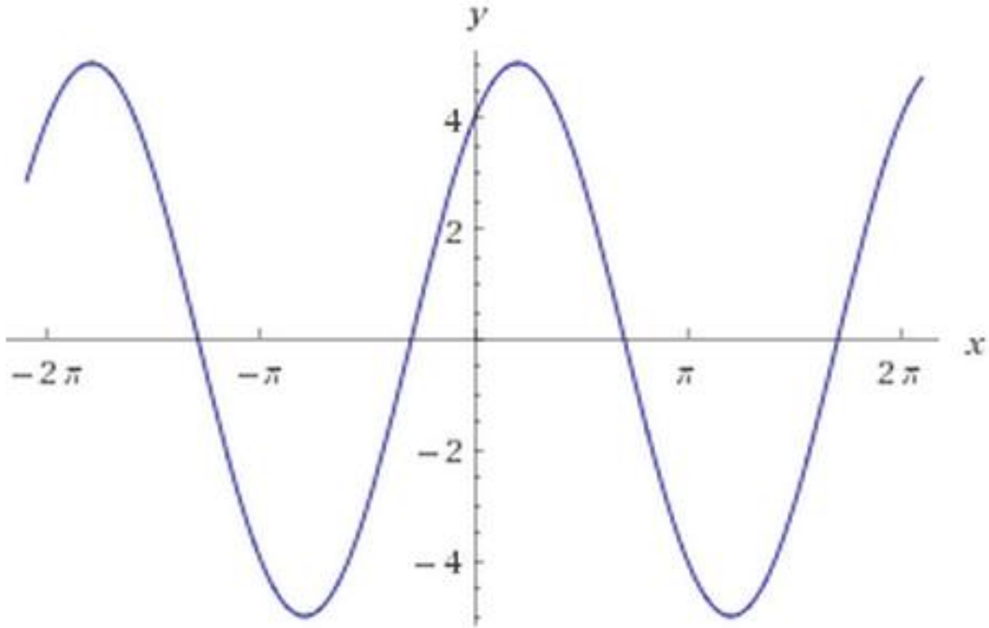
P2 Chapter 7: Trigonometric Equations

Adding Sine and Cosine

$a \sin\theta + b \cos\theta$

Here's a sketch of $y = 3\sin x + 4\cos x$.
What do you notice?



It's a sin graph that seems to be translated on the x-axis and stretched on the y axis. This suggests we can represent it as $y = R\sin(x + \alpha)$, where α is the horizontal translation and R the stretch on the y-axis.

$a \sin\theta + b\cos\theta$

Q Put $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$ giving α in degrees to 1dp.

STEP 1: Expanding:

$$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$$

STEP 2: Comparing coefficients:

$$R\cos\alpha = 3 \qquad R\sin\alpha = 4$$

STEP 3: Using the fact that $R^2\sin^2\alpha + R^2\cos^2\alpha = R^2$:

$$R = \sqrt{3^2+4^2} = 5$$

STEP 4: Using the fact that $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$:

$$\tan\alpha = \frac{4}{3}$$

$$\alpha = 53.1°$$

STEP 5: Put values back into original expression.

$$3\sin x + 4\cos x \equiv 5\sin(x + 53.1°)$$

Fro Tip: I recommend you follow this procedure every time – I've tutored students who've been taught a 'shortcut' (usually skipping Step 1), and they more often than not make a mistake.

If $R\cos\alpha = 3$ and $R\sin\alpha = 4$ then $R^2\cos^2\alpha = 3^2$ and $R^2\sin^2\alpha = 4^2$.

$$R^2\sin^2\alpha + R^2\cos^2\alpha = 3^2 + 4^2$$
$$R^2(\sin^2\alpha + \cos^2\alpha) = 3^2 + 4^2$$
$$R^2 = 3^2 + 4^2$$
$$R = \sqrt{3^2+4^2}$$

(You can write just the last line in exams)

Test Your Understanding

Q Put $\sin x + \cos x$ in the form $R\sin(x+\alpha)$ giving α in terms of π.

?

Q Put $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x-\alpha)$ giving α in terms of π.

?

Test Your Understanding

Q Put $\sin x + \cos x$ in the form $R\sin(x+\alpha)$ giving α in terms of π.

$$R\sin(x+\alpha) \equiv R\sin x\cos\alpha + R\cos x\sin\alpha$$
$$R\cos\alpha = 1 \qquad R\sin\alpha = 1$$
$$R = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\tan\alpha = 1$$
$$\alpha = \frac{\pi}{4}$$
$$\sin x + \cos x \equiv \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

Q Put $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x-\alpha)$ giving α in terms of π.

$$R\sin(x-\alpha) \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$$
$$R\cos\alpha = 1 \qquad R\sin\alpha = \sqrt{3}$$
$$R = 2$$
$$\tan\alpha = \sqrt{3} \quad so \quad \alpha = \frac{\pi}{3}$$
$$sinx - \sqrt{3}\cos x \equiv 2\sin\left(x-\frac{\pi}{3}\right)$$

Further Examples

Q

Put $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $0 < \alpha < 90°$
Hence solve, for $0 < \theta < 360$, the equation $2\cos\theta + 5\sin\theta = 3$

?

Fro Tip: This is an exam favourite!

Q

(Without using calculus), find the maximum value of $12\cos\theta + 5\sin\theta$, and give the smallest positive value of θ at which it arises.

?

Further Examples

Q Put $2\cos\theta + 5\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $0 < \alpha < 90°$
Hence solve, for $0 < \theta < 360$, the equation $2\cos\theta + 5\sin\theta = 3$

$$2\cos\theta + 5\sin\theta \equiv \sqrt{29}\cos(\theta - 68.2°)$$

Therefore:

$$\sqrt{29}\cos(\theta - 68.2°) = 3$$
$$\cos(\theta - 68.2°) = \frac{3}{\sqrt{29}}$$
$$\theta - 68.2° = -56.1\ldots°, 56.1\ldots°$$
$$\theta = 12.1°, 124.3°$$

Fro Tip: This is an exam favourite!

Q (Without using calculus), find the maximum value of $12\cos\theta + 5\sin\theta$, and give the smallest positive value of θ at which it arises.

Use either $R\sin(\theta + \alpha)$ or $R\cos(\theta - \alpha)$ before that way the + sign in the middle matches up.

$$\equiv 13\cos(\theta - 22.6°)$$

cos is at most 1,thus the expression has value at most 13.
This occurs when $\theta - 22.6 = 0$ (as $\cos 0 = 1$) thus $\theta = 22.6$

Quickfire Maxima

What is the maximum value of the expression and determine the smallest positive value of θ (in degrees) at which it occurs.

Expression	**Maximum**	**(Smallest) θ at max**
$20\sin\theta$	?	?
$5-10\sin\theta$	?	?
$3\cos(\theta+20°)$	?	?
$\dfrac{2}{10+3\sin(\theta-30)}$	?	?

Quickfire Maxima

What is the maximum value of the expression and determine the smallest positive value of θ (in degrees) at which it occurs.

Expression	**Maximum**	**(Smallest) θ at max**
$20\sin\theta$	20	$90°$
$5 - 10\sin\theta$	15	$270°$
$3\cos(\theta + 20°)$	3	$340°$
$\dfrac{2}{10 + 3\sin(\theta - 30)}$	$\dfrac{2}{7}$	$300°$

Further Test Your Understanding

Edexcel C3 Jan 2013 Q4

4. (*a*) Express $6 \cos \theta + 8 \sin \theta$ in the form $R \cos (\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

(*b*)
$$\mathrm{p}(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \quad 0 \leq \theta \leq 2\pi.$$

Calculate

(i) the maximum value of $\mathrm{p}(\theta)$,

(ii) the value of θ at which the maximum occurs.

(4)

?

Further Test Your Understanding

Edexcel C3 Jan 2013 Q4

4. (*a*) Express $6\cos\theta + 8\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 3 decimal places.

(4)

(*b*)
$$\mathrm{p}(\theta) = \frac{4}{12 + 6\cos\theta + 8\sin\theta}, \quad 0 \le \theta \le 2\pi.$$

Calculate

(i) the maximum value of $\mathrm{p}(\theta)$,

(ii) the value of θ at which the maximum occurs.

(4)

(a)	$R^2 = 6^2 + 8^2 \Rightarrow R = 10$	M1A1	
	$\tan\alpha = \frac{8}{6} \Rightarrow \alpha = \text{awrt } 0.927$	M1A1	
			(4)
(b)(i)	$\mathrm{p}(x) = \frac{4}{12 + 10\cos(\theta - 0.927)}$		
	$\mathrm{p}(x) = \frac{4}{12 - 10}$	M1	
	Maximum $= 2$	A1	
			(2)
(b)(ii)	$\theta - \text{'their } \alpha\text{'} = \pi$	M1	
	$\theta = \text{awrt } 4.07$	A1	
			(2)
			(8 marks)

Exercise 7.5

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Homework Exercise

Unless otherwise stated, give all angles to 1 decimal place and write non-integer values of R in surd form.

1 Given that $5\sin\theta + 12\cos\theta \equiv R\sin(\theta + \alpha)$, find the value of R, $R > 0$, and the value of $\tan\alpha$.

2 Given that $\sqrt{3}\sin\theta + \sqrt{6}\cos\theta \equiv 3\cos(\theta - \alpha)$, where $0 < \alpha < 90°$, find the value of α.

3 Given that $2\sin\theta - \sqrt{5}\cos\theta \equiv -3\cos(\theta + \alpha)$, where $0 < \alpha < 90°$, find the value of α.

4 **a** Show that $\cos\theta - \sqrt{3}\sin\theta$ can be written in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

b Hence sketch the graph of $y = \cos\theta - \sqrt{3}\sin\theta$, $0 < \theta < \frac{\pi}{2}$, giving the coordinates of points of intersection with the axes.

5 **a** Express $7\cos\theta - 24\sin\theta$ in the form $R\cos(\theta + \alpha)$, with $R > 0$ and $0 < \alpha < 90°$.

b The graph of $y = 7\cos\theta - 24\sin\theta$ meets the y-axis at P. State the coordinates of P.

c Write down the maximum and minimum values of $7\cos\theta - 24\sin\theta$.

d Deduce the number of solutions, in the interval $0 < \theta < 360°$, of the following equations:

i $7\cos\theta - 24\sin\theta = 15$ **ii** $7\cos\theta - 24\sin\theta = 26$ **iii** $7\cos\theta - 24\sin\theta = -25$

6 $f(\theta) = \sin\theta + 3\cos\theta$

Given $f(\theta) = R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$.

a Find the value of R and the value of α. **(4 marks)**

b Hence, or otherwise, solve $f(\theta) = 2$ for $0 \leqslant \theta < 360°$. **(3 marks)**

Homework Exercise

7 a Express $\cos 2\theta - 2\sin 2\theta$ in the form $R\cos(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of α to 3 decimal places. **(4 marks)**

b Hence, or otherwise, solve for $0 \leqslant \theta < \pi$, $\cos 2\theta - 2\sin 2\theta = -1.5$, rounding your answers to 2 decimal places. **(4 marks)**

8 Solve the following equations, in the intervals given in brackets.

a $6\sin x + 8\cos x = 5\sqrt{3}$, $[0, 360°]$

b $2\cos 3\theta - 3\sin 3\theta = -1$, $[0, 90°]$

c $8\cos\theta + 15\sin\theta = 10$, $[0, 360°]$

d $5\sin\frac{x}{2} - 12\cos\frac{x}{2} = -6.5$, $[-360°, 360°]$

9 a Express $3\sin 3\theta - 4\cos 3\theta$ in the form $R\sin(3\theta - \alpha)$, with $R > 0$ and $0 < \alpha < 90°$. **(3 marks)**

b Hence write down the minimum value of $3\sin 3\theta - 4\cos 3\theta$ and the value of θ at which it occurs. **(3 marks)**

c Solve, for $0 \leqslant \theta < 180°$, the equation $3\sin 3\theta - 4\cos 3\theta = 1$. **(3 marks)**

10 a Express $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$ in the form $a\sin 2\theta + b\cos 2\theta + c$, where a, b and c are constants to be found. **(3 marks)**

b Hence find the maximum and minimum values of $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta$. **(4 marks)**

c Solve $5\sin^2\theta - 3\cos^2\theta + 6\sin\theta\cos\theta = -1$ for $0 \leqslant \theta < 180°$, rounding your answers to 1 decimal place. **(4 marks)**

Homework Exercise

11 A class were asked to solve $3\cos\theta = 2 - \sin\theta$ for $0 \leqslant \theta < 360°$. One student expressed the equation in the form $R\cos(\theta - \alpha) = 2$, with $R > 0$ and $0 < \alpha < 90°$, and correctly solved the equation.

a Find the values of R and α and hence find her solutions.

Another student decided to square both sides of the equation and then form a quadratic equation in $\sin\theta$.

b Show that the correct quadratic equation is $10\sin^2\theta - 4\sin\theta - 5 = 0$.

c Solve this equation, for $0 \leqslant \theta < 360°$.

d Explain why not all of the answers satisfy $3\cos\theta = 2 - \sin\theta$.

12 **a** Given $\cot\theta + 2 = \operatorname{cosec}\theta$, show that $2\sin\theta + \cos\theta = 1$. **(4 marks)**

b Solve $\cot\theta + 2 = \operatorname{cosec}\theta$ for $0 \leqslant \theta < 360°$. **(3 marks)**

13 **a** Given $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$, show that $\cos\theta + \sqrt{3}\sin\theta = 2$. **(4 marks)**

b Solve $\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1)\sin\theta = 2$ for $0 \leqslant \theta \leqslant 2\pi$. **(2 marks)**

Homework Exercise

14 **a** Express $9\cos\theta + 40\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$.
Give the value of α to 3 decimal places. **(4 marks)**

b $g(\theta) = \dfrac{18}{50 + 9\cos\theta + 40\sin\theta}, \ 0 \leqslant \theta \leqslant 360°$

Calculate:

i the minimum value of $g(\theta)$ **(2 marks)**

ii the smallest positive value of θ at which the minimum occurs. **(2 marks)**

15 $p(\theta) = 12\cos 2\theta - 5\sin 2\theta$
Given that $p(\theta) = R\cos(2\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$,

a find the value of R and the value of α. **(3 marks)**

b Hence solve the equation $12\cos 2\theta - 5\sin 2\theta = -6.5$ for $0 \leqslant \theta < 180°$. **(5 marks)**

c Express $24\cos^2\theta - 10\sin\theta\cos\theta$ in the form $a\cos 2\theta + b\sin 2\theta + c$, where a, b and c are constants to be found. **(3 marks)**

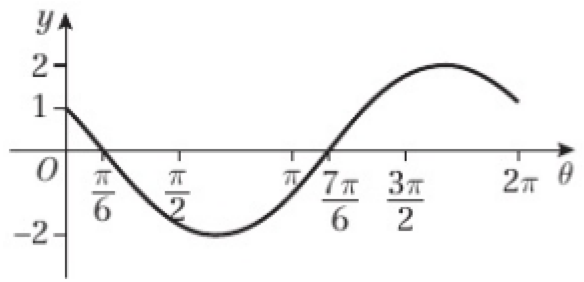
d Hence, or otherwise, find the minimum value of $24\cos^2\theta - 10\sin\theta\cos\theta$. **(2 marks)**

Homework Answers

1 $R = 13$; $\tan\alpha = \frac{12}{5}$ **2** $35.3°$ **3** $41.8°$

4 a $\cos\theta - \sqrt{3}\sin\theta \equiv R\cos(\theta + \alpha)$ gives $R = 2$, $\alpha = \frac{\pi}{3}$

b $y = 2\cos\left(\theta + \frac{\pi}{3}\right)$



5 a $25\cos(\theta + 73.7°)$ **b** $(0, 7)$

c $25, -25$ **d i** 2 **ii** 0 **iii** 1

6 a $R = \sqrt{10}$, $\alpha = 71.6°$ **b** $\theta = 69.2°, 327.7°$

7 a $\sqrt{5}\cos(2\theta + 1.107)$ **b** $\theta = 0.60, 1.44$

8 a $6.9°, 66.9°$ **b** $16.6°, 65.9°$

c $8.0°, 115.9°$ **d** $-165.2°, 74.8°$

9 a $5\sin(3\theta - 53.1°)$

b Minimum value is -5,
when $3\theta - 53.1° = 270° \Rightarrow \theta = 107.7°$

c $21.6°, 73.9°, 141.6°$

10 a $5\left(\frac{1 - \cos 2\theta}{2}\right) - 3\left(\frac{1 + \cos 2\theta}{2}\right) + 3\sin 2\theta$

$\equiv 1 + 3\sin 2\theta - 4\cos 2\theta$, so $a = 3$, $b = -4$, $c = 1$

b Maximum $= 6$, minimum $= -4$ **c** $14.8°, 128.4°$

11 a $R = \sqrt{10}$, $\alpha = 18.4°$, $\theta = 69.2°, 327.7°$

b $9\cos^2\theta = 4 - 4\sin\theta + \sin^2\theta$

$\Rightarrow 9(1 - \sin^2\theta) = 4 - 4\sin\theta + \sin^2\theta$

So $10\sin^2\theta - 4\sin\theta - 5 = 0$

c $69.2°, 110.8°, 212.3°, 327.7°$

d When you square you are also solving $3\cos\theta = -(2 - \sin\theta)$. The other two solutions are for this equation.

12 a $\frac{\cos\theta}{\sin\theta} \times \sin\theta + 2\sin\theta = \frac{1}{\sin\theta} \times \sin\theta \Rightarrow$

$\cos\theta + 2\sin\theta = 1$

b $\theta = 126.9°$ (1 d.p.)

13 a $\sqrt{2}\cos\theta\cos\frac{\pi}{4} + \sqrt{2}\sin\theta\sin\frac{\pi}{4} + \sqrt{3}\sin\theta - \sin\theta = 2$

$\Rightarrow \cos\theta + \sin\theta - \sin\theta + \sqrt{3}\sin\theta = 2$

$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = 2$

b $\frac{\pi}{3}$

14 a $R = 41$, $\alpha = 77.320°$ **b i** $\frac{18}{91}$ **ii** $77.320°$

15 a $R = 13$, $\alpha = 22.6°$ **b** $\theta = 48.7°, 108.7°$

c $a = 12$, $b = -5$, $c = 12$ **d** minimum value $= -1$