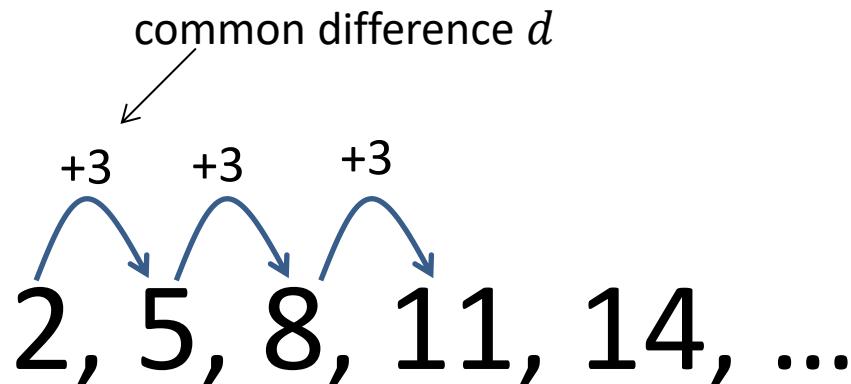


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## P2 Chapter 3: Sequences and Series

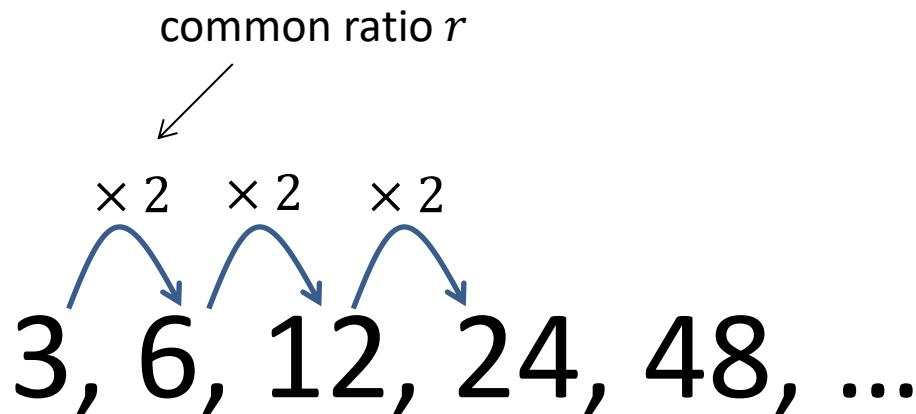
### Arithmetic Series

# Types of sequences



This is an ...

Arithmetic Sequence



An arithmetic sequence is one which has a common difference between terms.

Geometric Sequence  
(We will explore these later in the chapter)

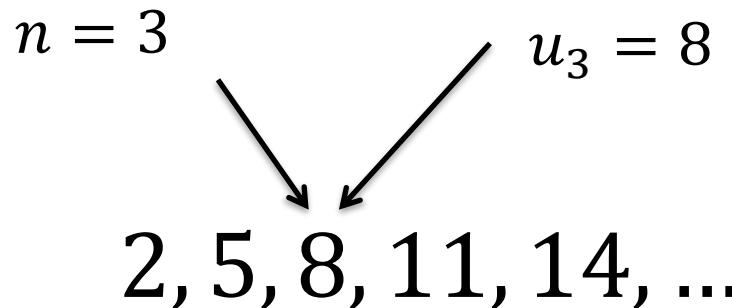
$1, 1, 2, 3, 5, 8, \dots$

This is the **Fibonacci Sequence**. The terms follow a **recurrence relation** because each term can be generated using the previous ones. We will encounter recurrence relations later in the chapter.

# The fundamentals of sequences

$u_n$  The  $n^{\text{th}}$  **term**. So  $u_3$  would refer to the 3<sup>rd</sup> term.

$n$  The **position** of the term in the sequence.



# $n^{\text{th}}$ term of an arithmetic sequence

We use  $a$  to denote the **first term**.  $d$  is the **difference** between terms, and  $n$  is the **position** of the term we're interested in. Therefore:

1<sup>st</sup> Term

2<sup>nd</sup> Term

3<sup>rd</sup> Term

...

$n^{\text{th}}$  term

$$a \quad a + d \quad a + 2d \quad \dots \quad a + (n - 1)d$$



**$n^{\text{th}}$  term of arithmetic sequence:**

$$u_n = a + (n - 1)d$$

## Example 1

The  $n^{\text{th}}$  term of an arithmetic sequence is

$$u_n = 55 - 2n.$$

- Write down the first 3 terms of the sequence.
- Find the first term in the sequence that is negative.

a)  $u_1 = 55 - 2(1) = 53$ ,  $u_2 = 51$ ,  $u_3 = 49$

b)  $55 - 2n < 0 \rightarrow n > 27.5 \therefore n = 28$

$u_{28} = 55 - 2(28) = -1$

## Example 2

Find the  $n^{\text{th}}$  term of each arithmetic sequence.

- 6, 20, 34, 48, 62
- 101, 94, 87, 80, 73

**For Tip:** Always write out  $a =$ ,  $d =$ ,  $n =$  first.

a)  $a = 6, d = 14$

$$\begin{aligned} u_n &= 6 + 14(n - 1) \\ &= 14n - 8 \end{aligned}$$

b)  $a = 101, d = -7$

$$\begin{aligned} u_n &= 101 - 7(n - 1) \\ &= 108 - 7n \end{aligned}$$

# Further Examples

[Textbook] A sequence is generated by the formula  $u_n = an + b$  where  $a$  and  $b$  are constants to be found.

Given that  $u_3 = 5$  and  $u_8 = 20$ , find the values of the constants  $a$  and  $b$ .

?

For which values of  $x$  would the expression  $-8, x^2$  and  $17x$  form the first three terms of an arithmetic sequence.

?

# Further Examples

[Textbook] A sequence is generated by the formula  $u_n = an + b$  where  $a$  and  $b$  are constants to be found.

Given that  $u_3 = 5$  and  $u_8 = 20$ , find the values of the constants  $a$  and  $b$ .

$$u_3 = 5 \rightarrow a + 2d = 5$$

$$u_8 = 20 \rightarrow a + 7d = 20$$

Solving simultaneously,  $a = 3, b = -4$

For which values of  $x$  would the expression  $-8, x^2$  and  $17x$  form the first three terms of an arithmetic sequence.

**Remember that an arithmetic sequence is one where there is a common difference between terms.**

$$x^2 + 8 = 17x - x^2$$

$$2x^2 - 17x + 8 = 0 \rightarrow (2x - 1)(x - 8) = 0$$

$$x = \frac{1}{2}, x = 8$$

# Test Your Understanding

## Edexcel C1 May 2014(R) Q10

Xin has been given a 14 day training schedule by her coach.

Xin will run for  $A$  minutes on day 1, where  $A$  is a constant.

She will then increase her running time by  $(d + 1)$  minutes each day, where  $d$  is a constant.

- (a) Show that on day 14, Xin will run for

$$(A + 13d + 13) \text{ minutes.} \quad (2)$$

Yi has also been given a 14 day training schedule by her coach.

Yi will run for  $(A - 13)$  minutes on day 1.

She will then increase her running time by  $(2d - 1)$  minutes each day.

Given that Yi and Xin will run for the same length of time on day 14,

- (b) find the value of  $d$ . (3)

(a).

?

(b)

?

# Test Your Understanding

Edexcel C1 May 2014(R) Q10

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Given that Yi and Xin will run for the same length of time on day 14,

- (b) find the value of  $d$ .

(3)

- (a.) Attempts to use  $a + (n-1)d$  with  $a=A$  and " $d$ " =  $d+1$  and  $n = 14$   
 $A + 13(d+1) = A + 13d + 13 *$

- M1  
A1\*

- (b) Calculates time for Yi on Day 14 =  $(A - 13) + 13(2d - 1)$   
 Sets times equal  $A + 13d + 13 = (A - 13) + 13(2d - 1) \Rightarrow d = \dots$   
 $d = 3$

- M1  
M1  
A1 cso

# Exercise 3.1

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## Extension

- 1 [STEP I 2004 Q5] The positive integers can be split into five distinct arithmetic progressions, as shown:

- A: 1, 6, 11, 16, ...
- B: 2, 7, 12, 17, ...
- C: 3, 8, 13, 18, ...
- D: 4, 9, 14, 19, ...
- E: 5, 10, 15, 20, ...

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

- i) Prove that there are no positive integers  $x$  and  $y$  such that  $x^2 + 5y = 243723$
- ii) Prove also that there are no positive integers  $x$  and  $y$  such that  $x^4 + 2y^4 = 26081974$

?

# Exercise 3.1

## Pearson Pure Mathematics Year 2/AS Page 17

### Extension

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- A: 1, 6, 11, 16, ...
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- i) Prove that there are no positive integers  $x$  and  $y$  such that  $x^2 + 5y = 243723$
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A: $5n - 4$	B: $5n - 3$	C: $5n - 2$
D: $5n - 1$	E: $5n$	

Term in B + Term in C (note that the positions in each sequence could be different, so use  $n$  and  $m$ ):

$$\begin{aligned} & 5n - 3 + 5m - 2 \\ &= 5(n + m - 1) \end{aligned}$$

hence a term in E.

Square of term in B:

$$\begin{aligned} & (5n - 3)^2 = 25n^2 - 30n + 9 \\ &= 5(5n^2 - 6n + 2) - 1 \end{aligned}$$

hence a term in D.

Square of term in C:

$$\begin{aligned} & (5n - 2)^2 = 25n^2 - 20n + 4 \\ &= 5(5n^2 - 4n + 1) - 1 \end{aligned}$$

hence also a term in D.

i) 243723 is a term in C. Note that whatever sequence  $x^2$  is in, adding 5y will keep it in the same sequence as it is a multiple of 5. So we need to prove no term squared can give a term in C.

We have already proved a term in B or in C, squared, gives a term in D. We can similarly show a term in A, squared, stays in A, and a term in E squared stays in E. Finally, a term in D squared gives a term in A. Since the sequences A, B, C, D, E clearly span all positive integers, there cannot be any integer solutions to the equation.

# Homework Exercise

1 For each sequence:

i write down the first 4 terms of the sequence

ii write down  $a$  and  $d$ .

a  $u_n = 5n + 2$

b  $u_n = 9 - 2n$

c  $u_n = 7 + 0.5n$

d  $u_n = n - 10$

2 Find the  $n$ th terms and the 10th terms in the following arithmetic progressions:

a 5, 7, 9, 11, ...

b 5, 8, 11, 14, ...

c 24, 21, 18, 15, ...

d -1, 3, 7, 11, ...

e  $x, 2x, 3x, 4x, \dots$

f  $a, a + d, a + 2d, a + 3d, \dots$

3 Calculate the number of terms in each of the following arithmetic sequences.

a 3, 7, 11, ..., 83, 87

b 5, 8, 11, ..., 119, 122

c 90, 88, 86, ..., 16, 14

d 4, 9, 14, ..., 224, 229

e  $x, 3x, 5x, \dots, 35x$

f  $a, a + d, a + 2d, \dots, a + (n - 1)d$

## Problem-solving

Find an expression for  $u_n$  and set it equal to the final term in the sequence. Solve the equation to find the value of  $n$ .

4 The first term of an arithmetic sequence is 14. The fourth term is 32. Find the common difference.

5 A sequence is generated by the formula  $u_n = pn + q$  where  $p$  and  $q$  are constants to be found. Given that  $u_6 = 9$  and  $u_9 = 11$ , find the constants  $p$  and  $q$ .

# Homework Exercise

- 6 For an arithmetic sequence  $u_3 = 30$  and  $u_9 = 9$ . Find the first negative term in the sequence.
- 7 The 20th term of an arithmetic sequence is 14. The 40th term is -6. Find the value of the 10th term.
- 8 The first three terms of an arithmetic sequence are  $5p$ , 20 and  $3p$ , where  $p$  is a constant. Find the 20th term in the sequence.
- 9 The first three terms in an arithmetic sequence are  $-8$ ,  $k^2$ ,  $17k \dots$   
Find two possible values of  $k$ . (3 marks)
- 10 An arithmetic sequence has first term  $k^2$  and common difference  $k$ , where  $k > 0$ . The fifth term of the sequence is 41. Find the value of  $k$ , giving your answer in the form  $p + q\sqrt{5}$ , where  $p$  and  $q$  are integers to be found. (4 marks)

## Problem-solving

You will need to make use of the condition  $k > 0$  in your answer.

## Challenge

The  $n$ th term of an arithmetic sequence is  $u_n = \ln a + (n - 1) \ln b$  where  $a$  and  $b$  are integers.  $u_3 = \ln 16$  and  $u_7 = \ln 256$ . Find the values of  $a$  and  $b$ .

# Homework Answers

- |           |                  |               |                |                          |                    |                          |          |    |          |    |          |     |
|-----------|------------------|---------------|----------------|--------------------------|--------------------|--------------------------|----------|----|----------|----|----------|-----|
| <b>1</b>  | <b>a</b>         | <b>i</b>      | 7, 12, 17, 22  | <b>ii</b>                | $a = 7, d = 5$     |                          |          |    |          |    |          |     |
|           | <b>b</b>         | <b>i</b>      | 7, 5, 3, 1     | <b>ii</b>                | $a = 7, d = -2$    |                          |          |    |          |    |          |     |
|           | <b>c</b>         | <b>i</b>      | 7.5, 8, 8.5, 9 | <b>ii</b>                | $a = 7.5, d = 0.5$ |                          |          |    |          |    |          |     |
|           | <b>d</b>         | <b>i</b>      | -9, -8, -7, -6 | <b>ii</b>                | $a = -9, d = 1$    |                          |          |    |          |    |          |     |
| <b>2</b>  | <b>a</b>         | $2n + 3, 23$  | <b>b</b>       | $3n + 2, 32$             |                    |                          |          |    |          |    |          |     |
|           | <b>c</b>         | $27 - 3n, -3$ | <b>d</b>       | $4n - 5, 35$             |                    |                          |          |    |          |    |          |     |
|           | <b>e</b>         | $nx, 10x$     | <b>f</b>       | $a + (n - 1)d, a + 9d$   |                    |                          |          |    |          |    |          |     |
| <b>3</b>  | <b>a</b>         | 22            | <b>b</b>       | 40                       | <b>c</b>           | 39                       | <b>d</b> | 46 | <b>e</b> | 18 | <b>f</b> | $n$ |
| <b>4</b>  | $d = 6$          |               | <b>5</b>       | $p = \frac{2}{3}, q = 5$ | <b>6</b>           | -1.5                     |          |    |          |    |          |     |
| <b>7</b>  | 24               |               | <b>8</b>       | -70                      | <b>9</b>           | $k = \frac{1}{2}, k = 8$ |          |    |          |    |          |     |
| <b>10</b> | $-2 + 3\sqrt{5}$ |               |                |                          |                    |                          |          |    |          |    |          |     |

## Challenge

$$a = 4, b = 2$$