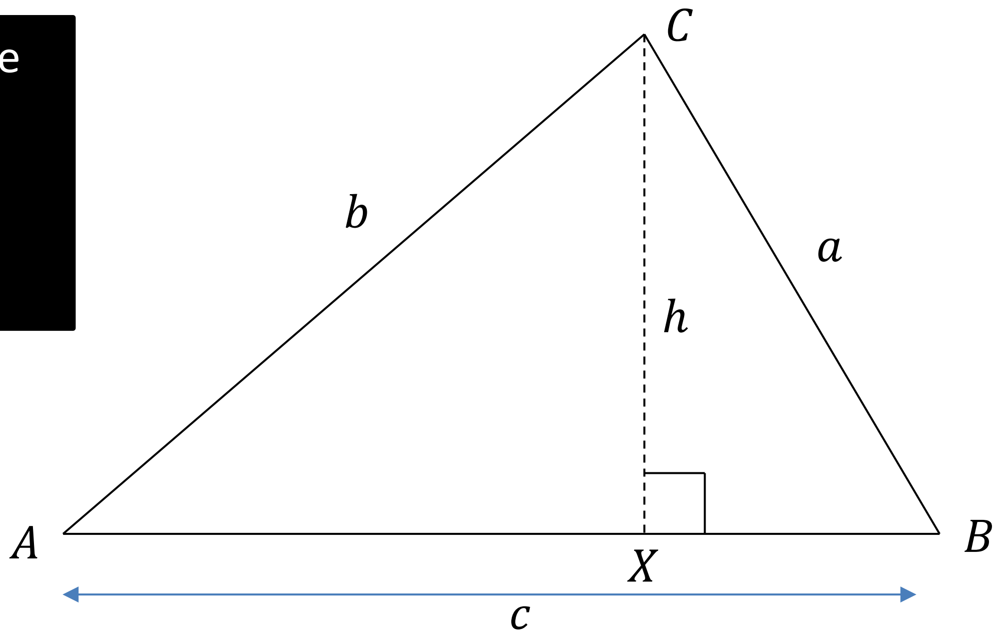
# P1 Chapter 9: Trigonometric Ratios

## Sine Rule

# Proof of Sine Rule

The idea is that we can use the common length of $\Delta ACX$ and $\angle XBC$, i.e. $h$, to connect the two triangles, and therefore connect their angles/length.




Find the area of the triangle using $h = b \sin A$ :

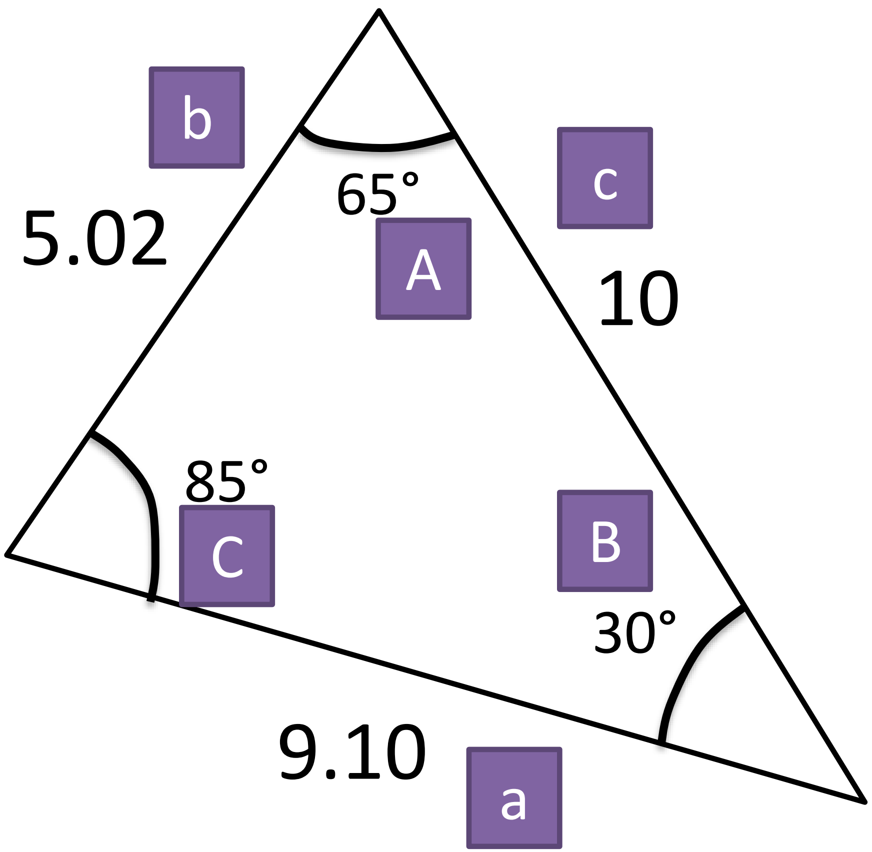
$$Area = \frac{1}{2} cb \sin A$$

Now repeat taking each side of the triangle as the base in turn:

$$Area = \frac{1}{2} cb \sin A = \frac{1}{2} ba \sin C = \frac{1}{2} ac \sin B$$

Divide through by $\frac{1}{2} abc$: $\quad \Rightarrow \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

# The Sine Rule




For this triangle, try calculating each side divided by the sin of its opposite angle. What do you notice in all three cases?

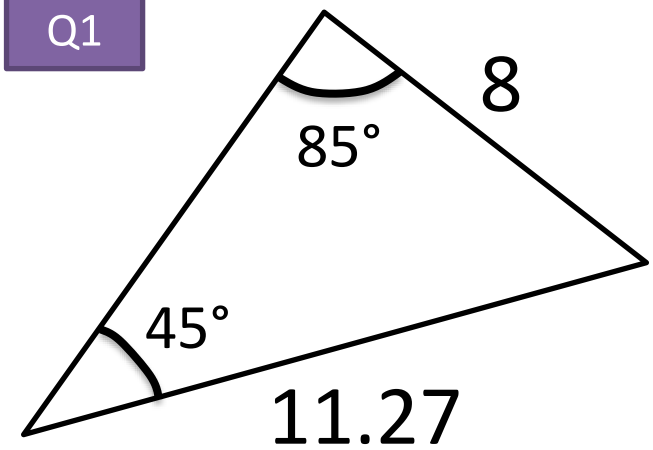
Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

| You have | You want | Use |
|---|---|---|
| #1: Two angle-side opposite pairs | Missing angle or side in one pair | Sine rule |

# Examples

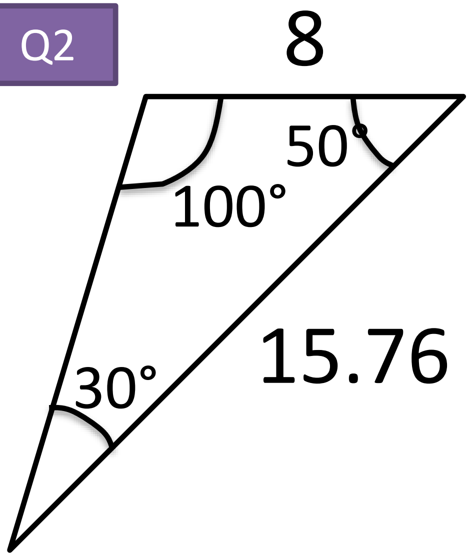
Q1




$$\frac{x}{\sin 85} = \frac{8}{\sin 45}$$

$$x = \frac{8 \sin 85}{\sin 45} = 11.27$$

Q2




$$\frac{x}{\sin 100} = \frac{8}{\sin 30}$$

$$x = \frac{8 \sin 100}{\sin 30} = 15.76$$

| You have | You want | Use |
|---|---|---|
| #1: Two angle-side opposite pairs | Missing angle or side in one pair | Sine rule |

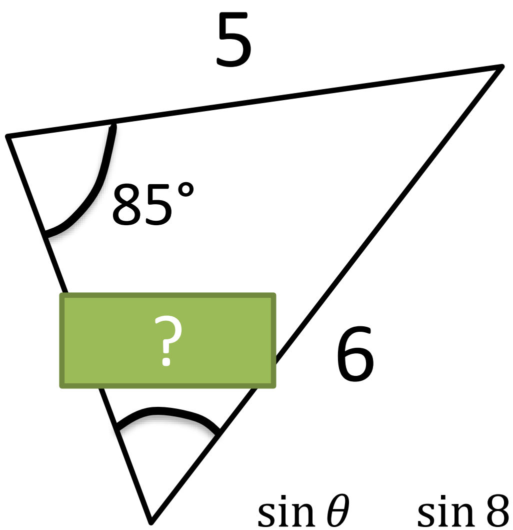
# Examples

When you have a missing angle, it's better to reciprocate to get:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

i.e. in general put the missing value in the numerator.

Q3



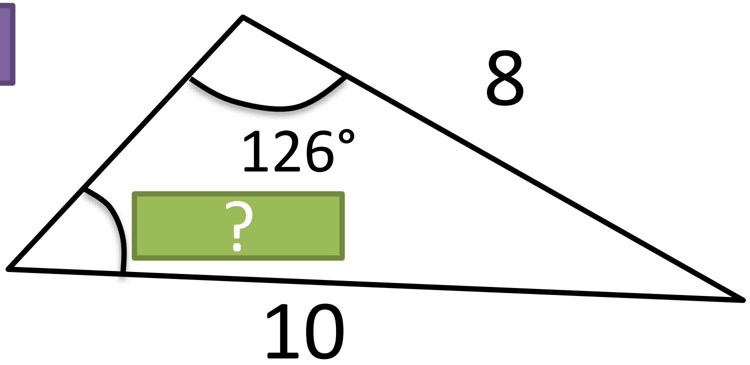

$$\frac{\sin\theta}{5} = \frac{\sin 85}{6}$$

$$\sin\theta = \frac{5\sin 85}{6}$$

$$\theta = \sin^{-1}\left(\frac{5\sin 85}{6}\right)$$

$$= 56.11°$$

Q4



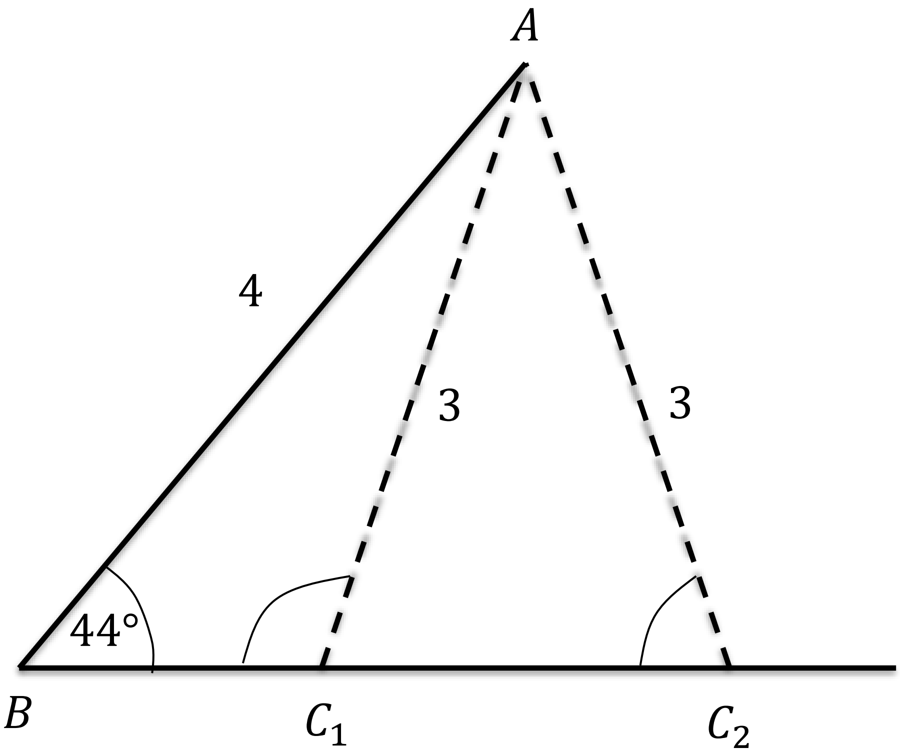

$$\frac{\sin\theta}{8} = \frac{\sin 126°}{10}$$

$$\sin\theta = \frac{8\sin 126}{10}$$

$$\theta = \sin^{-1}\left(\frac{8\sin 126}{10}\right)$$

$$= 40.33°$$

# The ‘Ambiguous Case’

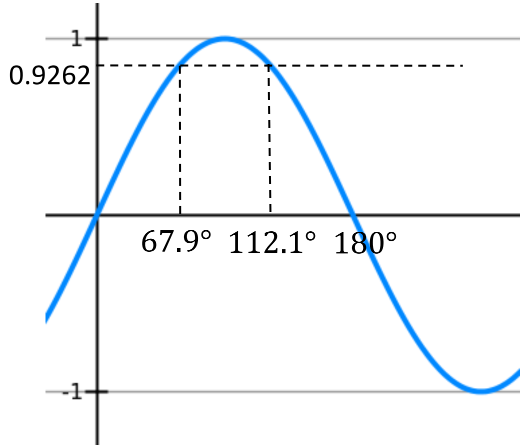



Suppose you are told that $AB = 4$, $AC = 3$ and $\angle ABC = 44°$. What are the possible values of $\angle ACB$?

$C$ is somewhere on the horizontal line. There’s two ways in which the length could be 3. Using the sine rule:

$$\frac{\sin \boldsymbol{C}}{\mathbf{4}} = \frac{\sin \mathbf{44}}{\mathbf{3}}$$

$$\boldsymbol{C} = \sin^{-1}(\mathbf{0.9262})$$



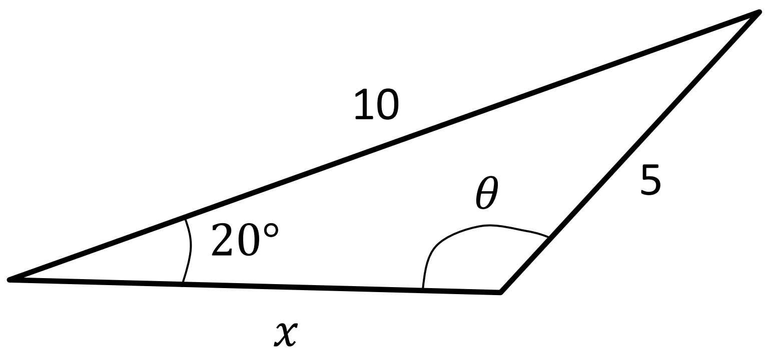

Your calculator will give the acute angle of 67.9° (i.e. $C_2$). But if we look at a graph of sin, we can see there’s actually a second value for $\sin^{-1}(0.9262)$, corresponding to angle $C_1$.

✎ The sine rule produces two possible solutions for a missing angle:

$$\sin\theta = \sin(180° - \theta)$$

Whether we use the acute or obtuse angle depends on context.

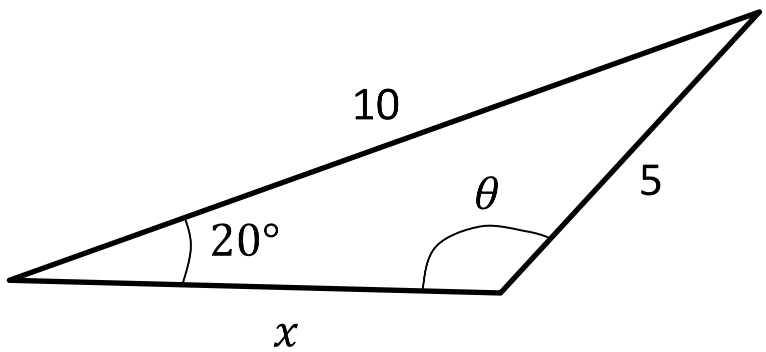
# Test Your Understanding




Given that the angle $\theta$ is obtuse, determine $\theta$ and hence determine the length of $x$.

?

# Test Your Understanding




Given that the angle $\theta$ is obtuse, determine $\theta$ and hence determine the length of $x$.

$$\frac{\sin\theta}{10} = \frac{\sin 20°}{5}$$

$$\sin^{-1}\left(\frac{10\sin 20°}{5}\right) = 43.1602°$$

$$\therefore \theta = 180° - 43.1602° = 136.8398°$$

The other angle is:

$$180° - 136.8398° - 20° = 23.1602°$$

Using sine rule again:

$$\frac{x}{\sin 23.1602°} = \frac{5}{\sin 20°}$$

$$x = 5.75\ (3sf)$$

# Exercise 9.2

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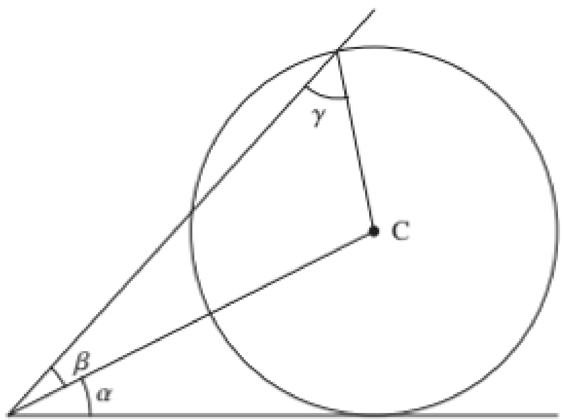
**Extension**

1 *[MAT 2011 1E]*
The circle in the diagram has centre $C$. Three angles $\alpha, \beta, \gamma$ are also indicated.




The angles $\alpha, \beta, \gamma$ are related by the equation:

A) $\cos\alpha = \sin(\beta + \gamma)$
B) $\sin\beta = \sin\alpha \sin\gamma$
C) $\sin\beta(1 - \cos\alpha) = \sin\gamma$
D) $\sin(\alpha + \beta) = \cos\gamma \sin\alpha$

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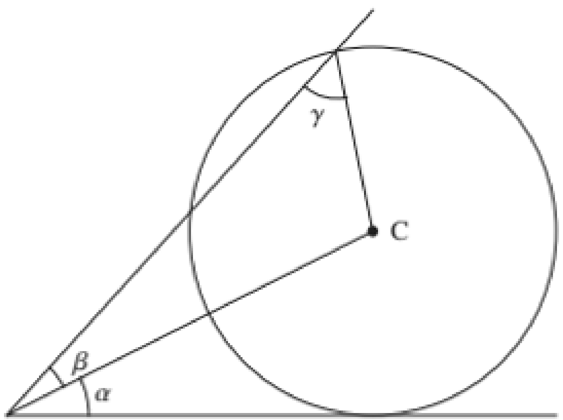


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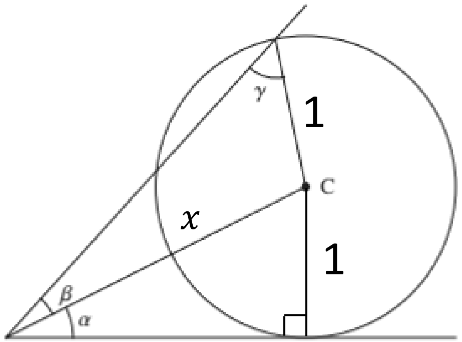
**If we draw a vertical line down from $C$, we have two triangles with a common length. This common lengths allows us to relate the two triangles. Let the radius be 1.**




**Using bottom triangle:**

$$\mathbf{1 = x \sin\alpha \quad \rightarrow \quad x = \frac{1}{\sin\alpha}}$$

**Using sine rule on top:**

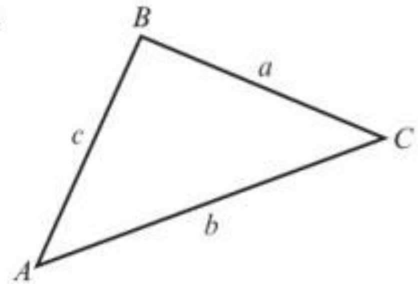
$$\mathbf{\frac{x}{\sin\gamma} = \frac{1}{\sin\beta}}$$

**Substituting in $x$ from the first equation, and rearranging, we obtain (B).**

# Homework Exercise

**Give answers to 3 significant figures, where appropriate.**

**1** In each of parts **a** to **d**, the given values refer to the general triangle.




**a** Given that $a = 8$ cm, $A = 30°$, $B = 72°$, find $b$.

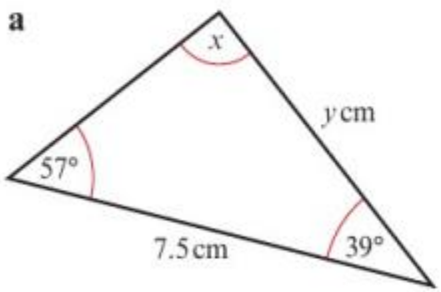
**b** Given that $a = 24$ cm, $A = 110°$, $C = 22°$, find $c$.

**c** Given that $b = 14.7$ cm, $A = 30°$, $C = 95°$, find $a$.

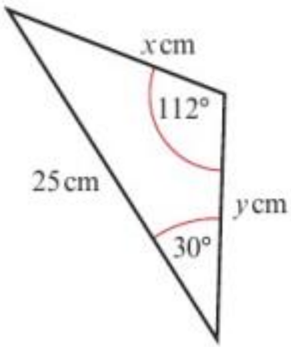
**d** Given that $c = 9.8$ cm, $B = 68.4°$, $C = 83.7°$, find $a$.

**2** In each of the following triangles calculate the values of $x$ and $y$.

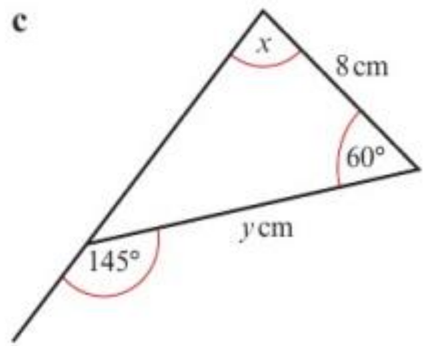
**a**



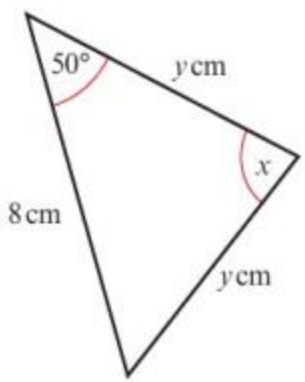

**b**



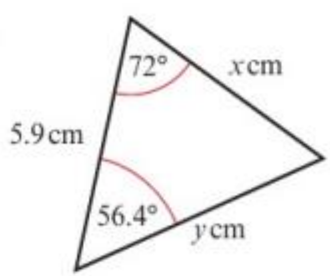

**c**



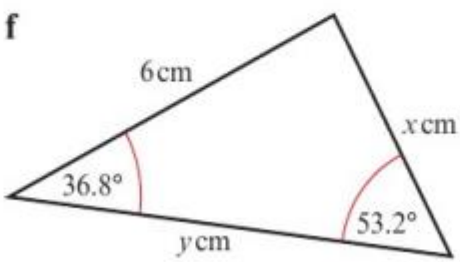

**d**




**e**

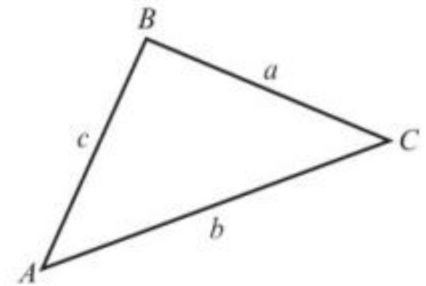



**f**



# Homework Exercise

**3** In each of the following sets of data for a triangle $ABC$, find the value of $x$.




**a** $AB = 6\text{ cm}$, $BC = 9\text{ cm}$, $\angle BAC = 117°$, $\angle ACB = x$

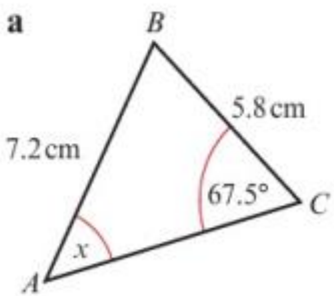
**b** $AC = 11\text{ cm}$, $BC = 10\text{ cm}$, $\angle ABC = 40°$, $\angle CAB = x$

**c** $AB = 6\text{ cm}$, $BC = 8\text{ cm}$, $\angle BAC = 60°$, $\angle ACB = x$

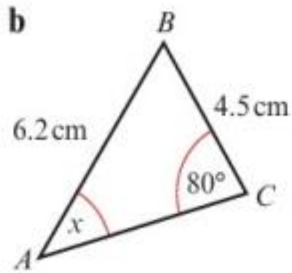
**d** $AB = 8.7\text{ cm}$, $AC = 10.8\text{ cm}$, $\angle ABC = 28°$, $\angle BAC = x$

**4** In each of the diagrams shown below, work out the size of angle $x$.

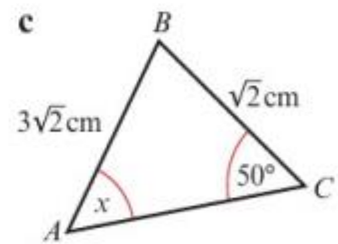
**a**



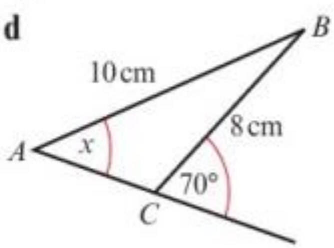

**b**



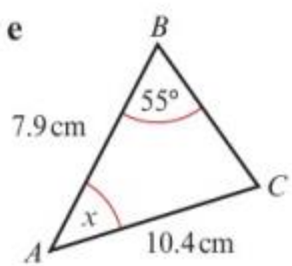

**c**



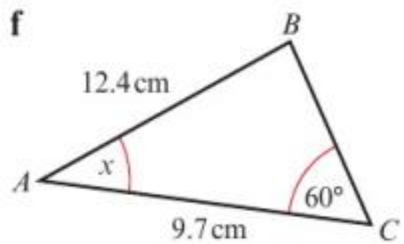

**d**




**e**




**f**



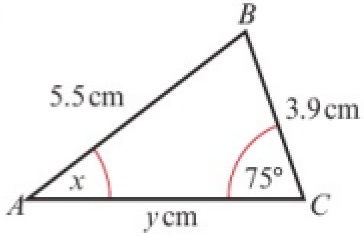

**5** In $\triangle PQR$, $QR = \sqrt{3}\text{ cm}$, $\angle PQR = 45°$ and $\angle QPR = 60°$. Find **a** $PR$ and **b** $PQ$.

**6** In $\triangle PQR$, $PQ = 15\text{ cm}$, $QR = 12\text{ cm}$ and $\angle PRQ = 75°$. Find the two remaining angles.

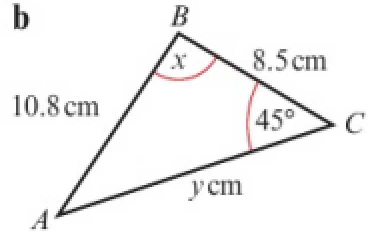
# Homework Exercise

**7** In each of the following diagrams work out the values of $x$ and $y$.

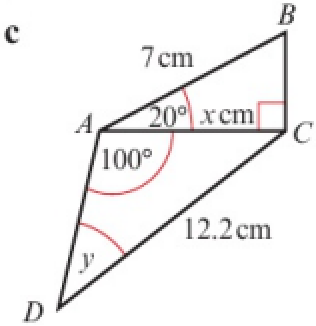
**a**



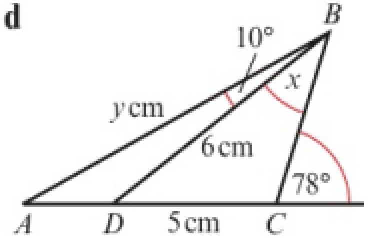

**b**



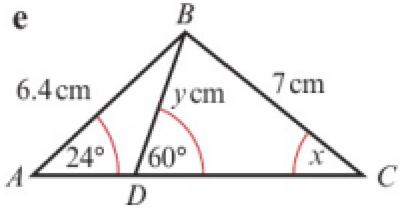

**c**



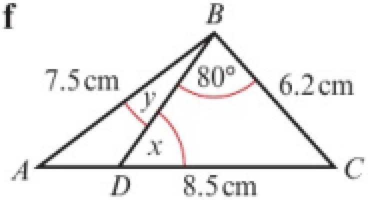

**d**




**e**




**f**




**8** Town $B$ is 6 km, on a bearing of 020°, from town $A$. Town $C$ is located on a bearing of 055° from town $A$ and on a bearing of 120° from town $B$. Work out the distance of town $C$ from:

**a** town $A$ **b** town $B$

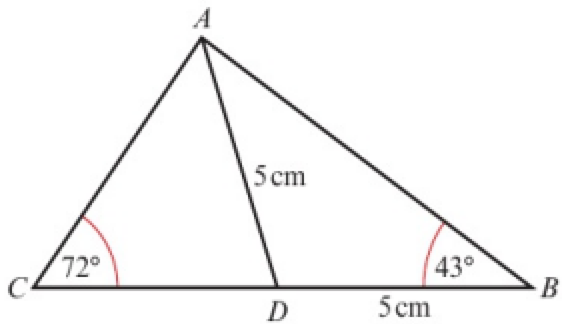
**Problem-solving**

Draw a sketch to show the information.

**9** In the diagram $AD = DB = 5$ cm, $\angle ABC = 43°$ and $\angle ACB = 72°$.
Calculate:

**a** $AB$

**b** $CD$



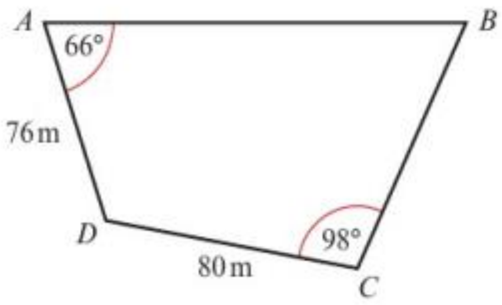
# Homework Exercise

10 A zookeeper is building an enclosure for some llamas. The enclosure is in the shape of a quadrilateral as shown.

If the length of the diagonal $BD$ is 136 m

**a** find the angle between the fences $AB$ and $BC$

**b** find the length of fence $AB$




11 In $\triangle ABC$, $AB = x$ cm, $BC = (4 - x)$ cm, $\angle BAC = y$ and $\angle BCA = 30°$.

Given that $\sin y = \frac{1}{\sqrt{2}}$, show that

$x = 4(\sqrt{2} - 1)$ **(5 marks)**

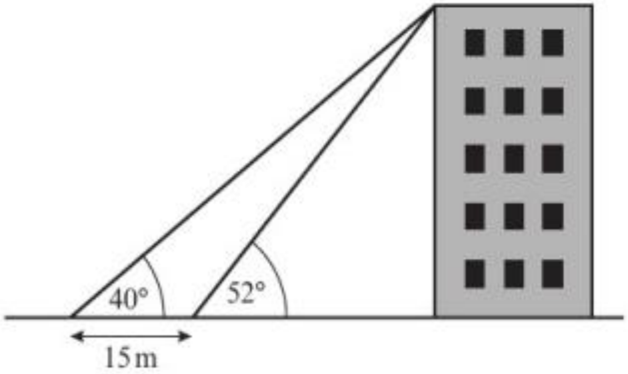
**Problem-solving**

You can use the value of sin $y$ directly in your calculation. You don't need to work out the value of $y$.

12 A surveyor wants to determine the height of a building. She measures the angle of elevation of the top of the building at two points 15 m apart on the ground.

**a** Use this information to determine the height of the building. **(4 marks)**

**b** State one assumption made by the surveyor in using this mathematical model. **(1 mark)**



# Homework Answers

**1** **a** 15.2 cm **b** 9.57 cm **c** 8.97 cm **d** 4.61 cm

**2** **a** $x = 84°$, $y = 6.32$

**b** $x = 13.5$, $y = 16.6$

**c** $x = 85°$, $y = 13.9$

**d** $x = 80°$, $y = 6.22$ (isosceles triangle)

**e** $x = 6.27$, $y = 7.16$

**f** $x = 4.49$, $y = 7.49$ (right-angled)

**3** **a** 36.4° **b** 35.8° **c** 40.5° **d** 130°

**4** **a** 48.1° **b** 45.6° **c** 14.8° **d** 48.7°

**e** 86.5° **f** 77.4°

**5** **a** 1.41 cm ($\sqrt{2}$ cm) **b** 1.93 cm

**6** $QPR = 50.6°$, $PQR = 54.4°$

**7** **a** $x = 43.2°$, $y = 5.02$ cm **b** $x = 101°$, $y = 15.0$ cm

**c** $x = 6.58$ cm, $y = 32.1°$ **d** $x = 54.6°$, $y = 10.3$ cm

**e** $x = 21.8°$, $y = 3.01$ **f** $x = 45.9°$, $y = 3.87°$

**8** **a** 6.52 km **b** 3.80 km

**9** **a** 7.31 cm **b** 1.97 cm

**10** **a** 66.3° **b** 148 m

**11** Using the sine rule, $x = \dfrac{4\sqrt{2}}{2+\sqrt{2}}$; rationalising

$$x = \frac{4\sqrt{2}(2-\sqrt{2}}{2} = 4\sqrt{2} - 4 = 4(\sqrt{2} - 1).$$

**12** **a** 36.5 m

**b** That the angles have been measured from ground level