Homework: Eigenvectors, Eigenvalues, Diagonalization, Orthogonal Basis and Complements

DS5020

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Generalized Cosine

Find $\cos\theta$ where θ is the angle between:

(a) $u = (1, 3, -5, 4)$ and $v = (2, -3, 4, 1)$ in $\mathbf{R}^4$,

(b) $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, where $\langle A, B \rangle = \mathrm{tr}(B^T A)$.

Use $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\|\|v\|}$

Orthogonality

Find k so that $u = (1, 2, k, 3)$ and $v = (3, k, 7, -5)$ in $\mathbf{R}^4$ are

orthogonal.

Orthogonal Complement: Basis

Let W be the subspace of $\mathbf{R}^5$ spanned by $u = (1, 2, 3, -1, 2)$

and $v = (2, 4, 7, 2, -1)$. Find a basis of the orthogonal complement $W^{\perp}$ of W.

Orthogonal Basis:

Let S consist of the following vectors in $\mathbf{R}^4$:

$$u_1 = (1, 1, 0, -1),\ \ u_2 = (1, 2, 1, 3),\ \ u_3 = (1, 1, -9, 2),\ \ u_4 = (16, -13, 1, 3)$$

(a) Show that S is orthogonal and a basis of $\mathbf{R}^4$.

(b) Find the coordinates of an arbitrary vector $v = (a, b, c, d)$ in $\mathbf{R}^4$ relative to the basis S.

Projection

Suppose $w \neq 0$. Let v be any vector in V. Show that

$$c = \frac{\langle v, w \rangle}{\langle w, w \rangle} = \frac{\langle v, w \rangle}{\|w\|^2}$$

is the unique scalar such that $v' = v - cw$ is orthogonal to w.

Eigendecomposition 2×2 Matrix

. Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

(a) Find all eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal, and P^{-1}.

(c) Find A^6 and $f(A)$, where $t^4 - 3t^3 - 6t^2 + 7t + 3$.

(d) Find a "real cube root" of B—that is, a matrix B such that $B^3 = A$ and B has real eigenvalues.

(a) First find the characteristic polynomial $\Delta(t)$ of A:

$$\Delta(t) = t^2 - \mathrm{tr}(A)\, t + |A| = t^2 - 5t + 4 = (t-1)(t-4)$$

The roots $\lambda = 1$ and $\lambda = 4$ of $\Delta(t)$ are the eigenvalues of A. We find corresponding eigenvectors.

(i) Subtract $\lambda = 1$ down the diagonal of A to obtain the matrix $M = A - \lambda I$, where the corresponding homogeneous system $MX = 0$ yields the eigenvectors belonging to $\lambda = 1$. We have

$$M = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \qquad \text{corresponding to} \qquad \begin{array}{r} x + 2y = 0 \\ x + 2y = 0 \end{array} \qquad \text{or} \qquad x + 2y = 0$$

The system has only one independent solution; for example, $x = 2, y = -1$. Thus, $v_1 = (2, -1)$ is an eigenvector belonging to (and spanning the eigenspace of) $\lambda = 1$.

(ii) Subtract $\lambda = 4$ down the diagonal of A to obtain

$$M = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \qquad \text{corresponding to} \qquad \begin{array}{r} -2x + 2y = 0 \\ x - \ \ y = 0 \end{array} \qquad \text{or} \qquad x - y = 0$$

The system has only one independent solution; for example, $x = 1, y = 1$. Thus, $v_2 = (1, 1)$ is an eigenvector belonging to $\lambda = 4$.

(b) Let P be the matrix whose columns are v_1 and v_2. Then

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \qquad \text{where} \qquad P^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(c) Using the diagonal factorization $A = PDP^{-1}$, and $1^6 = 1$ and $4^6 = 4096$, we get

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4096 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1366 & 2230 \\ 1365 & 2731 \end{bmatrix}$$

Also, $f(1) = 2$ and $f(4) = 1$. Hence,

$$f(A) = Pf(D)P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

(d) Here $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt[3]{4} \end{bmatrix}$ is the real cube root of D. Hence the real cube root of A is

$$B = P\sqrt[3]{D}P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt[3]{4} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 + \sqrt[3]{4} & -2 + 2\sqrt[3]{4} \\ -1 + \sqrt[3]{4} & 1 + 2\sqrt[3]{4} \end{bmatrix}$$

Diagonalization Symmetric Matrix

. Let $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$. Find an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal.