M2 Chapter 8: Further Kinematics

Chapter Practice

Key Points

1 If a particle starts from the point with position vector $\mathbf{r}_0$ and moves with constant velocity $\mathbf{v}$, then its displacement from its initial position at time t is $\mathbf{v}t$ and its position vector $\mathbf{r}$ is given by $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$.

2 For an object moving in a plane with constant acceleration:

- $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
- $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

where

- $\mathbf{u}$ is the initial velocity
- $\mathbf{a}$ is the acceleration
- $\mathbf{v}$ is the velocity at time t
- $\mathbf{r}$ is the displacement at time t.

3 If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$

and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$

4 $\mathbf{v} = \int \mathbf{a}\,dt$ and $\mathbf{r} = \int \mathbf{v}\,dt$

Chapter Exercises

1 A constant force $\mathbf{F}$ N acts on a particle of mass 4 kg for 5 seconds. The particle was initially at rest, and after 5 seconds it has velocity $6\mathbf{i} - 8\mathbf{j}\,\mathrm{m\,s^{-1}}$. Find $\mathbf{F}$.

2 A force $2\mathbf{i} - \mathbf{j}$ N acts on a particle of mass 2 kg. If the initial velocity of the particle is $\mathbf{i} + 3\mathbf{j}\,\mathrm{m\,s^{-1}}$, find the distance of the particle from its initial position after 3 seconds.

3 In this question $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors due east and north respectively. At 2 pm the coastguard spots a rowing dinghy 500 m due south of a fixed observation point. The dinghy has constant velocity $(2\mathbf{i} + 3\mathbf{j})\,\mathrm{m\,s^{-1}}$.

a Find, in terms of t, the displacement vector of the dinghy relative to the observation point t seconds after 2 pm.

b Find the distance of the dinghy from the observation point at 2.05 pm.

4 In this question $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors due east and north respectively. At 8 am two ships A and B have position vectors $\mathbf{r}_A = (\mathbf{i} + 3\mathbf{j})$ km and $\mathbf{r}_B = (5\mathbf{i} - 2\mathbf{j})$ km relative to a fixed origin, O. Their velocities are $\mathbf{v}_A = (2\mathbf{i} - \mathbf{j})\,\mathrm{km\,h^{-1}}$ and $\mathbf{v}_B = (-\mathbf{i} + 4\mathbf{j})\,\mathrm{km\,h^{-1}}$ respectively.

a Write down the position vectors of A and B t hours later. **(3 marks)**

b Show that t hours after 8 am the displacement vector of B relative to A is given by

$$((4 - 3t)\mathbf{i} + (-5 + 5t)\mathbf{j})\,\mathrm{km}$$ **(2 marks)**

c Show that the two ships do not collide. **(3 marks)**

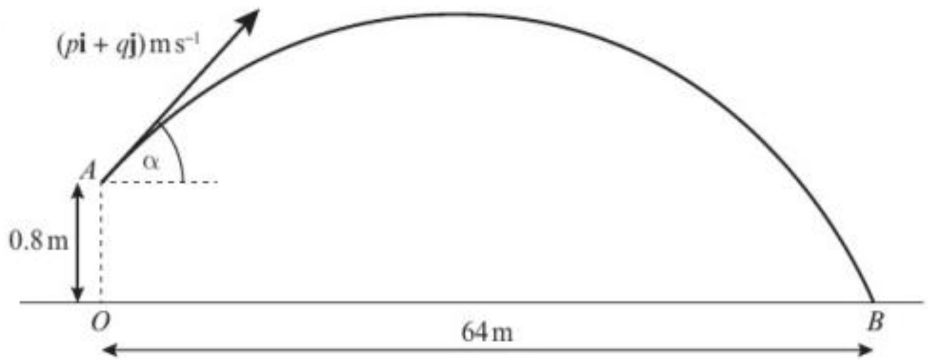
d Find the distance between A and B at 10 am. **(3 marks)**

Chapter Exercises

5 A particle is projected with velocity $(8\mathbf{i} + 10\mathbf{j})\,\text{m s}^{-1}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors horizontally and vertically respectively, from a point O at the top of a cliff and moves freely under gravity. Six seconds after projection, the particle strikes the sea at the point S. Calculate:

a the horizontal distance between O and S

b the vertical distance between O and S.

6 

A cricket ball is hit from a point A with velocity of $(p\mathbf{i} + q\mathbf{j})\,\text{m s}^{-1}$, at an angle α above the horizontal. $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors horizontally and vertically upwards respectively. The point A is 0.8 m vertically above the point O, which is on horizontal ground.

The ball takes 4 seconds to travel from A to B, where B is on the ground and $OB = 64$ m, as shown in the diagram. By modelling the motion of the ball as that of a particle moving freely under gravity,

a find the value of p and the value of q **(5 marks)**

b find the initial speed of the ball **(2 marks)**

c find the exact value of $\tan\alpha$ **(1 mark)**

d find the length of time for which the cricket ball is at least 5 m above the ground. **(6 marks)**

e State an additional physical factor which may be taken into account in a refinement of the above model to make it more realistic. **(1 mark)**

Chapter Exercises

7 A particle P moves in a straight line in such a way that, at time t seconds, its velocity, $\mathbf{v}\,\mathrm{m\,s^{-1}}$, is given by

$$\mathbf{v} = \begin{cases} t\sqrt{14 + 2t^2}, & 0 \leqslant t \leqslant 5 \\ \dfrac{1000}{t^2}, & t > 5 \end{cases}$$

When $t = 0$, P is at the point O. Calculate the displacement of P from O:

a when $t = 5$ **(3 marks)**

b when $t = 6$. **(3 marks)**

8 A particle P of mass 0.4 kg is moving in a straight line under the action of a single variable force of magnitude F N. At time t seconds (where $t \geqslant 0$) the displacement x m of P from a fixed point O is given by $x = 2t + \frac{k}{t+1}$, where k is a constant. Given that when $t = 0$, the velocity of P is $6\,\mathrm{m\,s^{-1}}$, find:

a the value of k **(5 marks)**

b the distance of P from O when $t = 0$ **(1 mark)**

c the value of F when $t = 3$. **(4 marks)**

9 A ball, attached to the end of an elastic string, is moving in a vertical line. The motion of the ball is modelled as a particle B moving along a vertical line so that its displacement, x m, from a fixed point O on the line at time t seconds is given by $x = 0.6\cos\left(\frac{\pi t}{3}\right)$. Find:

a the distance of B from O when $t = \frac{1}{2}$ **(2 marks)**

b the smallest positive value of t for which B is instantaneously at rest **(4 marks)**

c the magnitude of the acceleration of B when $t = 1$. Give your answer to 3 significant figures. **(3 marks)**

10 A light spot S moves along a straight line on a screen. At time $t = 0$, S is at a point O. At time t seconds (where $t \geqslant 0$) the distance, x cm, of S from O is given by $x = 4t\mathrm{e}^{-0.5t}$. Find:

a the acceleration of S when $t = \ln 4$ **(5 marks)**

b the greatest distance of S from O. **(2 marks)**

11 Two particles P and Q move in a plane so that at time t seconds, where $t \geqslant 0$, P and Q have position vectors $\mathbf{r}_P$ metres and $\mathbf{r}_Q$ metres respectively, relative to a fixed origin O, where

$$\mathbf{r}_P = (3t^2 + 4)\mathbf{i} + \left(2t - \tfrac{1}{2}\right)\mathbf{j}$$

$$\mathbf{r}_Q = (t + 6)\mathbf{i} + \frac{3t^2}{2}\mathbf{j}$$

Find:

a the velocity vectors of P and Q at time t seconds **(5 marks)**

b the speed of P when $t = 2$ **(2 marks)**

c the value of t at the instant when the particles are moving parallel to one another. **(4 marks)**

d Show that the particles collide and find the position vector of their point of collision. **(6 marks)**

12 At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = (3t^2 - 4)\mathbf{i} + (8 - 4t^2)\mathbf{j}$$

a Show that the acceleration of P is a constant.

b Find the magnitude of the acceleration of P and the size of the angle which the acceleration makes with $\mathbf{j}$.

13 At time t seconds, a particle P has position vector $\mathbf{r}$ m with respect to a fixed origin O, where

$$\mathbf{r} = 2\cos 3t\mathbf{i} - 2\sin 3t\mathbf{j}$$

a Find the velocity of P when $t = \frac{\pi}{6}$ **(5 marks)**

b Show that the magnitude of the acceleration of P is constant. **(4 marks)**

14 A particle of mass 0.5 kg is acted upon by a variable force $\mathbf{F}$. At time t seconds, the velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$ is given by $\mathbf{v} = (4ct - 6)\mathbf{i} + (7 - c)t^2\mathbf{j}$, where c is a constant.

a Show that $\mathbf{F} = (2c\mathbf{i} + (7 - c)t\mathbf{j})$ N. **(4 marks)**

b Given that when $t = 5$ the magnitude of $\mathbf{F}$ is 17 N, find the possible values of c. **(5 marks)**

15 At time t seconds (where $t \geqslant 0$) the particle P is moving in a plane with acceleration $\mathbf{a}\,\mathrm{m\,s^{-2}}$, where $\mathbf{a} = (8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}$.

When $t = 2$, the velocity of P is $(16\mathbf{i} + 3\mathbf{j})\,\mathrm{m\,s^{-1}}$. Find:

a the velocity of P after t seconds **(3 marks)**

b the value of t when P is moving parallel to $\mathbf{i}$. **(4 marks)**

16 A particle P moves so that its acceleration $\mathbf{a}\,\mathrm{m\,s^{-2}}$ at time t seconds, where $t \geqslant 0$, is given by

$$\mathbf{a} = 4t\mathbf{i} + 5t^{-\frac{1}{2}}\mathbf{j}$$

When $t = 0$, the velocity of P is $10\mathbf{i}\,\mathrm{m\,s^{-1}}$.

Find the speed of P when $t = 5$. **(6 marks)**

Chapter Exercises

17 In this question $\mathbf{i}$ and $\mathbf{j}$ are horizontal unit vectors due east and due north respectively.

A clockwork train is moving on a flat, horizontal floor. At time $t = 0$, the train is at a fixed point O and is moving with velocity $3\mathbf{i} + 13\mathbf{j}\,\text{m s}^{-1}$. The velocity of the train at time t seconds is $\mathbf{v}\,\text{m s}^{-1}$, and its acceleration, $\mathbf{a}\,\text{m s}^{-2}$, is given by $\mathbf{a} = 2t\mathbf{i} + 3\mathbf{j}$.

a Find $\mathbf{v}$ in terms of t. **(3 marks)**

b Find the value of t when the train is moving in a north-east direction. **(3 marks)**

Challenge

1 A particle moves on the positive x-axis such that its displacement, s m, from O at time t seconds is given by

$$s = (20 - t^2)\sqrt{t+1},\ t \geqslant 0$$

a State the initial displacement of the particle.

b Show that the particle changes direction exactly once and determine the time at which this occurs.

c Find the exact speed of the particle when it crosses O.

2 Relative to a fixed origin O, the particle R has position vector $\mathbf{r}$ metres at time t seconds, where

$$\mathbf{r} = (6\sin\omega t)\mathbf{i} + (4\cos\omega t)\mathbf{j}$$

and ω is a positive constant.

a Find $\dot{\mathbf{r}}$ and hence show that $v^2 = 2\omega^2(13 + 5\cos 2\omega t)$, where $v\,\text{m s}^{-1}$ is the speed of R at time t seconds.

b Deduce that $4\omega \leqslant v \leqslant 6\omega$.

c At the instant when $t = \frac{\pi}{3\omega}$, find the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$, giving your answer in degrees to one decimal place.

Chapter Answers

1 $4.8\mathbf{i} - 6.4\mathbf{j}$
2 10.1 m
3 a $2t\mathbf{i} + (-500 + 3t)\mathbf{j}$ b 721 m
4 a $(1 + 2t)\mathbf{i} + (3 - t)\mathbf{j}$, $(5 - t)\mathbf{i} + (-2 + 4t)\mathbf{j}$
b $\mathbf{r}_{BA} = \mathbf{r}_B - \mathbf{r}_A = (5 - t)\mathbf{i} + (3 - t)\mathbf{j} - ((1 + 2t)\mathbf{i} + (4t - 2)\mathbf{j})$
$= (4 - 3t)\mathbf{i} + (5 - 5t)\mathbf{j}$
c For A and B to collide $\mathbf{r}_A = \mathbf{r}_B$.
Equating $\mathbf{i} \rightarrow t = \frac{4}{3}$, equating $\mathbf{j} \rightarrow t = 1$. Times are not the same therefore the ships do not collide.
d 5.39 km
5 a 48 m b 120 m (2 s.f.)
6 a $p = 16$, $q = 19.4$ b $25.1\,\text{m}\,\text{s}^{-1}$
c $\frac{97}{80}$ d 3.50 s (3 s.f.)
e e.g. weight of the ball, air resistance
7 a 76.6 m (3 s.f.) b 110 m (3 s.f.)
8 a $k = -4$ b 4 m c 0.05
9 a $0.3\sqrt{3}$ m b $t = 3$ c $0.329\,\text{m}\,\text{s}^{-2}$ (3 s.f.)
10 a $(\ln 2 - 2)\,\text{m}\,\text{s}^{-2}$ in the direction of x increasing.
b $\frac{8}{e}$ m
11 a V_P is $(6t\mathbf{i} + 2\mathbf{j})\,\text{m}\,\text{s}^{-1}$ and V_Q is $(\mathbf{i} + 3t\mathbf{j})\,\text{m}\,\text{s}^{-1}$
b $12.2\,\text{m}\,\text{s}^{-1}$ (3 s.f.)
c $t = \frac{1}{3}$
d Equate **i**-components and solve to get $t = 1$.
Equate **j**-components and solve to get $t = \frac{1}{3}$ or 1.
So $t = 1$ and $r = (7\mathbf{i} + \frac{3}{2}\mathbf{j})$ m

12 a Differentiate: $\mathbf{v} = 6t\mathbf{i} - 8t\mathbf{j}$, $\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$ so constant
b $10\,\text{m}\,\text{s}^{-2}$
143.1° (nearest 0.1°)
13 a $-6\mathbf{i}\,\text{m}\,\text{s}^{-1}$
b Differentiate: $\mathbf{v} = -6\sin 3t\mathbf{i} - 6\cos 3t\mathbf{j}$,
$\mathbf{a} = -18\cos 3t\mathbf{i} + 18\sin 3t\mathbf{j}$
$|\mathbf{a}| = \sqrt{18^2(\cos^2 t + \sin^2 t)} = 18\,\text{m}\,\text{s}^{-2}$ so constant
14 a Differentiate: $\mathbf{a} = 4c\mathbf{i} + (14 - 2c)t\mathbf{j}$,
$\mathbf{F} = m\mathbf{a} = \frac{1}{2}(4c\mathbf{i} + (14 - 2c)t\mathbf{j})$
b $4, \frac{234}{29} \approx 8.07$
15 a $((2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j})\,\text{m}\,\text{s}^{-1}$
b $t = \frac{7}{4}$
16 $10\sqrt{41}\,\text{m}\,\text{s}^{-1}$
17 a $\mathbf{v} = (2t^2 + 3)\mathbf{i} + (3t + 13)\mathbf{j}$ b 3.11 s (3 s.f.)

Challenge

1 a 20
b $v = 0$ when $t = 1.64$ s (or -2.44 s) so only changes direction once
c $-2\sqrt{20}(\sqrt{20} + 1)^{\frac{1}{2}}$
2 a Differentiate: $\dot{\mathbf{r}} = (6\omega\cos\omega t)\mathbf{i} - (4\omega\sin\omega t)\mathbf{j}$
$|\dot{\mathbf{r}}|^2 = (36\omega^2\cos^2\omega t)\mathbf{i} + (16\omega^2\sin^2\omega t)\mathbf{j}$
use $\sin^2\omega t + \cos^2\omega t = 1$ and $2\cos^2\omega t = \cos 2\omega t + 1$
b $v = \sqrt{26\omega^2 + 10\omega^2\cos 2\omega t}$ max when $\cos 2\omega t = 1$ and min when $\cos 2\omega t = -1$
c 109.8° (1 d.p.)