P2 Chapter 2: Graphing Functions

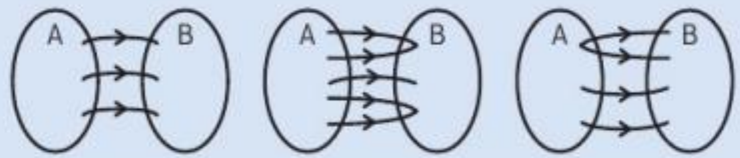
Chapter Practice

1 A modulus function is, in general, a function of the type $y = |f(x)|$.

- When $f(x) \geqslant 0$, $|f(x)| = f(x)$
- When $f(x) < 0$, $|f(x)| = -f(x)$

2 To sketch the graph of $y = |ax + b|$, sketch $y = ax + b$ then reflect the section of the graph below the x-axis in the x-axis.

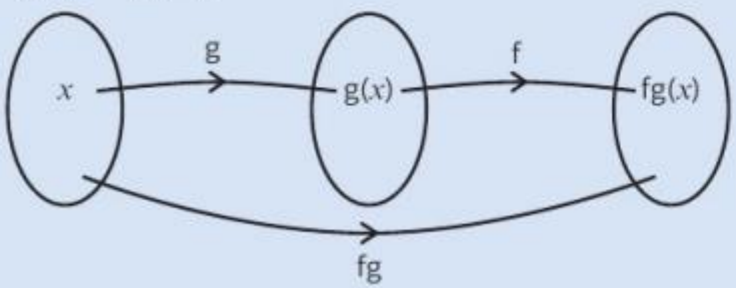
3 A mapping is a **function** if every input has a distinct output. Functions can either be **one-to-one** or **many-to-one**.



one-to-one function | many-to-one function | not a function

4 $fg(x)$ means apply g first, then apply f.

$fg(x) = f(g(x))$



5 Functions $f(x)$ and $f^{-1}(x)$ are inverses of each other. $ff^{-1}(x) = x$ and $f^{-1}f(x) = x$.

6 The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of each another in the line $y = x$.

Key Points

7 The domain of $f(x)$ is the range of $f^{-1}(x)$.

8 The range of $f(x)$ is the domain of $f^{-1}(x)$.

9 To sketch the graph of $y = |f(x)|$
- Sketch the graph of $y = f(x)$
- Reflect any parts where $f(x) < 0$ (parts below the x-axis) in the x-axis
- Delete the parts below the x-axis

10 To sketch the graph of $y = f(|x|)$
- Sketch the graph of $y = f(x)$ for $x \geqslant 0$
- Reflect this in the y-axis

11 $f(x + a)$ is a horizontal translation of $-a$.

12 $f(x) + a$ is a vertical translation of $+a$.

13 $f(ax)$ is a horizontal stretch of scale factor $\frac{1}{a}$

14 $af(x)$ is a vertical stretch of scale factor a.

15 $f(-x)$ reflects $f(x)$ in the y-axis.

16 $-f(x)$ reflects $f(x)$ in the x-axis.

Chapter Exercises

1 **a** On the same axes, sketch the graphs of $y = 2 - x$ and $y = 2|x + 1|$.

b Hence, or otherwise, find the values of x for which $2 - x = 2|x + 1|$.

2 The equation $|2x - 11| = \frac{1}{2}x + k$ has exactly two distinct solutions.

Find the range of possible values of k. **(4 marks)**

3 Solve $|5x - 2| = -\frac{1}{4}x + 8$. **(4 marks)**

4 **a** On the same set of axes, sketch $y = |12 - 5x|$ and $y = -2x + 3$. **(3 marks)**

b State with a reason whether there are any solutions to the equation
$|12 - 5x| = -2x + 3$ **(2 marks)**

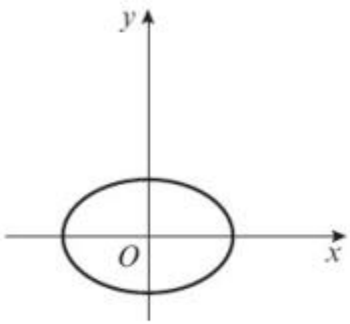
Chapter Exercises

5 For each of the following mappings:

i state whether the mapping is one-to-one, many-to-one or one-to-many

ii state whether the mapping could represent a function.

a



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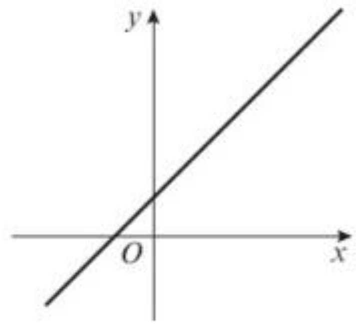

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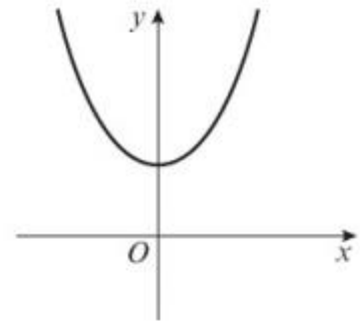

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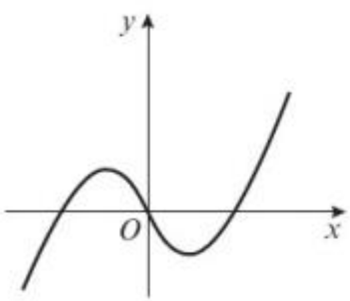

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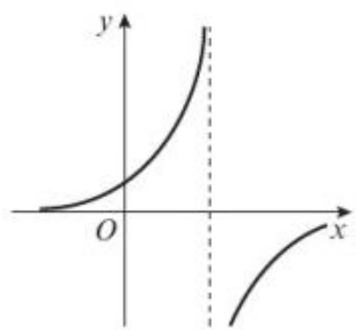

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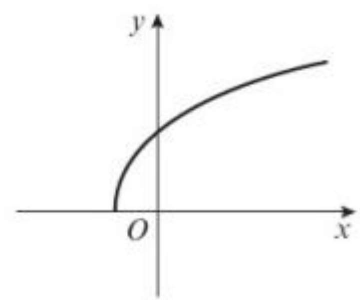


6 The function f(x) is defined by

$$f(x) = \begin{cases} -x, & x \leqslant 1 \\ x - 2, & x > 1 \end{cases}$$

a Sketch the graph of f(x) for $-2 \leqslant x \leqslant 6$. **(4 marks)**

b Find the values of x for which $f(x) = -\frac{1}{2}$ **(3 marks)**

Chapter Exercises

7 The functions p and q are defined by

p: $x \mapsto x^2 + 3x - 4, x \in \mathbb{R}$

q: $x \mapsto 2x + 1, x \in \mathbb{R}$

a Find an expression for pq(x). **(2 marks)**

b Solve pq(x) = qq(x). **(3 marks)**

8 The function g(x) is defined as $g(x) = 2x + 7$, $\{x \in \mathbb{R}, x \geqslant 0\}$.

a Sketch $y = g(x)$ and find the range. **(3 marks)**

b Determine $y = g^{-1}(x)$, stating its domain. **(3 marks)**

c Sketch $y = g^{-1}(x)$ on the same axes as $y = g(x)$, stating the relationship between the two graphs. **(2 marks)**

9 The function f is defined by

$$f: x \mapsto \frac{2x+3}{x-1}, \quad \{x \in \mathbb{R}, x > 1\}$$

a Find $f^{-1}(x)$. **(4 marks)**

b Find: **i** the range of $f^{-1}(x)$

ii the domain of $f^{-1}(x)$ **(2 marks)**

Chapter Exercises

10 The functions f and g are given by

$$f: x \mapsto \frac{x}{x^2 - 1} - \frac{1}{x + 1}, \quad \{x \in \mathbb{R}, x > 1\}$$

$$g: x \mapsto \frac{2}{x}, \quad \{x \in \mathbb{R}, x > 0\}$$

a Show that $f(x) = \dfrac{1}{(x - 1)(x + 1)}$ **(3 marks)**

b Find the range of $f(x)$. **(1 mark)**

c Solve $gf(x) = 70$. **(4 marks)**

11 The following functions $f(x)$, $g(x)$ and $h(x)$ are defined by

$$f(x) = 4(x - 2), \quad \{x \in \mathbb{R}, x \geqslant 0\}$$

$$g(x) = x^3 + 1, \quad \{x \in \mathbb{R}\}$$

$$h(x) = 3^x, \quad \{x \in \mathbb{R}\}$$

a Find $f(7)$, $g(3)$ and $h(-2)$.

b Find the range of $f(x)$ and the range of $g(x)$.

c Find $g^{-1}(x)$.

d Find the composite function $fg(x)$.

e Solve $gh(a) = 244$.

12 The function $f(x)$ is defined by $f: x \mapsto x^2 + 6x - 4$, $x \in \mathbb{R}$, $x > a$, for some constant a.

a State the least value of a for which f^{-1} exists. **(4 marks)**

b Given that $a = 0$, find f^{-1}, stating its domain. **(4 marks)**

13 The functions f and g are given by

$$f : x \mapsto 4x - 1, \{x \in \mathbb{R}\}$$

$$g : x \mapsto \frac{3}{2x-1}, \left\{x \in \mathbb{R}, x \neq \frac{1}{2}\right\}$$

Find in its simplest form:

a the inverse function f^{-1} **(2 marks)**

b the composite function gf, stating its domain **(3 marks)**

c the values of x for which $2f(x) = g(x)$, giving your answers to 3 decimal places. **(4 marks)**

14 The functions f and g are given by

$$f : x \mapsto \frac{x}{x-2}, \quad \{x \in \mathbb{R}, x \neq 2\}$$

$$g : x \mapsto \frac{3}{x}, \quad \{x \in \mathbb{R}, x \neq 0\}$$

a Find an expression for $f^{-1}(x)$. **(2 marks)**

b Write down the range of $f^{-1}(x)$. **(1 mark)**

c Calculate gf(1.5). **(2 marks)**

d Use algebra to find the values of x for which $g(x) = f(x) + 4$. **(4 marks)**

15 The function $n(x)$ is defined by

$$n(x) = \begin{cases} 5 - x, & x \leqslant 0 \\ x^2, & x > 0 \end{cases}$$

a Find $n(-3)$ and $n(3)$.

b Solve the equation $n(x) = 50$.

16 $g(x) = \tan x, -180° \leqslant x \leqslant 180°$

a Sketch the graph of $y = g(x)$.

b Sketch the graph of $y = |g(x)|$.

c Sketch the graph of $y = g(|x|)$.

17 The diagram shows the graph of $f(x)$.

The points $A(4, -3)$ and $B(9, 3)$ are turning points on the graph.

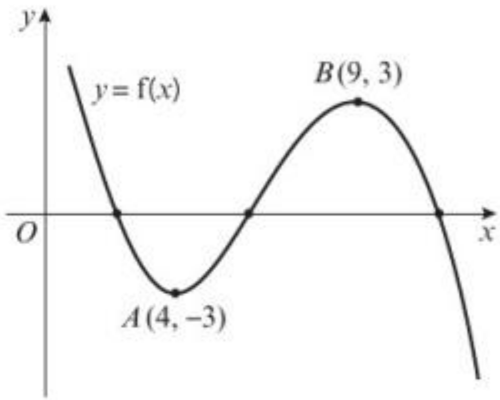
Sketch on separate diagrams, the graphs of

a $y = f(2x) + 1$ **(3 marks)**

b $y = |f(x)|$ **(3 marks)**

c $y = -f(x - 2)$ **(3 marks)**

Indicate on each diagram the coordinates of any turning points on your sketch.



18 Functions f and g are defined by

$$f: x \mapsto 4 - x, \qquad \{x \in \mathbb{R}\}$$

$$g: x \mapsto 3x^2, \qquad \{x \in \mathbb{R}\}$$

a Write down the range of g. **(1 mark)**

b Solve $gf(x) = 48$. **(4 marks)**

c Sketch the graph of $y = |f(x)|$ and hence find the values of x for which $|f(x)| = 2$. **(4 marks)**

Chapter Exercises

19 The function f is defined by $f: x \mapsto |2x - a|$, $\{x \in \mathbb{R}\}$, where a is a positive constant.

a Sketch the graph of $y = f(x)$, showing the coordinates of the points where the graph cuts the axes. **(3 marks)**

b On a separate diagram, sketch the graph of $y = f(2x)$, showing the coordinates of the points where the graph cuts the axes. **(2 marks)**

c Given that a solution of the equation $f(x) = \frac{1}{2}x$ is $x = 4$, find the two possible values of a. **(4 marks)**

20 a Sketch the graph of $y = |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph meets the axes. **(3 marks)**

b Using algebra solve, for x in terms of a, $|x - 2a| = \frac{1}{3}x$. **(4 marks)**

c On a separate diagram, sketch the graph of $y = a - |x - 2a|$, where a is a positive constant. Show the coordinates of the points where the graph cuts the axes. **(4 marks)**

21 a Sketch the graph of $y = |2x + a|$, $a > 0$, showing the coordinates of the points where the graph meets the coordinate axes. **(3 marks)**

b On the same axes, sketch the graph of $y = \frac{1}{x}$ **(2 marks)**

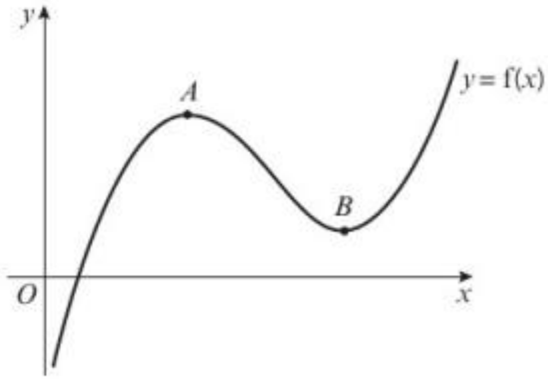
c Explain how your graphs show that there is only one solution of the equation $x|2x + a| - 1 = 0$ **(2 marks)**

d Find, using algebra, the value of x for which $x|2x + a| - 1 = 0$. **(3 marks)**

22 The diagram shows part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^2 - 7x + 5\ln x + 8, \quad x > 0$$

The points A and B are the stationary points of the curve.

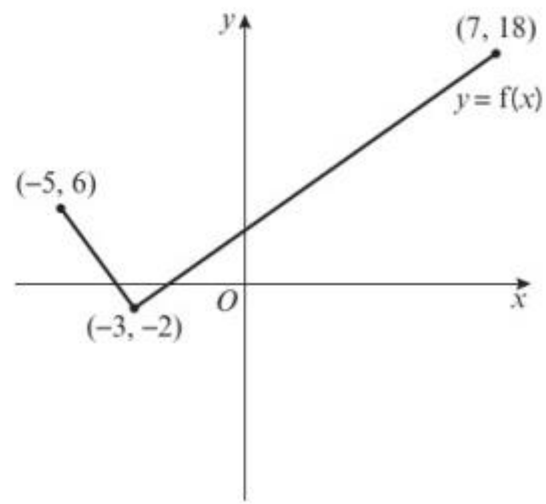


a Using calculus and showing your working, find the coordinates of the points A and B. **(4 marks)**

b Sketch the curve with equation $y = -3\mathrm{f}(x - 2)$. **(3 marks)**

c Find the coordinates of the stationary points of the curve with equation $y = -3\mathrm{f}(x - 2)$. State, without proof, which point is a maximum and which point is a minimum. **(3 marks)**

23 The function f has domain $-5 \leqslant x \leqslant 7$ and is linear from $(-5, 6)$ to $(-3, -2)$ and from $(-3, -2)$ to $(7, 18)$.
The diagram shows a sketch of the function.



a Write down the range of f. **(1 mark)**

b Find $\mathrm{ff}(-3)$. **(2 marks)**

c Sketch the graph of $y = |\mathrm{f}(x)|$, marking the points at which the graph meets or cuts the axes. **(3 marks)**

The function g is defined by $\mathrm{g}: x \mapsto x^2 - 7x + 10$.

d Solve the equation $\mathrm{fg}(x) = 2$. **(3 marks)**

24 The function p is defined by

p: $x \mapsto -2|x + 4| + 10$

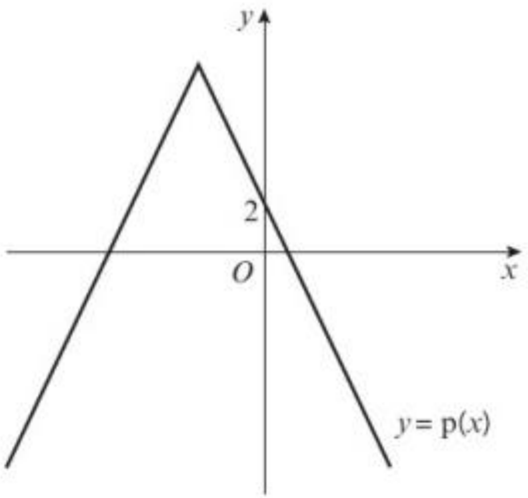
The diagram shows a sketch of the graph.

a State the range of p. **(1 mark)**

b Give a reason why p^{-1} does not exist. **(1 mark)**

c Solve the inequality $p(x) > -4$. **(4 marks)**

d State the range of values of k for which the equation $p(x) = -\frac{1}{2}x + k$ has no solutions. **(4 marks)**



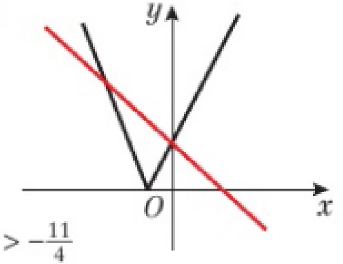
Challenge

a Sketch, on a single diagram, the graphs of $y = a^2 - x^2$ and $y = |x + a|$, where a is a constant and $a > 1$.

b Write down the coordinates of the points where the graph of $y = a^2 - x^2$ cuts the coordinate axes.

c Given that the two graphs intersect at $x = 4$, calculate the value of a.

1 a

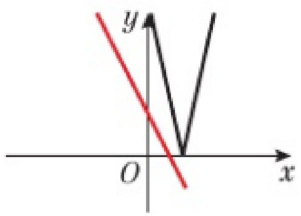


b $x = 0, x = -4$

2 $k > -\frac{11}{4}$

3 $x = -\frac{24}{19}$ and $x = \frac{40}{21}$

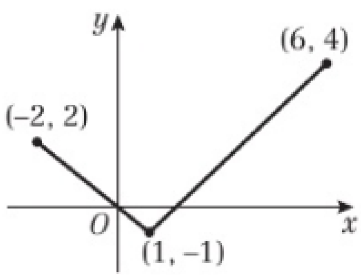
4 a



b The graphs do not intersect, so there are no solutions.

5 a i one-to-many **ii** not a function
b i one-to-one **ii** function
c i many-to-one **ii** function
d i many-to-one **ii** function
e i one-to-one
ii not a function
f i one-to-one
ii not a function

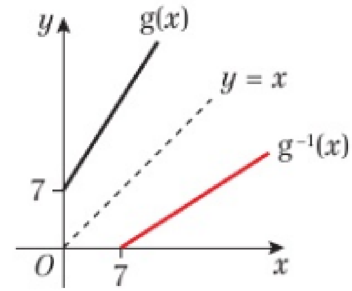
6 a



b $\frac{1}{2}$ and $1\frac{1}{2}$

7 a $\text{pq}(x) = 4x^2 + 10x$ **b** $x = \dfrac{-3 \pm \sqrt{21}}{4}$

8 a Range $\text{g}(x) \geqslant 7$



b $\text{g}^{-1}(x) = \dfrac{x-7}{2}, \{x \in \mathbb{R}, x \geqslant 7\}$

c $\text{g}^{-1}(x)$ is a reflection of $\text{g}(x)$ in the line $y = x$

9 a $\text{f}^{-1}(x) = \dfrac{x+3}{x-2}, \{x \in \mathbb{R}, x > 2\}$

b i Range $\text{f}^{-1}(x) > 1$ **ii** $\{x \in \mathbb{R}, x > 2\}$

10 a $$\text{f}(x) = \frac{x}{x^2-1} - \frac{1}{x+1} = \frac{x}{(x-1)(x+1)} - \frac{1}{x+1}$$
$$= \frac{x}{(x-1)(x+1)} - \frac{x-1}{(x-1)(x+1)} = \frac{1}{(x-1)(x+1)}$$

b $\text{f}(x) > 0$ **c** $x = 6$

11 a $20, 28, \frac{1}{9}$ **b** $\text{f}(x) \geqslant -8, \text{g}(x) \in \mathbb{R}$

c $\text{g}^{-1}(x) = \sqrt[3]{x-1}, \{x \in \mathbb{R}\}$

d $4(x^3 - 1)$ **e** $a = \frac{5}{3}$

12 a $a = -3$ **b** $\text{f}^{-1}: x \mapsto \sqrt{x+13} - 3, x > -4$

13 a $\text{f}^{-1}(x) = \dfrac{x+1}{4}, \{x \in \mathbb{R}\}$

b $\text{gf}(x) = \dfrac{3}{8x-3}, \{x \in \mathbb{R}, x \neq \frac{3}{8}\}$

c -0.076 and 0.826 (3 d.p.)

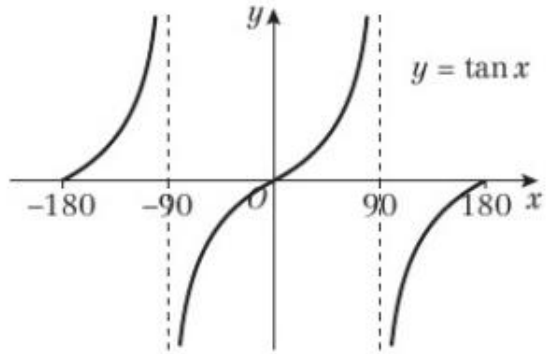
14 a $f^{-1}(x) = \dfrac{2x}{x-1}$, $\{x \in \mathbb{R}, x \neq 1\}$

b Range $f^{-}1(x) \in \mathbb{R}$, $f^{-1}(x) \neq 2$

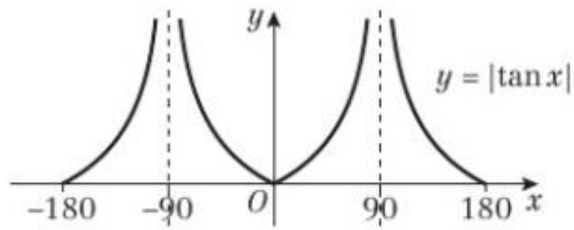
c -1 **d** $1, \frac{6}{5}$

15 a 8, 9 **b** -45 and $5\sqrt{2}$

16 a



b



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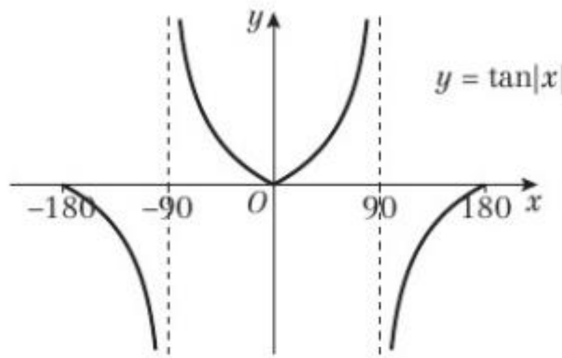
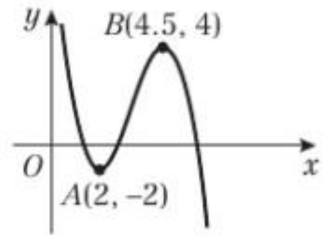


17 a



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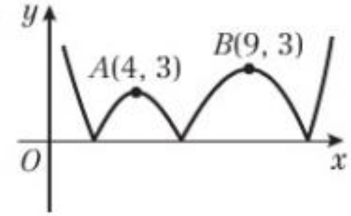

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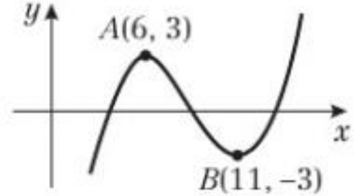
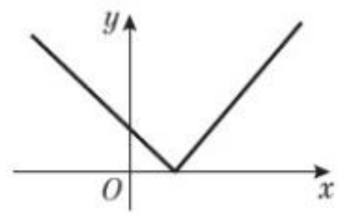


18 a $g(x) \geqslant 0$ **b** $x = 0, x = 8$

c



$x = 2$ and $x = 6$

19 a Positive $|x|$ graph with vertex at $\left(\frac{a}{2}, 0\right)$ and y-intercept at $(0, a)$.

b Positive $|x|$ graph with vertex at $\left(\frac{a}{4}, 0\right)$ and y-intercept at $(0, a)$.

c $a = 6, a = 10$

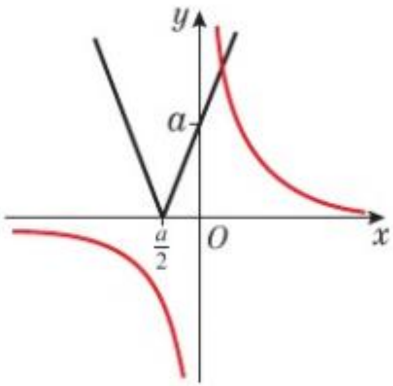
Chapter Answers

20 a Positive $|x|$ graph with vertex at $(2a, 0)$ and y-intercept at $(0, 2a)$.

b $x = \frac{3a}{2}$, $x = 3a$

c Negative $|x|$ graph with x-intercepts at $(a, 0)$ and $(3a, 0)$ and y-intercept at $(0, -a)$.

21 a, b

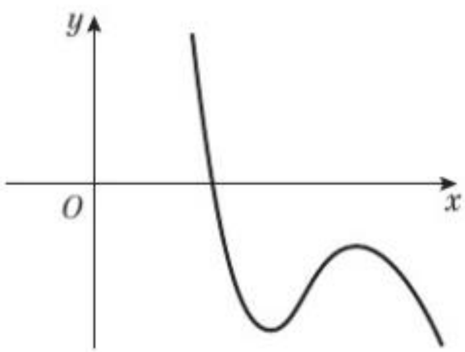


c One intersection point

d $x = \frac{-a + \sqrt{(a^2 + 8)}}{4}$

22 a $(1, 2)$, $(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4})$

b



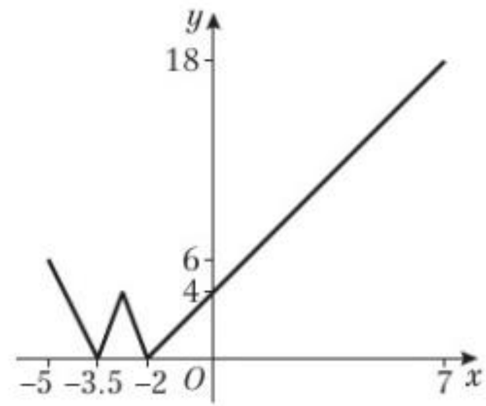
c $(3, -6)$, Minimum

$(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2})$, Maximum

23 a $-2 \leqslant f(x) \leqslant 18$

b 0

c



d $x = \frac{7 \pm \sqrt{5}}{2}$

24 a $p(x) \leqslant 10$

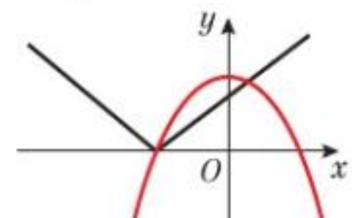
b Original function is many-to-one, therefore the inverse is one-to-many, which is not a function.

c $-11 < x < 3$

d $k > 8$

Challenge

a



b $(-a, 0)$, $(a, 0)$, $(0, a^2)$

c $a = 5$