M2 Chapter 8: Further Kinematics

Vector Differentiation

Differentiating Vectors

Suppose that $\boldsymbol{v} = \begin{pmatrix} t^2 \\ \sin t \end{pmatrix}$. What would be the acceleration?

?

✎ If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ then $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$

and $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt} = \ddot{\boldsymbol{r}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}$

Notational note: Dot notation is a short-hand for differentiation with respect to time: $\dot{x} = \frac{dx}{dt}$
Its use is common in Physics.

[Textbook] A particle P of mass 0.8kg is acted on by a single force $\mathbf{F}$ N. Relative to a fixed origin O, the position vector of P at time t seconds is $\boldsymbol{r}$ metres, where

$$\boldsymbol{r} = 2t^3\boldsymbol{i} + 50t^{-\frac{1}{2}}\boldsymbol{j}, \qquad t \geq 0$$

Find:

(a) the speed of P when $t = 4$
(b) the acceleration of P as a vector when $t = 2$
(c) $\mathbf{F}$ when $t = 2$.

a ?

b ?

c ?

Differentiating Vectors

Suppose that $\boldsymbol{v} = \begin{pmatrix} t^2 \\ \sin t \end{pmatrix}$. What would be the acceleration?

We can simply differentiate the $\boldsymbol{i}$ and $\boldsymbol{j}$ components independently:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \begin{pmatrix} 2t \\ \cos t \end{pmatrix}$$

✎ If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ then $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j}$

and $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt} = \ddot{\boldsymbol{r}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j}$

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(a) the speed of P when $t = 4$

(b) the acceleration of P as a vector when $t = 2$

(c) $\mathbf{F}$ when $t = 2$.

a

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \begin{pmatrix} 6t^2 \\ -25t^{-\frac{3}{2}} \end{pmatrix} \text{ ms}^{-1}$$

When $t = 4, \boldsymbol{v} = \begin{pmatrix} 96 \\ -\frac{25}{8} \end{pmatrix}$

Speed $= \sqrt{96^2 + \left(\frac{25}{8}\right)^2} = 96.1\text{ms}^{-1}$

b

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \begin{pmatrix} 12t \\ \frac{75}{2}t^{-\frac{3}{2}} \end{pmatrix} \text{ms}^{-2}$$

c

When $t = 2, \boldsymbol{a} = \begin{pmatrix} 24 \\ 6.6291\ldots \end{pmatrix}$ ms^{-2}

$$\boldsymbol{F} = m\boldsymbol{a} = 0.8\begin{pmatrix} 24 \\ 6.6291\ldots \end{pmatrix} = \begin{pmatrix} 19.2 \\ 5.30 \end{pmatrix} \text{ N}$$

Exercise 8.4

Pearson Stats/Mechanics Year 2

Pages 72-73

Homework Exercise

1 At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = (3t - 4)\mathbf{i} + (t^3 - 4t)\mathbf{j},\ t \geqslant 0$$

Find:

a the velocity of P when $t = 3$

b the acceleration of P when $t = 3$.

2 A particle P of mass 3 grams moving in a plane is acted on by a force $\mathbf{F}$ N. Its velocity at time t seconds is given by $\mathbf{v} = (t^2\mathbf{i} + (2t - 3)\mathbf{j})\ \mathrm{m\,s^{-1}}$, $t \geqslant 0$.

Find $\mathbf{F}$ when $t = 4$.

3 In this question $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors east and north respectively.

A particle P is moving in a plane. At time t seconds, the position vector of P, $\mathbf{r}$ m, relative to a fixed origin O is given by $\mathbf{r} = 5e^{-3t}\mathbf{i} + 2\mathbf{j}$, $t \geqslant 0$.

a Find the time at which the particle is directly north-east of O.

b Find the speed of the particle at this time.

c Explain why the particle is always moving directly west.

4 At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = 4t^2\mathbf{i} + (24t - 3t^2)\mathbf{j},\ t \geqslant 0$$

a Find the speed of P when $t = 2$. **(3 marks)**

b Show that the acceleration of P is a constant and find the magnitude of this acceleration. **(3 marks)**

Homework Exercise

5 A particle P is initially at a fixed origin O. At time $t = 0$, P is projected from O and moves so that, at time t seconds after projection, its position vector $\mathbf{r}$ m relative to O is given by

$$\mathbf{r} = (t^3 - 12t)\mathbf{i} + (4t^2 - 6t)\mathbf{j},\ t \geqslant 0$$

Find:

a the speed of projection of P **(5 marks)**

b the value of t at the instant when P is moving parallel to $\mathbf{j}$ **(3 marks)**

c the position vector of P at the instant when P is moving parallel to $\mathbf{j}$. **(3 marks)**

The motion of the particle is due to it being acted on by a single variable force, $\mathbf{F}$ N.

d Given that the mass of the particle is 0.5 kg, find the magnitude of $\mathbf{F}$ when $t = 5$ s. **(4 marks)**

6 A particle P is moving in a plane. At time t seconds, the position vector of P, $\mathbf{r}$ m, is given by $\mathbf{r} = (3t^2 - 6t + 4)\mathbf{i} + (t^3 + kt^2)\mathbf{j}$, where k is a constant.

When $t = 3$, the speed of P is $12\sqrt{5}\ \text{m s}^{-1}$.

a Find the two possible values of k. **(6 marks)**

b For each of these values of k, find the magnitude of the acceleration of P when $t = 1.5$. **(4 marks)**

7 Relative to a fixed origin O, the position vector of a particle P at time t seconds is $\mathbf{r}$ metres, where

$$\mathbf{r} = 6t^2\mathbf{i} + t^{\frac{5}{2}}\mathbf{j},\ t \geqslant 0$$

At the instant when $t = 4$, find:

a the speed of P **(5 marks)**

b the acceleration of P, giving your answer as a vector. **(2 marks)**

Homework Exercise

8 A particle P moves in a horizontal plane. At time t seconds, the position vector of P is $\mathbf{r}$ metres relative to a fixed origin O where $\mathbf{r}$ is given by

$$\mathbf{r} = (18t - 4t^3)\mathbf{i} + ct^2\mathbf{j},\ t \geqslant 0,$$

where c is a positive constant. When $t = 1.5$, the speed of P is $15\,\mathrm{m\,s^{-1}}$. Find:

a the value of c **(6 marks)**

b the acceleration of P when $t = 1.5$. **(3 marks)**

9 At time t seconds, a particle P has position vector $\mathbf{r}$ metres relative to a fixed origin O, where

$$\mathbf{r} = (2t^2 - 3t)\mathbf{i} + (5t + t^2)\mathbf{j},\ t \geqslant 0$$

Show that the acceleration of P is constant and find its magnitude. **(5 marks)**

10 A particle P moves in a horizontal plane. At time t seconds, the position vector of P is $\mathbf{r}$ metres relative to a fixed origin O, and $\mathbf{r}$ is given by $\mathbf{r} = (20t - 2t^3)\mathbf{i} + kt^2\mathbf{j}$, $t \geqslant 0$, where k is a positive constant. When $t = 2$, the speed of P is $16\,\mathrm{m\,s^{-1}}$. Find:

a the value of k **(6 marks)**

b the acceleration of P at the instant when it is moving parallel to $\mathbf{j}$. **(4 marks)**

Homework Answers

1 **a** $(3\mathbf{i} + 23\mathbf{j})\,\mathrm{m\,s^{-1}}$ **b** $18\mathbf{j}\,\mathrm{m\,s^{-2}}$

2 $(0.024\mathbf{i} + 0.006\mathbf{j})\,\mathrm{N}$

3 **a** $0.305\,\mathrm{s}$ **b** $6\,\mathrm{m\,s^{-1}}$

c $\mathbf{i}$-component of velocity is negative, $\mathbf{j}$-component of velocity = 0

4 **a** $20\,\mathrm{m\,s^{-1}}$ **b** $10\,\mathrm{m\,s^{-2}}$

b $a = 8\mathbf{i} - 6\mathbf{j}$, no dependency on t therefore constant. $|a| = 10\,\mathrm{m\,s^{-2}}$

5 **a** $6\sqrt{5}\,\mathrm{m\,s^{-1}}$ **b** $t = 2$

c $(-16\mathbf{i} + 4\mathbf{j})\,\mathrm{m}$ **d** $15.2\,\mathrm{N}$ (3 s.f.)

6 **a** $k = -0.5, -8.5$

b $10\,\mathrm{m\,s^{-2}}$ for both values of k

7 **a** $52\,\mathrm{m\,s^{-1}}$ **b** $(12\mathbf{i} + \frac{15}{2}\mathbf{j})\,\mathrm{m\,s^{-2}}$

8 **a** 4 **b** $(-36\mathbf{i} + 8\mathbf{j})\,\mathrm{m\,s^{-2}}$

9 $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$, no t dependency so constant. $|\mathbf{a}| = 2\sqrt{5}\,\mathrm{m\,s^{-2}}$

10 **a** $\sqrt{15}$ **b** $(-4\sqrt{30}\mathbf{i} + 2\sqrt{15}\mathbf{j})\,\mathrm{m\,s^{-2}}$