P1 Chapter 8: Binomial Expansion

Pascal's Triangle

Starter

a) Expand $(a+b)^0$
b) Expand $(a+b)^1$
c) Expand $(a+b)^2$
d) Expand $(a+b)^3$
e) Expand $(a+b)^4$

?
?
?
?
?

What do you notice about:

The coefficients: ?
The powers of a and b: ?

Starter

a) Expand $(a+b)^0$ $\quad 1$

b) Expand $(a+b)^1$ $\quad 1a + 1b$

c) Expand $(a+b)^2$ $\quad 1a^2 + 2ab + 1b^2$

d) Expand $(a+b)^3$ $\quad 1a^3 + 3a^2b + 3ab^2 + 1b^3$

e) Expand $(a+b)^4$ $\quad 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

What do you notice about:

The coefficients: They follow Pascal's triangle (more on next slide).

The powers of a and b: Power of a decreases each time (starting at the power)
Power of b increases each time (starting at 0)

More on Pascal's Triangle

The second number of each row tells us what row we should use for an expansion.

So if we were expanding $(2 + x)^4$, the power is 4, so we use this row.

In Pascal's Triangle, each term (except for the 1s) is the sum of the two terms above.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 1 1

Fro Tip: I <u>highly recommend</u> memorising each row up to what you see here.

We'll see later WHY each row gives us the coefficients in an expansion of $(a + b)^n$

Example

Find the expansion of $(2+3x)^4$

Next have descending or ascending powers of one of the terms, going between 0 and 4 (note that if the power is 0, the term is 1, so we need not write it).

$$
\begin{aligned}
(2+3x)^4 = \ & 1\,(2^4) \\
& +4\,(2^3)(3x)^1 \\
& +6\,(2^2)(3x)^2 \\
& +4\,(2^1)(3x)^3 \\
& +1\,(3x)^4
\end{aligned}
$$

First fill in the correct row of Pascal's triangle.

And do the same with the second term but with powers going the opposite way, noting again that the 'power of 0' term does not appear.

Simplify each term (ensuring any number in a bracket is raised to the appropriate power)

$$= 16 + 96x + 216x^2 + 216x^3 + 81x^4$$

Fro Tip: Initially write *one line per term* for your expansion (before you simplify at the end), as we have done above. There will be less faffing trying to ensure you have enough space for each term.

Another Example

$(1-2x)$ is the same as $(1+(-2x))$, so we expand as before, but use $-2x$ for the second term.

$$
\begin{aligned}
(1-2x)^3 = \quad & 1\,(1^3) \\
& +3\,(1^2)(-2x)^1 \\
& +3\,(1)\;\;(-2x)^2 \\
& +1\qquad\;\;(-2x)^3
\end{aligned}
$$

$$= 1 - 6x + 12x^2 - 8x^3$$

Fro Tip: If one of the terms in the original bracket is negative, the terms in your expansion will <u>oscillate between positive and negative</u>. If they don't (e.g. two consecutive negatives), you've done something wrong!

Getting a single term in the expansion

The coefficient of x^2 in the expansion of $(2 - cx)^5$ is 720. Find the possible value(s) of the constant c.

The '5' row in Pascal's triangle is 1 5 10 10 5 1. If we count the 1 as the '0th term', we want the 2nd term, which is 10.

Since we want the x^2 term:

- The power of $(-cx)$ must be ?
- The power of 2 must be ?

Therefore term is: ?

Getting a single term in the expansion

The coefficient of x^2 in the expansion of $(2 - cx)^5$ is 720.
Find the possible value(s) of the constant c.

The '5' row in Pascal's triangle is 1 5 10 10 5 1. If we count the 1 as the '0th term', we want the 2nd term, which is 10.

Since we want the x^2 term:

- The power of $(-cx)$ must be **2.**
- The power of 2 must be **3 (the two powers must add up to 5).**

Therefore, term is: $10(-cx)^2(2^3) = 80c^2x^2$

$$\therefore 80c^2 = 720$$
$$c^2 = 9$$
$$c = \pm 3$$

Test Your Understanding

Edexcel C2

(*a*) Find the first 3 terms, in ascending powers of x, of the binomial expansion of

$$(2 + kx)^7$$

where k is a constant. Give each term in its simplest form. **(4)**

Given that the coefficient of x^2 is 6 times the coefficient of x,

(*b*) find the value of k. **(2)**

(a)	?
(b)	?

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(a)	$(7 \times \ldots \times x)$ or $(21 \times \ldots \times x^2)$ The 7 or 21 can be in 'unsimplified' form.	M1
	$(2+kx)^7 = 2^7 + 2^6 \times 7 \times kx + 2^5 \times \binom{7}{2} k^2x^2$	
	$= 128; \quad +448kx, \quad +672k^2x^2$ [or $672(kx)^2$]	B1; A1, A1
	(If $672kx^2$ follows $672(kx)^2$, isw and allow A1)	(4)
(b)	$6 \times 448k = 672k^2$	M1
	$k = 4$ (Ignore $k = 0$, if seen)	A1 (2) [6]

Exercise 8.1

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Pages 62

Extension

1 *[MAT 2009 1J]*
The number of pairs of positive integers x, y which solve the equation:

$$x^3 + 6x^2y + 12xy^2 + 8y^3 = 2^{30}$$

is:

A) 0
B) 2^6
C) $2^9 - 1$
D) $2^{10} + 2$

?

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is:

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The LHS is the binomial expansion of $(\boldsymbol{x + 2y})^3$, therefore:

$$\boldsymbol{(x + 2y)^3 = 2^{30}}$$
$$\boldsymbol{x + 2y = 2^{10}}$$
$$\boldsymbol{x = 2(2^9 - y)}$$

In order for $\boldsymbol{x}$ to be a positive integer, $\boldsymbol{y}$ can be between 1 and $\boldsymbol{2^9 - 1}$. The answer is C.

Homework Exercise

1 State the row of Pascal's triangle that would give the coefficients of each expansion:

a $(x+y)^3$ **b** $(3x-7)^{15}$ **c** $\left(2x+\frac{1}{2}\right)^n$ **d** $(y-2x)^{n+4}$

2 Write down the expansion of:

a $(x+y)^4$ **b** $(p+q)^5$ **c** $(a-b)^3$ **d** $(x+4)^3$

e $(2x-3)^4$ **f** $(a+2)^5$ **g** $(3x-4)^4$ **h** $(2x-3y)^4$

3 Find the coefficient of x^3 in the expansion of:

a $(4+x)^4$ **b** $(1-x)^5$ **c** $(3+2x)^3$ **d** $(4+2x)^5$

e $(2+x)^6$ **f** $\left(4-\frac{1}{2}x\right)^4$ **g** $(x+2)^5$ **h** $(3-2x)^4$

4 Fully expand the expression $(1+3x)(1+2x)^3$.

Problem-solving

Expand $(1+2x)^3$, then multiply each term by 1 and by $3x$.

5 Expand $(2+y)^3$. Hence or otherwise, write down the expansion of $(2+x-x^2)^3$ in ascending powers of x.

6 The coefficient of x^2 in the expansion of $(2+ax)^3$ is 54. Find the possible values of the constant a.

Homework Exercise

7 The coefficient of x^3 in the expansion of $(2 - x)(3 + bx)^3$ is 45. Find possible values of the constant b.

8 Work out the coefficient of x^2 in the expansion of $(p - 2x)^3$. Give your answer in terms of p.

9 After 5 years, the value of an investment of £500 at an interest rate of X% per annum is given by:

$$500\left(1 + \frac{X}{100}\right)^5$$

Find an approximation for this expression in the form $A + BX + CX^2$, where A, B and C are constants to be found. You can ignore higher powers of X.

Challenge

Find the constant term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^3$.

Homework Exercise

1 **a** 4th row **b** 16th row
c $(n+1)$th row **d** $(n+5)$th row

2 **a** $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
b $p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$
c $a^3 - 3a^2b + 3ab^2 - b^3$
d $x^3 + 12x^2 + 48x + 64$
e $16x^4 - 96x^3 + 216x^2 - 216x + 81$
f $a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32$
g $81x^4 - 432x^3 + 864x^2 - 768x + 256$
h $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

3 **a** 16 **b** -10 **c** 8 **d** 1280
e 160 **f** -2 **g** 40 **h** -96

4 $1 + 9x + 30x^2 + 44x^3 + 24x^4$

5 $8 + 12y + 6y^2 + y^3$, $8 + 12x - 6x^2 - 11x^3 + 3x^4 + 3x^5 - x^6$

6 ± 3

7 $\frac{5}{2}, -1$

8 $12p$

9 $500 + 25X + \frac{X^2}{2}$

Challenge

$\frac{3}{4}$