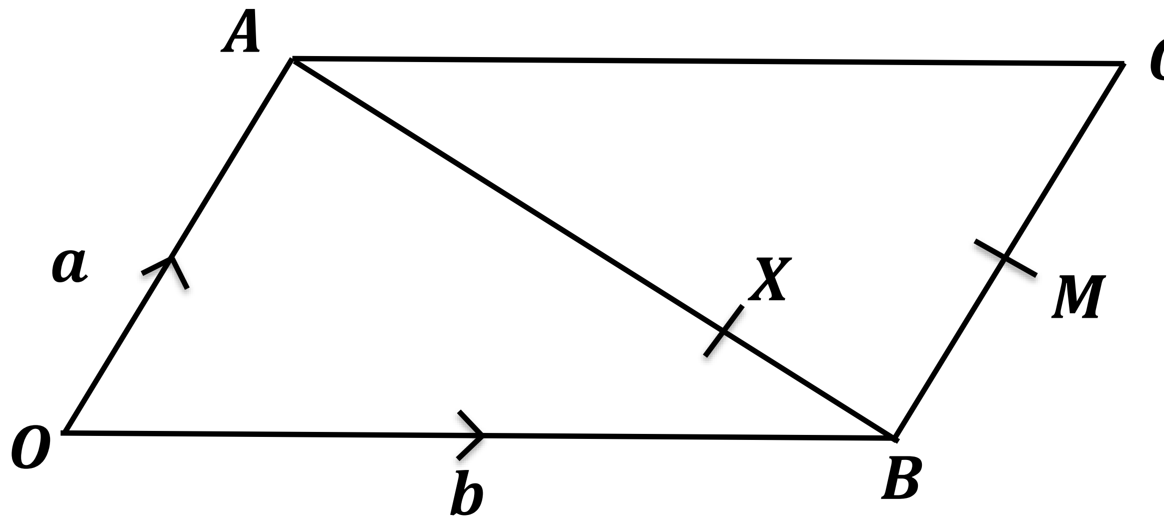
P1 Chapter 11: Vectors

Geometric Problems

Solving Geometric Problems



X is a point on AB such that $AX:XB = 3:1$. M is the midpoint of BC. Show that $\overrightarrow{XM}$ is parallel to $\overrightarrow{OC}$.

$\overrightarrow{OC} = a + b$

$\overrightarrow{XM} = \frac{1}{4}(-a+b) + \frac{1}{2}a = \frac{1}{4}a + \frac{1}{4}b$

$= \frac{1}{4}(a+b)$

$\overrightarrow{XM}$ **<u>is a multiple of</u>** $\overrightarrow{OC}$ $\therefore$ parallel.

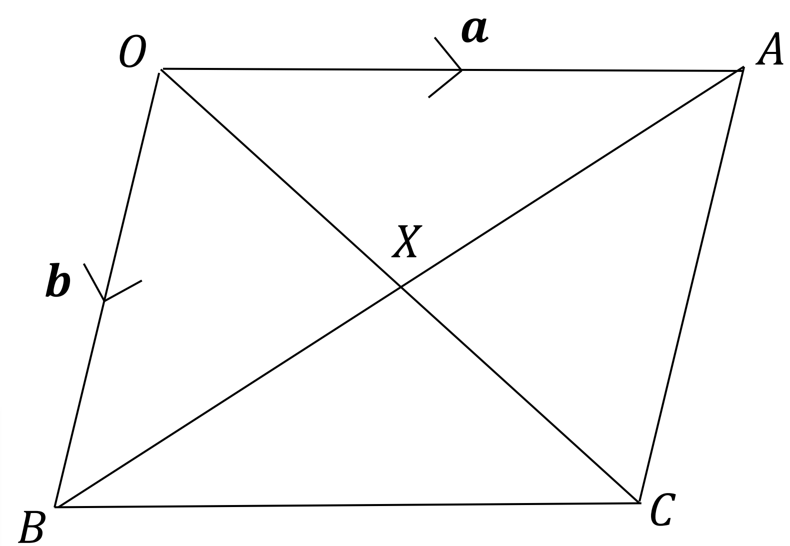
For any proof question always find the vectors involved first, in this case $\overrightarrow{XM}$ and $\overrightarrow{OC}$.

The key is to factor out a scalar such that we see the same vector.

The magic words here are "*is a multiple of*".

Introducing Scalars and Comparing Coefficients

Remember when we had **identities** like: $ax^2 + 3x \equiv 2x^2 + bx$
we could **compare coefficients**, so that $a = 2$ and $3 = b$.
We can do the same with (non-parallel) vectors!



$OACB$ is a parallelogram, where $\overrightarrow{OA} = a$ and $\overrightarrow{OB} = b$. The diagonals OC and AB intersect at a point X. Prove that the diagonals bisect each other.
(Hint: Perhaps find $\overrightarrow{OX}$ in two different ways?)

By considering the route $O \to B \to X$:

$$\begin{aligned}\overrightarrow{OX} &= \boldsymbol{b} + \lambda\,\overrightarrow{BA}\\ &= \boldsymbol{b} + \lambda(-\boldsymbol{b} + \boldsymbol{a}) = \boldsymbol{b} - \lambda\boldsymbol{b} + \lambda\boldsymbol{a}\\ &= \lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}\end{aligned}$$

We don't know what fraction of the way across $\overrightarrow{BA}$ the point X is, so let λ be the fraction of the way across. We're hoping that $\lambda = \frac{1}{2}$, so that X is exactly halfway across and therefore bisects BA.

Similarly, considering the line OC:

$$\overrightarrow{OX} = \mu\,\overrightarrow{OC} = \mu(\boldsymbol{a} + \boldsymbol{b}) = \mu\boldsymbol{a} + \mu\boldsymbol{b}$$

$$\therefore \mu\boldsymbol{a} + \mu\boldsymbol{b} = \lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}$$

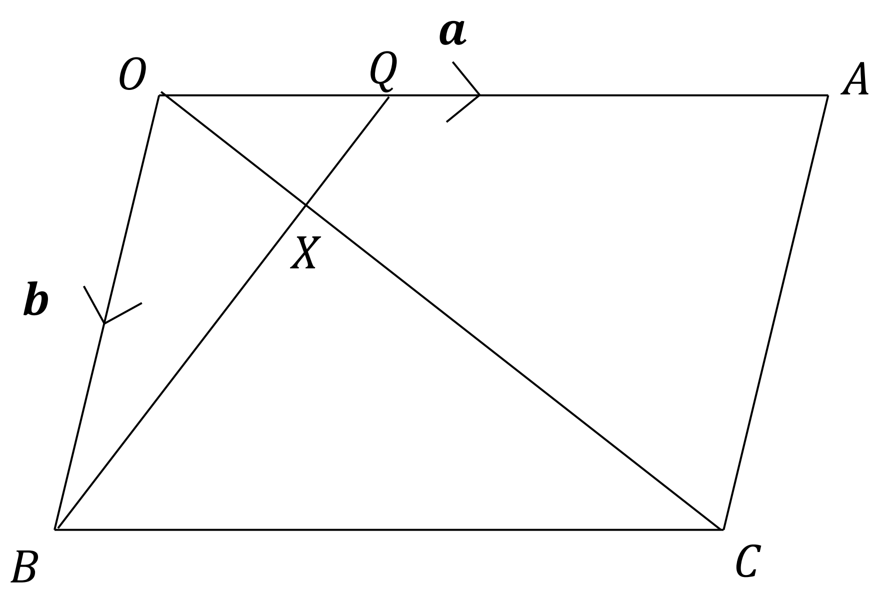
We need to use a different scalar constant, this time μ. It is common to use the letters λ and μ for scalars.

Comparing coefficients of $\boldsymbol{a}$: $\mu = \lambda$
Comparing coefficients of $\boldsymbol{b}$: $\mu = 1 - \lambda$

$$\therefore \mu = \lambda = \frac{1}{2}$$

$\therefore X$ is the midpoint of each of the diagonals.

Test Your Understanding



In the above diagram, $\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{OQ} = \frac{1}{3}\boldsymbol{a}$. We wish to find the ratio $OX: XC$.

a) If $\overrightarrow{OX} = \lambda\,\overrightarrow{OC}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and λ.

b) If $\overrightarrow{BX} = \mu\,\overrightarrow{BQ}$, find an expression for $\overrightarrow{OX}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and μ.

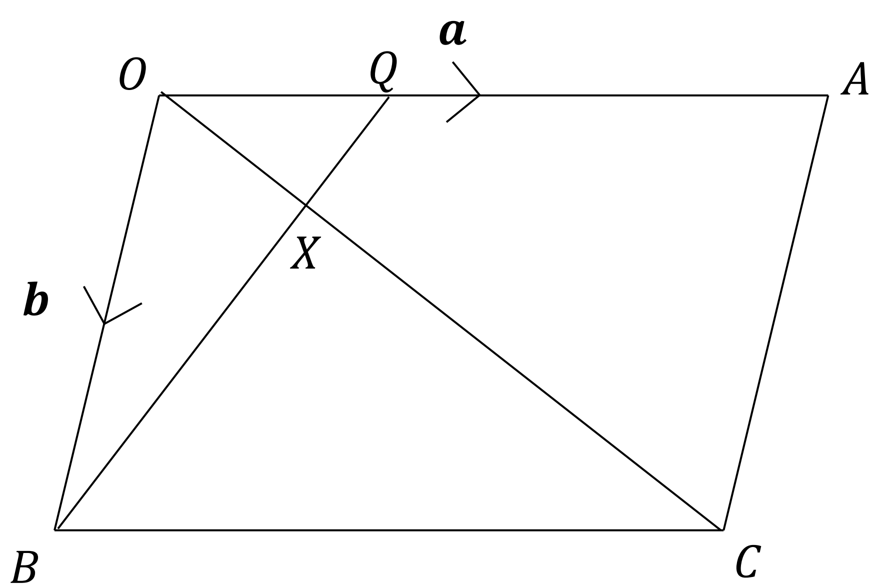
c) By comparing coefficients or otherwise, determine the value of λ, and hence the ratio $OX: XC$.

a ?

b ?

c ?

Test Your Understanding



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c) By comparing coefficients or otherwise, determine the value of λ, and hence the ratio $OX: XC$.

a $\quad \overrightarrow{OX} = \lambda\, \overrightarrow{OC} = \lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$

b $\quad \overrightarrow{OX} = \boldsymbol{b} + \mu\, \overrightarrow{BQ} = \boldsymbol{b} + \mu\left(-\boldsymbol{b} + \frac{1}{3}\boldsymbol{a}\right) = \frac{1}{3}\mu\boldsymbol{a} + (1-\mu)\boldsymbol{b}$

Expand and collect $\boldsymbol{a}$ terms and collect $\boldsymbol{b}$ terms, so that we can compare coefficients later.

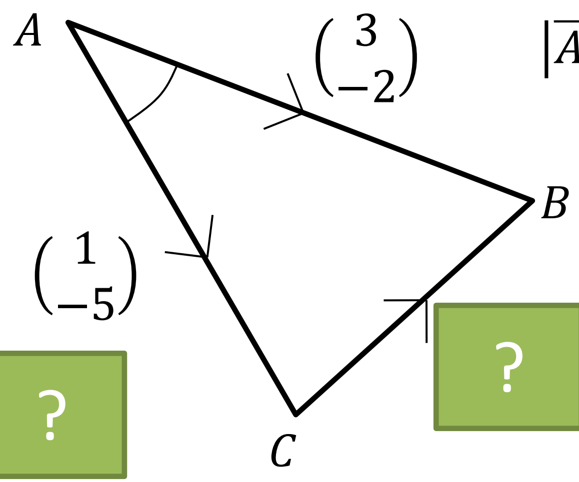
c $\quad$ Comparing coefficients: $\lambda = \frac{1}{3}\mu$ and $\lambda = 1 - \mu$, $\therefore \lambda = \frac{1}{4}$

If $\overrightarrow{OX} = \frac{1}{4}\, \overrightarrow{OC}$, then $OX: XC = 1: 3$.

Area of a Triangle

$\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{AC} = \boldsymbol{i} - 5\boldsymbol{j}$. Determine $\angle BAC$.

Strategy: ?



$|\overrightarrow{AB}| =$?

$|\overrightarrow{CB}| =$?

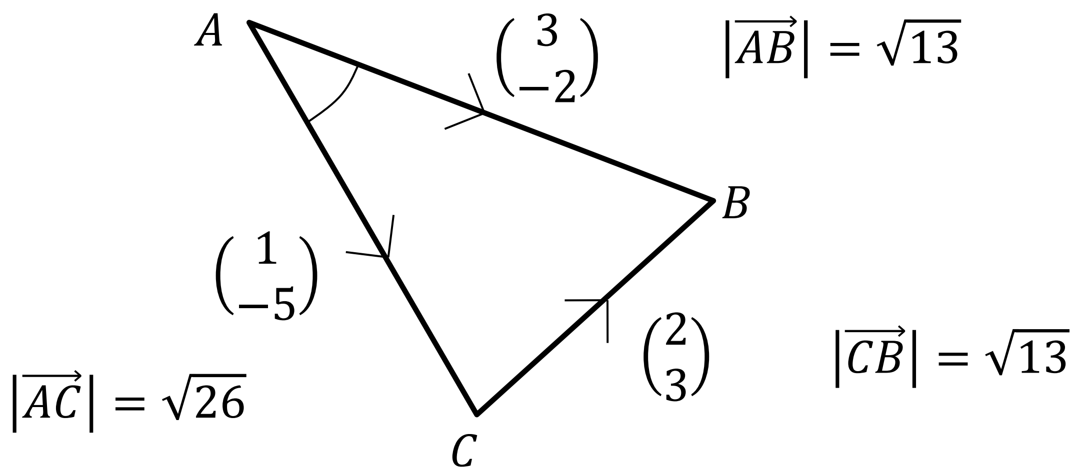
$|\overrightarrow{AC}| =$?

?

Area of a Triangle

$\overrightarrow{AB} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\overrightarrow{AC} = \boldsymbol{i} - 5\boldsymbol{j}$. Determine $\angle BAC$.

Strategy: Find 3 lengths of triangle then use cosine rule to find angle.



A clever student might at this point realise that we can divide all the lengths by $\sqrt{13}$ without changing $\angle BAC$, giving a $1:1:\sqrt{2}$ triangle (one of our 'special' triangles!), and thus instantly getting $\angle BAC = 45°$.
But let's use a more general method of using the **cosine rule**:

$$a = \sqrt{13}, b = \sqrt{13}, c = \sqrt{26}$$
$$13 = 13 + 26 - 2 \times \sqrt{13} \times \sqrt{26} \times \cos(A)$$
$$\cos(A) = \frac{1}{\sqrt{2}} \quad \rightarrow \quad A = 45°$$

Exercise 11.5

Pearson Pure Mathematics Year 1/AS
Pages 89-90

Extension

1 *[STEP 2010 Q7]*
Relative to a fixed origin O, the points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\boldsymbol{c}$ given by

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b},$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\boldsymbol{p}$ and the lines OB and AC meet at the point Q with position vector $\boldsymbol{q}$. Show that

$$\boldsymbol{p} = \frac{\alpha a}{1 - \beta}$$

and write down q in terms of α, β and b.
Show further that the point R with position vector r given by

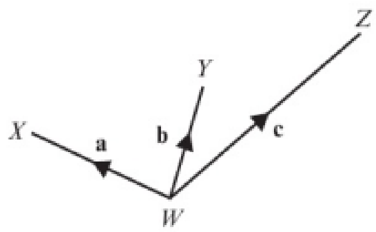
$$\boldsymbol{r} = \frac{\alpha\boldsymbol{a} + \beta\boldsymbol{b}}{\alpha + \beta},$$

lies on the lines OC and AB.
The lines OB and PR intersect at the point S. Prove that $\frac{OQ}{BQ} = \frac{OS}{BS}$.

Click here for the solution:
http://www.mathshelper.co.uk/STEP%202010%20Solutions.pdf
(go to Q7)

Homework Exercise

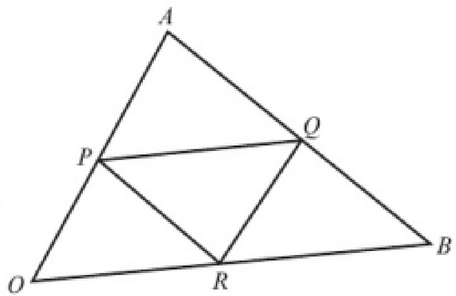
1 In the diagram, $\overrightarrow{WX} = \mathbf{a}$, $\overrightarrow{WY} = \mathbf{b}$ and $\overrightarrow{WZ} = \mathbf{c}$. It is given that $\overrightarrow{XY} = \overrightarrow{YZ}$.
Prove that $\mathbf{a} + \mathbf{c} = 2\mathbf{b}$.



2 OAB is a triangle. P, Q and R are the midpoints of OA, AB and OB respectively.
OP and OR are equal to $\mathbf{p}$ and $\mathbf{r}$ respectively.

a Find **i** $\overrightarrow{OB}$ **ii** $\overrightarrow{PQ}$

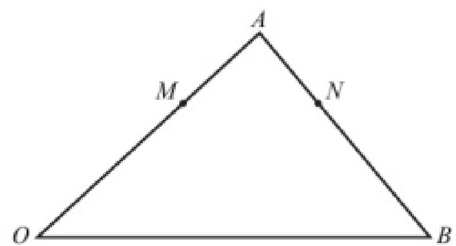
b Hence prove that triangle PAQ is similar to triangle OAB.



3 OAB is a triangle. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
The point M divides OA in the ratio $2:1$.
MN is parallel to OB.

a Express the vector $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

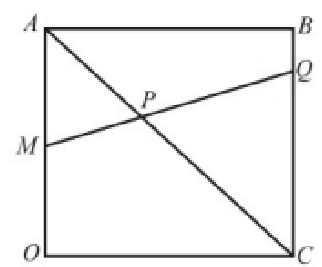
b Show that $AN : NB = 1:2$



4 $OABC$ is a square. M is the midpoint of OA, and Q divides BC in the ratio $1:3$.
AC and MQ meet at P.

a If $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$, express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.

b Show that P divides AC in the ratio $2:3$.



Homework Exercise

5 In triangle ABC the position vectors of the vertices A, B and C are $\begin{pmatrix}5\\8\end{pmatrix}$, $\begin{pmatrix}4\\3\end{pmatrix}$ and $\begin{pmatrix}7\\6\end{pmatrix}$. Find:

a $|\overrightarrow{AB}|$ **b** $|\overrightarrow{AC}|$ **c** $|\overrightarrow{BC}|$

d the size of $\angle BAC$, $\angle ABC$ and $\angle ACB$ to the nearest degree.

6 OPQ is a triangle.

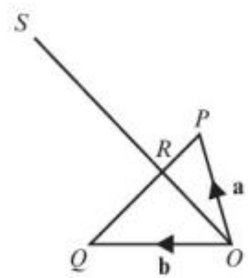
$2\overrightarrow{PR} = \overrightarrow{RQ}$ and $3\overrightarrow{OR} = \overrightarrow{OS}$

$\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OQ} = \mathbf{b}$.

a Show that $\overrightarrow{OS} = 2\mathbf{a} + \mathbf{b}$.

b Point T is added to the diagram such that $\overrightarrow{OT} = -\mathbf{b}$.

Prove that points T, P and S lie on a straight line.

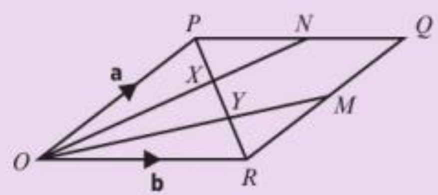


Problem-solving

To show that T, P and S lie on the same straight line you need to show that any **two** of the vectors $\overrightarrow{TP}$, $\overrightarrow{TS}$ or $\overrightarrow{PS}$ are parallel.

Challenge

$OPQR$ is a parallelogram.



N is the midpoint of PQ and M is the midpoint of QR.

$\overrightarrow{OP} = \mathbf{a}$ and $\overrightarrow{OR} = \mathbf{b}$. The lines ON and OM intersect the diagonal PR at points X and Y respectively.

a Explain why $\overrightarrow{PX} = -j\mathbf{a} + j\mathbf{b}$, where j is a constant.

b Show that $\overrightarrow{PX} = (k - 1)\mathbf{a} + \frac{1}{2}k\mathbf{b}$, where k is a constant.

c Explain why the values of j and k must satisfy these simultaneous equations:

$k - 1 = -j$

$\frac{1}{2}k = j$

d Hence find the values of j and k.

e Deduce that the lines ON and OM divide the diagonal PR into 3 equal parts.

Homework Answers

1 $\overrightarrow{XY} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{YZ} = \mathbf{c} - \mathbf{b}$, so $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{b}$.
Hence $\mathbf{a} + \mathbf{c} = 2\mathbf{b}$.

2 **a** **i** $2\mathbf{r}$ **ii** $\mathbf{r}$

b Sides of triangle OAB are twice the length of sides of triangle PAQ and angle A is common to both SAS.

3 **a** $\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$

b $\overrightarrow{AN} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{NB} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$
so $AN : NB = 1 : 2$.

4 **a** $\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{c}$

b $\overrightarrow{AP} = -\mathbf{a} + \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{c} = \frac{2}{5}(\mathbf{c} - \mathbf{a})$,
$\overrightarrow{PC} = \mathbf{c} - (\frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{c}) = \frac{3}{5}(\mathbf{c} - \mathbf{a})$ so $AP : PC = 2 : 3$

5 **a** $\sqrt{26}$ **b** $2\sqrt{2}$ **c** $3\sqrt{2}$

d $\angle BAC = 56°$, $\angle ABC = 34°$, $\angle ACB = 90°$

6 **a** $\overrightarrow{OR} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$,
$\overrightarrow{OS} = 3\overrightarrow{OR} = 3(\frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}) = 2\mathbf{a} + \mathbf{b}$

b $\overrightarrow{TP} = \mathbf{a} + \mathbf{b}$, $\overrightarrow{PS} = \frac{1}{3}(\mathbf{b} - \mathbf{a}) + \frac{2}{3}(2\mathbf{a} + \mathbf{b}) = \mathbf{a} + \mathbf{b}$
$\overrightarrow{\text{TP}}$ is parallel (and equal) to $\overrightarrow{PS}$ and they have a point, P, in common so T, P and S lie on a straight line.

Challenge:

a $\overrightarrow{PR} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{PX} = j(\mathbf{b} - \mathbf{a}) = -j\mathbf{a} + j\mathbf{b}$

b $\overrightarrow{ON} = \mathbf{a} + \frac{1}{2}\mathbf{b}$, $\overrightarrow{PX} = -\mathbf{a} + k(\mathbf{a} + \frac{1}{2}\mathbf{b}) = (k - 1)\mathbf{a} + \frac{1}{2}k\mathbf{b}$

c Coefficients of **a** and **b** must be the same in both expressions for $\overrightarrow{PX}$
Coefficients of **a**: $k - 1 = -j$; Coefficients of **b**: $j = \frac{1}{2}k$

d Solving simultaneously gives $j = \frac{1}{3}$ and $k = \frac{2}{3}$

e $\overrightarrow{PX} = \frac{1}{3}\overrightarrow{PR}$.
By symmetry, $\overrightarrow{PX} = \overrightarrow{YR} = \overrightarrow{XY}$, so ON and OM divide PR into 3 equal parts.