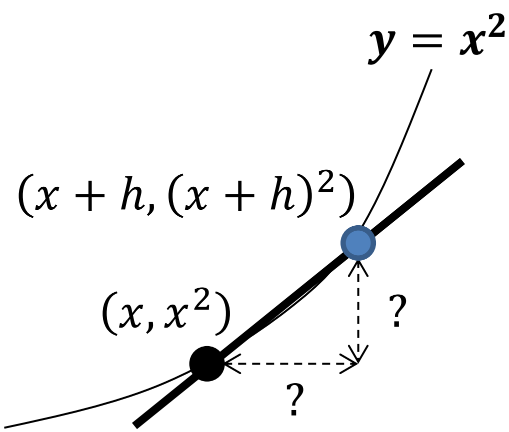
P1 Chapter 12: Differentiation

Limiting Chords

Finding the Gradient Function

Improving chords gives us a numerical method to get the gradient **at a particular $\boldsymbol{x}$**, but, doesn't give us the gradient function in general. Let's use exactly the same method, but keep x general, and make the 'small change' (which was previously 0.01) 'h':



$$gradient = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x$$

As always, gradient is change in y over change in x.

The lim means *"the limit of the following expression as h tends towards 0"*.
For example, $\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$, because as x "tends towards" infinity, the "limiting" value of the expression is 0.

The h disappears as h tends towards 0, i.e. we can effectively treat it as 0 at this point.

And voila, we got the $2x$ we saw earlier!

Finding the Gradient Function

✏ The gradient function, or derivative, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

We will soon see an easier/quicker way to differentiate expressions like $y = x^3 - x$ without using 'limits'. But this method, known as **differentiating by first principles**, is now in the A Level syllabus, and you could be tested on it!

Notation Note:

Whether we use $\frac{dy}{dx}$ or $f'(x)$ for the gradient function depends on whether we use $y =$ or $f(x) =$ to start with:

"Leibniz's notation"

$$y = x^2 \quad \rightarrow \quad \frac{dy}{dx} = 2x$$

$$f(x) = x^2 \quad \rightarrow \quad f'(x) = 2x$$

"Lagrange's notation"

Advanced Notation Note:

Rather than h for the small change in x, the formal notation is δx. So actually:

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

So we in fact have 3 symbols for "change in"!

- Δx: any change in x (as seen in Chp5: $m = \frac{\Delta y}{\Delta x}$)
- δx: a small change in x
- dx: an infinitesimally small change in x

There's in fact a third way to indicate the gradient function, notation used by Newton: (but not used at A Level)

$$y = x^2 \quad \rightarrow \quad \dot{y} = 2x$$

So the estimated gradient using some point close by was $\frac{\delta y}{\delta x}$, but in the 'limit' as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$

Example

The point A with coordinates $(4,16)$ lies on the curve with equation $y = x^2$.
At point A the curve has gradient g.

a) Show that $g = \lim_{h \to 0}(8 + h)$

b) Deduce the value of g.

a

?

b

?

Example

The point A with coordinates $(4,16)$ lies on the curve with equation $y = x^2$.
At point A the curve has gradient g.

a) Show that $g = \lim_{h \to 0}(8 + h)$

b) Deduce the value of g.

a

$$g = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

Use the "differentiation by first principles" formula.

$$= \lim_{h \to 0} \frac{(4+h)^2 - 4^2}{h}$$

Function is $f(x) = x^2$

$$= \lim_{h \to 0} \frac{16 + 8h + h^2 - 16}{h}$$

$$= \lim_{h \to 0} \frac{8h + h^2}{h}$$

$$= \lim_{h \to 0}(8 + h)$$

b

$$g = 8$$

As $h \to 0$, clearly the limiting value of $8 + h$ is 8.

Test Your Understanding

Prove **from first principles** that the derivative of x^4 is $4x^3$.

?

Helping Hand:
"Row 4" of Pascal's Triangle is:
1 4 6 4 1
You're welcome.

Test Your Understanding

Prove **from first principles** that the derivative of x^4 is $4x^3$.

Helping Hand:
"Row 4" of Pascal's Triangle is:
1 4 6 4 1
You're welcome.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\
&= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 \\
&= 4x^3
\end{aligned}
$$

As $h \to 0$, any term involving a multiplication by h will become 0.

Exercise 12.2

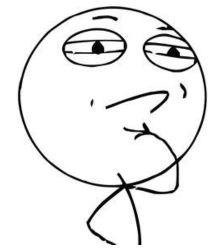
Pearson Pure Mathematics Year 1/AS
Page 93

Reminder:
$f'(x)$ and $\frac{dy}{dx}$ both mean the gradient function, also known as the derivative of y.

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Just for your interest…

Why couldn't we just immediately make h equal to 0 in $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$?



Not another one of these…

If we just stick $h=0$ in straight away:

$$\lim_{h\to 0}\frac{(x+0)^2-x^2}{0}=\lim_{h\to 0}\frac{0}{0}$$

Wait, uh oh…

$\frac{0}{0}$ is known as an **indeterminate form**.

Whereas we know what happens with expressions like $\frac{1}{0}$ (i.e. its value tends towards infinity), indeterminate forms are bad because their values are ambiguous, and prevent lim expressions from being evaluated.

Consider $\frac{0}{0}$ for example: 0 divided by anything usually gives 0, but anything divided by 0 is usually infinity. We can see these conflict.

Can you guess some other indeterminate forms, i.e. expressions whose value is ambiguous?

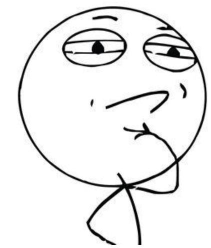
?

Thankfully, when indeterminate forms appear in lim expressions, there are variety of techniques to turn the expression into one that doesn't have any indeterminate forms.

One simple technique, that worked in our example, is to expand and simplify. This gave us $\lim_{h\to 0}(2x+h)$ and $2x+0$ clearly is fine. Other techniques are more advanced. This is important when sketching harder functions:

Just for your interest...

Why couldn't we just immediately make h equal to 0 in $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$?



Not another one of these...

If we just stick $h = 0$ in straight away:

$$\lim_{h \to 0} \frac{(x+0)^2 - x^2}{0} = \lim_{h \to 0} \frac{0}{0}$$

Wait, uh oh...

$\frac{0}{0}$ is known as an **indeterminate form**.

Whereas we know what happens with expressions like $\frac{1}{0}$ (i.e. its value tends towards infinity), indeterminate forms are bad because their values are ambiguous, and prevent lim expressions from being evaluated.

Consider $\frac{0}{0}$ for example: 0 divided by anything usually gives 0, but anything divided by 0 is usually infinity. We can see these conflict.

Can you guess some other indeterminate forms, i.e. expressions whose value is ambiguous?

$$\frac{\mathbf{0}}{\mathbf{0}} \quad \frac{\infty}{\infty} \quad \mathbf{0} \times \infty \quad \infty - \infty \quad \mathbf{0^0} \quad \mathbf{1}^{\infty} \quad \infty^{\mathbf{0}}$$

Thankfully, when indeterminate forms appear in lim expressions, there are variety of techniques to turn the expression into one that doesn't have any indeterminate forms.

One simple technique, that worked in our example, is to expand and simplify. This gave us $\lim_{h \to 0}(2x + h)$ and $2x + 0$ clearly is fine. Other techniques are more advanced. This is important when sketching harder functions:

Exercise 12.2

Pearson Pure Mathematics Year 1/AS
Page 1

Homework Exercise

1 For the function $f(x) = x^2$, use the definition of the derivative to show that:

a $f'(2) = 4$ b $f'(-3) = -6$ c $f'(0) = 0$ d $f'(50) = 100$

2 $f(x) = x^2$

a Show that $f'(x) = \lim_{h \to 0}(2x + h)$. b Hence deduce that $f'(x) = 2x$.

3 The point A with coordinates $(-2, -8)$ lies on the curve with equation $y = x^3$.
At point A the curve has gradient g.

a Show that $g = \lim_{h \to 0}(12 - 6h + h^2)$. b Deduce the value of g.

4 The point A with coordinates $(-1, 4)$ lies on the curve with equation $y = x^3 - 5x$.
The point B also lies on the curve and has x-coordinate $(-1 + h)$.

a Show that the gradient of the line segment AB is given by $h^2 - 3h - 2$.

b Deduce the gradient of the curve at point A.

Problem-solving

Draw a sketch showing points A and B and the chord between them.

5 Prove, from first principles, that the derivative of $6x$ is 6. **(3 marks)**

Homework Exercise

6 Prove, from first principles, that the derivative of $4x^2$ is $8x$. **(4 marks)**

7 $f(x) = ax^2$, where a is a constant. Prove, from first principles, that $f'(x) = 2ax$. **(4 marks)**

Challenge

$f(x) = \frac{1}{x}$

a Given that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, show that $f'(x) = \lim_{h \to 0} \frac{-1}{x^2 + xh}$

b Deduce that $f'(x) = -\frac{1}{x^2}$

Homework Answers

1 **a** $f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{(2 + h)^2 - 2^2}{h}$

$= \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} (4 + h) = 4$

b $f'(-3) = \lim_{h \to 0} \frac{f(-3 + h) - f(-3)}{h} = \lim_{h \to 0} \frac{(-3 + h)^2 - 3^2}{h}$

$= \lim_{h \to 0} \frac{-6h + h^2}{h} = \lim_{h \to 0} (-6 + h) = -6$

c $f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 - 0^2}{h} = \lim_{h \to 0} h = 0$

d $f'(50) = \lim_{h \to 0} \frac{f(50 + h) - f(50)}{h} = \lim_{h \to 0} \frac{(50 + h)^2 - 50^2}{h}$

$= \lim_{h \to 0} \frac{100h + h^2}{h} = \lim_{h \to 0} (100 + h) = 100$

2 **a** $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h)$

b As $h \to 0$, $f'(x) = \lim_{h \to 0} (2x + h) = 2x$

3 **a** $g = \lim_{h \to 0} \frac{(-2 + h)^3 - (-2)^3}{h}$

$= \lim_{h \to 0} \frac{-8 + 3(-2)^2h + 3(-2)h^2 + h^3 + 8}{h}$

$= \lim_{h \to 0} \frac{12h - 6h^2 + h^3}{h} = \lim_{h \to 0} (12 - 6h + h^2)$

b $g = 12$

4 **a** Gradient of $AB = \frac{(-1 + h)^3 - 5(-1 + h) - 4}{(-1 + h) - (-1)}$

$= \frac{-1 + 3h = 3h^2 + h^3 + 5 - 5h - 4}{h}$

$= \frac{h^3 - 3h^2 - 2h}{h} = h^2 - 3h - 2$

b gradient $= -2$

5 $\frac{dy}{dx} = \lim_{h \to 0} \frac{6(x + h) - 6x}{h} = \lim_{h \to 0} \frac{6h}{h} = 6$

6 $\frac{dy}{dx} = \lim_{h \to 0} \frac{4(x + h)^2 - 4x^2}{h} = \lim_{h \to 0} \frac{8xh + 4h^2}{h}$

$= \lim_{h \to 0} (8x + 4h) = 8x$

7 $\frac{dy}{dx} = \lim_{h \to 0} \frac{a(x + h)^2 - ax^2}{h} = \lim_{h \to 0} \frac{(a - a)x^2 + 2axh + ah^2}{h}$

$= \lim_{h \to 0} \frac{2axh + ah^2}{h} = \lim_{h \to 0} (2ax + ah) = 2ax$

Challenge

a $f'(x) = \lim_{h \to 0} \frac{\frac{1}{x + h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x + h)}{xh(x + h)} = \lim_{h \to 0} \frac{-1}{x(x + h)}$

$= \lim_{h \to 0} \frac{-1}{x^2 + xh}$

b $f'(x) = \lim_{h \to 0} \frac{-1}{x(x + h)} = \frac{-1}{x^2 + xh} = \frac{-1}{x^2 + 0} = -\frac{1}{x^2}$