x^2}$. From one variable calculus we know that

$$|\Omega| = \int_1^{+\infty} \frac{dx}{x^2} = 1,$$

so the integral is convergent.

(b) *If the integral is divergent, then Ω is unbounded* ☒ ☐

It it equivalent to: Ω bounded $\Rightarrow |\Omega| < \infty$; this is true, since if Ω is bounded and measurable, then $\Omega \subset B_R(0)$ for some sufficiently big $R > 0$ and this implies that $|\Omega| \le |B_R(0)| < \infty$.

(c) *If there exists $\varepsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ so that $B_\varepsilon(x_j) \subset \Omega$ for every $j \in \mathbb{N}$, then the integral is divergent* ☒ ☐

Since $(x_j)_{j\in\mathbb{N}}$ is unbounded, there exists a subsequence $(x_{j'})_{j'}$ so that the balls if $B_\varepsilon(x_{j'})$, are pairwise disjoint; since $\bigcup_{j'} B_\varepsilon(x_{j'}) \subseteq \Omega$, by additivity of the integral this implies $|\Omega| \ge N|B_\varepsilon(0)|$ for every $N \in \mathbb{N}$, and so $|\Omega|$ cannot be finite.

(d) *If the integral is unbounded, then there exists $\varepsilon > 0$ and an unbounded sequence $(x_j)_{j\in\mathbb{N}}$ so that $B_\varepsilon(x_j) \subset \Omega$ for every $j \in \mathbb{N}$* ☐ ☒

Consider for instance $\Omega = \{(x,y) : x > 1, 0 < y < \frac{1}{x}\} \subset \mathbb{R}^2$, i.e. the subgraph of $\frac{1}{x}$. From one variable calculus we know that

$$|\Omega| = \int_1^{+\infty} \frac{dx}{x} = +\infty,$$

so the integral is divergent. But since Ω shrinks at infinity, for no $\varepsilon > 0$ it is possible to find such unbounded sequence.

Multiple Choice 13.2 True or false? Motivate your answer.

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a non-negative, continuous function. Then

		True	False
(a)	If $\lim_{x\to\infty} f(x) = 0$, the improper integral $\int_{\mathbb{R}^n} f\,dx$ exists and is finite	☐	☐
(b)	If the improper integral $\int_{\mathbb{R}^n} f\,dx$ exists and is finite, then $\lim_{x\to\infty} f(x) = 0$	☐	☐
(c)	If $\lim_{x\to\infty} f(x)$ does not exist, then the improper integral $\int_{\mathbb{R}^n} f\,dx$ is not finite	☐	☐
(d)	If $\lim_{x\to\infty} f(x)$ exists and is nonzero, the improper integral $\int_{\mathbb{R}^n} f\,dx$ is not finite.	☐	☐

Solution. It is

Then

		True	False
(a)	*If* $\lim_{x\to\infty} f(x) = 0$*, the improper integral* $\int_{\mathbb{R}^n} f\,dx$ *exists and is finite*	☐	☒

A counterexample already in $n = 1$ is $f(x) = \frac{1}{1+|x|}$.

		True	False
(b)	*If the improper integral* $\int_{\mathbb{R}^n} f\,dx$ *exists and is finite, then* $\lim_{x\to\infty} f(x) = 0$	☐	☒

Consider for instance a infinite-sawtooth-like function $f : \mathbb{R} \to [0, 1]$ consisting of isosceles triangles in $[j, j + 2^{-j}]$, $j \in \mathbb{N}$ each of height 1 and 0 otherwise (see picture below): it is an elementary matter to see that $\int_{\mathbb{R}} f\,dx$ is finite but $\lim_{x\to\infty} f(x)$ does not exist.

		True	False
(c)	*If* $\lim_{x\to\infty} f(x)$ *does not exist, then the improper integral* $\int_{\mathbb{R}^n} f\,dx$ *is not finite*	☐	☒

It is the same as (b).

		True	False
(d)	*If* $\lim_{x\to\infty} f(x)$ *exists and is nonzero, the improper integral* $\int_{\mathbb{R}^n} f\,dx$ *is not finite.*	☒	☐

If $\lim_{x\to\infty} f(x) = c > 0$, then there exists $R > 0$ so that $f(x) > c/2$ if $|x| > R$, thus $\int_{\mathbb{R}^n} f\,dx \geq \int_{\mathbb{R}^n \setminus B_R} \frac{c}{2}\,dx = +\infty$.

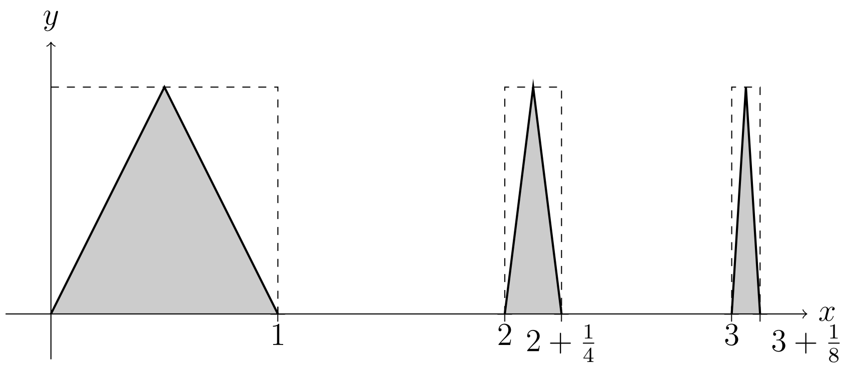


Figure 1: Sawtooth-like function

Exercise 13.1 Compute the following integrals over the specified domains:

(a) $\displaystyle\int_D \sqrt{x^2+y^2}\,dx\,dy, \qquad D=\{(x,y): x^2+y^2-2x\le 0\},$

(b) $\displaystyle\int_{B_R^+(0)} z\,dx\,dy\,dz, \qquad B_R^+(0)=\{(x,y,z): x^2+y^2+z^2\le R^2,\ z\ge 0\},$

(c) $\displaystyle\int_{D_2} xy\,dx\,dy, \qquad D_2=\{(x,y): 0\le x\le y\le 1\},$

(d) $\displaystyle\int_{D_3} xyz\,dx\,dy\,dz, \qquad D_3=\{(x,y,z): 0\le x\le y\le z\le 1\},$

Solution. (a) Since

$$x^2+y^2-2x=(x-1)^2+y^2-1,$$

D is the disk with center $(1,0)$ and radius 1. In polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ it is

$$(x-1)^2+y^2-1=r^2-2r\cos\theta\le 0,$$

and hence D is expressed in polar coordinates by

$$D=\{(r\cos\theta,r\sin\theta): -\pi/2\le\theta\le\pi/2,\ 0\le r\le 2\cos\theta\}.$$

Therefore, by the change of variables formula, we have

$$\int_D\sqrt{x^2+y^2}dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2drd\theta=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta)^3d\theta.$$

To compute this integral, with the help of the trigonometric identity

$$(\cos\theta)^3 = \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos\theta,$$

one concludes

$$\int_D \sqrt{x^2+y^2}\,dx\,dy = \frac{32}{9}.$$

(b) Using spherical coordinates $x = r\cos\theta\sin\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\varphi$, the integration domain in spherical coordinates reads

$$B_R^+(0) = \{(r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) : 0 \le r \le R,\ -\pi \le \theta \le \pi,\ 0 \le \varphi \le \pi/2\}.$$

Thus changing variables in the integral and using $\sin(2\varphi) = 2\cos\varphi\sin\varphi$, we get

$$\begin{aligned}
\int_{B_3^+(0,R)} z\,dxdydz &= \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}}\int_0^R r^3\cos(\varphi)\sin(\varphi)drd\varphi d\theta \\
&= 2\pi\left(\int_0^R r^3dr\right)\left(\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2\varphi)d\varphi\right) \\
&= 2\pi\left[\frac{r^4}{4}\right]_0^R\left[-\frac{1}{4}\cos(2\varphi)\right]_0^{\frac{\pi}{2}} \\
&= 2\pi\cdot\frac{R^4}{4}\cdot\frac{1}{2} = \frac{\pi R^4}{4}.
\end{aligned}$$

(c) We may write the integral as iterated integral and compute:

$$\int_{D_2} xy\,dx\,dy = \int_0^1 y\left(\int_0^y x\,dx\right)dy = \int_0^1 y\frac{y^2}{2}\,dy = \frac{1}{2}\left[\frac{y^4}{4}\right]_0^1 = \frac{1}{8}.$$

(d) If for every fixed $z \in [0,1]$ we set

$$D(z) = \{(x,y) : 0 \le x \le y \le z\},$$

then the integral can be written as

$$\int_{D_3} xyz\,dx\,dy\,dz = \int_0^1 z\left(\int_{D(z)} xy\,dx\,dy\right)dz$$

to compute $\int_{D(z)} xy\,dx\,dy$,

we note that it is $D(z) = zD_2$, where D_2 is as in (c). So with a change of variables $x = zu$ and $y = zv$ ($\Longrightarrow dx\,dy = z^2 du\,dv$) we obtain, with (c)

$$\int_{D(z)} xy\,dx\,dy = \int_{D_2} (zu)(zv)z^2\,du\,dv = z^4 \int_{D_2} uv\,du\,dv = \frac{1}{8}z^4,$$

and in turn

$$\int_{D_3} xyz\,dx\,dy\,dz = \frac{1}{8}\int_0^1 z^5\,dz = \frac{1}{48}.$$

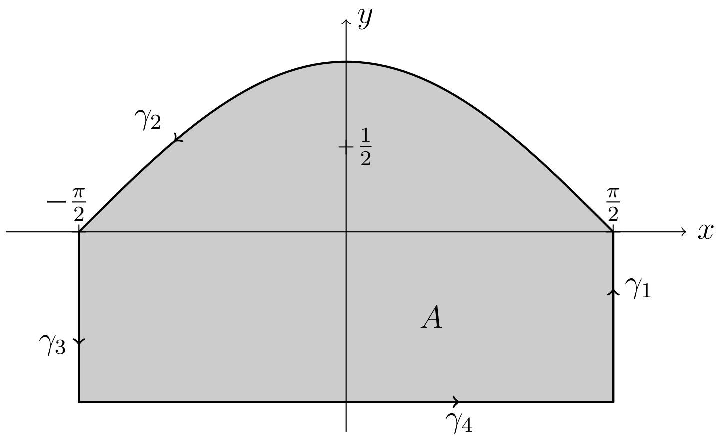
Exercise 13.2 Compute the following line integrals, draw the corresponding domain and compute them first directly, then using Green's theorem (the curves are always oriented counter-clockwise).

(a) $\displaystyle\int_{\partial A} (x^2, y^2) \cdot d\vec{s}, \qquad A = \Big\{(x, y) \in \mathbb{R}^2 \mid -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2},\ -1 \le y \le \cos x\Big\}$

(b) $\displaystyle\int_{\partial B} (xy, e^x) \cdot d\vec{s}, \qquad B = \Big\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ -1 \le y \le 1 - x\Big\}$

Solution. (a) Parametrizations for each side of A are as follows:

$$\begin{aligned}
\gamma_1 &\colon [-1, 0] \to \mathbb{R}^2, & \gamma_1(t) &= (\tfrac{\pi}{2}, t), & \dot\gamma_1(t) &= (0, 1),\\
\gamma_2 &\colon [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to \mathbb{R}^2, & \gamma_2(t) &= (-t, \cos t), & \dot\gamma_2(t) &= (-1, -\sin t),\\
\gamma_3 &\colon [0, 1] \to \mathbb{R}^2, & \gamma_3(t) &= (-\tfrac{\pi}{2}, -t), & \dot\gamma_3(t) &= (0, -1),\\
\gamma_4 &\colon [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to \mathbb{R}^2, & \gamma_4(t) &= (t, -1). & \dot\gamma_4(t) &= (1, 0),
\end{aligned}$$



Thus

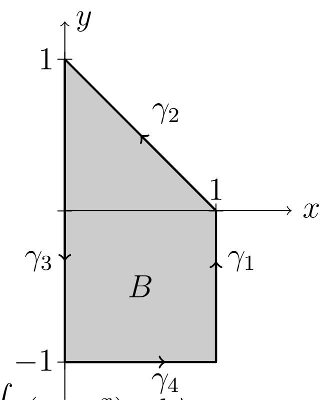
$$\begin{aligned}
&\int_{\partial A}(x^2,y^2)\cdot d\vec{s} \\
&= \int_{\gamma_1}(x^2,y^2)\cdot d\vec{s} + \int_{\gamma_2}(x^2,y^2)\cdot d\vec{s} + \int_{\gamma_3}(x^2,y^2)\cdot d\vec{s} + \int_{\gamma_4}(x^2,y^2)\cdot d\vec{s} \\
&= \int_{-1}^{0} t^2\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2(-1) + (\cos^2 t)(-\sin t)\,dt + \int_0^1 t^2(-1)\,dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\,dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 t)(-\sin t)\,dt = \Big[\tfrac{1}{3}\cos^3 t\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0.
\end{aligned}$$

With Green's theorem we get likewise

$$\int_{\partial A}(x^2,y^2)\cdot d\vec{s} = \int_A\Big(\frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial y}x^2\Big)\,d\mu = \int_A\Big(0-0\Big)\,d\mu = 0.$$

(b) Parametrizations for each side of ∂B are as follows:

$$\begin{aligned}
\gamma_1\colon [-1,0] &\mapsto \mathbb{R}^2, & \gamma_1(t) &= (1,t), & \dot\gamma_1(t) &= (0,1), \\
\gamma_2\colon [0,1] &\mapsto \mathbb{R}^2, & \gamma_2(t) &= (1-t,t), & \dot\gamma_2(t) &= (-1,1), \\
\gamma_3\colon [-1,1] &\mapsto \mathbb{R}^2, & \gamma_3(t) &= (0,-t), & \dot\gamma_3(t) &= (0,-1), \\
\gamma_4\colon [0,1] &\mapsto \mathbb{R}^2, & \gamma_4(t) &= (t,-1), & \dot\gamma_4(t) &= (1,0).
\end{aligned}$$



Thus

$$\begin{aligned}
&\int_{\partial B}(xy,\mathrm{e}^x)\cdot d\vec{s} \\
&= \int_{\gamma_1}(xy,\mathrm{e}^x)\cdot d\vec{s} + \int_{\gamma_2}(xy,\mathrm{e}^x)\cdot d\vec{s} + \int_{\gamma_3}(xy,\mathrm{e}^x)\cdot d\vec{s} + \int_{\gamma_4}(xy,\mathrm{e}^x)\cdot d\vec{s} \\
&= \int_{-1}^{0} e^1\,dt + \int_0^1 (1-t)t(-1) + e^{1-t}\,dt + \int_{-1}^{1} e^0(-1)\,dt + \int_0^1 -t\,dt \\
&= e - (\tfrac{1}{2}-\tfrac{1}{3}) - (1-e^1) - 2 - \tfrac{1}{2} = 2e - \tfrac{11}{3}.
\end{aligned}$$

With Green's theorem we get likewise

$$\begin{aligned}
&\int_{\partial B}(xy,\mathrm{e}^x)\cdot d\vec{s} \\
&= \int_B\Big(\frac{\partial}{\partial x}e^x - \frac{\partial}{\partial y}(xy)\Big)\,d\mu = \int_B e^x - x\,d\mu = \int_0^1\int_{-1}^{1-x} e^x - x\,dy\,dx \\
&= \int_0^1 (2-x)(e^x-x)\,dx = \Big[(2-x)(e^x-\tfrac{1}{2}x^2)\Big]_0^1 + \int_0^1 (e^x - \tfrac{1}{2}x^2)\,dx \\
&= (e-\tfrac{1}{2}) - 2 + (e-1) - \tfrac{1}{6} = 2e - \tfrac{11}{3}.
\end{aligned}$$

Exercise 13.3

(a) For $0 < a < b$ determine whether the improper integral

$$\int_{E(a,b)} x^2 y \mathrm{e}^{-(xy)^2}\, dx\, dy \quad \text{where } E(a,b) = [a,b] \times [0,+\infty)$$

is convergent, and if so compute it.

(b) Use (a) to determine whether the improper integral

$$\int_{E} x^2 y \mathrm{e}^{-(xy)^2}\, dx\, dy \quad \text{where } E = [0,+\infty) \times [0,+\infty)$$

is convergent (no need to compute it).

Solution. (a) For every $x \neq 0$, from one variable calculus we have

$$\int_0^{+\infty} y \mathrm{e}^{-x^2y^2}\, dy = -\frac{1}{2x^2}\Big[\mathrm{e}^{-x^2y^2}\Big]_{y=0}^{y=+\infty} = \frac{1}{2x^2},$$

and

$$\int_a^b x^2 \Big\{ \int_0^{+\infty} y\mathrm{e}^{-x^2y^2}\, dy \Big\} dx = \frac{1}{2}\int_a^b dx = \frac{1}{2}(b-a),$$

which is finite. Since the integrand is positive, this implies that the improper integral is convergent with value $\frac{1}{2}(b-a)$.

(b) Since the integrand is positive, if $\int_E x^2 y \mathrm{e}^{-(xy)^2}\, dx\, dy$ were convergent it would be, for every $a < b$,

$$\int_E x^2 y \mathrm{e}^{-(xy)^2}\, dx\, dy \geq \int_{E(a,b)} x^2 y \mathrm{e}^{-(xy)^2}\, dx\, dy = \frac{1}{2}(b-a),$$

but the right hand side is divergent as $a \to 0$ and $b \to +\infty$. So the integral is divergent.

Multiple Choice 4.1 True or false? Motivate your answer.

Consider the Cauchy problem

$$\begin{cases} y' = \sqrt{|y|} & \text{for } t > 0, \\ y(0) = 0, \end{cases}$$

You notice that that $y \equiv 0$ solves the problem. Consequently, without any further computation, you can say that there exists a sufficiently small half-interval $I = [0, \varepsilon)$, $\varepsilon > 0$, where 0 is the only solution.

☐ True ☐ False

Solution. False. Since the equation is not linear, our existence and uniqueness results does not apply, and in fact another solution can be found by separating the variables. Suppose indeed that $y \not\equiv 0$, we see that

$$\frac{dy}{dx} = \sqrt{|y|} \quad \Rightarrow \quad \int \frac{dy}{\sqrt{|y|}} = \int dx \quad \Rightarrow \quad |y| = \left(\frac{x+c}{2}\right)^2,$$

where $c \in \mathbb{R}$. Imposing the boundary condition $y(0) = 0$ gives $c = 0$ and in fact $y(x) = \frac{x^2}{4}$ is another solution of the Cauchy problem.

Multiple Choice 4.2 Choose the correct statement. Motivate your answer.

Consider for $n \geq 2$ a function

$$f : (a, b) \to \mathbb{R}^n, \qquad f(t) = (f_1(t), \dots, f_n(t)).$$

In order for f to be differentiable:

(a) it is *necessary*, but in general not sufficient, that each f_j is differentiable. ☐

(b) it is *sufficient*, but in general not necessary, that each f_j is differentiable. ☐

(c) it is *necessary and sufficient* that each f_j's is differentiable. ☐

Solution. The correct choice is

(c) it is *necessary and sufficient* that each f_j's is differentiable. ☒

Indeed, differentiability of f at $t_0 \in (a, b)$ means that

$$f'(t_0) = \lim_{t \to 0} \frac{f(t_0 + t) - f(t_0)}{t}$$

exists and is finite. Since

$$\begin{aligned}
&\lim_{t\to 0}\frac{f(t_0+t)-f(t_0)}{t}\\
&=\lim_{t\to 0}\left(\frac{f_1(t_0+t)-f_1(t_0)}{t},\dots,\frac{f_n(t_0+t)-f_n(t_0)}{t}\right)\\
&=\left(\lim_{t\to 0}\frac{f_1(t_0+t)-f_1(t_0)}{t},\dots,\lim_{t\to 0}\frac{f_n(t_0+t)-f_n(t_0)}{t}\right),
\end{aligned}$$

it follows that f is differentiable at t_0 if and only if each of the f_j's is.

Exercise 4.1 Find the general solution of the ODE:

$$y^{(4)}+2y''+y=f(x),$$

when

(a) $f(x)=\sin x$,

(b) $f(x)=\mathrm{e}^{2x}$,

(c) $f(x)=\sin x+\mathrm{e}^{2x}$.

Solution. The characteristic polynomial of the homogeneous ODE is $\lambda^4+2\lambda^2+1=(\lambda^2+1)^2$. It roots are $\pm i$, each with multiplicity 2. So the general solution of the homogeneous ODE is

$$y_h(x)=A\sin x+B\cos x+Cx\sin x+Dx\cos x,\quad A,B,C,D\in\mathbb{R}.$$

We now need to find a particular solution for the specific f's.

(a) We make the Ansatz

$$y_p(x)=ax^2\sin x+bx^2\cos x,$$

with the coefficients a and b to be determined. We see that

$$\begin{aligned}
y_p''(x)&=-ax^2\sin x-bx^2\cos x+4ax\cos x-4bx\sin x+2a\sin x+2b\cos x,\\
y_p^{(4)}(x)&=ax^2\sin x+bx^2\cos x-8ax\cos x-8bx\sin x-12a\sin x-12b\cos x,
\end{aligned}$$

and it must be

$$y_p^{(4)}+2y_p''+y_p=(-8a)\sin x+(-8b)\cos x\overset{!}{=}\sin x,$$

so $a = -\frac{1}{8}$, $b = 0$. The general solution is then

$$\begin{aligned} y_a(x) &= y_h(x) + y_p(x) \\ &= A \sin x + B \cos x + Cx \sin x + Dx \cos x - \frac{x^2}{8} \sin x, \end{aligned}$$

for arbitrary $A, B, C, D \in \mathbb{R}$.

(b) We make the Ansatz:

$$y_p(x) = a\mathrm{e}^{2x}, \quad a \in \mathbb{R}.$$

Substituting y_p in the equation gives

$$y_p^{(4)} + 2y_p'' + y_p = 25a\mathrm{e}^{2x} \stackrel{!}{=} \mathrm{e}^{2x}$$

so it must be $a = \frac{1}{25}$. The general solution is then:

$$\begin{aligned} y_a(x) &= y_h(x) + y_p(x) \\ &= A \sin x + B \cos x + Cx \sin x + Dx \cos x + \frac{1}{25}\mathrm{e}^{2x}, \end{aligned}$$

for $A, B, C, D \in \mathbb{R}$.

(c) Since the equation is linear, from (a), (b) and the superposition principle we deduce immediately that the general solution is

$$\begin{aligned} y_a(x) &= y_{(\mathbf{a})}(x) + y_{(\mathbf{b})}(x) \\ &= A \sin x + B \cos x + Cx \sin x + Dx \cos x - \frac{x^2}{8} \sin x + \frac{1}{25}\mathrm{e}^{2x}, \end{aligned}$$

for arbitrary $A, B, C, D \in \mathbb{R}$.

Exercise 4.2 Solve the following ODE with the method of the variation of constants:

$$y'' + 4y = \frac{1}{\sin(2x)}.$$

Solution. (a) The characteristic polynomial of the homogeneous problem is

$$z^2 + 4 = (z + 2i)(z - 2i),$$

so the general solution of the homogeneous part is

$$y_h(x) = \lambda_1 \cos(2x) + \lambda_2 \sin(2x)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$. To find a particular solution, we then suppose

$$y_p(x) \stackrel{!}{=} \lambda_1(x)\cos(2x) + \lambda_2(x)\sin(2x)$$

where now λ_1, λ_2 are two functions to be determined. Inserting y_p in the equation gives

$$\begin{cases} \lambda_1'(x)\cos(2x) + \lambda_2'(x)\sin(2x) = 0, \\ -2\lambda_1'(x)\sin(2x) + 2\lambda_2'(x)\cos(2x) = \dfrac{1}{\sin(2x)}. \end{cases} \tag{1}$$

For all $x \in \mathbb{R}$, the matrix

$$R(x) = \begin{pmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{pmatrix}$$

is invertible ($\det R(x) = 2\cos^2(2x) + 2\sin^2(2x) = 2 \neq 0$) and

$$R(x)^{-1} = \frac{1}{2}\begin{pmatrix} 2\cos(2x) & -\sin(2x) \\ 2\sin(2x) & \cos(2x) \end{pmatrix} = \begin{pmatrix} \cos(2x) & -\frac{1}{2}\sin(2x) \\ \sin(2x) & \frac{1}{2}\cos(2x) \end{pmatrix}. \tag{2}$$

Then (1) and (2) give

$$\begin{cases} \lambda_1'(x) = -\dfrac{1}{2} \\ \lambda_2'(x) = \dfrac{\cos(2x)}{2\sin(2x)}. \end{cases} \implies \begin{cases} \lambda_1(x) = -\dfrac{x}{2}, \\ \lambda_2(x) = \dfrac{1}{4}\log|\sin(2x)|. \end{cases}$$

We conclude that the general solution is

$$y(x) = \lambda_1\cos(2x) + \lambda_2\sin(2x) - \frac{x}{2}\cos(2x) + \frac{1}{4}\log|\sin(2x)|\sin(2x),$$

for $\lambda_1, \lambda_2 \in \mathbb{R}$.

Exercise 4.3 Solve the following ODE/Cauchy problems. If you apply the method of separation of variables, be careful not to divide by zero!

(a) $y' - y = \sin x$,

(b) $\begin{cases} y' = (x+y)^2, & \text{for } x \in (-\pi/2, \pi/2), \\ y(0) = 1. \end{cases}$

(c) $\begin{cases} y' = \sqrt{\dfrac{1-y^2}{1-x^2}}, & \text{for } x \in (-\pi/2, \pi/2), \\ y(0) = 0. \end{cases}$

(d) $yy' - (1+y)x^2 = 0$,

Note for (d): you will not be able to write explicitly every solution (this often happens when dealing with nonconstant coefficient ODEs). Is suffices that you find an implicit relation for y that does not involve its derivatives.

Solution. (a) We multiply both hand-sides by $\rho(x) = \mathrm{e}^{-x}$ and deduce that

$$\begin{aligned}
&\mathrm{e}^{-x} y'(x) - \mathrm{e}^{-x} y = \mathrm{e}^{-x} \sin x \\
&\Rightarrow \frac{d}{dx}\left(y(x)\mathrm{e}^{-x}\right) = \mathrm{e}^{-x} \sin x \\
&\Rightarrow y(x) = \mathrm{e}^{x} \int \mathrm{e}^{-x} \sin x \, dx + C\mathrm{e}^{x} \quad C \in \mathbb{R}.
\end{aligned}$$

Through integration by parts the integral is seen to be

$$\int \mathrm{e}^{-x} \sin x \, dx = -\frac{1}{2}\mathrm{e}^{-x}(\sin x + \cos x),$$

thus we deduce that the solutions are

$$y(x) = -\frac{1}{2}(\sin x + \cos x) + C\mathrm{e}^{x}, \quad C \in \mathbb{R}.$$

(b) With the substitution $z = x + y$, i.e. $z' = 1 + y'$. the ODE becomes $z' = z^2 + 1$. By means of separation of variables, we get

$$\frac{dz}{dx} = z^2 + 1 \Rightarrow \int \frac{dz}{z^2+1} = \int dx \Rightarrow \arctan z = x + C, \quad C \in \mathbb{R}.$$

Thus the general solution of the ODE reads: $z(x) = \tan(x + C)$, $C \in \mathbb{R}$, that is

$$y(x) = \tan(x + C) - x, \quad C \in \mathbb{R},$$

and since it must be $\tan C = 1$, the solution of the problem is

$$y(x) = \tan\left(x + \frac{\pi}{4}\right) - x.$$

(c) We notice that $y \equiv 1$ and $y \equiv -1$ are constant solutions of the ODE and they do not solve the problem. In the other cases, we may separate the variables:

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}} \quad \Longrightarrow \quad \int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}},$$

which means that $\arcsin y = \arcsin x + C$ for $C \in \mathbb{R}$. Consequently, we deduce that the nonconstant solutions of the ODE are

$$y(x) = \sin\left(\arcsin x + C\right), C \in \mathbb{R},$$

(or equivalently $y(x) = x \cos C + (\sqrt{1-x^2}) \sin C$). Thus, the solution of the problem is

$$y(x) = x.$$

(d) We see that $y \equiv -1$ is a solution of the ODE. We now look for nonconstant solutions. By separating the variables we see that

$$\begin{aligned}
& y \, \frac{dy}{dx} = (1+y)x^2 \\
\Rightarrow & \int \frac{y}{1+y} \, dy = \int x^2 \, dx \\
\Rightarrow & \int \left(1 - \frac{1}{1+y}\right) dy = \int x^2 \, dx,
\end{aligned}$$

so we deduce the following implicit relation for the nonconstant solutions:

$$y - \log|1+y| = \frac{x^3}{3} + C, \quad C \in \mathbb{R}.$$

Multiple Choice 9.1 True or false? Motivate your answer.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice differentiable function with a critical point p_0, whose Hessian matrix at p_0 is

$$\mathrm{Hess}_f(p_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then:

		True	False
(a)	p_0 *cannot* be a local maximum	☐	☐
(b)	p_0 *cannot* be a local minimum	☐	☐
(c)	p_0 *cannot* be a saddle point	☐	☐
(d)	none of the above.	☐	☐

Solution. It is

		True	False
(a)	p_0 *cannot* be a local maximum	☒	☐
(b)	p_0 *cannot* be a local minimum	☐	☒
(c)	p_0 *cannot* be a saddle point	☐	☒
(d)	none of the above	☐	☒

Indeed, denoting $p_0 = (x_0, y_0, z_0)$, the restriction $\varphi(y, z) = f(x_0, y, z)$ has (y_0, z_0) as critical point and $\mathrm{Hess}_\varphi(y_0, z_0)$ is positive definite, whence φ has a local minimum at (y_0, z_0). Therefore p_0 cannot be a local maximum for f.

But otherwise it is a simple matter to cook up examples where p_0 is either a local minimum or a saddle point: $f(x, y, z) = x^4 + \frac{1}{2}y^2 + z^2$ has in $p_0 = 0$ its absolute minimum (since $f > 0$ in $\mathbb{R}^3 \setminus \{0\}$), while $g(x, y, z) = x^3 + \frac{1}{2}y^2 + z^2$ has in $p_0 = 0$ a saddle point (since, if it where a local minimum, 0 would be a local minimum for the 1-variable function $g(x, 0, 0) = x^3$, but this latter is a saddle point instead).

Multiple Choice 9.2 True or false? Motivate your answers.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function and consider its restriction over the square $Q = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Then:

		True	False
(a)	If f has a local max/min/saddle at x_0 in Q, then $df(x_0) = 0$	☐	☐
(b)	Let $x_0 \in Q$ be a point such that $df(x_0) = 0$, then f has a local max/min/saddle at x_0.	☐	☐

Solution. It is

		True	False
(a)	If f has a local max/min/saddle at $x_0 \in Q$, then $df(x_0) = 0$	☐	☒
(b)	Let $x_0 \in Q$ be a point such that $df(x_0) = 0$, then f has a local max/min/saddle at x_0.	☒	☐

Consider for instance $f(x, y) = x$: clearly $f \leq 1$ and so f has a maximum at each point in $(1, y)$; however $df(1, y) = (1, 0) \neq 0$, so (a) is false.

However, if $df(x_0) = 0$, this means that f has a min/max/saddle at x_0 in $\mathbb{R}^2$, and thus so it is for the restriction of f to Q.

Exercise 9.1 For each of the following functions, determine their critical points and find those for which the 2nd derivative test applies, determining in such case whether they are local maxima, local minima or saddle points.

(a) $f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x^3 + y^3 - 3xy,$

(b) $f: \mathbb{R}^3 \to \mathbb{R}, \quad f(x, y, z) = (x^3 - 3x - y^2)z^2 + z^3,$

(c) $f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = xy^2 - \cos(x).$

Solution. (a) The differential of f is

$$df(x, y) = \Big(\tfrac{\partial f}{\partial x}(x, y), \tfrac{\partial f}{\partial y}(x, y)\Big) = \Big(3x^2 - 3y, 3y^2 - 3x\Big),$$

so its critical points are the solution to the system

$$\begin{cases} x^2 = y, \\ y^2 = x. \end{cases}$$

This means that

$$y = x^2 = \left(y^2\right)^2 = y^4 \;\Leftrightarrow\; (y^3 - 1)y = 0 \;\Leftrightarrow\; y \in \{1, 0\}.$$

For $y = 1$ it follows $x = 1$, For $y = 0$ it follows $x = 0$. So the critical points are $(1, 1)$ and $(0, 0)$. The Hessian matrix is

$$\operatorname{Hess}_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y \partial y} \end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}.$$

At the critical points we have

$$\operatorname{Hess}_f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}, \qquad \operatorname{Hess}_f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}.$$

Since $\det(\operatorname{Hess}_f(1,1)) = 27 > 0$ and $\frac{\partial^2 f}{\partial x \partial x}(1,1) = 6 > 0$, $\operatorname{Hess}_f(1,1)$ is positive definite and thus $(1, 1)$ is a local minimum. Since $\det(\operatorname{Hess}_f(0,0)) = -9 < 0$, $\operatorname{Hess}_f(0,0)$ is indefinite and so $(0, 0)$ is a saddle point.

(b) The differential of f is

$$df(x,y,z) = \left(3(x^2-1)z^2,\ -2yz^2,\ 2z(x^3-3x-y^2)+3z^2\right)$$

so the critical points are the solutions to

$$\begin{cases} 3(x^2-1)z^2 = 0, \\ -2yz^2 = 0, \\ 2z(x^3-3x-y^2)+3z^2 = 0. \end{cases}$$

Clearly, every point with $z = 0$ is a solution to the system. When $z \neq 0$, we have the system

$$\begin{cases} (x^2-1) = 0, \\ y = 0, \\ 2(x^3-3x-y^2)+3z = 0. \end{cases}$$

whose solutions are then $(1, 0, \frac{4}{3})$ and $(-1, 0, -\frac{4}{3})$. To sum up the critical points are

$$(x, y, 0) \text{ for every } x, y \in \mathbb{R}, \quad (1, 0, 4/3), \quad (-1, 0, -4/3).$$

We need to determine their type. The Hessian of f is

$$\operatorname{Hess}_f(x,y,z) = \begin{pmatrix} 6xz^2 & 0 & 6z(x^2-1) \\ 0 & -2z^2 & -4yz \\ 6z(x^2-1) & -4yz & 2(x^3-3x-y^2)+6z \end{pmatrix},$$

hence

$$\mathrm{Hess}_f(1,0,4/3) = \begin{pmatrix} 6(4/3)^2 & 0 & 0 \\ 0 & -2(4/3)^2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

which is clearly indefinite, so $(1,0,4/3)$ is a saddle point; then

$$\mathrm{Hess}_f(-1,0,-4/3) = \begin{pmatrix} -6(4/3)^2 & 0 & 0 \\ 0 & -2(4/3)^2 & 0 \\ 0 & 0 & -4 \end{pmatrix},$$

which is clearly negative definite, so $(-1,0,-4/3)$ is a local maximum; finally

$$\mathrm{Hess}_f(x,y,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2(x^3-3x-y^2) \end{pmatrix},$$

is semidefinite, so the 2nd derivative test does not apply in this case.

(c) The differential of f is

$$df(x,y) = \left(\tfrac{\partial f}{\partial x}(x,y), \tfrac{\partial f}{\partial y}(x,y)\right) = \left(y^2+\sin(x), 2xy\right).$$

so the critical points are the solutions to the system

$$\begin{cases} y^2+\sin(x) = 0, \\ 2xy = 0. \end{cases}$$

From the 2nd equation it follows that $x=0$ or $y=0$. If $x=0$, the 1st equation yields $y=0$. If $y=0$ the 1st equation yields $x=k\pi$ with $k\in\mathbb{Z}$. The set of critical points is then $\{(k\pi,0) \mid k\in\mathbb{Z}\}$. The Hessian is

$$\mathrm{Hess}_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x\partial x} & \frac{\partial^2 f}{\partial y\partial x} \\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y\partial y} \end{pmatrix} = \begin{pmatrix} \cos(x) & 2y \\ 2y & 2x \end{pmatrix}.$$

At $(k\pi,0)\in\mathbb{R}^2$ with $k\in\mathbb{Z}$ we get

$$\mathrm{Hess}_f(k\pi,0) = \begin{pmatrix} \cos(k\pi) & 0 \\ 0 & 2k\pi \end{pmatrix} = \begin{pmatrix} (-1)^k & 0 \\ 0 & 2k\pi \end{pmatrix},$$

which is diagonal, and hence we deduce that

$k>0$	even	$\Rightarrow$ both EV positive	$\Rightarrow$ local minima
$k>0$	odd	$\Rightarrow$ EV with different sign	$\Rightarrow$ saddle
$k=0$		$\Rightarrow$ one EV vanishes	$\Rightarrow$ cannot conclude
$k<0$	even	$\Rightarrow$ EV with different sign	$\Rightarrow$ saddle
$k<0$	odd	$\Rightarrow$ both EV negative	$\Rightarrow$ local maxima.

Exercise 9.2 Consider a system of N particles in $\mathbb{R}^n$, that is N points $a_1, \dots, a_N$ with masses $m_1, \dots, m_N$. Prove that the expression

$$I(x) = \sum_{i=1}^{N} m_i |x - a_i|^2$$

has a unique global minimum, and find it explicitly. Such point C is called the *center of mass* of the system.

Solution. The gradient of I is

$$\nabla I(x) = \sum_{i=1}^{N} 2m_i \begin{pmatrix} x_1 - (a_i)_1 \\ \cdots \\ x_n - (a_i)_n \end{pmatrix} = \sum_{i=1}^{N} 2m_i (x - a_i),$$

which vanishes exactly when

$$\begin{cases} \sum_i m_i x_1 = \sum_i m_i (a_i)_1, \\ \qquad\cdots \\ \sum_i m_i x_n = \sum_i m_i (a_i)_n \end{cases}$$

and hence, such critical point is

$$C = \frac{\sum_{i=1}^{N} m_i a_i}{\sum_{i=1}^{N} m_i}.$$

Now, I is *always positive*, *continuous* and $\lim_{x\to\infty} I(x) = +\infty$, which means that its global minimum has to be attained at some point in $\mathbb{R}^n$, which will then be critical. But the only critical point is C, so this has to be the global minimum.

One may also see this by computing the Hessian of I:

$$\mathrm{Hess}_I(x) \equiv \sum_i 2m_i \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = \Big(\sum_i 2m_i\Big) \mathbb{I}_{n\times n},$$

which is then positive definite everywhere. So C has to be a local minimum, and hence also the global minimum since $\lim_{x\to\infty} I(x) = +\infty$.

Exercise 9.3 Let $0 \neq a \in \mathbb{R}^n$ be fixed and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{a \cdot x}{|x|^2 + 1}, \quad x \in \mathbb{R}^n,$$

where "$\cdot$" denotes the usual scalar product.

(a) Prove that f attains its global maximum and global minimum.

(b) Compute the global extrema of f.

Solution. (a) Note that $f(a) = \frac{|a|^2}{|a|^2+1} > 0$ and $f(-a) = \frac{-|a|^2}{|a|^2+1} < 0$, and finally

$$|f(x)| \leq \frac{|a||x|}{|x|^2 + 1} \longrightarrow 0 \quad \text{as } x \to \infty. \tag{1}$$

Consequently, since f is continuous, f does not tend to its maximum or minimum at $x \to \infty$, and these are instead attained at some $x_{\max}, x_{\min} \in \mathbb{R}^n$.

(b) Since $f(a) > 0$, the maximum has to be positive and hence it must lie in the half-space $H = \{x \in \mathbb{R}^n \mid x \cdot a \geq 0\}$. We can write every $x \in H$ as

$$x = ta + u$$

for some $t > 0$ and $u \in \mathbb{R}^n$ $a \perp u$. Then

$$f(x) = \frac{a \cdot x}{|x|^2 + 1} = \frac{t|a|^2}{t^2|a|^2 + |u|^2 + 1},$$

and we notice that

$$f(x) \leq \frac{t|a|^2}{t^2|a|^2 + 1} = \frac{|a|}{2} \left(\frac{2}{t|a| + \frac{1}{t|a|}} \right) \leq \frac{|a|}{2}, \tag{2}$$

with equality if and only if

$$u = 0 \quad \text{and} \quad \frac{2}{t|a| + \frac{1}{t|a|}} = 1.$$

Since $t|a| > 0$, we can find the value of t in the second expression:

$$\begin{aligned} 2 = t|a| + \frac{1}{t|a|} &\Leftrightarrow 2t|a| = (t|a|)^2 + 1 \\ &\Leftrightarrow 0 = (t|a|)^2 + 1 - 2t|a| = (t|a| - 1)^2 \\ &\Leftrightarrow t|a| = 1 \Leftrightarrow t = \frac{1}{|a|}. \end{aligned}$$

Consequently, the maximum of f is attained when

$$u = 0 \quad \text{und} \quad t = \frac{1}{|a|} \quad \text{i.e.} \quad x_{\max} = \frac{a}{|a|}.$$

An entirely analogous discussion yields that the minimum of f is attained at $x_{\min} = -\frac{a}{|a|}$ an.

Multiple Choice 10.1 Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and consider its *Jacobian matrix*, namely the 3×3 matrix of its 1st derivatives:

$$DV = \begin{pmatrix} \partial_{x_1} V_1 & \partial_{x_2} V_1 & \partial_{x_3} V_1 \\ \partial_{x_1} V_2 & \partial_{x_2} V_2 & \partial_{x_3} V_2 \\ \partial_{x_1} V_3 & \partial_{x_2} V_3 & \partial_{x_3} V_3 \end{pmatrix}.$$

Knowing that, in this matrix, there are three *distinct* coinciding pairs of elements, is, in general,

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for V to be conservative.

Solution. It is

(a) *necessary*, but not sufficient, ☒

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

Indeed since $\mathbb{R}^3$ is star-shaped V is conservative if any only if $\partial_i V_j = \partial_j V_i$ for $i, j \in \{1, 2, 3\}$, which means that it has to be

$$\begin{pmatrix} \partial_{x_1} V_2 \\ \partial_{x_1} V_3 \\ \partial_{x_2} V_3 \end{pmatrix} = \begin{pmatrix} \partial_{x_2} V_1 \\ \partial_{x_3} V_1 \\ \partial_{x_3} V_2 \end{pmatrix}$$

So if this holds, V is conservative. But consider for instance $V(x, y, z) = (x + y, y, z)$: since $\partial_2 V_1 \neq \partial_1 V_2$, V is not conservative however its Jacobian matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see for instance that the couples of elements with entries $(1, 1) - (2, 2)$, $(2, 2) - (3, 3)$ and $(2, 3) - (3, 2)$ have the same values.

Multiple Choice 10.2 Choose the correct statement. Motivate your answer.

Let $V : \mathbb{R}^3 \to \mathbb{R}^3$ and V be as in the previous question.

Knowing that the matrix DV is *symmetric*, is

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☐

(d) neither necessary nor sufficient, ☐

for V to be conservative.

Solution. It is

(a) *necessary*, but not sufficient, ☐

(b) *sufficient*, but not necessary, ☐

(c) *necessary and sufficient*, ☒

(d) neither necessary nor sufficient, ☐

Indeed since $\mathbb{R}^3$ is star-shaped V is conservative if any only if $\partial_i V_j = \partial_j V_i$ for $i, j \in \{1, 2, 3\}$, and this precisely means that DV is symmetric.

Exercise 10.1 In each of the following, find a parametrization of the curve γ and compute the line integral $\int_\gamma F \cdot d\vec{s}$.

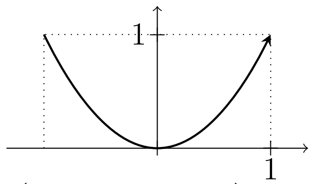
(a) $F(x, y) = (x + y, x - y)$ and γ runs through the parabola $\{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$ from the point $(-1, 1)$ to the point $(1, 1)$.

(b) $F(x, y) = (0, xy^2)$ and γ runs through the half-circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4, y \geq 0\}$ in counter-clockwise direction.

(c) $F(x, y) = (x^2 + y^2, x^2 - y^2)$ and γ runs trough the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ in counter-clockwise direction.

Solution. (a) We choose $\gamma\colon [-1,1] \to \mathbb{R}^2$ with $\gamma(t) = (t, t^2)$. Then

$$\begin{aligned}\int_\gamma F \cdot d\vec{s} &= \int_{-1}^{1} F(\gamma(t)) \cdot \gamma'(t)\, dt \\ &= \int_{-1}^{1} \begin{pmatrix} t+t^2 \\ t-t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt \\ &= \int_{-1}^{1} \Big((t+t^2) + 2t(t-t^2)\Big)\, dt = \int_{-1}^{1} \Big(-2t^3 + 3t^2 + t\Big)\, dt = 2.\end{aligned}$$

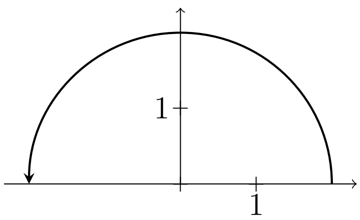


(b) We choose $\gamma\colon [0,\pi] \to \mathbb{R}^2$ with $\gamma(t) = (2\cos t, 2\sin t)$. Then

$$\begin{aligned}\int_\gamma F \cdot d\vec{s} &= \int_0^\pi F(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^\pi \begin{pmatrix} 0 \\ (2\cos t)(2\sin t)^2 \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt \\ &= \int_0^\pi (2\cos t)^2(2\sin t)^2\, dt = 2\int_0^\pi 1 - \cos(4t)\, dt = 2\Big[t - \tfrac{1}{4}\sin(4t)\Big]_0^\pi = 2\pi,\end{aligned}$$

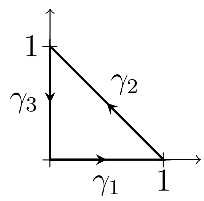
where we used the trigonometric identities

$$\begin{aligned} \sin(2\alpha) &= 2(\sin\alpha)(\cos\alpha), \\ \cos(2\alpha) &= (\cos\alpha)^2 - (\sin\alpha)^2 \\ &= 1 - 2(\sin\alpha)^2, \\ \Rightarrow\ 8(\sin t)^2(\cos t)^2 &= 2(\sin(2t))^2 = 1 - \cos(4t). \end{aligned}$$



(c) We parametrize the triangle piecewise-linearly by

$$\gamma_1\colon [0,1] \to \mathbb{R}^2,\quad t \mapsto \begin{pmatrix} t \\ 0 \end{pmatrix} \qquad \gamma_2\colon [0,1] \to \mathbb{R}^2,\quad t \mapsto \begin{pmatrix} 1-t \\ t \end{pmatrix} \qquad \gamma_3\colon [0,1] \to \mathbb{R}^2,\quad t \mapsto \begin{pmatrix} 0 \\ 1-t \end{pmatrix}$$



The line integral over each piece of the curve is

$$\begin{aligned}\int_{\gamma_1} F \cdot d\vec{s} &= \int_0^1 \begin{pmatrix} t^2 \\ t^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = \int_0^1 t^2\, dt = \frac{1}{3}, \\ \int_{\gamma_2} F \cdot d\vec{s} &= \int_0^1 \begin{pmatrix} (1-t)^2 + t^2 \\ (1-t)^2 - t^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} dt = \int_0^1 -2t^2\, dt = -\frac{2}{3}, \\ \int_{\gamma_3} F \cdot d\vec{s} &= \int_0^1 \begin{pmatrix} (1-t)^2 \\ -(1-t)^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} dt = \int_0^1 (1-t)^2\, dt = \frac{1}{3},\end{aligned}$$

so, by additivity of the lintegral, we conclude

$$\int_\gamma F \cdot d\vec{s} = \int_{\gamma_1} F \cdot d\vec{s} + \int_{\gamma_2} F \cdot d\vec{s} + \int_{\gamma_3} F \cdot d\vec{s} = \frac{1}{3} - \frac{2}{3} + \frac{1}{3} = 0.$$

Exercise 10.2 In each of the following, determine whether the vector field $V\colon \mathbb{R}^3 \to \mathbb{R}^3$ admits a potential and compute the line integral $\int_\gamma V \cdot d\vec{s}$ along the curve $\gamma\colon [0,1] \to \mathbb{R}^3$, $\gamma(t) = (t^3, t^2+t, t)$.

$$\text{(a)} \quad V(x,y,z) = \begin{pmatrix} 2xy^3 \\ 3x^2y^2 + 2yz \\ y^2 \end{pmatrix}, \qquad \text{(b)} \quad V(x,y,z) = \begin{pmatrix} x+z \\ x+y+z \\ x+z \end{pmatrix}.$$

Solution. Recall that, to determine whether there exists a potential, namely a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ so that $\nabla f = V$, the necessary condition is

$$\frac{\partial}{\partial x_i} V_j = \frac{\partial}{\partial x_j} V_i \quad \forall\, i,j \in \{1,2,3\}, \tag{$\star$}$$

where we denoted $(x_1, x_2, x_3) = (x, y, z)$. Since $\mathbb{R}^3$ is star-shaped the condition is also sufficient.

(a) It is a simple computation to see that ($\star$) is satisfied, so V admits a potential f. To find, one, we see that

$$\begin{aligned} \frac{\partial f}{\partial x} \overset{!}{=} 2xy^3 \ &\Rightarrow f(x,y,z) = x^2y^3 + g(y,z) \\ &\Rightarrow \frac{\partial f}{\partial y} = 3x^2y^2 + \frac{\partial g}{\partial y} \overset{!}{=} 3x^2y^2 + 2yz \\ &\Rightarrow g(y,z) = y^2z + h(z) \\ &\Rightarrow \frac{\partial f}{\partial z} = y^2 + h'(z) \overset{!}{=} y^2. \end{aligned}$$

So h is constant. Hence $f(x,y,z) = x^2y^3 + y^2z$ is a potential for V. For the integral along γ we then have

$$\int_\gamma v \cdot d\vec{s} = f(\gamma(1)) - f(\gamma(0)) = f(1,2,1) - f(0,0,0) = 12.$$

(b) Since $\partial_2 V_1 = 0 \neq 1 = \partial_1 V_2$, V does not fulfill ($\star$) and so it does not admit a potential. We then compute the line integral with its definition:

$$\begin{aligned}
\int_\gamma V \cdot d\vec{s} &= \int_0^1 v(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^1 \begin{pmatrix} t^3+t \\ t^3+t^2+2t \\ t^3+t \end{pmatrix} \cdot \begin{pmatrix} 3t^2 \\ 2t+1 \\ 1 \end{pmatrix} dt \\
&= \int_0^1 (t^3+t)(3t^2+1)\, dt + \int_0^1 2t^4+3t^3+5t^2+2t\, dt \\
&= \left[\tfrac{1}{2}(t^3+t)^2\right]_0^1 + \left[\tfrac{2}{5}t^5+\tfrac{3}{4}t^4+\tfrac{5}{3}t^3+t^2\right]_0^1 \\
&= 2+\tfrac{2}{5}+\tfrac{3}{4}+\tfrac{5}{3}+1 = \tfrac{349}{60}.
\end{aligned}$$

Exercise 10.3 The following vector field describes, according to the *Biot-Savart law*, the magnetic field generated by an infinitely long, constant-current electric wire displaced along the z-axis:

$$B(x,y,z) = \frac{\mu_0 I}{2\pi} \frac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \text{defined for } (x,y) \neq 0,$$

where μ_0 and I are, respectively, the magnetic constant and I the (also constant) current.

(a) Prove that it is

$$\frac{\partial}{\partial x_i} B_j = \frac{\partial}{\partial x_j} B_i \quad \forall\, i,j \in \{1,2,3\},$$

where we denoted $(x_1,x_2,x_3) = (x,y,z)$.

(b) Consider the curves $\gamma_m : [0, 2\pi m] \to \mathbb{R}^3$, $\gamma_m(t) = (\cos(t), \sin(t), 0)$ for $m \in \mathbb{Z}$, and compute the line integrals $\int_{\gamma_m} B \cdot d\vec{s}$.

(c) Does B admit a potential in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$?

Solution. (a) This is a direct calculation.

(b) We see that

$$\begin{aligned}
\int_{\gamma_m} B \cdot d\vec{s} &= \frac{\mu_0 I}{2\pi} \int_0^{2\pi m} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} dt \\
&= \frac{\mu_0 I}{2\pi} \int_0^{2\pi m} dt = \mu_0 I m.
\end{aligned}$$

(c) No, since the γ_m's are loops but the line integral are not zero. Note that this is not in conflict with (a) since $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ is not star-shaped.

Multiple Choice 5.1 True or false? Motivate your answer.

Let $m, n \in \mathbb{N}$ and $L : \mathbb{R}^m \to \mathbb{R}^n$ be a *linear* function. Then if L is continuous at one point $x_0 \in \mathbb{R}^m$, it is continuous at every point in $\mathbb{R}^m$.

☐ True ☐ False

Solution. True. Let $a \in \mathbb{R}^m$. Since L is linear and continuous at x_0, we see that

$$\begin{aligned}\lim_{x\to 0} \|L(a+x) - L(a)\| &= \lim_{x\to 0} \|L(a) + L(x) - L(a)\| \\ &= \lim_{x\to 0} \|L(x)\| \\ &= \lim_{x\to 0} \|L(x_0+x) - L(x_0)\| = 0,\end{aligned}$$

so L is continuous at a.

Multiple Choice 5.2 Choose the correct statement. Motivate your answer.

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(0,0) = 0$. For f to be continuous at $(0,0)$, the fact that

$$\lim_{x\to 0} f(x,0) = 0 \quad \text{and} \quad \lim_{y\to 0} f(0,y) = 0$$

(a) is *necessary*, but in general not sufficient. ☐

(b) is *sufficient*, but in general not necessary. ☐

(c) is *necessary and sufficient.* ☐

Note: Looking at Exercise 5.1 may be useful!

Solution. The correct choice is

(a) is *necessary*, but in general not sufficient. ☒

The fact that is it necessary follows because if f is continuous, then the restrictions to f along the lines $f(0,\cdot)$ and $f(\cdot,0)$ are continuous.

The fact that this needs not to be sufficient in general is illustrated by the example provided in Exercise 5.1 below.

Exercise 5.1 Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Prove that f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) Prove that, for every fixed $y \in \mathbb{R}$, there holds

$$\lim_{x \to 0} f(x,y) = 0,$$

and that or every fixed $x \in \mathbb{R}$, there holds

$$\lim_{y \to 0} f(x,y) = 0.$$

(c) Consider now the parabola $P = \{(x, x^2) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Prove that

$$\lim_{\substack{(x,y)\to 0 \\ (x,y) \in P}} f(x,y) = \lim_{x \to 0} f(x, x^2) = \frac{1}{2}.$$

Without any further computation, what can you say about

$$\lim_{(x,y)\to 0} f(x,y) \ldots ?$$

Is f continuous at $(0,0)$?

Solution. (a) f is the ratio of two continuous functions, and the denominator never vanishes in $\mathbb{R}^2 \setminus \{(0,0)\}$. Hence f is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$.

(b) If $y = 0$, then $f(x,0) \equiv 0$. If $y \neq 0$, we have

$$\lim_{x\to 0} \frac{x^2 y}{x^4 + y^2} = \lim_{x \to 0} \frac{x^2 y}{y^2} = \lim_{x \to 0} \frac{x^2}{y} = 0,$$

thus $\lim_{x\to 0} f(x,y) = 0$.

Similarly, if $x = 0$, then $f(0,y) \equiv 0$. If $x \neq 0$, we have

$$\lim_{y\to 0} \frac{x^2 y}{x^4 + y^2} = \lim_{y \to 0} \frac{x^2 y}{x^4} = \lim_{y \to 0} \frac{y}{x^2} = 0,$$

thus $\lim_{y\to 0} f(x,y) = 0$.

(c) We see that

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

Consequently $\lim_{(x,y)\to(0,0)} f(x,y)$ cannot exist: if it did, for every sequence $(x_k,y_k)_{k\in\mathbb{N}}$, $\lim_{k\to\infty} f(x_k,y_k)$ had to be the same, but (b) and the above show that, if we take $(x_k,y_k) = (\frac{1}{k},0)$, then the limit along such sequence is 0, and if we take $(x_k,y_k) = (\frac{1}{k},\frac{1}{k^2})$ the limit along such sequence is $\frac{1}{2}$. In particular then f cannot be continuous at $(0,0)$.

Exercise 5.2 Compute the limits

(a) $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2},$$

(b) $$\lim_{(x,y)\to(1,0)} \frac{y^2\log x}{(x-1)^2+y^2},$$

(c) $$\lim_{(x,y,z)\to\infty} f(x,y,z),$$
where $f(x,y,z) = x^4+y^2+z^2-x^3+xyz-x+4$.

Note: similarly as in one variable, "$\lim_{(x,y,z)\to\infty} f(x,y,z) = \alpha$" means that for every $\varepsilon>0$ there exists $M>0$, so that if $\|(x,y,z)\|\geq M$, then $|f(x,y,z)-\alpha|\leq\varepsilon$.

Solution. (a) From one variable calculus we have $\lim_{t\to 0}\frac{\sin t}{t} = 1$ or in other words that

$$\sin t = t + o(t) \quad \text{as } t\to 0,$$

whence

$$\sin(x^2+y^2) = x^2+y^2+o(x^2+y^2) \quad \text{as } (x,y)\to 0,$$

where $\lim_{(x,y)\to 0}\frac{o(x^2+y^2)}{x^2+y^2} = 0$. Consequently then

$$\begin{aligned}\lim_{(x,y)\to 0}\frac{\sin(x^2+y^2)}{x^2+y^2} &= \lim_{(x,y)\to 0}\left(\frac{x^2+y^2+o(x^2+y^2)}{x^2+y^2}\right)\\ &= \lim_{(x,y)\to 0}\left(1+\frac{o(x^2+y^2)}{x^2+y^2}\right) = 1.\end{aligned}$$

Hence the limit in question is equal to 1.

(b) Writing

$$\frac{y^2 \log x}{(x-1)^2 + y^2} = \left(\frac{\log x}{x-1}\right)\frac{(x-1)y^2}{(x-1)^2+y^2},$$

we have, from one variable calculus, that

$$\lim_{(x,y)\to(1,0)} \frac{\log x}{x-1} = \lim_{x\to 1}\frac{\log x}{x-1} = 1,$$

so are left with checking the limiting behaviour of $\frac{(x-1)y^2}{(x-1)^2+y^2}$. With a change of variables, we see that

$$\begin{aligned}\lim_{(x,y)\to(1,0)} \frac{(x-1)y^2}{(x-1)^2+y^2} &= \lim_{(u,v)\to(0,0)} \frac{uv^2}{u^2+v^2} \\ &= \lim_{(u,v)\to(0,0)} \frac{uv^2}{u^2+v^2} \\ &= \lim_{(u,v)\to(0,0)} u\frac{v^2}{u^2+v^2} = 0,\end{aligned}$$

since (for instance) u tends to 0 and $\frac{v^2}{u^2+v^2}$ is bounded as $(u,v)\to(0,0)$. Hence the limit in question is equal to 0.

(c) The limit does not exist. On the one hand, on the x-axis we have

$$\lim_{x\to\infty} f(x,0,0) = \lim_{x\to\infty} x^4 - x^3 - x + 4 = +\infty,$$

however if we consider for instance the sequence $(\sqrt{k}, -k, k)$, $k\in\mathbb{N}$, we see that

$$\lim_{k\to\infty} f(\sqrt{k},-k,k) = \lim_{k\to\infty} 3k^2 - k^{3/2} - k^{5/2} - k^{1/2} + 4 = -\infty,$$

So the limit cannot exist.

Exercise 5.3 Which of the following subsets of the Euclidean space are compact?

(a) $A = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 < 2020\}$;

(b) $B = \{(a,b,c)\in\mathbb{R}^3 \mid a,b,c \text{ are integers and } a^2+b^2+c^2 < 2020\}$;

(c) $C = \{(x,f(x))\in\mathbb{R}^2 \mid x\in(0,1], f(x) = \sin(\frac{1}{x})\}$;

(d) $D = \{(x,y)\in\mathbb{R}^2 \mid x\geq 0 \text{ and } y\geq 0\}$;

(e) $E = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\leq 2\}$.

Solution. Recall that a subset in the Euclidean space is compact if and only if it is bounded and closed.

(a) A is not compact, since it is not closed: for instance the the sequence $(x_k, y_k) = \left(\sqrt{2020 - \frac{1}{k}}, 0\right)$ $k \geq 1$ is contained in A and has limit $(\sqrt{2020}, 0)$ that does not belong to A.

(b) B is bounded and consists of finitely many points, hence it is compact.

(c) C is not compact since it is not closed. The sequence $(\frac{1}{2k\pi}, 0)$, $k \geq 1$ is contained C $(\frac{1}{2k\pi}, 0) \to (0, 0)$ for $k \to +\infty$, but $(0, 0) \notin C$.

(d) D is not bounded, hence cannot be compact.

(e) E is closed and bounded, hence compact.

Multiple Choice 3.1 True or false? Motivate your answers.

Consider the ODE $y'(x) = \mathrm{e}^{y(x)}$. Then:

		True	False
(a)	Any solution with $y(0) \geq 0$ satisfies $y(t) > 0$ for $t > 0$.	☐	☐
(b)	For any $a, b \in \mathbb{R}$ there always is a solution with $y(0) = a$ and $y(1) = b$.	☐	☐
(c)	Any solution is increasing.	☐	☐

Solution.

		True	False
(a)	*Any solution with $y(0) \geq 0$ satisfies $y(t) > 0$ for $t > 0$.*	☒	☐

From the fundamental theorem of calculus, we have

$$y(t) = y(0) + \int_0^t \mathrm{e}^{y(\tau)}\, d\tau,$$

so since $\mathrm{e}^x > 0$ for every $x \in \mathbb{R}$, so must be y for $t > 0$.

		True	False
(b)	*For any $a, b \in \mathbb{R}$ there always is a solution with $y(0) = a$ and $y(1) = b$.*	☐	☒

From (a) it follows that if we chose $a \geq 0$ and $b < 0$, there will be no solution satisfying $y(1) < 0$.

		True	False
(b)	*Every solution must be increasing.*	☒	☐

From one variable calculus this is true if and only if $y' > 0$, which is true from the properties of the exponential.

Multiple Choice 3.2 Choose the correct statement(s). Motivate your answers.

The ODE $y^{(4)} + y = 0$ has:

(a) Only periodic solutions. ☐

(b) Some nonzero periodic solutions. ☐

(c) No nonzero periodic solutions. ☐

(d) No nonzero solution y so that $\lim_{t\to+\infty} y(t)$ exists and is finite. ☐

(e) Some nonzero solution y so that $\lim_{t\to+\infty} y(t)$ exists and is finite. ☐

Solution. The characteristic polynomial of the equation is: $\lambda^4+1=(\lambda^2+i)(\lambda^2-i)$, so its roots are $\pm\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$ and $\pm\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$, hence the general solution of the equation is

$$y(t)=\mathrm{e}^{\frac{\sqrt{2}}{2}t}\Big(c_1\cos\Big(\frac{\sqrt{2}}{2}t\Big)+c_2\sin\Big(\frac{\sqrt{2}}{2}t\Big)\Big)+\mathrm{e}^{-\frac{\sqrt{2}}{2}t}\Big(c_3\cos\Big(\frac{\sqrt{2}}{2}t\Big)+c_4\sin\Big(\frac{\sqrt{2}}{2}t\Big)\Big),$$

for $c_1,\dots,c_4\in\mathbb{R}$. With this, we have:

(a) *Only periodic solutions.* ☐

(b) *Some nonzero periodic solutions.* ☐

(c) *No nonzero periodic solutions.* ☒

because of the exponential factors. Next:

(d) No nonzero solution y so that $\lim_{t\to+\infty}y(t)$ exists and is finite. ☐

(e) Some nonzero solution y so that $\lim_{t\to+\infty}y(t)$ exists and is finite. ☒

since it suffices to chose any solution with $c_1=c_2=0$.

Exercise 3.1 (A glance at systems) Let

$$y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1y'+a_0y=0,\quad t\in I\subseteq\mathbb{R}\tag{$*$}$$

be a linear, homogeneous ODE of order $n\geq 2$ with constant coefficients.

(a) Show that, by setting $z_1=y$, $z_2=y'$, $\dots$, $z_n=y^{(n-1)}$, equation $(*)$ can be seen as a first-order *system* of ODEs:

$$\boldsymbol{z}'=A\boldsymbol{z},\quad t\in I\tag{$\star$}$$

where $A\in M_{n\times n}(\mathbb{R})$ is a $n\times n$ matrix. Write down explicitly the expression for A.

(b) Prove that the characteristic polynomial of the ODE $(*)$ is the characteristic polynomial of the matrix A.

(c) Show that

$$\boldsymbol{\zeta}(t)=\mathrm{e}^{\lambda t}\boldsymbol{u}$$

is a solution of the homogeneous problem $(\star)$ if and only if λ is an eigenvalue of A and $\boldsymbol{u}$ is a corresponding eigenvector.

(d) *Fact:* Similarly as for ODE of order n, one can prove that for a system like ($\star$), the set of solutions is vector space of dimension n.

Assuming that all the eigenvalues of A are distinct, use (c) and the fact above to find an explicit expression for the general solution of homogeneous system.

Solution. (a) By construction $\boldsymbol{z}$ satisfies

$$\boldsymbol{z}' = \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_{n-1}' \\ z_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix},$$

and thus

$$A = \left(\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \mathbb{I}_{n-1} \\ \hline -a_0 & -a_1 \ \cdots \ -a_{n-1} \end{array}\right),$$

where $\mathbb{I}_{n-1}$ denotes the $(n-1) \times (n-1)$ identity matrix.

(b) We proceed by induction on n. The assertion is trivial for $n = 1$. For the inductive step $(n-1) \to n$, we see that the characteristic polynomial of A is

$$p_A(\lambda) = \det(\lambda \mathbb{I}_n - A) = \det \begin{pmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & \lambda & -1 \\ a_0 & a_1 & a_2 & & \cdots & \lambda + a_{n-1}, \end{pmatrix}$$

so choosing the first column to expand the expression we see that

$$p_A(\lambda) = \lambda \det \begin{pmatrix} \lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & \cdots & & 0 & \lambda & -1 \\ a_1 & a_2 & a_3 & & \cdots & \lambda + a_{n-1}, \end{pmatrix} + (-1)^{2(n-1)} a_0,$$

and, by inductive hypothesis, the determinant in the expression above is $\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_1$. In other words

$$p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + \lambda a_1 + a_0.$$

(c) We have $\boldsymbol{\zeta}'(t) = \lambda e^{\lambda t}\boldsymbol{u}$, thus since $e^{\lambda t}$ is never zero,

$$\boldsymbol{\zeta}'(t) = A\boldsymbol{\zeta}(t) \iff \lambda e^{\lambda t}\boldsymbol{u} = A[e^{\lambda t}\boldsymbol{u}] \iff \lambda\boldsymbol{u} = A\boldsymbol{u},$$

and the last equality is precisely what defines eigenvalues and eigenvectors.

(d) From linear algebra, we know that, for two distinct eigenvalues, two (nonzero) corresponding eigenvectors are linearly independent. So, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ are corresponding nonzero eigenvectors, the general solution for $(\star)$ has the form

$$c_1 e^{\lambda_1 t}\boldsymbol{u}_1 + c_2 e^{\lambda_2 t}\boldsymbol{u}_2 + \ldots + c_n e^{\lambda_n t}\boldsymbol{u}_n,$$

for arbitrary constants $c_i \in \mathbb{R}$.

Exercise 3.2 Find the general solution of the following ODE and, when specified, the solutions fulfilling the indicated requirements. The solutions must always be expressed in real form.

(a) $y'' - 3y' + 2y = 0$,

(b) $y'' - 4y' = 0$ and the solutions that are always positive,

(c) $y'' - y' + y = 0$ and the solutions satisfying $y(0) = 0$, $y'(0) = 1$,

(d) $y^{(4)} + 1 = 0$ and the solutions so that are even i.e. $y(t) = y(-t)$.

Solution. (a) The characteristic polynomial is $\lambda^2 - 3\lambda + 2$; its roots are 1 and 2, so the general solution is

$$y(t) = c_1 e^t + c_2 e^{2t},$$

for arbitrary $c_1, c_2 \in \mathbb{R}$.

(b) The characteristic polynomial is $\lambda^2 - 4\lambda$; its roots are 0 and 4, so the general solution is

$$y(t) = c_1 + c_2 e^{4t},$$

for arbitrary $c_1, c_2 \in \mathbb{R}$. To find those that are always positive, since $\lim_{t\to-\infty} e^{4t} = 0$, it must be $c_1 > 0$, and since $\lim_{t\to+\infty} e^{4t} = \operatorname{sign}(c_2)\cdot\infty$, so it must be $c_2 > 0$. Thus the only positive solutions are those so that $c_1, c_2 > 0$.

(c) The characteristic polynomial is $\lambda^2 - \lambda + 1$; its roots are $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, hence the general solution is

$$y(t) = \mathrm{e}^{\frac{t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right),$$

for arbitrary $c_1, c_2 \in \mathbb{R}$. Substituting the conditions $y(0) = 0$, $y'(0) = 1$ gives $c_1 = 0$, $c_2 = \frac{2}{\sqrt{3}}$, so the particular solution is

$$y_p(t) = \frac{2}{\sqrt{3}}\mathrm{e}^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right).$$

(d) It suffices apply the fundamental theorem of calculus four times:

$$\begin{aligned}
&y^{(4)}(t) = -1 \\
\Rightarrow\, &y^{(3)}(t) = -t + c_1 \\
\Rightarrow\, &y^{(2)}(t) = -\frac{t^2}{2} + c_1 t + c_2 \\
\Rightarrow\, &y^{(1)}(t) = -\frac{t^3}{3!} + c_1\frac{t^2}{2} + c_2 t + c_3 \\
\Rightarrow\, &y(t) = -\frac{t^4}{4!} + c_1\frac{t^3}{3!} + c_2\frac{t^2}{2!} + c_3 t + c_4, \quad c_1, \dots, c_4 \in \mathbb{R}.
\end{aligned}$$

From this expression, we see that for a solution be even it it is necessary and sufficient that $c_1 = 0 = c_3$.

Exercise 3.3 Solve the following Cauchy problems:

(a) $$\begin{cases} 2t^2 y' - y = 0, & \text{for } t \geq 1, \\ y(1) = 1. \end{cases}$$

(b) $$\begin{cases} y' - y = y \log x + 1 + \log x, & \text{for } x > 2, \\ y(2) = 3. \end{cases}$$

Find the values $a \in \mathbb{R}$ so that the problem

(c) $$\begin{cases} y'' - (a+1)y' + ay = 0 \text{ in } \mathbb{R}, \\ y(t) \text{ is bounded for } t > 0, \end{cases}$$

has nonzero solutions, and write them explicitly.

Solution. (a) From the condition $y(1) \neq 0$ we may suppose that $y \not\equiv 0$, and write

$$\frac{y'}{y} = \frac{1}{2t^2},$$

and thus use the method of separation of variables to deduce that

$$\log|y| = -\frac{1}{2t} + c \quad \Longleftrightarrow \quad |y| = \mathrm{e}^{-\frac{1}{2t}+c},$$

and hence that the (nonzero) general solution of the ODE is

$$y(t) = K\mathrm{e}^{-\frac{1}{2t}}, \quad K \in \mathbb{R} \setminus \{0\}.$$

Imposing the boundary condition gives $K = \mathrm{e}^{\frac{1}{2}}$, so we conclude

$$y(t) = \mathrm{e}^{\frac{1}{2}-\frac{1}{2t}}.$$

(b) We notice first that the equation can be written as

$$y' = (\log x + 1)(y + 1)$$

The right-hand side vanishes identically for $y \equiv -1$ which is a solution of the ODE (but not of the given problem). Otherwise we may apply the separation of variables:

$$\int \frac{dy}{y+1} = \int (\log x + 1)\, dx = x \log x + C, \quad C \in \mathbb{R}$$

and hence

$$|y+1| = \mathrm{e}^C \mathrm{e}^{x \log x} = \mathrm{e}^C x^x,$$

thus yielding

$$y(x) = Kx^x - 1, \quad K \in \mathbb{R} \setminus \{0\},$$

and requiring that $y(2) = 3$ yields $K = 1$. So the solution is

$$y(x) = x^x - 1.$$

(c) The characteristic polynomial is $\lambda^2 - (a+1)\lambda + a$, hence, If $a = 1$, the general solution of the ODE is $y(t) = c_1\mathrm{e}^t + c_2 t\mathrm{e}^t$, while if $a \neq 1$, it is $y(t) = c_1\mathrm{e}^{at} + c_2\mathrm{e}^t$, for $c_1, c_2 \in \mathbb{R}$. Nonzero solution bounded in $(0, +\infty)$ thus exist only when $a \leq 0$, and in this care they are

$$y(t) = c_1\mathrm{e}^{at}, \quad \forall\, c_1 \neq 0.$$

Multiple Choice 1.1 True or false? Motivate your answers.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^n$, $n \geq 1$, $f(x) = (f_1(x), \ldots, f_n(x))$ be a function. Then:

		True	False
(a)	If one of the f_j's is injective, then f is injective.	☐	☐
(b)	If f is injective, at least one of the f_j's is injective.	☐	☐
(c)	If every f_j is surjective, then f is surjective.	☐	☐
(d)	If f is surjective, then every f_j is surjective.	☐	☐

Solution.

		True	False
(a)	*If one of the f_j's is injective, then f is injective.*	☒	☐

Suppose that, say, f_1 is injective. By definition of injectivity, we have that if $x_1 \neq x_2$ then $f_1(x_1) \neq f_1(x_2)$, and consequently $f(x_1) \neq f(x_2)$, so f is injective.

		True	False
(b)	*If f is injective, at least one of the f_j's is injective.*	☐	☒

Consider for instance $f : [0, 2\pi) \to \mathbb{R}^2$, $f(x) = (\cos x, \sin x)$ (a pararmetrization of the circle). Neither sin nor cos are injective in $[0, 2\pi)$ but

$$\cos x_1 = \cos x_2 \quad \iff \quad x_1 = x_2 + 2k\pi \text{ or } x_1 = -x_2 + 2k\pi$$

and

$$\sin x_1 = \sin x_2 \quad \iff \quad x_1 = x_2 + 2k\pi \text{ or } x_1 = \pi - x_2 + 2k\pi,$$

for $k \in \mathbb{Z}$. Since in our case $x_1, x_2 \in [0, 2\pi)$, we have that both condition are satisfied if and only of $x_1 = x_2$.

		True	False
(c)	*If every f_j is surjective, then f is surjective.*	☐	☒

Take for instance $f(x) = (x, x, \ldots, x)$, $x \in \mathbb{R}$: each f_j is the identity and so is surjective from $\mathbb{R} \to \mathbb{R}$, but $(1, 0, \ldots 0) \notin f(\mathbb{R})$.

		True	False
(d)	*If f is surjective, then every f_j is surjective.*	☒	☐

If one of the f_j's were not surjective, say f_1, then there is $p_1 \in \mathbb{R}$ so that $p_1 \notin f_1(I)$. But then $(p_1, \ldots, p_2) \notin f(I)$ for any $p_2, \ldots, p_n$, so f cannot be surjective.

Multiple Choice 1.2 Let $f : [a, b] \to \mathbb{R}^n$, $n \geq 1$ be continuous and differentiable in (a, b). Then there is $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The above statement is

☐ True ☐ False

Motivate your answer.

Solution. False. If $n = 1$, this is true and is the mean value theorem, but when $n \geq 2$ this does not hold, in general. Cosndier for instance $f : [0, 2\pi] \to \mathbb{R}^2$, $f(t) = (\cos t, \sin t)$: Then $f'(t) \neq 0$ for every t (in fact, $|f'(t)| \equiv 1$), but $f(0) - f(2\pi) = 0$. So the assertion cannot hold.

Exercise 1.1 For each of the following expressions, determine whether they are ODE, and in such case, whether they are linear/nonlinear, homogeneous/inhomogeneous and their order.

(a) $y'(x) = y(x)(1 - y(x))$.

(b) $a_2\, x^2\, y''(x) + a_1\, x\, y'(x) + a_0\, y(x) = 1$, where $a_0, a_1, a_2 \in \mathbb{R}$.

(c) $y''(x) + y(x) = y(2x)$.

(d) $y(x) = xy'(x) + f(y'(x))$, where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

(e) $y''(x) + 2y'(x) + y(x) = \cos x$.

Solution. (a) Nonlinear, 1st order ODE.

(b) Inhomogeneous linear of 2nd order if $a_2 \neq 0$, 1st order if $a_2 = 0 \neq a_1$, or "0th order" if $a_2 = 0 = a_1$.

(c) Not an ODE for the presence of $2x$ on the argument of right-hand side

(d) It depends on f: if f is affine, i.e. $f(y) = ay + b$ for $a, b \in \mathbb{R}$ then it is linear, and in this case it is of 1st order, inhomogeneous if $b \neq 0$ and homogeneous if $b = 0$. Otherwise it is nonlinear of 1st order.

(e) Inhomogeneous linear of 2nd order.

Exercise 1.2 Consider the ODE in $y : \mathbb{R} \to \mathbb{R}$:

$$y'' + 2y' + y = 0.$$

(a) Verify that e^{-x} and $x\mathrm{e}^{-x}$ are solutions of the equation.

(b) Verify that $a\mathrm{e}^{-x} + x\mathrm{e}^{-x}$ is again a solution of the equation for every $a \in \mathbb{R}$.

(c) Can you find all the (twice continuously differentiable) functions $\alpha : \mathbb{R} \to \mathbb{R}$ so that $\alpha(x)\mathrm{e}^{-x} + x\mathrm{e}^{-x}$ is a solution of the equation?

Solution. (a) & (b): recalling $\frac{d}{dx}(\mathrm{e}^{-x}) = -\mathrm{e}^{-x}$, it suffices to plug in the functions in the equation to verify the assertions.

(c) Calling $f(x) = \alpha(x)\mathrm{e}^{-x} + x\mathrm{e}^{-x}$, we have:

$$\begin{aligned} f'(x) &= (\alpha'(x) - \alpha(x))\mathrm{e}^{-x} + (1 - x)\mathrm{e}^{-x}, \\ f''(x) &= (\alpha''(x) - 2\alpha'(x) + \alpha(x))\mathrm{e}^{-x} + (-2 + x)\mathrm{e}^{-x}, \end{aligned}$$

hence

$$f''(x) + 2f'(x) + f(x) = \alpha''(x)\mathrm{e}^{-x},$$

and thus f solves the equation if and only if $\alpha'' = 0$. From the fundamental theorem of calculus, it then follows that $\alpha(x) = ax + b$ for some constants $a, b \in \mathbb{R}$.

Exercise 1.3 Find an ODE of the specified order solved by the given function:

(a) $\varphi(t) = \frac{1}{1-t}$, of 1st order,

(b) $\varphi(t) = c_1 \cos t + c_2 \sin t$, of 2nd order, where $c_1, c_2 \in \mathbb{R}$,

(c) $\varphi(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{-t}$, of 2nd order, where $c_1, c_2 \in \mathbb{R}$.

Solution. (a) Since $\varphi'(t) = \frac{1}{(1-t)^2}$, one such ODE is $y' = y^2$.

(b) Since $\varphi'(t) = -c_1 \sin t + c_2 \cos t$ and $\varphi''(t) = -c_1 \cos t - c_2 \sin t$, one such ODE is $y'' + y = 0$.

(c) Since $\varphi'(t) = c_1\mathrm{e}^t - c_2\mathrm{e}^{-t}$ and $\varphi''(t) = c_1\mathrm{e}^t + c_2\mathrm{e}^{-t}$, one such ODE is $y'' - y = 0$.

Multiple Choice 6.1 The following subsets are open, closed, bounded, compact? Motivate your answers.

(a) $S = (0,1) \cup \left(2, 2+\frac{1}{2}\right) \cup \left(3, 3+\frac{1}{3}\right) \cup \left(4, 4+\frac{1}{4}\right) \cup \cdots \subset \mathbb{R}$

☐Open ☐Closed ☐Bounded ☐Compact

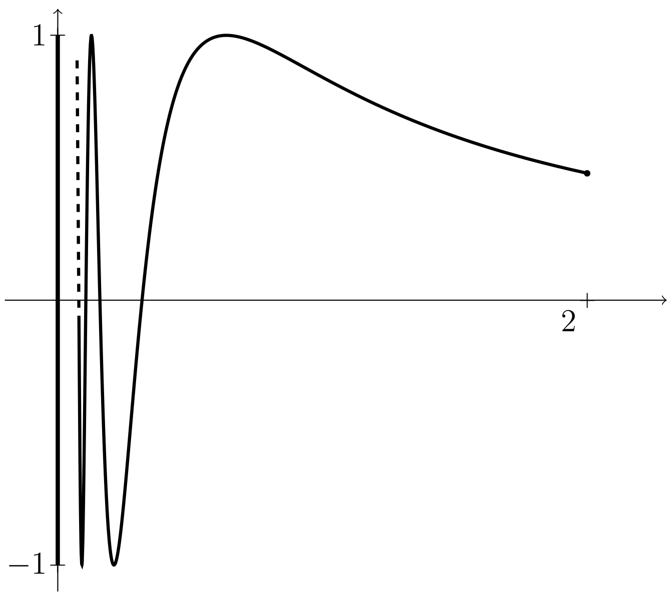
(b) $S = (0,1) \times \{0\} \subset \mathbb{R}^2$

☐Open ☐Closed ☐Bounded ☐Compact

(c) $S = \bigcap_{n=1}^{\infty} B_{1+\frac{1}{n}}(0) \subset \mathbb{R}^n$, where $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$

☐Open ☐Closed ☐Bounded ☐Compact

(d) $S = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x \in (0,2]\right\} \cup \{0\} \times [-1,1] \subset \mathbb{R}^2$



☐Open ☐Closed ☐Bounded ☐Compact

(e) $S = f^{-1}(\{1\}) \subset \mathbb{R}^3$, where $f(x,y,z) = z^2 - x^2 - y^2$

☐Open ☐Closed ☐Bounded ☐Compact

(f) $S = g^{-1}((-2,2]) \subset \mathbb{R}^3$, where $g(x,y,z) = x^2 + y^2 + z^2 + 1$

☐Open ☐Closed ☐Bounded ☐Compact

Solution.

(a) ☒Open ☐Closed ☐Bounded ☐Compact
It is the union of open segments.

(b) ☐Open ☐Closed ☒Bounded ☐Compact

Clearly bounded, not open since it does not contain any ball in $\mathbb{R}^2$ and not closed since (for instance) the sequence $\left(1 - \frac{1}{n}, 0\right)$, $n \geq 1$ is contained in S and has limit $(1, 0) \notin S$.

(c) ☐Open ☒Closed ☒Bounded ☒Compact

In fact it is $S = \overline{B_1(0)}$: the inclusion "$\supseteq$" is obvious, while for the opposite one "$\subseteq$" note that if $x \in S$ then $|x| < 1 + \frac{1}{n}$ for every n, hence $|x| \leq 1$.

(d) ☐Open ☒Closed ☒Bounded ☒Compact

Let (x_n, y_n) be a sequence in S converging in $\mathbb{R}^2$ to (x, y). We distinguish two cases:

- if $x > 0$, then also for n big enough it must be $x_n > 0$, and this means that for n big enough it is $(x_n, y_n) = \left(x_n, \sin\left(\frac{1}{x_n}\right)\right) \to \left(x, \sin\left(\frac{1}{x}\right)\right) \in S$.
- If $x = 0$ then since $y_n \in [-1, 1]$ it must be $y \in [-1, 1]$, and also in this case $(x, y) \in S$.

So S is closed, and being clearly also bounded it is compact.

(e) ☐Open ☒Closed ☐Bounded ☐Compact

S is closed since f is continuous and $\{1\}$ is closed, but it is not compact, for instance since it contains the unbounded sequence $(\sqrt{n}, 0, \sqrt{n+1})$, $n \geq 0$.

(f) ☐Open ☒Closed ☒Bounded ☒Compact

Note first of all that $g^{-1}(\{a\}) = \varnothing$ for $a < 1$, hence it is

$$\begin{aligned} S &= g^{-1}([1,2]) \\ &= \bigcup_{a \in [1,2]} \{(x,y,z) : x^2 + y^2 + z^2 + 1 = a\} \\ &= \bigcup_{a \in [1,2]} \partial B_{\sqrt{a-1}}(0) = \overline{B_1(0)}, \end{aligned}$$

where $\partial B_r(0)$ is the sphere of radius r and center 0. So S closed and bounded and hence compact.

Multiple Choice 6.2 True or false? Motivate your answers (For the definition of directional derivative, see Exercise 6.2 below).

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. For f to be continuous at $(0, 0)$:

		True	False
(a)	it is sufficient that, along *some* direction $v \neq 0$, the directional derivative $D_v f(0, 0)$ exists.	☐	☐
(b)	it is sufficient that *both* the partial derivatives $\partial_x f(0, 0)$ and $\partial_y f(0, 0)$ exist.	☐	☐
(c)	it is sufficient that the directional derivatives $D_v f(0, 0)$ along *every* direction $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ exists.	☐	☐

Solution. All the statements are false. We consider again the example given in the Exercise 5.1, namely:

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We had seen that f is not continuous at 0. However, $D_v f(0, 0)$ exists along every direction: indeed, for $t \neq 0$ the difference quotient is:

$$\frac{f((0, 0) + tv) - f(0, 0)}{t} = \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2},$$

whence taking the limit as $t \to 0$ we see that

$$D_v f(0, 0) = \begin{cases} \dfrac{v_1^2}{v_2} & \text{if } v_2 \neq 0, \\ 0 & \text{if } v_2 = 0. \end{cases}$$

Exercise 6.1 Determine the domain and compute the (1st order) partial derivatives of the following functions:

(a) $f(x, y) = \pi x^2$

(b) $f(x, y) = e^{xy}$

(c) $f(x, y) = x^y$

(d) $f(x, y) = \dfrac{x - y}{x^2 + y^2}$

(e) $f(x, y) = x^2 y \sin(xy)$

(f) $f(x, y, z) = xy^2 z^3 + y$.

Solution. (a) $f(x,y) = \pi x^2$ is defined in $\mathbb{R}^2$ and there holds $\frac{\partial f}{\partial x}(x,y) = 2\pi x$ and $\frac{\partial f}{\partial y}(x,y) = 0$.

(b) $f(x,y) = e^{xy}$ is defined in $\mathbb{R}^2$ and there holds $\frac{\partial f}{\partial x}(x,y) = ye^{xy}$ and $\frac{\partial f}{\partial y}(x,y) = xe^{xy}$.

(c) $f(x,y) = x^y$ is defined for $x > 0$ and $y \in \mathbb{R}$, hence its domain is $(0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$ and there holds

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= yx^{y-1}, \\ \frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y} e^{y \log x} = e^{y\log x}(\log x) = x^y(\log x). \end{aligned}$$

(d) $f(x,y) = \frac{x-y}{x^2+y^2}$ is defined for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and there holds

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= \frac{(x^2+y^2) - 2x(x-y)}{(x^2+y^2)^2} = \frac{-x^2+2xy+y^2}{(x^2+y^2)^2}, \\ \frac{\partial f}{\partial y}(x,y) &= \frac{-(x^2+y^2) - 2y(x-y)}{(x^2+y^2)^2} = \frac{-x^2-2xy+y^2}{(x^2+y^2)^2}. \end{aligned}$$

(e) $f(x,y) = x^2 y \sin(xy)$ is defined for $(x,y) \in \mathbb{R}^2$ and there holds

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= 2xy\sin(xy) + x^2y^2\cos(xy), \\ \frac{\partial f}{\partial y}(x,y) &= x^2\sin(xy) + x^3 y\cos(xy). \end{aligned}$$

(f) $f(x,y,z) = xy^2z^3 + y$ is defined for $(x,y,z) \in \mathbb{R}^3$ and there holds

$$\frac{\partial f}{\partial x}(x,y,z) = y^2z^3, \quad \frac{\partial f}{\partial y}(x,y,z) = 2xyz^3 + 1, \quad \frac{\partial f}{\partial z}(x,y,z) = 3xy^2z^2.$$

Exercise 6.2 Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function and $v \in \mathbb{R}^n$ a vector. When it exists, the limit

$$D_v f(x) = \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}$$

is called *directional derivative* of f along v at the point $x \in \mathbb{R}^n$. In particular, along the coordinate directions $e_i = (0,\dots,0,1,0,\dots,0)$ we have $D_{e_i} f = \frac{\partial f}{\partial x_i} = \partial_{x_i} f$.

Compute, using the definition above, the directional derivative of the folloing functions along the direction $v, w \in \mathbb{R}^2$ at the given point:

(a) $f(x,y) = \cos(xy) + x^2, \quad (x,y) = (\pi, 3), \quad v = (1,1), \quad w = (2,0)$

(b) $f(x,y) = 2x^2y + 3xy + y, \quad (x,y) = (2,1), \quad v = (1,1), \quad w = (1,2).$

Compute now the usual partial derivatives at the same point. What do you notice?

Solution. $D_v f(x)$ exists if and only if the function $\varphi\colon [-\delta,\delta] \to \mathbb{R}$ given by $\varphi(t) = f(x+tv)$ is differentiable at $t=0$,since

$$\varphi'(0) = \lim_{h\to 0} \frac{\varphi(h)-\varphi(0)}{h} = \lim_{h\to 0} \frac{f(x+hv)-f(x)}{h} = D_v f(x).$$

With this, we have:

(a) $\varphi(t) = f(\pi + tv_1, 3 + tv_2) = \cos\Big((\pi+tv_1)(3+tv_2)\Big) + (\pi+tv_1)^2$ is clearly differentiable with

$$\begin{aligned}
\varphi'(t) &= -\Big(v_1(3+tv_2) + v_2(\pi+tv_1)\Big)\sin\Big((\pi+tv_1)(3+tv_2)\Big) + 2(\pi+tv_1)v_1, \\
\varphi'(0) &= -(3v_1 + \pi v_2)\sin(3\pi) + 2\pi v_1 = 2\pi v_1.
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
D_{(1,1)}f(\pi,3) &= 2\pi, \\
D_{(2,0)}f(\pi,3) &= 4\pi, \\
\frac{\partial f}{\partial x}(\pi,3) = D_{(1,0)}f(\pi,3) &= 2\pi, \\
\frac{\partial f}{\partial y}(\pi,3) = D_{(0,1)}f(\pi,3) &= 0,
\end{aligned}$$

and we notice that

$$D_{(v_1,v_2)}f(\pi,3) = 2\pi v_1 + 0 v_2 = v_1 \cdot \frac{\partial f}{\partial x}(\pi,3) + v_2 \cdot \frac{\partial f}{\partial y}(\pi,3).$$

(b) In this case we have

$$\begin{aligned}
\varphi(t) &= f(2+tv_1, 1+tv_2) \\
&= 2(2+tv_1)^2(1+tv_2) + 3(2+tv_1)(1+tv_2) + (1+tv_2),
\end{aligned}$$

whose derivative is

$$\begin{aligned}
\varphi'(t) &= 4v_1(2+tv_1)(1+tv_2) + 2v_2(2+tv_1)^2 + 3v_1(1+tv_2) + 3v_2(2+tv_1) + v_2 \\
\varphi'(0) &= 8v_1 + 8v_2 + 3v_1 + 6v_2 + v_2 = 11v_1 + 15v_2.
\end{aligned}$$

Hence

$$\begin{aligned} D_{(1,1)}f(2,1) &= 26, \\ D_{(1,2)}f(2,1) &= 41, \\ \frac{\partial f}{\partial x}(2,1) = D_{(1,0)}f(2,1) &= 11, \\ \frac{\partial f}{\partial y}(2,1) = D_{(0,1)}f(2,1) &= 15. \end{aligned}$$

Again we see that

$$D_{(v_1,v_2)}f(2,1) = 11v_1 + 15v_2 = v_1 \cdot \frac{\partial f}{\partial x}(2,1) + v_2 \cdot \frac{\partial f}{\partial y}(2,1).$$

Exercise 6.3 Compute the mixed 2nd partial derivatives i.e.

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

of the following functions, and compare them.

(a) $f(x,y) = x^2 e^{y \sin x}$,

(b) $f(x,y) = \begin{cases} \dfrac{xy}{2}\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Note: For (b), at the point $(0,0)$ use the definition of derivative as limit of difference quotient, or use Exercise 6.2.

Solution. (a) The first partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= (2x + x^2 y \cos x)e^{y \sin x}, \\ \frac{\partial f}{\partial y}(x,y) &= (x^2 \sin x)e^{y \sin x}, \end{aligned}$$

thus

$$\begin{aligned} \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y) &= \Big(x^2 \cos x + (2x + x^2 y \cos x)\sin x\Big)e^{y \sin x}, \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y) &= \Big(2x \sin x + x^2 \cos x + x^2(\sin x)y(\cos x)\Big)e^{y \sin x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y), \end{aligned}$$

and we see that they coincide.

(b) When $(x, y) \neq (0, 0)$ we have

$$f(x,y) = \frac{xy}{2} \cdot \frac{x^2 - y^2}{x^2 + y^2} = \frac{xy}{2} \cdot \frac{x^2 + y^2 - 2y^2}{x^2 + y^2} = \frac{xy}{2} - \frac{xy^3}{x^2 + y^2},$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{y}{2} - \frac{y^3(x^2 + y^2) - 2x\,xy^3}{(x^2 + y^2)^2} = \frac{y}{2} - \frac{y^5 - x^2y^3}{(x^2 + y^2)^2},$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x}{2} - \frac{3xy^2(x^2 + y^2) - 2y\,xy^3}{(x^2 + y^2)^2} = \frac{x}{2} - \frac{3x^3y^2 + xy^4}{(x^2 + y^2)^2},$$

$$\begin{aligned}\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y) &= \frac{1}{2} - \frac{(5y^4 - 3x^2y^2)(x^2 + y^2) - 4y(y^5 - x^2y^3)}{(x^2 + y^2)^3} \\ &= \frac{1}{2} - \frac{6x^2y^4 + y^6 - 3x^4y^2}{(x^2 + y^2)^3},\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y) &= \frac{1}{2} - \frac{(9x^2y^2 + y^4)(x^2 + y^2) - 4x(3x^3y^2 + xy^4)}{(x^2 + y^2)^3} \\ &= \frac{1}{2} - \frac{-3x^4y^2 + 6x^2y^4 + y^6}{(x^2 + y^2)^3} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y).\end{aligned}$$

Thus also in this case $\partial^2_{xy} f = \partial^2_{xy} f$ in $\mathbb{R}^2 \setminus \{(0,0)\}$.

At the point $(0, 0)$ we compute the partial derivatives using the definition with different quotients. Since for every $h \in \mathbb{R}$ there holds $f(h, 0) = 0 = f(0, h)$, we have

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0, \\ \frac{\partial f}{\partial y}(0,0) &= 0.\end{aligned}$$

Next, since $\partial_x f(0, h) = \frac{h}{2} - \frac{h^5}{h^4} = -\frac{h}{2}$ and $f_y(h, 0) = \frac{h}{2} - 0 = \frac{h}{2}$, we have

$$\begin{aligned}\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) &= \lim_{h \to 0} \frac{\partial_x f(0,h) - \partial_x f(0,0)}{h} = \lim_{h \to 0} \frac{-\frac{h}{2} - 0}{h} = -\frac{1}{2}, \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) &= \lim_{h \to 0} \frac{\partial_y f(h,0) - \partial_y f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h}{2} - 0}{h} = \frac{1}{2}.\end{aligned}$$

Thus, at $(0, 0)$ the partial derivative do not coincide.

Multiple Choice 2.1 True or false? Motivate your answers.

Consider the ODE $y''(x) - \sigma^2 y(x) = 0$, where $\sigma > 0$. Then:

		True	False
(a)	It has at least one bounded solution.	☐	☐
(b)	If y is a nonzero solution, then y is unbounded.	☐	☐
(c)	For any two values $a, b \in \mathbb{R}$, there is only one solution with $y(0) = a$ and $y(1) = b$.	☐	☐
(d)	There is no solution which is surjective from $\mathbb{R}$ to $\mathbb{R}$.	☐	☐

Solution.

		True	False
(a)	*It has at least one bounded solution.*	☒	☐
(b)	*If y is a nonzero solution, then y is unbounded.*	☒	☐
(c)	*For any two values $a, b \in \mathbb{R}$, there is only one solution with $y(0) = a$ and $y(1) = b$.*	☒	☐
(d)	*There is no solution which is surjective from $\mathbb{R}$ to $\mathbb{R}$.*	☐	☒

(a) *It has at least one bounded solution.*

$y \equiv 0$ is one such solution.

(b) *If y is a nonzero solution, then y is unbounded.*

The characteristic polynomial of the ODE is $\lambda^2 - \sigma^2$, so its roots are $\lambda = \pm\sigma$ and the general solution of the equation is then $y(x) = A\mathrm{e}^{\sigma x} + B\mathrm{e}^{-\sigma x}$. So if $A \neq 0$ y is unbounded at $+\infty$, and if $B \neq 0$ y is unbounded at $-\infty$.

(c) *For any two values $a, b \in \mathbb{R}$, there is only one solution with $y(0) = a$ and $y(1) = b$.*

If we impose the conditions on the general solution, we obtain

$$A + B = a \quad \text{and} \quad A\mathrm{e}^{\sigma} + B\mathrm{e}^{-\sigma} = b$$

and this system has a unique solution in (A, B) (for instance because the associated determinant is $e^{-\sigma} - \mathrm{e}^{\sigma} \neq 0$ for every σ).

(d) *There is no solution which is surjective from $\mathbb{R}$ to $\mathbb{R}$.*

It suffices to choose a solution with $A > 0 > B$: since it is continuous with limit $-\infty$ as $x \to -\infty$ and $+\infty$ as $x \to +\infty$, by the intermediate value theorem it is surjective from $\mathbb{R}$ to $\mathbb{R}$.

Multiple Choice 2.2 Choose the correct statement(s). Motivate your answers.

A particular solution of the ODE $y'' + 2y' + y = \mathrm{e}^{-x}$ is:

(a) $k\mathrm{e}^{-x}$ for some constant k. ☐

(b) $kx\mathrm{e}^{-x}$ for some constant k. ☐

(c) $kx^2\mathrm{e}^{-x}$ for some constant k. ☐

(d) $k\mathrm{e}^{-x^2}$ for some constant k. ☐

Solution. Directly plugging in the ODE each of the functions above, one sees that only (c) is correct (with $k = \frac{1}{2}$).

Exercise 2.1 A piece of mass (of mass m) connected to a coil spring that can stretch along its length. If $k > 0$ denotes the spring constant, the equation of motion of such system is given, according to Hooke and Newton's laws, by

$$m\ddot{x}(t) = -kx(t), \tag{$\dagger$}$$

where $x = x(t)$ denotes the position in time of the piece of mass along the vertical direction. Call $\omega = \sqrt{\frac{k}{m}}$. Find the solution of ($\dagger$):

(a) with initial position $x(0) = 1$ and initial velocity $\dot{x}(0) = 2\omega$.

(b) with initial position $x(0) = 1$ and final position $x(\frac{\pi}{2\omega}) = 1$.

(c) Is is possible to find a solution so that $x(t) \to -\infty$ as $t \to +\infty$?

Solution. Setting $\omega^2 = \frac{k}{m}$, we may rewrite the equation in normal form

$$\ddot{x} + \omega^2 x = 0.$$

The characteristic polynomial of the equation is $p(\lambda) = \lambda^2 + \omega^2$ whose roots are $\lambda_{1,2} = \pm i\omega$. Thus the general solution of the ODE is

$$x(t) = C_1\cos(\omega t) + C_2\sin(\omega t), \quad \text{for arbitrary } C_1, C_2 \in \mathbb{R}.$$

Consequently

(a) With $x(0) = 1$ and $\dot{x}(0) = 2\omega$ we get:

$$\begin{aligned} x(0) = 1 \quad &\Rightarrow \quad C_1\cos(0) + C_2\sin(0) = C_1 = 1, \\ \dot{x}(0) = 2\omega \quad &\Rightarrow \quad -C_1\omega\sin(0) + C_2\omega\cos(0) = C_2\omega = 2\omega. \end{aligned}$$

Hence it is $C_1 = 1$ and $C_2 = 2$. The required solution is then

$$x(t) = \cos(\omega t) + 2\sin(\omega t).$$

(b) With $x(0) = 1$ and $x(\frac{\pi}{2\omega}) = 1$ we get:

$$x(0) = 1 \quad \Rightarrow \quad C_1 \cos(0) + C_2 \sin(0) = C_1 = 1,$$
$$x\left(\frac{\pi}{2\omega}\right) = 1 \quad \Rightarrow \quad C_1 \cos\left(\frac{\pi}{2}\right) + C_2 \sin\left(\frac{\pi}{2}\right) = C_2 = 1.$$

Hence it is $C_1 = C_2 = 1$. The required solution is then

$$x(t) = \cos(\omega t) + \sin(\omega t).$$

(c) No, since every solution is periodic.

Exercise 2.2 It is observed that the populations of certain species of bacteria grow, when there is plenty of food and space, with a rate proportional to the number of present individuals. So, if $y(t)$ represents the size of the bacteria population with respect to time $t \geq 0$, it satisfies

$$\begin{cases} y'(t) = \kappa y(t) & \text{for } t > 0, \\ y(0) = y_0, \end{cases} \tag{$\circ$}$$

where y_0 represents the initial size of the population and $\kappa > 0$ is a constant determined by the biology of the bacteria in consideration.

(a) Find the solution of the problem ($\circ$). Looking at the solution, can you guess what is this kind of growth called?

(b) Suppose you are in a lab where the technology to observe the population delivers one picture per ε seconds, for some small $\varepsilon > 0$. Explain why ($\circ$) is replaced by

$$\begin{cases} \dfrac{y(t+\varepsilon) - y(t)}{\varepsilon} = \kappa y(t) & \text{for } t = 0, \varepsilon, 2\varepsilon, \ldots \\ \qquad\qquad y(0) = y_0, \end{cases} \tag{$\diamond$}$$

and find the solution of the problem ($\diamond$).

(c) How does the solution to ($\diamond$) behave as $\varepsilon \to 0$?

Solution. (a) The general solution of the differential equation (found for instance by separation of variables) is ce^{kt}, for arbitrary $c \in \mathbb{R}$. Since it must be $y(0) = y_0$, the solution to ($\circ$) is

$$y(t) = y_0 e^{kt}.$$

So, we have an exponential kind of growth.

(b) Since the derivative represents the infinitesimal rate of change of y, namely

$$y'(t) = \lim_{\varepsilon \to 0} \frac{y(t+\varepsilon) - y(t)}{\varepsilon},$$

in a model where we measure only discrete intervals of times we will replace y by the actual rate of change:

$$y'(t) \rightsquigarrow \frac{y(t+\varepsilon) - y(t)}{\varepsilon},$$

and consequently arrive to ($\diamond$). To solve the problem, note first that we may write

$$y(t+\varepsilon) = (k\varepsilon + 1)y(t) \quad \text{for every } t,$$

and hence, by iteration, with $y(0) = y_0$ obtain

$$\begin{aligned} y(t+\varepsilon) &= (k\varepsilon + 1)y(t) \\ &= (k\varepsilon + 1)^2 y(t-\varepsilon) \\ &= (k\varepsilon + 1)^3 y(t-2\varepsilon) \\ &\vdots \\ &= (k\varepsilon + 1)^{\frac{t}{\varepsilon}+1} y_0, \end{aligned}$$

that is

$$y(t) = (k\varepsilon + 1)^{\frac{t}{\varepsilon}} y_0.$$

(c) From one variable calculus, we see that

$$\lim_{\varepsilon \to 0} (k\varepsilon + 1)^{\frac{t}{\varepsilon}} = \lim_{x \to +\infty} \left(\frac{k}{x} + 1\right)^{tx} = \left(\lim_{x \to +\infty} \left(\frac{1}{x} + 1\right)^{x}\right)^{kt} = \mathrm{e}^{kt},$$

so we deduce that the discrete-time model approaches the continuous-time one as $\varepsilon \to 0$.

Exercise 2.3 Consider the differential equation

$$xy' = 2y - 3xy^2 \qquad \text{for } x > 0,$$

in the unknown $y = y(x)$.

(a) Rewrite the equation using the change of variable $y = \frac{1}{u}$.

(b) Solve the equation in the new variable u and then write the solution in the original variable y.

(c) There is one solution that cannot be obtained with the procedure (a) & (b). What is it, and why?

Solution. (a) With $y(x) = \frac{1}{u(x)}$, it is $y'(x) = -\frac{u'(x)}{u(x)^2}$, hence the equation becomes

$$-x\frac{u'(x)}{u(x)^2} = \frac{2}{u(x)} - \frac{3x}{u(x)^2} \quad \Longleftrightarrow \quad xu'(x) + 2u(x) = 3x.$$

(b) Since $x \neq 0$ the equation reads

$$u'(x) + \frac{2}{x}u(x) = 3.$$

To solve it, we multiply both hand-sides by $\exp\left(\int \frac{2}{x}dx\right) = \mathrm{e}^{2\log|x|} = x^2$, so to deduce

$$\frac{d}{dx}(x^2u(x)) = 3x^2,$$

and hence,

$$x^2u(x) = x^3 + c \quad \Longleftrightarrow \quad u(x) = \frac{x^3+c}{x^2},$$

where $c \in \mathbb{R}$ is an arbitrary constant, thus yielding

$$y(x) = \frac{x^2}{x^3+c} \qquad \text{for abitrary } c \in \mathbb{R}.$$

(c) To make sense, the change of variable $y = \frac{1}{u} \Leftrightarrow u = \frac{1}{y}$, needs y to be not identically 0. In fact however, we see that $y \equiv 0$ is also a solution.

Multiple Choice 12.1 Choose the correct statement. Motivate your answer.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous functions and let $B_r(0) \subset \mathbb{R}^n$ be the ball of radius $r > 0$ centered ad the origin. The integral

$$\int_{B_r(0)} f(x)\,dx$$

can also be written as

(a) $r^n \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$ ☐

(b) $\frac{1}{r^n} \int_{B_1(0)} f(rx)\,dx$ ☐

(c) $r^n \int_{B_1(0)} f(rx)\,dx$ ☐

(d) $\frac{1}{r^n} \int_{B_1(0)} f\left(\frac{1}{r}x\right) dx$ ☐

where we denoted $rx = (rx_1, \ldots, rx_n)$ and similarly for $\frac{1}{r}x$.

Solution. The correct choice is

(c) $r^n \int_{B_1(0)} f(rx)\,dx$ ☒

Indeed, the dilation that transforms $B_r(0)$ into $B_1(0)$ is defined by $y = \frac{1}{r}x$, hence the change of variables formula gives

$$y = rx \quad \Longrightarrow \quad dx_1 \cdots dx_n = r^n dy_1 \cdots dy_n.$$

Multiple Choice 12.2 Choose the correct statement. Motivate your answer.

Recall from one variable calculus that the improper integral

$$\int_{-1}^{1} \frac{1}{|x|^\alpha} dx$$

is convergent if and only if $\alpha < 1$. The following analogue in 3 dimensions:

$$\int_{B_1(0)} \frac{1}{|x|^\alpha} dx, \quad B_1(0) \subset \mathbb{R}^3,$$

is convergent if and only if

(a) $\alpha < 1$ ☐

(b) $\alpha < 3$ ☐

(c) $\alpha < \frac{3+1}{2} = 2$ ☐

(d) $\alpha < 3!$ ☐

Solution. The correct choice is

(b) $\alpha < 3$ ☒

Indeed, using spherical coordinates

$$x = r(\cos\vartheta\sin\varphi, \sin\vartheta\sin\varphi, \cos\varphi), \quad dx = r^2 \sin\varphi\, dr\, d\vartheta\, d\varphi$$

the integral transforms into

$$\begin{aligned}\int_{B_1(0)} \frac{1}{|x|^\alpha} dx &= \int_{B_1(0)} \frac{1}{r^\alpha} r^2 \sin\varphi\, dr\, d\vartheta\, d\varphi \\ &= \int_0^1 \int_0^{2\pi} \int_0^\pi \frac{1}{r^{\alpha-2}} \sin\varphi\, d\varphi\, d\vartheta\, dr \\ &= \int_0^1 \frac{1}{r^{\alpha-2}} dr \cdot C,\end{aligned}$$

where and $C > 0$ is a constant. From the one variable case, we deduce then that such expression is finite if and only if $\alpha - 2 < 1$, that is $\alpha < 3$.

Note: Polar coordinates ($n = 2$) and spherical coordinates ($n = 3$) have a generalization to any dimension n. In this case, the answer to the analogue question is $\alpha < n$.

Exercise 12.1 For each of the following integrals:

– Determine the domain of integration $\Omega \subset \mathbb{R}^2$,

– Rewrite them exchanging the integration order,

– Compute them using this new integration order.

(a) $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx,$ (b) $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx,$

(c) $\displaystyle\int_0^2 \int_x^2 2y^2 \sin(xy)\, dy\, dx,$ (d) $\displaystyle\int_0^{2\sqrt{\log(3)}} \int_{y/2}^{\sqrt{\log(3)}} e^{x^2}\, dx\, dy.$

Solution. (a) The integration domain is $\Omega = \{(x,y) : 0 \le x \le y \le \pi\}$. Thus

$$\begin{aligned}\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\, dy\, dx &= \int_0^\pi \int_0^y \frac{\sin(y)}{y}\, dx\, dy = \int_0^\pi \frac{\sin(y)}{y} \int_0^y 1\, dx\, dy \\ &= \int_0^\pi \frac{\sin(y) y}{y}\, dy = \int_0^\pi \sin(y)\, dy = [-\cos(y)]_{y=0}^\pi \\ &= 1 + 1 = 2.\end{aligned}$$

(b) The domain is $\Omega = \{(x,y) : 0 \le x \le 3 \text{ and } \sqrt{x/3} \le y \le 1\}$. Thus

$$\frac{x}{3} \le y^2 \le 1 \quad \text{i.e.} \quad x \le 3y^2 \le 3$$

and hence, we have equivalently

$$\Omega = \{(x,y) : 0 \le y \le 1 \text{ and } 0 \le x \le 3y^2\}$$

And the integral can be rewritten and computed as

$$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx = \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy = \int_0^1 e^{y^3} 3y^2\, dy = \left[e^{y^3}\right]_{y=0}^1 = e - 1.$$

(c) It is $\Omega = \{(x,y) : 0 \le x \le y \le 2\}$, hence

$$\begin{aligned}&\int_0^2 \int_x^2 2y^2 \sin(xy)\, dy\, dx \\ &= \int_0^2 2y \int_0^y y \sin(xy)\, dx\, dy = \int_0^2 2y[-\cos(xy)]_{x=0}^y\, dy \\ &= \int_0^2 2y(1 - \cos(y^2))\, dy = \int_0^2 2y\, dy - \int_0^2 2y \cos(y^2)\, dy \\ &= [y^2]_0^2 - [\sin(y^2)]_0^2 = 4 - (\sin(4) - \sin(0)) = 4 - \sin(4).\end{aligned}$$

(d) It is $\Omega = \{(x,y) : \frac{1}{2} y \le x \le \sqrt{\log(3)} \text{ and } 0 \le y \le 2\sqrt{\log(3)}\}$, which we may rewrite as

$$0 \le y \le 2x \quad \text{and} \quad 0 \le x \le \sqrt{\log(3)}.$$

Hence we get

$$\begin{aligned}\int_0^{2\sqrt{\log(3)}} \int_{y/2}^{\sqrt{\log(3)}} e^{x^2}\, dx\, dy &= \int_0^{\sqrt{\log(3)}} \int_0^{2x} e^{x^2}\, dy\, dx \\ &= \int_0^{\sqrt{\log(3)}} e^{x^2} \int_0^{2x} 1\, dy\, dx \\ &= \int_0^{\sqrt{\log(3)}} e^{x^2} 2x\, dx \\ &= \left[e^{x^2}\right]_0^{\sqrt{\log(3)}} = e^{\log(3)} - 1 \\ &= 3 - 1 = 2.\end{aligned}$$

Exercise 12.2 Let $K = Z_1 \cap Z_2$ be the intersection of the cylinders

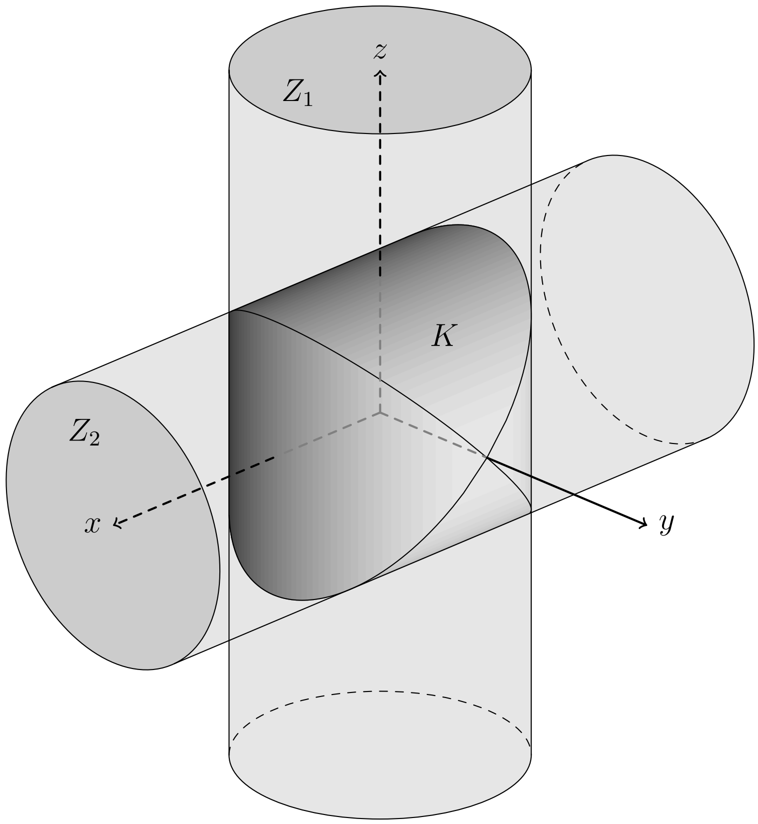
$$Z_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1\}, \qquad Z_2 = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 \leq 1\}.$$

(a) Draw K.

(b) Compute the *volume* of K.

(c) If the function $\rho\colon K \to \mathbb{R}$ given by $\rho(x, y, z) = 1 + x^2 + z^2$ represent the *mass density* of K, compute its *mass* $m(K) = \int_K \rho\, d\mu$.

Solution. (i) A sketch is as follows:



(ii) The condition defining K are given by $x^2 \leq 1 - y^2$ and $z^2 \leq 1 - y^2$. Hence

$$\begin{aligned} K &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 \leq 1 - y^2 \text{ and } z^2 \leq 1 - y^2\} \\ &= \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq y \leq 1 \text{ and } -\sqrt{1 - y^2} \leq x, z \leq \sqrt{1 - y^2}\}. \end{aligned}$$

The volume of K is then given by

$$\begin{aligned} \mathrm{Vol}(K) &= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1 \, dz \right) dx \right) dy \\ &= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1 \, dx \right) \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1 \, dz \right) dy \\ &= \int_{-1}^{1} \left(2\sqrt{1 - y^2} \right)^2 dy = 4 \int_{-1}^{1} (1 - y^2) \, dy = \frac{16}{3}. \end{aligned}$$

(iii) By the discussion above, we see that

$$
\begin{aligned}
m(K) = \int_K \rho \, d\mu &= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1 + x^2 + z^2 \, dz \right) dx \right) dy \\
&= \mathrm{Vol}(K) + \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 \, dz \right) dx \right) dy \\
&\qquad + \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} z^2 \, dz \right) dx \right) dy \\
&= \mathrm{Vol}(K) + \int_{-1}^{1} \left(2\sqrt{1-y^2} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2 \, dx \right) dy \\
&\qquad + \int_{-1}^{1} \left(2\sqrt{1-y^2} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} z^2 \, dz \right) dy \\
&= \mathrm{Vol}(K) + \frac{8}{3} \int_{-1}^{1} \left(1-y^2\right)^2 dy = \frac{16}{3} + \frac{8}{3} \int_{-1}^{1} (1 - 2y^2 + y^4) \, dy \\
&= \frac{8}{3} \left(2 + 2 - \frac{4}{3} + \frac{2}{5} \right) = \frac{368}{45},
\end{aligned}
$$

where, since ρ is continuous, we made use of the theorem of Fubini.

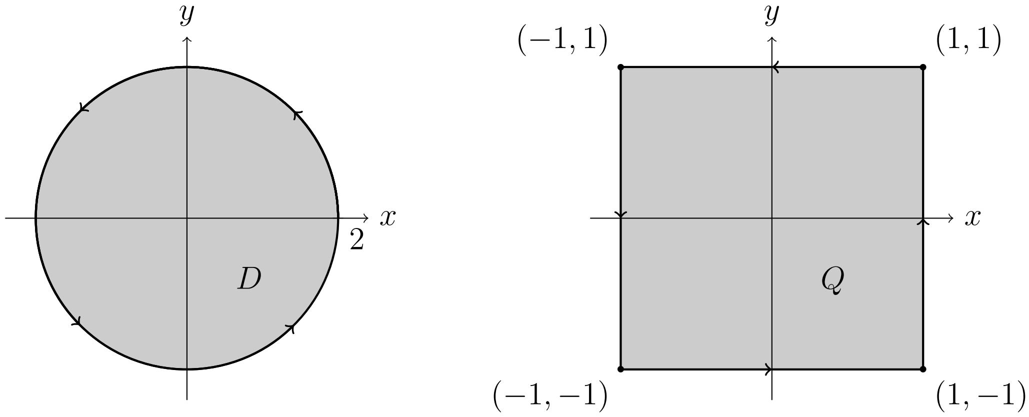
Exercise 12.3 The "curl" of a vector field in $\mathbb{R}^2$ is, by definition, the function

$$\mathrm{curl}(v) = \frac{\partial}{\partial x} v_2 - \frac{\partial}{\partial y} v_1.$$

Consider the vector field $v(x, y) = (y^2, x)$.

(a) Compute the line integral of v along the circle of radius 2 centered at the origin and along the square of vertices $(\pm 1, \pm 1)$, both oriented counter-clockwise (see the picture).

(b) Now compute the double integral of $\mathrm{curl}(v)$ over the disk D and the square Q enclosed by the curves in (b). What do you notice?



Solution. (a) Parametrizing the circle ∂D with $\gamma : [0, 2\pi) \to \partial D$, $\gamma(t) = 2(\cos\varphi, \sin\varphi)$, we see that

$$\int_{\partial D} v \cdot d\vec{s} = \int_0^{2\pi} \begin{pmatrix} 4(\sin\varphi)^2 \\ 2\cos\varphi \end{pmatrix} \cdot \begin{pmatrix} -2\sin\varphi \\ 2\cos\varphi \end{pmatrix} d\varphi$$
$$= \int_0^{2\pi} \Big(-8(\sin\varphi)^3 + 4(\cos\varphi)^2 \Big) d\varphi = 4\pi.$$

As for ∂Q, parametrizations for each side are given by

$$q_1(t) = (1-t)(1,1) + t(-1,1) = (1-2t, 1),$$
$$q_2(t) = (1-t)(-1,1) + t(-1,-1) = (-1, 1-2t),$$
$$q_3(t) = (1-t)(-1,-1) + t(1,-1) = (-1+2t, -1),$$
$$q_3(t) = (1-t)(1,-1) + t(1,1) = (1, -1+2t),$$

and the result is

$$\int_{\partial Q} v \cdot d\vec{s} = \sum_{j=1}^{4} \int_0^1 v(q_j(t)) \cdot q_j'(t)\, dt = 4.$$

(b) The curl of v is $\operatorname{curl}(v) = \dfrac{\partial}{\partial x} v_2 - \dfrac{\partial}{\partial y} v_1 = 1 - 2y$.

For D, using polar coordinates we compute

$$\begin{aligned}
\int_\gamma v\ d\gamma &= \int_D (1-2y)dxdy \\
&= \int_0^{2\pi}\int_0^2 (1-2r\sin(\varphi))r\ drd\varphi \\
&= \int_0^{2\pi}\int_0^2 r - 2r^2\sin(\varphi)\ drd\varphi \\
&= 2\pi\left[\frac{r^2}{2}\right]_0^2 - 2\int_0^2 r^2dr\int_0^{2\pi}\sin(\varphi)d\varphi = 4\pi,
\end{aligned}$$

and we see that it coincides with the line integral $\int_{\partial D} v\cdot d\vec{s}$.

As for Q, we see that

$$\begin{aligned}
\int_Q (1-2y)dxdy &= \int_{-1}^1\int_{-1}^1 (1-2y)\ dxdy \\
&= 2\int_{-1}^1 (1-2y)\ dy = 2(2-[y^2]_{y=-1}^1) = 4.
\end{aligned}$$

Once again this coincides with $\int_{\partial Q} v\cdot d\vec{s}$

Multiple Choice 8.1 True or False? Motivate your answers.

The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big), & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

at the point $(0,0)$ is:

		True	False
(a)	discontinuous	☐	☐
(b)	continuous	☐	☐
(c)	differentiable	☐	☐
(d)	C^1 (i.e. continuously differentiable).	☐	☐

Solution. It is

		True	False
(a)	discontinuous	☐	☒
(b)	continuous	☒	☐
(c)	differentiable	☒	☐
(d)	C^1 (i.e. continuously differentiable).	☐	☒

We are going to show that f is differentiable, and hence continuous, at $(0,0)$ but not of class C^1. For $(x,y) \neq (0,0)$ the partial derivatives are

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= 2x\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) + (x^2+y^2)\cos\Big(\frac{1}{\sqrt{x^2+y^2}}\Big)\cdot\frac{2x}{-2(x^2+y^2)^{\frac{3}{2}}} \\ &= 2x\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) - \frac{x}{\sqrt{x^2+y^2}}\cos\Big(\frac{1}{\sqrt{x^2+y^2}}\Big),\end{aligned}$$

$$\frac{\partial f}{\partial y}(x,y) = 2y\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) - \frac{y}{\sqrt{x^2+y^2}}\cos\Big(\frac{1}{\sqrt{x^2+y^2}}\Big),$$

that are clearly continuous away from $(0,0)$. To see that f is differentiable at $(0,0)$ we see that

$$\left|\frac{f(x,y)-f(0,0)}{\|(x,y)\|}\right| = \left|\frac{(x^2+y^2)\sin\Big(\frac{1}{\sqrt{x^2+y^2}}\Big) - 0}{\sqrt{x^2+y^2}}\right| \le \sqrt{x^2+y^2} \xrightarrow{(x,y)\to(0,0)} 0.$$

which implies f is differentiable with $df(0,0) = (0,0)$.

However, the partial derivatives of f are not continuous at $(0,0)$, since for $x > 0$ we have

$$\frac{\partial f}{\partial x}(x,0) = 2x\sin\left(\frac{1}{\sqrt{x^2}}\right) - \frac{x}{\sqrt{x^2}}\cos\left(\frac{1}{\sqrt{x^2}}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

The term $-\cos(\frac{1}{x})$ is divergent $(x,0) \to (0,0)$. Consequently, $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$ and f is not of class C^1.

Multiple Choice 8.2 Choose the correct statement. Motivate your answer.

Recall that a *critical point* of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is an $x_0 \in \mathbb{R}^n$ so that $df(x_0) = 0$. At such point, the tangent plane to the graph of f is:

(a) not defined ☐

(b) horizontal (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis) ☐

(c) vertical (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis) ☐

(d) none of the above, in general. ☐

Solution. The correct answer is

(b) horizontal (looking at $\mathbb{R}^3$ in the usual way with upward-pointing z-axis) ☒

Indeed, (see also Exercise 8.1 below) the equation of the plane is just

$$z = f(x_0) + df(x_0) \cdot (x,y) = f(x_0)$$

which means that it is parallel to the x-y plane, and therefore horizontal.

Exercise 8.1 Let $\mathcal{G} = \{(x, y, f(x,y)) : (x,y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$ be the graph of the function

$$f \colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^{-(x^2+y^2-2x+3y+2)}.$$

(a) Find the equation of the tangent plane E to $\mathcal{G}$ at the point $(0,0,e^{-2})$, both in *Cartesian form*, i.e. with an equation:

$$E = \{(x,y,z) \in \mathbb{R}^3 : \text{“equation in } x,y,z\text{”}\},$$

and in *parametric form* i.e. with a function:

$$\varphi \colon \mathbb{R}^2 \to E \subset \mathbb{R}^3, \quad \varphi(s,t) = \big(x(s,t), y(s,t), z(s,t)\big).$$

(b) Use a plotting software of your choice to verify that φ actually plots a plane that is tangent to $\mathcal{G}$ as above.

(c) Find all the points in $\mathcal{G}$ where the tangent plane is parallel to the x-y plane $\Pi = \{(x, y, 0) : (x, y) \in \mathbb{R}^2\}$.

Solution. Recall that, in general, the tangent plane for $\mathcal{G}$ at (x_0, y_0) is given by

$$E = \left\{(x, y, A(x, y)) : (x, y) \in \mathbb{R}^2\right\},$$

where A is the affine approximation of f:

$$\begin{aligned} A(x, y) &= f(x_0, y_0) + df(x_0) \cdot (x, y) \\ &= f(x_0, y_0) + \frac{\partial}{\partial x} f(x_0, y_0)\,(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0). \end{aligned}$$

consequently, a paremetrization and a Cartesian equation E are

$$\varphi(s, t) = (s, t, A(s, t)), \quad \text{and} \quad z = A(x, y).$$

(a) The partial derivatives of f are

$$\frac{\partial f}{\partial x}(x, y) = (-2x + 2)\, f(x, y), \qquad \frac{\partial f}{\partial y}(x, y) = (-2y - 3)\, f(x, y),$$

consequently, we have $df(0, 0) = (2e^{-2}, -3e^{-2})$ and the affine approximation of f is

$$A(x, y) = f(0, 0) + df(0, 0) \cdot (x, y) = e^{-2} + 2e^{-2}x - 3e^{-2}y.$$

Thus, it is

$$\varphi(s, t) = (s, t, e^{-2} + 2e^{-2}s - 3e^{-2}t)$$

and

$$E = \{(x, y, z) : z = e^{-2} + 2e^{-2}x - 3e^{-2}y\}.$$

(b) This is left to the student.

(c) The tangent plane in (x_0, y_0) is parallel to Π if and only if A is constant, hence if and only if $df(x_0, y_0) \equiv 0$. From the computation in (a), since $f > 0$ this means that $-2x_0 + 2 = 0$ and $2y_0 - 3 = 0$, i.e. $(x_0, y_0) = (1, -\frac{3}{2})$. Consequently, E is parallel to Π at the point

$$\left(1, -\tfrac{3}{2}, f(1, -\tfrac{3}{2})\right) = \left(1, -\tfrac{3}{2}, e^{\frac{5}{4}}\right) \in \mathcal{G}.$$

Exercise 8.2 Consider the function

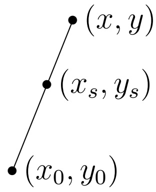
$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = e^x \sin(y).$$

(a) Compute the Taylor polynomials of 1st and 2nd order of f at $(x_0, y_0) = (0, \frac{\pi}{2})$; approximate with each of them the value of f at $(x_1, y_1) = (0, \frac{\pi}{2} + \frac{1}{4})$. Compare the results approximating numerically the value of $f(x_1, y_1)$ with a software of your choice.

(b) Similarly as for the one variable case, one can prove that if a function is C^2, one can write

$$f(x) = f(x_0) + df(x_0) \cdot (x - x_0) + R_1 f(x,y),$$

where $R_1 f$ is the *rest*, given by



$$\begin{aligned} R_1 f(x,y) = {} & \frac{1}{2}\frac{\partial^2 f}{\partial x \partial x}(x_s, y_s) \cdot (x - x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial y}(x_s, y_s) \cdot (y - y_0)^2 \\ & + \frac{\partial^2 f}{\partial x \partial y}(x_s, y_s) \cdot (x - x_0)(y - y_0) \end{aligned}$$

where $(x_s, y_s) = \Big(x_0 + s(x - x_0), y_0 + s(y - y_0)\Big)$ for some $s \in [0,1]$ (see e.g. Satz 7.5.2 of Struwe's script).

With this information, quantify how precise in the linear approximation in the ball $B_{\frac{1}{4}}(0, \frac{\pi}{2})$ by giving an upper bound for the corresponding error.

Solution. (a) The partial derivatives of f are $f(x,y) = e^x \sin(y)$ sind

$$\begin{aligned} &\frac{\partial f}{\partial x}(x,y) = e^x \sin(y), \quad && \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = e^x \sin(y), \quad && \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y) = e^x \cos(y), \\ &\frac{\partial f}{\partial y}(x,y) = e^x \cos(y), \quad && \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y) = e^x \cos(y), \quad && \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = -e^x \sin(y). \end{aligned}$$

Consequently, at $(x_0, y_0) = (0, \frac{\pi}{2}) \in \mathbb{R}^2$ we have

$$\begin{aligned} &\frac{\partial f}{\partial x}(0, \tfrac{\pi}{2}) = 1, \qquad && \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(0, \tfrac{\pi}{2}) = 1, \qquad && \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0, \tfrac{\pi}{2}) = 0, \\ &\frac{\partial f}{\partial y}(0, \tfrac{\pi}{2}) = 0, \qquad && \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0, \tfrac{\pi}{2}) = 0, \qquad && \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(0, \tfrac{\pi}{2}) = -1. \end{aligned}$$

Since moreover $f(0, \frac{\pi}{2}) = 1$ sowie, the taylor polynomials are

$$\begin{aligned} T_1 f(x,y) &= f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) \\ &= 1 + x, \\ T_2 f(x,y) &= T_1 f(x,y) + \frac{1}{2}\frac{\partial^2 f}{\partial x \partial x}(x_0,y_0)\cdot(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial y}(x_0,y_0)\cdot(y-y_0)^2 \\ &\quad + \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\cdot(x-x_0)(y-y_0) \\ &= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}\Big(y - \frac{\pi}{2}\Big)^2. \end{aligned}$$

Evaluating them at $(0, \frac{\pi}{2} + \frac{1}{4})$ gives

$$T_1 f(0, \tfrac{\pi}{2} + \tfrac{1}{4}) = 1, \qquad T_2 f(0, \tfrac{\pi}{2} + \tfrac{1}{4}) = 1 - \tfrac{1}{2}(\tfrac{1}{4})^2 = \tfrac{31}{32} = 0.96875.$$

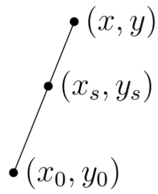
The value approximated numerically by a program is

$$f(0, \tfrac{\pi}{2} + \tfrac{1}{4}) = \sin(\tfrac{\pi}{2} + \tfrac{1}{4}) \approx 0.96891,$$

so the approximation of T_2 is relatively good.

(b) Since $f(x,y) = T_1 f(x,y) + R_1 f(x,y)$, where the rest $R_1 f$ is given by

$$\begin{aligned} R_1 f(x,y) = &\frac{1}{2}\frac{\partial^2 f}{\partial x \partial x}(x_s,y_s)\cdot(x-x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y \partial y}(x_s,y_s)\cdot(y-y_0)^2 \\ &+ \frac{\partial^2 f}{\partial x \partial y}(x_s,y_s)\cdot(x-x_0)(y-y_0) \end{aligned}$$



where $(x_s,y_s) := \Big(x_0 + s(x-x_0), y_0 + s(y-y_0)\Big)$ for some $s \in [0,1]$. To estimate this rest, we see that

$$|\partial^2_{xx} f(x,y)|, |\partial^2_{xy}(x,y)|, |\partial^2 f_{yy}(x,y)| \le e^x,$$

since $|\sin x| \le 1$ und $|\cos x| \le 1$. So for $(x_0,y_0) = (0, \frac{\pi}{2})$ and $(x,y) \in B_{\frac{1}{4}}(x_0,y_0)$ we have $|x - x_0| \le \frac{1}{4}$, $|y - y_0| \le \frac{1}{4}$ and $x_0 + s(x-x_0) \le \frac{1}{4}$, so

$$|R_1(x,y)| \le 4 \cdot \left(\frac{1}{2}e^{\frac{1}{4}} \cdot \left(\frac{1}{4}\right)^2\right) = \frac{1}{8}e^{\frac{1}{4}} \approx 0.1605.$$

Exercise 8.3 Compute the Taylor polynomials of the following functions at the given point and of the given order.

(a) $f(x,y) = \frac{1}{1-xy}$, at $(0,0)$, $2n-$th order with $n \geq 1$.

(b) $f(x,y) = \arctan(x^2 y)$, at $(0,0)$, 2nd order.

(c) $f(z) = \log(|z|^2+1)$ ($z \in \mathbb{C} \simeq \mathbb{R}^2$), at $z=0$, $2n-$th order with $n \geq 1$.

(d) $f(x_1,\dots,x_n) = \prod_{i=1}^n x_i$, at $x_0 = (2,\dots,2)$ 2nd order.

Solution. All the polynomials can be computed directly by working out each of the partial derivatives; we try to give below some alternative methods. We denote with $T_n f$ the required polynomial.

(a) Recall that for $|b| < 1$ one has the geometric series formula: $\frac{1}{1-b} = \sum_{n=0}^{\infty} b^n$. Consequently, we can write, as $(x,y) \to (0,0)$,

$$\frac{1}{1-xy} = \sum_{n\in\mathbb{N}} (xy)^n = \sum_{n=0}^{N} (xy)^n + \sum_{n=N+1}^{\infty} (xy)^n = \sum_{n=0}^{N} (xy)^n + o(|(x,y)|^{2N}),$$

consequently, it has to be, for every $n \in \mathbb{N}$,

$$T_{2n} f(x,y) = 1 + \sum_{k=1}^{n} (xy)^k.$$

(b) Recall that $\arctan(t) = t + O(t^3)$, so $\arctan(x^2 y) = x^2 y + O(x^6 y^3)$ and in particular

$$T_2 f(x,y) = 0.$$

(c) Recall that for any $|t| < 1$ we have

$$\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k,$$

consequently $|z| < 1$

$$\log(1+|z|^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} |z|^{2k},$$

and thus for $z = x+iy \in \mathbb{C} \simeq \mathbb{R}^2$ we have

$$T_{2n} f(z) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} |z|^{2k} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (x^2+y^2)^k.$$

(d) Note that for any $i \neq j$ we have

$$\partial_{x_i} f(x) = \prod_{k \neq i} x_k, \quad \partial^2_{x_i} f(x) = 0, \quad \partial^2_{x_i, x_j} f(x) = \prod_{k \neq i,j} x_k,$$

hence, for $1 \leq i \neq j \leq n$,

$$f(2) = 2^n, \quad \partial_{x_i} f(2) = 2^{n-1}, \quad \partial^2_{x_i,x_j} f(2) = 2^{n-2},$$

Thus writing $x_0 = (2, \ldots, 2)$ and $y = x - x_0$ we have

$$\begin{aligned} T_2 f(x) &= f(x_0) + df(x_0) \cdot y + \frac{1}{2} y \cdot \mathrm{Hess} f(x_0) \cdot y \\ &= 2^n + 2^{n-1} \sum_{i=1}^n y_i + 2^{n-2} \sum_{1 \leq i < j \leq n} y_i y_j. \end{aligned}$$

(In the terms of order 2, we have a coefficient $\frac{1}{2}$ in Taylor formula, but we have two symmetric terms $\partial_i \partial_j$ and $\partial_j \partial_i$ for $1 \leq i < j \leq n$. We combine this 2 terms and end up with $(\frac{1}{2} + \frac{1}{2}) \partial^2_{x_i,x_j} f(2)$.)