

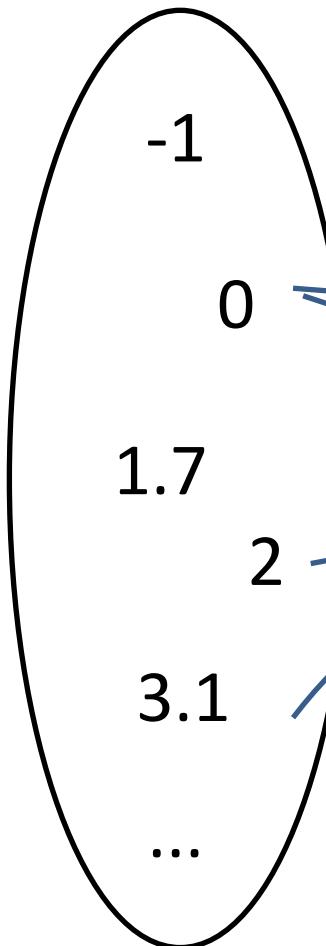
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## P2 Chapter 2: Graphing Functions

### 3D Functions as Maps

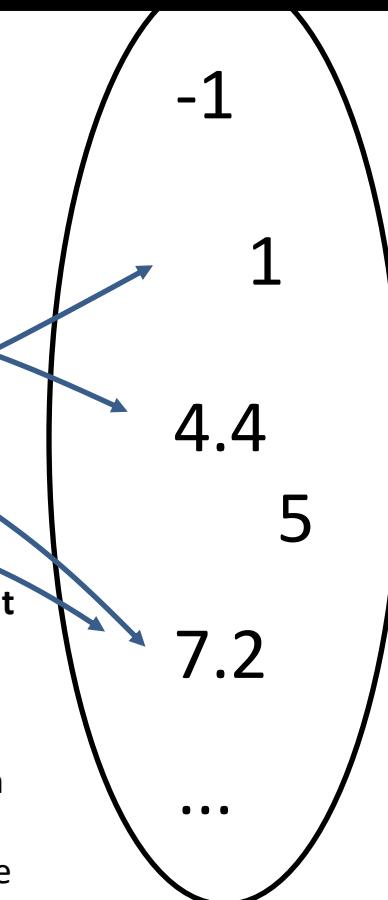
# What is a mapping?

Inputs



A **mapping** is something which maps one set of numbers to another.

Outputs



The **domain** is the set of possible inputs.

The **range** is the set of possible outputs.

# What is a function?

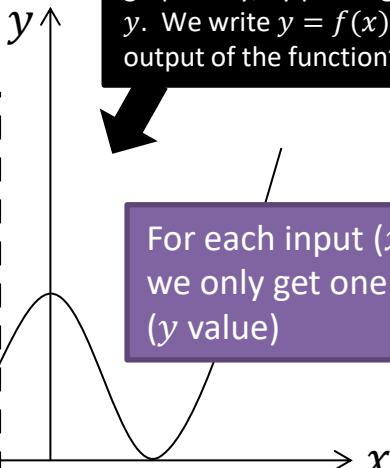
✍ A function is: a mapping such that every element of the domain is mapped to exactly one element of the range.

Notation:  $f(x) = 2x + 1$        $f: x \rightarrow 2x + 1$

$f(x)$  refers to the output of the function.

## Function?

**Tip:** Use the 'vertical ray test'. If a vertically fired ray can hit the curve multiple times, it is NOT a function.

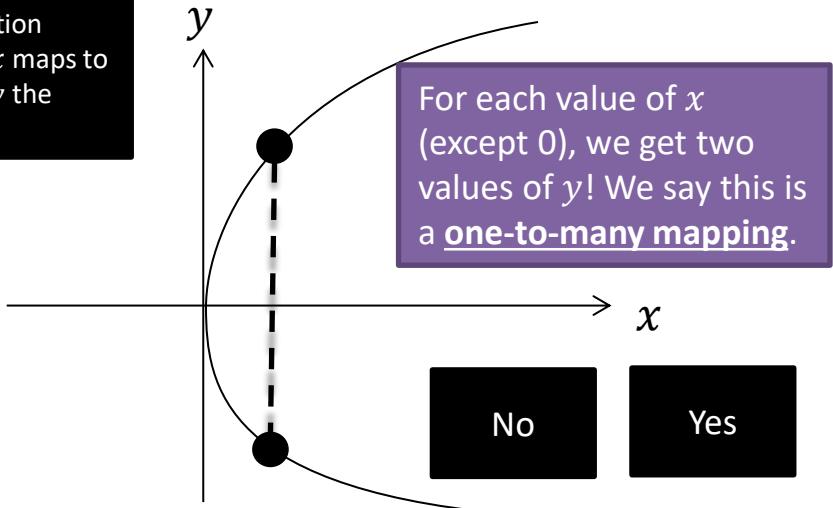


No

Yes

**Note:** We can illustrate a mapping/function graphically, by plotting a point  $(x, y)$  if  $x$  maps to  $y$ . We write  $y = f(x)$  to mean "make  $y$  the output of the function".

For each input ( $x$  value), we only get one output ( $y$  value)



No

Yes

For each value of  $x$  (except 0), we get two values of  $y$ ! We say this is a one-to-many mapping.

$$f(x) = 2^x$$

Domain:  $x \in \mathbb{R}$   
(i.e. all real values)

No

Yes

$$f(x) = \sqrt{x}$$

No

Yes

Domain:  $x \in \mathbb{R}$

We can't square root a negative number, but the input set is  $\mathbb{R}$ , so some inputs don't map to a value.

$$f(x) = \pm\sqrt{x}$$

Domain:  $x \geq 0$

No

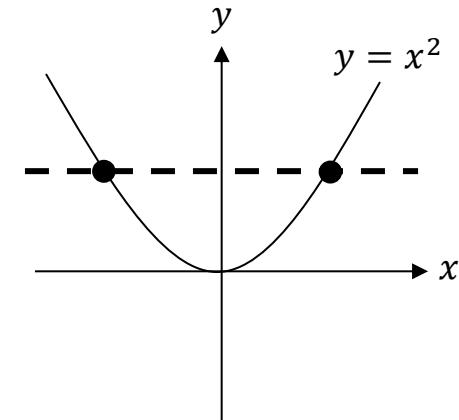
Yes

$f(4) = 2$  but  $f(4) = -2$  also. This is one-to-many so not a function.

# One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

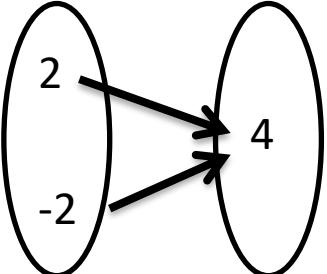
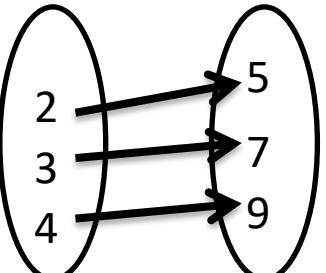
Type	Description	Example
Many-to-one function	?	?
One-to-one function	?	?

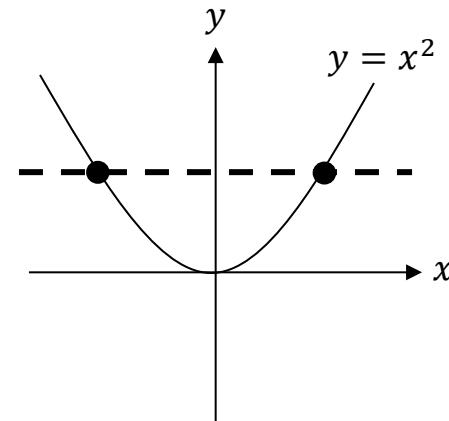


You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

# One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

Type	Description	Example
Many-to-one function	Multiple inputs can map to the same output. 	$f(x) = x^2$ e.g. $f(2) = 4$ $f(-2) = 4$
One-to-one function	Each output has one input and vice versa. 	$f(x) = 2x + 1$



You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

# Further Examples

It is often helpful to sketch the function to reason about the range.

[Textbook] Find the range of each of the following functions.

- a)  $f(x) = 3x - 2$ , domain  $\{1,2,3,4\}$
- b)  $g(x) = x^2$ , domain  $\{x \in \mathbb{R}, -5 \leq x \leq 5\}$
- c)  $h(x) = \frac{1}{x}$ , domain  $\{x \in \mathbb{R}, 0 < x \leq 3\}$

State if the functions are one-to-one or many-to-one.

We use  $x$  to refer to the input, and  $f(x)$  to refer to the output.  
**Thus your ranges should be in terms of  $f(x)$ .**

a

b

c

?

?

?

# Further Examples

It is often helpful to sketch the function to reason about the range.

[Textbook] Find the range of each of the following functions.

- $f(x) = 3x - 2$ , domain  $\{1,2,3,4\}$
- $g(x) = x^2$ , domain  $\{x \in \mathbb{R}, -5 \leq x \leq 5\}$
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State if the functions are one-to-one or many-to-one.

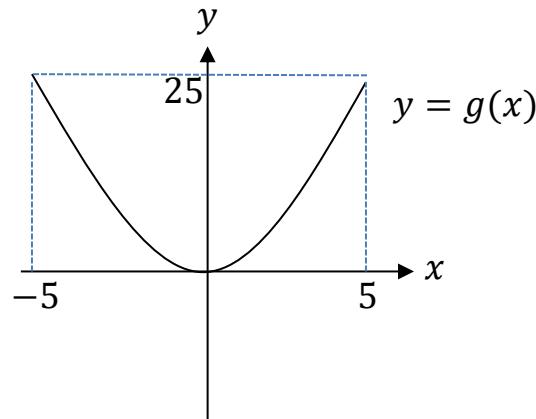
We use  $x$  to refer to the input, and  $f(x)$  to refer to the output.  
Thus your ranges should be in terms of  $f(x)$ .

a

$$\begin{aligned}f(1) &= 1 \\f(2) &= 4 \\f(3) &= 7 \\f(4) &= 10\end{aligned}$$

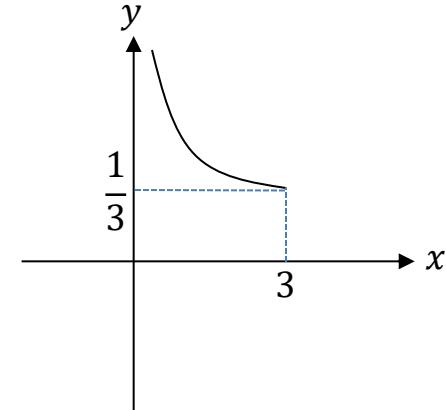
Therefore range is  $\{1,4,7,10\}$   
 $f(x)$  is one-to-one.

b



Using graph, range is  
 $0 \leq g(x) \leq 25$   
 $g(x)$  is many-to-one.

c



Using graph, range is  
 $h(x) \geq \frac{1}{3}$   
 $h(x)$  is one-to-one.

# Piecewise Functions

A ‘piecewise function’ is one which is defined in parts: we can use different rules for different intervals within the domain.

[Textbook] The function  $f(x)$  is defined by

$$f: x \rightarrow \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geq 1 \end{cases}$$

- Sketch  $y = f(x)$ , and state the range of  $f(x)$ .
- Solve  $f(x) = 19$

a

?

b

?

# Piecewise Functions

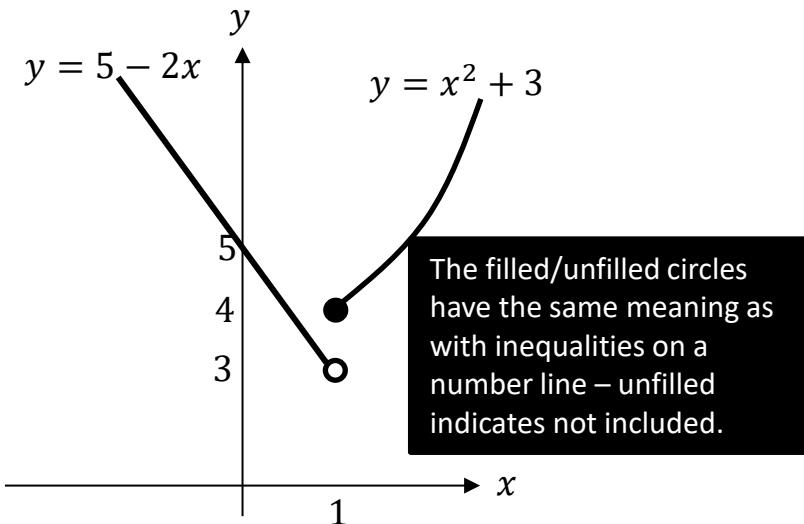
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- Sketch  $y = f(x)$ , and state the range of  $f(x)$ .
- Solve  $f(x) = 19$

a



$f(x) > 3$   
(as the 3 is not included)

b

Using the graph, the range is  $f(x) > 3$

When  $x \geq 1$ :

$$x^2 + 3 = 19$$

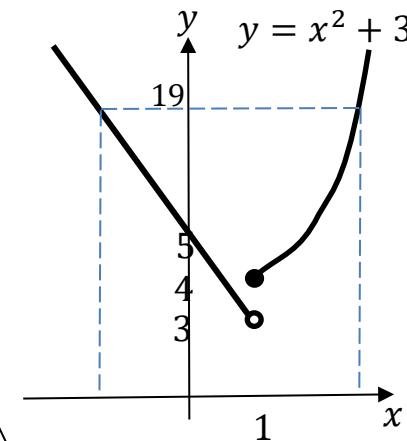
$$x = \pm 4$$

$$x = 4$$

When  $x < 1$ :

$$5 - 2x = 19$$

$$x = -7$$



$x = -2$  corresponds to a part of the curve which was never used.

# Test Your Understanding

Edexcel C4 June 2012 Q6a

The function  $f$  is defined by

$$f: x \rightarrow e^x + 2, \quad x \in \mathbb{R}$$

State the range of  $f$ .

?

Edexcel C4 June 2010 Q4d

The function  $g$  is defined by

$$g: x \rightarrow x^2 - 4x + 1, \quad x \in \mathbb{R}, 0 \leq x \leq 5$$

Find the range of  $g$ .

**Hint:** Identify the minimum point first, as this may or may not affect the range.

**Extra Hint:** Carefully consider the stated domain.

?

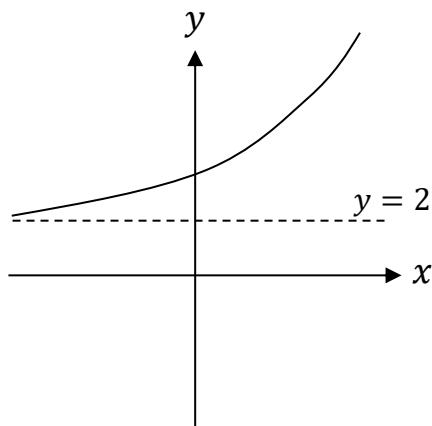
# Test Your Understanding

Edexcel C4 June 2012 Q6a

The function  $f$  is defined by

$$f: x \rightarrow e^x + 2, \quad x \in \mathbb{R}$$

State the range of  $f$ .



$$f(x) > 2$$

Notice the range doesn't include 2, as the line never reaches the asymptote.

Edexcel C4 June 2010 Q4d

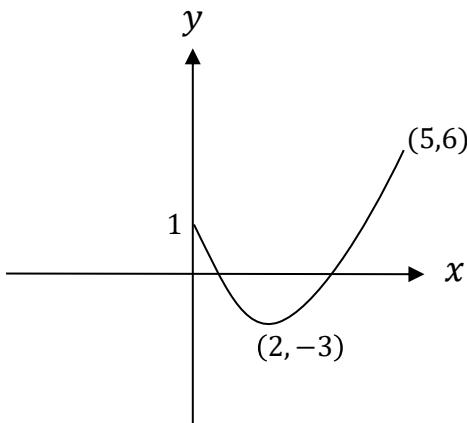
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Find the range of  $g$ .

**Hint:** Identify the minimum point first, as this may or may not affect the range.

**Extra Hint:** Carefully consider the stated domain.



$$\begin{aligned}x^2 - 4x + 1 \\&= (x - 2)^2 - 4 + 1 \\&= (x - 2)^2 - 3\end{aligned}$$

So minimum point is  
(2, -3)  
At two end points of curve:

$$\begin{aligned}f(0) &= 1 \\f(5) &= 6\end{aligned}$$

Therefore range:  
 $-3 \leq f(x) \leq 6$

# Summary of Domain/Range

It is important that you can identify the range for common graphs, using a suitable sketch:

---

$$f(x) = x^2, \quad x \in \mathbb{R}$$

**Range:**  $f(x) \geq 0$

$$f(x) = \frac{1}{x}, \quad x \in \mathbb{R}, x \neq 0$$

**Range:**  $f(x) \neq 0$

$$f(x) = \ln x, \quad x \in \mathbb{R}, x > 0$$

**Range:**  $f(x) \in \mathbb{R}$

$$f(x) = e^x, \quad x \in \mathbb{R},$$

**Range:**  $f(x) > 0$

$$f(x) = x^2 + 2x + 9, \quad x \in \mathbb{R}$$

**Range:**  $f(x) \geq 8$

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Be careful in noting the domain – it may be ‘restricted’, which similarly restricts the range. Again, use a sketch!

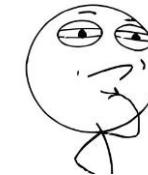
$$f(x) = x^2, \quad x \in \mathbb{R}, -1 \leq x \leq 4$$

**Range:**  $0 \leq f(x) \leq 16$

# Just for your interest...

What is the difference between the notation

$$f(x) = 2x + 1 \text{ and } f: x \rightarrow 2x + 1?$$



$f: x \rightarrow 2x + 1$  means “the value of  $f$  is a mapping from  $x$  to  $2x + 1$ ”.

You're used to variables, e.g.  $x$ , representing numerical values. But we've also seen that the value of a variable can be a vector, e.g.  $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ , sets, e.g.  $A = \{1, 2, 3\}$  and so on. So when we use  $f$  on its own, its ‘value’ is a mapping, in this case with the value  $x \rightarrow 2x + 1$ .

**This notation therefore places more emphasis on the value of  $f$ , and its ‘value’ as a mapping.**

$f(x) = 2x + 1$  means “the output of  $f$  is  $2x + 1$ ”.

It's easy to think that the notation “ $f(x)$ ” refers to the function. It doesn't! The  $f$  is the function, and the “ $(x)$ ” appendage obtains the output of the function when the input is  $x$ . Therefore  $f(x)$  refers specifically to the output of the function, which is why we write the range of a function in terms of  $f(x)$  (and not in terms of  $f$ ).  
**This notation therefore places more emphasis on the output of  $f$ .**

Consequence 1

Consequence 2

To solve an equation means to find the values of the variables, e.g. the “solution” of  $2x + 1 = 5$  is  $x = 2$ .

To solve a **functional equation** means to find the ‘values’ of  $f$ .

Solve  $f(x + y) = f(x)f(y)$

One solution to this equation is  $f: x \rightarrow 2^x$  because  $f(x + y) = 2^{x+y}$  and  $f(x)f(y) = 2^x 2^y = 2^{x+y}$ . To fully solve this functional equation means to find **all** functions which satisfy the equation.

See <http://www.drfrostmaths.com/resources/resource.php?rid=165>

**A bit of Computer Science!**

In many programming languages, we can pass functions as the parameters of a method, when a variable is allowed to have a function as its value.

We could code a function `map` which takes a list, say  $a$ , and applies a function  $f$  to each item of this list.  
e.g. `map (x→x+1, [1, 2, 3])` would output `[2, 3, 4]`.

```
function map(f, a) {  
    let b be a new list  
    for(i from 1 to size(a)) {  
        bi = f(ai)  
    }  
    return b  
}
```

# Exercise 2.2

Pearson Pure Mathematics Year 2/AS

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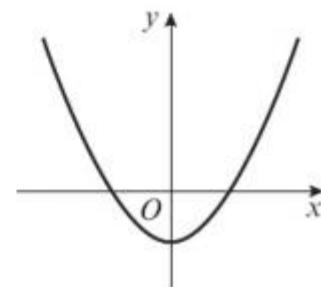
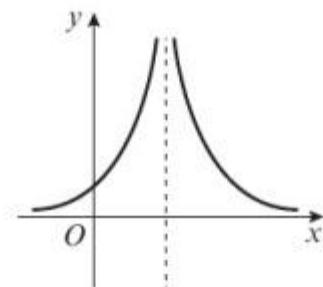
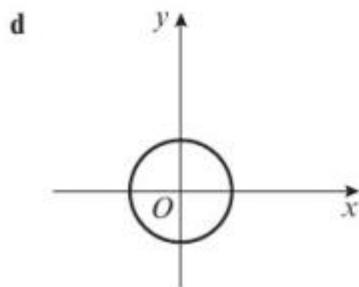
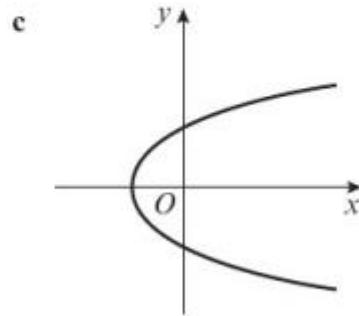
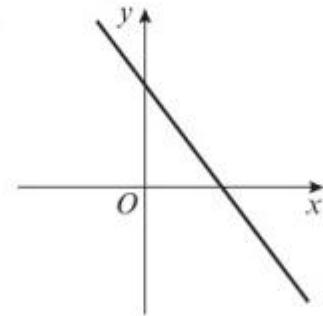
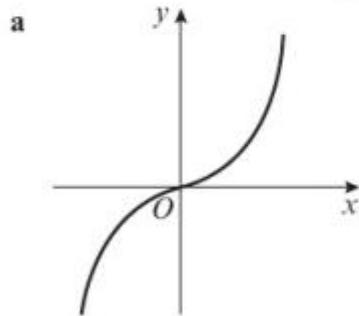
# Homework Exercise

1 For each of the following functions:

- draw the mapping diagram
  - state if the function is one-to-one or many-to-one
  - find the range of the function.
- a  $f(x) = 5x - 3$ , domain  $\{x = 3, 4, 5, 6\}$
- b  $g(x) = x^2 - 3$ , domain  $\{x = -3, -2, -1, 0, 1, 2, 3\}$
- c  $h(x) = \frac{7}{4 - 3x}$ , domain  $\{x = -1, 0, 1\}$

2 For each of the following mappings:

- State whether the mapping is one-to-one, many-to-one or one-to-many.
- State whether the mapping could represent a function.



# Homework Exercise

3 Calculate the value(s) of  $a$ ,  $b$ ,  $c$  and  $d$  given that:

a  $p(a) = 16$  where  $p: x \mapsto 3x - 2$ ,  $x \in \mathbb{R}$

b  $q(b) = 17$  where  $q: x \mapsto x^2 - 3$ ,  $x \in \mathbb{R}$

c  $r(c) = 34$  where  $r: x \mapsto 2(2^x) + 2$ ,  $x \in \mathbb{R}$

d  $s(d) = 0$  where  $s: x \mapsto x^2 + x - 6$ ,  $x \in \mathbb{R}$

4 For each function:

i represent the function on a mapping diagram, writing down the elements in the range

ii state whether the function is one-to-one or many-to-one.

a  $f(x) = 2x + 1$  for the domain  $\{x = 1, 2, 3, 4, 5\}$

b  $g: x \mapsto \sqrt{x}$  for the domain  $\{x = 1, 4, 9, 16, 25, 36\}$

c  $h(x) = x^2$  for the domain  $\{x = -2, -1, 0, 1, 2\}$

d  $j: x \mapsto \frac{2}{x}$  for the domain  $\{x = 1, 2, 3, 4, 5\}$

e  $k(x) = e^x + 3$  for the domain  $\{x = -2, -1, 0, 1, 2\}$

**Notation**

Remember,  $\sqrt{x}$  means the positive square root of  $x$ .

5 For each function:

i sketch the graph of  $y = f(x)$

ii state the range of  $f(x)$

iii state whether  $f(x)$  is one-to-one or many-to-one.

a  $f: x \mapsto 3x + 2$  for the domain  $\{x \geq 0\}$

b  $f(x) = x^2 + 5$  for the domain  $\{x \geq 2\}$

c  $f: x \mapsto 2 \sin x$  for the domain  $\{0 \leq x \leq 180^\circ\}$

d  $f: x \mapsto \sqrt{x+2}$  for the domain  $\{x \geq -2\}$

e  $f(x) = e^x$  for the domain  $\{x \geq 0\}$

f  $f(x) = 7 \log x$ , for the domain,  $\{x \in \mathbb{R}, x > 0\}$

# Homework Exercise

- 6 The following mappings  $f$  and  $g$  are defined on all the real numbers by

$$f(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x \geq 4 \end{cases} \quad g(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x > 4 \end{cases}$$

- a Explain why  $f(x)$  is a function and  $g(x)$  is not. b Sketch  $y = f(x)$ .  
c Find the values of: i  $f(3)$  ii  $f(10)$  d Solve  $f(a) = 90$ .

- 7 The function  $s$  is defined by

$$s(x) = \begin{cases} x^2 - 6, & x < 0 \\ 10 - x, & x \geq 0 \end{cases}$$

- a Sketch  $y = s(x)$ .  
b Find the value(s) of  $a$  such that  $s(a) = 43$ .  
c Solve  $s(x) = x$ .

## Problem-solving

The solutions of  $s(x) = x$  are the values in the domain that get mapped to themselves in the range.

- 8 The function  $p$  is defined by

$$p(x) = \begin{cases} e^{-x}, & -5 \leq x < 0 \\ x^3 + 4, & 0 \leq x \leq 4 \end{cases}$$

- a Sketch  $y = p(x)$ . (3 marks)

- b Find the values of  $a$ , to 2 decimal places, such that  $p(a) = 50$ . (4 marks)

# Homework Exercise

- 9 The function  $h$  has domain  $-10 \leq x \leq 6$ , and is linear from  $(-10, 14)$  to  $(-4, 2)$  and from  $(-4, 2)$  to  $(6, 27)$ .

a Sketch  $y = h(x)$ . (2 marks)

b Write down the range of  $h(x)$ . (1 mark)

c Find the values of  $a$ , such that  $h(a) = 12$ . (4 marks)

## Problem-solving

The graph of  $y = h(x)$  will consist of two line segments which meet at  $(-4, 2)$ .

- 10 The function  $g$  is defined by  $g(x) = cx + d$  where  $c$  and  $d$  are constants to be found.

Given  $g(3) = 10$  and  $g(8) = 12$  find the values of  $c$  and  $d$ .

- 11 The function  $f$  is defined by  $f(x) = ax^3 + bx - 5$  where  $a$  and  $b$  are constants to be found.

Given that  $f(1) = -4$  and  $f(2) = 9$ , find the values of the constants  $a$  and  $b$ .

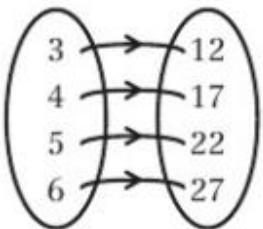
- 12 The function  $h$  is defined by  $h(x) = x^2 - 6x + 20$  and has domain  $x \geq a$ . Given that  $h(x)$  is a one-to-one function find the smallest possible value of the constant  $a$ . (6 marks)

## Problem-solving

First complete the square for  $h(x)$ .

# Homework Answers

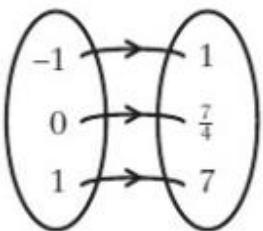
1 a i



ii one-to-one

iii  $\{f(x) = 12, 17, 22, 27\}$

c i



ii one-to-one

iii  $\{h(x) = 1, \frac{7}{4}, 7\}$

2 a i one-to-one

ii function

b i one-to-one

ii function

c i one-to-many

ii not a function

d i one to many

ii not a function

e i one to one

ii not valid at the asymptote, so not a function.

f i many to one

ii function

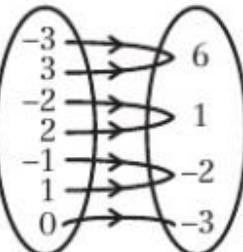
3 a 6

b  $\pm 2\sqrt{5}$

c 4

d 2, -3

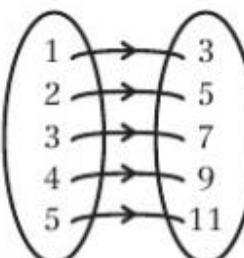
b i



ii many-to-one

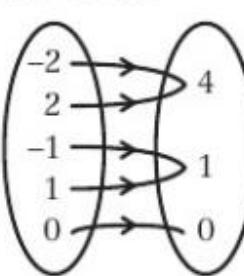
iii  $\{g(x) = -3, -2, 1, 6\}$

4 a i



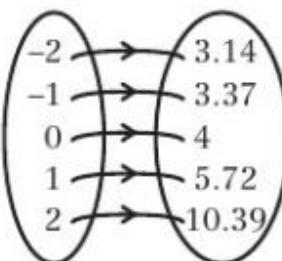
ii one-to-one

c i



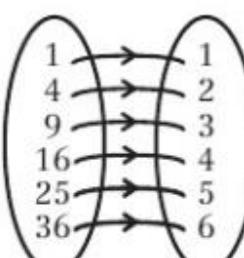
ii many-to-one

e i



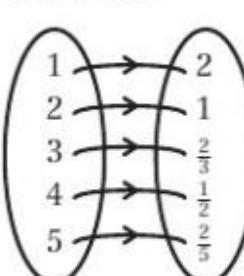
ii one-to-one

b i



ii one-to-one

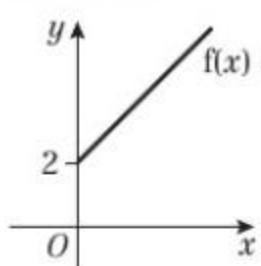
d i



ii one-to-one

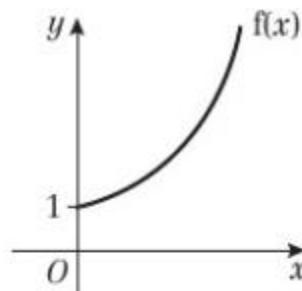
# Homework Answers

5 a i



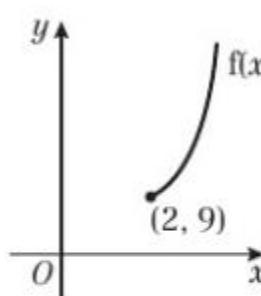
- ii  $f(x) \geq 2$   
iii one-to-one

5 e i



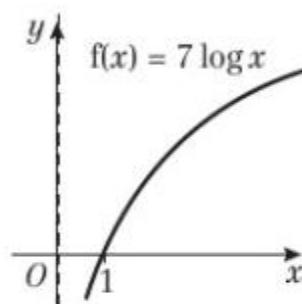
- ii  $f(x) \geq 1$   
iii one-to-one

b i



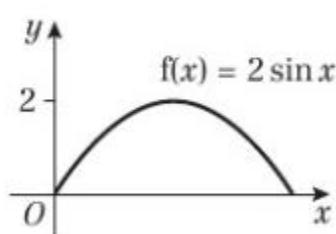
- ii  $f(x) \geq 9$   
iii one-to-one

f i



- ii  $f(x) \in \mathbb{R}$   
iii one-to-one

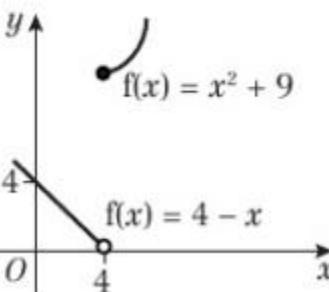
c i



- ii  $0 \leq f(x) \leq 2$   
iii many-to-one

6 a  $g(x)$  is not a function because it is not defined for  $x = 4$

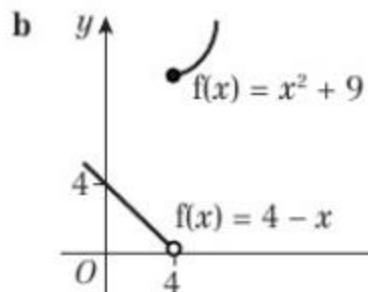
b



- c i 1      ii 109  
d  $a = -86$  or  $a = 9$

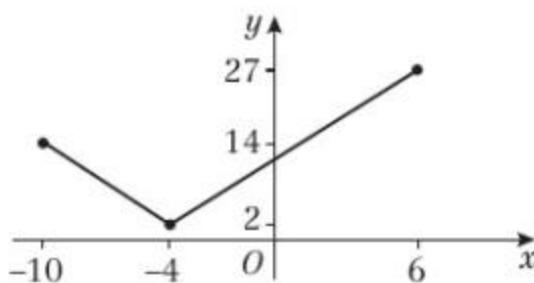
# Homework Answers

- 6 a  $g(x)$  is not a function because it is not defined for  $x = 4$



- c i 1 ii 109  
d  $a = -86$  or  $a = 9$

- 9 a



- b Range  $\{2 \leq h(x) \leq 27\}$

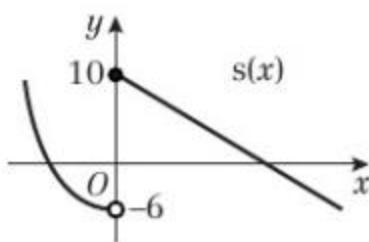
- c  $a = -9, a = 0$

10  $c = \frac{2}{5}, d = \frac{44}{5}$

11  $a = 2, b = -1$

12  $a = 3$

- 7 a
- b  $-7$   
c  $-2$  and  $5$



- 8 a
- 
- The graph shows a function on the interval  $[-5, 4]$ . The function is a curve that starts at  $(-5, 0)$ , goes down to a local minimum at  $(0, 1)$ , and then goes up to a local maximum at  $(1, 4)$ .
- b  $a = -3.91$  or  $a = 3.58$