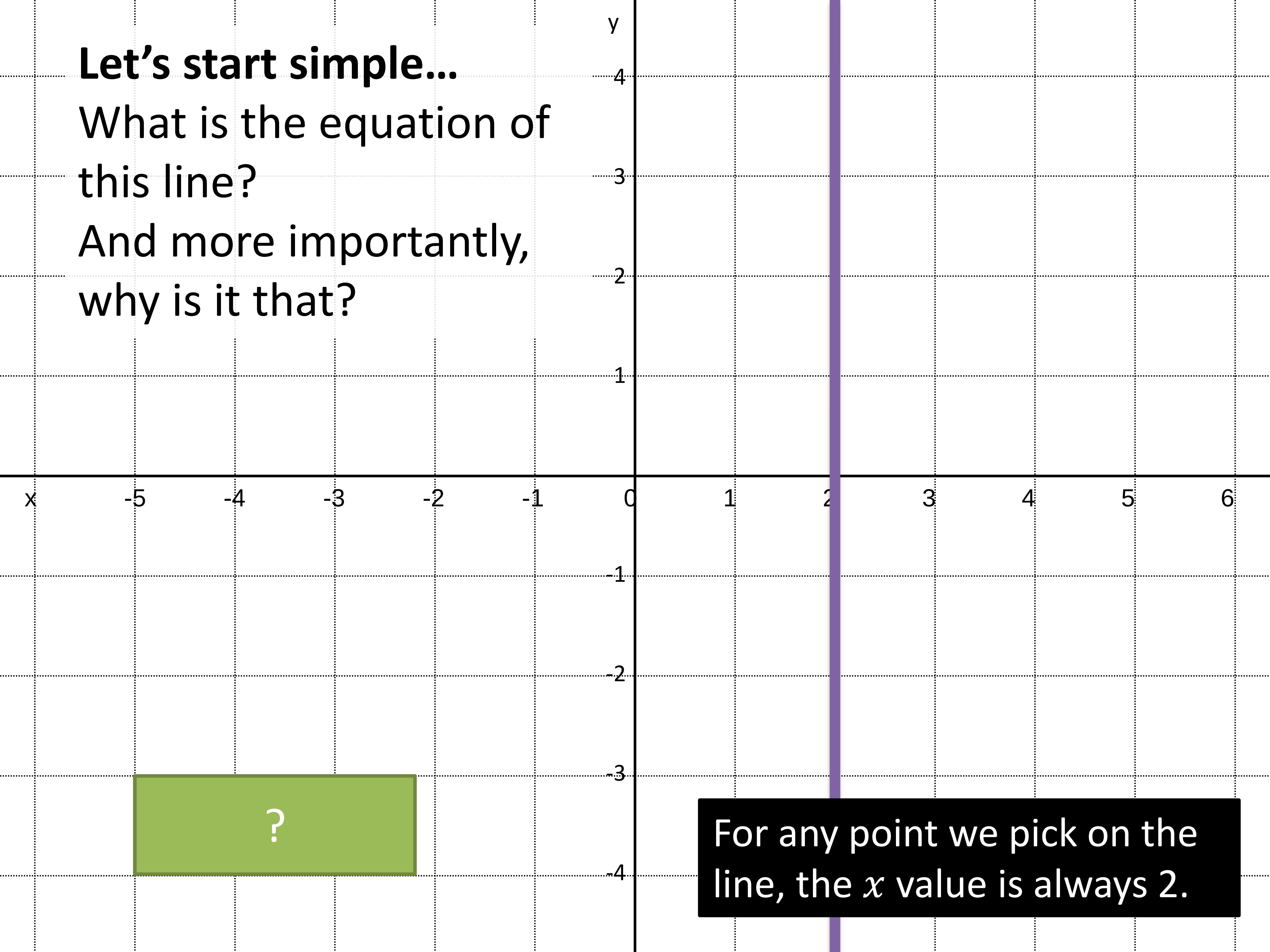
P1 Chapter 5: Linear Graphs

Linear Formulae

Let’s start simple…

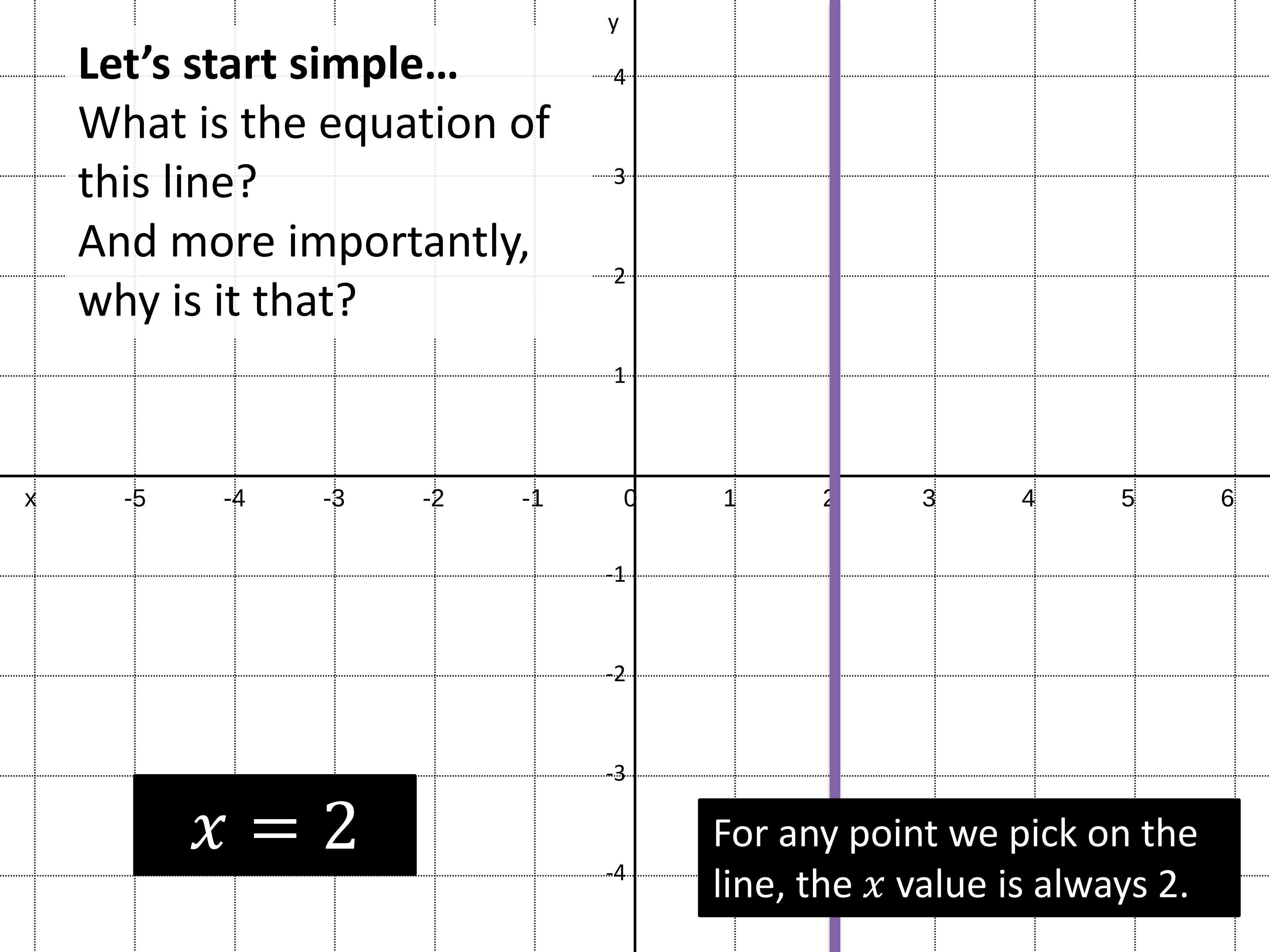
What is the equation of this line?

And more importantly, why is it that?



?

For any point we pick on the line, the x value is always 2.



Let’s start simple…

What is the equation of this line?

And more importantly, why is it that?

$$x = 2$$

For any point we pick on the line, the x value is always 2.

Lines and Equations of Lines

A line consists of the set of all points which satisfy some formula in terms of x and/or y.

A line L_1 is defined by the set of points $\{ (x, y) : y = 3x + 2 \}$.

A curved line L_2 is defined by the set of points $\{ (x, y) : y = x^2 \}$.

A straight line L_1 is defined by the set of points $\{ (x, y) : y = mx + c \}$ where m and c are real-number constants.

This chapter will focus on straight lines of the last example type.

Examples

This means we can **substitute** the values of a coordinate into our equation whenever we know the point lies on the line.

The point $(5, a)$ lies on the line with equation $y = 3x + 2$. Determine the value of a.

?

Find the coordinate of the point where the line $2x + y = 5$ cuts the x-axis.

?

Examples

This means we can **substitute** the values of a coordinate into our equation whenever we know the point lies on the line.

The point $(5, a)$ lies on the line with equation $y = 3x + 2$. Determine the value of a.

Substituting in x and y value:

$$a = 3(5) + 2$$
$$a = 17$$

Find the coordinate of the point where the line $2x + y = 5$ cuts the x-axis.

On the x-axis, $y = 0$. Substituting:

$$2x + 0 = 5$$
$$x = \frac{5}{2} \quad \rightarrow \quad \left(\frac{5}{2}, 0\right)$$

Test Your Understanding

Determine where the line $x + 2y = 3$ crosses the:

a) y-axis: ?

b) x-axis: ?

What mistakes do you think it's easy to make?

- ?
-

Test Your Understanding

Determine where the line $x + 2y = 3$ crosses the:

a) y-axis: **Let $\boldsymbol{x = 0}$.**

$$\boldsymbol{2y = 3 \quad \rightarrow \quad y = \frac{3}{2}}$$

$$\boldsymbol{\left(0, \frac{3}{2}\right)}$$

b) x-axis: **Let $\boldsymbol{y = 0}$**

$$\boldsymbol{x + 0 = 3}$$

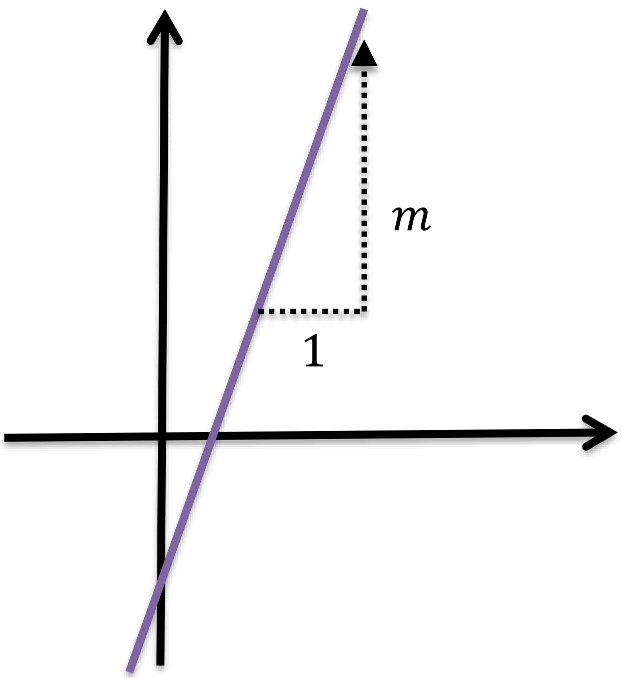
$$\boldsymbol{(3, 0)}$$

What mistakes do you think it's easy to make?

- **Mixing up x/y: Putting answer as $(0, 3)$ rather than $(3, 0)$.**
- **Setting $y = 0$ to find the y-intercept, or $x = 0$ to find the x-intercept.**

Recap of gradient

The steepness of a line is known as the **gradient**.
It tells us what y changes by as x increases by 1.



So if the y value increased by 6 as the x value increased by 2, what is y increasing by for each unit increase of x?
How would that give us a suitable formula for the gradient m?

$$m = \frac{\Delta y}{\Delta x}$$

Δ is the (capital) Greek letter "delta" and means **"change in"**.

Textbook Note:

The textbook uses $m = \frac{y_2 - y_1}{x_2 - x_1}$ for two points (x_1, y_1) and (x_2, y_2). Reasons I don't use it for non-algebraic coordinates:

- Students often get the y_1 and y_2 the wrong way round (or with the x's)
- Students often make sign errors when dealing with negatives, e.g. $(-3) - (-4)$
- It can't be done as easily mentally,
- Students see it as "yet another formula to learn" when really all you need is to appreciate is what gradient is, i.e. "y change per x change".

Examples

Find the gradient of the line that goes through the points:

1 $(1,4) \quad (3,10)$ $m =$?

2 $(5,7) \quad (8,1)$ $m =$?

3 $(2,2) \quad (-1,10)$ $m =$?

4 Show that the points $A(3,4), B(5,5), C(11,8)$ all lie on a straight line.

?

Examples

Find the gradient of the line that goes through the points:

1 $(1,4) \quad (3,10)$ $\qquad m = 3$

2 $(5,7) \quad (8,1)$ $\qquad m = -2$

3 $(2,2) \quad (-1,10)$ $\qquad m = -\frac{8}{3}$

4 Show that the points $\mathrm{A}(3,4), B(5,5), C(11,8)$ all lie on a straight line.

$$\boldsymbol{m_{AB} = \frac{1}{2} \qquad m_{BC} = \frac{3}{6} = \frac{1}{2}}$$

Gradients the same ∴ 'collinear'.

If points are 'collinear' they lie on the same line.

Further Example

The line joining $(2, -5)$ to $(4, a)$ has gradient -1. Work out the value of a.

?

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The line joining $(2, -5)$ to $(4, a)$ has gradient -1. Work out the value of a.

$$\frac{a - -5}{4 - 2} = -1$$

$$\frac{a + 5}{2} = -1$$

$$a = -7$$

The gradient-intercept form for straight lines is:

$$\underbrace{y = \overbrace{m}^{}x + c}$$

Gradient

y-intercept

Why does it work?

?

$y = mx + c$

The gradient-intercept form for straight lines is:

$$y = mx + c$$

Gradient $\quad$ y-intercept

Why does it work?

- **The y-intercept by definition is the y value when $x = 0$. Substituting:**
  $y = (m \times 0) + c = c$ as expected.
- **By the definition of gradient, if we increase x by 1, y should increase by m:**
  $$\boldsymbol{y = m(x+1) + c = mx + m + c}$$
  which indeed has increased by m.

$y = mx + c$

Determine the gradient and y-intercept of the line with equation $4x - 3y + 5 = 0$

?

Make y the subject so we have the form
$$y = mx + c$$
Put y on the side it's positive.

Divide **each** term by 3; don't write $y = \frac{4x+5}{3}$ otherwise it's not in the form $y = mx + c$

This is algebra, so use improper fractions, and not mixed numbers or recurring decimals.

$y = mx + c$

Determine the gradient and y-intercept of the line with equation $4x - 3y + 5 = 0$

$$3y = 4x + 5$$

$$y = \frac{4}{3}x + \frac{5}{3}$$

$$\therefore m = \frac{4}{3}, \qquad c = \frac{5}{3}$$

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$ax + by + c = 0$

Another common form is $\boldsymbol{ax + by + c = 0}$, where a, b, c are integers.
This is known as the '**standard**' form.

Express $y = \frac{1}{3}x - \frac{2}{3}$ in the form $ax + by + c = 0$, where a, b, c are integers.

?

We'll see on the next slide WHY we might want to put an equation in this form over $y = mx + c$...

$ax + by + c = 0$

Another common form is $\boldsymbol{ax + by + c = 0}$, where a, b, c are integers.
This is known as the '**standard**' form.

Express $y = \frac{1}{3}x - \frac{2}{3}$ in the form $ax + by + c = 0$, where a, b, c are integers.

$$3y = x - 2$$
$$x - 3y - 2 = 0$$

We don't want fractions, so multiply by an appropriate number.

Put everything on either side of equation.
$-x + 3y + 2 = 0$ would also be OK.

We'll see on the next slide WHY we might want to put an equation in this form over $y = mx + c$...

Test Your Understanding

Express $y = \frac{2}{5}x + \frac{3}{5}$ in the form $ax + by + c = 0$, where a, b, c are integers.

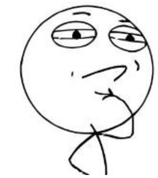
?

Test Your Understanding

Express $y = \frac{2}{5}x + \frac{3}{5}$ in the form $ax + by + c = 0$, where a, b, c are integers.

$$5y = 2x + 3$$
$$2x - 5y + 3 = 0$$

Why might we want to put a straight-line equation in the form $ax + by + c = 0$?

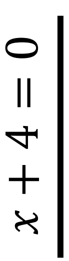


$y = mx + c$
"Slope-Intercept Form"

$ax + by + c = 0$
"Standard Form"

Coverage

$y = mx + c$ doesn't allow you to represent vertical lines. Standard form allows us to do this by just making b zero.



Symmetry

In general, the '**linear combination**' of two variables x and y is $ax + by$, i.e. "some amount of x and some amount of y". There is a greater elegance and symmetry to this form over $y = mx + c$ because x and y appear similarly within the expression.

Usefulness

This more 'elegant' form also means it ties in with vectors and matrices. For vectors, you will learn about the '**dot product**' of two vectors:

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = ax + by$$

Since $ax + by + c = 0$, we can represent a straight line using:

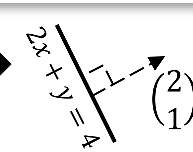
$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + c = 0 \qquad (1)$$

We can extend to 3D points to get the equation of a **plane**:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + d = 0 \qquad (2)$$

Conveniently, in equation (1), the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is **perpendicular to the line**. And in equation (2), the vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is perpendicular to the plane. Nice!

$$2x + y = 4 \;\Rightarrow\; \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 4 \;\Rightarrow$$



Exercise 5.1

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Homework Exercise

1 Work out the gradients of the lines joining these pairs of points:

a $(4, 2), (6, 3)$ **b** $(-1, 3), (5, 4)$ **c** $(-4, 5), (1, 2)$

d $(2, -3), (6, 5)$ **e** $(-3, 4), (7, -6)$ **f** $(-12, 3), (-2, 8)$

g $(-2, -4), (10, 2)$ **h** $(\frac{1}{2}, 2), (\frac{3}{4}, 4)$ **i** $(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{2}{3})$

j $(-2.4, 9.6), (0, 0)$ **k** $(1.3, -2.2), (8.8, -4.7)$ **l** $(0, 5a), (10a, 0)$

m $(3b, -2b), (7b, 2b)$ **n** $(p, p^2), (q, q^2)$

2 The line joining $(3, -5)$ to $(6, a)$ has a gradient 4. Work out the value of a.

3 The line joining $(5, b)$ to $(8, 3)$ has gradient -3. Work out the value of b.

4 The line joining $(c, 4)$ to $(7, 6)$ has gradient $\frac{3}{4}$. Work out the value of c.

5 The line joining $(-1, 2d)$ to $(1, 4)$ has gradient $-\frac{1}{4}$. Work out the value of d.

6 The line joining $(-3, -2)$ to $(2e, 5)$ has gradient 2. Work out the value of e.

7 The line joining $(7, 2)$ to $(f, 3f)$ has gradient 4. Work out the value of f.

8 The line joining $(3, -4)$ to $(-g, 2g)$ has gradient -3. Work out the value of g.

9 Show that the points $A(2, 3)$, $B(4, 4)$ and $C(10,7)$ can be joined by a straight line.

Problem-solving
Find the gradient of the line joining the points A and B and the line joining the points A and C.

10 Show that the points $A(-2a, 5a)$, $B(0, 4a)$ and points $C(6a, a)$ are collinear. **(3 marks)**

Notation Points are collinear if they all lie on the same straight line.

Homework Answers

1 **a** $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $-\frac{3}{5}$ **d** 2
e -1 **f** $\frac{1}{2}$ **g** $\frac{1}{2}$ **h** 8
i $\frac{2}{3}$ **j** -4 **k** $-\frac{1}{3}$ **l** $-\frac{1}{2}$
m 1 **n** $\frac{q^2 - p^2}{q - p} = q + p$

2 7
3 12
4 $4\frac{1}{3}$
5 $2\frac{1}{4}$
6 $\frac{1}{4}$
7 26
8 -5
9 Gradient of AB = gradient of $BC = 0.5$; point B is common
10 Gradient of AB = gradient of $BC = -0.5$; point B is common

Homework Exercise

1 Work out the gradients of these lines:

a $y = -2x + 5$ **b** $y = -x + 7$ **c** $y = 4 + 3x$

d $y = \frac{1}{3}x - 2$ **e** $y = -\frac{2}{3}x$ **f** $y = \frac{5}{4}x + \frac{2}{3}$

g $2x - 4y + 5 = 0$ **h** $10x - 5y + 1 = 0$ **i** $-x + 2y - 4 = 0$

j $-3x + 6y + 7 = 0$ **k** $4x + 2y - 9 = 0$ **l** $9x + 6y + 2 = 0$

2 These lines cut the y-axis at $(0, c)$. Work out the value of c in each case.

a $y = -x + 4$ **b** $y = 2x - 5$ **c** $y = \frac{1}{2}x - \frac{2}{3}$

d $y = -3x$ **e** $y = \frac{6}{7}x + \frac{7}{5}$ **f** $y = 2 - 7x$

g $3x - 4y + 8 = 0$ **h** $4x - 5y - 10 = 0$ **i** $-2x + y - 9 = 0$

j $7x + 4y + 12 = 0$ **k** $7x - 2y + 3 = 0$ **l** $-5x + 4y + 2 = 0$

3 Write these lines in the form $ax + by + c = 0$.

a $y = 4x + 3$ **b** $y = 3x - 2$ **c** $y = -6x + 7$

d $y = \frac{4}{5}x - 6$ **e** $y = \frac{5}{3}x + 2$ **f** $y = \frac{7}{3}x$

g $y = 2x - \frac{4}{7}$ **h** $y = -3x + \frac{2}{9}$ **i** $y = -6x - \frac{2}{3}$

j $y = -\frac{1}{3}x + \frac{1}{2}$ **k** $y = \frac{2}{3}x + \frac{5}{6}$ **l** $y = \frac{3}{5}x + \frac{1}{2}$

4 The line $y = 6x - 18$ meets the x-axis at the point P. Work out the coordinates of P.

5 The line $3x + 2y = 0$ meets the x-axis at the point R. Work out the coordinates of R.

6 The line $5x - 4y + 20 = 0$ meets the y-axis at the point A and the x-axis at the point B. Work out the coordinates of A and B.

Homework Exercise

7 A line l passes through the points with coordinates (0, 5) and (6, 7).

a Find the gradient of the line.

b Find an equation of the line in the form $ax + by + c = 0$.

8 A line l cuts the x-axis at (5, 0) and the y-axis at (0, 2).

a Find the gradient of the line. **(1 mark)**

b Find an equation of the line in the form $ax + by + c = 0$. **(2 marks)**

9 Show that the line with equation $ax + by + c = 0$ has gradient $-\frac{a}{b}$ and cuts the y-axis at $-\frac{c}{b}$

10 The line l with gradient 3 and y-intercept (0, 5) has the equation $ax - 2y + c = 0$.
Find the values of a and c. **(2 marks)**

Problem-solving

Try solving a similar problem with numbers first:

Find the gradient and y-intercept of the straight line with equation $3x + 7y + 2 = 0$.

11 The straight line l passes through (0, 6) and has gradient −2. It intersects the line with equation $5x - 8y - 15 = 0$ at point P. Find the coordinates of P. **(4 marks)**

12 The straight line l_1 with equation $y = 3x - 7$ intersects the straight line l_2 with equation $ax + 4y - 17 = 0$ at the point $P(-3, b)$.

a Find the value of b. **(1 mark)**

b Find the value of a. **(2 marks)**

Challenge

Show that the equation of a straight line through $(0, a)$ and $(b, 0)$ is $ax + by - ab = 0$.

Homework Answers

1 **a** -2 **b** -1 **c** 3 **d** $\frac{1}{3}$
e $-\frac{2}{3}$ **f** $\frac{5}{4}$ **g** $\frac{1}{2}$ **h** 2
i $\frac{1}{2}$ **j** $\frac{1}{2}$ **k** -2 **l** $-\frac{3}{2}$

2 **a** 4 **b** -5 **c** $-\frac{2}{3}$ **d** 0
e $\frac{7}{5}$ **f** 2 **g** 2 **h** -2
i 9 **j** -3 **k** $\frac{3}{2}$ **l** $-\frac{1}{2}$

3 **a** $4x - y + 3 = 0$ **b** $3x - y - 2 = 0$
c $6x + y - 7 = 0$ **d** $4x - 5y - 30 = 0$
e $5x - 3y + 6 = 0$ **f** $7x - 3y = 0$
g $14x - 7y - 4 = 0$ **h** $27x + 9y - 2 = 0$
i $18x + 3y + 2 = 0$ **j** $2x + 6y - 3 = 0$
k $4x - 6y + 5 = 0$ **l** $6x - 10y + 5 = 0$

4 $(3, 0)$

5 $(0, 0)$

6 $(0, 5)$, $(-4, 0)$

7 **a** $\frac{1}{3}$ **b** $x - 3y + 15 = 0$

8 **a** $-\frac{2}{5}$ **b** $2x + 5y - 10 = 0$

9 $ax + by + c = 0$

$by = -ax - c$

$y = \left(-\frac{a}{b}\right)x - \left(\frac{c}{b}\right)$

10 $a = 6$, $c = 10$

11 $P(3,0)$

12 **a** -16 **b** -27

Challenge

Gradient $= -\frac{a}{b}$; y-intercept $= a$. So $y = -\frac{a}{b}x + a$

Rearrange to give $ax + by - ab = 0$