P2 Chapter 3: Sequences and Series

Chapter Practice

Key Points

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8 The Greek capital letter 'sigma' is used to signify a sum. You write it as $\sum$. You write limits on the top and bottom to show which terms you are summing.

9 A recurrence relation of the form $u_{n+1} = f(u_n)$ defines each term of a sequence as a function of the previous term.

10 A sequence is **increasing** if $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.

A sequence is **decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

A sequence is **periodic** if the terms repeat in a cycle. For a periodic sequence there is an integer k such that $u_{n+k} = u_n$ for all $n \in \mathbb{N}$. The value k is called the **order** of the sequence.

1 A geometric series has third term 27 and sixth term 8.

   a Show that the common ratio of the series is $\frac{2}{3}$ **(2 marks)**

   b Find the first term of the series.   **(2 marks)**

   c Find the sum to infinity of the series.   **(2 marks)**

   d Find the difference between the sum of the first 10 terms of the series and the sum to infinity. Give your answer to 3 significant figures. **(2 marks)**

2 The second term of a geometric series is 80 and the fifth term of the series is 5.12.

   a Show that the common ratio of the series is 0.4.   **(2 marks)**

   Calculate:

   b the first term of the series   **(2 marks)**

   c the sum to infinity of the series, giving your answer as an exact fraction   **(1 mark)**

   d the difference between the sum to infinity of the series and the sum of the first 14 terms of the series, giving your answer in the form $a \times 10^n$, where $1 \leqslant a < 10$ and n is an integer. **(2 marks)**

3 The nth term of a sequence is u_n, where $u_n = 95\left(\frac{4}{5}\right)^n$, $n = 1, 2, 3, \ldots$

   a Find the values of u_1 and u_2.   **(2 marks)**

   Giving your answers to 3 significant figures, calculate:

   b the value of u_{21} **(1 mark)**

   c $\displaystyle\sum_{n=1}^{15} u_n$ **(2 marks)**

   d the sum to infinity of the series whose first term is u_1 and whose nth term is u_n.   **(1 mark)**

4 A sequence of numbers $u_1, u_2, \ldots, u_n, \ldots$ is given by the formula $u_n = 3\left(\frac{2}{3}\right)^n - 1$ where n is a positive integer.

   a Find the values of u_1, u_2 and u_3. **(2 marks)**

   b Show that $\sum\limits_{n=1}^{15} u_n = -9.014$ to 4 significant figures. **(2 marks)**

   c Prove that $u_{n+1} = \dfrac{2u_n - 1}{3}$ **(2 marks)**

5 The third and fourth terms of a geometric series are 6.4 and 5.12 respectively. Find:

   a the common ratio of the series, **(2 marks)**

   b the first term of the series, **(2 marks)**

   c the sum to infinity of the series. **(2 marks)**

   d Calculate the difference between the sum to infinity of the series and the sum of the first 25 terms of the series. **(2 marks)**

6 The price of a car depreciates by 15% per annum. Its price when new is £20 000.

   a Find the value of the car after 5 years. **(2 marks)**

   b Find when the value will be less than £4000. **(3 marks)**

7 The first three terms of a geometric series are $p(3q + 1)$, $p(2q + 2)$ and $p(2q - 1)$, where p and q are non-zero constants.

   a Show that one possible value of q is 5 and find the other possible value. **(2 marks)**

   b Given that $q = 5$, and the sum to infinity of the series is 896, find the sum of the first 12 terms of the series. Give your answer to 2 decimal places. **(4 marks)**

8 a Prove that the sum of the first n terms in an arithmetic series is

$$S = \frac{n}{2}(2a + (n-1)d)$$

where a = first term and d = common difference. **(3 marks)**

b Use this to find the sum of the first 100 natural numbers. **(2 marks)**

9 Find the least value of n for which $\sum_{r=1}^{n}(4r-3) > 2000$. **(2 marks)**

10 The sum of the first two terms of an arithmetic series is 47.
The thirtieth term of this series is −62. Find:

a the first term of the series and the common difference **(3 marks)**

b the sum of the first 60 terms of the series. **(2 marks)**

11 a Find the sum of the integers which are divisible by 3 and lie between 1 and 400. **(3 marks)**

b Hence, or otherwise, find the sum of the integers, from 1 to 400 inclusive, which are **not** divisible by 3. **(2 marks)**

12 A polygon has 10 sides. The lengths of the sides, starting with the shortest, form an arithmetic series. The perimeter of the polygon is 675 cm and the length of the longest side is twice that of the shortest side. Find the length of the shortest side of the polygon. **(4 marks)**

13 Prove that the sum of the first $2n$ multiples of 4 is $4n(2n+1)$. **(4 marks)**

14 A sequence of numbers is defined, for $n \geqslant 1$, by the recurrence relation $u_{n+1} = ku_n - 4$, where k is a constant. Given that $u_1 = 2$:

   a find expressions, in terms of k, for u_2 and u_3. **(2 marks)**

   b Given also that $u_3 = 26$, use algebra to find the possible values of k. **(2 marks)**

15 The fifth term of an arithmetic series is 14 and the sum of the first three terms of the series is -3.

   a Use algebra to show that the first term of the series is -6 and calculate the common difference of the series. **(3 marks)**

   b Given that the nth term of the series is greater than 282, find the least possible value of n. **(3 marks)**

16 The fourth term of an arithmetic series is $3k$, where k is a constant, and the sum of the first six terms of the series is $7k + 9$.

   a Show that the first term of the series is $9 - 8k$. **(3 marks)**

   b Find an expression for the common difference of the series in terms of k. **(2 marks)**

   Given that the seventh term of the series is 12, calculate:

   c the value of k **(2 marks)**

   d the sum of the first 20 terms of the series. **(2 marks)**

17 A sequence is defined by the recurrence relation

$$a_{n+1} = \frac{1}{a_n}, \ a_1 = p$$

   a Show that the sequence is periodic and state its order. **(2 marks)**

   b Find $\sum_{r=1}^{1000} a_n$ in terms of p. **(2 marks)**

18 A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$
$$a_{n+1} = 2a_n + 6, \ n \geqslant 1$$

where k is an integer.

    a Given that the sequence is increasing for the first 3 terms, show that $k > p$, where p is an integer to be found. **(2 marks)**

    b Find a_4 in terms of k. **(2 marks)**

    c Show that $\sum_{r=1}^{4} a_r$ is divisible by 3. **(3 marks)**

19 The first term of a geometric series is 130. The sum to infinity of the series is 650.

    a Show that the common ratio, r, is $\frac{4}{5}$ **(3 marks)**

    b Find, to 2 decimal places, the difference between the 7th and 8th terms. **(2 marks)**

   **c** Calculate the sum of the first 7 terms. **(2 marks)**

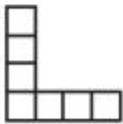The sum of the first n terms of the series is greater than 600.

    d Show that $n > \dfrac{-\log 13}{\log 0.8}$ **(4 marks)**

20 The adult population of a town is 25 000 at the beginning of 2012.
A model predicts that the adult population of the town will increase by 2% each year, forming a geometric sequence.

a Show that the predicted population at the beginning of 2014 is 26 010. **(1 mark)**

The model predicts that after n years, the population will first exceed 50 000.

b Show that $n > \dfrac{\log 2}{\log 1.02}$ **(3 marks)**

c Find the year in which the population first exceeds 50 000. **(2 marks)**

d Every member of the adult population is modelled to visit the doctor once per year. Calculate the number of appointments the doctor has from the beginning of 2012 to the end of 2019. **(4 marks)**

e Give a reason why this model for doctors' appointments may not be appropriate. **(1 mark)**

21 Kyle is making some patterns out of squares. He has made 3 rows so far.

a Find an expression, in terms of n, for the number of squares required to make a similar arrangement in the nth row. **(3 marks)**

Row 1

b Kyle counts the number of squares used to make the pattern in the kth row. He counts 301 squares. Write down the value of k. **(1 mark)**

Row 2

c In the first q rows, Kyle uses a total of p squares.

i Show that $q^2 + 2q - p = 0$. **(3 marks)**

Row 3

ii Given that $p > 1520$, find the minimum number of rows that Kyle makes. **(3 marks)**

22 A convergent geometric series has first term a and common ratio r. The second term of the series is -3 and the sum to infinity of the series is 6.75.

    a Show that $27r^2 - 27r - 12 = 0$.   **(4 marks)**

    b Given that the series is convergent, find the value of r.   **(2 marks)**

    c Find the sum of the first 5 terms of the series, giving your answer to
       2 decimal places.   **(3 marks)**

Challenge

A sequence is defined by the recurrence relation $u_{n+2} = 5u_{n+1} - 6u_n$.

a Prove that any sequence of the form $u_n = p \times 3^n + q \times 2^n$, where p and q are constants, satisfies this recurrence relation.

Given that $u_1 = 5$ and $u_2 = 12$,

b find an expression for u_n in terms of n only.

c Hence determine the number of digits in u_{100}.

1   **a** $ar^2 = 27, ar^5 = 8 \Rightarrow r^3 = \frac{8}{27} \Rightarrow r = \frac{2}{3}$

    b 60.75   **c** 182.25   **d** 3.16

2   **a** $ar = 80, ar^4 = 5.12$

      $\Rightarrow r^3 = \frac{8}{125} \Rightarrow r = \frac{2}{5} = 0.4$

    b 200   **c** $333\frac{1}{3}$ **d** 8.95×10^{-4}

3   **a** 76, 60.8   **b** 0.876   **c** 367   **d** 380

4   **a** $1, \frac{1}{3}, -\frac{1}{9}$

    b $\sum\limits_{n=1}^{15}\left(3\left(\frac{2}{3}\right)^n - 1\right) = \sum\limits_{n=1}^{15} 3\left(\frac{2}{3}\right)^n - \sum\limits_{r=1}^{15} 1$

      $\sum\limits_{n=1}^{15} 3\left(\frac{2}{3}\right)^n = \dfrac{2\left(1 - \left(\frac{2}{3}\right)^{15}\right)}{\frac{1}{3}} = 5.9863$

      $\sum\limits_{r=1}^{15} 1 = 15$

      $5.9863 - 15 = -9.014$

    c $u_{n+1} = 3\left(\frac{2}{3}\right)^{n+1} - 1 = 3 \times \frac{2}{3}\left(\frac{2}{3}\right)^n - 1 = \frac{1}{3}\left(2 \times 3\left(\frac{2}{3}\right)^n - 3\right)$

      $= \dfrac{2u_n - 1}{3}$

5   **a** 0.8   **b** 10   **c** 50   **d** 0.189

6   **a** £8874.11   **b** after 9.9 years

7   **a** $\dfrac{p(2q + 2)}{p(3q + 1)} = \dfrac{p(2q - 1)}{p(2q + 2)}$

      $(2q + 2)^2 = (2q - 1)(3q + 1)$

      $4q^2 + 8q + 4 = 6q^2 - q - 1$

      $0 = 2q^2 - 9q - 5 = (q - 5)(2q + 1) \Rightarrow q = 5 \text{ or } -\frac{1}{2}$

    b 867.62

8   **a** $S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n - 2)d)$

      $+ (a + (n - 1)d)$ (1)

      $S_n = (a + (n - 1)d) + (a + (n - 2)d) + \ldots (a + 2d)$

      $+ (a + d) + a$ (2)

      Adding (1) and (2):

      $2 \times S_n = n(2a + (n - 1)d) \Rightarrow S_n = \frac{n}{2}(2a + (n - 1)d)$

    b 5050

9   32

10 **a** $a = 25, d = -3$ **b** -3810

11 **a** 26733   **b** 53467

12 45 cm

13 $S_{2n} = \dfrac{2n}{2}(2(4) + (2n - 1)4) = n(4 + 8n) = 4n(2n + 1)$

14 **a** $u_2 = 2k - 4, u_3 = 2k^2 - 4k - 4$ **b** 5, −3

15 **a** $a + 4d = 14, \frac{3}{2}(2a + 2d) = -3$

      $3a + 3d = -3, 3a + 12d = 42$

      $9d = 45 \Rightarrow d = 5 \Rightarrow a = -6$

    b 59

16 a $a + 3d = 3k$, $3(2a + 5d) = 7k + 9 \Rightarrow$
$6a + 15d = 7k + 9$

$6a + 15\left(\dfrac{3k - a}{3}\right) = 7k + 9$

$6a + 15k - 5a = 7k + 9 \Rightarrow a = 9 - 8k$

b $\dfrac{11k - 9}{3}$ **c** 1.5 **d** 415

17 a $a_1 = p$, $a_2 = \dfrac{1}{p}$, $a_3 = \dfrac{1}{\frac{1}{p}} = 1 \times \dfrac{p}{1} = p$

$a_1 = a_3 \Rightarrow$ Sequence is periodic, order 2

b $500\left(p + \dfrac{1}{p}\right)$

18 a $a_1 = k$, $a_2 = 2k + 6$, $a_3 = 2(2k + 6) + 6 = 4k + 18$
$a_1 < a_2 < a_3 \Rightarrow k < 2k + 6 < 4k + 18 \Rightarrow k > -6$

b $a_4 = 8k + 42$

c $a_4 = 8k + 42$

$\displaystyle\sum_{r=1}^{4} a_r = k + 2k + 6 + 4k + 18 + 8k + 42$

$= 15k + 66 = 3(5k + 22)$

therefore divisible by 3

19 a $a = 130$

$S_\infty = \dfrac{130}{1 - r} = 650 \Rightarrow 130 = 650 - 650r$

$-520 = -650r \Rightarrow r = \dfrac{-520}{-650} = \dfrac{4}{5}$

b 6.82

c 513.69 (2 d.p.)

d $\dfrac{130(1 - (0.8)^n)}{0.2} > 600 \Rightarrow 1 - (0.8)^n > \dfrac{12}{13}$

$(0.8)^n < \dfrac{1}{13} \Rightarrow n \log(0.8) < -\log 13 \Rightarrow n > \dfrac{-\log 13}{\log 0.8}$

20 a $25000 \times 1.02^2 = 26010$

b $25000 \times 1.02^n > 50000$

$1.02^n > 2 \Rightarrow n \log 1.02 > \log 2 \Rightarrow n > \dfrac{\log 2}{\log 1.02}$

c 2047

d 214574

e People may visit the doctor more frequently than once a year, some may not visit at all, depends on state of health

21 a $2n + 1$ **b** 150

c i $S_q = \dfrac{q}{2}(2(3) + (q-1)2) = 2q + q^2$

$S_q = p \Rightarrow q^2 + 2q - p = 0$

ii 39

22 a $ar = -3$, $\dfrac{a}{1-r} = 6.75$

$\Rightarrow -\dfrac{3}{r} \times \dfrac{1}{1-r} = 6.75 \Rightarrow \dfrac{-3}{r-r^2} = 6.75$

$6.75r - 6.75r^2 + 3 = 0$

$27r^2 - 27r - 12 = 0$

b $-\frac{1}{3}$ series is convergent so $|r| < 1$

c 6.78

Challenge

a $u_{n+2} = 5u_{n+1} - 6u_n.$

$= 5\left[p(3^{n+1}) + q(2^{n+1})\right] - 6\left[p(3^n) + q(2^n)\right]$

$= 5\left[p\left(\dfrac{1}{3}\right)(3^{n+2}) + q\left(\dfrac{1}{2}\right)(2^{n+2})\right]$

$-6\left[p\left(\dfrac{1}{3}\right)^2(3^{n+2}) + q\left(\dfrac{1}{2}\right)^2(2^{n+2})\right]$

$= \left(\dfrac{5}{3}p - \dfrac{6}{9}p\right)(3^{n+2}) + \left(\dfrac{5}{2}q - \dfrac{6}{4}q\right)(2^{n+2})$

$= p(3^{n+2}) + q(2^{n+2})$

b $u_n = \left(\dfrac{2}{3}\right)(3^n) + \left(\dfrac{3}{2}\right)(2^n)$ or e.g. $u_n = 2(3^{n-1}) + 3(2^{n-1})$

c $u_{100} = 3.436 \times 10^{47}$ (4 s.f.) so contains 48 digits.