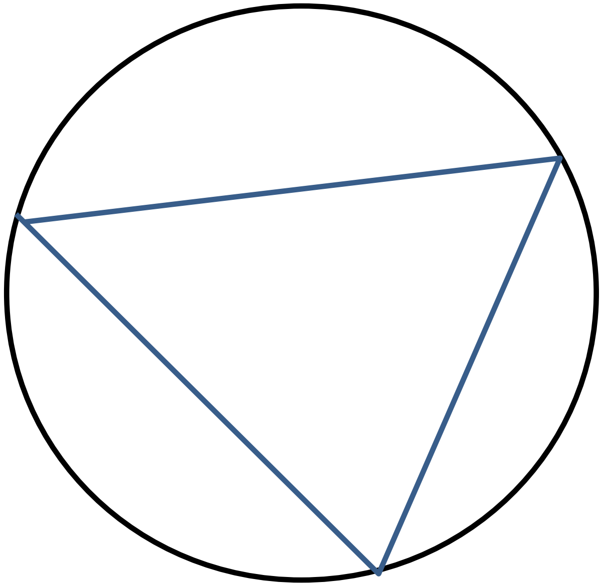
P1 Chapter 6: Circles

Inscribed Triangles

Triangles in Circles

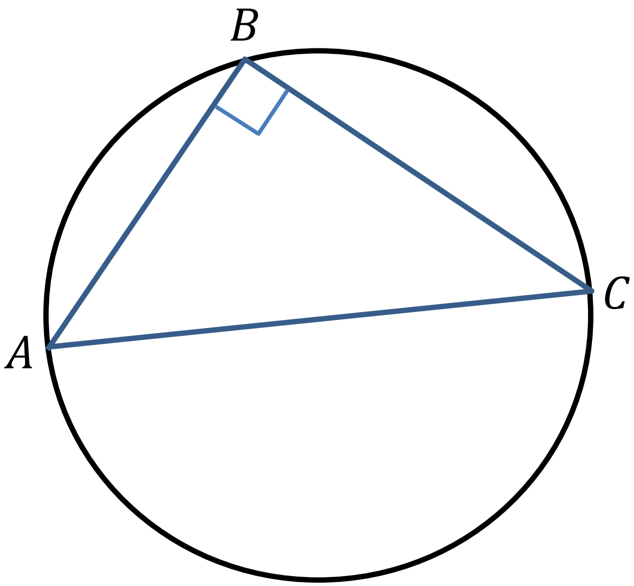


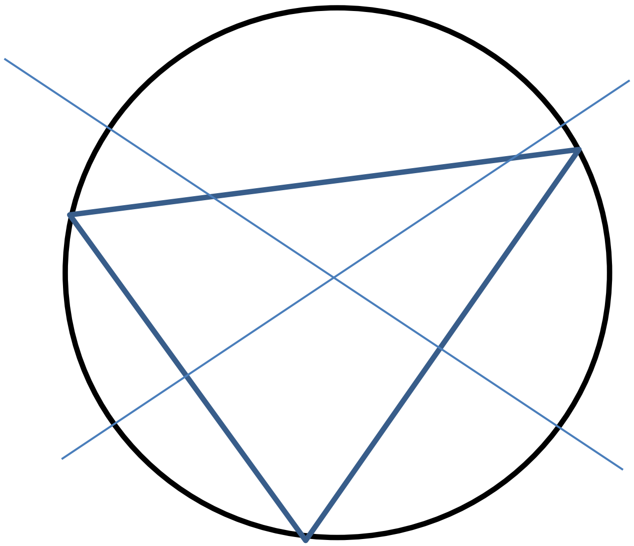
We'd say:

- The triangle **inscribes** the circle.
  (A shape inscribes another if it is inside and its boundaries touch but do not intersect the outer shape)

- The circle **circumscribes** the triangle.
- If the circumscribing shape is a circle, it is known as the **circumcircle** of the triangle.
- The centre of a circumcircle is known as the **circumcentre**.

Triangles in Circles





If $\angle ABC = 90°$ then:

- $\boldsymbol{AC}$ **is the diameter of the circumcircle of triangle** $\boldsymbol{ABC}$**.**

Similarly if AC is the diameter of a circle:

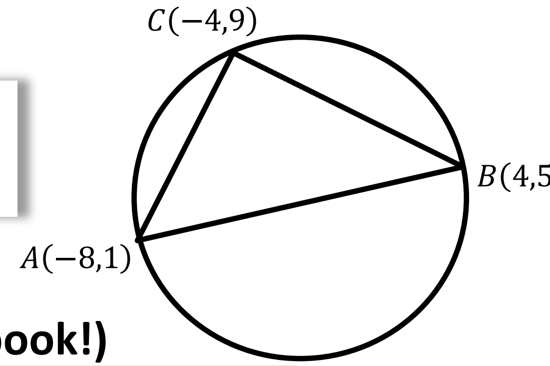
- $\boldsymbol{\angle ABC = 90°}$ **therefore** $\boldsymbol{AB}$ **is perpendicular to** $\boldsymbol{BC}$**.**
- $\boldsymbol{AB^2 + BC^2 = AC^2}$

Given three points/a triangle we can find the centre of the circumcircle by:

- **Finding the equation of the perpendicular bisectors of two different sides.**
- **Find the point of intersection of the two bisectors.**

Example

[Textbook] The points $A(-8,1), B(4,5), C(-4,9)$ lie on a circle.
a) Show that AB is a diameter of the circle.



Method 1:

?

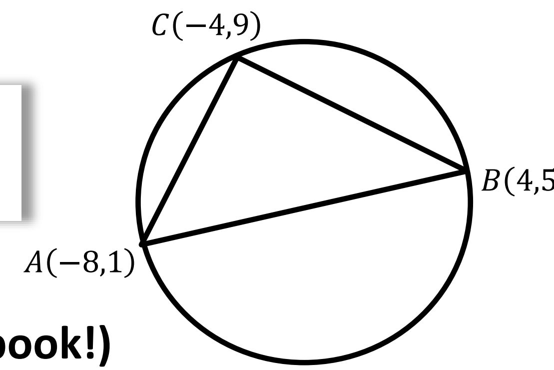
Method 2: (not in textbook!)

?

b) Hence find the equation of the circle.

?

Example



[Textbook] The points $A(-8,1), B(4,5), C(-4,9)$ lie on a circle.
a) Show that AB is a diameter of the circle.

Method 1:
Show that $AC^2 + BC^2 = AB^2$

$$AC = \sqrt{4^2 + 8^2} = \sqrt{80}$$
$$BC = \sqrt{8^2 + 4^2} = \sqrt{80}$$
$$AB = \sqrt{12^2 + 4^2} = \sqrt{160}$$
$$80 + 80 = 160$$

Therefore AB is the diameter.

Method 2: (not in textbook!)
Show that AC is perpendicular to BC.

$$m_{AC} = \frac{8}{4} = 2$$
$$m_{CB} = -\frac{4}{8} = -\frac{1}{2}$$
$$2 \times -\frac{1}{2} = -1$$

$\therefore AC$ is perpendicular to BC
Therefore AB is the diameter.

b) Hence find the equation of the circle.

Centre is midpoint of AB: $M(-2,3)$
Radius: $r = AM = \sqrt{6^2 + 2^2} = \sqrt{40}$
$$(x+2)^2 + (y-3)^2 = 40$$

Example

The points $A(0,2), B(2,0), C(8,18)$ lie on the circumference of a circle. Determine the equation of the circle.

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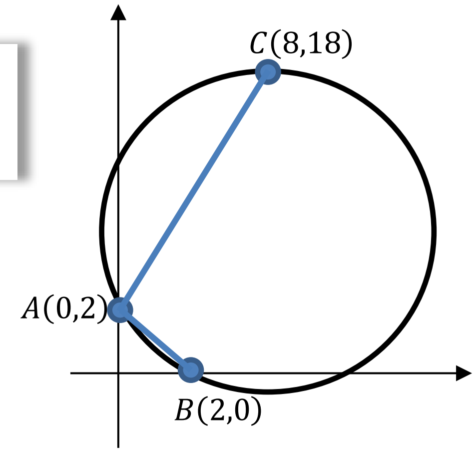
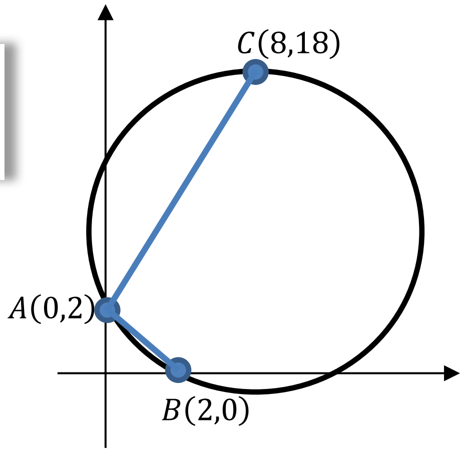


Example

The points $A(0,2), B(2,0), C(8,18)$ lie on the circumference of a circle. Determine the equation of the circle.



Perpendicular bisector of AB:
By inspection, $y = x$

Perpendicular bisector of AC:
Midpoint: $(4,10)$

$$m_{AC} = \frac{16}{8} = 2 \quad \therefore m_{\perp} = -\frac{1}{2}$$

$$y - 10 = -\frac{1}{2}(x - 4)$$

Solving simultaneously with $y = x$:

$$x - 10 = -\frac{1}{2}(x - 4)$$

$$2x - 20 = -x + 4$$

$$3x = 24$$

$$x = 8 \quad \therefore y = 8$$

Centre is $(8,8)$.
Using A and centre of circle: $r = \sqrt{8^2 + 6^2} = 10$
Equation of circle: $(x - 8)^2 + (y - 8)^2 = 100$

Exercise 6.5

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Extension:

1 *[STEP 2009 Q8 Edited]* If equation of the circle C is $(x-2t)^2+(y-t)^2=t^2$, where t is a positive number, it can be shown that C touches the line $y=0$ as well as the line $3y=4x$.

Find the equation of the incircle of the triangle formed by the lines $y=0$, $3y=4x$ and $4y+3x=15$.
Note: The incircle of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

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Note: The incircle of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

Solution: $\boldsymbol{(x-2)^2+(y-1)^2=1}$

Homework Exercise

1 The points $U(-2, 8)$, $V(7, 7)$ and $W(-3, -1)$ lie on a circle.

a Show that triangle UVW has a right angle.

b Find the coordinates of the centre of the circle.

c Write down an equation for the circle.

2 The points $A(2, 6)$, $B(5, 7)$ and $C(8, -2)$ lie on a circle.

a Show that AC is a diameter of the circle.

b Write down an equation for the circle.

c Find the area of the triangle ABC.

3 The points $A(-3, 19)$, $B(9, 11)$ and $C(-15, 1)$ lie on the circumference of a circle.

a Find the equation of the perpendicular bisector of

i AB **ii** AC

b Find the coordinates of the centre of the circle.

c Write down an equation for the circle.

4 The points $P(-11, 8)$, $Q(-6, -7)$ and $R(4, -7)$ lie on the circumference of a circle.

a Find the equation of the perpendicular bisector of

i PQ **ii** QR

b Find an equation for the circle.

5 The points $R(-2, 1)$, $S(4, 3)$ and $T(10, -5)$ lie on the circumference of a circle C. Find an equation for the circle.

Problem-solving

Use headings in your working to keep track of what you are working out at each stage.

Homework Exercise

6 Consider the points $A(3, 15)$, $B(-13, 3)$, $C(-7, -5)$ and $D(8, 0)$.

a Show that ABC is a right-angled triangle.

b Find the equation of the circumcircle.

c Hence show that A, B, C and D all lie on the circumference of this circle.

7 The points $A(-1, 9)$, $B(6, 10)$, $C(7, 3)$ and $D(0, 2)$ lie on a circle.

a Show that $ABCD$ is a square.

b Find the area of $ABCD$.

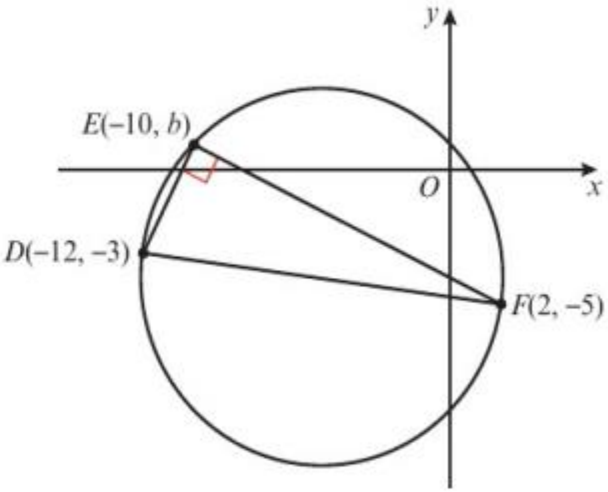
c Find the centre of the circle.

8 The points $D(-12, -3)$, $E(-10, b)$ and $F(2, -5)$ lie on the circle C as shown in the diagram.

Given that $\angle DEF = 90°$ and $b > 0$

a show that $b = 1$ **(5 marks)**

b find an equation for C. **(4 marks)**



9 A circle has equation $x^2 + 2x + y^2 - 24y - 24 = 0$

a Find the centre and radius of the circle. **(3 marks)**

b The points $A(-13, 17)$ and $B(11, 7)$ both lie on the circumference of the circle. Show that AB is a diameter of the circle. **(3 marks)**

c The point C lies on the negative x-axis and the angle $ACB = 90°$. Find the coordinates of C. **(3 marks)**

Homework Answers

1 **a** $WV^2 = WU^2 + UV^2$
b (2, 3)
c $(x - 2)^2 + (y - 3)^2 = 41$

2 **a** $AC^2 = AB^2 + BC^2$
b $(x - 5)^2 + (y - 2)^2 = 25$
c 15

3 **a** **i** $y = \frac{3}{2}x + \frac{21}{2}$ **ii** $y = -\frac{2}{3}x + 4$
b (−3, 6)
c $(x + 3)^2 + (y - 6)^2 = 169$

4 **a** **i** $y = \frac{1}{3}x + \frac{10}{3}$ **ii** $x = -1$
b $(x + 1)^2 + (y - 3)^2 = 125$

5 $(x - 3)^2 + (y + 4)^2 = 50$

6 **a** $AB^2 + BC^2 = AC^2$
$AB^2 = 400$, $BC^2 = 100$, $AC^2 = 500$
b $(x + 2)^2 + (y - 5)^2 = 125$
c $D(8, 0)$ satisfies the equation of the circle.

7 **a** $AB = BC = CD = DA = \sqrt{50}$
b 50
c (3,6)

8 **a** $DE^2 = b^2 + 6b + 13$
$EF^2 = b^2 + 10b + 169$
$DF^2 = 200$
So $b^2 + 6b + 13 + b^2 + 10b + 169 = 200$
$(b + 9)(b - 1) = 0$; as $b > 0$, $b = 1$
b $(x + 5)^2 + (y + 4)^2 = 50$

9 **a** Centre (−1, 12) and radius = 13
b Use distance formula to find $AB = 26$. This is twice radius, so AB is the diameter. Other methods possible.
c $C(-6, 0)$