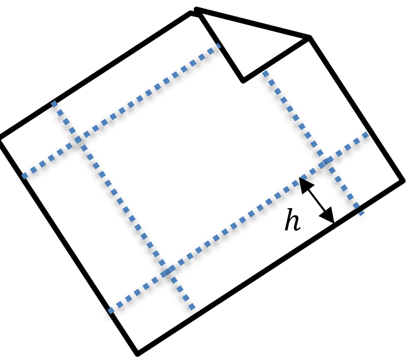
P1 Chapter 12: Differentiation

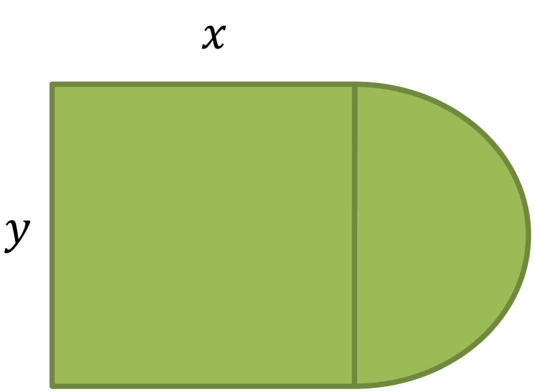
Modelling with Derivatives

Optimisation Problems/Modelling



We have a sheet of A4 paper, which we want to fold into a cuboid. What height should we choose for the cuboid in order to maximise the volume?

These are examples of **optimisation problems**: we're trying to maximise/minimise some quantity by choosing an appropriate value of a variable that we can control.

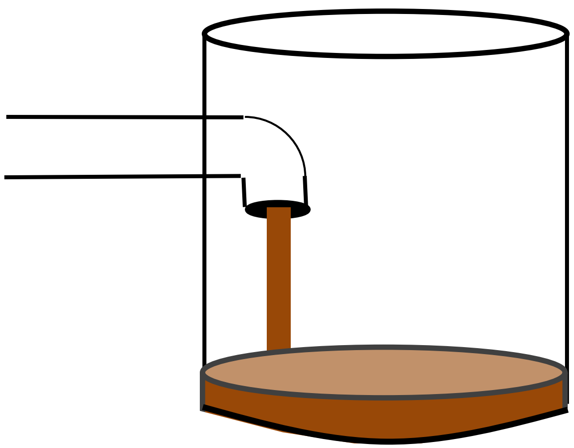


We have 50m of fencing, and want to make a bear pen of the following shape, such that the area is maximised. What should we choose x and y to be?

‘Rate of change’

Up to now we’ve had y in terms of x, where $\frac{dy}{dx}$ means “the **<u>rate</u>** at which y changes with respect to x”.

But we can use similar gradient function notation for other **physical quantities**.



A sewage container fills at a rate of 20 cm^3 per second.

How could we use appropriate notation to represent this?

Real Life Maths!

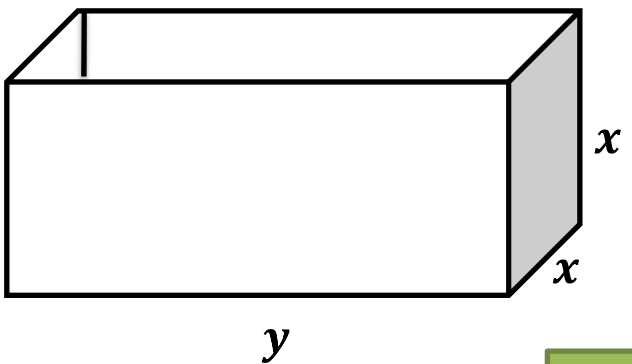
$$\frac{dV}{dt} = 20\ cm^3/s$$

“The rate at which the volume V changes with respect to time t.”

Example Optimisation Problem

Optimisation problems in an exam usually follow the following pattern:

- There are 2 variables involved (you may have to introduce one yourself), typically lengths.
- There are expressions for **two different physical quantities**:
  - One is a **constraint**, e.g. "the surface area is 20cm^2".
  - The **other we wish to maximise/minimise**, e.g. "we wish to maximise the volume".
- We use the constraint to **eliminate one of the variables** in the latter equation, so that it is then **just in terms of one variable**, and we can then use differentiation to find the turning point.



[Textbook] A large tank in the shape of a cuboid is to be made from 54m^2 of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two of the opposite vertical faces are squares.

a) Show that the volume, V m^3, of the tank is given by

$$V = 18x - \frac{2}{3}x^3.$$

We need to introduce a second variable ourselves so that we can find expressions for the surface area and volume.

?

These are the two equations mentioned in the guidance: one for surface area and one volume.

b) Given that x can vary, use differentiation to find the maximum or minimum value of V.

?

Once we have the 'optimal' value of x, we sub it back into V to get the best possible volume.

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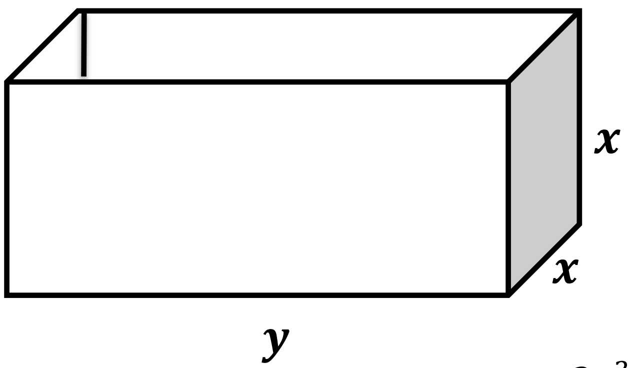

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$$2x^2 + 3xy = 54$$
$$V = x^2 y$$

But we want V just in terms of x:

$$y = \frac{54 - 2x^2}{3x} \quad \rightarrow \quad V = x^2\left(\frac{54 - 2x^2}{3x}\right)$$

$$V = \frac{54x^2 - 2x^4}{3x} = 18x - \frac{2}{3}x^3$$

b) Given that x can vary, use differentiation to find the maximum or minimum value of V.

$$\frac{dV}{dt} = 18 - 2x^2 = 0 \quad \therefore x = 3$$

$$V = 18(3) - \frac{2}{3}(3)^3 = 36$$

Once we have the 'optimal' value of x, we sub it back into V to get the best possible volume.

Test Your Understanding

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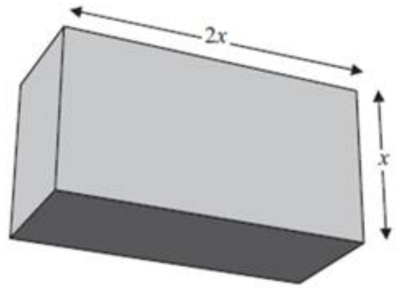

Figure 2

A cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm, as shown in Figure 2.

The volume of the cuboid is 81 cubic centimetres.

(*a*) Show that the total length, L cm, of the twelve edges of the cuboid is given by

$$L = 12x + \frac{162}{x^2}.$$

(3)

(*b*) Use calculus to find the minimum value of L.

(6)

(*c*) Justify, by further differentiation, that the value of L that you have found is a minimum.

(2)

(b) ? b

(c) ? c

(a) ? a

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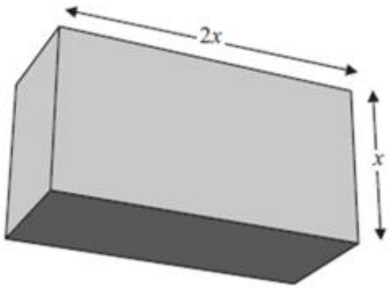

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(b)	$\frac{dL}{dx} = 12 - \frac{324}{x^3}$ $\left\{= 12 - 324x^{-3}\right\}$	Either $12x \to 12$ or $\frac{162}{x^2} \to \frac{\pm\lambda}{x^3}$	M1
		Correct differentiation (need not be simplified).	A1 aef
	$\left\{\frac{dL}{dx} =\right\} 12 - \frac{324}{x^3} = 0 \Rightarrow x^3 = \frac{324}{12}; = 27 \Rightarrow x = 3$	$L' = 0$ and "their $x^3 = \pm$ value" or "their $x^{-3} = \pm$ value"	M1;
		$x = \sqrt[3]{27}$ or $x = 3$	A1 **cso**
	$\{x = 3,\}$ $L = 12(3) + \frac{162}{3^2} = 54$ (cm)	Substitute candidate's value of $x (\neq 0)$ into a formula for L.	ddM1
		54	A1 **cao**

(c)	$\{\text{For } x = 3\}$, $\frac{d^2L}{dx^2} = \frac{972}{x^4} > 0 \Rightarrow$ Minimum	Correct ft L'' and considering sign.	M1
		$\frac{972}{x^4}$ and > 0 and conclusion.	A1 **[2]**

(a)	$\{V =\}\ 2x^2y = 81$	$2x^2y = 81$	B1 oe
	$\{L = 2(2x + x + 2x + x) + 4y \Rightarrow L = 12x + 4y\}$		
	$y = \frac{81}{2x^2} \Rightarrow L = 12x + 4\left(\frac{81}{2x^2}\right)$	Making y the subject of their expression and substitute this into the correct L formula.	M1
	So, $L = 12x + \frac{162}{x^2}$ **AG**	Correct solution only.	A1 **cso**

Exercise 12.11

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Extension

1 [STEP I 2006 Q2] A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze.
Prove that $A \leq 14\pi a^2$ and determine the minimum value of A.

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Prove that $A \leq 14\pi a^2$ and determine the minimum value of A.

Let the goat be tethered a distance x from a corner. Therefore, the goat can graze an area

$$A = \frac{16a^2\pi}{2} + \frac{(4a-x)^2\pi}{4} + \frac{(2a-x)^2\pi}{4} + \frac{(2a+x)^2\pi}{4} + \frac{(x)^2\pi}{4} = \frac{\pi}{4}\left(56a^2 + 4x^2 - 8ax\right)$$

So the area grazed $A = \pi\left[13a^2 + (x-a)^2\right]$. This is minimised when $x = a$, and maximised when $x = 0$ or $2a$ (since $0 \leqslant x \leqslant 2a$), hence $13\pi a^2 \leqslant A \leqslant 14\pi a^2$.

Homework Exercise

1 Find $\frac{d\theta}{dt}$ where $\theta = t^2 - 3t$.

2 Find $\frac{dA}{dr}$ where $A = 2\pi r$.

3 Given that $r = \frac{12}{t}$, find the value of $\frac{dr}{dt}$ when $t = 3$.

4 The surface area, A cm^2, of an expanding sphere of radius r cm is given by $A = 4\pi r^2$. Find the rate of change of the area with respect to the radius at the instant when the radius is 6 cm.

5 The displacement, s metres, of a car from a fixed point at time t seconds is given by $s = t^2 + 8t$. Find the rate of change of the displacement with respect to time at the instant when $t = 5$.

6 A rectangular garden is fenced on three sides, and the house forms the fourth side of the rectangle.

a Given that the total length of the fence is 80 m, show that the area, A, of the garden is given by the formula $A = y(80 - 2y)$, where y is the distance from the house to the end of the garden.

b Given that the area is a maximum for this length of fence, find the dimensions of the enclosed garden, and the area which is enclosed.

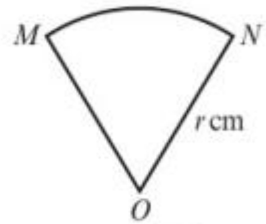
7 A closed cylinder has total surface area equal to 600π.

a Show that the volume, V cm^3, of this cylinder is given by the formula $V = 300\pi r - \pi r^3$, where r cm is the radius of the cylinder.

b Find the maximum volume of such a cylinder.

Homework Exercise

8 A sector of a circle has area $100\,\text{cm}^2$.

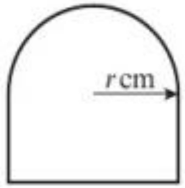


a Show that the perimeter of this sector is given by the formula

$$P = 2r + \frac{200}{r},\ r > \sqrt{\frac{100}{\pi}}$$

b Find the minimum value for the perimeter.

9 A shape consists of a rectangular base with a semicircular top, as shown.



a Given that the perimeter of the shape is 40 cm, show that its area, $A\,\text{cm}^2$, is given by the formula

$$A = 40r - 2r^2 - \frac{\pi r^2}{2}$$

where r cm is the radius of the semicircle. **(2 marks)**

b Hence find the maximum value for the area of the shape. **(4 marks)**

10 The shape shown is a wire frame in the form of a large rectangle split by parallel lengths of wire into 12 smaller equal-sized rectangles.

a Given that the total length of wire used to complete the whole frame is 1512 mm, show that the area of the whole shape, $A\,\text{mm}^2$, is given by the formula

$$A = 1296x - \frac{108x^2}{7}$$

where x mm is the width of one of the smaller rectangles. **(4 marks)**

b Hence find the maximum area which can be enclosed in this way. **(4 marks)**

Homework Answers

1 $2t - 3$ **2** 2π **3** $-\frac{4}{3}$

4 48π **5** 18

6 **a** Let x = width of garden.

$x + 2y = 80$

$A = xy = 2y$

b $20\,\text{m} \times 40\,\text{m}$, $800\,\text{m}^2$

7 **a** $2\pi r^2 + 2\pi rh = 600\pi \Rightarrow h = \dfrac{300 - r^2}{r}$

$V = \pi r^2 h = \pi r(300 - r^2) = 300\pi r - \pi r^3$

b $2000\pi\,\text{cm}^3$

8 **a** Let θ = angle of sector.

$\pi r^2 \times \dfrac{\theta}{360} = 100 \Rightarrow \theta = \dfrac{36\,000}{\pi r^2}$

$P = 2r + 2\pi r \times \dfrac{\theta}{360} = 2r + \dfrac{200\pi r}{\pi r^2}$

$= 2r + \dfrac{100}{r}$

$\theta < 2\pi \Rightarrow \text{Area} < \pi r^2$, so $\pi r^2 > 100$

$\therefore r > \sqrt{\dfrac{100}{\pi}}$

b $40\,\text{cm}$

9 **a** Let h = height of rectangle.

$P = \pi r + 2r + 2h = 40 \Rightarrow 2h = 40 - 2r - \pi r$

$A = \dfrac{\pi}{2}r^2 + 2rh = \dfrac{\pi}{2}r^2 + r(40 - 2r - \pi r)$

$= 40r - 2r^2 - \dfrac{\pi}{2}r^2$

b $\dfrac{800}{4 + \pi}\,\text{cm}^2$

10 **a** $18x + 14y = 1512 \Rightarrow y = \dfrac{1512 - 18x}{14}$

$A = 12xy = 12x\left(\dfrac{1512 - 18x}{14}\right)$

$= 1296x - \dfrac{108x^2}{7}$

b $27\,216\,\text{mm}^2$