P1 Chapter 13: Integration

Chapter Practice

Key Points

1 If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = x^n$, then $f(x) = \frac{1}{n+1}x^{n+1} + c, n \neq -1$.

2 If $\frac{dy}{dx} = kx^n$, then $y = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

Using function notation, if $f'(x) = kx^n$, then $f(x) = \frac{k}{n+1}x^{n+1} + c, n \neq -1$.

When integrating polynomials, apply the rule of integration separately to each term.

3 $\int f'(x)dx = f(x) + c$

4 $\int(f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

5 To find the constant of integration, c

- Integrate the function
- Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$ into the integrated function
- Solve the equation to find c

Key Points

6 If $f'(x)$ is the derivative of $f(x)$ for all values of x in the interval $[a, b]$, then the definite integral is defined as $\int_a^b f'(x)\,dx = [f(x)]_a^b = f(b) - f(a)$

7 The area between a positive curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b y\,dx$$

where $y = f(x)$ is the equation of the curve.

8 When the area bounded by a curve and the x-axis is below the x-axis, $\int y\,dx$ gives a negative answer.

9 You can use definite integration together with areas of trapeziums and triangles to find more complicated areas on graphs.

Chapter Exercises

1 Find:

a $\int(x+1)(2x-5)\,\mathrm{d}x$ **b** $\int(x^{\frac{1}{3}}+x^{-\frac{1}{3}})\,\mathrm{d}x$

2 The gradient of a curve is given by $f'(x) = x^2 - 3x - \frac{2}{x^2}$. Given that the curve passes through the point (1, 1), find the equation of the curve in the form $y = f(x)$.

3 Find:

a $\int(8x^3 - 6x^2 + 5)\,\mathrm{d}x$ **b** $\int(5x+2)x^{\frac{1}{2}}\,\mathrm{d}x$

4 Given $y = \dfrac{(x+1)(2x-3)}{\sqrt{x}}$, find $\int y\,\mathrm{d}x$.

5 Given that $\dfrac{\mathrm{d}x}{\mathrm{d}t} = (t+1)^2$ and that $x = 0$ when $t = 2$, find the value of x when $t = 3$.

6 Given that $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$:

a show that $y = x^{\frac{2}{3}} + Ax^{\frac{1}{3}} + B$, where A and B are constants to be found. **(2 marks)**

b hence find $\int y\,\mathrm{d}x$. **(3 marks)**

7 Given that $y^{\frac{1}{2}} = 3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}}$ $(x > 0)$:

a find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ **(2 marks)**

b find $\int y\,\mathrm{d}x$. **(3 marks)**

8 $\int\left(\frac{a}{3x^3} - ab\right)dx = -\frac{2}{3x^2} + 14x + c$

Find the value of a and the value of b.

9 A rock is dropped off a cliff. The height in metres of the rock above the ground after t seconds is given by the function $f(t)$. Given that $f(0) = 70$ and $f'(t) = -9.8t$, find the height of the rock above the ground after 3 seconds.

10 A cyclist is travelling along a straight road. The distance in metres of the cyclist from a fixed point after t seconds is modelled by the function $f(t)$, where $f'(t) = 5 + 2t$ and $f(0) = 0$.

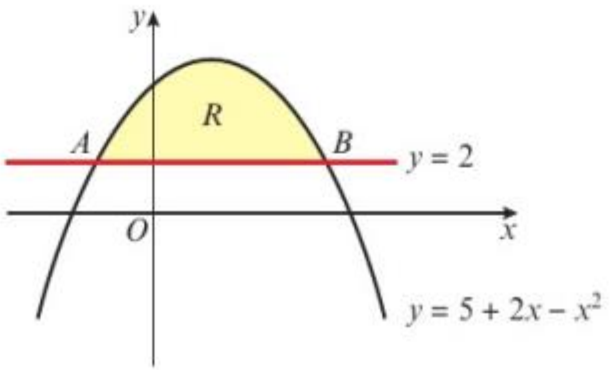
a Find an expression for $f(t)$.

b Calculate the time taken for the cyclist to travel 100 m.

11 The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.

a Find the x-coordinates of A and B.

b The shaded region R is bounded by the curve and the line. Find the area of R.



Chapter Exercises

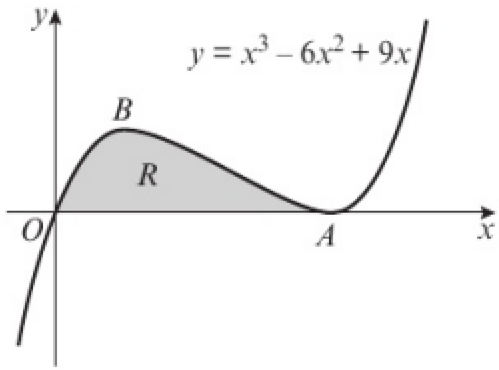
12 **a** Find $\int(x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)\mathrm{d}x$. **(4 marks)**

b Use your answer to part **a** to evaluate

$$\int_1^4 (x^{\frac{1}{2}} - 4)(x^{-\frac{1}{2}} - 1)\mathrm{d}x$$

giving your answer as an exact fraction. **(2 marks)**

13 The diagram shows part of the curve with equation $y = x^3 - 6x^2 + 9x$. The curve touches the x-axis at A and has a local maximum at B.



a Show that the equation of the curve may be written as $y = x(x - 3)^2$, and hence write down the coordinates of A. **(2 marks)**

b Find the coordinates of B. **(2 marks)**

c The shaded region R is bounded by the curve and the x-axis. Find the area of R. **(6 marks)**

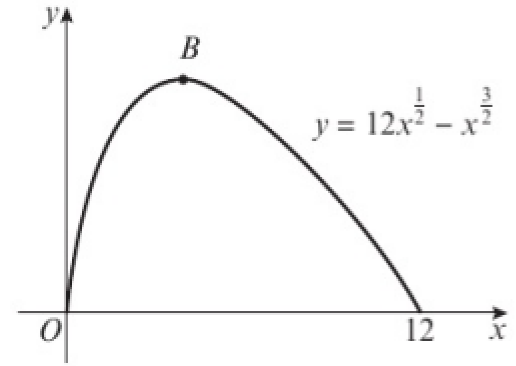
14 Consider the function $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$, $x > 0$.

a Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. **(2 marks)**

b Find $\int y\,\mathrm{d}x$. **(3 marks)**

c Hence show that $\int_1^3 y\,\mathrm{d}x = A + B\sqrt{3}$, where A and B are integers to be found. **(2 marks)**

15 The diagram shows a sketch of the curve with equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$ for $0 \leqslant x \leqslant 12$.

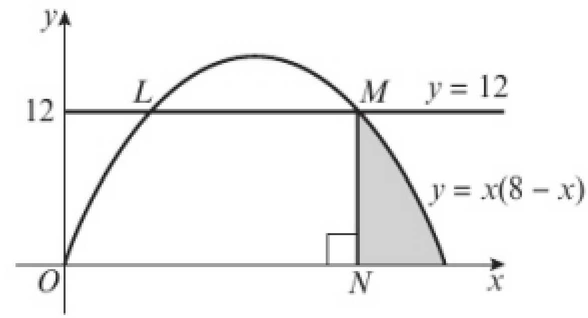


a Show that $\dfrac{dy}{dx} = \dfrac{3}{2}x^{-\frac{1}{2}}(4 - x)$. **(2 marks)**

b At the point B on the curve the tangent to the curve is parallel to the x-axis. Find the coordinates of the point B. **(2 marks)**

c Find, to 3 significant figures, the area of the finite region bounded by the curve and the x-axis. **(6 marks)**

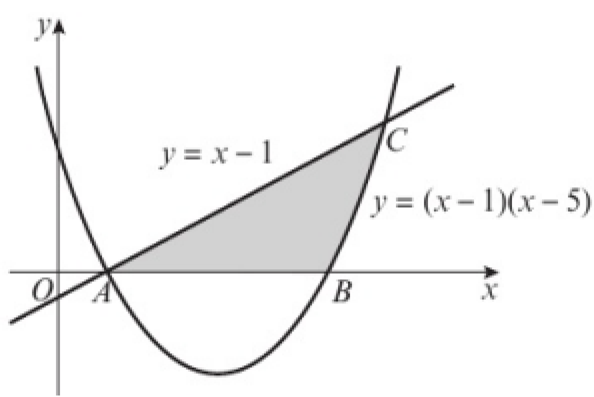
16 The diagram shows the curve C with equation $y = x(8 - x)$ and the line with equation $y = 12$ which meet at the points L and M.



a Determine the coordinates of the point M. **(2 marks)**

b Given that N is the foot of the perpendicular from M on to the x-axis, calculate the area of the shaded region which is bounded by NM, the curve C and the x-axis. **(6 marks)**

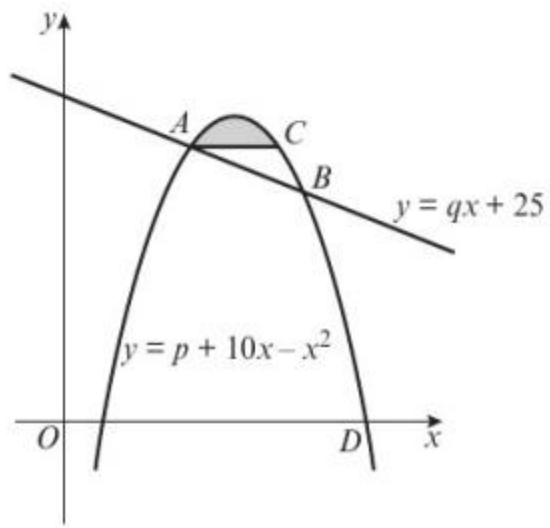
17 The diagram shows the line $y = x - 1$ meeting the curve with equation $y = (x - 1)(x - 5)$ at A and C. The curve meets the x-axis at A and B.



a Write down the coordinates of A and B and find the coordinates of C. **(4 marks)**

b Find the area of the shaded region bounded by the line, the curve and the x-axis. **(6 marks)**

18 The diagram shows part of the curve with equation $y = p + 10x - x^2$, where p is a constant, and part of the line l with equation $y = qx + 25$, where q is a constant. The line l cuts the curve at the points A and B. The x-coordinates of A and B are 4 and 8 respectively. The line through A parallel to the x-axis intersects the curve again at the point C.



a Show that $p = -7$ and calculate the value of q. **(3 marks)**

b Calculate the coordinates of C. **(2 marks)**

c The shaded region in the diagram is bounded by the curve and the line segment AC. Using integration and showing all your working, calculate the area of the shaded region. **(6 marks)**

19 Given that $f(x) = \frac{9}{x^2} - 8\sqrt{x} + 4x - 5$, $x > 0$, find $\int f(x)\,dx$. **(5 marks)**

20 Given that A is constant and $\int_4^9 \left(\frac{3}{\sqrt{x}} - A\right) dx = A^2$ show that there are two possible values for A and find these values. **(5 marks)**

21 $f'(x) = \frac{(2 - x^2)^3}{x^2}$, $x \neq 0$

a Show that $f'(x) = 8x^{-2} - 12 + Ax^2 + Bx^4$, where A and B are constants to be found.

b Find $f''(x)$. **(3 marks)**

Given that the point $(-2, 9)$ lies on the curve with equation $y = f(x)$,

c find $f(x)$. **(5 marks)**

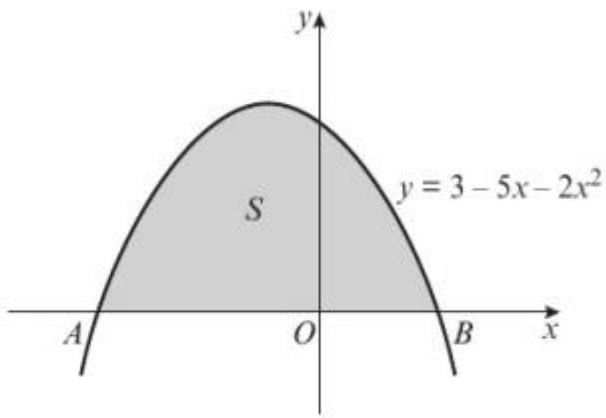
Chapter Exercises

22 The finite region S, which is shown shaded, is bounded by the x-axis and the curve with equation $y = 3 - 5x - 2x^2$.

The curve meets the x-axis at points A and B.

a Find the coordinates of point A and point B. **(2 marks)**

b Find the area of the region S. **(4 marks)**



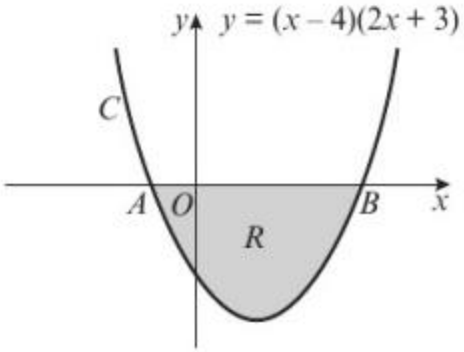
23 The graph shows a sketch of part of the curve C with equation $y = (x - 4)(2x + 3)$.

The curve C crosses the x-axis at the points A and B.

a Write down the x-coordinates of A and B. **(1 mark)**

The finite region R, shown shaded, is bounded by C and the x-axis.

b Use integration to find the area of R. **(6 marks)**

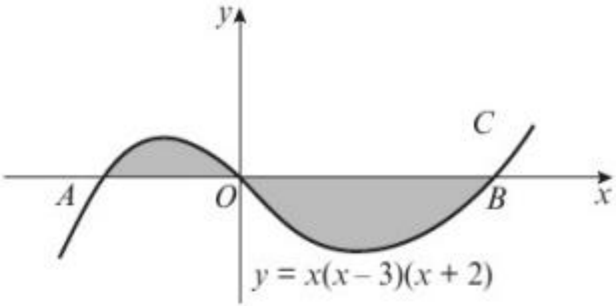


Chapter Exercises

24 The graph shows a sketch of part of the curve C with equation $y = x(x - 3)(x + 2)$.

The curve crosses the x-axis at the origin O and the points A and B.

a Write down the x-coordinates of the points A and B. **(1 mark)**



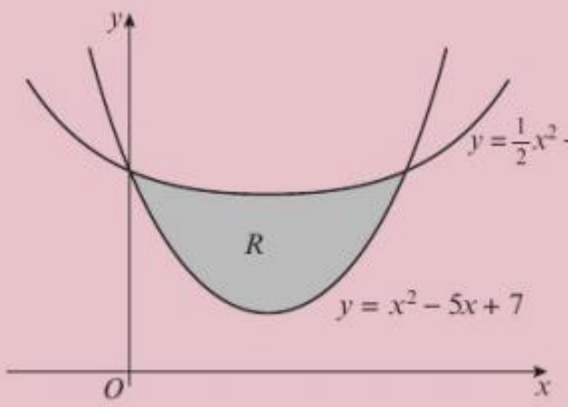
The finite region shown shaded is bounded by the curve C and the x-axis.

b Use integration to find the total area of this region. **(7 marks)**

Challenge

The curve with equation $y = x^2 - 5x + 7$ cuts the curve with equation $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$. The shaded region R is bounded by the curves as shown.

Find the exact area of R.



1 **a** $\frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x + c$ **b** $\frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}} + c$

2 $\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + \frac{1}{6}$

3 **a** $2x^4 - 2x^3 + 5x + c$ **b** $2x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + c$

4 $\frac{4}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + c$

5 $x = \frac{1}{3}t^3 + t^2 + t - 8\frac{2}{3}$; $x = 12\frac{1}{3}$

6 **a** $A = 6, B = 9$ **b** $\frac{3}{5}x^{\frac{5}{3}} + \frac{9}{2}x^{\frac{4}{3}} + 9x + c$

7 **a** $\frac{9}{2}x^{-\frac{1}{2}} - 8x^{-\frac{3}{2}}$ **b** $6x^{\frac{3}{2}} + 32x^{\frac{1}{2}} - 24x + c$

8 $a = 4, b = -3.5$

9 25.9 m

10 **a** $f(t) = 5t + t^2$ **b** 7.8 seconds

11 **a** $-1, 3$ **b** $10\frac{2}{3}$

12 **a** $-\frac{2x^{\frac{3}{2}}}{3} + 5x - 8\sqrt{x} + c$ **b** $\frac{7}{3}$

13 **a** $(3, 0)$ **b** $(1, 4)$ **c** $6\frac{3}{4}$

14 **a** $\frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$ **b** $2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c$ **c** $A = 6, B = -2$

15 **a** $\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$

b $(4, 16)$ **c** 133 (3 sf)

16 **a** $(6, 12)$ **b** $13\frac{1}{3}$

17 **a** $A(1, 0), B(5, 0), C(6, 5)$ **b** $10\frac{1}{6}$

18 **a** $q = -2$ **b** $C(6, 17)$ **c** $1\frac{1}{3}$

19 $-\frac{9}{x} - \frac{16x^{\frac{3}{2}}}{3} + 2x^2 - 5x + c$

20 $A = -6$ or 1

21 **a** $f'(x) = \frac{(2 - x^2)(4 - 4x^2 + x^4)}{x^2}$

$= 8x^{-2} - 12 + 6x^2 - x^4$

b $f''(x) = -16x^{-3} + 12x - 4x^3$

c $f(x) = -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} - \frac{47}{5}$

22 **a** $(-3, 0)$ and $(\frac{1}{2}, 0)$ **b** $14\frac{7}{24}$

23 **a** $(-\frac{3}{2}, 0)$ and $(4, 0)$ **b** $55\frac{11}{24}$

24 **a** -2 and 3 **b** $21\frac{1}{12}$

Challenge

$10\frac{5}{12}$