P2 Chapter 4: Binomial Terms

Basic Binomials

Pure Year 1 Recap

Remember that for small integer n you could use a row of Pascal Triangle for the Binomial coefficients, descending powers of the first term and ascending powers of the second.
If the first term is 1, we can ignore the powers of 1.

$(1+x)^5 =$?

$(1+2x)^4 =$?

$=$

$(1-3x)^3 =$?

$=$

Do you remember the simple way to find your Binomial coefficients?

Hopefully you can see the pattern by this point.

$\binom{n}{1} =$? $\binom{n}{2} =$? $\binom{n}{3} =$? $\binom{n}{4} =$?

$\binom{10}{3} =$? $\binom{-1}{2} =$? $\binom{-2}{3} =$?

$\binom{0.5}{2} =$?

Note: You can work out a 'choose' value, in the same way, when the top number is negative or fractional, but your calculator can not do this directly.

Pure Year 1 Recap

Remember that for small integer n you could use a row of Pascal Triangle for the Binomial coefficients, descending powers of the first term and ascending powers of the second.
If the first term is 1, we can ignore the powers of 1.

$$\begin{aligned}(1+x)^5 &= 1+5x+10x^2+10x^3+5x^4+x^5\\ (1+2x)^4 &= 1+4(2x)+6(2x)^2+4(2x)^3+(2x)^4\\ &= 1+8x+24x^2+32x^3+16x^5\\ (1-3x)^3 &= 1+3(-3x)+3(-3x)^2+(-3x)^3\\ &= 1-9x+27x^2-27x^3\end{aligned}$$

Do you remember the simple way to find your Binomial coefficients?

Hopefully you can see the pattern by this point.

$$\binom{n}{1} = \boldsymbol{n} \qquad \binom{n}{2} = \frac{\boldsymbol{n(n-1)}}{\boldsymbol{2}!} \qquad \binom{n}{3} = \frac{\boldsymbol{n(n-1)(n-2)}}{\boldsymbol{3}!} \qquad \binom{n}{4} = \frac{\boldsymbol{n(n-1)(n-2)(n-3)}}{\boldsymbol{4}!}$$

$$\binom{10}{3} = \frac{\mathbf{10\times 9\times 8}}{\mathbf{6}} = \mathbf{120} \qquad \binom{-1}{2} = \frac{\mathbf{-1\times -2}}{\mathbf{2}} = \mathbf{1} \qquad \binom{-2}{3} = \frac{\mathbf{-2\times -3\times -4}}{\mathbf{6}} = \mathbf{-4}$$

$$\binom{0.5}{2} = \frac{\mathbf{0.5\times -0.5}}{\mathbf{2}} = -\frac{\mathbf{1}}{\mathbf{8}}$$

Note: You can work out a 'choose' value, in the same way, when the top number is negative or fractional, but your calculator can not do this directly.

Binomial Expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$$

Binomial expansions, when n is either negative or fractions, are infinitely long.

Use the binomial expansion to find the first four terms of $\frac{1}{1+x}$

?

And the first four terms of $\sqrt{1-3x}$

?

Binomial Expansion

✎ $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + {}^nC_r\, x^n$

Binomial expansions, when n is either negative or fractions, are infinitely long.

Use the binomial expansion to find the first four terms of $\frac{1}{1+x}$

$$\frac{1}{1+x} = (1+x)^{-1}$$
$$= 1 + (-1)x + \frac{-1 \times -2}{2!}x^2 + \frac{-1 \times -2 \times -3}{3!}x^3 + \cdots$$
$$= \mathbf{1 - x + x^2 - x^3 + \cdots}$$

And the first four terms of $\sqrt{1-3x}$

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$= \mathbf{1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

When are infinite expansions valid?

Our expansion might be an infinite number of terms. If so, the result must converge

$$\frac{1}{1+x} = (1+x)^{-1}$$
$$= 1 + (-1)x + \frac{-1 \times -2}{2!}x^2 + \frac{-1 \times -2 \times -3}{3!}x^3 + \cdots$$
$$= \mathbf{1 - x + x^2 - x^3 + \cdots}$$

What would happen in the expansion if:

a) $x > 1$: ?

b) $0 < x < 1$: ?

c) $-1 < x < 0$: ?

d) $x = 1$: ?

Therefore requirement on x:

?

When are infinite expansions valid?

Our expansion might be an infinite number of terms. If so, the result must converge

$$\begin{aligned}\frac{1}{1+x} &= (1+x)^{-1} \\ &= 1 + (-1)x + \frac{-1\times -2}{2!}x^2 + \frac{-1\times -2\times -3}{3!}x^3 + \cdots \\ &= \mathbf{1 - x + x^2 - x^3 + \cdots}\end{aligned}$$

What would happen in the expansion if:

a) $x > 1$: e.g. $1 - 2 + 2^2 - 2^4 + \cdots$ This diverges, so expansion is not valid.

b) $0 < x < 1$: e.g $1 - 0.5 + 0.5^2 - 0.5^3 + \cdots$ This converges, as the terms become smaller each time.

c) $-1 < x < 0$: Again, this converges.

d) $x = 1$: This would suggest $1 - 1 + 1 - 1 + 1 - \cdots = \frac{1}{2}$! Indeed in some branches of mathematics this result is valid, but for the purposes of A Level this expansion is not valid.

Therefore requirement on x:

$\boldsymbol{-1 < x < 1}$

which we'd typically write as $|\boldsymbol{x}| < \mathbf{1}$

When are infinite expansions valid?

Expansions are allowed to infinite. However, the result must converge

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$= \mathbf{1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

This time, what do you think needs to be between -1 and 1 for the expansion to be valid?

?

✎ An infinite expansion $(1+x)^n$ is valid if $|x| < 1$

Quickfire Examples:

Expansion of $(1+2x)^{-1}$ valid if: ?

Expansion of $(1-x)^{-2}$ valid if: ?

Expansion of $\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}$ valid if: ?

Expansion of $\left(1-\frac{2}{3}x\right)^{-1}$ valid if:

?

When are infinite expansions valid?

Expansions are allowed to infinite. However, the result must converge

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2}\times -\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2}\times -\frac{1}{2}\times -\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$= \mathbf{1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

This time, what do you think needs to be between -1 and 1 for the expansion to be valid?

$$|\mathbf{3x}| < \mathbf{1}$$
$$\rightarrow |\mathbf{x}| < \mathbf{\frac{1}{3}}$$

✐ An infinite expansion $(1+x)^n$ is valid if $|x| < 1$

Quickfire Examples:

Expansion of $(1+2x)^{-1}$ valid if: $|\mathbf{2x}| < \mathbf{1} \quad \rightarrow \quad |\mathbf{x}| < \mathbf{\frac{1}{2}}$

Expansion of $(1-x)^{-2}$ valid if: $|\mathbf{x}| < \mathbf{1}$

Expansion of $\left(1+\frac{1}{4}x\right)^{\frac{1}{2}}$ valid if: $\left|\mathbf{\frac{1}{4}x}\right| < \mathbf{1} \quad \rightarrow \quad |\mathbf{x}| < \mathbf{4}$

Expansion of $\left(1-\frac{2}{3}x\right)^{-1}$ valid if:

$$\left|\mathbf{\frac{2}{3}x}\right| < \mathbf{1} \quad \rightarrow \quad |\mathbf{x}| < \mathbf{\frac{3}{2}}$$

Combining Expansions

Edexcel C4 June 2013 Q2

(*a*) Use the binomial expansion to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1 + x + \frac{1}{2}x^2, \qquad |x| < 1$$

(6)

Firstly express as a product:

?

How many terms do we need in each expansion?

?

Completing:

?

Combining Expansions

Edexcel C4 June 2013 Q2

(*a*) Use the binomial expansion to show that

$$\sqrt{\left(\frac{1+x}{1-x}\right)} \approx 1 + x + \frac{1}{2}x^2, \qquad |x| < 1$$

(6)

Firstly express as a product: $$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

How many terms do we need in each expansion?

We don't need beyond x^2 in each, because in the product other terms would have a power higher than 2.

Completing:

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}x^2 \quad = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{-\frac{1}{2} \times -\frac{3}{2}}{2!}(-x)^2 = 1 + \frac{1}{2}x + \frac{3}{8}x^2 - \cdots$$

$$\left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right) = 1 + \frac{1}{2}x + \frac{1}{2}x + \frac{1}{4}x - \frac{1}{8}x^2 + \frac{3}{8}x^2 + \cdots$$

$$= 1 + x + \frac{1}{2}x^2$$

Test Your Understanding

Find the binomial expansion of $\frac{1}{(1+4x)^2}$ up to an including the term in x^3.
State the values of x for which the expansion is valid.

?

?

C4 Edexcel Jan 2010

1. (*a*) Find the binomial expansion of

$$\sqrt{(1-8x)}, \qquad |x| < \frac{1}{8},$$

in ascending powers of x up to and including the term in x^3, simplifying each term. **(6)**

(*b*) Show that, when $x = \frac{1}{100}$, the exact value of $\sqrt{(1-8x)}$ is $\frac{\sqrt{23}}{5}$. **(2)**

(*c*) Substitute $x = \frac{1}{100}$ into the binomial expansion in part (*a*) and hence obtain an approximation to $\sqrt{23}$. Give your answer to 5 decimal places. **(3)**

Test Your Understanding

Find the binomial expansion of $\frac{1}{(1+4x)^2}$ up to an including the term in x^3.

State the values of x for which the expansion is valid.

$$(1+4x)^{-2} = 1 - 8x + 48x^2 - 256x^3 + \cdots$$

Expand valid if $|4x| < 1$

$$\Rightarrow |x| < \frac{1}{4}$$

C4 Edexcel Jan 2010

1. (*a*) Find the binomial expansion of

$$\sqrt{(1-8x)}, \quad |x| < \frac{1}{8},$$

in ascending powers of x up to and including the term in x^3, simplifying each term. **(6)**

(*b*) Show that, when $x = \frac{1}{100}$, the exact value of $\sqrt{(1-8x)}$ is $\frac{\sqrt{23}}{5}$. **(2)**

(*c*) Substitute $x = \frac{1}{100}$ into the binomial expansion in part (*a*) and hence obtain an approximation to $\sqrt{23}$. Give your answer to 5 decimal places. **(3)**

(a) $(1-8x)^{\frac{1}{2}} = 1 - 4x - 8x^2; -32x^3 - \ldots$

(b) $\sqrt{(1-8x)} = \sqrt{\frac{92}{100}} = \sqrt{\frac{23}{25}} = \frac{\sqrt{23}}{5}$

(c) $= 1 - 4(0.01) - 8(0.01)^2 - 32(0.01)^3$

$= 1 - 0.04 - 0.0008 - 0.000\,032 = 0.959\,168$

$\sqrt{23} = 5 \times 0.959\,168$

$= 4.795\,84$

Accuracy of an approximation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

If $x = 0.01$, how accurate would the approximation $1 - x + x^2$ by for the value of $\frac{1}{1+x}$?

?

Accuracy of an approximation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

If $x = 0.01$, how accurate would the approximation $1 - x + x^2$ by for the value of $\frac{1}{1+x}$?

Fairly accurate, as the values of x^3 and beyond will be very small (consider 0.01^3 and so on).

Common Errors

$$\sqrt{1-3x} = (1-3x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}(-3x) + \frac{\frac{1}{2}\times-\frac{1}{2}}{2!}(-3x)^2 + \frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{3!}(-3x)^3 + \cdots$$
$$= \mathbf{1 - \frac{3}{2}x - \frac{9}{8}x^2 - \frac{27}{16}x^3 - \cdots}$$

What errors do you think are easy to make?

- Sign errors, e.g. $(-3x)^2 = -9x^2$
- Not putting brackets around the $-3x$, e.g. $-3x^2$ instead of $(-3x)^2$
- Dividing by say 3 instead of 3!

Exercise 4.1

Pearson Pure Mathematics Year 2/AS
Page 29

Extension

1 [STEP I 2011 Q6] Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of $\frac{1-x+2x^2}{(1-x)^3}$ is r^2+1 and hence find the sum of the series

$$1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\frac{37}{64}+\cdots$$

(ii) Find the sum of the series

$$1+2+\frac{9}{4}+2+\frac{25}{16}+\frac{9}{8}+\frac{49}{64}$$

Homework Exercise

1 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1+x)^{-4}$ **b** $(1+x)^{-6}$ **c** $(1+x)^{\frac{1}{2}}$

d $(1+x)^{\frac{5}{3}}$ **e** $(1+x)^{-\frac{1}{4}}$ **f** $(1+x)^{-\frac{3}{2}}$

2 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1+3x)^{-3}$ **b** $\left(1+\frac{1}{2}x\right)^{-5}$ **c** $(1+2x)^{\frac{3}{4}}$

d $(1-5x)^{\frac{7}{3}}$ **e** $(1+6x)^{-\frac{2}{3}}$ **f** $\left(1-\frac{3}{4}x\right)^{-\frac{5}{3}}$

3 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $\dfrac{1}{(1+x)^2}$ **b** $\dfrac{1}{(1+3x)^4}$ **c** $\sqrt{1-x}$

d $\sqrt[3]{1-3x}$ **e** $\dfrac{1}{\sqrt{1+\frac{1}{2}x}}$ **f** $\dfrac{\sqrt[3]{1-2x}}{1-2x}$

Hint In part **f**, write the fraction as a single power of $(1-2x)$.

Homework Exercise

4 $f(x) = \dfrac{1+x}{1-2x}$

a Show that the series expansion of $f(x)$ up to and including the x^3 term is $1 + 3x + 6x^2 + 12x^3$. **(4 marks)**

b State the range of values of x for which the expansion is valid. **(1 mark)**

Hint First rewrite $f(x)$ as $(1+x)(1-2x)^{-1}$.

5 $f(x) = \sqrt{1+3x}$, $-\dfrac{1}{3} < x < \dfrac{1}{3}$

a Find the series expansion of $f(x)$, in ascending powers of x, up to and including the x^3 term. Simplify each term. **(4 marks)**

b Show that, when $x = \dfrac{1}{100}$, the exact value of $f(x)$ is $\dfrac{\sqrt{103}}{10}$ **(2 marks)**

c Find the percentage error made in using the series expansion in part **a** to estimate the value of $f(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

6 In the expansion of $(1+ax)^{-\frac{1}{2}}$ the coefficient of x^2 is 24.

a Find the possible values of a.

b Find the possible coefficients of the x^3 term.

7 Show that if x is small, the expression $\sqrt{\dfrac{1+x}{1-x}}$ is approximated by $1 + x + \frac{1}{2}x^2$.

Notation 'x is small' means we can assume the expansion is valid for the x values being considered, as high powers become insignificant compared to the first few terms.

Homework Exercise

8 $h(x) = \dfrac{6}{1+5x} - \dfrac{4}{1-3x}$

a Find the series expansion of $h(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(6 marks)**

b Find the percentage error made in using the series expansion in part **a** to estimate the value of $h(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

c Explain why it is not valid to use the expansion to find $h(0.5)$. **(1 mark)**

9 **a** Find the binomial expansion of $(1-3x)^{\frac{3}{2}}$ in ascending powers of x up to and including the x^3 term, simplifying each term. **(4 marks)**

b Show that, when $x = \dfrac{1}{100}$, the exact value of $(1-3x)^{\frac{3}{2}}$ is $\dfrac{97\sqrt{97}}{1000}$ **(2 marks)**

c Substitute $x = \dfrac{1}{100}$ into the binomial expansion in part **a** and hence obtain an approximation to $\sqrt{97}$. Give your answer to 5 decimal places. **(3 marks)**

Challenge

$h(x) = \left(1 + \dfrac{1}{x}\right)^{-\frac{1}{2}}, |x| > 1$

a Find the binomial expansion of $h(x)$ in ascending powers of x up to and including the x^2 term, simplifying each term.

b Show that, when $x = 9$, the exact value of $h(x)$ is $\dfrac{3\sqrt{10}}{10}$

c Use the expansion in part **a** to find an approximate value of $\sqrt{10}$. Write your answer to 2 decimal places.

Hint Replace x with $\dfrac{1}{x}$

Homework Answers

1 a i $1 - 4x + 10x^2 - 20x^3 \ldots$ ii $|x| < 1$
b i $1 - 6x + 21x^2 - 56x^3 \ldots$ ii $|x| < 1$
c i $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \ldots$ ii $|x| < 1$
d i $1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} \ldots$ ii $|x| < 1$
e i $1 - \frac{x}{4} + \frac{5x^2}{32} - \frac{15x^3}{128} \ldots$ ii $|x| < 1$
f i $1 - \frac{3x}{2} + \frac{15x^2}{8} - \frac{35x^3}{16} \ldots$ ii $|x| < 1$

2 a i $1 - 9x + 54x^2 - 270x^3 \ldots$ ii $|x| < \frac{1}{3}$
b i $1 - \frac{5x}{2} + \frac{15x^2}{4} - \frac{35x^3}{8} \ldots$ ii $|x| < 2$
c i $1 + \frac{3x}{2} - \frac{3x^2}{8} + \frac{5x^3}{16} \ldots$ ii $|x| < \frac{1}{2}$
d i $1 - \frac{35x}{3} + \frac{350x^2}{9} - \frac{1750x^3}{81} \ldots$ ii $|x| < \frac{1}{5}$
e i $1 - 4x + 20x^2 - \frac{320x^3}{3} \ldots$ ii $|x| < \frac{1}{6}$
f i $1 + \frac{5x}{4} + \frac{5x^2}{4} + \frac{55x^3}{48} \ldots$ ii $|x| < \frac{4}{3}$

3 a i $1 - 2x + 3x^2 - 4x^3 \ldots$ ii $|x| < 1$
b i $1 - 12x + 90x^2 - 540x^3 \ldots$ ii $|x| < \frac{1}{3}$
c i $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} \ldots$ ii $|x| < 1$
d i $1 - x - x^2 - \frac{5x^3}{3} \ldots$ ii $|x| < \frac{1}{3}$
e i $1 - \frac{x}{4} + \frac{3x^2}{32} - \frac{5x^3}{128} \ldots$ ii $|x| < 2$
f i $1 + \frac{4x}{3} + \frac{20x^2}{9} + \frac{320x^3}{81} \ldots$ ii $|x| < \frac{1}{2}$

4 a Expansion of $(1 - 2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + \ldots$
Multiply by $(1 + x) = 1 + 3x + 6x^2 + 12x^3 + \ldots$
b $|x| < \frac{1}{2}$

5 a $1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3$
b $f(x) = \sqrt{\frac{103}{100}} = \frac{\sqrt{103}}{\sqrt{100}} = \frac{\sqrt{103}}{10}$
c $3.1 \times 10^{-6}\%$

6 a $\alpha = \pm 8$ b ± 160

7 For small values of x ignore powers of x^3 and higher.
$(1 + x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots,\ (1 - x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \ldots$
$\sqrt{\frac{1 + x}{1 - x}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x}{2} + \frac{x^2}{4} + \frac{3x^2}{8} + \ldots = 1 + x + \frac{x^2}{2}$

8 a $2 - 42x + 114x^2$
b 0.052%
c The expansion is only valid for $|x| < \frac{1}{5}$. $|0.5|$ is not less than $\frac{1}{5}$.

9 a $1 - \frac{9}{2}x + \frac{27}{8}x^2 + \frac{27}{16}x^3$
b $0.97^{\frac{3}{2}} = \left(\frac{\sqrt{97}}{10}\right)^3 = \frac{97\sqrt{97}}{1000}$
c 9.84886

Challenge

a $1 - \frac{1}{2x} + \frac{3}{8x^2}$
b $h(x) = \left(\frac{10}{9}\right)^{-\frac{1}{2}} = \left(\frac{9}{10}\right)^{\frac{1}{2}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$
c 3.16