P1 Chapter 1: Algebra

Rationalising Denominators

Rationalising The Denominator

Here's a surd. What could we multiply it by such that it's no longer an irrational number?

$$\sqrt{5} \times \sqrt{5} = 5$$

$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

In this fraction, the denominator is irrational. '**Rationalising the denominator**' means making the denominator a rational number.

What could we multiply this fraction by to both rationalise the denominator, but leave the value of the fraction unchanged?

Side note: There's two reasons why we might want to do this:

1. For aesthetic reasons, it makes more sense to say "half of root 2" rather than "one root two-th of 1". It's nice to divide by something whole!
2. It makes it easier for us to add expressions involving surds.

Examples

$$\frac{3}{\sqrt{2}} = ?$$

$$\frac{6}{\sqrt{3}} = ?$$

$$\frac{7}{\sqrt{7}} = ?$$

$$\frac{15}{\sqrt{5}} + \sqrt{5} = ?$$

Test Your Understanding:

$$\frac{12}{\sqrt{3}} = ?$$

$$\frac{2}{\sqrt{6}} = ?$$

$$\frac{4\sqrt{2}}{\sqrt{8}} = ?$$

Examples

$$\frac{3}{\sqrt{2}} = \frac{\mathbf{3\sqrt{2}}}{\mathbf{2}}$$

$$\frac{6}{\sqrt{3}} = \frac{\mathbf{6\sqrt{3}}}{\mathbf{3}} = \mathbf{2\sqrt{3}}$$

$$\frac{7}{\sqrt{7}} = \frac{\mathbf{7\sqrt{7}}}{\mathbf{7}} = \sqrt{\mathbf{7}}$$

$$\frac{15}{\sqrt{5}} + \sqrt{5} = \mathbf{3\sqrt{5}} + \sqrt{\mathbf{5}} = \mathbf{4\sqrt{5}}$$

Test Your Understanding:

$$\frac{12}{\sqrt{3}} = \mathbf{4\sqrt{3}}$$

$$\frac{2}{\sqrt{6}} = \frac{\sqrt{\mathbf{6}}}{\mathbf{3}}$$

$$\frac{4\sqrt{2}}{\sqrt{8}} = \frac{\mathbf{16}}{\mathbf{8}} = \mathbf{2}$$

More Complex Denominators

You've seen 'rationalising a denominator', the idea being that we don't like to divide things by an irrational number.

But what do we multiply the top and bottom by if we have a more complicated denominator?

$$\frac{1}{\sqrt{2}+1} \times \frac{\sqrt{\mathbf{2}}-\mathbf{1}}{\sqrt{\mathbf{2}}-\mathbf{1}} = \frac{\sqrt{\mathbf{2}}-\mathbf{1}}{\mathbf{1}} = \sqrt{\mathbf{2}}-\mathbf{1}$$

We basically use the same expression but with the sign reversed (this is known as the *conjugate*). That way, we obtain the difference of two squares. Since $(a+b)(a-b)=a^2-b^2$, any surds will be squared and thus we'll end up with no surds in the denominator.

More Examples

$$\frac{3}{\sqrt{6}-2} \times \boxed{?} = \boxed{?}$$

You can explicitly expand out $(\sqrt{6}-2)(\sqrt{6}+2)$ in the denominator, but remember that $(a-b)(a+b) = a^2 - b^2$ so we can mentally obtain $6 - 4 = 2$
Just remember: ‘difference of two squares’!

$$\frac{4}{\sqrt{3}+1} \times \boxed{?} = \boxed{?} = \boxed{?}$$

$$\frac{3\sqrt{2}+4}{5\sqrt{2}-7} \times \boxed{?} = \boxed{?}$$

$$= $$

More Examples

$$\frac{3}{\sqrt{6}-2} \times \frac{\sqrt{6}+2}{\sqrt{6}+2} = \frac{3\sqrt{6}+6}{2}$$

You can explicitly expand out $(\sqrt{6}-2)(\sqrt{6}+2)$ in the denominator, but remember that $(a-b)(a+b) = a^2 - b^2$ so we can mentally obtain $6-4=2$
Just remember: 'difference of two squares'!

$$\frac{4}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{4\sqrt{3}-4}{2} = 2\sqrt{3}-2$$

$$\frac{3\sqrt{2}+4}{5\sqrt{2}-7} \times \frac{\mathbf{5}\sqrt{\mathbf{2}}+\mathbf{7}}{\mathbf{5}\sqrt{\mathbf{2}}+\mathbf{7}} = \frac{\mathbf{30}+\mathbf{21}\sqrt{\mathbf{2}}+\mathbf{20}\sqrt{\mathbf{2}}+\mathbf{28}}{\mathbf{1}}$$

$$= \mathbf{58}+\mathbf{41}\sqrt{\mathbf{2}}$$

Test Your Understanding

Rationalise the denominator and simplify

$$\frac{4}{\sqrt{5}-2}$$

?

Rationalise the denominator and simplify

$$\frac{2\sqrt{3}-1}{3\sqrt{3}+1}$$

?

AQA IGCSE FM June 2013 Paper 1

Solve $y\left(\sqrt{3}-1\right)=8$

Give your answer in the form $a+b\sqrt{3}$ where a and b are integers.

?

Test Your Understanding

Rationalise the denominator and simplify

$$\frac{4}{\sqrt{5}-2}$$

$$\mathbf{8+4\sqrt{5}}$$

Rationalise the denominator and simplify

$$\frac{2\sqrt{3}-1}{3\sqrt{3}+1}$$

$$\frac{2\sqrt{3}-1}{3\sqrt{3}+1}\times\frac{3\sqrt{3}-1}{3\sqrt{3}-1}$$
$$=\frac{18-2\sqrt{3}-3\sqrt{3}+1}{27-1}$$
$$=\frac{19-5\sqrt{3}}{26}$$

AQA IGCSE FM June 2013 Paper 1

Solve $y(\sqrt{3}-1)=8$

Give your answer in the form $a+b\sqrt{3}$ where a and b are integers.

$$y=\frac{8}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$
$$=\frac{8\sqrt{3}+8}{2}=4+4\sqrt{3}$$

Writing surd expressions in power form

This is not in the textbook, but a common type of question in exams, often asked in conjunction with differentiation (a later chapter).

Express $\frac{(2+\sqrt{x})^2}{x}$ as powers of x.

?

Test Your Understanding

Express $\frac{(x+\sqrt{x})^2}{2\sqrt{x}}$ as powers of x.

?

Writing surd expressions in power form

This is not in the textbook, but a common type of question in exams, often asked in conjunction with differentiation (a later chapter).

Express $\frac{(2+\sqrt{x})^2}{x}$ as powers of x.

$$\frac{4+4\sqrt{x}+x}{x}$$

Expand.

$$=\frac{4+4x^{\frac{1}{2}}+x}{x^1}$$

Write any roots as powers of x.

$$=\frac{4}{x^1}+\frac{4x^{\frac{1}{2}}}{x^1}+\frac{x}{x}$$

$$=4x^{-1}+4x^{-\frac{1}{2}}+1$$

Split fraction (some may wish to do this step mentally)

Test Your Understanding

Express $\frac{(x+\sqrt{x})^2}{2\sqrt{x}}$ as powers of x.

$$\frac{x^2+2x\sqrt{x}+x}{2\sqrt{x}}$$

$$=\frac{x^2+2x^{\frac{3}{2}}+x}{2x^{\frac{1}{2}}}$$

$$=\frac{1}{2}x^{\frac{3}{2}}+x+\frac{1}{2}x^{\frac{1}{2}}$$

Exercise

1 Rationalise the denominator and simplify the following:

a $\dfrac{1}{\sqrt{5}+2} =$?

b $\dfrac{\sqrt{3}}{\sqrt{3}-1} =$?

c $\dfrac{\sqrt{5}+1}{\sqrt{5}-2} =$?

d $\dfrac{2\sqrt{3}-1}{3\sqrt{3}+4} =$?

e $\dfrac{5\sqrt{5}-2}{2\sqrt{5}-3} =$?

2 Expand and simplify:

$$(\sqrt{5}+3)(\sqrt{5}-2)(\sqrt{5}+1) = \text{?}$$

3 Rationalise the denominator, giving your answer in the form $a+b\sqrt{3}$.

$$\frac{3\sqrt{3}+7}{3\sqrt{3}-5} = \text{?}$$

4 Solve $x(4-\sqrt{6}) = 10$ giving your answer in the form $a+b\sqrt{6}$.

?

5 Solve $y(1+\sqrt{2})-\sqrt{2} = 3$

$$y = \frac{3+\sqrt{2}}{1+\sqrt{2}} = \text{?}$$

6 Simplify:

$$\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}+\sqrt{a}} = \text{?}$$

Exercise

1 Rationalise the denominator and simplify the following:

a $\dfrac{1}{\sqrt{5}+2} = \mathbf{\sqrt{5}-2}$

b $\dfrac{\sqrt{3}}{\sqrt{3}-1} = \mathbf{\dfrac{3+\sqrt{3}}{2}}$

c $\dfrac{\sqrt{5}+1}{\sqrt{5}-2} = \mathbf{7+3\sqrt{5}}$

d $\dfrac{2\sqrt{3}-1}{3\sqrt{3}+4} = \mathbf{2-\sqrt{3}}$

e $\dfrac{5\sqrt{5}-2}{2\sqrt{5}-3} = \mathbf{4+\sqrt{5}}$

2 Expand and simplify:

$$\left(\sqrt{5}+3\right)\left(\sqrt{5}-2\right)\left(\sqrt{5}+1\right) = \mathbf{4}$$

3 Rationalise the denominator, giving your answer in the form $a + b\sqrt{3}$.

$$\frac{3\sqrt{3}+7}{3\sqrt{3}-5} = \mathbf{31+18\sqrt{3}}$$

4 Solve $x\left(4-\sqrt{6}\right) = 10$ giving your answer in the form $a + b\sqrt{6}$.

$$x = \frac{10}{4-\sqrt{6}} = \mathbf{4+\sqrt{6}}$$

5 Solve $y\left(1+\sqrt{2}\right) - \sqrt{2} = 3$

$$y = \frac{3+\sqrt{2}}{1+\sqrt{2}} = \mathbf{2\sqrt{2}-1}$$

6 Simplify:

$$\frac{\sqrt{a+1}-\sqrt{a}}{\sqrt{a+1}+\sqrt{a}} = \mathbf{2a+1-2\sqrt{a}\sqrt{a+1}}$$

A final super hard puzzle

☠ Solve $\frac{\sqrt[4]{9}}{\sqrt[5]{27}} = \sqrt[x]{3}$

?

A final super hard puzzle

☠ Solve $\frac{\sqrt[4]{9}}{\sqrt[5]{27}} = \sqrt[x]{3}$

$$\frac{\sqrt[4]{3^2}}{\sqrt[5]{3^3}} = \frac{\left(3^2\right)^{\frac{1}{4}}}{\left(3^3\right)^{\frac{1}{5}}} = \frac{3^{\frac{1}{2}}}{3^{\frac{3}{5}}} = 3^{-\frac{1}{10}}$$

But $\sqrt[x]{3} = 3^{\frac{1}{x}}$

$$\therefore \frac{1}{x} = -\frac{1}{10} \quad \rightarrow \quad x = -10$$

Exercise 1.6

Pearson Pure Mathematics Year 2/AS
Pages 5

Homework Exercise

1 Simplify:

a $\frac{1}{\sqrt{5}}$ **b** $\frac{1}{\sqrt{11}}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\frac{\sqrt{3}}{\sqrt{15}}$

e $\frac{\sqrt{12}}{\sqrt{48}}$ **f** $\frac{\sqrt{5}}{\sqrt{80}}$ **g** $\frac{\sqrt{12}}{\sqrt{156}}$ **h** $\frac{\sqrt{7}}{\sqrt{63}}$

2 Rationalise the denominators and simplify:

a $\frac{1}{1+\sqrt{3}}$ **b** $\frac{1}{2+\sqrt{5}}$ **c** $\frac{1}{3-\sqrt{7}}$ **d** $\frac{4}{3-\sqrt{5}}$ **e** $\frac{1}{\sqrt{5}-\sqrt{3}}$

f $\frac{3-\sqrt{2}}{4-\sqrt{5}}$ **g** $\frac{5}{2+\sqrt{5}}$ **h** $\frac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}$ **i** $\frac{11}{3+\sqrt{11}}$ **j** $\frac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

k $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$ **l** $\frac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}$ **m** $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

3 Rationalise the denominators and simplify:

a $\frac{1}{(3-\sqrt{2})^2}$ **b** $\frac{1}{(2+\sqrt{5})^2}$ **c** $\frac{4}{(3-\sqrt{2})^2}$

d $\frac{3}{(5+\sqrt{2})^2}$ **e** $\frac{1}{(5+\sqrt{2})(3-\sqrt{2})}$ **f** $\frac{2}{(5-\sqrt{3})(2+\sqrt{3})}$

4 Simplify $\frac{3-2\sqrt{5}}{\sqrt{5}-1}$ giving your answer in the form $p+q\sqrt{5}$, where p and q are rational numbers. **(4 marks)**

Problem-solving

You can check that your answer is in the correct form by writing down the values of p and q and checking that they are rational numbers.

Homework Answers

1 **a** $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{11}}{11}$ **c** $\frac{\sqrt{2}}{2}$

d $\frac{\sqrt{5}}{5}$ **e** $\frac{1}{2}$ **f** $\frac{1}{4}$

g $\frac{\sqrt{13}}{13}$ **h** $\frac{1}{3}$

2 **a** $\frac{1-\sqrt{3}}{-2}$ **b** $\sqrt{5}-2$ **c** $\frac{3+\sqrt{7}}{2}$

d $3+\sqrt{5}$ **e** $\frac{\sqrt{5}+\sqrt{3}}{2}$ **f** $\frac{(3-\sqrt{2})(4+\sqrt{5})}{11}$

g $5(\sqrt{5}-2)$ **h** $5(4+\sqrt{14})$ **i** $\frac{11(3-\sqrt{11})}{-2}$

j $\frac{5-\sqrt{21}}{-2}$ **k** $\frac{14-\sqrt{187}}{3}$ **l** $\frac{35+\sqrt{1189}}{6}$

m -1

3 **a** $\frac{11+6\sqrt{2}}{49}$ **b** $9-4\sqrt{5}$ **c** $\frac{44+24\sqrt{2}}{49}$

d $\frac{81-30\sqrt{2}}{529}$ **e** $\frac{13+2\sqrt{2}}{161}$ **f** $\frac{7-3\sqrt{3}}{11}$

4 $-\frac{7}{4}+\frac{\sqrt{5}}{4}$