P1 Chapter 11: 3D Vectors

Solving Geometric Problems

Geometric Problems

For more general problems involving vectors, often **drawing a diagram** helps!

[Textbook] A, B, C and D are the points $(2, -5, -8)$, $(1, -7, -3)$, $(0, 15, -10)$ and $(2, 19, -20)$ respectively.

a. Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$, giving your answers in the form $p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k}$.
b. Show that the lines AB and DC are parallel and that $\overrightarrow{DC} = 2\overrightarrow{AB}$.
c. Hence describe the quadrilateral $ABCD$.

a ?

b ?

c ?

[Textbook] P, Q and R are the points $(4, -9, -3)$, $(7, -7, -7)$ and $(8, -2, 0)$ respectively. Find the coordinates of the point S so that $PQRS$ forms a parallelogram.

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b. Show that the lines AB and DC are parallel and that $\overrightarrow{DC} = 2\overrightarrow{AB}$.

c. Hence describe the quadrilateral $ABCD$.

a $\overrightarrow{AB} = -\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$

$\overrightarrow{DC} = -2\boldsymbol{i} - 4\boldsymbol{j} + 10\boldsymbol{k}$

b $\overrightarrow{DC} = 2(-\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}) = 2\overrightarrow{AB}$

They are multiples ∴ parallel.

c **AB and DC are parallel but different in length. Therefore $ABCD$ is a trapezium.**

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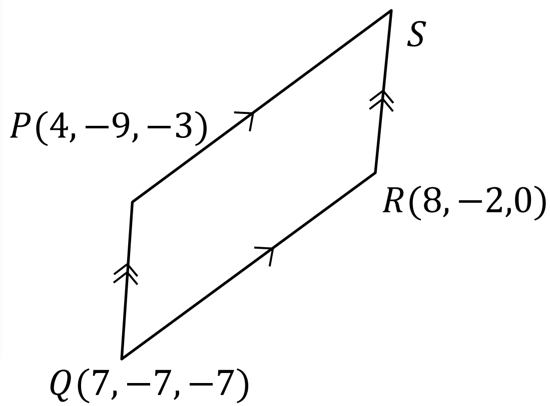


(Draw a diagram, recalling that the letters go in a clockwise or anticlockwise order)

$$\overrightarrow{QP} = \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix}$$

$$\therefore \overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{RS}$$

$$= \overrightarrow{OR} + \overrightarrow{QP}$$

$$= \begin{pmatrix} 8 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 4 \end{pmatrix}$$

$\boldsymbol{S(5, -4, 4)}$

This is basically just saying "whatever we move from Q to P, we do the same movement starting from R"

Comparing Coefficients

There are many contexts in maths where we can 'compare coefficients', e.g.

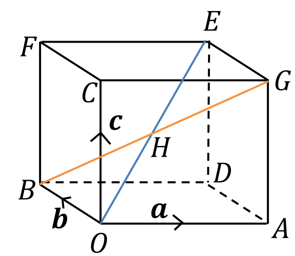
$3x^2 + 5x \equiv A(x^2+1) + Bx + C$
Comparing x^2 terms: $3 = A$

We can do the same with vectors:

[Textbook] Given that $3\boldsymbol{i} + (p+2)\boldsymbol{j} + 120\boldsymbol{k} = p\boldsymbol{i} - q\boldsymbol{j} + 4pqr\boldsymbol{k}$, find the values of p, q and r.

?

[Textbook] The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G. Vectors a, b and c are the position vectors of the vertices A, B and C respectively. Prove that the diagonals OE and BG bisect each other.



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find the values of p, q and r.

Comparing i: $\quad 3 = p$
Comparing j: $\quad p + 2 = -q \quad \therefore q = -5$
Comparing k: $\quad 120 = 4pqr \quad \therefore r = -2$

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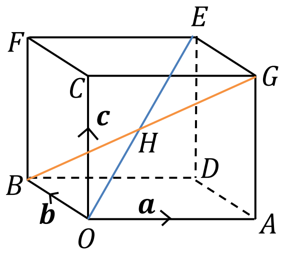


The strategy behind this type of question is to find the point of intersection in 2 ways, and compare coefficients.

Suppose there is a point of intersection H of OE and BG.

We can get to H in two ways:
$\overrightarrow{OH} = r\,\overrightarrow{OE}$ for some scalar r.
$\overrightarrow{OH} = \overrightarrow{OB} + \overrightarrow{BH} = \overrightarrow{OB} + s\,\overrightarrow{BG}$ for some scalar s.

$$\overrightarrow{OH} = r(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{b} + s(\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c})$$
$$r\boldsymbol{a} + r\boldsymbol{b} + r\boldsymbol{c} = s\boldsymbol{a} + (1-s)\boldsymbol{b} + s\boldsymbol{c}$$

Comparing coefficients, $r = s$ and $r = 1 - s$
Adding: $2r = 1 \quad \therefore r = s = \frac{1}{2}$
Therefore lines bisect each other.

Exercise 12C

Pearson Pure Mathematics Year 2/AS
Pages 107-108

Homework Exercise

1 The points A, B and C have position vectors $\begin{pmatrix}1\\-4\\8\end{pmatrix}$, $\begin{pmatrix}4\\4\\7\end{pmatrix}$ and $\begin{pmatrix}10\\0\\30\end{pmatrix}$ relative to a fixed origin, O.

a Show that:

i $|\overrightarrow{OA}| = |\overrightarrow{OB}|$ **ii** $|\overrightarrow{AC}| = |\overrightarrow{BC}|$

b Hence describe the quadrilateral $OACB$.

2 The points A, B and C have coordinates (2, 1, 5), (4, 4, 3) and (2, 7, 5) respectively.

a Show that triangle ABC is isosceles.

b Find the area of triangle ABC.

c Find a point D such that $ABCD$ is a parallelogram.

3 The points A, B, C and D have coordinates (7, 12, −1), (11, 2, −9), (14, −14, 3) and (8, 1, 15) respectively.

a Show that AB and CD are parallel, and find the ratio $AB : CD$ in its simplest form.

b Hence describe the quadrilateral $ABCD$.

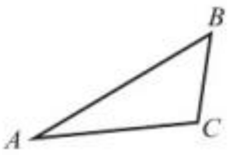
4 Given that $(3a + b)\mathbf{i} + \mathbf{j} + ac\mathbf{k} = 7\mathbf{i} - b\mathbf{j} + 4\mathbf{k}$, find the values of a, b and c.

5 The points A and B have position vectors $10\mathbf{i} - 23\mathbf{j} + 10\mathbf{k}$ and $p\mathbf{i} + 14\mathbf{j} - 22\mathbf{k}$ respectively, relative to a fixed origin O, where p is a constant.

Given that $\triangle OAB$ is isosceles, find **three** possible positions of point B.

6 The diagram shows a triangle ABC.

Given that $\overrightarrow{AB} = 7\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{BC} = -\mathbf{i} + 5\mathbf{k}$



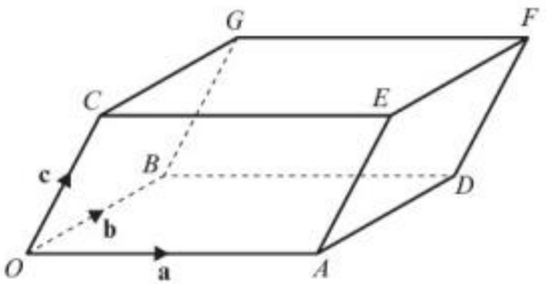
a find the area of triangle ABC. **(7 marks)**

The point D is such that $\overrightarrow{AD} = 3\overrightarrow{AB}$, and the point E is such that $\overrightarrow{AE} = 3\overrightarrow{AC}$.

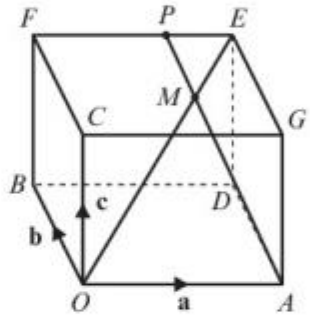
b Find the area of triangle ADE. **(2 marks)**

Homework Exercise

7 A parallelepiped is a three-dimensional figure formed by six parallelograms. The diagram shows a parallelepiped with vertices O, A, B, C, D, E, F, and G.
a, **b** and **c** are the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.
Prove that the diagonals OF and AG bisect each other.



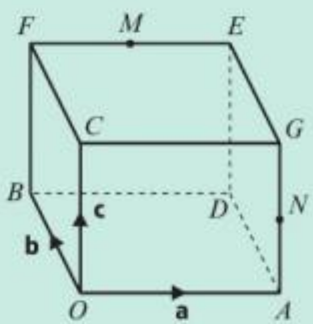
8 The diagram shows a cuboid whose vertices are O, A, B, C, D, E, F and G. **a**, **b** and **c** are the position vectors of the vertices A, B and C respectively. The point M lies on OE such that $OM : ME = 3 : 1$. The straight line AP passes through point M. Given that $AM : MP = 3 : 1$, prove that P lies on the line EF and find the ratio $FP : PE$.



Challenge

1 **a**, **b** and **c** are the vectors $\begin{pmatrix}1\\0\\4\end{pmatrix}$, $\begin{pmatrix}2\\0\\-3\end{pmatrix}$ and $\begin{pmatrix}-5\\3\\1\end{pmatrix}$ respectively. Find scalars p, q and r such that $p\mathbf{a} + q\mathbf{b} + r\mathbf{c} = \begin{pmatrix}28\\-12\\-4\end{pmatrix}$

2 The diagram shows a cuboid with vertices O, A, B, C, D, E, F and G. M is the midpoint of FE and N is the midpoint of AG.
a, **b** and **c** are the position vectors of the vertices A, B and C respectively.
Prove that the lines OM and BN trisect the diagonal AF.



Hint Trisect means divide into three equal parts.

Homework Answers

1 **a** **i** $|\overrightarrow{OA}| = 9$; $|\overrightarrow{OB}| = 9 \Rightarrow |\overrightarrow{OA}| = |\overrightarrow{OB}|$

ii $\overrightarrow{AC} = \begin{pmatrix} 9 \\ 4 \\ 22 \end{pmatrix}$, $|\overrightarrow{AC}| = \sqrt{581}$; $\overrightarrow{BC} = \begin{pmatrix} 6 \\ -4 \\ 23 \end{pmatrix}$, $|\overrightarrow{BC}| = \sqrt{581}$

Therefore $|\overrightarrow{AC}| = |\overrightarrow{BC}|$

b $OACB$ is a kite

2 **a** $\overrightarrow{AB} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \Rightarrow |\overrightarrow{AB}| = \sqrt{17}$

$\overrightarrow{AC} = 6\mathbf{j} \Rightarrow |\overrightarrow{AC}| = 6$

$\overrightarrow{BC} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \Rightarrow |\overrightarrow{BC}| = \sqrt{17}$

$|\overrightarrow{AB}| = |\overrightarrow{BC}|$, so ABC is isosceles.

b $6\sqrt{2}$ **c** (4, 10, 3), (0, 4, 7) or (4, −2, 3)

3 **a** $\overrightarrow{AB} = 4\mathbf{i} - 10\mathbf{j} - 8\mathbf{k} = 2(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = -6\mathbf{i} + 15\mathbf{j} + 12\mathbf{k} = -3(2\mathbf{i} - 5\mathbf{j} - 4\mathbf{k})$

$\overrightarrow{CD} = -\frac{3}{2}\overrightarrow{AB}$, so AB is parallel to CD

$AB : CD = 2 : 3$

b $ABCD$ is a trapezium

4 $a = \frac{8}{3}$, $b = -1$, $c = \frac{3}{2}$

5 (7, 14, −22), (−7, 14, −22) and $\left(\frac{1813}{20}, 14, -22\right)$

6 **a** 18.67 (2 d.p.) **b** 168.07 (2 d.p.)

7 Let H = point of intersection of OF and AG.

$\overrightarrow{OH} = r\overrightarrow{OF} = \overrightarrow{OA} + s\overrightarrow{AG}$

$\overrightarrow{OF} = \mathbf{a} + \mathbf{b} + \mathbf{c}$, $\overrightarrow{AG} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$

So $r(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} + s(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$r = 1 - s = s \Rightarrow r = s = \frac{1}{2}$, so $\overrightarrow{OH} = \frac{1}{2}\overrightarrow{OF}$ and $\overrightarrow{AH} = \frac{1}{2}\overrightarrow{AG}$.

8 Show that $\overrightarrow{FP} = \frac{2}{3}\mathbf{a}$ (multiple methods possible)

Show that $\overrightarrow{PE} = \frac{1}{3}\mathbf{a}$ (multiple methods possible)

Therefore FP and PE are parallel, so P lies on FE

$FP{:}PE = 2{:}1$

Homework Answers

Challenge

1 $p = \frac{24}{11}, q = \frac{32}{11}, r = -4$

2 $\overrightarrow{OM} = \frac{1}{2}\mathbf{a} + \mathbf{b} + \mathbf{c}, \overrightarrow{BN} = \mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}, \overrightarrow{AF} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$

Let $\overrightarrow{OM}$ and $\overrightarrow{AF}$ intersect at X: $\overrightarrow{AX} = r\overrightarrow{AF} = r(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{OX} = s\overrightarrow{OM} = s\left(\frac{1}{2}\mathbf{a} + \mathbf{b} + \mathbf{c}\right)$ for scalars r and s

$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \mathbf{a} + r(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\Rightarrow s\left(\frac{1}{2}\mathbf{a} + \mathbf{b} + \mathbf{c}\right) = \mathbf{a} + r(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

Comparing coefficients in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ gives $r = s = \frac{2}{3}$

Let $\overrightarrow{BN}$ and $\overrightarrow{AF}$ intersect at Y: $\overrightarrow{AY} = p\overrightarrow{AF} = p(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{BY} = q\overrightarrow{BN} = q\left(\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}\right)$ for scalars p and q

$\overrightarrow{BY} = \overrightarrow{BA} + \overrightarrow{AY} = \mathbf{a} - \mathbf{b} + p(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\Rightarrow q\left(\mathbf{a} - \mathbf{b} + \frac{1}{2}\mathbf{c}\right) = \mathbf{a} - \mathbf{b} + p(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

Comparing coefficients in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ gives $p = \frac{1}{3}, q = \frac{2}{3}$

$\overrightarrow{AX} = \frac{2}{3}\overrightarrow{AF}, \overrightarrow{AY} = \frac{1}{3}\overrightarrow{AF}$

So the line segments OM and BN trisect the diagonal AF.