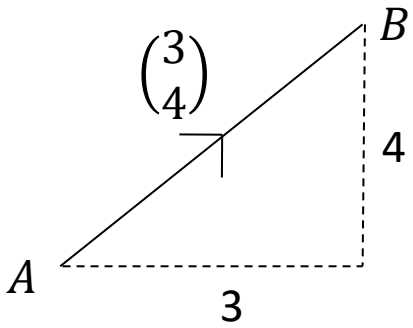
P1 Chapter 11: Vectors

Magnitude and Direction

Magnitude of a Vector

The magnitude $|a|$ of a vector $\vec{a}$ is its length.

If $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow |\vec{a}| = \sqrt{x^2 + y^2}$



$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 4^2} = 5$$

$$\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| = \sqrt{2}$$

$$\left|\begin{pmatrix} -5 \\ -12 \end{pmatrix}\right| = 13$$

$\vec{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $|\vec{a}| =$?

$\vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $|\vec{b}| =$?

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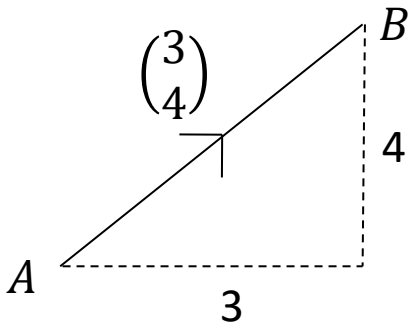

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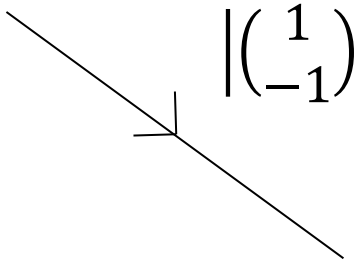
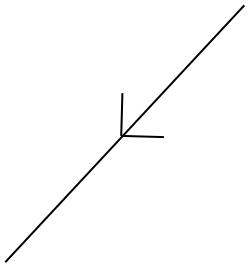


$$\left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| = \sqrt{2}$$



$$\left|\begin{pmatrix} -5 \\ -12 \end{pmatrix}\right| = 13$$

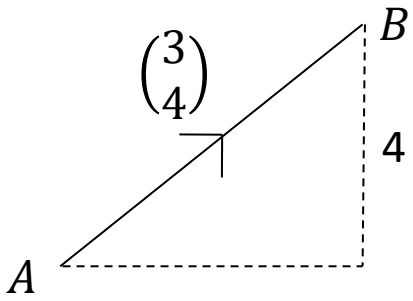
$$\vec{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad |\vec{a}| = \sqrt{4^2 + 1^2} = \sqrt{17}$$

$$\vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad |\vec{b}| = \sqrt{2^2 + 0^2} = 2$$

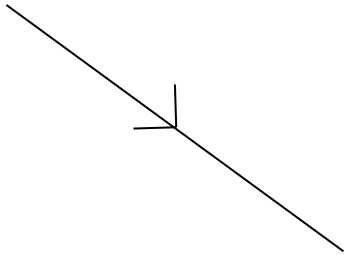
Direction of a Vector

The direction of a vector $\vec{a}$ is the angle it makes with the x -axis.

If $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \tan(\theta) = \frac{y}{x}$



$$\tan(\theta) = \frac{4}{3} \Rightarrow \theta = 53.1$$



$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \tan(\theta) = \frac{-1}{1} \Rightarrow \theta = -45$$

$\vec{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\tan(\theta) =$?

$\vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\tan(\theta) =$?

Direction of a Vector

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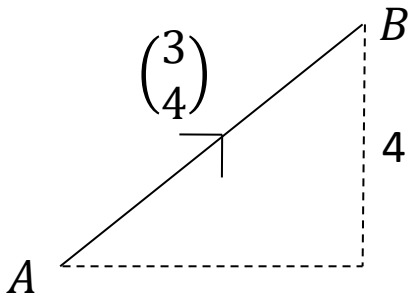
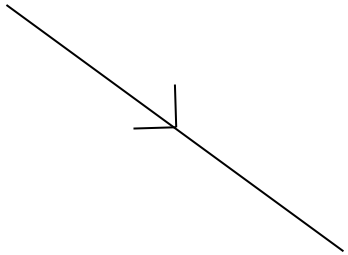


$$\tan(\theta) = \frac{4}{3} \Rightarrow \theta = 53.1$$



$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \tan(\theta) = \frac{-1}{1} \Rightarrow \theta = -45$$

$\vec{a} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ $\tan(\theta) = \frac{-1}{4} \Rightarrow \theta = -14.0$

$\vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\tan(\theta) = \frac{0}{2} \Rightarrow \theta = 0$

Unit Vectors

A unit vector is a vector whose magnitude is 1

There's certain operations on vectors that require the vectors to be 'unit' vectors. We just scale the vector so that its magnitude is now 1.

$$\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$
$$|\vec{a}| = \sqrt{3^2 + 4^2} = 5$$
$$\hat{\boldsymbol{a}} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

If $\vec{a}$ is a vector, then the unit vector $\hat{\boldsymbol{a}}$ in the same direction is

$$\hat{\boldsymbol{a}} = \frac{\vec{a}}{|\vec{a}|}$$

Test Your Understanding: Convert the following vectors to unit vectors.

$$\vec{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$$
$$|\vec{a}| = \sqrt{\mathbf{12^2 + 5^2}} = \mathbf{13}$$
$$\therefore \hat{\boldsymbol{a}} = \begin{pmatrix} \frac{\mathbf{12}}{\mathbf{13}} \\ -\frac{\mathbf{5}}{\mathbf{13}} \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$|\vec{b}| = \sqrt{\mathbf{1^2 + 1^2}} = \sqrt{\mathbf{2}}$$
$$\therefore \hat{\boldsymbol{b}} = \begin{pmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \\ \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \end{pmatrix}$$

Unit Vectors

A unit vector is a vector whose magnitude is 1

There's certain operations on vectors that require the vectors to be 'unit' vectors. We just scale the vector so that its magnitude is now 1.

$\vec{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

?

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$$\hat{\boldsymbol{a}} = \frac{\vec{a}}{|\vec{a}|}$$

Test Your Understanding: Convert the following vectors to unit vectors.

$\vec{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$

?

$\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

?

Exercise 11.3

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Pages 87-88

Homework Exercise

1 Find the magnitude of each of these vectors.

a $3\mathbf{i} + 4\mathbf{j}$ **b** $6\mathbf{i} - 8\mathbf{j}$ **c** $5\mathbf{i} + 12\mathbf{j}$ **d** $2\mathbf{i} + 4\mathbf{j}$

e $3\mathbf{i} - 5\mathbf{j}$ **f** $4\mathbf{i} + 7\mathbf{j}$ **g** $-3\mathbf{i} + 5\mathbf{j}$ **h** $-4\mathbf{i} - \mathbf{j}$

2 $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{c} = 5\mathbf{i} - \mathbf{j}$. Find the exact value of the magnitude of:

a $\mathbf{a} + \mathbf{b}$ **b** $2\mathbf{a} - \mathbf{c}$ **c** $3\mathbf{b} - 2\mathbf{c}$

3 For each of the following vectors, find the unit vector in the same direction.

a $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}$ **b** $\mathbf{b} = 5\mathbf{i} - 12\mathbf{j}$ **c** $\mathbf{c} = -7\mathbf{i} + 24\mathbf{j}$ **d** $\mathbf{d} = \mathbf{i} - 3\mathbf{j}$

4 Find the angle that each of these vectors makes with the positive x-axis.

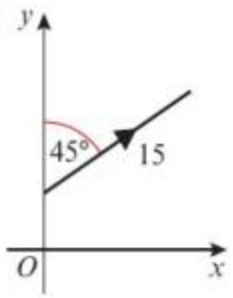
a $3\mathbf{i} + 4\mathbf{j}$ **b** $6\mathbf{i} - 8\mathbf{j}$ **c** $5\mathbf{i} + 12\mathbf{j}$ **d** $2\mathbf{i} + 4\mathbf{j}$

5 Find the angle that each of these vectors makes with $\mathbf{j}$.

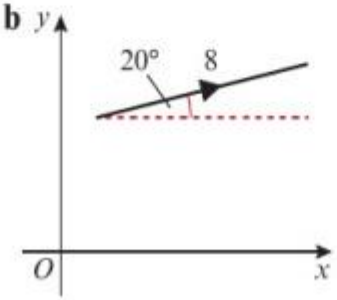
a $3\mathbf{i} - 5\mathbf{j}$ **b** $4\mathbf{i} + 7\mathbf{j}$ **c** $-3\mathbf{i} + 5\mathbf{j}$ **d** $-4\mathbf{i} - \mathbf{j}$

6 Write these vectors in $\mathbf{i}$, $\mathbf{j}$ and column vector form.

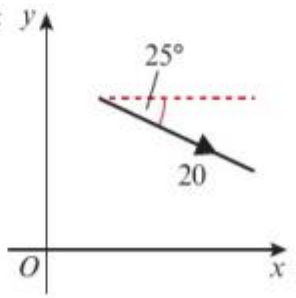
a



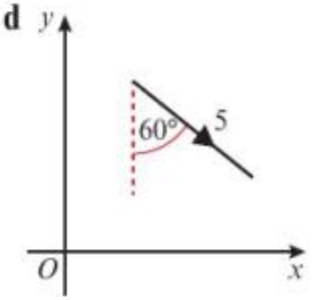
b



c



d



Homework Exercise

7 Draw a sketch for each vector and work out the exact value of its magnitude and the angle it makes with the positive x-axis to one decimal place.

a $3\mathbf{i} + 4\mathbf{j}$ **b** $2\mathbf{i} - \mathbf{j}$ **c** $-5\mathbf{i} + 2\mathbf{j}$

8 Given that $|2\mathbf{i} - k\mathbf{j}| = 2\sqrt{10}$, find the exact value of k. **(3 marks)**

9 Vector $\mathbf{a} = p\mathbf{i} + q\mathbf{j}$ has magnitude 10 and makes an angle θ with the positive x-axis where $\sin\theta = \frac{3}{5}$. Find the possible values of p and q.

(4 marks)

Problem-solving

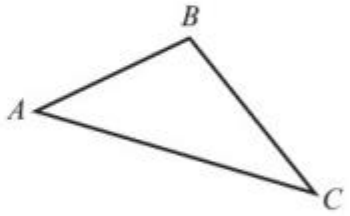
Make sure you consider all the possible cases.

10 In triangle ABC, $\overrightarrow{AB} = 4\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{AC} = 6\mathbf{i} - 4\mathbf{j}$.

a Find the angle between $\overrightarrow{AB}$ and $\mathbf{i}$.

b Find the angle between $\overrightarrow{AC}$ and $\mathbf{i}$.

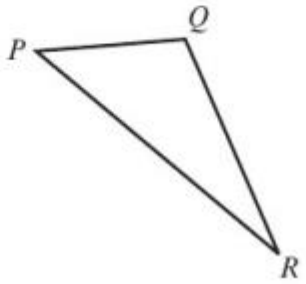
c Hence find the size of $\angle BAC$, in degrees, to one decimal place.



11 In triangle PQR, $\overrightarrow{PQ} = 4\mathbf{i} + \mathbf{j}$, $\overrightarrow{PR} = 6\mathbf{i} - 8\mathbf{j}$.

a Find the size of $\angle QPR$, in degrees, to one decimal place. **(5 marks)**

b Find the area of triangle PQR. **(2 marks)**



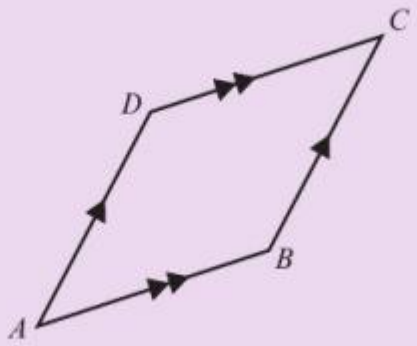
Hint The area of a triangle is $\frac{1}{2}ab\sin\theta$.

← Section 9.3

Homework Exercise

Challenge

In the diagram below $\overrightarrow{AB} = p\mathbf{i} + q\mathbf{j}$ and $\overrightarrow{AD} = r\mathbf{i} + s\mathbf{j}$.
$ABCD$ is a parallelogram.
Prove that the area of $ABCD$ is $ps - qr$.



Problem-solving

Draw the parallelogram on a coordinate grid, and choose a position for the origin that will simplify your calculations.

When multiplying two vectors together would the resulting object be another vector or something else?

When multiplying two vectors together would the *magnitude* of the resulting object be equal to the area of parallelogram?

Would the answer to either of the above questions ever be used in physics to describe the real universe!

Homework Answers

1 **a** 5 **b** 10 **c** 13
d 4.47 (3 s.f.) **e** 5.83 (3 s.f.) **f** 8.06 (3 s.f.)
g 5.83 (3 s.f.) **h** 4.12 (3 s.f.)

2 **a** $\sqrt{26}$ **b** $5\sqrt{2}$ **c** $\sqrt{101}$

3 **a** $\frac{1}{5}\begin{pmatrix}4\\3\end{pmatrix}$ **b** $\frac{1}{13}\begin{pmatrix}5\\-12\end{pmatrix}$
c $\frac{1}{25}\begin{pmatrix}-7\\24\end{pmatrix}$ **d** $\frac{1}{\sqrt{10}}\begin{pmatrix}1\\-3\end{pmatrix}$

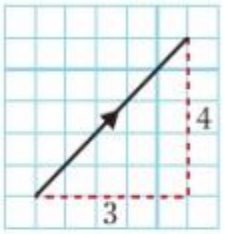
4 **a** 53.1° above **b** 53.1° below
c 67.4° above **d** 63.4° above

5 **a** 149° to the right **b** 29.7° to the right
c 31.0° to the left **d** 104° to the left

6 **a** $\frac{15\sqrt{2}}{2}\mathbf{i} + \frac{15\sqrt{2}}{2}\mathbf{j}, \begin{pmatrix}\frac{15\sqrt{2}}{2}\\\frac{15\sqrt{2}}{2}\end{pmatrix}$ **b** $7.52\mathbf{i} + 2.74\mathbf{j}, \begin{pmatrix}7.52\\2.74\end{pmatrix}$

c $18.1\mathbf{i} - 8.45\mathbf{j}, \begin{pmatrix}18.1\\-8.45\end{pmatrix}$ **d** $\frac{5\sqrt{3}}{2}\mathbf{i} - 2.5\mathbf{j}, \begin{pmatrix}\frac{5\sqrt{3}}{2}\\-2.5\end{pmatrix}$

7 **a** $|3\mathbf{i} + 4\mathbf{j}| = 5$, 53.1° above



b $|2\mathbf{i} - \mathbf{j}| = \sqrt{5}$, 26.6° below

c $|-5\mathbf{i} + 2\mathbf{j}| = \sqrt{29}$, 158.2° above

8 $k = \pm 6$

9 $p = \pm 8, q = 6$

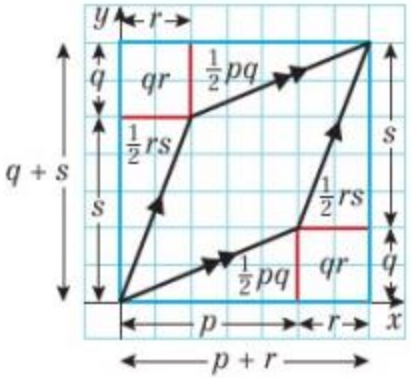
Homework Answers

10 **a** 36.9° **b** 33.7° **c** 70.6°

11 **a** 67.2° **b** 19.0

Challenge

Possible solution:



Area of parallelogram = area of large rectangle – 2(area of small rectangle) – 2 (area triangle 1) – 2(area triangle 2)

Area of parallelogram = $(p + r)(q + s) - 2qr - 2(\frac{1}{2}pq)$

$- 2(\frac{1}{2}rs) = ps - qr$