P1 Chapter 3: Inequalities

Linear Inequalities

Solutions sets

The solution(s) to an equation may be:

A single value:	$2x + 1 = 5$
Multiple values:	$x^2 + 3x + 2 = 0$
An infinitely large set of values:	$x > 3$
No (real) values!	$x^2 = -1$
Every value!	$x^2 + x = x(x + 1)$

The point is that you shouldn't think of the solution to an equation/inequality as an 'answer', but a **set** of values, which might just be a set of 1 value (known as a singleton set), a set of no values (i.e. the empty set ∅), or an infinite set (in the last example above, this was $\mathbb{R}$)

✏ The solutions to an equation are known as the **solution set**.

Solutions sets

For simultaneous equations, the same is true, except each 'solution' in the solution set is an assignment to **multiple** variables.
All equations have to be satisfied **at the same time, i.e. 'simultaneously'**.

Scenario	**Example**	**Solution Set**
A single solution:	?	?
Two solutions:	?	?
No solutions:	?	?
Infinitely large set of solutions:	?	?

Solutions sets

For simultaneous equations, the same is true, except each 'solution' in the solution set is an assignment to **multiple** variables.
All equations have to be satisfied **at the same time, i.e. 'simultaneously'**.

Scenario	Example	Solution Set
A single solution:	$x + y = 9$ $x - y = 1$	**Solution 1:** $\boldsymbol{x = 5, y = 4}$ To be precise here, the solution set is of size 1, but this solution is an assignment to multiple variables, i.e. a pair of values.
Two solutions:	$x^2 + y^2 = 10$ $x + y = 4$	**Solution 1:** $\boldsymbol{x = 3, y = 1}$ **Solution 2:** $\boldsymbol{x = 1, y = 3}$ This time we have two solutions, each an x, y pair.
No solutions:	$x + y = 1$ $x + y = 3$	The solution set is empty, i.e. $\emptyset$, as both equation can't be satisfied at the same time.
Infinitely large set of solutions:	$x + y = 1$ $2x + 2y = 2$	**Solution 1:** $\boldsymbol{x = 0, y = 1}$ **Solution 2:** $\boldsymbol{x = 1, y = 0}$ **Solution 3:** $\boldsymbol{x = 2, y = -1}$ **Solution 4:** $\boldsymbol{x = 0.5, y = 0.5}$ **…** Infinite possibilities!

Set Builder Notation

Recall that a **set** is a **collection of values** such that:

a) The **order of values does not matter**.

b) There are **no duplicates**.

Remark: Sets seem sensible for listing solutions to an equation, as the order doesn't matter.

Recap from GCSE:

- We use curly braces to list the values in a set, e.g. $A = \{1,4,6,7\}$
- If A and B are sets then $A \cap B$ is the **intersection** of A and B, giving a set which has the elements in A **and** B.
- $A \cup B$ is the **union** of A and B, giving a set which has the elements in A **or** in B.
- $\emptyset$ is the empty set, i.e. the set with nothing in it.
- Sets can also be infinitely large. $\mathbb{N}$ is the set of natural numbers (all positive integers), $\mathbb{Z}$ is the set of all integers (including negative numbers and 0) and $\mathbb{R}$ is the set of all real numbers (including all possible decimals).
- We write $x \in A$ to mean "x is a member of the set A". So $x \in \mathbb{R}$ would mean "x is a real number".

$$\{1,2,3\} \cap \{3,4,5\} = \{\mathbf{3}\}$$
$$\{1,2,3\} \cup \{3,4,5\} = \{\mathbf{1, 2, 3, 4, 5}\}$$
$$\{1,2\} \cap \{3,4\} = \emptyset$$

Set Builder Notation

It is possible to construct sets without having to explicitly list its values. We use:

The | or : means "such that".

$$\{expr \mid condition\}$$

or

$$\{expr : condition\}$$

Can you guess what sets the following give?

(In words "All numbers $2x$ <u>such that</u> x is an integer)

$$\{2x : x \in \mathbb{Z}\} = \boxed{?}$$

$$\{2^x : x \in \mathbb{N}\} = \boxed{?}$$

$$\{xy : x, y \text{ are prime}\} = \boxed{?}$$

We previously talked about 'solutions sets', so set builder notation is very useful for specifying the set of solutions!

Set Builder Notation

It is possible to construct sets without having to explicitly list its values. We use:

The | or : means "such that".

$$\{expr \mid condition\}$$
or
$$\{expr : condition\}$$

Can you guess what sets the following give?

(In words "All numbers $2x$ <u>such that</u> x is an integer)

$$\{2x : x \in \mathbb{Z}\} = \{0,2,-2,4,-4,6,-6,\dots\}$$

i.e. The set of all even numbers!

$$\{2^x : x \in \mathbb{N}\} = \{2,4,8,16,32,\dots\}$$

$$\{xy : x, y \; are \; prime\} = \{4,6,10,14,15,\dots\}$$

i.e. All possible products of two primes.

We previously talked about 'solutions sets', so set builder notation is very useful for specifying the set of solutions!

Set Builder Notation

Can you use set builder notation to specify the following sets?

All odd numbers. ?

All (real) numbers greater than 5. ?

All (real) numbers less than 5 **or** greater than 7. ?

All (real) numbers between 5 and 7 inclusive. ?

Set Builder Notation

Can you use set builder notation to specify the following sets?

All odd numbers.

$$\{2x + 1 \ : \ \ x \in \mathbb{Z}\}$$

All (real) numbers greater than 5.

$$\{x: \ \ x > 5\}$$

Technically it should be $\{x: \ \ x > 5, x \in \mathbb{R}\}$ but the $x > 5$ by default implies **real numbers** greater than 5.

All (real) numbers less than 5 **or** greater than 7.

$$\{x: \ \ x < 5\} \cup \{x: x > 7\}$$

We combine the two sets together.

All (real) numbers between 5 and 7 inclusive.

$$\{x: \ \ 5 \leq x \leq 7\}$$

While we could technically write $\{x: x \geq 5\} \cap \{x: x \leq 7\}$, we tend to write multiple required conditions within the same set.

Recap of linear inequalities

Inequality	Solution Set
$2x + 1 > 5$	?
$3(x - 5) \geq 5 - 2(x - 8)$	?
$-x \geq 2$	?

Combining Inequalities:

If $x < 3$ and $2 \leq x < 4$, what is the combined solution set?

? Hint

? Solution

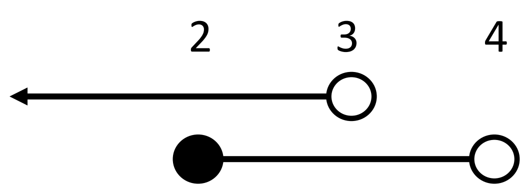
Recap of linear inequalities

Inequality	Solution Set
$2x + 1 > 5$	$\{x : x > 2\}$
$3(x - 5) \geq 5 - 2(x - 8)$	$\{x : x \geq 7.2\}$
$-x \geq 2$	$\{x : x \leq -2\}$

Fro Note: Multiplying or both sides of an inequality by a negative number reverses the direction.

Combining Inequalities:

If $x < 3$ and $2 \leq x < 4$, what is the combined solution set?



If both inequalities have to be satisfied, we have to be on both lines. Place your finger vertically and scan across.

$$2 \leq x < 3$$

Test Your Understanding

Edexcel C1 May 2010 Q3

Find the set of values of x for which

(*a*) $3(x-2) < 8 - 2x$,

(2)

(*b*) $(2x-7)(1+x) < 0$,

(3)

(*c*) both $3(x-2) < 8 - 2x$ **and** $(2x-7)(1+x) < 0$.

(1)

(a)	?
(b)	?
(c)	?

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(*c*) both $3(x-2) < 8 - 2x$ **and** $(2x-7)(1+x) < 0$.

(1)

(a)	$3x-6<8-2x \rightarrow \quad 5x<14$	(Accept $5x-14<0$ (o.e.)	M1	
	$x<2.8$ or $\frac{14}{5}$ or $2\frac{4}{5}$	(condone $\leq$)	A1	(2)
(b)	Critical values are $x=\frac{7}{2}$ and -1		B1	
	Choosing "inside" $-1<x<\frac{7}{2}$		M1 A1	(3)
(c)	$-1<x<2.8$		B1ft	(1)
	Accept any exact equivalents to -1, 2.8, 3.5			**6**

Deal with inequalities with a division by x

Find the set of values for which $\frac{6}{x} > 2, \quad x \neq 0$

Why can't we just multiply both sides by x?

?

? Solution

Spec Note: This is an example in the textbook, although it is ambiguous whether this type of question is in the new specification. Dealing with an x in the denominator within an inequality is a skill previously in the old Further Pure 2 module. But you never really know!

Deal with inequalities with a division by x

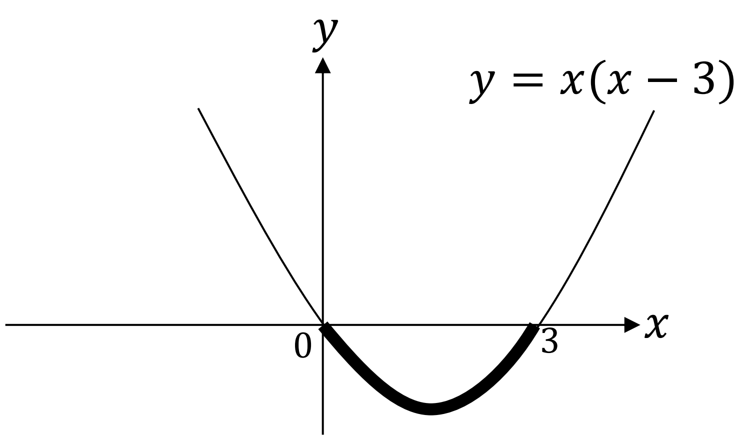
Find the set of values for which $\frac{6}{x} > 2, \quad x \neq 0$

Spec Note: This is an example in the textbook, although it is ambiguous whether this type of question is in the new specification. Dealing with an x in the denominator within an inequality is a skill previously in the old Further Pure 2 module. But you never really know!

Why can't we just multiply both sides by x?
We earlier saw that multiplying by a negative number would flip the inequality, but multiplying by a positive number would not. Since we don't know x, we don't know whether the inequality would flip or not!

Once solution is to sketch $y = \frac{6}{x}$ and $y = 2$, find the points of intersection and reason about the graph (see next section, "Inequalities on Graphs"), but an easier way is to **multiply both sides by x^2**, because it is **guaranteed to be positive**:

$$\mathbf{6x > 2x^2}$$
$$\mathbf{2x^2 - 6x < 0}$$
$$\mathbf{x^2 - 3x < 0}$$
$$\mathbf{x(x-3) < 0}$$
$$\mathbf{0 < x < 3}$$



Exercise 3.4

Pearson Pure Mathematics Year 1/AS

Page 19

Homework Exercise

1 Find the set of values of x for which:

a $2x - 3 < 5$

b $5x + 4 \geqslant 39$

c $6x - 3 > 2x + 7$

d $5x + 6 \leqslant -12 - x$

e $15 - x > 4$

f $21 - 2x > 8 + 3x$

g $1 + x < 25 + 3x$

h $7x - 7 < 7 - 7x$

i $5 - 0.5x \geqslant 1$

j $5x + 4 > 12 - 2x$

2 Find the set of values of x for which:

a $2(x - 3) \geqslant 0$

b $8(1 - x) > x - 1$

c $3(x + 7) \leqslant 8 - x$

d $2(x - 3) - (x + 12) < 0$

e $1 + 11(2 - x) < 10(x - 4)$

f $2(x - 5) \geqslant 3(4 - x)$

g $12x - 3(x - 3) < 45$

h $x - 2(5 + 2x) < 11$

i $x(x - 4) \geqslant x^2 + 2$

j $x(5 - x) \geqslant 3 + x - x^2$

k $3x + 2x(x - 3) \leqslant 2(5 + x^2)$

l $x(2x - 5) \leqslant \dfrac{4x(x + 3)}{2} - 9$

Homework Exercise

3 Use set notation to describe the set of values of x for which:

a $3(x-2) > x-4$ and $4x+12 > 2x+17$

b $2x-5 < x-1$ and $7(x+1) > 23-x$

c $2x-3 > 2$ and $3(x+2) < 12+x$

d $15-x < 2(11-x)$ and $5(3x-1) > 12x+19$

e $3x+8 \leqslant 20$ and $2(3x-7) \geqslant x+6$

f $5x+3 < 9$ or $5(2x+1) > 27$

g $4(3x+7) \leqslant 20$ or $2(3x-5) \geqslant \dfrac{7-6x}{2}$

Challenge

$A = \{x : 3x+5 > 2\}$ $\qquad B = \left\{x : \dfrac{x}{2}+1 \leqslant 3\right\}$ $\qquad C = \{x : 11 < 2x-1\}$

Given that $A \cap (B \cup C) = \{x : p < x \leqslant q\} \cup \{x : x > r\}$, find the values of p, q and r.

Homework Answers

1 **a** $x < 4$ **b** $x \geqslant 7$ **c** $x > 2\frac{1}{2}$ **d** $x \leqslant -3$
e $x < 11$ **f** $x < 2\frac{3}{5}$ **g** $x > -12$ **h** $x < 1$
i $x \leqslant 8$ **j** $x > 1\frac{1}{7}$

2 **a** $x \geqslant 3$ **b** $x < 1$ **c** $x \leqslant -3\frac{1}{4}$ **d** $x < 18$
e $x > 3$ **f** $x \geqslant 4\frac{2}{5}$ **g** $x < 4$ **h** $x > -7$
i $x \leqslant -\frac{1}{2}$ **j** $x \geqslant \frac{3}{4}$ **k** $x \geqslant -\frac{10}{3}$ **l** $x \geqslant \frac{9}{11}$

3 **a** $\{x: x > 2\frac{1}{2}\}$ **b** $\{x: 2 < x < 4\}$
c $\{x: 2\frac{1}{2} < x < 3\}$ **d** No values
e $x = 4$ **f** $\{x: x < 1.2\} \cup \{x: x > 2.2\}$
g $\left\{x : x \leqslant -\frac{2}{3}\right\} \cup \left\{x : x \geqslant \frac{3}{2}\right\}$

Challenge

$p = -1$, $q = 4$, $r = 6$