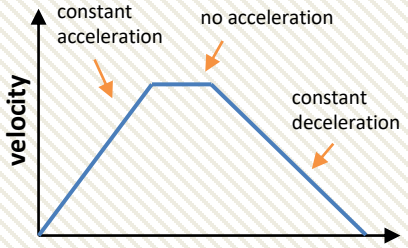
M1 Chapter 11: Variable Acceleration

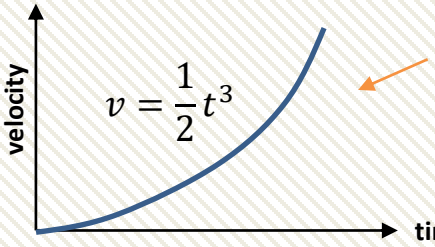
Changing Velocity

Functions of time

Up to now, the acceleration has always been constant in any particular period of time…



However, it's possible to specify either the displacement, velocity or acceleration as any function of time (i.e. an expression in terms of t). This allows the acceleration to constantly change.



Our velocity-time graph can be any shape we want! We can use an expression in terms of t to give a certain shape.

The velocity-time graph of a body is shown above, where $v = \frac{1}{2}t^3$.

(a) What is the velocity after 4 seconds have elapsed?

(b) How many seconds have elapsed when the velocity of the body is 108 ms^{-1}?

a $v = \frac{1}{2} \times 4^3 = 32$ ms^{-1}

b $108 = \frac{1}{2}t^3$

$t^3 = 216 \quad \rightarrow \quad t = 6$ s

[Textbook] A body moves in a straight line such that its velocity, v ms^{-1}, at time t seconds is given by $v = 2t^2 - 16t + 24$. Find

(a) The initial velocity

(b) The values of t when the body is instantaneously at rest.

(c) The value of t when the velocity is 64 ms^{-1}.

(d) The greatest speed of the body in the interval $0 \leq t \leq 5$.

a When $t = 0, v = 24$ ms^{-1}.

b $0 = 2t^2 - 16t + 24$

$t^2 - 8t + 12 = 0$

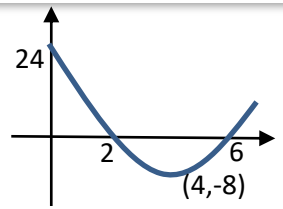
$\rightarrow \quad t = 2$ or $t = 6$.

c $64 = 2t^2 - 16t + 24$

$t^2 - 8t - 20 = 0$

$\rightarrow \quad t = 10$ or ~~$t = -2$~~

d



By the symmetry of a quadratic graph, minimum occurs when $t = 4$.

$v = 2(4^2) - 16(4) + 24 = -8$ ms^{-1}

By inspection, greatest velocity is 24 ms^{-1} within the range $0 \leq t \leq 5$.

Exercise 11.1 Functions of time

Pearson Stats/Mechanics Year 1
Pages 77-78

Using Differentiation

In Chapter 9, we saw that velocity v is the rate of change of displacement s (i.e. the gradient). But in Pure, we know that we can use differentiation to find the gradient function:

$$v = \frac{ds}{dt}$$

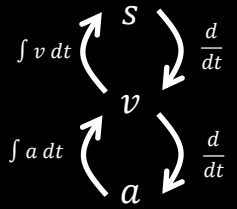
velocity is the rate of change of displacement

and similarly…

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

acceleration is the rate of change of velocity

Memory Tip: I picture interchanging between s, v, a as differentiating to go downwards and integrating to go upwards:



(We will do integration a bit later)

[Textbook] A particle P is moving on the x-axis. At time t seconds, the displacement x metres from O is given by $x = t^4 - 32t + 14$. Find:

(a) the velocity of P when $t = 3$

(b) The value of t when P is instantaneously at rest

(c) The acceleration of P when $t = 1.5$

a $v = \frac{dx}{dt} = 4t^3 - 32$

When $t = 3, v = 4(3^3) - 32 = 76$ ms^{-2}

b $4t^3 - 32 = 0 \quad \rightarrow \quad t = 2$

c $a = \frac{dv}{dt} = 12t^2$

When $t = 1.5, a = 12(1.5^2) = 27$

Test Your Understanding

Pudding the Cat's displacement from a house, in metres, is $t^3 - \frac{3}{2}t^2 - 36t$ where t is in seconds.

(a) Determine the velocity of the cat when $t = 2$.

(b) At what time will the cat be instantaneously at rest?

(c) What is the cat's acceleration after 5 seconds?

a ?

b ?

c ?

Test Your Understanding

Pudding the Cat's displacement from a house, in metres, is $t^3 - \frac{3}{2}t^2 - 36t$ where t is in seconds.
(a) Determine the velocity of the cat when $t = 2$.
(b) At what time will the cat be instantaneously at rest?
(c) What is the cat's acceleration after 5 seconds?

a

$s = t^3 - \frac{3}{2}t^2 - 36t$

$v = 3t^2 - 3t - 36$

When $t = 2$, $v = 3(2^2) - 3(2) - 36 = -30$ ms^{-1}

b

When $v = 0$, $3t^2 - 3t - 36 = 0$

$t^2 - t - 12 = 0$

$(t + 3)(t - 4) = 0$

$t = 4$ s

c

$a = 6t - 3$

When $t = 5$, $a = 30 - 3 = 27$ ms^{-2}

Exercise 11.2 Using Differentiation

Pearson Stats/Mechanics Year 1
Pages 78-79

Homework Exercise

1 A body moves in a straight line such that its displacement, s metres, at time t seconds is given by $s = 9t - t^3$. Find:

a s when $t = 1$ **b** the values of t when $s = 0$.

2 A particle P moves on the x-axis. At time t seconds the displacement s metres is given by $s = 5t^2 - t^3$. Find:

a the change in displacement between $t = 2$ and $t = 4$

b the change in displacement in the third second.

Hint The third second is the time between $t = 2$ and $t = 3$.

3 A particle moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v = 3 + 5t - t^2$ for $t \geqslant 0$. Find:

a the velocity of the particle when $t = 1$

b the greatest speed of the particle in the interval $0 \leqslant t \leqslant 4$

c the velocity of the particle when $t = 7$ and describe the direction of motion of the particle at this time.

4 At time $t = 0$, a toy car is at point P. It moves in a straight line from point P and then returns to P. Its distance from P, s m, at time t seconds can be modelled by $s = \frac{1}{5}(4t - t^2)$. Find:

a the maximum displacement **b** the time taken for the toy car to return to P

c the total distance travelled **d** the values of t for which the model is valid.

5 A body moves in a straight line such that its velocity, $v\,\text{m s}^{-1}$, at time t seconds is given by $v = 3t^2 - 10t + 8$, for $t \geqslant 0$. Find:

a the initial velocity

b the values of t when the body is instantaneously at rest

c the values of t when the velocity is $5\,\text{m s}^{-1}$

d the greatest speed of the body in the interval $0 \leqslant t \leqslant 2$.

Homework Exercise

6 A particle P moves on the x-axis. At time t seconds the velocity of P is $v\,\text{m s}^{-1}$ in the direction of x increasing, where $v = 8t - 2t^2$. When $t = 0$, P is at the origin O. Find:

a the time taken for the particle to come to instantaneous rest **(2 marks)**

b the greatest speed of the particle in the interval $0 \leqslant t \leqslant 4$. **(3 marks)**

7 At time $t = 0$, a particle moves in a straight horizontal line from a point O, then returns to the starting point. The distance, s metres, from the point O at time t seconds is given by:

$$s = 3t^2 - t^3, \quad 0 \leqslant t \leqslant T$$

Given that the model is valid when $s \geqslant 0$, find the value of T. Explain your answer. **(3 marks)**

8 A particle P moves on the x-axis. At time t seconds the velocity of P is $v\,\text{m s}^{-1}$ in the direction of x increasing, where:

$$v = \tfrac{1}{5}(3t^2 - 10t + 3), \qquad x \geqslant 0$$

a Find the values of t when P is instantaneously at rest. **(3 marks)**

b Determine the greatest speed of P in the interval $0 \leqslant t \leqslant 3$. **(4 marks)**

9 Find an expression for **i** the velocity and **ii** the acceleration of a particle given that the displacement is given by:

a $s = 4t^4 - \dfrac{1}{t}$ **b** $x = \dfrac{2}{3}t^3 + \dfrac{1}{t^2}$ **c** $s = (3t^2 - 1)(2t + 5)$ **d** $x = \dfrac{3t^4 - 2t^3 + 5}{2t}$

10 A particle is moving in a straight line. At time t seconds, its displacement, x m, from a fixed point O on the line is given by $x = 2t^3 - 8t$. Find:

a the velocity of the particle when $t = 3$ **b** the acceleration of the particle when $t = 2$.

Homework Exercise

11 A particle P is moving on the x-axis. At time t seconds (where $t \geqslant 0$), the velocity of P is $v\,\text{m s}^{-1}$ in the direction of x increasing, where $v = 12 - t - t^2$.

Find the acceleration of P when P is instantaneously at rest.

12 A particle is moving in a straight line. At time t seconds, its displacement, x m, from a fixed point O on the line is given by $x = 4t^3 - 39t^2 + 120t$.

Find the distance between the two points where P is instantaneously at rest.

13 A particle P moves in a straight line. At time t seconds the acceleration of P is $a\,\text{m s}^{-2}$ and the velocity $v\,\text{m s}^{-1}$ is given by $v = kt - 3t^2$, where k is a constant.

The initial acceleration of P is $4\,\text{m s}^{-2}$.

a Find the value of k. **(3 marks)**

b Using the value of k found in part **a**, find the acceleration when P is instantaneously at rest. **(3 marks)**

14 The print head on a printer moves such that its displacement s cm from the side of the printer at time t seconds is given by:

$$\tfrac{1}{4}(4t^3 - 15t^2 + 12t + 30),\ 0 \leqslant t \leqslant 3$$

Find the distance between the points when the print head is instantaneously at rest, in cm to 1 decimal place. **(6 marks)**

Homework Answers

1 **a** $8\,\text{m}$ **b** $t = 0$ and $t = \pm 3$

2 **a** $4\,\text{m}$ **b** $6\,\text{m}$

3 **a** $7\,\text{m s}^{-1}$ **b** $9.25\,\text{m s}^{-1}$

c $-11\,\text{m s}^{-1}$; body is travelling in opposite direction.

4 **a** $0.8\,\text{m}$ **b** $4\,\text{s}$

c $1.6\,\text{m}$ **d** $0 \leqslant t \leqslant 4$

5 **a** $8\,\text{m s}^{-1}$ **b** $t = \frac{4}{3}$ and $t = 2$

c $t = \frac{1}{3}$ and $t = 3$ **d** $8\,\text{m s}^{-1}$

6 **a** $4\,\text{s}$ **b** $8\,\text{m s}^{-1}$

7 $T = 3$: returns to starting point and $s = 0$ when $t = 0$ and $t = 3$.

8 **a** $t = \frac{1}{3}$ and $t = 3$ **b** $\frac{16}{15}\,\text{m s}^{-1}$

9 **a** **i** $v = 16t^3 + \frac{1}{t^2}$ **ii** $a = 48t^2 - \frac{2}{t^3}$

b **i** $v = 2t^2 - \frac{2}{t^3}$ **ii** $a = 4t + \frac{6}{t^4}$

c **i** $v = 18t^2 + 30t - 2$ **ii** $a = 36t + 30$

d **i** $v = \frac{9t^2}{2} - 2t - \frac{5}{2t^2}$ **ii** $a = 9t - 2 + \frac{5}{t^3}$

10 **a** $46\,\text{m s}^{-1}$ **b** $24\,\text{m s}^{-2}$

11 $7\,\text{m s}^{-2}$ in the direction of x decreasing.

12 $6.75\,\text{m}$

13 **a** $k = 4$ **b** $a = -4\,\text{m s}^{-2}$

14 $1.7\,\text{cm}$