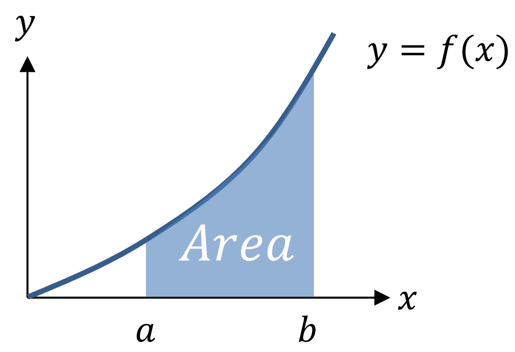
P1 Chapter 13: Integration

Areas Under Curves

Areas under curves

Earlier we saw that the definite integral $\int_a^b f(x)\ dx$ gives the **area** between a positive curve $y = f(x)$, the **x-axis**, and the lines $x = a$ and $x = b$.
(We'll see why this works in a sec)



Find the area of the finite region between the curve with equation $y = 20 - x - x^2$ and the x-axis.

?

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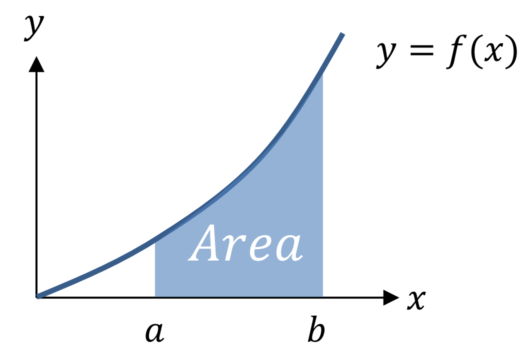

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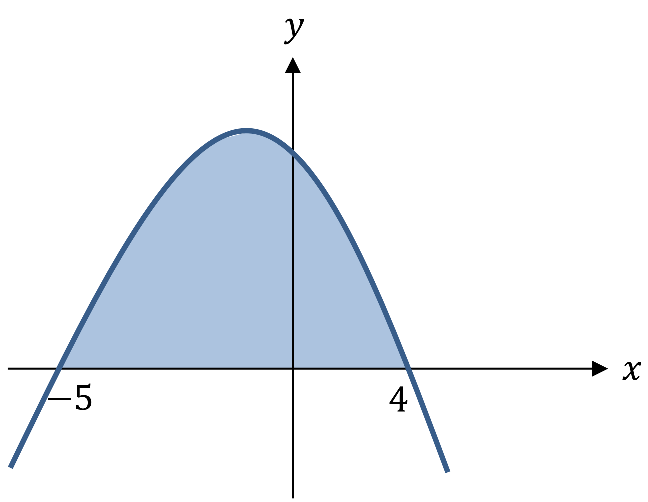

Factorise in order to find roots:

$$20 - x - x^2 = 0$$
$$x^2 + x - 20 = 0$$
$$(x+5)(x-4) = 0$$
$$x = -5 \text{ or } x = 4$$

Therefore area between curve and x-axis is:

$$\int_{-5}^{4} 20 - x - x^2\, dx = \left[20x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-5}^{4}$$
$$= \left(80 - 8 - \frac{64}{3}\right) - \left(-100 - \frac{25}{2} + \frac{125}{3}\right)$$
$$= \frac{243}{2}$$

Test Your Understanding

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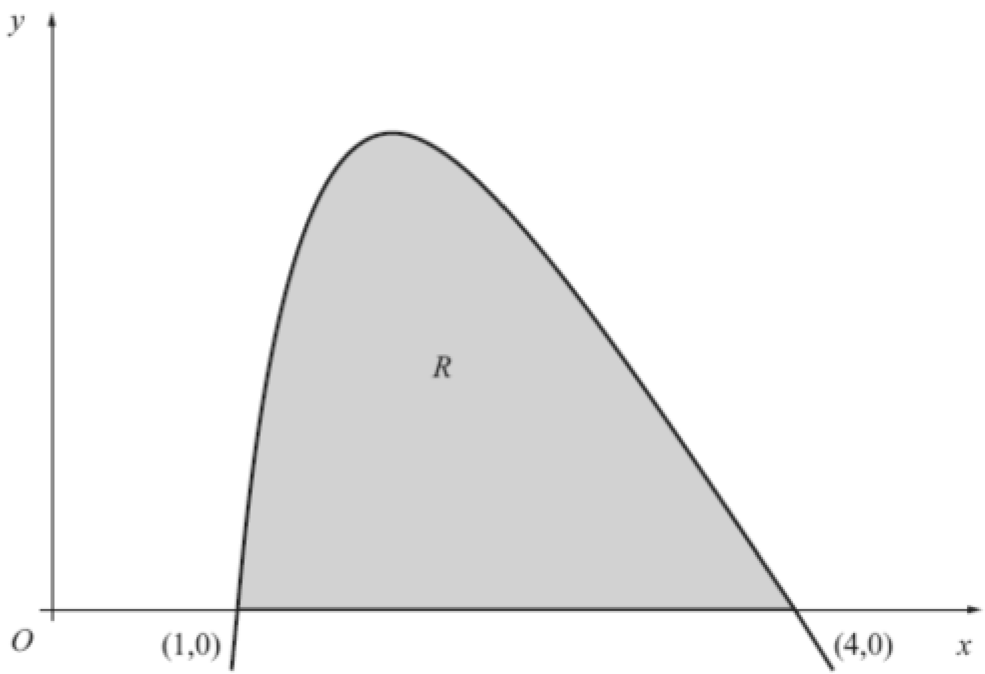

Figure 2

The finite region R, as shown in Figure 2, is bounded by the x-axis and the curve with equation

$$y = 27 - 2x - 9\sqrt{x} - \frac{16}{x^2}, \qquad x > 0.$$

The curve crosses the x-axis at the points $(1, 0)$ and $(4, 0)$.

(*c*) Use integration to find the exact value for the area of R.

(6)

?

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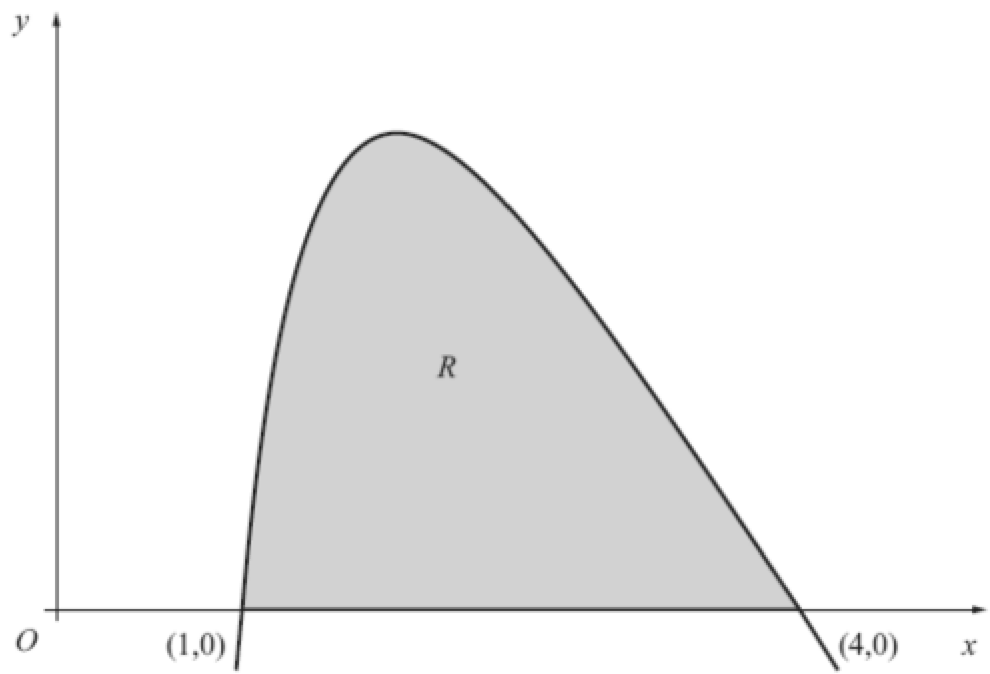

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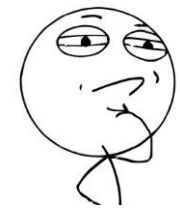
(6)

Solution: 12

Just for your interest…

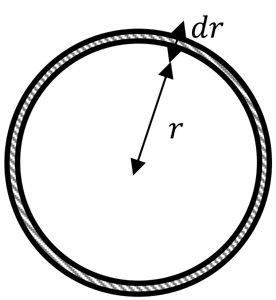
Why does integrating a function give you the area under the graph?



Part 1:

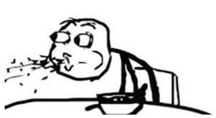
You're already familiar with the idea that gradient is the rate at which a quantity changes, and we consider an infinitesimally small change in the variable.
You could consider the gradient as the little bit you're adding on each time.

Here's some practical examples using formulae you covered in your younger years!

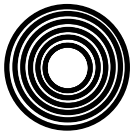


If you wanted to consider the rate at which the area of a circle increases with respect to the radius, consider a small change dr in the radius.
The change in area is an infinitely thin ring, which looks like you've drawn a circumference.
So what is this rate?

$$A = \pi r^2$$
$$\therefore \frac{dA}{dr} = 2\pi r$$



OMG THAT <u>IS</u> THE CIRCUMFERENCE

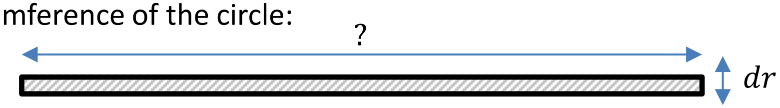


So to draw a shaded circle (which has area!), it's a bit like repeatedly drawing the circumference of a circle with gradually increasing radius.
Since the circumference is what the <u>area is increasing by each time</u>, the circumference is the gradient of the area of the circle, and conversely (by the definition of integration being the opposite of differentiation), the area is the integral of the circumference.

<u>You might be rightly upset that you can't add a length to an area</u> (only an area to an area, innit).
But by considering the infinitely thin width dr of the circumference you're drawing, it does have area!

$$\frac{dA}{dr} = 2\pi r \quad \rightarrow \quad dA = 2\pi r\, dr$$

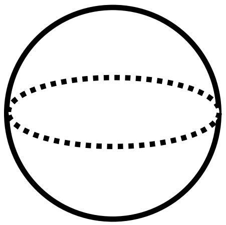
i.e. the change in area, dA, is $2\pi r \times dr$
If we 'roll out' the added area we're adding each time, this forms a rectangle, whose length hopefully corresponds to the circumference of the circle:



If the added area is $2\pi r\, dr$ and the thickness is dr, then the length is $\frac{2\pi r\, dr}{dr} = 2\pi r$ as expected.

Part 2:

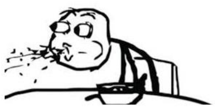
It gets better!



Consider the volume of a sphere:

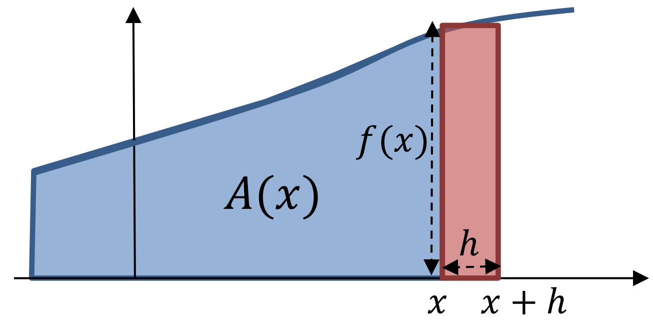
$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$



OMG THAT'S THE SURFACE AREA

This works for a similar reason to before.



And the same principle applies to area under a graph. If $A(x)$ gives the area up to x, then what we're 'adding on each time' (i.e. the gradient) is sort of like drawing a line of length $f(x)$ at the right-most end each time (or more accurately, an infinitely thin rectangle of area $f(x) \times h$). Thus if $f(x)$ is the gradient of the area, then conversely the area is the integral of $f(x)$.

This gives us an **intuitive** sense of why it works, but we need a formal proof:

Using the diagram above, the area up to $x + h$, i.e. $A(x + h)$, is approximately the area up to x plus the added rectangular strip:

$$A(x + h) \approx A(x) + (f(x) \times h)$$

Rearranging:

$$f(x) \approx \frac{A(x + h) - A(x)}{h}$$

In the limit, as the rectangle becomes infinitely thin, this becomes an equality:

$$f(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h}$$

But we know from differentiating by first principles that:

$$A'(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h}$$

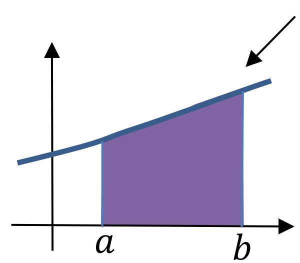
$$\therefore f(x) = A'(x)$$

And thus we have proven that the gradient of the area function is indeed $f(x)$ (and hence the integral of $f(x)$ the area).

But we're missing one final bit: Why does $\int_b^a f(x)\,dx$ give the area between $x = a$ and $x = b$?

Since $f(x) = A'(x)$, the area function $A(x)$ is the integral of $f(x)$. Thus:

$$\int_b^a f(x)\,dx = [A(x)]_a^b \quad = A(b) - A(a)$$



The area between a and b is the area up to b, i.e. $A(b)$, with the area up to a, i.e. $A(a)$, cut out. This gives $A(b) - A(a)$ as expected. Because we obtained $A(x)$ by an integration, we have a constant of integration $+c$. But because of the subtraction $A(b) - A(a)$, these constants in each of $A(b)$ and $A(a)$ cancel, which explains why the constant is not present in definite integration.

This is known as the Fundamental Theorem of Calculus.

Exercise 13.5

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Extension

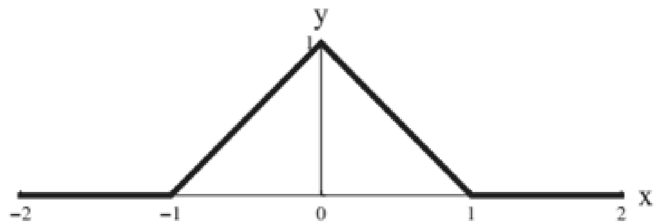
1 *[MAT 2007 1H]* Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,dx + \int_1^2 2f(x)\,dx = 7$$
$$\int_0^2 f(x)\,dx + \int_1^2 f(x)\,dx = 1$$

It follows that $\int_0^2 f(x)\ dx$ equals what?

?

2 *[MAT 2011 1G]*
A graph of the function $y = f(x)$ is sketched on the axes below:



What is the value of $\int_{-1}^{1} f(x^2 - 1)\ dx$?

?

Exercise 13.5

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$$\int_0^2 f(x)\,dx + \int_1^2 f(x)\,dx = 1$$

It follows that $\int_0^2 f(x)\ dx$ equals what?

Because the area between $x = 0$ and 2 is the sum of the area between 0 and 1, and between 1 and 2, it follows that $\int_0^2 f(x)\,dx = \int_1^2 f(x)\,dx + \int_1^2 f(x)\,dx$. Also note that $\int kf(x)\,dx = k\int f(x)\ dx$

Letting $a = \int_0^1 f(x)\,dx$ and $b = \int_1^2 f(x)\,dx$ purely for convenience, then:

$$3a + 2b = 7$$
$$(a+b) + b = 1$$

Solve, $a = 3, b = -1, \therefore a + b = 2$

2 *[MAT 2011 1G]*
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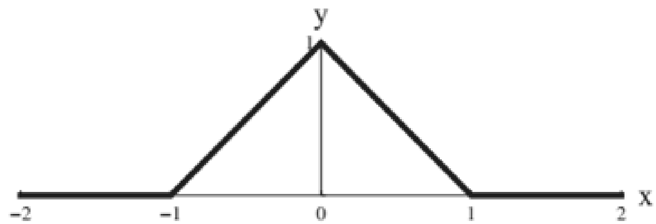


What is the value of $\int_{-1}^1 f(x^2-1)\ dx$?

We first need to reflect on what part of the function f we're actually using. x in the integral varies between -1 and 1, thus $x^2 - 1$, i.e. the input of f, varies between -1 and 0. We're therefore only using the left half of the graph, and thus $f(x) = x + 1$

$$\therefore \int_{-1}^1 f(x^2-1)\ dx = \int_{-1}^1 (x^2-1) + 1\,dx$$
$$= \left[\frac{1}{3}x^3\right]_{-1}^1 = \left(\frac{1}{3}\right) - \left(-\frac{1}{3}\right) = \frac{2}{3}$$

Homework Exercise

1 Find the area between the curve with equation $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$ in each of the following cases:

a $\mathrm{f}(x) = -3x^2 + 17x - 10;$ $a = 1,$ $b = 3$

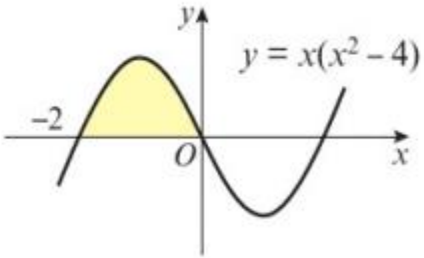
b $\mathrm{f}(x) = 2x^3 + 7x^2 - 4x;$ $a = -3,$ $b = -1$

c $\mathrm{f}(x) = -x^4 + 7x^3 - 11x^2 + 5x;$ $a = 0,$ $b = 4$

d $\mathrm{f}(x) = \dfrac{8}{x^2};$ $a = -4,$ $b = -1$

Hint For part **c**, $\mathrm{f}(x) = -x(x-1)^2(x-5)$

2 The sketch shows part of the curve with equation $y = x(x^2 - 4)$. Find the area of the shaded region.

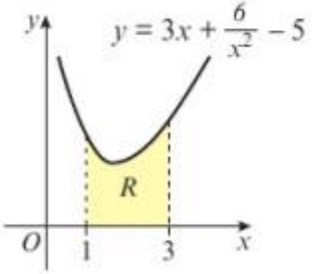


3 The diagram shows a sketch of the curve with equation

$y = 3x + \dfrac{6}{x^2} - 5,\ x > 0.$

The region R is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 3$.

Find the area of R.



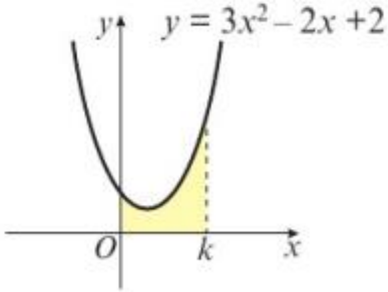
Homework Exercise

4 Find the area of the finite region between the curve with equation $y = (3 - x)(1 + x)$ and the x-axis.

5 Find the area of the finite region between the curve with equation $y = x(x - 4)^2$ and the x-axis.

6 Find the area of the finite region between the curve with equation $y = 2x^2 - 3x^3$ and the x-axis.

7 The shaded area under the graph of the function $f(x) = 3x^2 - 2x + 2$, bounded by the curve, the x-axis and the lines $x = 0$ and $x = k$, is 8. Work out the value of k.



Problem-solving

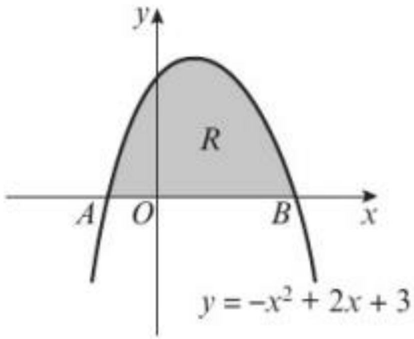
$\int_0^k (3x^2 - 2x + 2)\,dx = 8$

8 The finite region R is bounded by the x-axis and the curve with equation $y = -x^2 + 2x + 3$, $x \geqslant 0$.

The curve meets the x-axis at points A and B.

a Find the coordinates of point A and point B. **(2 marks)**

b Find the area of the region R. **(4 marks)**

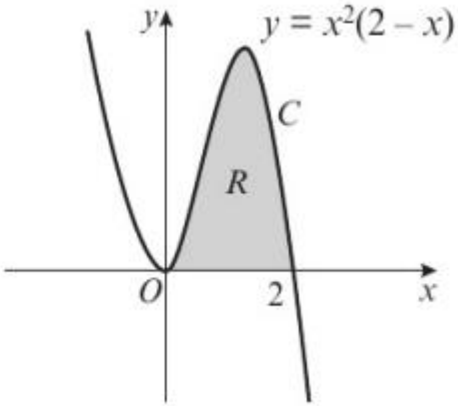


Homework Exercise

9 The graph shows part of the curve C with equation $y = x^2(2 - x)$.

The region R, shown shaded, is bounded by C and the x-axis.

Use calculus to find the exact area of R. **(5 marks)**



Watch out If a question says "use calculus" then you need to use integration or differentiation, and show clear algebraic working.

Homework Answers

1 **a** 22 **b** $36\frac{2}{3}$ **c** $48\frac{8}{15}$ **d** 6

2 4 **3** 6 **4** $10\frac{2}{3}$

5 $21\frac{1}{3}$ **6** $\frac{4}{81}$ **7** $k = 2$

8 **a** $(-1, 0)$ and $(3, 0)$ **b** $10\frac{2}{3}$

9 $1\frac{1}{3}$