Equations for MAS-413
Machine Technical Systems (SI - Units)

Assembled By
Graduate Student

Strandene, Jan Christian

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Beta Version

Department of Engineering

Faculty of Engineering and Science

Abstract

This booklet is written as a collection of assorted formulas, with no definite sources. The booklet may be freely shared without further consent by the author, but may not be published or in any way shared with any monetary- exchange.

All formulas and Equations in this booklet are based on SI-units.

Version note: The booklet is in a draft-phase, and is written speedily in preparation for exams. Therefore, there may be grammatical errors and inconsistencies throughout. There may also be some parts missing.

For a more in dept understanding of the formulas and equations in this booklet, I can recommend the book **"Machine Component Design"** by Robert C. Juvinall and Kurt M. Marshek. Isbn : 978-1-118-09226-2 (International student version)

Contents

1 Basics

1.1 Velocity

Angular and Peripheral velocity :

$$\omega = 2 \cdot \pi \cdot f = 2 \cdot \pi \cdot \frac{n}{60} \tag{1.1}$$
$$v = 2 \cdot \pi \cdot r \cdot \frac{n}{60} \tag{1.2}$$

Were:

f	=	Frequency	[Hz]
n	=	Rotations	[RPM]
r	=	Radius to the periphery	[m]

1.2 Forces and Moments

$$\Sigma F = m \cdot x'' = m \cdot a \tag{1.3}$$
$$M = F \cdot \Delta x \tag{1.4}$$

Were :

F	=	Force	[N]
M	=	Moment, Torque	[Nm]
x'', a	=	Acceleration	[m/s^2]
Δx	=	Distance to were force is applied	[m]

1.2.1 Equilibrium

If a system is to be considered stationary Forces and Moment have to equal 0

$$\Sigma F_{x,y,z} = 0 \tag{1.5}$$
$$\Sigma M = 0 \tag{1.6}$$

Forces to be considered in any of the axis and moment in chosen point

1.3 Stresses

1.3.1 Normal and Bending

$$\sigma_n = \frac{F}{A} \tag{1.7}$$

$$\sigma_b = \frac{M_b \cdot y}{I} \tag{1.8}$$

Were :

F	=	Force	[N]
M	=	Moment, Torque	[Nm]
σ_n	=	Normal Stress	[Pa]
σ_b	=	Bending Stress	[Pa]

1.3.2 Shear-stress

$$\tau = \frac{V \cdot s}{I \cdot t} \tag{1.9}$$

Solid Round section (center) :

$$\tau = \frac{4 \cdot V}{3 \cdot A} \tag{1.10}$$

Solid Rectangular section (Center) :

$$\tau = \frac{3 \cdot V}{2 \cdot A} \tag{1.11}$$

Thin-walled Round Section (center) :

$$\tau = \frac{2 \cdot V}{A} \tag{1.12}$$

Which ia applicable for were the wall is $\approx 10\%$ of the outer diameter or less
The shear-stress is larges at the center of the Cross-Section, and lowers at the periphery of the Cross-Section

Where:

V	=	Shear Force	[N]
s	=	First Moment of Area	
I	=	Moment of Inertia	$[mm^4]$
t	=	Width of the area where the stress is found	$[mm^2]$
A	=	Area	$[mm^2]$
τ	=	Shear-stress	[Pa]

1.3.3 Torsional Stress

$$\tau = \frac{T \cdot R}{J} \tag{1.13}$$

Were :

T	=	Applied Torque	[Nm]
R	=	Radius of the Cross-Section	[mm]
J	=	Polar Moment of Area	$[mm^4]$

1.3.4 von Mises (Equivalent Stress)

2D-Mechanics :

$$\sigma_{ea} = \frac{1}{2}\sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \cdot \sigma_y + 3\tau_{xy}^2} \tag{1.14}$$

3D-Mechanics

$$\sigma_{ea} = \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2) + \tau_{yz}^2 + \tau_{zx}^2} \tag{1.15}$$

NOTE: For some cases, it's possible to make the shear-stress negligible, because the stress is highest at the center of the cross-section, whereas the equivalent stress is commonly greatest at the periphery.

2 | General Force Analysis

2.1 Force in Gears effect on shafts

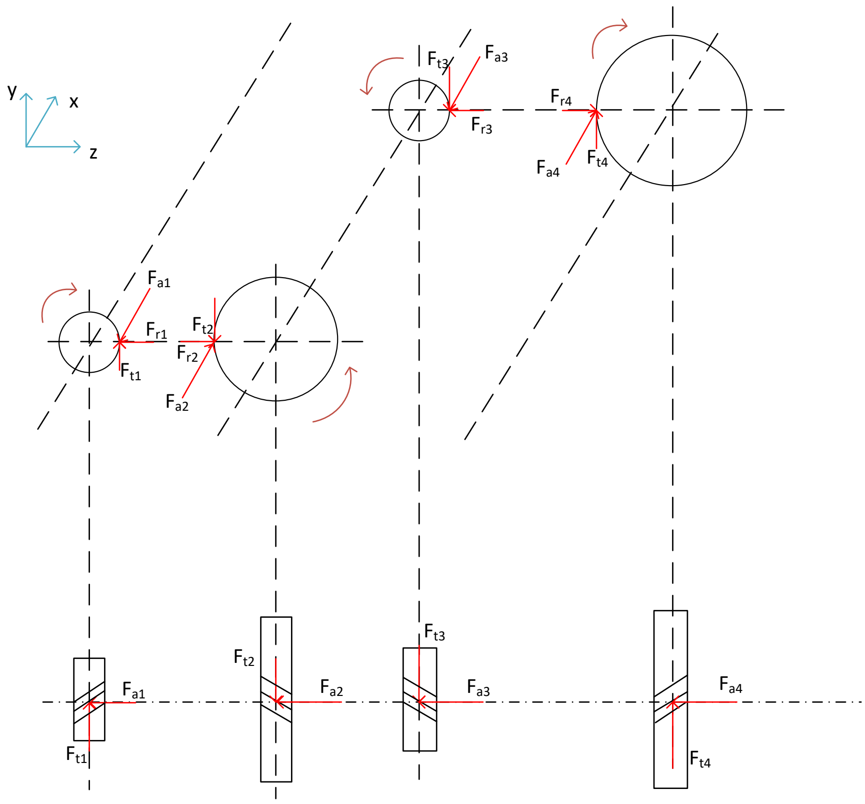
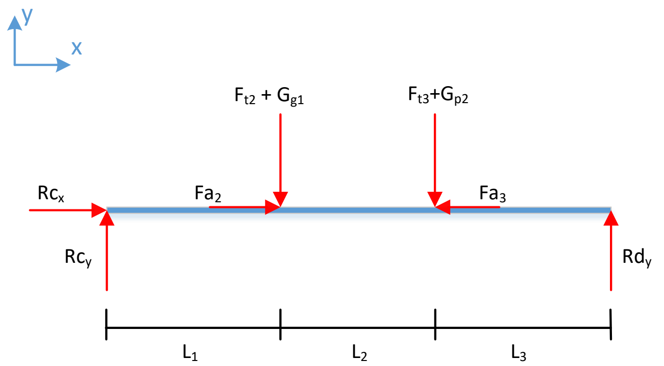


Figure 2.1: Exampe of gearbox configuration :
From left to right : Shaft A, Shaft B, Shaft C

Example on Force analysis of the forces in a Shaft. Here are shaft B used as an example



$$\overset{\curvearrowright}{\Sigma M_C} = (F_{t2}+G_{g1})\cdot(L1) + (F_{t3}+G_{p2})\cdot(L1+L2) - R_{dy}\cdot(L1+L2+L3) = 0$$

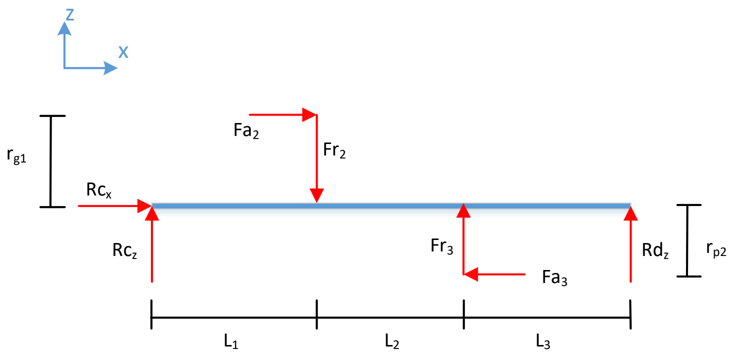
$$R_{dy} = \frac{(F_{t2}+G_{g1})\cdot(L1) + (F_{t3}+G_{p2})\cdot(L1+L2)}{(L1+L2+L3)} \tag{2.1}$$

$$\Sigma F_y = R_{cy} - F_{f2} - F_{t3} + R_{dy} - G_{g1} - G_{g2} = 0$$

$$R_{cy} = F_{t2} + F_{t3} - R_{dy} + G_{g1} + G_{p2} \tag{2.2}$$

$$\Sigma F_x = R_{cx} + F_{a2} - F_{a3} = 0$$

$$R_{cx} = F_{a3} - F_{a2} \tag{2.3}$$



$$\overset{\curvearrowright}{\Sigma M_C} = F_{r2}\cdot(L1) + F_{a2}\cdot\left(\frac{D_{p2}}{2}\right) - F_{r3}\cdot(L1+L2) + F_{a3}\cdot\left(\frac{D_{p3}}{2}\right) - F_{dz}(L1+L2+L3) = 0$$

$$R_{dz} = \frac{F_{r2}\cdot(L1) + F_{a2}\cdot\left(\frac{D_{p2}}{2}\right) - F_{r3}\cdot(L1+L2) + F_{a3}\cdot\left(\frac{D_{p3}}{2}\right)}{(L1+L2+L3)} \tag{2.4}$$

$$\Sigma F_Z = R_{cz} - F_{r2} + F_{r3} + R_{dz} = 0$$

$$R_{cz} = F_{r2} - F_{r3} - R_{dz} \tag{2.5}$$

3 | Spur and Helical Gears

3.1 Number of Teeth

The factorized number of teeth in meshing gears should ideally not have a higher common number than 1. Choosing prime numbers are a good way to start.

Some prime numbers : 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199

$$Z = Dp \cdot m \tag{3.1}$$

Where:

m	=	module	[mm]
Dp	=	Pitch Diameter	[mm]
Z	=	number of teeth	

3.1.1 Module

Spur

$$m = \frac{Dp}{Z} \tag{3.2}$$

$$Dp = m \cdot Z \tag{3.3}$$

Helical

$$Dp = \frac{m \cdot Z}{cos(\beta)} \tag{3.4}$$

Where:

m	=		module	[mm]
Dp	=		Pitch Diameter	[mm]
Z	=		number of teeth	
$cos(\beta)$	=	Helical angle	[deg, rad]	

Module correction for Helical Gears

$$m = \frac{m_n}{cos(\beta)} \tag{3.5}$$

Where:

m	=	module	[mm]
m_n	=	Normal module (spur gear)	[mm]
β	=	helical angle	[deg]

3.2 Gear Ratio

$$\frac{\omega_p}{\omega_g} = -\frac{d_g}{d_p} \tag{3.6}$$

Where:

ω	=	Angular Velocity	
d	=	Pitch Diameter	[mm]
p	=	Pinion(small cog)	
g	=	Gear (big cog)	

- sign becouse of change of rotation

$$\begin{gathered} i_{tot} = i_1 \cdot i_2 \cdot ... \cdot i_n \\ i = \frac{Z_g}{Z_p} = \frac{N_g}{N_p} = \frac{d_g}{d_p} = \frac{n_p}{n_g} = \frac{\omega_p}{\omega_g} \end{gathered} \tag{3.7}$$

Where:

ω	=	Angular Velocity	
n	=	rotational speed	[RPM]
d	=	Pitch Diameter	[mm]
r	=	Pitch Radius	[mm]
Z	=	Number of teeth	
N	=	Number of teeth	
i	=	Gear ratio	
i_{tot}	=	Total Gear Ratio	

3.3 Distance Between Gears and CR-factor

$$CR = \left(\frac{b \cdot tan(\beta)}{Pt}\right)^{(*)} + \frac{\sqrt{r_{ap}^2 - r_{bp}^2} + \sqrt{r_{ag}^2 - r_{bg}^2} - c \cdot sin(\alpha)}{Pb} \tag{3.8}$$

$(*)$ = Factor for helical gears. NOT used in spur gears

$$P_t = \frac{\pi \cdot m}{cos(\beta)} \tag{3.9}$$

$$P_b = \frac{\pi}{Z} \cdot D_p \tag{3.10}$$

$$r_a = \frac{D_p}{2} + h_t \tag{3.11}$$

$$r_b = \frac{D_p}{2} - h_f \tag{3.12}$$

$$h_t = 1 \cdot m \tag{3.13}$$

$$h_f = 1.25 \cdot m \tag{3.14}$$

$$C = \frac{d_p + d_g}{2} = r_p + r_g \tag{3.15}$$

Max possible Addendum:

$$r_{a,max} = \sqrt{r_b^2 + C^2 \cdot sin^2(\phi)} \tag{3.16}$$

Where:

d	=	Pitch Diameter	[mm]
r	=	Pitch Radius	[mm]
p	=	Pinion(small cog)	
g	=	Gear (big cog)	
C	=	Center distance	[mm]
β	=	Helical angle	[deg]
P_t	=	Transverse Circular Pitch	[mm]
P_b	=	Base Pitch	[mm]
r_a	=	Addendum Radius	[mm]
r_b	=	Dedendum Radius	[mm]
α	=	Pressure angle	[deg]
ϕ	=	Pressure angle	[deg]
m	=	module	[mm]
Z	=	Number of teeth	
D_p	=	Pitch Diameter	[mm]
h_t	=	Distance from pitch radius to addendum	[mm]
h_f	=	Distance from pitch radius to dedendum	[mm]

3.4 With of Gear

$$b = \lambda \cdot m \tag{3.17}$$

Where:

m	=	module	[mm]
λ	=	with factor	
b	=	With of gear	

With of gear is not standardized but is normally between $\lambda = 9 and 14$

3.5 Tooth Profile

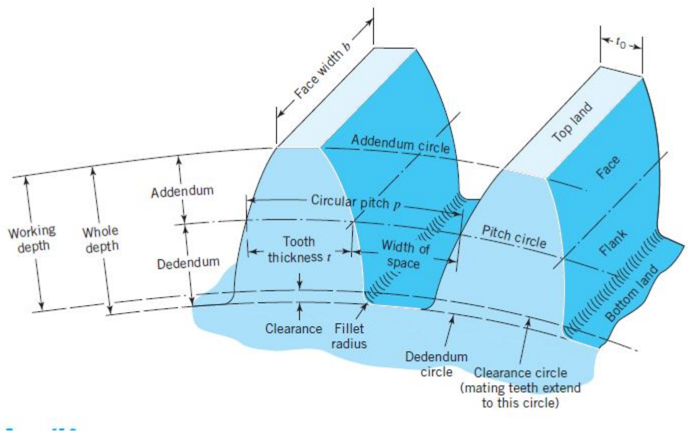


Figure 3.1: [1]

F_t	=	Tangential Force Component	[N]
F_r	=	Radial Force Component	[N]
F_a	=	Axial Force Component	[N]
ϕ	=	Angel of tooth profile	[deg]
β	=	Angle of Helical angle	[deg]

ϕ is normally 20°

$$h_t = 1 \cdot m \tag{3.18}$$
$$h_f = 1.25 \cdot m \tag{3.19}$$
$$h = 2.25 \cdot m \tag{3.20}$$
$$t = \frac{P}{2} - 0.5 \cdot m \tag{3.21}$$
$$\text{with of space} = \frac{p}{2} + 0.5 \cdot m \tag{3.22}$$

Where :

h_t	=	Hight from pitch diameter to top (addendum)	[mm]
h_f	=	Hight from pitch diameter to bottom (didendum)	[mm]
h	=	Whole Dept of tooth	[mm]
m	=	module of gear	[mm]
t	=	Thickness of tooth	[mm]
p	=	pitch diameter	[mm]

Pitch Diameter	$d = m \cdot N$
Addendum	$d_t = m \cdot N + 2 \cdot h_t$

3.6 Force Analysis

3.6.1 Basic Force equations

$$r = \frac{m \cdot Z}{2} \tag{3.23}$$

$$F_r = F_t \cdot tan(\phi) \tag{3.24}$$

$$v = \frac{\pi \cdot d \cdot n}{60} \tag{3.25}$$

$$P = F_t \cdot v \tag{3.26}$$

$$\eta_{tot} = \frac{P_{out}}{P_{in}} = \frac{T_{out}}{T_{in} \cdot i_{tot}} \tag{3.27}$$

$$F_t = \frac{P}{v} = \left[\frac{Trasmittedpower[W]}{Rotationalspeed[\frac{m}{s}]}\right] \tag{3.28}$$

$$F_t = \frac{T}{r} = \left[\frac{Torque[Nm]}{Pitchradius[m]}\right] \tag{3.29}$$

$$T_{in} = \frac{P \cdot \eta}{v} = \frac{P \cdot \eta \cdot 60}{2\pi \cdot n} \tag{3.30}$$

For helical gears :

$$F_t = \frac{T}{r} = \frac{P \cdot \eta}{\omega \cdot r} \tag{3.31}$$

$$r = \frac{m \cdot Z}{2 \cdot cos(\beta)} \tag{3.32}$$

$$\omega = \frac{2\pi \cdot n}{60} \tag{3.33}$$

$$F_r = F_t \cdot \frac{tan(\alpha)}{cos(\beta)} \tag{3.34}$$

$$F_a = F_t \cdot tan(\beta) \tag{3.35}$$

T	=	Torque	[Nm]
ϕ, α	=	pressure angle	[deg]
β	=	Helical angle	[deg]
F_t	=	Tangential Force	[Nm]
F_r	=	Radial Force	[Nm]
F_a	=	Axial Force	[Nm]
v	=	Rotational speed	[$\frac{m}{s}$]
d	=	Pitch diameter	[m]
n	=	Rotational speed	[RPM]
η, k	=	Load factor	
ω	=	Angular velocity	[rad/s]
P	=	Power	[Watt]

3.6.2 Stress in Tooth

The tooth have stress concentrations on the top of the tooth and in the base angles. F_t is giving a bending stress while F_r gives a compression stress. These stresses have to be combined to find the resulting stress σ_{res}, σ_{tot}

3.6.3 Bending Stress

$$\sigma_b = \frac{F_{tn} \cdot k_a \cdot k_v}{b \cdot m} \cdot \gamma \tag{3.36}$$

$$F_{tn} = \frac{T}{r} \tag{3.37}$$

$$k_v = \frac{A + v}{A} \tag{3.38}$$

Where :

F_{tn}	=	Theoretical tangential Force	[N]
T	=	Torque in shaft	[Nm]
r	=	Pitch Radius	[m]
k_a	=	External dynamic load factor (Tab3.1b)	
k_v	=	Inner dynamic factor from the speed of the gear (Eq: 3.38)	
A	=	Operating factor (Tab3.1a)	[m/s]
v	=	Peripheral speed	[m/s]
b	=	Width of the tooth	[mm]
m	=	Module of the gear	[mm]
γ	=	Tooth form-factor (Tab3.2 or Fig3.2)	

Table 3.1: Operational and Dynamic factors

(a) Operations-Factor (A)

Type of Cog	A
Highly accurately machined cog	10
Accurately machined cog	5
Normal cog in open operation	3

(b) External Dynamic Load-Factor

Running-hours per day	**No Jerks**	**Light Jerks**	**Havy Jerks**
8 - 10	1	1.25	1.75
2 - 3	0.8	1	1.5

Table 3.2: Tooth-form-factor

Number of teeth	γ	**Number of teeth**	γ
10	N/A	40	2.45
20	2.90	60	2.30
25	2.73	80	2.24
30	2.60	100	2.21

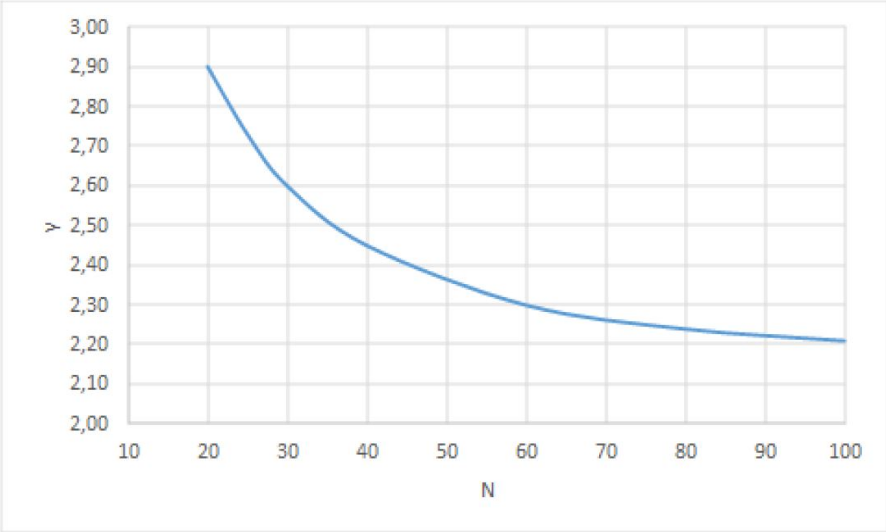


Figure 3.2: Tooth-Form-Factor. Fig made by Jørgen Bruun Tørnby

Maximum bending stress :

$$\sigma_b = \frac{\sigma_{b,lim}}{v_b} \tag{3.39}$$

Where :

$\sigma_{b,lim}$	=	The maximum allowable stress for unlimited lifetime (Tab 4.12)
v_b	=	Safety factor (normally 1.7)

3.6.4 Contact Stress

$$\sigma_o = f_w \cdot f_c \cdot \sqrt{\frac{F_{tn} \cdot k_a \cdot k_v}{b \cdot d_1}\left(\frac{i+1}{i}\right)} \tag{3.40}$$

$$\sigma_b = b \cdot d_1 \tag{3.41}$$

$$f_w = \sqrt{0.35E} = 271 \tag{3.42}$$

$$f_c = 1.76 \tag{3.43}$$

Where :

f_w	=	Material factor	$[N/mm^2]$
f_c	=	Edge-form factor (involute gears $\alpha = 20° \Rightarrow 1.76$)	
d_1	=	Pitch Diameter of the pinion	
i	=	Gear ratio	
E	=	Modulus of Elasticity	

Maximum Contact stress :

$$\sigma_o = \frac{\sigma_{o,lim}}{V_o} \cdot k_L \cdot Z_v \tag{3.44}$$

Velocity-Factor

Z_v an be found by using by Table3.3 or Fig3.3

Table 3.3: Velocity-Factor

Peripheral Velocity$[m/s]$	Z_v	Peripheral Velocity$[m/s]$	Z_v
0.25	0.835	6	0.960
1	0.842	7	0.980
2	0.855	8	1.000
3	0.877	9	1.015
4	0.905	10	1.033
5	0.932	12	1.058

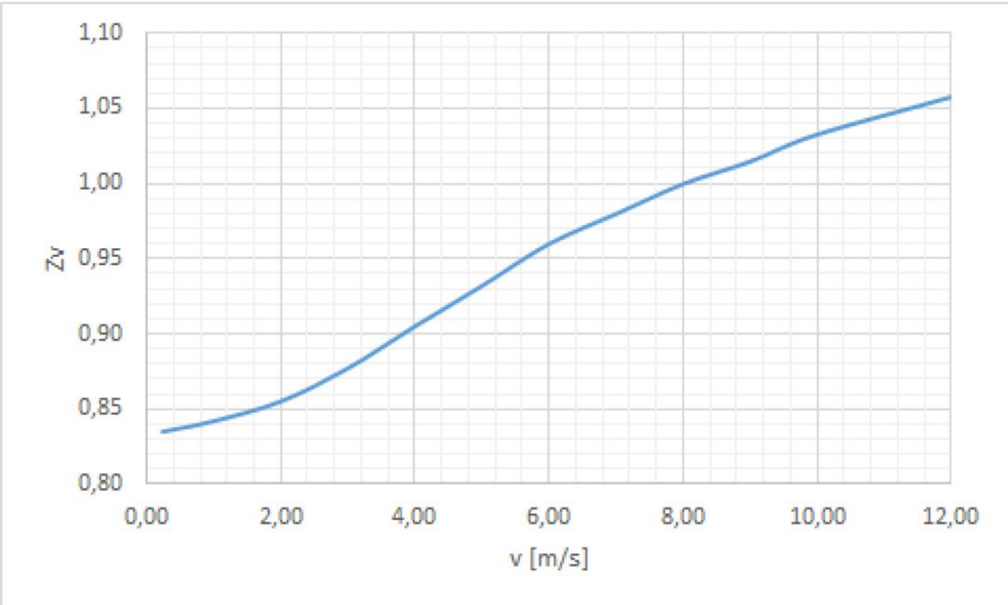


Figure 3.3: Velocity-factor. Fig made by Jørgen Bruun Tørnby

Where :

$\sigma_{o,lim}$	=	The maximum allowable contact stress for unlimited lifetime (Tab 4.12)
V_o	=	Safety factor (normally 1.25)
k_L	=	Lubrication Factor. Dependent on the viscosity of the oil (tab 4.10)
Z_v	=	Speed factor. Dependent on the peripheral speed of the gears (tab 4.11)

3.6.5 Module From Stresses

Dependent on Bending Stress

$$m = 1.26 \cdot \sqrt[3]{\frac{T \cdot k_a \cdot k_v}{Z \cdot \lambda \cdot \sigma_b} \cdot \gamma} \cdot (cos(\beta))^{(*)} \tag{3.45}$$

$(*)$Only for helical gears

Dependent on Contact Stress

$$m = 1.84 \cdot \sqrt[3]{\frac{T \cdot k_a \cdot k_v}{K \cdot Z^2 \cdot \lambda} \left(\frac{i+1}{i}\right)} \tag{3.46}$$

$$K = \frac{(\sigma_{o,lim})^2}{0.35 \cdot E} \tag{3.47}$$

Where :

K	=	Edge strength factor	$[N/mm^2]$
E	=	Modulus of Elastically	$[N/mm^2], [MPa]$
Z	=	Number of Teeth	
λ	=	Width factor of cog	
K_a	=	External dynamic load factor 3.1b	
K_v	=	Inner dynamic factor from the speed of the gear (Eq:3.38)	
i	=	gear ratio	
σ_b	=	bending stress	[Pa]
T	=	Torque, Moment	[Nm]
γ	=	Tooth form factor (Tab 3.2)	
β	=	Helical angle	[deg, rad]
m	=	module	[mm]

3.6.6 Tabels

Tabell 4.6 Belastningsfaktor K_a

Driv-mekanisme	Driftstimer per dag	Jevn	Lette støt	Tunge støt
Elektromotor eller hydromotor	8-10 timer	1	1,25	1,75
	2-3 timer	0,8	1	1,50

jevn	lette støt	tunge støt
lette transportenheter	betongblandere	steinknusere
blandeapparater	blandeverk	veihøvler
sorteringsbånd	traverskraner	presser
vinsjer	svingkraner	kulemøller
sentrifugalpumper	stempelpumper med flere sylindre	sakser
		grabber
		mudderapparater
		gravemaskiner

Tabell 4.7 Driftsfaktor A

Tannhjulstype	$A\,(m/s)$
Svært nøyaktig tilvirket (slipt) tannhjul	10
Nøyaktig tilvirket tannhjul	5
Vanlig tannhjul, inkl. åpen drift	3

Tabell 4.10 Smøreoljefaktor

Viskositet i °E	5	9	13,5	19	26
K_L	0,9	0,95	1	1,05	1,1

Tabell 4.8

Antall tenner z	Tannformfaktor γ	Antall tenner z	Tannformfaktor γ
15	–	40	2,45
20	2,90	60	2,30
25	2,73	80	2,24
30	2,60	100	2,21

Tabell 4.11 Hastighetsfaktor

v i m/s	0,25	1	2	3	4	5
Z_v	0,835	0,842	0,855	0,877	0,905	0,932
v i m/s	6	7	8	9	10	12
Z_v	0,960	0,980	1	1,015	1,033	1,058

Tabell 4.12

Material-betegnelse ifølge DIN	Brinell-hardhet i kgf/mm^2	σ_B i N/mm^2	$\sigma_{b\,lim}$ * i N/mm^2	$\sigma_{o\,lim}$ * i N/mm^2
Fe 430	125	450	160	430
Fe 590	180	650	210	620
C 45 N	185	650	220	540
C 60 N	220	800	250	610
34 Cr 4 V	260	900	300	715
42 CrMo 4 V	280	950	310	760
16 MnCr 5	650	1000	410	1600
15 CrNi 6	650	1600	410	1900

* Empirisk bestemte grenseverdier for ubegrenset levetid

4 | Gear Trains

4.1 Planetary Gearbox

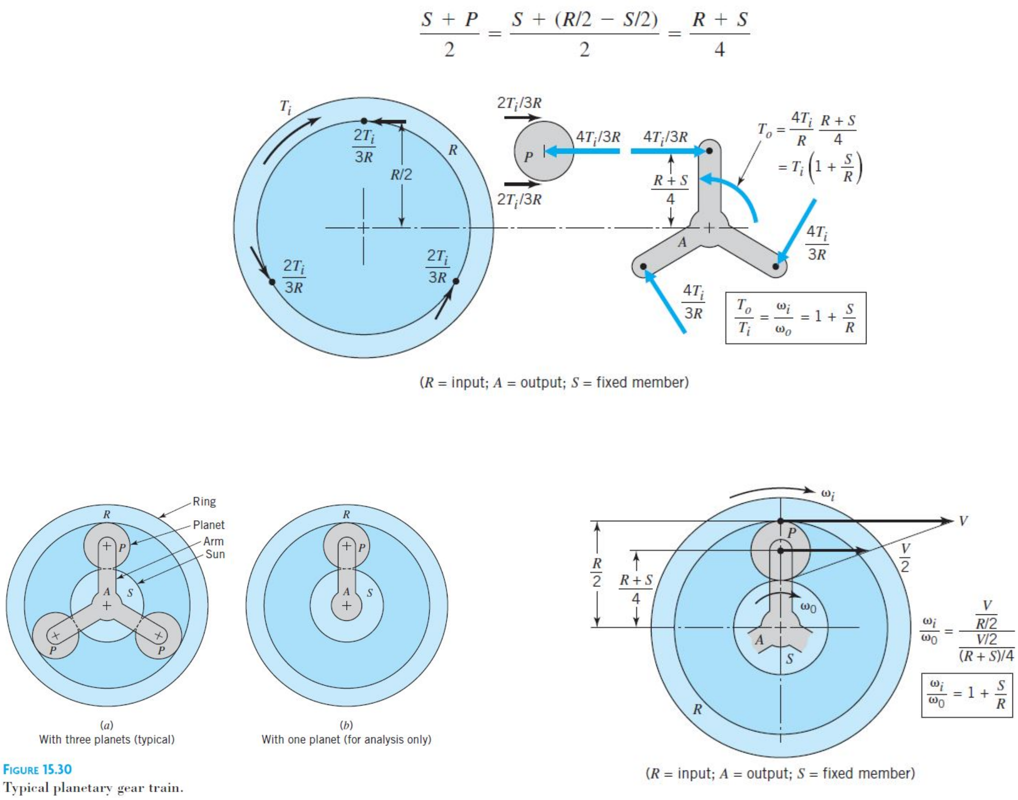


Figure 4.1: [1]

Tree alternative arrangement :

A	-	Reaction Member, Arm
R	-	Ring gear
S	-	Sun gear
P	-	Planet gears

Relation fot the number of teeth :

$$R = 2P + S \tag{4.1}$$

Neutral :
Behavior:
The gear is disengaged, disabling the connection between input and output.
Configuration:
One of the components is free (no constraints) while the other two are respectively input and output.

Direct-Drive
Behavior:
The input-velocity and output-velocities are equal, as with the power (excluding the efficiency-factor).
Configuration:
Two of the components are inputs, while the last is the output.
Gear-ratio:

$$i = 1 \tag{4.2}$$

Reduction (direct):
Behavior:
The gearing reduces the velocity from the input, while maintaining the direction of rotation for the input and output.
Configuration:
S - Input
A - Output
R - Stationary
Gear-ratio:

$$i = 1 + \frac{R}{S} \tag{4.3}$$

Overdrive (direct):

Behaviour:
The gearing increases the velocity from the input, while maintaining the direction of rotation for the input and output.
Configuration:
S - Stationary
A - Input
R - Output
Gear-ratio:

$$i = \frac{1}{1 + \frac{S}{R}} \tag{4.4}$$

Reduction (reverse):
Behavior:
The gearing reduces the velocity from the input, while reversing the direction of rotation for the input and output.
Configuration:
S - Input
A - Stationary
R - Output
Gear-ratio :

$$i = -\frac{R}{S} \tag{4.5}$$

4.1.1 Force Analyzes

R, S, P : diameter (or number of teeth)

Ring :

$$T_i = F_t \cdot Arm \tag{4.6}$$

$$F_t = \frac{2 \cdot T_i}{3 \cdot R} \tag{4.7}$$

Planet :

$$2F = \frac{4 \cdot T_i}{3 \cdot R} \tag{4.8}$$

Arm :

$$T_{o,arm} = 3\left(\frac{4 \cdot T_i}{3 \cdot R}\right)\left(\frac{R+S}{4}\right) = T_i\left(1 + \frac{S}{R}\right) \tag{4.9}$$

$$arm : \frac{S+P}{2} = \frac{S\frac{R-S}{2}}{2} = \frac{R+S}{4} \tag{4.10}$$

4.2 Bevel Gears

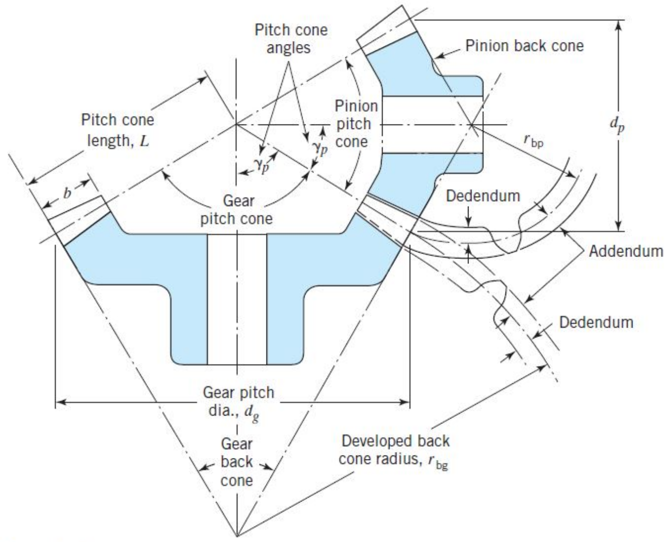


FIGURE 16.9
Bevel gear terminology.

Figure 4.2: [1]

Size and shape of the teeth are defined at the large end, were they intersect the back cones. NOTE : The pitch cone and the back cone elements are perpendicular.

Number of teeth :

$$N'_p = \frac{2\pi \cdot r_{bp}}{p} \quad , \quad N_g = \frac{2\pi \cdot r_{bg}}{p} \tag{4.11}$$

OR ;

$$N'_p = 2 \cdot r_{bp} \cdot P \quad , \quad N_g = 2 \cdot r_{bg} \cdot P \tag{4.12}$$

Gear ratio :

$$\frac{\omega_p}{\omega_g} = \frac{N_g}{N_p} = \frac{d_g}{d_p} = tan(\gamma_g) = cot(\gamma_p) \tag{4.13}$$

Two limits on the face width :

$$b \leq 10 \cdot m \tag{4.14}$$

$$b \leq \frac{L}{3} \tag{4.15}$$

Where:

N	=	Number of teeth	
m	=	module of the imaginary gear	[mm]
p	=	Circular pitch of both the imaginary spur gears and bevel gear	[mm]
P	=	Diametrical Pitch	[mm]
L	=	Pitch cone length	[mm]

4.2.1 Force Analysis

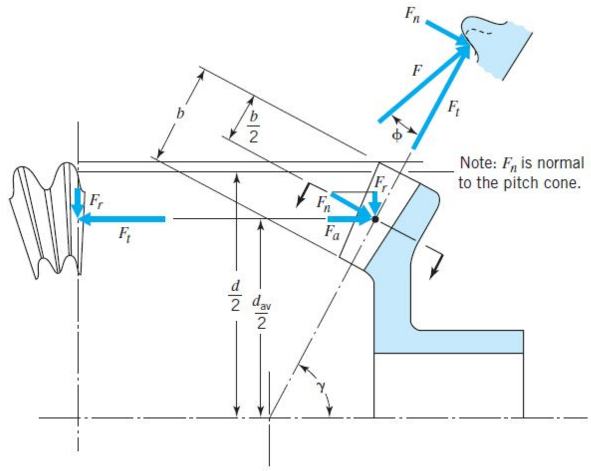


FIGURE 16.12
Resolution of resultant bevel gear-tooth force F.

Figure 4.3: [1]

Resultant force F is shown applied to the tooth at the pitch-cone surface and midway along the tooth width b (see fig.)

$$d_{av} = d - b \cdot sin(\gamma) \tag{4.16}$$

$$v_{av} = \pi \cdot d_{av} \cdot n \tag{4.17}$$

$$F_t = \frac{\dot{W}}{v_{avg}} = \frac{2 \cdot T}{d_{avg}} \tag{4.18}$$

$$F_a = F_t \cdot tan(\phi) \cdot sin(\gamma) \tag{4.19}$$

$$F_r = F_t \cdot tan(\phi) \cdot cos(\gamma) \tag{4.20}$$

d	=	pitch diameter	[m]
d_{avg}	=	center of force diameter	[m]
v_{avg}	=	tangential speed	[m/s]
n	=	number of revolution / sec	[r/s]
$\dot{W}$	=	Effect , Power	[Watts]
F_t	=	Tangential Force Component	[N]
F_a	=	Axial Force Component	[N]
F_r	=	Radial Force Component	[N]

4.2.2 Spiral Bevel Gears

$$F_a = \frac{F_t}{cos(\psi)}(tan(\phi) \cdot sin(\gamma) \mp sin(\psi) \cdot cos(\gamma)) \tag{4.21}$$

$$F_r = \frac{F_t}{cos(\psi)}(tan(\phi) \cdot cos(\gamma) \pm sin(\psi) \cdot sin(\gamma)) \tag{4.22}$$

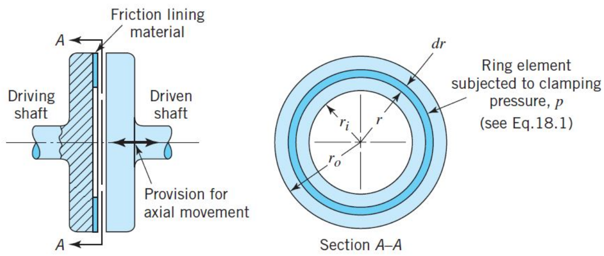
NOTE : upper sign applies to a driving pinion, with right-hand spiral rotating clockwise.

ψ	=	Spiral Angle, usually 35°	[deg]
ϕ	=	Pressure Angle, Usually 20°	[deg]

5 | Clutches and Brakes

5.1 Clutches

5.1.1 Flat-Clutch



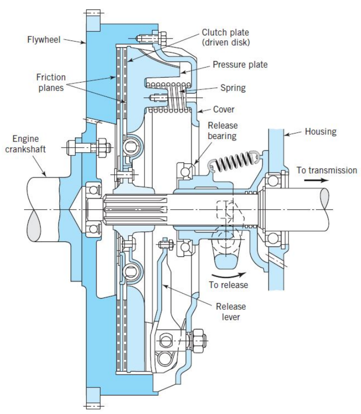


FIGURE 18.2
Automotive-type disk clutch. (Courtesy Borg-Warner Corporation.)

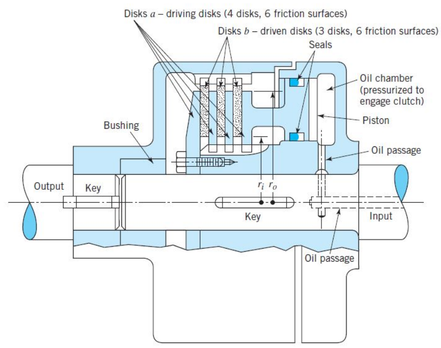


FIGURE 18.3
Multiple-disk clutch, hydraulically operated.

Figure 5.1: [1]

Total normal Force (axial) :

$$dF = p(2 \cdot \pi \cdot r \cdot dr)$$

$$F = \int_{r_i}^{r_o} 2\pi r \cdot dr = \pi \cdot p(r_o^2 - r_i^2) \tag{5.1}$$

Total Friction Torque :

$$dT = f \cdot dF \cdot r$$

$$T = \int_{r_i}^{r_o} 2\pi \cdot p \cdot f \cdot r^2 \cdot dr = \frac{2}{3}\pi \cdot p \cdot f \cdot (r_o^3 - r_i^3) \tag{5.2}$$

Where :

p	=	Uniform level of interface pressure	[N/m]
r	=	radius	[m]
i	=	inner	
o	=	outer	
f, μ	=	Friction coefficient	

With N-friction interface :

$$T = \frac{2}{3} \cdot p \cdot \pi \cdot f \cdot (r_o^3 - r_i^3) \cdot N \tag{5.3}$$

Torque capacity as a function of axial clamping force :

$$F = \pi \cdot p \cdot (r_o^2 - r_i^2) \Rightarrow p = \frac{F}{\pi \cdot (r_o^2 - r_i^2)} \tag{5.4}$$

$$T = \frac{2}{3} \cdot \pi \cdot \left(\frac{F}{\pi \cdot (r_o^2 - r_i^2)} \right) \cdot f \cdot (r_o^3 - r_i^3) \cdot N \tag{5.5}$$

$$T = \frac{2 \cdot F \cdot f \cdot (r_o^3 - r_i^3) \cdot N}{3 \cdot (r_o^2 - r_i^2)} \tag{5.6}$$

$$\tag{5.7}$$

Relation between r_i and r_o :

$$r_i = \sqrt{\frac{1}{3}} \cdot r_o = 0.58 \cdot r_o \tag{5.8}$$

Uniform rate of wear interface

$$F = 2 \cdot \pi \cdot p_{max} \cdot r_i \cdot (r_o - r_i) \tag{5.9}$$

$$T = \pi \cdot p_{max} \cdot r_i \cdot f \cdot (r_o^2 - r_i^2) \cdot N \tag{5.10}$$

$$T = F \cdot f \cdot \left(\frac{r_o + r_i}{2} \right) \cdot N \tag{5.11}$$

Were :

T	=	Torque, Moment	[Nmm],[Nm]
F	=	Force	[N]
r_i	=	Inner Radius	[mm],[m]
r_o	=	Outer Radius	[mm],[m]
f, μ	=	Friction coefficient	
p_{max}	=	Max pressure on disks	[Pa]
p	=	Pressure on disks	[Pa]
N	=	Number of Friction surfaces/interfaces	

5.1.2 Cone Clutch and Brakes

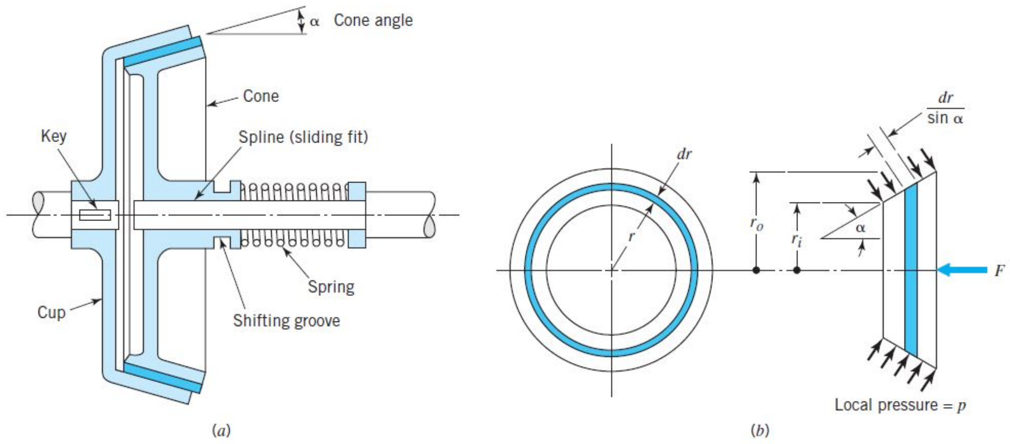


FIGURE 18.5
Cone clutch—parts *a* and *b* are not to the same scale.

Figure 5.2: [1]

$$r_m = r_o - \frac{1}{2} \cdot W \cdot sin(\alpha) \tag{5.12}$$

$$r_m = r_i + \frac{1}{2} \cdot W \cdot sin(\alpha) \tag{5.13}$$

$$p = \frac{F}{\pi(r_o^2 - r_i^2)} \tag{5.14}$$

Surface area :

$$dA = 2\pi \cdot r \cdot \frac{dr}{sin(\alpha)}$$

$$A = \int_{r_i}^{r_o} 2\pi \cdot r \cdot \frac{dr}{sin(\alpha)} = \frac{\pi}{sin(\alpha)} \cdot \left(r_o^2 - r_i^2\right) \tag{5.15}$$

Normal Force :

$$dN = p \cdot \frac{2\pi \cdot r \cdot dr}{sin(\alpha)}$$

$$N = \int_{r_i}^{r_o} p \cdot \frac{2\pi \cdot r}{sin(\alpha)} \cdot dr =$$

$$N = \frac{2\pi \cdot p}{sin(\alpha)} \cdot \left(r_o^2 - r_i^2\right) \tag{5.16}$$

Torque :

$$dT = d \cdot N \cdot f \cdot r = \frac{2\pi \cdot p \cdot f \cdot r^2}{sin(\alpha)} \cdot dr \tag{5.17}$$

Assuming Uniform Pressure :

$$T = \frac{2}{3}\pi \cdot p \cdot f \cdot \frac{(r_0^3 - r_i^3)}{sin(\alpha)} = \frac{2 \cdot f \cdot F \cdot (r_o^3 - r_i^3)}{3 \cdot (r_o^2 - r_i^2) \cdot sin(\alpha)} \tag{5.18}$$

Assuming Uniform Wear :

$$T = \frac{\pi \cdot p_{max} \cdot r_i \cdot f \cdot (r_o^2 - r_i^2)}{sin(\alpha)} = \frac{f \cdot F \left(\frac{r_o + r_i}{2}\right)}{sin(\alpha)} = \frac{f \cdot F\,(r_m)}{sin(\alpha)} \tag{5.19}$$

$$p_{max} = \frac{F \cdot r_m}{\pi \cdot r_i \cdot (r_o^2 - r_i^2)} = \frac{F}{2 \cdot \pi \cdot r_i \cdot (r_o - r_i)} \tag{5.20}$$

Angle α :
Commonly $8^\circ < \alpha < 15^\circ$

If $\alpha < 8^\circ$ The clutch will be hard to disengage (easily self-locking)
$\alpha = 12^\circ$ is considered an optimum angle for a conical Clutch

Where :

F	=	Force	[N]
T	=	Torque	[Nm]
r	=	Radius	[m]
i	=	inner	
o	=	outer	
A	=	Area of friction material	[m^2]
α	=	Angle of cone	[deg]
p	=	Uniform level of interface pressure	[N/m]
f, μ	=	Friction coefficient	
N	=	Normal Force	[N]

5.2 Drum Brakes

5.2.1 Short-Shoe Drum-Brakes

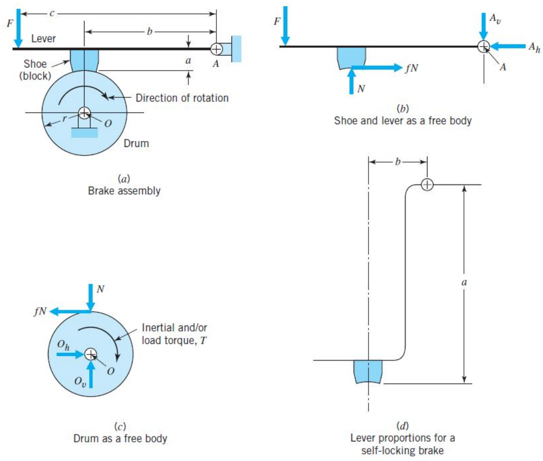


FIGURE 18.6
"Short-shoe" drum brake.

Figure 5.3: [1]

$$\mu \cdot N = f \cdot N = \text{Friction Force} \tag{5.21}$$

$$p = \frac{F}{A} = \frac{\text{Applied Force}}{\text{Projected area}} \tag{5.22}$$

Reaction-forces on the shoe can be calculated as one centralized force acting on the center.

Moment about pivot A :

$$\Sigma \overset{\circlearrowleft}{M} = 0; -F \cdot c - f \cdot N \cdot a + N \cdot b = 0 \tag{5.23}$$

$$N = \frac{F \cdot c}{b - f \cdot a} \tag{5.24}$$

Moment about point O :

$$\Sigma \overset{\circlearrowleft}{M} = 0; T - f \cdot N \cdot r = 0 \tag{5.25}$$

$$T = f \cdot N \cdot r = \frac{f \cdot F \cdot c \cdot r}{b - f \cdot a} \tag{5.26}$$

Counter-Clockwise Rotation :

$$T = \frac{f \cdot F \cdot c \cdot r}{b + f \cdot a} \tag{5.27}$$

Self energizing because the moment of the friction force $f \cdot N \cdot a$ assists the applied force F in applying the brake. If the rotational direction are changed this will on longer be the case.

Self-locking : (Denominator is zero/negative)

$$b - f \cdot a \leqslant 0 \tag{5.28}$$
$$b \leqslant f \cdot a \tag{5.29}$$

5.2.2 Long-Shoe Drum-Brakes

If a brake-shoe contacts the drum over an area of about 45° or more, errors introduced by using the short-shoe equations are usually significant. So more accurate equations have to be used.

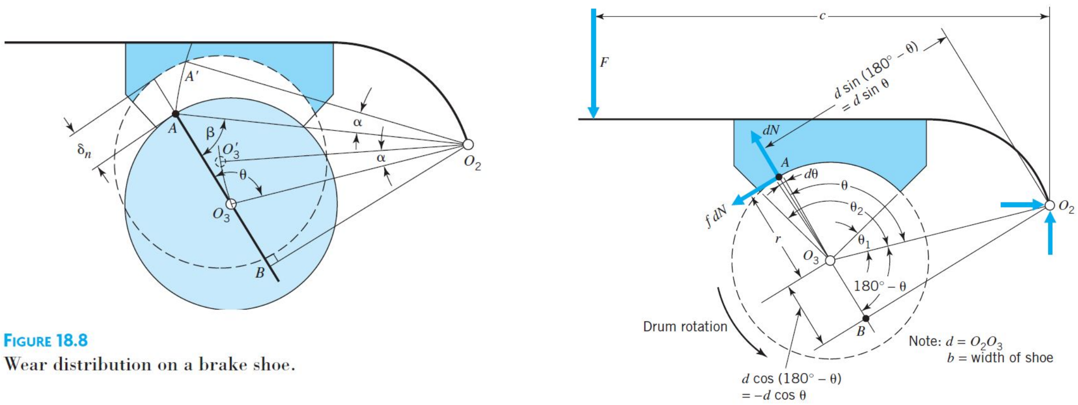


Figure 5.4: [1]

$$\Sigma M_{02} = 0; F \cdot c + M_n + M_f = 0 \tag{5.30}$$
$$M_n = -\frac{p_{max} \cdot b \cdot r \cdot d}{4(sin(\theta)_{max})}\left(2(\theta_2 - \theta_1) - sin(2\theta_2) + sin(2\theta_1)\right) \tag{5.31}$$
$$M_f = \frac{f \cdot p_{max} \cdot r \cdot b}{Sin(\theta)_{max}}\left(r(cos(\theta_1) - cos(\theta_2)) + \frac{d}{4}\left(cos(2\theta_2) - cos(2\theta_1)\right)\right) \tag{5.32}$$
$$T = -\frac{r^2 \cdot f \cdot b \cdot p_{max}}{sin(\theta)_{max}}\left(cos(\theta_1) - cos(\theta_2)\right) \tag{5.33}$$
$$P = T \cdot \omega \tag{5.34}$$
$$\omega = RPM \cdot \left(\frac{2\pi}{60}\right) \tag{5.35}$$

All values of θ is measured in RAD
If θ is $>$ 90° : $sin(\theta)_{max} = 1$
Brake is self-locking if : $M_f \geq M_n$

where:

M_n	=	Moment of normal forces	[Nm]
M_f	=	Moment of friction forces	[Nm]
T	=	Moments action on the drum	[Nm]
d	=	Distance from O_2 to O_3	[m]
b	=	Width of the shoe in contact with drum	[m]
r	=	Radius of drum	[m]
p_{max}	=	Max allowable normal pressure	[Pa]
P	=	Power	[Watt]
ω	=	revolutions	[rad/s]

TABLE 18.3 Typical Values of Pressure Times Rubbing Velocity Used in Industrial Shoe Brakes

	pV	
Operating Conditions	**(psi)(ft/min)**	**(kPa)(m/s)**
Continuous, poor heat dissipation	30,000	1050
Occasional, poor heat dissipation	60,000	2100
Continuous, good heat dissipation as in an oil bath	85,000	3000

Figure 5.5: [1]

5.3 Band Brakes

5.3.1 Standard

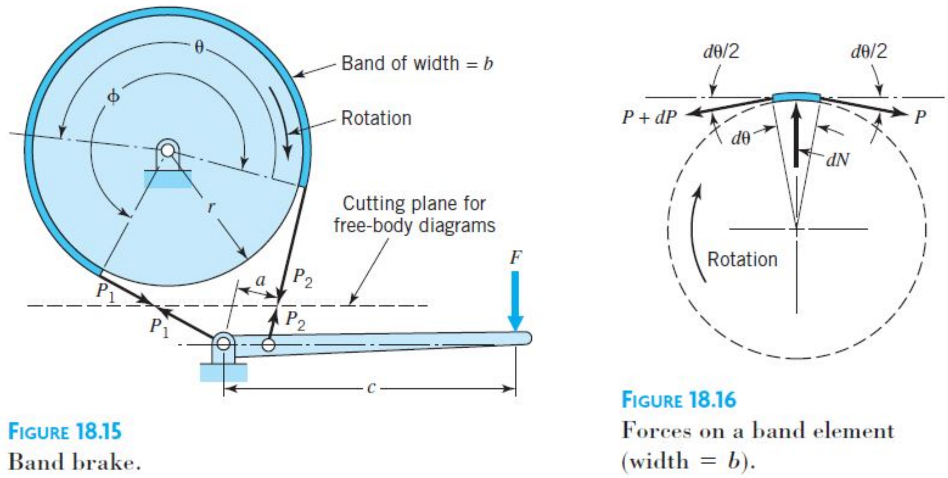


Figure 5.6: [1]

Drum :

$$Torque = T = (P_1 - P_2) \cdot r \tag{5.36}$$

Lever Arm :

$$\begin{aligned} F \cdot c - P_2 \cdot a &= 0 \\ F &= P_2 \cdot \frac{a}{c} \end{aligned} \tag{5.37}$$

Eytelweins Equation :

$$\frac{P_1}{P_2} = e^{f \cdot \theta} \tag{5.38}$$

Maximum Normal Pressure :

$$P_1 = P_{max} \cdot r \cdot b \tag{5.39}$$

Power transmitted :

$$W = T \cdot \omega = T \cdot \frac{2\pi \cdot n}{60} \tag{5.40}$$

Where :

F	=	Force	[N]
T	=	Torque	[Nm]
r	=	Radius of drum	[m]
b	=	Width of drum	[m]
f, μ	=	Friction coefficient	
θ	=	Angle of drum in contact with band	[Rad]
P_1	=	Tight side of band	
P_2	=	Slack side of band	
W	=	Power	[Watt]
n	=	rotational speed	[RPM]
ω	=	rotational velocity	[rad/s]

Band : Made of steel, lined with a woven friction material.

5.3.2 Differential Band Brakes

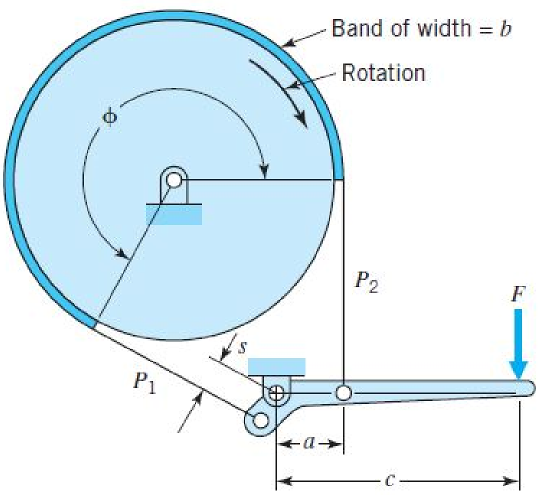


FIGURE 18.17
Differential band brake.

Figure 5.7: [1]

$$F \cdot c - P_2 \cdot a + P_1 \cdot s = 0 \tag{5.41}$$

$$T = (P_1 - P_2) \cdot r \tag{5.42}$$

$$P_1 = P_{max} \cdot r \cdot b \tag{5.43}$$

$$\frac{P_1}{P_2} \cdot e^{\theta \cdot f} \tag{5.44}$$

Where :

F	=	Force	[N]
T	=	Torque	[Nm]
r	=	Radius of drum	[m]
b	=	Width of drum	[m]
f, μ	=	Friction coefficient	
θ	=	Angle of drum in contact with band	[Rad]
P_1	=	Tight side of band	
P_2	=	Slack side of band	

5.3.3 Internal Drum brakes

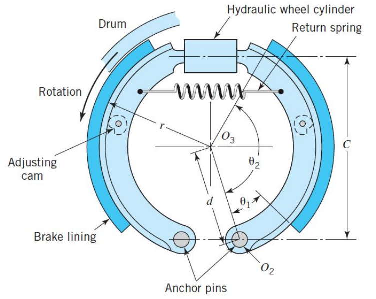


FIGURE 18.13
Automotive-type drum brake (internal shoes).

(a) Will function the same regardless of the rotational direction

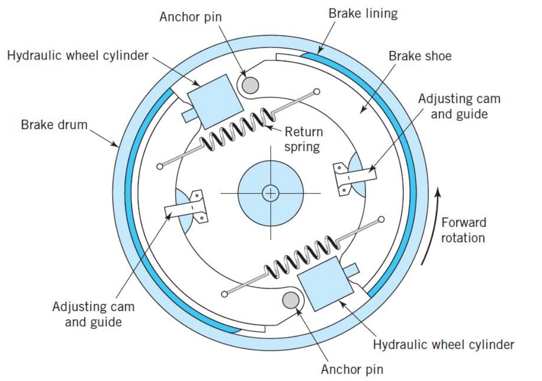


FIGURE 18.14
Automotive brake with two hydraulic wheel cylinders. Both shoes are self-energizing for forward car motion. (Courtesy Chrysler Corporation.)

(b) Will function different depending on rotation.

Figure 5.8: [1]

Calculation is the same as external, but Force and Torque equations have to be adjusted. Same limits on long- and short-shoe.

5.4 Caliper Disk Brakes

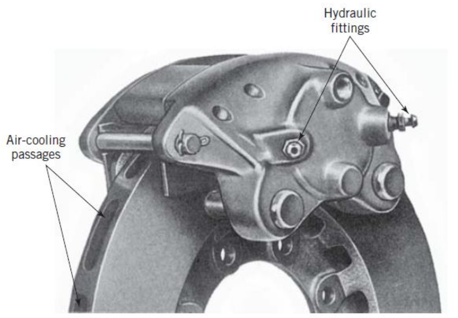
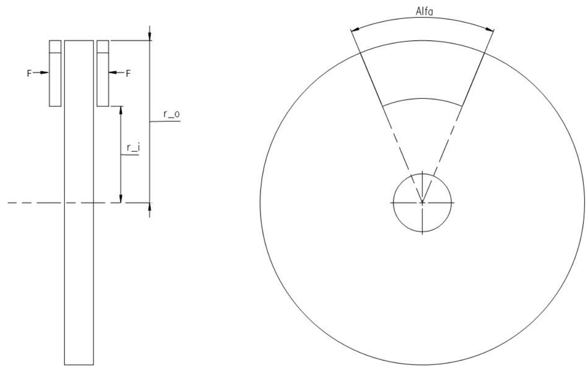


FIGURE 18.4
Caliper disk brake, hydraulically operated. (Courtesy Auto Specialities Manufacturing Company.)

(a) Example on a automotive brake caliper [1]



(b) Distances for Calculations

Uniform Pressure :

$$F = p \cdot \frac{1}{2} \cdot \left(r_o^2 - r_i^2\right) \cdot \alpha \tag{5.45}$$

$$p = \frac{F}{\frac{1}{2} \cdot (r_o^2 - r_i^2) \cdot \alpha} \tag{5.46}$$

Brake Torque :

$$M = f \cdot p \cdot \frac{1}{3}(r_0^3 - r_i^3) \cdot \alpha \tag{5.47}$$

$$M = \frac{2}{3} \frac{(r_o^2 + r_o \cdot r_i + r_i^2)}{r_o + r_i} \cdot f \cdot F \tag{5.48}$$

$$M(r_o = r_i = r) = f \cdot F \cdot r[\text{Simplefication}] \tag{5.49}$$

Uniform Ware :

$$p(r) = p_{max} \cdot \frac{r_i}{r} \tag{5.50}$$

$$F = p_{max} \cdot r_i \cdot (r_o - r_i) \cdot \alpha \tag{5.51}$$

$$p_{max} = \frac{F}{r_i \cdot (r_o - r_i) \cdot \alpha} \tag{5.52}$$

$$M = f \cdot p_{max} \frac{1}{2} \cdot r_i \cdot (r_0^2 - r_i^2) \cdot \alpha \tag{5.53}$$

$$M = \frac{1}{2}(r_o + r_i) \cdot f \cdot F \tag{5.54}$$

Were :

F	=	Force applied to brake shoes	[N]
p	=	Pressure applied by the brake shoes	[Pa]
M, T	=	Moment / Brake torque	[Nm]
f, μ	=	Friction coefficient	
p_{max}	=	Maximum allowable pressure	[Pa]
r	=	Radius to were force is applied	[m]
r_i	=	Radius to inner side of brake shoe	[m]
r_o	=	Radius to outer side of brake shoe	[m]
α	=	Angle were brake shoes contacts with disk	[Rad or Deg]

6 Belts

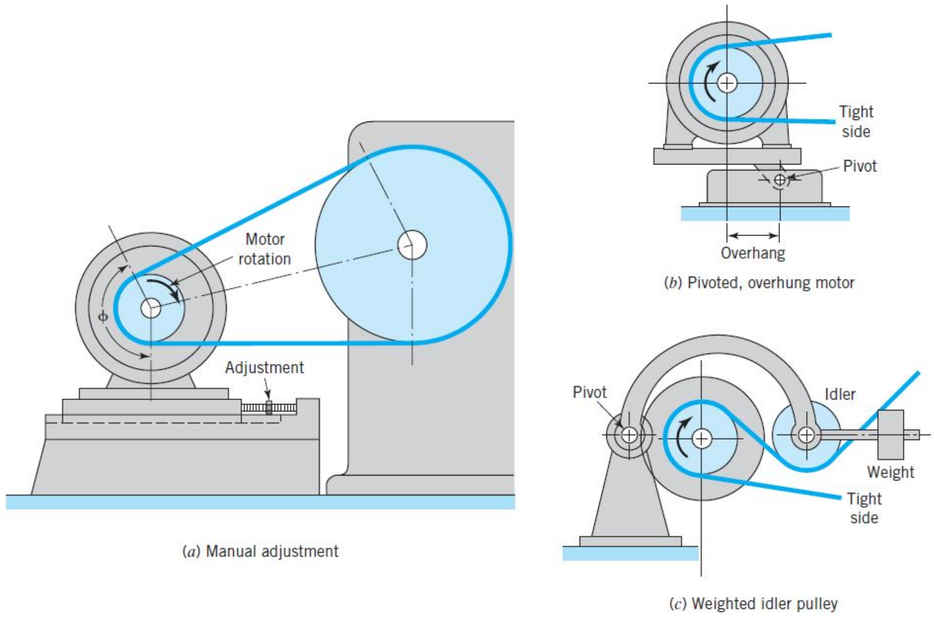


Figure 6.1: [1]

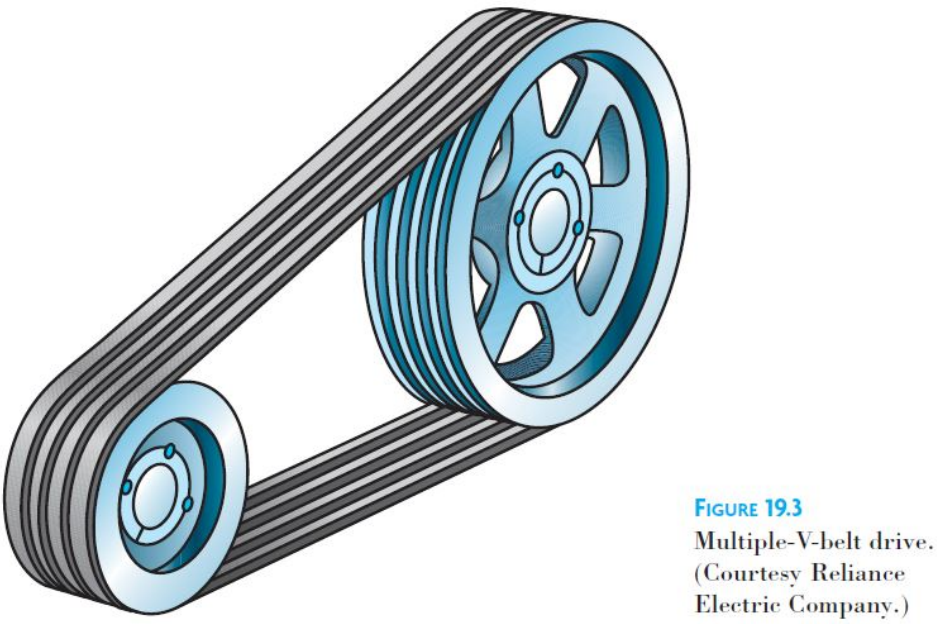


Figure 6.2: [1]

6.1 Flat Belts

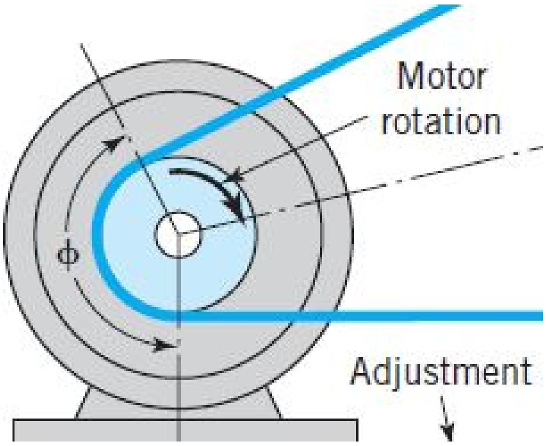


Figure 6.3: [1]

Torque :

$$T = (P_1 - P_2) \cdot r \tag{6.1}$$

$$\frac{P_1}{P_2} = e^{f \cdot \phi} \tag{6.2}$$

Required initial belt tension :

$$P_i = \frac{P_1 + P_2}{2} \tag{6.3}$$

Operation at relative high speeds :

Centrifugal force acting on the belt creates a tension P_c :

$$P_c = m' \cdot v_2 = m' \cdot \omega^2 \cdot r^2 \tag{6.4}$$

$$\frac{P1 - P_c}{P_2 - P_c} = e^{f \cdot \phi} \tag{6.5}$$

Were :

T	=	Torque in the pulley	[Nm]
P_1	=	Force in the Tight part of the belt	[N]
P_2	=	Force in the Slack part of the belt	[N]
r	=	Radius of pulley	[m]
m’	=	Mass pr. unit length of belt	
v	=	Velocity of the belt	[m/s]

6.2 V - Belts

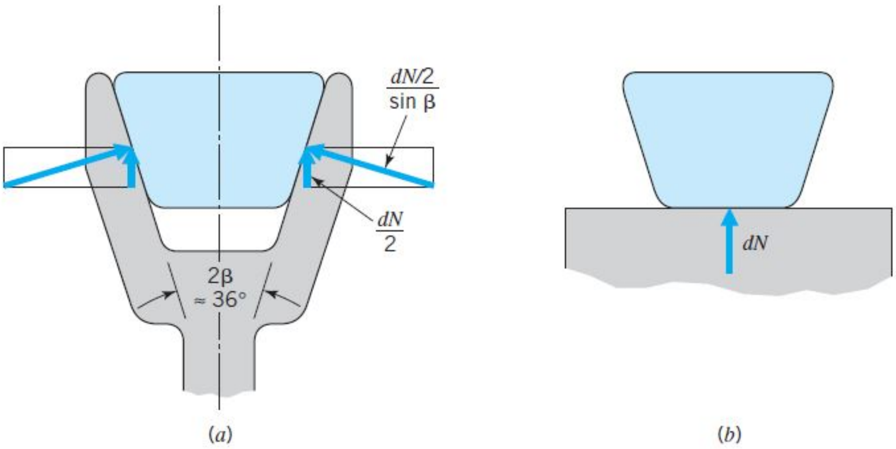


FIGURE 19.4
V-belt in sheave groove and on flat pulley rim.

Figure 6.4: [1]

V-belt VS flat-belt :

$$\frac{dN}{sin(\beta)} = 3.25 \cdot dN \tag{6.6}$$

V-belt can transfer 3.25 times more than a flat-belt drive with angle $2\beta = 36°$.

Calculations are the same for V-belts as for flat-belt, except for the relation of tensions:

Tension-relation :

$$\frac{P_1 - P_c}{P_2 - P_c} = e^{\frac{\mu \cdot \theta}{sin(\beta)}} \tag{6.7}$$

Were :

T	=	Torque in the pulley	[Nm]
P_1	=	Force in the Tight part of the belt	[N]
P_2	=	Force in the Slack part of the belt	[N]
P_c	=	Tension in belt	[N]
r	=	Radius of pulley	[m]
m'	=	Mass pr. unit length of belt	
v	=	Velocity of the belt	[m/s]

Bibliography

[1] K. M. M. Robert C. Juvinal, *Machine Component Design*. Wiley, 2012.

A Appendices

A.1 Deflection in Beams

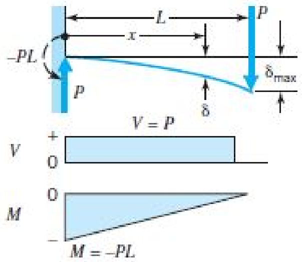
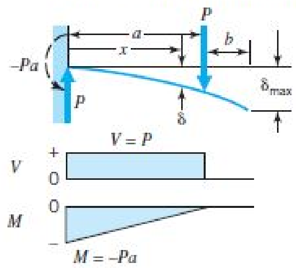
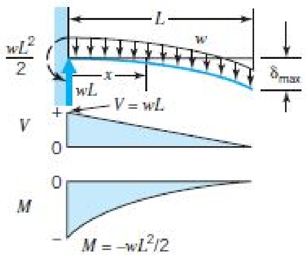
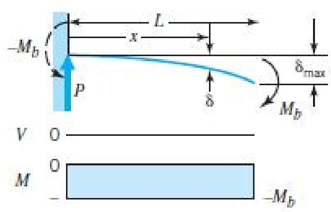
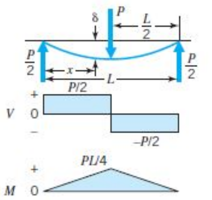
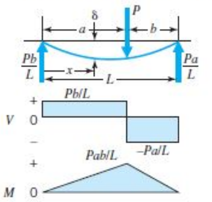
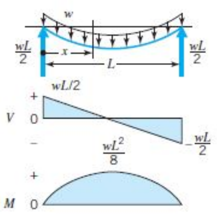
	Slope at Free End	Maximum Deflection	Deflection δ at Any Point x
1. Concentrated load at end 	$\theta = \frac{PL^2}{2EI}$	$\delta_{max} = \frac{PL^3}{3EI}$	$\delta = \frac{Px^2}{6EI}(3L - x)$
2. Concentrated load at any point 	$\theta = \frac{Pa^2}{2EI}$	$\delta_{max} = \frac{Pa^2}{6EI}(3L - a)$	For $0 \leq x \leq a$: $\delta = \frac{Px^2}{6EI}(3a - x)$ For $a \leq x \leq L$: $\delta = \frac{Pa^2}{6EI}(3x - a)$
3. Uniform load 	$\theta = \frac{wL^3}{6EI}$	$\delta_{max} = \frac{wL^4}{8EI}$	$\delta = \frac{wx^2}{24EI}(x^2 + 6L^2 - 4Lx)$
4. Moment load at free end 	$\theta = \frac{M_bL}{EI}$	$\delta_{max} = \frac{M_bL^2}{2EI}$	$\delta = \frac{M_bx^2}{2EI}$

Figure A.1: [1]

	Slope at Ends, θ	Maximum Deflection, δ_{max}	Deflection δ at Any Point x
1. Concentrated center load 	$\frac{PL^2}{16EI}$	At center: $\frac{PL^3}{48EI}$	For $0 \le x \le L/2$: $\frac{Px}{12EI}\left(\frac{3L^2}{4} - x^2\right)$
2. Concentrated load at any point 	At left end: $\frac{Pb(L^2 - b^2)}{6LEI}$	At $x = \sqrt{\frac{L^2 - b^2}{3}}$: $\frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}LEI}$	For $0 \le x \le a$: $\frac{Pbx}{6LEI}(L^2 - x^2 - b^2)$
3. Uniform load 	$\frac{wL^3}{24EI}$	$\frac{5wL^4}{384EI}$	$\frac{wx}{24EI}(L^3 - 2Lx^2 + x^3)$

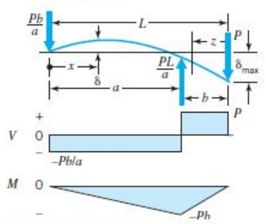
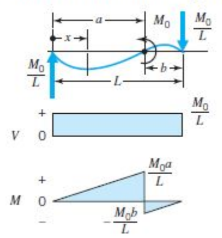
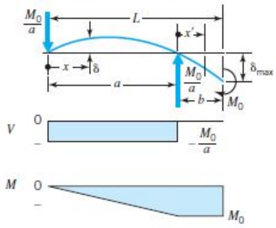
	Slope at Ends, θ	Maximum Deflection, δ_{max}	Deflection δ at Any Point x
4. Overhung load 	At left support: $\frac{Pab}{6EI}$ At right support: $\frac{Pab}{3EI}$ At load: $\frac{Pb}{6EI}(2L + b)$	$\delta_{max} = \frac{Pb^2L}{3EI}$	For $0 \le x \le a$: $\frac{Pbx}{6aEI}(x^2 - a^2)$ For $0 \le z \le b$: $\frac{P}{6EI}[z^3 - b(2L + b)z + 2b^2L]$
5. Moment load between support 	At left support: $\frac{-M_0}{6EIL}(2L^2 - 6aL + 3a^2)$ At load: $\frac{M_0}{EI}\left(\frac{L}{3} + \frac{a^2}{L} - a\right)$ At right support: $\frac{M_0}{6EIL}(L^2 - 3a^2)$	At load: $\frac{M_0a}{3EIL}(2a^2 - 3aL + L^2)$	For $0 \le x \le a$: $\frac{M_0x}{6EIL}(x^2 + 3a^2 - 6aL + 2L^2)$
6. Overhung moment load 	At left support: $\frac{M_0a}{6EI}$ At right support: $\frac{M_0a}{3EI}$ At load: $\frac{M_0(a + 3b)}{3EI}$	$\delta_{max} = \frac{M_0b}{6EI}(2L + b)$	For $0 \le x \le a$: $-\frac{M_0x}{6aEI}(a^2 - x^2)$ For $0 \le x' \le b$: $\frac{M_0}{6EI}(2ax' + 3x'^2)$

Figure A.2: [1]

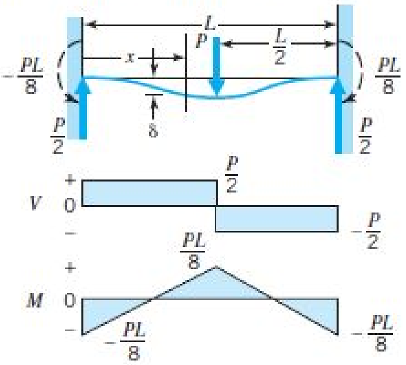
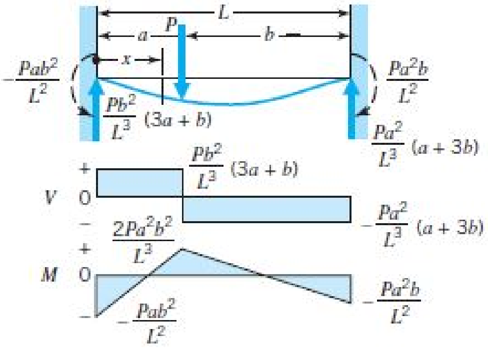
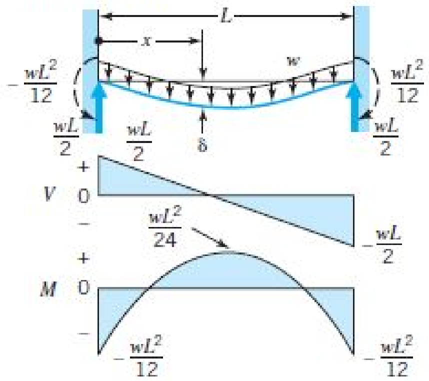
	Deflection δ	Deflection δ at Any Point x
1. Concentrated center load 	At center: $\delta_{max} = \frac{PL^3}{192EI}$	For $0 \le x \le L/2$: $\delta = \frac{Px^2}{48EI}(3L - 4x)$
2. Concentrated load at any point 	At load: $\delta = \frac{Pb^3a^3}{3EIL^3}$	For $0 \le x \le a$: $\delta = \frac{Pb^2x^2}{6EIL^3}[3aL - (3a + b)x]$
3. Uniform load 	At center: $\delta_{max} = \frac{wL^4}{384EI}$	For $0 \le x \le L$: $\delta = \frac{wx^2}{24EI}(L - x)^2$

Figure A.3: [1]