P1 Chapter 7: Algebraic Methods

Methods of Proof

Other Types of Proof

b. Proof by Exhaustion

This means breaking down the statement into **all possible smaller cases**, where we prove each individual case.
(This technique is sometimes known as 'case analysis')

Prove that $n^2 + n$ is even for all integers n.

?

c. Disproof by Counter-Example

While to prove a statement is true, we need to prove every possible case (potentially infinitely many!), **we only need one example to disprove** a statement.
This is known as a **counterexample**.

Disprove the statement:
"$n^2 - n + 41$ is prime for all integers n."

?

Other Types of Proof

b. Proof by Exhaustion

This means breaking down the statement into **all possible smaller cases**, where we prove each individual case.
(This technique is sometimes known as 'case analysis')

Prove that $n^2 + n$ is even for all integers n.

n is either even or odd.
If n is even:

$$\begin{aligned} n^2 + n &= even \times even + even \\ &= even + even \\ &= even \end{aligned}$$

If n is odd:

$$\begin{aligned} n^2 + n &= odd \times odd + odd \\ &= odd + odd \\ &= even \end{aligned}$$

$\therefore n^2 + n$ is even for all integers n.

c. Disproof by Counter-Example

While to prove a statement is true, we need to prove every possible case (potentially infinitely many!), **we only need one example to disprove** a statement.
This is known as a **counterexample**.

Disprove the statement:
"$n^2 - n + 41$ is prime for all integers n."

If $n = 41$, then we have $41^2 - 41 + 41$
$= 41^2$
Which is not prime as it has a factor of 41.
Thus the statement is not true.

Further Types of Proof

d. Proof by Induction

This proves a conjecture of an n-th term formulae given n=1 is known to be true.

These are only in Further Maths

c. Proof by Contradiction

The classic example is proving the square root of two is irrational.

Exercise 7.5

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Homework Exercise

1 Prove that when n is an integer and $1 \leqslant n \leqslant 6$, then $m = n + 2$ is not divisible by 10.

Hint You can try each integer for $1 \leqslant n \leqslant 6$.

2 Prove that every odd integer between 2 and 26 is either prime or the product of two primes.

3 Prove that the sum of two consecutive square numbers between 1^2 to 8^2 is an odd number.

4 Prove that all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9.

(4 marks)

5 Find a counter-example to disprove each of the following statements:

a If n is a positive integer then $n^4 - n$ is divisible by 4.

b Integers always have an even number of factors.

c $2n^2 - 6n + 1$ is positive for all values of n.

d $2n^2 - 2n - 4$ is a multiple of 3 for all integer values of n.

6 A student is trying to prove that $x^3 + y^3 < (x + y)^3$.

The student writes:

> $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
> which is less than $x^3 + y^3$ since
> $3x^2y + 3xy^2 > 0$

a Identify the error made in the proof. **(1 mark)**

b Provide a counter-example to show that the statement is not true. **(2 marks)**

Problem-solving For part **b** you need to write down suitable values of x and y and show that they do not satisfy the inequality.

7 Prove that for all real values of x

$$(x + 6)^2 \geqslant 2x + 11$$

(3 marks)

Homework Exercise

8 Given that a is a positive real number, prove that:

$$a + \frac{1}{a} \geqslant 2$$

Watch out Remember to state how you use the condition that a is positive.

(2 marks)

9 **a** Prove that for any positive numbers p and q:

$$p + q > \sqrt{4pq}$$

(3 marks)

b Show, by means of a counter-example, that this inequality does not hold when p and q are both negative. **(2 marks)**

Problem-solving Use jottings and work backwards to work out what expression to consider.

10 It is claimed that the following inequality is true for all negative numbers x and y:

$$x + y \geqslant \sqrt{x^2 + y^2}$$

The following proof is offered by a student:

$x + y \geqslant \sqrt{x^2 + y^2}$

$(x + y)^2 \geqslant x^2 + y^2$

$x^2 + y^2 + 2xy \geqslant x^2 + y^2$

$2xy > 0$ which is true because x and y are both negative, so xy is positive.

a Explain the error made by the student. **(2 marks)**

b By use of a counter-example, verify that the inequality is not satisfied if both x and y are negative. **(1 mark)**

c Prove that this inequality is true if x and y are both positive. **(2 marks)**

Homework Answers

1. 3, 4, 5, 6, 7 and 8 are not divisible by 10
2. 3, 5, 7, 11, 13, 17, 19, 23 are prime numbers. 9, 15, 21, 25, are the product of two prime numbers.
3. $2^2 + 3^2 =$ odd, $3^2 + 4^2 =$ odd, $4^2 + 5^2 =$ odd, $5^2 + 6^2 =$ odd, $6^2 + 7^2 =$ odd
4. $(3n)^3 = 27n^3 = 9n(3n^2)$ which is a multiple of 9
   $(3n + 1)^3 = 27n^3 + 27n^2 + 9n + 1 = 9n(3n^2 + 3n + 1) + 1$
   which is one more than a multiple of 9
   $(3n + 2)^3 = 27n^3 + 54n^2 + 36n + 8 = 9n(3n^2 + 6n + 4) + 8$
   which is one less than a multiple of 9
5. **a** For example, when $n = 2$, $2^4 - 2 = 14$, 14 is not divisible by 4.
   b Any square number
   c For example, when $n = \frac{1}{2}$
   d For example, when $n = 1$
6. **a** Assuming that x and y are positive
   b e.g. $x = 0$, $y = 0$
7. $(x + 5)^2 \geqslant 0$ for all real values of x, and
   $(x + 5)^2 + 2x + 11 = (x + 6)^2$, so $(x + 6)^2 \geqslant 2x + 11$
8. If $a^2 + 1 \geqslant 2a$ (a is positive, so multiplying both sides by a does not reverse the inequality), then
   $a^2 - 2a + 1 \geqslant 0$, and $(a - 1)^2 \geqslant 0$, which we know is true.
9. **a** $(p + q)^2 = p^2 + 2pq + q^2 = (p + q)^2 + 4pq$
   $(p - q)^2 \geqslant 0$ since it is a square, so $(p + q)^2 \geqslant 4pq$
   $p > 0, q > 0 \Rightarrow p + q > 0 \Rightarrow p + q \geqslant \sqrt{4pq}$
   b e.g. $p = q = -1$: $p + q = -2$, $\sqrt{4pq} = 2$
10. **a** Starts by assuming the inequality is true:
    i.e. negative $\geqslant$ positive
    b e.g. $x = y = -1$: $x + y = -2$, $\sqrt{x^2 + y^2} = \sqrt{2}$
    c $(x + y)^2 = x^2 + 2xy + y^2 \geqslant x^2 + y^2$ since $x > 0$, $y > 0 \Rightarrow 2xy > 0$
    As $x + y > 0$, can take square roots: $x + y \geqslant \sqrt{x^2 + y^2}$