P1 Chapter 2: Quadratics

Completing The Square

Completing the Square

"Completing the square" means putting a quadratic in the form $(x+a)^2+b$ or $a(x+b)^2+c$

The **underlying reason we do this is because x only appears once in the expression** (e.g. in $(x+2)^2+3$ vs x^2+4x+7), which makes it algebraically easier to handle. This has a number of consequences:

a. Solving Quadratics

If we have a completed square:

$$(x+4)^2-7=0$$

we saw at the start of the chapter how we could rearrange to make x the subject.

Indeed using the quadratic formula is actually solving the quadratic by completing the square – it's just someone has done the work for us already!

b. Sketching Quadratics

We'll see later that if $y=(x+a)^2+b$, then the minimum point is $(-a,b)$

c. In integration

In Further Maths, completing the square allows us to 'integrate' expressions like:

$$\int \frac{1}{x^2-4x+5}\,dx$$

(you will cover integration later this module)

Completing the Square Recap

Expand:

$$(x+\mathbf{9})^2 = \text{?}$$
$$(x-\mathbf{5})^2 = \text{?}$$

What do you notice about the relationship between the bold numbers?

?

Therefore, if we had x^2+12x, how could we write it in the form $(x+a)^2+b$?

$$x^2+12x = \text{?}$$

Since $(x+6)^2 = x^2+12x+36$, we want to discard the 36, so 'throw it away' by subtracting.

Further Examples:

$$x^2+8x = \text{?}$$
$$x^2-2x = \text{?}$$
$$x^2-6x+7 = \text{?}$$
$$= \text{?}$$

Notice that despite the a being negative, we still subtract after the bracket as $(-1)^2$ is positive.

Textbook Note:
The textbook uses the formula

$$x^2+bx = \left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

and similarly

$$ax^2+bx+c$$
$$= a\left(x+\frac{b}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right)$$

My personal judgement is that it's <u>not</u> worth memorising these and you should instead think through the steps. Even the textbook agrees!

Completing the Square Recap

Expand:

$$(x + \mathbf{9})^2 = x^2 + \mathbf{18}x + 81$$
$$(x - \mathbf{5})^2 = x^2 - \mathbf{10}x + 25$$

What do you notice about the relationship between the bold numbers?

The $\boldsymbol{a}$ in $(\boldsymbol{x} + \boldsymbol{a})^2$ is half the coefficient of $\boldsymbol{x}$ in the expansion.

Therefore, if we had $x^2 + 12x$, how could we write it in the form $(x + a)^2 + b$?

$$x^2 + 12x = (\boldsymbol{x} + \mathbf{6})^2 - \mathbf{36}$$

Since $(x + 6)^2 = x^2 + 12x + 36$, we want to discard the 36, so 'throw it away' by subtracting.

Further Examples:

$$x^2 + 8x = (\boldsymbol{x} + \mathbf{4})^2 - \mathbf{16}$$
$$x^2 - 2x = (\boldsymbol{x} - \mathbf{1})^2 - \mathbf{1}$$
$$x^2 - 6x + 7 = (\boldsymbol{x} - \mathbf{3})^2 - \mathbf{9} + \mathbf{7}$$
$$= (\boldsymbol{x} - \mathbf{3})^2 - \mathbf{2}$$

Notice that despite the a being negative, we still subtract after the bracket as $(-1)^2$ is positive.

Textbook Note:

The textbook uses the formula

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

and similarly

$$ax^2 + bx + c$$
$$= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

My personal judgement is that it's <u>not</u> worth memorising these and you should instead think through the steps. Even the textbook agrees!

Completing the Square

Express $2x^2 + 12x + 7$ in the form $a(x+b)^2 + c$

= ?

= ?

= ?

= ?

Factorise out coefficient of x^2. You can leave the constant term outside the bracket.

Complete the square inside the bracket (you should have two sets of brackets)

Expand out outer bracket.

Express $5 - 3x^2 + 6x$ in the form $a - b(x+c)^2$

= ?

= ?

= ?

= ?

= ?

It may help to write in the form $ax^2 + bx + c$ first.

Completing the Square

Express $2x^2 + 12x + 7$ in the form $a(x+b)^2 + c$

$$= 2(x^2 + 6x) + 7$$
$$= 2\left((x+3)^2 - 9\right) + 7$$
$$= 2(x+3)^2 - 18 + 7$$
$$= 2(x+3)^2 - 11$$

Factorise out coefficient of x^2. You can leave the constant term outside the bracket.

Complete the square inside the bracket (you should have two sets of brackets)

Expand out outer bracket.

Express $5 - 3x^2 + 6x$ in the form $a - b(x+c)^2$

$$= -3x^2 + 6x + 5$$
$$= -3(x^2 - 2x) + 5$$
$$= -3\left((x-1)^2 - 1\right) + 5$$
$$= -3(x-1)^2 + 3 + 5$$
$$= 8 - 3(x-1)^2$$

It may help to write in the form $ax^2 + bx + c$ first.

Test Your Understanding

Express $3x^2 - 18x + 4$ in the form $a(x+b)^2 + c$

$=$
$=$
$=$
$=$

?

Express $20x - 5x^2 + 3$ in the form $a - b(x+c)^2$

$=$
$=$
$=$
$=$
$=$

?

Test Your Understanding

Express $3x^2 - 18x + 4$ in the form $a(x+b)^2 + c$

$$
\begin{aligned}
&= 3(x^2 - 6x) + 4 \\
&= 3\left((x-3)^2 - 9\right) + 4 \\
&= 3(x-3)^2 - 27 + 4 \\
&= 3(x-3)^2 - 23
\end{aligned}
$$

Express $20x - 5x^2 + 3$ in the form $a - b(x+c)^2$

$$
\begin{aligned}
&= -5x^2 + 20x + 3 \\
&= -5(x^2 - 4x) + 3 \\
&= -5\left((x-2)^2 - 4\right) + 3 \\
&= -5(x-2)^2 + 20 + 3 \\
&= 23 - 5(x-2)^2
\end{aligned}
$$

Solving by Completing the Square

Solve the equation:

$$3x^2 - 18x + 4 = 0$$

? First step

? And the rest…

Note: Previously we factorised out the 3. This is because $3x^2 - 18x + 4$ on its own is an **expression,** so dividing by 3 (instead of factorising) would change the expression.

However, in an equation, we can divide both sides by 3 without affecting the solutions.

Solving by Completing the Square

Solve the equation:

$$3x^2 - 18x + 4 = 0$$

$$x^2 - 6x + \frac{4}{3} = 0$$

$$(x-3)^2 - 9 + \frac{4}{3} = 0$$

$$(x-3)^2 = \frac{23}{3}$$

$$x - 3 = \pm\sqrt{\frac{23}{3}}$$

$$x = 3 \pm \sqrt{\frac{23}{3}}$$

Note: Previously we factorised out the 3. This is because $3x^2 - 18x + 4$ on its own is an **expression,** so dividing by 3 (instead of factorising) would change the expression.

However, in an equation, we can divide both sides by 3 without affecting the solutions.

Proving the Quadratic Formula

If $ax^2 + bx + c = 0$, prove that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$ax^2 + bx + c = 0$$

?

Just use exactly the same method as you usual !

Proving the Quadratic Formula

If $ax^2 + bx + c = 0$, prove that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Just use exactly the same method as you usual !

Exercise 2.2

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Homework Exercise

> **Hint** In question **3d**, write the expression as $-4x^2 - 16x + 10$ then take a factor of -4 out of the first two terms to get $-4(x^2 + 4x) + 10$.

1 Complete the square for the expressions:

a $x^2 + 4x$ **b** $x^2 - 6x$ **c** $x^2 - 16x$ **d** $x^2 + x$ **e** $x^2 - 14$

2 Complete the square for the expressions:

a $2x^2 + 16x$ **b** $3x^2 - 24x$ **c** $5x^2 + 20x$ **d** $2x^2 - 5x$ **e** $8x - 2x^2$

3 Write each of these expressions in the form $p(x + q)^2 + r$, where p, q and r are constants to be found:

a $2x^2 + 8x + 1$ **b** $5x^2 - 15x + 3$ **c** $3x^2 + 2x - 1$ **d** $10 - 16x - 4x^2$ **e** $2x - 8x^2 + 10$

4 Given that $x^2 + 3x + 6 = (x + a)^2 + b$, find the values of the constants a and b. **(2 marks)**

5 Write $2 + 0.8x - 0.04x^2$ in the form $A - B(x + C)^2$, where A, B and C are constants to be determined. **(3 marks)**

6 Solve these quadratic equations by completing the square. Leave your answers in surd form.

a $x^2 + 6x + 1 = 0$ **b** $x^2 + 12x + 3 = 0$ **c** $x^2 + 4x - 2 = 0$ **d** $x^2 - 10x = 5$

7 Solve these quadratic equations by completing the square. Leave your answers in surd form.

a $2x^2 + 6x - 3 = 0$ **b** $5x^2 + 8x - 2 = 0$ **c** $4x^2 - x - 8 = 0$ **d** $15 - 6x - 2x^2 = 0$

Homework Exercise

8 $x^2 - 14x + 1 = (x + p)^2 + q$, where p and q are constants.

a Find the values of p and q. **(2 marks)**

b Using your answer to part **a**, or otherwise, show that the solutions to the equation $x^2 - 14x + 1 = 0$ can be written in the form $r \pm s\sqrt{3}$, where r and s are constants to be found. **(2 marks)**

9 By completing the square, show that the solutions to the equation $x^2 + 2bx + c = 0$ are given by the formula $x = -b \pm \sqrt{b^2 - c}$. **(4 marks)**

Problem-solving

Follow the same steps as you would if the coefficients were numbers.

Challenge

a Show that the solutions to the equation

$ax^2 + 2bx + c = 0$ are given by $x = -\frac{b}{a} \pm \sqrt{\frac{b^2 - ac}{a^2}}$.

b Hence, or otherwise, show that the solutions to the equation $ax^2 + bx + c = 0$ can be written as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hint Start by dividing the whole equation by a.

Links You can use this method to prove the quadratic formula. → Section 7.4

Homework Answers

1 **a** $(x+2)^2-4$ **b** $(x-3)^2-9$
c $(x-8)^2-64$ **d** $(x+\frac{1}{2})^2-\frac{1}{4}$
e $(x-7)^2-49$

2 **a** $2(x+4)^2-32$ **b** $3(x-4)^2-48$
c $5(x+2)^2-20$ **d** $2(x-\frac{5}{4})^2-\frac{25}{8}$
e $-2(x-2)^2+8$

3 **a** $2(x+2)^2-7$ **b** $5(x-\frac{3}{2})^2-\frac{33}{4}$
c $3(x+\frac{1}{3})^2-\frac{4}{3}$ **d** $-4(x+2)^2+26$
e $-8(x-\frac{1}{8})^2+\frac{81}{8}$

4 $a=\frac{3}{2}$, $b=\frac{15}{4}$

5 $A=6$, $B=0.04$, $C=-10$

6 **a** $x=-3\pm 2\sqrt{2}$ **b** $x=-6\pm\sqrt{33}$
c $x=-2\pm\sqrt{6}$ **d** $x=5\pm\sqrt{30}$

7 **a** $x=\frac{1}{2}(-3\pm\sqrt{15})$ **b** $x=\frac{1}{5}(-4\pm\sqrt{26})$
c $x=\frac{1}{8}(1\pm\sqrt{129})$ **d** $x=\frac{1}{2}(-3\pm\sqrt{39})$

8 **a** $p=-7$, $q=-48$
b $(x-7)^2=48$
$x=7\pm\sqrt{48}=7\pm 4\sqrt{3}$
$r=7$, $s=4$

9 $x^2+2bx+c=(x+b)^2-b^2+c$
$(x+b)^2=b^2-c$
$x=-b\pm\sqrt{b^2-c}$

Challenge

a $ax^2+2bx+c=0$

$$x^2+\frac{2b}{a}x+\frac{c}{a}=0$$

$$\left(x+\frac{b}{a}\right)^2-\frac{b^2}{a^2}+\frac{c}{a}=0$$

$$\left(x+\frac{b}{a}\right)^2=\frac{b^2-ac}{a^2}$$

$$x=-\frac{b}{a}\pm\sqrt{\frac{b^2-ac}{a^2}}$$

b $ax^2+bx+c=0$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$