Stats Yr2 Chapter 3: Distribution-N

Normal Hypothesis Testing

Hypothesis Testing on the Sample Mean

0 1 2 3 4 5 6 7 8 9

Imagine we have 10 children, one of each age between 0 and 9.
This is our population. There is a **known population mean** of $\mu = 4.5$

					$\bar{x}$
Sample 1:	1	3	7	8	4.75
Sample 2:	6	2	0	9	4.25

…

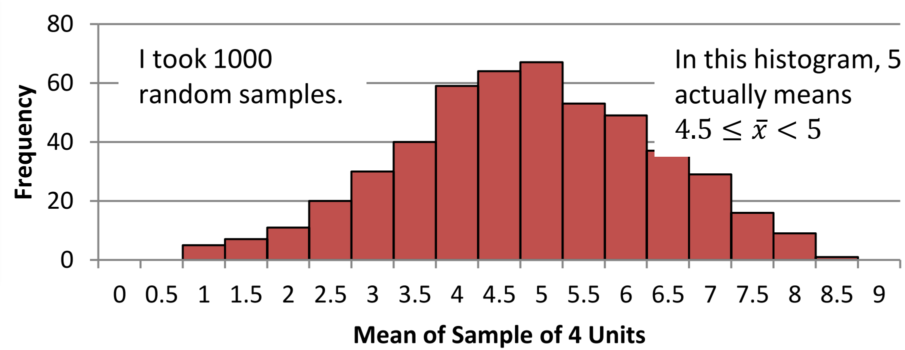
The mean of this sample is $\bar{x} = 4.75$. This sample mean $\bar{x}$ is close the true population mean μ, but is naturally going to vary as we consider different samples.

For a different sample of 4, we might obtain a different sample mean. What would happen if we took lots of different samples of 4, and found the mean $\bar{x}$ of each? How would these means be distributed?

Suppose we took a sample of 4 children.

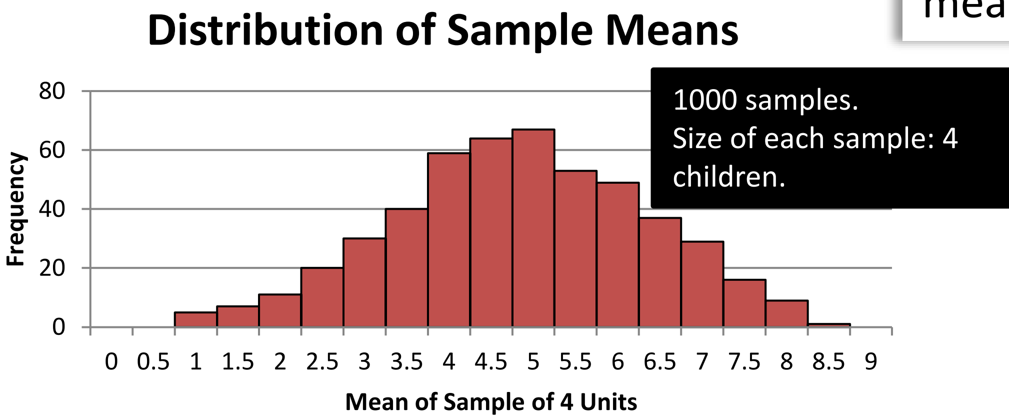
Sample mean $\bar{x}$	**Tally**
4.00	
4.25	$\vert$
4.50	
4.75	$\vert$
5.00	

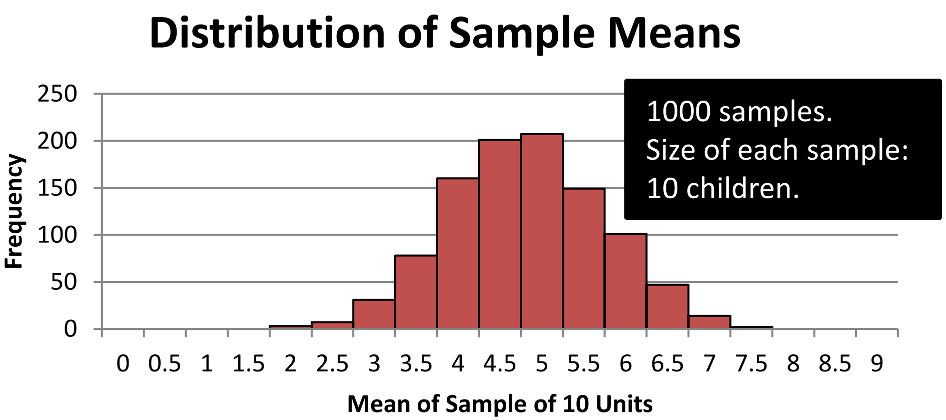
Distribution of Sample Means $\bar{X}$



Hypothesis Testing on the Sample Mean

$\bar{X}$ is our distribution across different sample means as we consider different samples.





✎ For a random sample of size n taken from a random variable X, the sample mean $\bar{X}$ is normally distributed with $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Question 1: What type of distribution is $\bar{X}$?
From the left it seems like it is approximately normally distributed!

Question 2: On average, what sample mean do we see? (i.e. the mean of the means!)
μ. The sample means $\bar{x}$ vary around the population mean μ, but on average is μ.

Question 3: Is the variance of $\bar{X}$ (i.e. how spread out the sample means are) the same as that of the variance of the population of children?
No! On the left, we can see that how spread out the sample means are depends on the sample size. If the sample size is small, the sample means are likely to vary quite a bit. But with a larger sample size, we expect the different $\bar{x}$ to be closer to the population mean μ.

Examples

[Textbook] A certain company sells fruit juice in cartons. The amount of juice in a carton has a normal distribution with a standard deviation of 3ml.
The company claims that the mean amount of juice per carton, μ, is 60ml. A trading inspector has received complaints that the company is overstating the mean amount of juice per carton and wishes to investigate this complaint. The trading inspector takes a random sample of 16 cartons and finds that the mean amount of juice per carton is 59.1ml.

Using a 5% level of significance, and stating your hypotheses clearly, test whether or not there is evidence to uphold this complaint.

H_0: ?
H_1: ?
Let X be ?
Assuming H_0, ?
$\therefore \bar{X} \sim$?

?

As always, state the distribution X and its distribution under H_0.

Use X to work out the distribution of the sample means, $\bar{X}$.

Note: Don't confuse X and $\bar{X}$. The X is the distribution over amounts of drink in each individual carton. $\bar{X}$ is the distribution over **sample means**, i.e. the possible sample means we see as we take samples of 16 cartons. X might not be normally distributed, but $\bar{X}$ will be.

Examples

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Using a 5% level of significance, and stating your hypotheses clearly, test whether or not there is evidence to uphold this complaint.

$H_0: \mu = 60$
$H_1: \mu < 60$
Let X be the amount of juice per carton.
Assuming H_0, $X \sim N(60, 3^2)$

$$\therefore \bar{X} \sim N\left(60, \frac{3^2}{16}\right) \quad \rightarrow \quad \bar{X} \sim N(60, 0.75^2)$$

$P(\bar{X} < 59.1) = 0.1151$
$0.1151 > 0.05$ so there is insufficient evidence to reject H_0 and conclude that the mean amount of juice in the whole population is less than 60 ml.

As always, state the distribution X and its distribution under H_0.

Use X to work out the distribution of the sample means, $\bar{X}$.

Note: Don't confuse X and $\bar{X}$. The X is the distribution over amounts of drink in each individual carton. $\bar{X}$ is the distribution over **sample means**, i.e. the possible sample means we see as we take samples of 16 cartons. X might not be normally distributed, but $\bar{X}$ will be.

Finding the critical region

[Textbook] A machine products bolts of diameter D where D has a normal distribution with mean 0.580 cm and standard deviation 0.015 cm. The machine is serviced and after the service a random sample of 50 bolts from the next production run is taken to see if the mean diameter of the bolts has changed from 0.580 cm. The distribution of the diameters of bolts after the service is still normal with a standard deviation of 0.015 cm.

(a) Find, at the 1% level, the critical region for this test, stating your hypotheses clearly.
The mean diameter of the sample of 50 bolts is calculated to be 0.587 cm.
(b) Comment on this observation in light of the critical region.

a

H_0: ?

Assuming H_0, ?

$\overline{D} \sim$?

$z = \pm$?

?

b

?

This is a two-tailed test so want the z value corresponding to the top 0.5% and bottom 0.5% respectively. Use table below or your calculator.

Use the usual formula $Z = \frac{X-\mu}{\sigma}$, except this time we use the standard deviation of $\bar{X}$: $\sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$

p	z	p	z
0.5000	0.0000	0.0500	1.6449
0.4000	0.2533	0.0250	1.9600
0.3000	0.5244	0.0100	2.3263
0.2000	0.8416	0.0050	2.5758
0.1500	1.0364	0.0010	3.0902
0.1000	1.2816	0.0005	3.2905

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(a) Find, at the 1% level, the critical region for this test, stating your hypotheses clearly.
The mean diameter of the sample of 50 bolts is calculated to be 0.587 cm.
(b) Comment on this observation in light of the critical region.

a $H_0: \mu = 0.580, H_1: \mu \neq 0.580$
Assuming H_0, $D \sim N(0.580, 0.015^2)$

$$\bar{D} \sim N\left(0.580, \frac{0.015^2}{50}\right) \rightarrow \bar{D} \sim N\left(0.580, \left(\frac{0.015}{\sqrt{50}}\right)^2\right)$$

$z = \pm 2.5758$

$$-2.5758 = \frac{\bar{d} - 0.580}{\frac{0.015}{\sqrt{50}}} \rightarrow \bar{d} = 0.5745$$

$$2.5758 = \frac{\bar{d} - 0.580}{\frac{0.015}{\sqrt{50}}} \rightarrow \bar{d} = 0.5854$$

Critical region is $\bar{D} \leq 0.575$ or $\bar{D} \geq 0.585$ $(3sf)$

b Observed value falls within critical region so sufficient evidence at 1% level that mean diameter has changed from 0.580cm.

This is a two-tailed test so want the z value corresponding to the top 0.5% and bottom 0.5% respectively. Use table below or your calculator.

Use the usual formula $Z = \frac{X-\mu}{\sigma}$, except this time we use the standard deviation of $\bar{X}$: $\sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$

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Test Your Understanding

Edexcel S3 June 2011 Q7a

Roastie's Coffee is sold in packets with a stated weight of 250 g. A supermarket manager claims that the mean weight of the packets is less than the stated weight. She weighs a random sample of 90 packets from their stock and finds that their weights have a mean of 248 g and a standard deviation of 5.4 g.

(*a*) Using a 5% level of significance, test whether or not the manager's claim is justified. State your hypotheses clearly.

(5)

(a)

?

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(*a*) Using a 5% level of significance, test whether or not the manager's claim is justified. State your hypotheses clearly.

(5)

(a)	$H_0: \mu=250$, $H_1: \mu<250$,	B1
	$z = \dfrac{248-250}{\dfrac{5.4}{\sqrt{90}}}$	M1
	$= -3.513...$	A1
	Critical value -1.6449	B1
	-3.513..<-1.6449 so sufficient evidence to reject H_0 Manager's claim is justified.	A1

Exercise 3.7

Pearson Stats/Mechanics Year 2
Pages 29-30

Homework Exercise

1 In each part, a random sample of size n is taken from a population having a normal distribution with mean μ and variance σ^2. Test the hypotheses at the stated levels of significance.

a $H_0: \mu = 21$, $H_1: \mu \neq 21$, $n = 20$, $\bar{x} = 21.2$, $\sigma = 1.5$, at the 5% level

b $H_0: \mu = 100$, $H_1: \mu < 100$, $n = 36$, $\bar{x} = 98.5$, $\sigma = 5.0$, at the 5% level

c $H_0: \mu = 5$, $H_1: \mu \neq 5$, $n = 25$, $\bar{x} = 6.1$, $\sigma = 3.0$, at the 5% level

d $H_0: \mu = 15$, $H_1: \mu > 15$, $n = 40$, $\bar{x} = 16.5$, $\sigma = 3.5$, at the 1% level

e $H_0: \mu = 50$, $H_1: \mu \neq 50$, $n = 60$, $\bar{x} = 48.9$, $\sigma = 4.0$, at the 1% level

2 In each part, a random sample of size n is taken from a population having a $N(\mu, \sigma^2)$ distribution. Find the critical regions for the test statistic $\bar{X}$ in the following tests.

a $H_0: \mu = 120$, $H_1: \mu < 120$, $n = 30$, $\sigma = 2.0$, at the 5% level

b $H_0: \mu = 12.5$, $H_1: \mu > 12.5$, $n = 25$, $\sigma = 1.5$, at the 1% level

c $H_0: \mu = 85$, $H_1: \mu < 85$, $n = 50$, $\sigma = 4.0$, at the 10% level

d $H_0: \mu = 0$, $H_1: \mu \neq 0$, $n = 45$, $\sigma = 3.0$, at the 5% level

e $H_0: \mu = -8$, $H_1: \mu \neq -8$, $n = 20$, $\sigma = 1.2$, at the 1% level

3 The times taken for a capful of stain remover to remove a standard chocolate stain from a baby's bib are normally distributed with a mean of 185 seconds and a standard deviation of 15 seconds. The manufacturers of the stain remover claim to have developed a new formula which will shorten the time taken for a stain to be removed. A random sample of 25 capfuls of the new formula are tested and the mean time for the sample is 179 seconds.

Test, at the 5% level, whether or not there is evidence that the new formula is an improvement.

Hint You are testing for an improvement, so use a **one-tailed** test.

Homework Exercise

4 The IQ scores of a population are normally distributed with a mean of 100 and a standard deviation of 15. A psychologist wishes to test the theory that eating chocolate before sitting an IQ test improves your score. A random sample of 80 people are selected and they are each given an identical bar of chocolate to eat before taking an IQ test.

a Find, at the 2.5% level, the critical region for this test, stating your hypotheses clearly.

The mean score on the test for the sample of 80 people was 102.5.

b Comment on this observation in light of the critical region.

5 The diameters of circular cardboard drinks mats produced by a certain machine are normally distributed with a mean of 9 cm and a standard deviation of 0.15 cm. After the machine is serviced a random sample of 30 mats is selected and their diameters are measured to see if the mean diameter has altered.

The mean of the sample was 8.95 cm.
Test, at the 5% level, whether there is significant evidence of a change in the mean diameter of mats produced by the machine.

Hint You are testing for an alteration in either direction, so use a **two-tailed** test.

6 Daily mean windspeed is modelled as being normally distributed with a standard deviation of 3.1 knots.

A random sample of 25 recorded daily mean windspeeds is taken at Heathrow in 2015.

Given that the mean of the sample is 12.2 knots, test at the 2.5% level of significance whether the mean of the daily mean windspeeds is greater than 9.5 knots.
State your hypotheses clearly. **(4 marks)**

Homework Exercise

7 A machine produces metal bolts of diameter D mm, where D is normally distributed with standard deviation 0.1 mm. Bolts with diameter either less than 5.1 mm or greater than 5.6 mm cannot be sold.

Given that 5% of bolts have a diameter in excess of 5.62 mm,

a find the probability that a randomly chosen bolt can be sold. **(5 marks)**

Twelve bolts are chosen.

b Find the probability that fewer than three cannot be sold. **(2 marks)**

A second machine produces bolts of diameter Y mm, where Y is normally distributed with standard deviation 0.08 mm.

A random sample of 20 bolts produced by this machine is taken and the sample mean of the diameters is found to be 5.52 mm.

c Stating your hypotheses clearly, and using a 2.5% level of significance, test whether the mean diameter of all the bolts produced by the machine is less than 5.7 mm. **(4 marks)**

8 The mass of European water voles, M grams, is normally distributed with standard deviation 12 grams.

Given that 2.5% of water voles have a mass greater than 160 grams,

a find the mean mass of a European water vole. **(3 marks)**

Eight water voles are chosen at random.

b Find the probability that at least 4 have a mass greater than 150 grams. **(3 marks)**

European water rats have mass, N grams, which is normally distributed with standard deviation 85 grams.

A random sample of 15 water rats is taken and the sample mean mass is found to be 875 grams.

c Stating your hypotheses clearly, and using a 10% level of significance, test whether the mean mass of all water rats is different from 860 grams. **(4 marks)**

Homework Answers

For Chapter 3, student answers may differ slightly from those shown here when calculators are used rather than table values.

1 **a** Not significant. Accept H_0.
b Significant. Reject H_0.
c Not significant. Accept H_0.
d Significant. Reject H_0.
e Not significant. Accept H_0.

2 **a** $\overline{X} < 119.39...$ or 119 (3 s.f.)
b $\overline{X} > 13.2$
c $\overline{X} < 84.3$
d $\overline{X} > 0.877$ or $\overline{X} < -0.877$
e $\overline{X} > -7.31$ or $\overline{X} < -8.69$

3 Result is significant so reject H_0. There is evidence that the new formula is an improvement.

4 **a** $\overline{X} \geqslant 103.29$
b $102.5 < 103.29$, so there is not enough evidence to reject the null hypothesis

5 Insufficient evidence; accept H_0.

6 H_0: $\mu = 9.5$, H_1: $\mu > 9.5$. Critical region is $\overline{X} \geqslant 10.715$. $\bar{x} = 12.2 > 10.715$, so reject H_0 and conclude that the mean daily windspeed is greater than 9.5 knots.

7 **a** 0.9256 **b** 0.9455
c H_0: $\mu = 5.7$, H_1: $\mu < 5.7$. There is sufficient evidence to suggest mean diameter less than 5.7 mm.

8 **a** 136.48 g **b** 0.01289
c H_0: $\mu = 860$, H_1: $\mu \neq 860$. Insufficient evidence to suggest mean mass is different to 860 g.