# P1 Chapter 7: Algebraic Methods

## Chapter Practice

# Key Points

1 When simplifying an algebraic fraction, factorise the numerator and denominator where possible and then cancel common factors.

2 You can use long division to divide a polynomial by $(x \pm p)$, where $p$ is a constant.

3 The **factor theorem** states that if $f(x)$ is a polynomial then:

- If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$
- If $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$

4 You can prove a mathematical statement is true by **deduction**. This means starting from known factors or definitions, then using logical steps to reach the desired conclusion.

5 In a mathematical proof you must

- State any information or assumptions you are using
- Show every step of your proof clearly
- Make sure that every step follows logically from the previous step
- Make sure you have covered all possible cases
- Write a statement of proof at the end of your working

# Key Points

**6** To prove an identity you should

- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step of your algebraic working

**7** You can prove a mathematical statement is true by **exhaustion**. This means breaking the statement into smaller cases and proving each case separately.

**8** You can prove a mathematical statement is not true by a **counter-example**. A counter-example is one example that does not work for the statement. You do not need to give more than one example, as one is sufficient to disprove a statement.

# Chapter Exercises

**1** Simplify these fractions as far as possible:

**a** $\dfrac{3x^4 - 21x}{3x}$

**b** $\dfrac{x^2 - 2x - 24}{x^2 - 7x + 6}$

**c** $\dfrac{2x^2 + 7x - 4}{2x^2 + 9x + 4}$

**2** Divide $3x^3 + 12x^2 + 5x + 20$ by $(x + 4)$.

**3** Simplify $\dfrac{2x^3 + 3x + 5}{x + 1}$

**4** **a** Show that $(x - 3)$ is a factor of $2x^3 - 2x^2 - 17x + 15$. **(2 marks)**

**b** Hence express $2x^3 - 2x^2 - 17x + 15$ in the form $(x - 3)(Ax^2 + Bx + C)$, where the values $A$, $B$ and $C$ are to be found. **(3 marks)**

**5** **a** Show that $(x - 2)$ is a factor of $x^3 + 4x^2 - 3x - 18$. **(2 marks)**

**b** Hence express $x^3 + 4x^2 - 3x - 18$ in the form $(x - 2)(px + q)^2$, where the values $p$ and $q$ are to be found. **(4 marks)**

**6** Factorise completely $2x^3 + 3x^2 - 18x + 8$. **(6 marks)**

# Chapter Exercises

**7** Find the value of $k$ if $(x - 2)$ is a factor of $x^3 - 3x^2 + kx - 10$. **(4 marks)**

**8** $f(x) = 2x^2 + px + q$. Given that $f(-3) = 0$, and $f(4) = 21$:

**a** find the value of $p$ and $q$ **(6 marks)**

**b** factorise $f(x)$. **(3 marks)**

**9** $h(x) = x^3 + 4x^2 + rx + s$. Given $h(-1) = 0$, and $h(2) = 30$:

**a** find the values of $r$ and $s$ **(6 marks)**

**b** factorise $h(x)$. **(3 marks)**

**10** $g(x) = 2x^3 + 9x^2 - 6x - 5$.

**a** Factorise $g(x)$. **(6 marks)**

**b** Solve $g(x) = 0$. **(2 marks)**

**11 a** Show that $(x - 2)$ is a factor of $f(x) = x^3 + x^2 - 5x - 2$. **(2 marks)**

**b** Hence, or otherwise, find the exact solutions of the equation $f(x) = 0$. **(4 marks)**

**12** Given that $-1$ is a root of the equation $2x^3 - 5x^2 - 4x + 3$, find the two positive roots. **(4 marks)**

**13** $f(x) = x^3 - 2x^2 - 19x + 20$

**a** Show that $(x + 4)$ is a factor of $f(x)$. **(3 marks)**

**b** Hence, or otherwise, find all the solutions to the equation
$x^3 - 2x^2 - 19x + 20 = 0$. **(4 marks)**

# Chapter Exercises

**14** $f(x) = 6x^3 + 17x^2 - 5x - 6$

**a** Show that $f(x) = (3x - 2)(ax^2 + bx + c)$, where $a$, $b$ and $c$ are constants to be found. **(2 marks)**

**b** Hence factorise $f(x)$ completely. **(4 marks)**

**c** Write down all the real roots of the equation $f(x) = 0$. **(2 marks)**

**15** Prove that $\dfrac{x - y}{\sqrt{x} - \sqrt{y}} \equiv \sqrt{x} + \sqrt{y}$.

**16** Use completing the square to prove that $n^2 - 8n + 20$ is positive for all values of $n$.

**17** Prove that the quadrilateral $A(1, 1)$, $B(3, 2)$, $C(4, 0)$ and $D(2, -1)$ is a square.

**18** Prove that the sum of two consecutive positive odd numbers less than ten gives an even number.

**19** Prove that the statement '$n^2 - n + 3$ is a prime number for all values of $n$' is untrue.

**20** Prove that $\left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right) \equiv x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$.

# Chapter Exercises

**21** Prove that $2x^3 + x^2 - 43x - 60 \equiv (x + 4)(x - 5)(2x + 3)$.

**22** The equation $x^2 - kx + k = 0$, where $k$ is a positive constant, has two equal roots.
Prove that $k = 4$. **(3 marks)**

**23** Prove that the distance between opposite edges of a regular hexagon of side length $\sqrt{3}$ is a rational value.

**24** **a** Prove that the difference of the squares of two consecutive even numbers is always divisible by 4.

**b** Is this statement true for odd numbers? Give a reason for your answer.

**25** A student is trying to prove that $1 + x^2 < (1 + x)^2$.
The student writes:

| |
|---|
| $(1 + x)^2 = 1 + 2x + x^2$. |
| So $1 + x^2 < 1 + 2x + x^2$. |

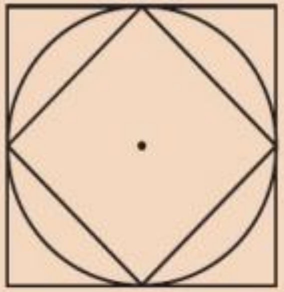
**a** Identify the error made in the proof. **(1 mark)**

**b** Provide a counter-example to show that the statement is not true. **(2 marks)**

# Chapter Exercises

**Challenge**

**1** The diagram shows two squares and a circle.



**a** Given that $\pi$ is defined as the circumference of a circle of diameter 1 unit, prove that $2\sqrt{2} < \pi < 4$.

**b** By similarly constructing regular hexagons inside and outside a circle, prove that $3 < \pi < 2\sqrt{3}$.

**2** Prove that if $f(x) = ax^3 + bx^2 + cx + d$ and $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.

# Chapter Answers

**1** **a** $x^3 - 7$ **b** $\dfrac{x+4}{x-1}$ **c** $\dfrac{2x-1}{2x+1}$

**2** $3x^2 + 5$

**3** $2x^2 - 2x + 5$

**4** **a** When $x = 3$, $2x^3 - 2x^2 - 17x + 15 = 0$
**b** $A = 2$, $B = 4$, $C = -5$

**5** **a** When $x = 2$, $x^3 + 4x^2 - 3x - 18 = 0$
**b** $p = 1$, $q = 3$

**6** $(x - 2)(x + 4)(2x - 1)$

**7** 7

**8** **a** $p = 1$, $q = -15$ **b** $(x + 3)(2x - 5)$

**9** **a** $r = 3$, $s = 0$ **b** $x(x + 1)(x + 3)$

**10** **a** $(x - 1)(x + 5)(2x + 1)$ **b** $-5, -\frac{1}{2}, 1$

**11** **a** When $x = 2$, $x^3 + x^2 - 5x - 2 = 0$
**b** $2, -\dfrac{3}{2} \pm \dfrac{\sqrt{5}}{2}$

**12** $\frac{1}{2}$, 3

**13** **a** When $x = -4$, $f(x) = 0$
**b** $(x + 4)(x - 5)(x - 1)$

**14** **a** $f(\frac{2}{3}) = 0$, therefore $(3x - 2)$ is a factor of $f(x)$
$a = 2$, $b = 7$ and $c = 3$
**b** $(3x - 2)(2x + 1)(x + 3)$
**c** $x = \frac{2}{3}, -\frac{1}{2}, -3$

**15** $\dfrac{x-y}{(\sqrt{x}-\sqrt{y})} \times \dfrac{(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})} = \dfrac{x\sqrt{x} + x\sqrt{y} - y\sqrt{x} - y\sqrt{y}}{x-y} = \sqrt{x} + \sqrt{y}$

**16** $n^2 - 8n + 20 = (n - 4)^2 + 4$, 4 is the minimum value so $n^2 - 8n + 20$ is always positive

**17** Gradient $AB = \frac{1}{2}$, gradient $BC = -2$, gradient $CD = \frac{1}{2}$, gradient $AD = -2$
$AB$ and $BC$, $BC$ and $CD$, $CD$ and $AD$ and $AB$ and $AD$ are all perpendicular
Length $AB = \sqrt{5}$, $BC = \sqrt{5}$, $CD = \sqrt{5}$ and $AD = \sqrt{5}$, all four sides are equal

**18** $1 + 3 =$ even, $3 + 5 =$ even, $5 + 7 =$ even, $7 + 9 =$ even

**19** For example when $n = 6$

**20** $\left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{\frac{-2}{3}}\right) = x^{\frac{7}{3}} + x^{\frac{1}{3}} - x^{\frac{1}{3}} - x^{-\frac{5}{3}} = x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$

**21** RHS $= (x + 4)(x - 5)(2x + 3) = (x + 4)(2x^2 - 7x - 15)$
$= 2x^3 + x^2 - 43x - 60 =$ LHS

**22** $x^2 - kx + k = 0$, $b^2 - 4ac = 0$, $k^2 - 4k = 0$, $k(k - 4) = 0$,
$k = 4$.

**23** The distance between opposite edges
$= 2\left((\sqrt{3})^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = 2\left(3 - \frac{3}{4}\right) = \frac{9}{2}$ which is rational.

**24 a** $(2n + 2)^2 - (2n)^2 = 8n + 4 = 4(2n + 1)$ is always divisible by 4.

**b** Yes, $(2n + 1)^2 - (2n - 1)^2 = 8n$ which is always divisible by 4.

**25 a** The assumption is that $x$ is positive

**b** $x = 0$

**Challenge**

**1 a** Perimeter of inside square $= 4\left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right) = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

Perimeter of outside square = 4,
therefore $2\sqrt{2} < \pi < 4$.

**b** Perimeter of inside hexagon = 3

Perimeter of outside hexagon $= 6 \times \frac{\sqrt{3}}{3} = 2\sqrt{3}$,
therefore $3 < \pi < 2\sqrt{3}$

**2** $ax^3 + bx^2 + cx + d \div (x - p) = ax^2 + (b + ap)x + (c + bp + ap^2)$ with remainder $d + cp + bp^2 + ap^3$
$f(p) = ap^3 + bp^2 + cp + d = 0$, which matches the remainder, so $(x - p)$ is a factor of $f(x)$.