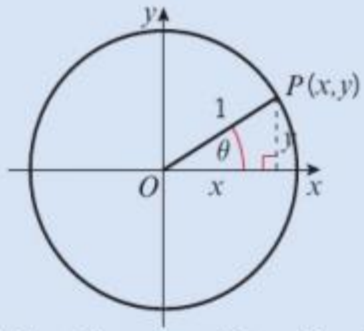
P1 Chapter 10: Trigonometry Equations

Chapter Practice

Key Points

1 For a point $P(x, y)$ on a unit circle such that OP makes an angle θ with the positive x-axis:

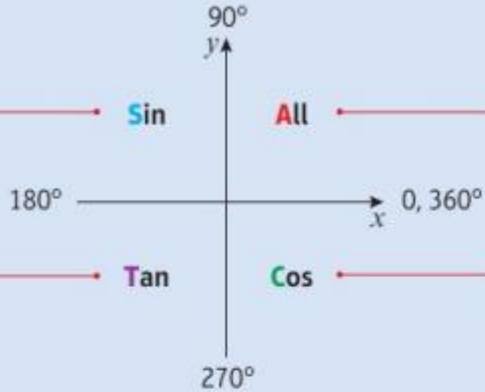
- $\cos\theta = x = x$-coordinate of P
- $\sin\theta = y = y$-coordinate of P
- $\tan\theta = \frac{y}{x} =$ gradient of OP



2 You can use the quadrants to determine whether each of the trigonometric ratios is positive or negative.

For an angle θ in the second quadrant, only $\sin\theta$ is positive.

For an angle θ in the first quadrant, $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all positive.



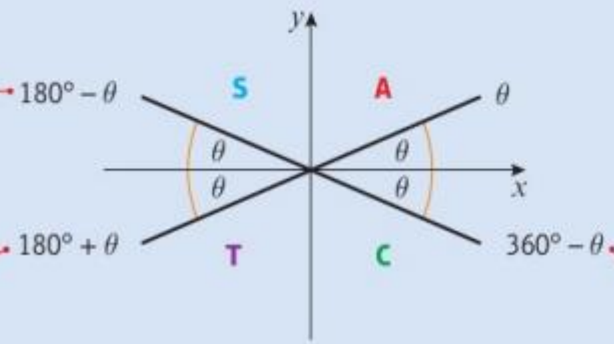
For an angle θ in the third quadrant, only $\tan\theta$ is positive.

For an angle θ in the fourth quadrant, only $\cos\theta$ is positive.

3 You can use these rules to find sin, cos or tan of any positive or negative angle using the corresponding **acute** angle made with the x-axis, θ.

$\sin(180° - \theta) = \sin\theta$
$\sin(180° + \theta) = -\sin\theta$
$\sin(360° - \theta) = -\sin\theta$

$\tan(180° - \theta) = -\tan\theta$
$\tan(180° + \theta) = \tan\theta$
$\tan(360° - \theta) = -\tan\theta$



$\cos(180° - \theta) = -\cos\theta$
$\cos(180° + \theta) = -\cos\theta$
$\cos(360° - \theta) = \cos\theta$

Key Points

4 The trigonometric ratios of 30°, 45° and 60° have exact forms, given below:

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\tan 45° = 1$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$

5 For all values of θ, $\sin^2\theta + \cos^2\theta \equiv 1$

6 For all values of θ such that $\cos\theta \neq 0$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

7
- Solutions to $\sin\theta = k$ and $\cos\theta = k$ only exist when $-1 \leqslant k \leqslant 1$
- Solutions to $\tan\theta = p$ exist for all values of p.

8 When you use the inverse trigonometric functions on your calculator, the angle you get is called the **principal value**.

9 Your calculator will give principal values in the following ranges:
- $\cos^{-1}$ in the range $0 \leqslant \theta \leqslant 180°$
- $\sin^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$
- $\tan^{-1}$ in the range $-90° \leqslant \theta \leqslant 90°$

Chapter Exercises

1 Write each of the following as a trigonometric ratio of an acute angle:

a $\cos 237°$ **b** $\sin 312°$ **c** $\tan 190°$

2 Without using your calculator, work out the values of:

a $\cos 270°$ **b** $\sin 225°$ **c** $\cos 180°$ **d** $\tan 240°$ **e** $\tan 135°$

3 Given that angle A is obtuse and $\cos A = -\sqrt{\frac{7}{11}}$, show that $\tan A = \frac{-2\sqrt{7}}{7}$

4 Given that angle B is obtuse and $\tan B = +\frac{\sqrt{21}}{2}$, find the exact value of: **a** $\sin B$ **b** $\cos B$

5 Simplify the following expressions:

a $\cos^4\theta - \sin^4\theta$ **b** $\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$

c $\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta$

6 **a** Given that $2(\sin x + 2\cos x) = \sin x + 5\cos x$, find the exact value of $\tan x$.

b Given that $\sin x \cos y + 3\cos x \sin y = 2\sin x \sin y - 4\cos x \cos y$, express $\tan y$ in terms of $\tan x$.

7 Prove that, for all values of θ:

a $(1 + \sin\theta)^2 + \cos^2\theta \equiv 2(1 + \sin\theta)$ **b** $\cos^4\theta + \sin^2\theta \equiv \sin^4\theta + \cos^2\theta$

8 Without attempting to solve them, state how many solutions the following equations have in the interval $0 \leqslant \theta \leqslant 360°$. Give a brief reason for your answer.

a $2\sin\theta = 3$ **b** $\sin\theta = -\cos\theta$

c $2\sin\theta + 3\cos\theta + 6 = 0$ **d** $\tan\theta + \frac{1}{\tan\theta} = 0$

Chapter Exercises

9 **a** Factorise $4xy - y^2 + 4x - y$. **(2 marks)**

b Solve the equation $4\sin\theta\cos\theta - \cos^2\theta + 4\sin\theta - \cos\theta = 0$, in the interval $0 \leqslant \theta \leqslant 360°$. **(5 marks)**

10 **a** Express $4\cos 3\theta - \sin(90° - 3\theta)$ as a single trigonometric function. **(1 mark)**

b Hence solve $4\cos 3\theta - \sin(90° - 3\theta) = 2$ in the interval $0 \leqslant \theta \leqslant 360°$. Give your answers to 3 significant figures. **(3 marks)**

11 Given that $2\sin 2\theta = \cos 2\theta$:

a Show that $\tan 2\theta = 0.5$. **(1 mark)**

b Hence find the values of θ, to one decimal place, in the interval $0 \leqslant \theta \leqslant 360°$ for which $2\sin 2\theta = \cos 2\theta$. **(4 marks)**

12 Find all the values of θ in the interval $0 \leqslant \theta \leqslant 360°$ for which:

a $\cos(\theta + 75°) = 0.5$,

b $\sin 2\theta = 0.7$, giving your answers to one decimal place.

13 Find the values of x in the interval $0 \leqslant x \leqslant 270°$ which satisfy the equation

$$\frac{\cos 2x + 0.5}{1 - \cos 2x} = 2$$ **(6 marks)**

14 Find, in degrees, the values of θ in the interval $0 \leqslant \theta \leqslant 360°$ for which
$2\cos^2\theta - \cos\theta - 1 = \sin^2\theta$
Give your answers to 1 decimal place, where appropriate. **(6 marks)**

Chapter Exercises

15 A teacher asks one of his students to solve the equation $2\sin 3x = 1$ for $-360° \leqslant x \leqslant 360°$.
The attempt is shown below:

$\sin 3x = \frac{1}{2}$
$3x = 30°$
$x = 10°$
Additional solution at $180° - 10° = 170°$

a Identify two mistakes made by the student. **(2 marks)**

b Solve the equation. **(2 marks)**

16 **a** Sketch the graphs of $y = 3\sin x$ and $y = 2\cos x$ on the same set of axes ($0 \leqslant x \leqslant 360°$).

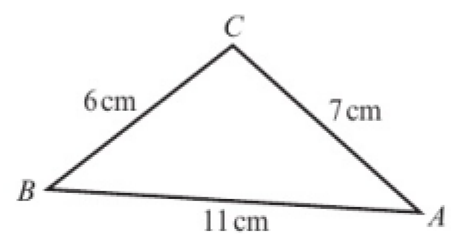
b Write down how many solutions there are in the given range for the equation $3\sin x = 2\cos x$.

c Solve the equation $3\sin x = 2\cos x$ algebraically, giving your answers to one decimal place.

17 The diagram shows the triangle ABC with $AB = 11$ cm, $BC = 6$ cm and $AC = 7$ cm.

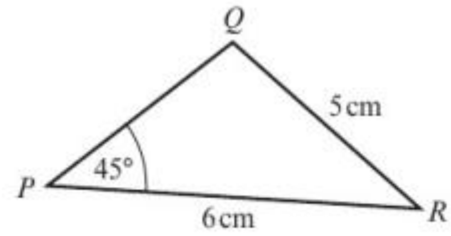
a Find the exact value of $\cos B$, giving your answer in simplest form. **(3 marks)**

b Hence find the exact value of $\sin B$. **(2 marks)**



Chapter Exercises

18 The diagram shows triangle PQR with $PR = 6$ cm, $QR = 5$ cm and angle $QPR = 45°$.



a Show that $\sin Q = \frac{3\sqrt{2}}{5}$ **(3 marks)**

b Given that Q is obtuse, find the exact value of $\cos Q$. **(2 marks)**

19 **a** Show that the equation $3\sin^2 x - \cos^2 x = 2$ can be written as $4\sin^2 x = 3$. **(2 marks)**

b Hence solve the equation $3\sin^2 x - \cos^2 x = 2$ in the interval $-180° \leqslant x \leqslant 180°$, giving your answers to 1 decimal place. **(7 marks)**

20 Find all the solutions to the equation $3\cos^2 x + 1 = 4\sin x$ in the interval $-360° \leqslant x \leqslant 360°$, giving your answers to 1 decimal place. **(6 marks)**

Challenge

Solve the equation $\tan^4 x - 3\tan^2 x + 2 = 0$ in the interval $0 \leqslant x \leqslant 360°$.

Chapter Answers

1 a $-\cos 57°$ b $-\sin 48°$ c $+\tan 10°$

2 a 0 b $-\frac{\sqrt{2}}{2}$ c -1

d $\sqrt{3}$ e -1

3 Using $\sin^2 A = 1 - \cos^2 A$, $\sin^2 A = 1 - \left(-\sqrt{\frac{7}{11}}\right)^2 = \frac{4}{11}$.

Since angle A is obtuse, it is in the second quadrant and sin is positive, so $\sin A = \frac{2}{\sqrt{11}}$.

Then $\tan A = \frac{\sin A}{\cos A} = \frac{2}{\sqrt{11}} \times \left(-\sqrt{\frac{11}{7}}\right) = -\frac{2}{\sqrt{7}} = -\frac{2}{7}\sqrt{7}$.

4 a $-\frac{\sqrt{21}}{5}$ b $-\frac{2}{5}$

5 a $\cos^2\theta - \sin^2\theta$ b $\sin^4 3\theta$ c 1

6 a 1 b $\tan y = \frac{4 + \tan x}{2\tan x - 3}$

7 a
$$\begin{aligned}\text{LHS} &= (1 + 2\sin\theta + \sin^2\theta) + \cos^2\theta\\ &= 1 + 2\sin\theta + 1\\ &= 2 + 2\sin\theta\\ &= 2(1 + \sin\theta) = \text{RHS}\end{aligned}$$

b
$$\begin{aligned}\text{LHS} &= \cos^4\theta + \sin^2\theta\\ &= (1 - \sin^2\theta)^2 + \sin^2\theta\\ &= 1 - 2\sin^2\theta + \sin^4\theta + \sin^2\theta\\ &= (1 - \sin^2\theta) + \sin^4\theta\\ &= \cos^2\theta + \sin^4\theta = \text{RHS}\end{aligned}$$

8 a No solutions: $-1 \leqslant \sin\theta \leqslant 1$

b 2 solutions: $\tan\theta = -1$ has two solutions in the interval.

c No solutions: $2\sin\theta + 3\cos\theta > -5$ so $2\sin\theta + 3\cos\theta + 6$ can never be equal to 0.

d No solutions: $\tan^2\theta = -1$ has no real solutions.

9 a $(4x - y)(y + 1)$ b $14.0°, 180°, 194°$

10 a $3\cos 3\theta$ b $16.1, 104, 136, 224, 256, 344$

11 a $2\sin 2\theta = \cos 2\theta \Rightarrow \frac{2\sin 2\theta}{\cos 2\theta} = 1$

$\Rightarrow 2\tan 2\theta = 1 \Rightarrow \tan 2\theta = 0.5$

b $13.3°, 103.3°, 193.3°, 283.3°$

12 a $225°, 345°$

b $22.2°, 67.8°, 202.2°, 247.8°$

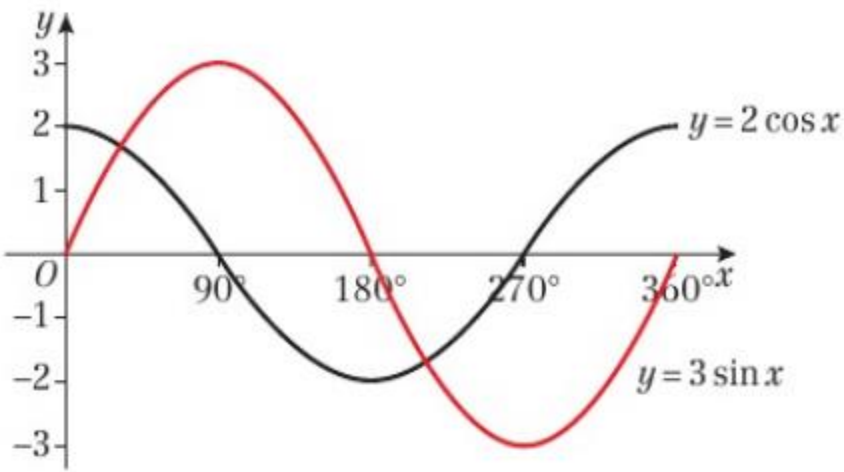
13 $30°, 150°, 210°$

14 $0°, 131.8°, 228.2°$

15 a Found additional solutions after dividing by three rather than before. Not applied the full interval for solutions.

b $-350°, -310°, -230°, -190°, -110°, -70°, 10°, 50°, 130°, 170°, 250°, 290°$

16 a



b 2 **c** 33.7°, 213.7°

17 a $\frac{9}{11}$ **b** $\frac{\sqrt{40}}{11}$

18 a Using sine rule: $\sin Q = \sin 45 \times \frac{6}{5} = \frac{\sqrt{2}}{2} \times \frac{6}{5} = \frac{3\sqrt{2}}{5}$

b $-\frac{\sqrt{7}}{5}$

19 a $3\sin^2 x - (1 - \sin^2 x) = 2$.
Rearrange to give $4\sin^2 x = 3$.

b –120°, –60°, 60°, 120°

20 –318.2°, –221.8°, 41.8°, 138.2°

Challenge
45°, 54.7°, 125.3°, 135°, 225°, 234.7°, 305.3°, 315°