P1 Chapter 14: Logarithms

Logarithm Equations

Solving equations with exponential terms

Solve $3^x = 20$

?

Solve $5^{4x-1} = 61$

?

Solving equations with exponential terms

Solve $3^x = 20$

Applying log_3 to each side of the equation:

$$x = \log_3 20 = 2.727 \ \ (to\ 3dp)$$

This is often said "Taking logs of both sides…"

Solve $5^{4x-1} = 61$

Applying log_5 to each side of the equation:

$$4x - 1 = \log_5 61$$

$$x = \frac{\log_5 61 + 1}{4} = 0.889\ (to\ 3dp)$$

Solving equations with exponential terms

Solve $3^x = 2^{x+1}$

Why can we not apply quite the same strategy here?

?

? Solution

Solving equations with exponential terms

Solve $3^x = 2^{x+1}$

Why can we not apply quite the same strategy here?
Because the exponential terms don't have the same base, so we can't apply the same log.

We 'take logs of'/apply log to both sides, but **we need not specify a base**. log on its own may either mean log_{10} (as per your calculator) or log_e (in academic circles, as well as on sites like WolframAlpha), but the point is, **the base does not matter, provided that the base is consistent on both sides**.

$$
\begin{aligned}
&\log 3^x = \log 2^{x+1} \\
&x \log 3 = (x+1) \log 2 \\
&x \log 3 = x \log 2 + \log 2 \\
&x \log 3 - x \log 2 = \log 2 \\
&x(\log 3 - \log 2) = \log 2 \\
&x = \frac{\log 2}{\log 3 - \log 2} \\
&\quad = 1.7095
\end{aligned}
$$

Logs in general are great for solving equations when the variable is in the power, because laws of logs allow us to move the power down.

This then becomes a GCSE-style 'changing the subject' type question. Just isolate x on one side and factorise out.

It doesn't matter what base you use to get the final answer as a decimal, provided that it's consistent. You may as well use the calculator's 'log' (no base) key.

Test Your Understanding

1 Solve $3^{2x-1} = 5$, giving your answer to 3dp.

?

3 Solve $2^x 3^{x+1} = 5$, giving your answer in exact form.

?

2 Solve $3^{x+1} = 4^{x-1}$, giving your answer to 3dp.

?

Test Your Understanding

1. Solve $3^{2x-1} = 5$, giving your answer to 3dp.

$$2x - 1 = \log_3 5$$
$$x = \frac{\log_3 5 + 1}{2} = 1.232$$

3. Solve $2^x 3^{x+1} = 5$, giving your answer in exact form.

$$\log 2^x 3^{x+1} = \log 5$$
$$\log 2^x + \log 3^{x+1} = \log 5$$
$$x \log 2 + (x+1)\log 3 = \log 5$$
$$x \log 2 + x \log 3 + \log 3 = \log 5$$
$$x(\log 2 + \log 3) = \log 5 - \log 3$$
$$x = \frac{\log 5 - \log 3}{\log 2 + \log 3}$$

which could be simplified to:

$$x = \frac{\log 5 - \log 3}{\log 6}$$

2. Solve $3^{x+1} = 4^{x-1}$, giving your answer to 3dp.

$$\log 3^{x+1} = \log 4^{x-1}$$
$$(x+1)\log 3 = (x-1)\log 4$$
$$x \log 3 + \log 3 = x \log 4 - \log 4$$
$$x \log 4 - x \log 3 = \log 3 + \log 4$$

$$x(\log 4 - \log 3) = \log 3 + \log 4$$
$$x = \frac{\log 3 + \log 4}{\log 4 - \log 3} = 8.638$$

Exercise 14.6

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Extension

1. [MAT 2011 1H] How many *positive* values x which satisfy the equation:
$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

?

2. [MAT 2013 1J] For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral
$$\int_0^n [2^x] \, dx$$
equals:

(A) $\log_2\big((2^n - 1)!\big)$
(B) $n\,2^n - \log_2\big((2^n)!\big)$
(C) $n\,2^n$
(D) $\log_2\big((2^n)!\big)$

(Warning: This one really is <u>very</u> challenging, even for MAT)

Exercise 14.6

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Extension

1 [MAT 2011 1H] How many *positive* values x which satisfy the equation:

$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

$$\boldsymbol{x = \left(2^3\right)^{\log_2 x} - \left(3^2\right)^{\log_3 x} - \left(2^2\right)^{\log_2 x} + 2}$$
$$\boldsymbol{x = 2^{3 \log_2 x} - 3^{2 \log_3 x} - 2^{2 \log_2 x} + 2}$$
$$\boldsymbol{x = 2^{\log_2 x^3} - 3^{\log_3 x^2} - 2^{\log_2 x^2} + 2}$$
$$\boldsymbol{x = x^3 - x^2 - x^2 + 2}$$
$$\boldsymbol{x^3 - 2x^2 - x + 2 = 0}$$
$$\boldsymbol{x^2(x-2) - 1(x-2) = 0}$$
$$\boldsymbol{\left(x^2 - 1\right)(x-2) = 0}$$
$$\boldsymbol{(x+1)(x-1)(x-2) = 0}$$

This has 2 positive solutions.

2 [MAT 2013 1J] For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral

$$\int_0^n [2^x] \ dx$$

equals:

(A) $\log_2\left((2^n - 1)!\right)$

(B) $n\,2^n - \log_2\left((2^n)!\right)$

(C) $n\,2^n$

(D) $\log_2\left((2^n)!\right)$

(Warning: This one really is <u>very</u> challenging, even for MAT)

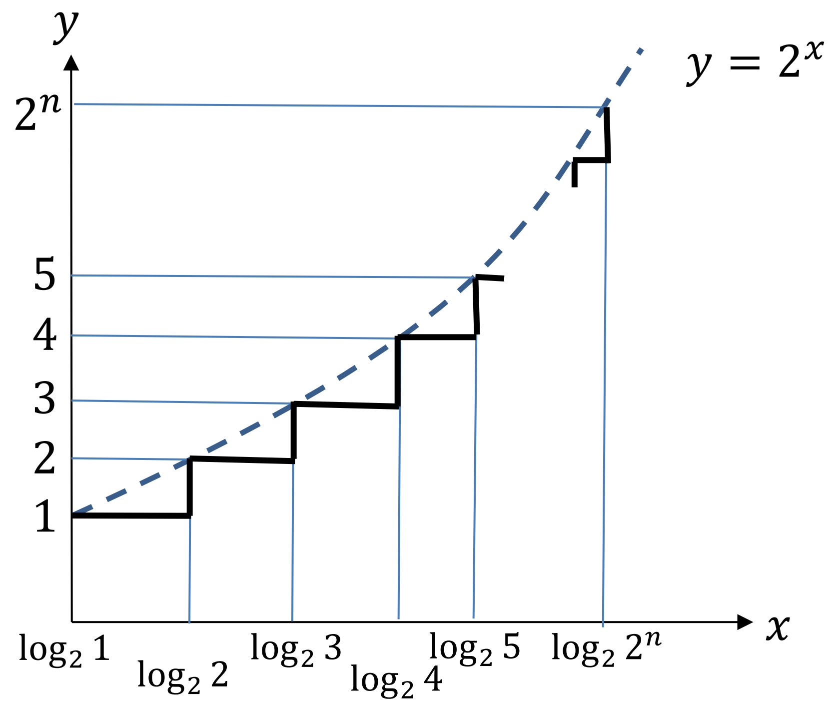
Solution to Extension Question 2

[MAT 2013 1J] For a real number x we denote by $[x]$ the largest integer less than or equal to x. Let n be a natural number. The integral

$$\int_0^n [2^x] \; dx$$

equals:

(A) $\log_2\big((2^n-1)!\big)$
(B) $n\,2^n - \log_2\big((2^n)!\big)$
(C) $n\,2^n$
(D) $\log_2\big((2^n)!\big)$



This biggest challenge is sketching the graph! Because of the rounding down, the graph jumps up 1 at a time, giving a bunch of rectangles. We can use logs to find the corresponding x values at which these jumps occur, which progressively become closer and closer together. The last y value is 2^n, thus the last x value is $\log_2 2^n = n$.

The area, using the rectangles, is thus:

$$\begin{aligned}
&1(\log_2 2 - \log_2 1) + 2(\log_2 3 - \log_2 2) + 3(\log_2 4 - \log_2 3) + \cdots + (2^n-1)(\log_2 2^n - \log_2(2^n-1)) \\
&= \log_2 2 - \log_2 1 + 2\log_2 3 - 2\log_2 2 + 3\log_2 4 - 3\log_2 3 + \cdots + (2^n-1)\log_2 2^n - (2^n-1)\,log_2(2^n-1) \\
&= -\log_2 2 - \log_2 3 - \cdots - \log_2(2^n-1) + (2^n-1)\,log_2 2^n \\
&= -(\log_2\big(2\times 3\times 4\times\cdots\times(2^n-1)\big) + n(2^n-1) \\
&= n\,2^n - log_2\big((2^n)!\big)
\end{aligned}$$

Homework Exercise

1 Solve, giving your answers to 3 significant figures.

a $2^x = 75$ **b** $3^x = 10$ **c** $5^x = 2$ **d** $4^{2x} = 100$

e $9^{x+5} = 50$ **f** $7^{2x-1} = 23$ **g** $11^{3x-2} = 65$ **h** $2^{3-2x} = 88$

2 Solve, giving your answers to 3 significant figures.

a $2^{2x} - 6(2^x) + 5 = 0$ **b** $3^{2x} - 15(3^x) + 44 = 0$

c $5^{2x} - 6(5^x) - 7 = 0$ **d** $3^{2x} + 3^{x+1} - 10 = 0$

e $7^{2x} + 12 = 7^{x+1}$ **f** $2^{2x} + 3(2^x) - 4 = 0$

g $3^{2x+1} - 26(3^x) - 9 = 0$ **h** $4(3^{2x+1}) + 17(3^x) - 7 = 0$

Hint $3^{x+1} = 3^x \times 3^1 = 3(3^x)$

Problem-solving

Consider these equations as functions of functions. Part **a** is equivalent to $u^2 - 6u + 5 = 0$, with $u = 2^x$.

3 Solve the following equations, giving your answers to 3 significant figures where appropriate.

a $3^{x+1} = 2000$ **(2 marks)**

b $\log_5(x - 3) = -1$ **(2 marks)**

4 **a** Sketch the graph of $y = 4^x$, stating the coordinates of any points where the graph crosses the axes. **(2 marks)**

b Solve the equation $4^{2x} - 10(4^x) + 16 = 0$. **(4 marks)**

Hint Attempt this question without a calculator.

5 Solve the following equations, giving your answers to four decimal places.

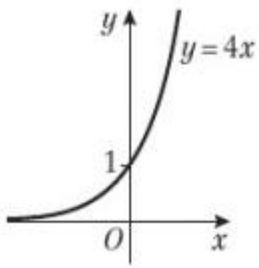
a $5^x = 2^{x+1}$ **b** $3^{x+5} = 6^x$ **c** $7^{x+1} = 3^{x+2}$

Hint Take logs of both sides.

Homework Answers

1	**a** 6.23		**b** 2.10		**c** 0.431	
	d 1.66		**e** –3.22		**f** 1.31	
	g 1.25		**h** –1.73			
2	**a** 0, 2.32		**b** 1.26, 2.18		**c** 1.21	
	d 0.631		**e** 0.565, 0.712		**f** 0	
	g 2		**h** –1			
3	**a** 5.92		**b** 3.2			

4 **a** (0, 1)



b $\frac{1}{2}, \frac{3}{2}$

5 **a** 0.7565 **b** 7.9248 **c** 0.2966