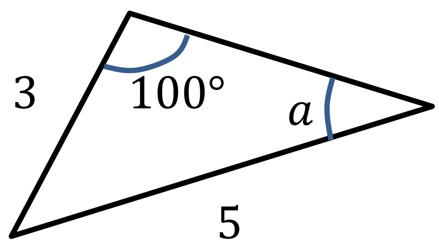
P1 Chapter 9: Trigonometric Ratios

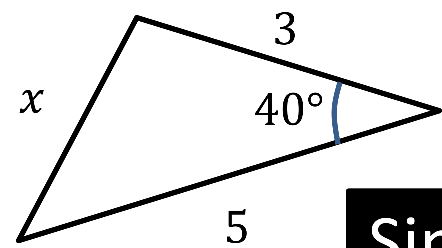
Solving Triangle Problems

Sin or cosine rule?

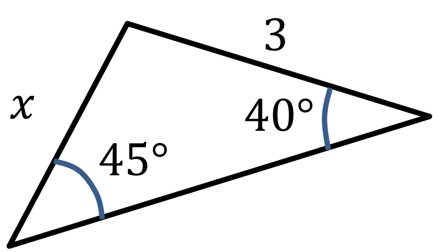
Recall that whenever we have **two “side-angle pairs**” involved, use sine rule. If there’s **3 sides** involved, we can use cosine rule. Sine rule is generally easier to use than cosine rule.



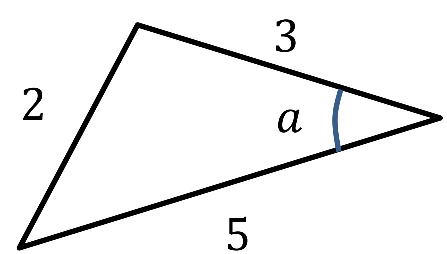
Sine | Cosine



Sine | Cosine



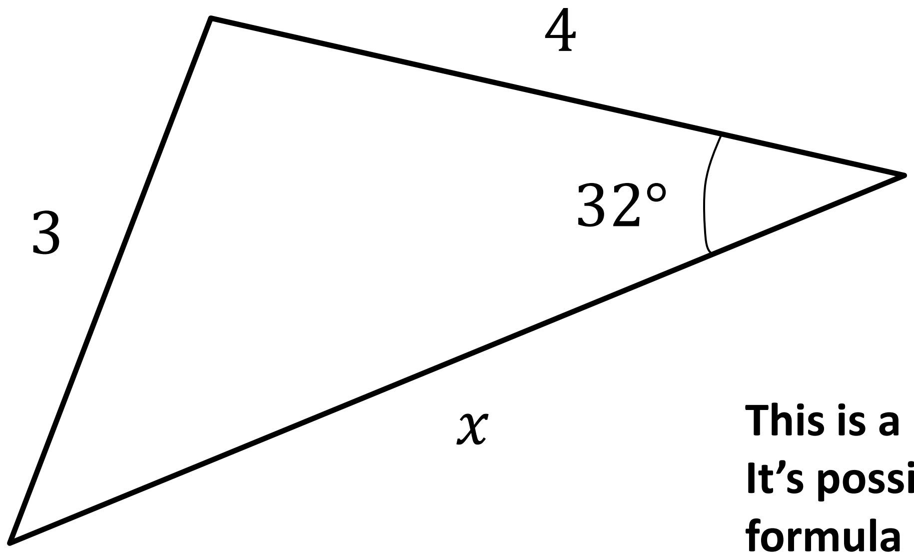
Sine | Cosine



Sine | Cosine

Using sine rule twice

You have	You want	Use
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	Sine rule twice



Given there is just one angle involved, you might attempt to use the cosine rule:

$$\mathbf{3^2 = x^2 + 4^2 - (2 \times x \times 4 \times \cos 32)}$$
$$\mathbf{9 = x^2 + 16 - 8x \cos 32}$$

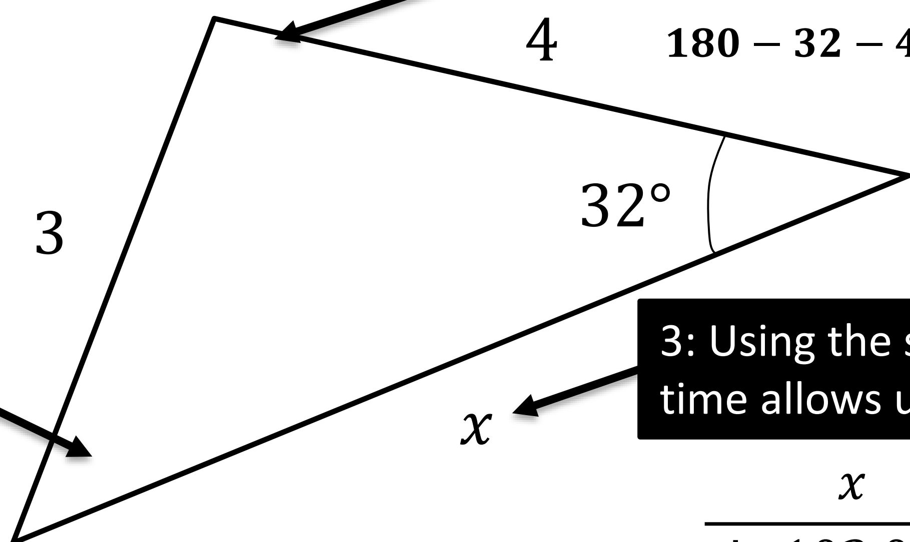
This is a quadratic equation!
It's possible to solve this using the quadratic formula (using $a = 1, b = -8\cos 32$, $c = 7$). However, this is a bit fiddly and not the primary method expected in the exam…

Using sine rule twice

You have	You want	Use
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	Sine rule twice

1: We could use the sine rule to find this angle.

$$\frac{\sin A}{4} = \frac{\sin 32}{3}$$
$$A = 44.9556°$$



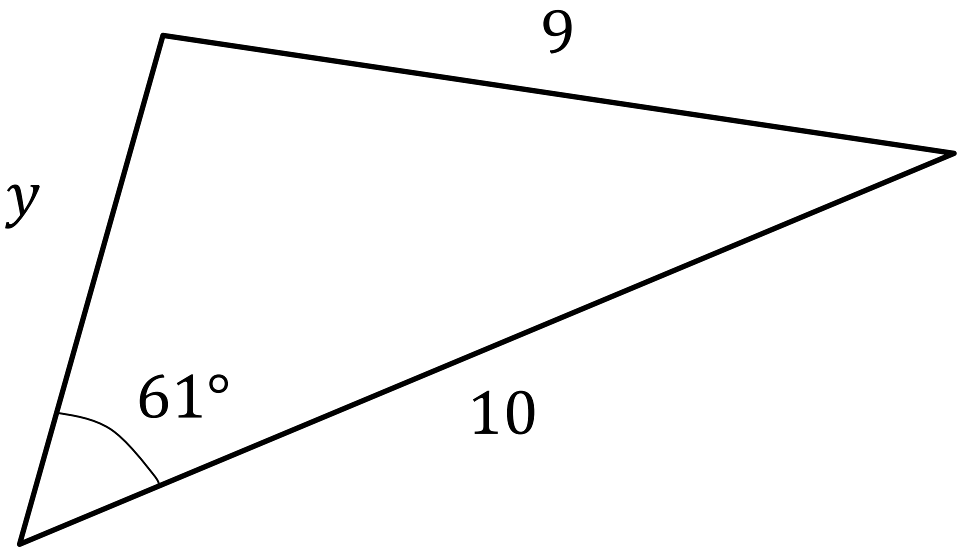
2: Which means we would then know this angle.

$$\mathbf{180 - 32 - 44.9556 = 103.0444}$$

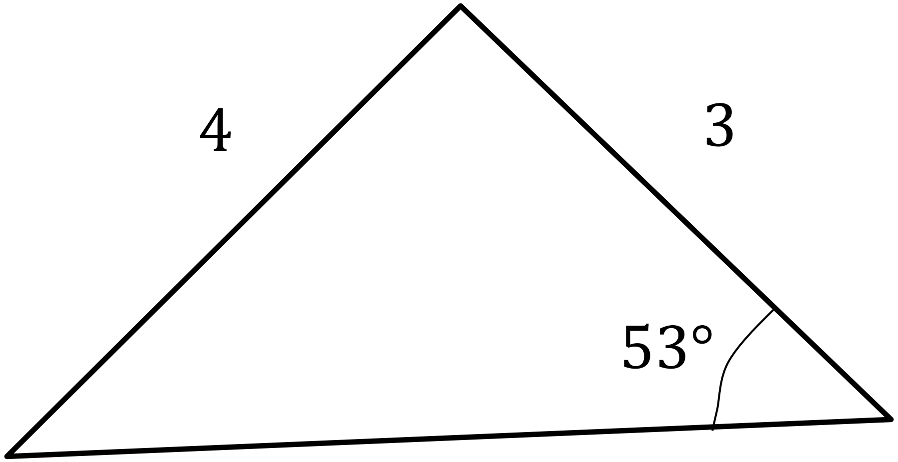
3: Using the sine rule a second time allows us to find x

$$\frac{x}{\sin 103.0444} = \frac{3}{\sin 32}$$
$$x = 5.52\ to\ 3sf$$

Test Your Understanding

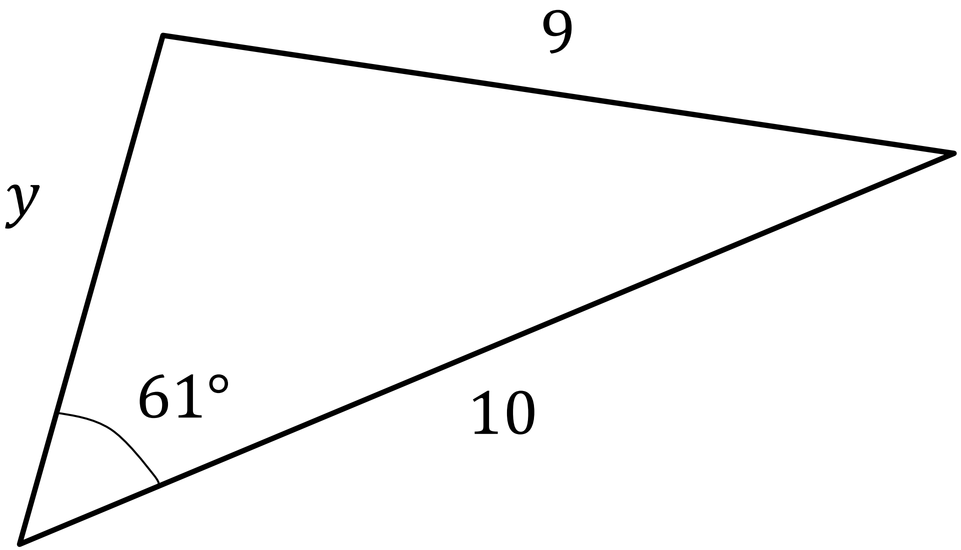


$y =$?

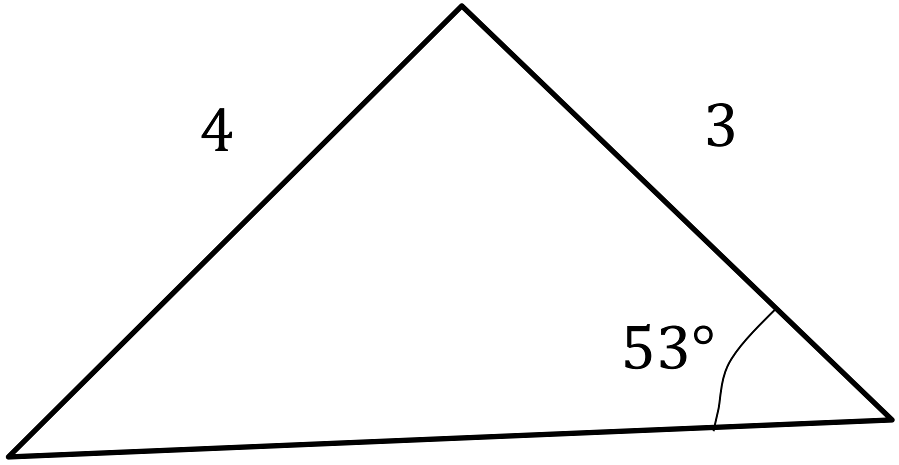


$Area =$?

Test Your Understanding



$y = 6.97$

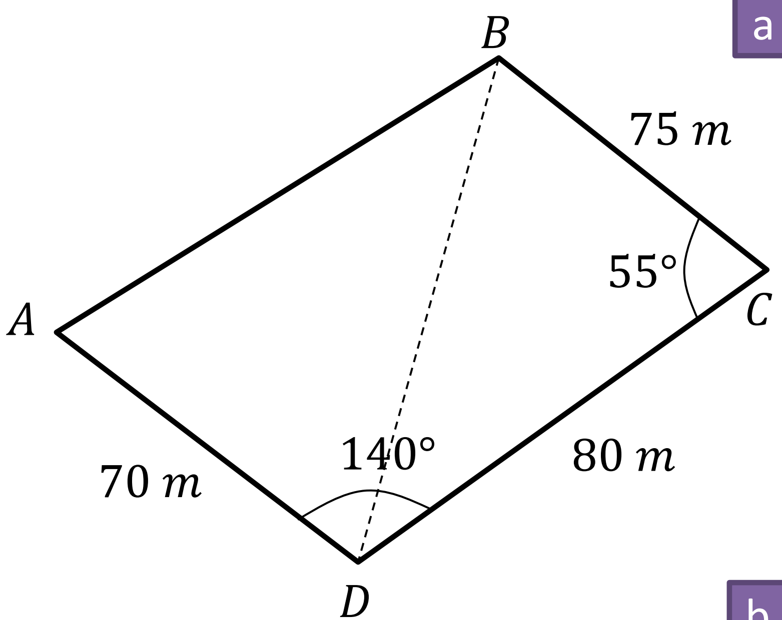


$Area = 6.00$

Problem Solving With Sine/Cosine Rule

[From Textbook] The diagram shows the locations of four mobile phone masts in a field, $BC = 75\ m$. $CD = 80m$, angle $BCD = 55°$ and angle $ADC = 140°$.
In order that the masts do not interfere with each other, they must be at least 70m apart.
Given that A is the minimum distance from D, find:

a) The distance A is from B
b) The angle BAD
c) The area enclosed by the four masts.



a ?

b ?

c ?

Problem Solving With Sine/Cosine Rule

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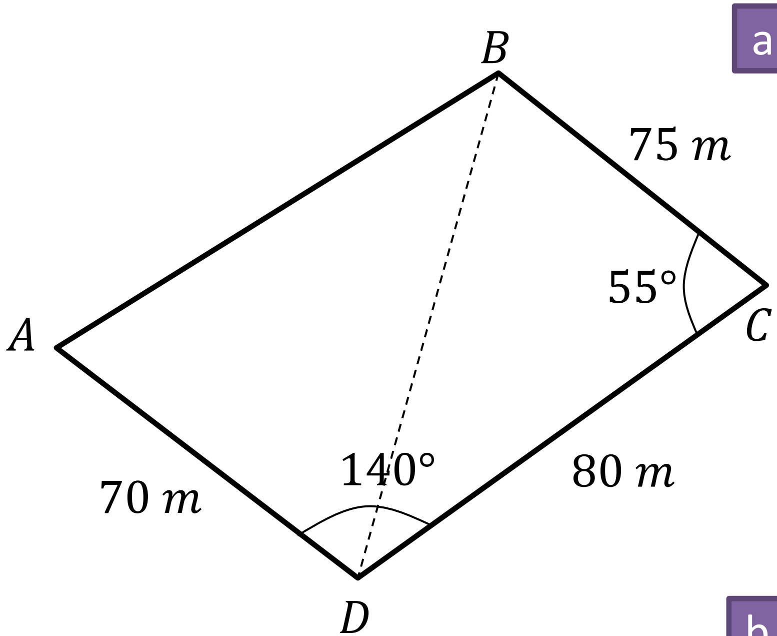

a Using triangle BCD:

$$BD^2 = 75^2 + 80^2 - 2 \times 75 \times 80 \times \cos 55°$$
$$BD = 71.708\ldots$$

Then use sine rule to find $\angle BDC$:

$$\frac{\sin(\angle BDC)}{75} = \frac{\sin(55)}{71.708} \quad \rightarrow \quad \angle BDC = 58.954$$
$$\therefore \angle BDA = 81.045\ldots$$

We can then use cosine rule on ΔABD:

$$AB^2 = 70^2 + 71.708^2 - 2 \times 70 \times 71.708 \times \cos(81.045)$$
$$AB = 92.1\ m\ (3sf)$$

b Using sine rule on ΔABD, $\angle BAD = 50.3°\ (3sf)$

c By adding areas of ΔABD and ΔBDC:

$$Area\ ABCD = 4940\ m^2\ \ (3sf)$$

Exercise 9.4

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1 [AEA 2009 Q5a] The sides of the triangle ABC have lengths $BC = a, AC = b$ and $AB = c$, where $a < b < c$. The sizes of the angles A, B and C form an arithmetic sequence.

(i) Show that the area of triangle ABC is $ac\frac{\sqrt{3}}{4}$.

Given that $a = 2$ and $\sin A = \frac{\sqrt{15}}{5}$, find

(ii) the value of b,

(iii) the value of c.

2 [STEP I 2006 Q8] *Note that the volume of a tetrahedron is equal to*

$$\frac{1}{3} \times area\ of\ base \times height$$

The points O, A, B, C have coordinates $(0,0,0), (a, 0,0), (0, b, 0)$ and $(0,0, c)$ respectively, where a, b, c are positive.

(i) Find, in terms of a, b, c the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that

$$\cos\theta = \frac{c^2}{\sqrt{(a^2 + c^2)(b^2 + c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

Solutions to extension problems on next slides.

Solution to Extension Problem 1

[AEA 2009 Q5a] The sides of the triangle ABC have lengths $BC = a, AC = b$ and $AB = c$, where $a < b < c$. The sizes of the angles A, B and C form an arithmetic sequence.

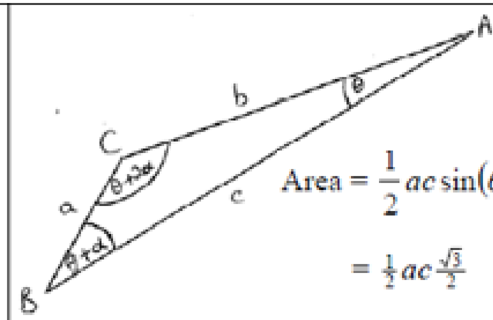
(i) Show that the area of triangle ABC is $ac\frac{\sqrt{3}}{4}$.

Given that $a = 2$ and $\sin A = \frac{\sqrt{15}}{5}$, find

(ii) the value of b,

(iii) the value of c.

(a)
(i)



$$\theta + (\theta+\alpha) + (\theta+2\alpha) = 180$$

$$3\theta + 3\alpha = 180$$

$$\therefore \hat{B} = (\theta+\alpha) = 60^\circ$$

$$\text{Area} = \frac{1}{2}ac\sin(\theta+\alpha)$$

$$= \tfrac{1}{2}ac\tfrac{\sqrt{3}}{2} = \frac{ac\sqrt{3}}{4} \quad (*)$$

(ii) Sine Rule $\qquad \dfrac{b}{\sin(\theta+\alpha)} = \dfrac{a}{\sin A}$ OR $\dfrac{1}{2}bc\sin A = \dfrac{ac\sqrt{3}}{4}$

$$\therefore \quad b = 2 \times \frac{5}{\sqrt{15}} \times \frac{\sqrt{3}}{2} = \sqrt{5}$$

(iii) Cosine Rule $\qquad b^2 = a^2 + c^2 - 2ac\cos(\theta+\alpha)$

$$5 = 4 + c^2 - 2 \times 2 \times c \ \frac{1}{2}$$

$$0 = c^2 - 2c - 1 \quad \text{OR} \quad c^2 - 2\sqrt{2} + 1 = 0$$

$$c = \frac{2 \pm \sqrt{4+4}}{2}$$

$$c = 1 + \sqrt{2} \quad \text{OR} \quad (3 + 2\sqrt{2})^{1/2}$$

Solution to Extension Problem 2

[STEP I 2006 Q8] Note that the volume of a tetrahedron is equal to $\frac{1}{3} \times area\ of\ base \times height$
The points O, A, B, C have coordinates $(0,0,0), (a,0,0), (0,b,0)$ and $(0,0,c)$ respectively, where a, b, c are positive.

(i) Find, in terms of a, b, c the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB = \theta$. Show that
$$\cos\theta = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$
and find, in terms of a, b and c, the area of triangle ABC.
Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies $\frac{1}{d^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

(i) The volume of $OABC = \frac{1}{3} \times$ the area of triangle $OAB \times OC = \frac{1}{6}abc$.

(ii) Using the scalar product with vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$,

This is FM content, but see a few lines below.

$$\sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \cos\theta = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix} = c^2 \Rightarrow \cos\theta = \frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}$$

The cosine rule ($AB^2 = AC^2 + BC^2 - 2 \times AC \times BC \times \cos\theta$) will also yield this result.

The area of triangle ABC will be $\frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sin\theta$

$$= \frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sqrt{1 - \left(\frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}\right)^2} \text{ (because } \sin^2\theta \equiv 1 - \cos^2\theta\text{)}$$

$$= \frac{1}{2} \times \sqrt{(a^2+c^2)(b^2+c^2) - c^4}$$

$$= \frac{1}{2} \times \sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

So $\frac{1}{3} \times \left(\frac{1}{2} \times \sqrt{a^2b^2 + b^2c^2 + c^2a^2}\right) \times d = \frac{1}{6}abc \Rightarrow \frac{1}{d^2} = \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2b^2c^2}$

which simplifies to the stated result.

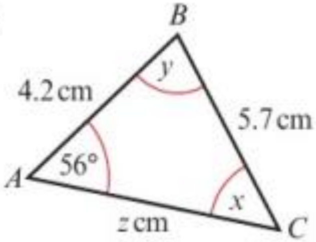
A similar result is true for the right-angled triangle PQR, in which X is the foot of the perpendicular from the right-angle Q to the hypotenuse PR: $\frac{1}{PQ^2} + \frac{1}{QR^2} = \frac{1}{QX^2}$

Homework Exercise

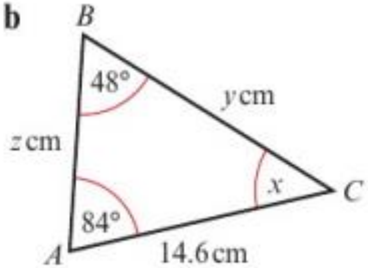
Try to use the most efficient method, and give answers to 3 significant figures.

1 In each triangle below find the values of x, y and z.

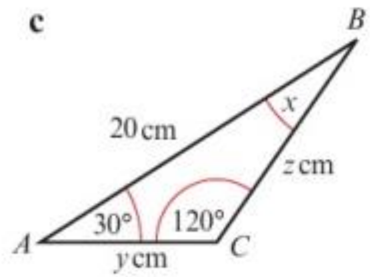
a



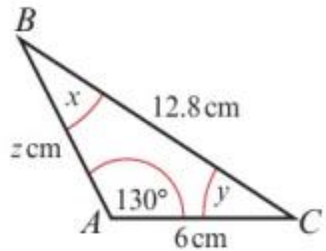
b



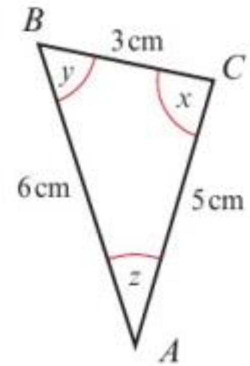
c



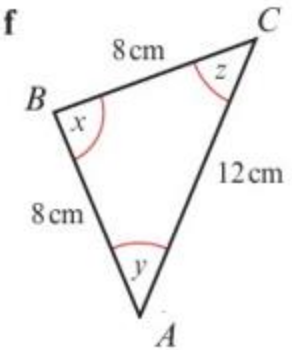
d



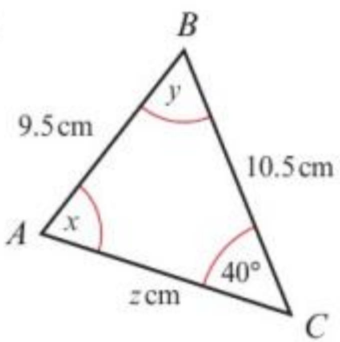
e



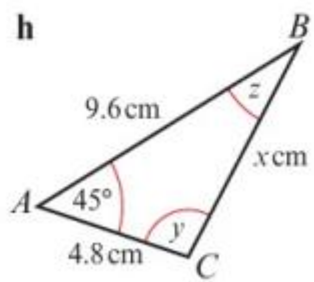
f



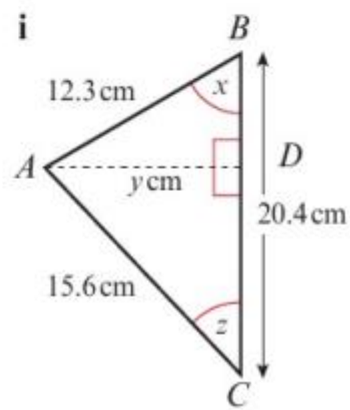
g



h



i



Homework Exercise

2 In $\triangle ABC$, calculate the size of the remaining angles, the lengths of the third side and the area of the triangle given that

a $\triangle BAC = 40°$, $AB = 8.5$ cm and $BC = 10.2$ cm

b $\triangle ACB = 110°$, $AC = 4.9$ cm and $BC = 6.8$ cm

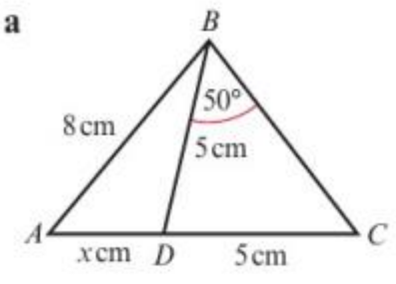
3 A hiker walks due north from A and after 8 km reaches B. She then walks a further 8 km on a bearing of 120° to C. Work out **a** the distance from A to C and **b** the bearing of C from A.

4 A helicopter flies on a bearing of 200° from A to B, where $AB = 70$ km. It then flies on a bearing of 150° from B to C, where C is due south of A. Work out the distance of C from A.

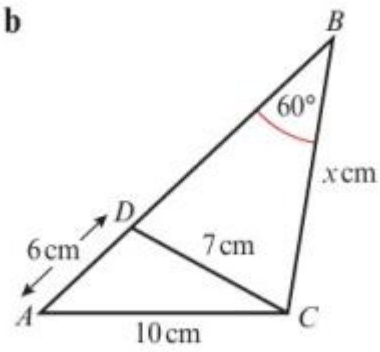
5 Two radar stations A and B are 16 km apart and A is due north of B. A ship is known to be on a bearing of 150° from A and 10 km from B. Show that this information gives two positions for the ship, and calculate the distance between these two positions.

6 Find x in each of the following diagrams:

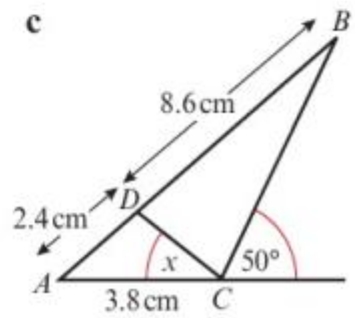
a



b



c



Homework Exercise

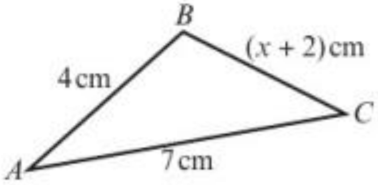
7 In $\triangle ABC$, $AB = 4\,\text{cm}$, $BC = (x + 2)\,\text{cm}$ and $AC = 7\,\text{cm}$.

a Explain how you know that $1 < x < 9$.

b Work out the value of x and the area of the triangle for the cases when

i $\angle ABC = 60°$ and

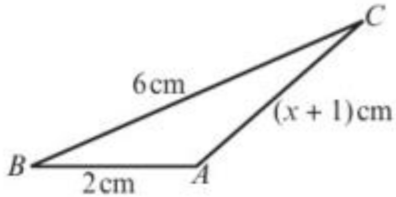
ii $\angle ABC = 45°$, giving your answers to 3 significant figures.



8 In the triangle, $\cos \angle ABC = \frac{5}{8}$

a Calculate the value of x.

b Find the area of triangle ABC.



9 In $\triangle ABC$, $AB = \sqrt{2}\,\text{cm}$, $BC = \sqrt{3}\,\text{cm}$ and $\angle BAC = 60°$. Show that $\angle ACB = 45°$ and find AC.

10 In $\triangle ABC$, $AB = (2 - x)\,\text{cm}$, $BC = (x + 1)\,\text{cm}$ and $\angle ABC = 120°$.

a Show that $AC^2 = x^2 - x + 7$.

b Find the value of x for which AC has a minimum value.

Problem-solving

Complete the square for the expression $x^2 - x + 7$ to find the minimum value of AC^2 and the value of x where it occurs.

11 Triangle ABC is such that $BC = 5\sqrt{2}\,\text{cm}$, $\angle ABC = 30°$ and $\angle BAC = \theta$, where $\sin\theta = \dfrac{\sqrt{5}}{8}$

Work out the length of AC, giving your answer in the form $a\sqrt{b}$, where a and b are integers.

12 The perimeter of $\triangle ABC = 15\,\text{cm}$. Given that $AB = 7\,\text{cm}$ and $\angle BAC = 60°$, find the lengths of AC and BC and the area of the triangle.

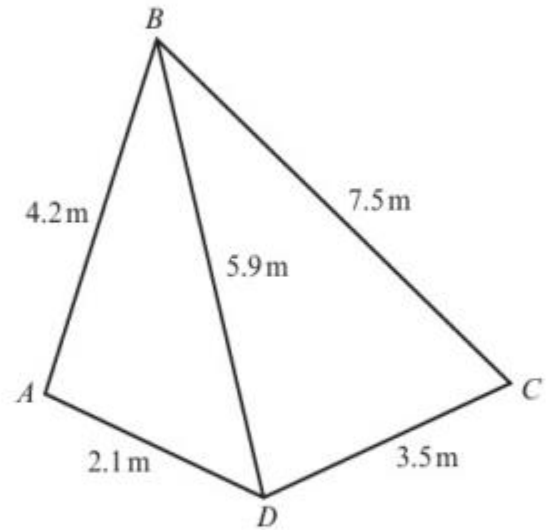
Homework Exercise

13 In the triangle ABC, $AB = 14$ cm, $BC = 12$ cm and $CA = 15$ cm.

a Find the size of angle C, giving your answer to 3 s.f. **(3 marks)**

b Find the area of triangle ABC, giving your answer in cm^2 to 3 s.f. **(3 marks)**

14 A flower bed is in the shape of a quadrilateral as shown in the diagram.



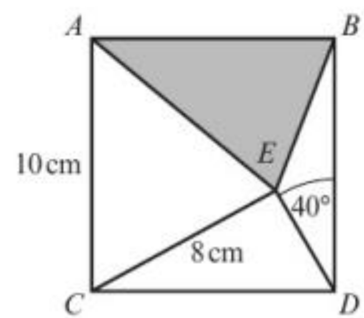
a Find the sizes of angles DAB and BCD. **(4 marks)**

b Find the total area of the flower bed. **(3 marks)**

c Find the length of the diagonal AC. **(4 marks)**

15 $ABCD$ is a square. Angle CED is obtuse.

Find the area of the shaded triangle. **(7 marks)**



Homework Answers

1 **a** $x = 37.7°$, $y = 86.3°$, $z = 6.86$
b $x = 48°$, $y = 19.5$, $z = 14.6$
c $x = 30°$, $y = 11.5$, $z = 11.5$
d $x = 21.0°$, $y = 29.0°$, $z = 8.09$
e $x = 93.8°$, $y = 56.3°$, $z = 29.9°$
f $x = 97.2°$, $y = 41.4°$, $z = 41.4°$
g $x = 45.3°$, $y = 94.7°$, $z = 14.7$
or $x = 135°$, $y = 5.27°$, $z = 1.36$
h $x = 7.07$, $y = 73.7°$, $z = 61.3°$
or $x = 7.07$, $y = 106°$, $z = 28.7°$
i $x = 49.8°$, $y = 9.39$, $z = 37.0°$

2 **a** $ABC = 108°$, $ACB = 32.4°$, $AC = 15.1$ cm
Area = 41.2 cm^2
b $BAC = 41.5°$, $ABC = 28.5°$, $AB = 9.65$ cm
Area = 15.7 cm^2

3 **a** 8 km **b** 060°

4 107 km

5 12 km

6 **a** 5.44 **b** 7.95 **c** 36.8°

7 **a** $AB + BC > AC \Rightarrow x + 6 > 7 \Rightarrow x > 1$;
$AC + AB > BC \Rightarrow 11 > x + 2 \Rightarrow x < 9$
b **i** $x = 6.08$ from $x^2 = 37$
Area = 14.0 cm^2
ii $x = 7.23$ from $x^2 - 4(\sqrt{2} - 1)x - (29 + 8\sqrt{2}) = 0$
Area = 13.1 cm^2

8 **a** $x = 4$ **b** 4.68 cm^2

9 $AC = 1.93$ cm

10 **a** $AC^2 = (2 - x)^2 + (x + 1)^2 - 2(2 - x)(x + 1)\cos 120°$
$= (4 - 4x + x^2) + (x^2 + 2x + 1) - 2(-x^2 + x + 2)\left(-\frac{1}{2}\right)$
$= x^2 - x + 7$
b $\frac{1}{2}$

11 $4\sqrt{10}$

12 $AC = 1\frac{2}{3}$ cm and $BC = 6$ cm
Area = 5.05 cm^2

13 **a** 61.3° **b** 78.9 cm^2

14 **a** $DAB = 136.3°$, $BCD = 50.1°$
b 13.1 m^2
c 5.15 m

15 34.2 cm^2