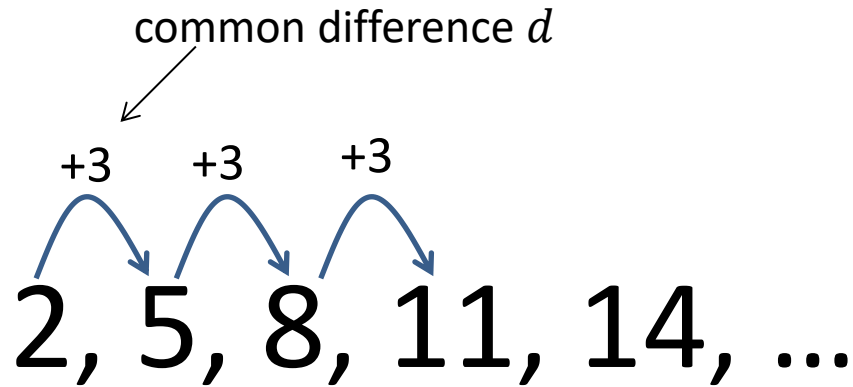

P2 Chapter 3: Sequences and Series


Arithmetic Series

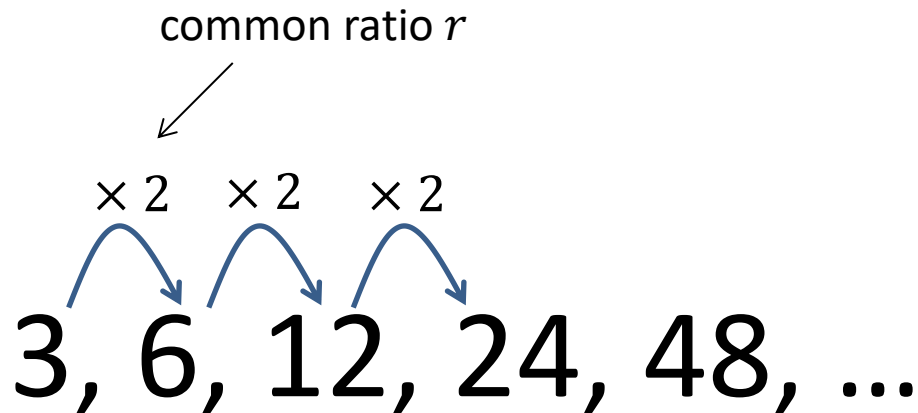
Types of sequences



This is an ...

Arithmetic Sequence

 An arithmetic sequence is one which has a common difference between terms.



Geometric Sequence
(We will explore these later in the chapter)

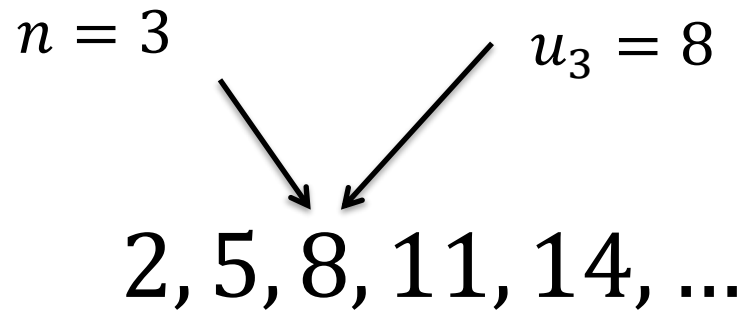
1, 1, 2, 3, 5, 8, ...

This is the **Fibonacci Sequence**. The terms follow a **recurrence relation** because each term can be generated using the previous ones. We will encounter recurrence relations later in the chapter.

The fundamentals of sequences

u_n The n^{th} **term**. So u_3 would refer to the 3rd term.

n The **position** of the term in the sequence.



n^{th} term of an arithmetic sequence

We use a to denote the **first term**. d is the **difference** between terms, and n is the **position** of the term we're interested in. Therefore:

1 st Term	2 nd Term	3 rd Term	...	n^{th} term
a	$a + d$	$a + 2d$...	$a + (n - 1)d$

 **n^{th} term of arithmetic sequence:**

$$u_n = a + (n - 1)d$$

Example 1

The n th term of an arithmetic sequence is

$$u_n = 55 - 2n.$$

- Write down the first 3 terms of the sequence.
- Find the first term in the sequence that is negative.

a) $u_1 = 55 - 2(1) = 53$, $u_2 = 51$, $u_3 = 49$

b) $55 - 2n < 0 \rightarrow n > 27.5 \therefore n = 28$
 $u_{28} = 55 - 2(28) = -1$

Example 2

Find the n th term of each arithmetic sequence.

- 6, 20, 34, 48, 62
- 101, 94, 87, 80, 73

a) $a = 6, d = 14$

$$\begin{aligned} u_n &= 6 + 14(n - 1) \\ &= 14n - 8 \end{aligned}$$

b) $a = 101, d = -7$

$$\begin{aligned} u_n &= 101 - 7(n - 1) \\ &= 108 - 7n \end{aligned}$$

Fro Tip: Always write out $a, d, n = \text{first}$.

Further Examples

[Textbook] A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.

Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b .

?

For which values of x would the expression -8 , x^2 and $17x$ form the first three terms of an arithmetic sequence.

?

Further Examples

[Textbook] A sequence is generated by the formula $u_n = an + b$ where a and b are constants to be found.

Given that $u_3 = 5$ and $u_8 = 20$, find the values of the constants a and b .

$$u_3 = 5 \rightarrow a + 2d = 5$$

$$u_8 = 20 \rightarrow a + 7d = 20$$

Solving simultaneously, $a = 3, b = -4$

For which values of x would the expression $-8, x^2$ and $17x$ form the first three terms of an arithmetic sequence.

Remember that an arithmetic sequence is one where there is a common difference between terms.

$$x^2 + 8 = 17x - x^2$$

$$2x^2 - 17x + 8 = 0 \rightarrow (2x - 1)(x - 8) = 0$$

$$x = \frac{1}{2}, x = 8$$

Test Your Understanding

Edexcel C1 May 2014(R) Q10

Xin has been given a 14 day training schedule by her coach.

Xin will run for A minutes on day 1, where A is a constant.

She will then increase her running time by $(d + 1)$ minutes each day, where d is a constant.

(a) Show that on day 14, Xin will run for

$$(A + 13d + 13) \text{ minutes.}$$

(2)

Yi has also been given a 14 day training schedule by her coach.

Yi will run for $(A - 13)$ minutes on day 1.

She will then increase her running time by $(2d - 1)$ minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

(b) find the value of d .

(3)

(a).

?

(b)

?

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(3)

- | | |
|------|---|
| (a). | Attempts to use $a + (n - 1)d$ with $a = A$ and " d " = $d + 1$ and $n = 14$
$A + 13(d + 1) = A + 13d + 13$ * |
| (b) | Calculates time for Yi on Day 14 = $(A - 13) + 13(2d - 1)$
Sets times equal $A + 13d + 13 = (A - 13) + 13(2d - 1) \Rightarrow d = \dots$
$d = 3$ |

M1
A1*

M1
M1
A1 cso

Exercise 3.1

Pearson Pure Mathematics Year 2/AS

Page 17

Extension

- 1** [STEP I 2004 Q5] The positive integers can be split into five distinct arithmetic progressions, as shown:

A: 1, 6, 11, 16, ...

B: 2, 7, 12, 17, ...

C: 3, 8, 13, 18, ...

D: 4, 9, 14, 19, ...

E: 5, 10, 15, 20, ...

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

- i) Prove that there are no positive integers x and y such that $x^2 + 5y = 243723$
- ii) Prove also that there are no positive integers x and y such $x^4 + 2y^4 = 26081974$

?

Exercise 3.1

Pearson Pure Mathematics Year 2/AS

Page 17

Extension

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- Prove that there are no positive integers x and y such that $x^2 + 5y = 243723$
- Prove also that there are no positive integers x and y such that $x^4 + 2y^4 = 26081974$

A: $5n - 4$

B: $5n - 3$

C: $5n - 2$

D: $5n - 1$

E: $5n$

Term in B + Term in C (note that the positions in each sequence could be different, so use n and m):

$$\begin{aligned} &5n - 3 + 5m - 2 \\ &= 5(n + m - 1) \end{aligned}$$

hence a term in E.

Square of term in B:

$$\begin{aligned} (5n - 3)^2 &= 25n^2 - 30n + 9 \\ &= 5(5n^2 - 6n + 2) - 1 \end{aligned}$$

hence a term in D.

Square of term in C:

$$\begin{aligned} (5n - 2)^2 &= 25n^2 - 20n + 4 \\ &= 5(5n^2 - 4n + 1) - 1 \end{aligned}$$

hence also a term in D.

i) 243723 is a term in C. Note that whatever sequence x^2 is in, adding $5y$ will keep it in the same sequence as it is a multiple of 5. So we need to prove no term squared can give a term in C.

We have already proved a term in B or in C, squared, gives a term in D. We can similarly show a term in A, squared, stays in A, and a term in E squared stays in E. Finally, a term in D squared gives a term in A. Since the sequences A, B, C, D, E clearly span all positive integers, there cannot be any integer solutions to the equation.

Homework Exercise

1 For each sequence:

i write down the first 4 terms of the sequence

ii write down a and d .

a $u_n = 5n + 2$

b $u_n = 9 - 2n$

c $u_n = 7 + 0.5n$

d $u_n = n - 10$

2 Find the n th terms and the 10th terms in the following arithmetic progressions:

a 5, 7, 9, 11, ...

b 5, 8, 11, 14, ...

c 24, 21, 18, 15, ...

d -1, 3, 7, 11, ...

e $x, 2x, 3x, 4x, \dots$

f $a, a + d, a + 2d, a + 3d, \dots$

3 Calculate the number of terms in each of the following arithmetic sequences.

a 3, 7, 11, ..., 83, 87

b 5, 8, 11, ..., 119, 122

c 90, 88, 86, ..., 16, 14

d 4, 9, 14, ..., 224, 229

e $x, 3x, 5x, \dots, 35x$

f $a, a + d, a + 2d, \dots, a + (n - 1)d$

Problem-solving

Find an expression for u_n and set it equal to the final term in the sequence. Solve the equation to find the value of n .

4 The first term of an arithmetic sequence is 14. The fourth term is 32. Find the common difference.

5 A sequence is generated by the formula $u_n = pn + q$ where p and q are constants to be found. Given that $u_6 = 9$ and $u_9 = 11$, find the constants p and q .

Homework Exercise

- 6 For an arithmetic sequence $u_3 = 30$ and $u_9 = 9$. Find the first negative term in the sequence.
- 7 The 20th term of an arithmetic sequence is 14. The 40th term is -6 . Find the value of the 10th term.
- 8 The first three terms of an arithmetic sequence are $5p$, 20 and $3p$, where p is a constant. Find the 20th term in the sequence.
- 9 The first three terms in an arithmetic sequence are -8 , k^2 , $17k \dots$
Find two possible values of k . (3 marks)
- 10 An arithmetic sequence has first term k^2 and common difference k , where $k > 0$. The fifth term of the sequence is 41. Find the value of k , giving your answer in the form $p + q\sqrt{5}$, where p and q are integers to be found. (4 marks)

Problem-solving

You will need to make use of the condition $k > 0$ in your answer.

Challenge

The n th term of an arithmetic sequence is $u_n = \ln a + (n - 1) \ln b$ where a and b are integers. $u_3 = \ln 16$ and $u_7 = \ln 256$. Find the values of a and b .

Homework Answers

- 1 a i 7, 12, 17, 22 ii $a = 7, d = 5$
 b i 7, 5, 3, 1 ii $a = 7, d = -2$
 c i 7.5, 8, 8.5, 9 ii $a = 7.5, d = 0.5$
 d i -9, -8, -7, -6 ii $a = -9, d = 1$
- 2 a $2n + 3, 23$ b $3n + 2, 32$
 c $27 - 3n, -3$ d $4n - 5, 35$
 e $nx, 10x$ f $a + (n - 1)d, a + 9d$
- 3 a 22 b 40 c 39 d 46 e 18 f n
- 4 $d = 6$ 5 $p = \frac{2}{3}, q = 5$ 6 -1.5
- 7 24 8 -70 9 $k = \frac{1}{2}, k = 8$
- 10 $-2 + 3\sqrt{5}$

Challenge

$$a = 4, b = 2$$