P1 Chapter 11: Vectors

Vector Algebra

Vector Basics

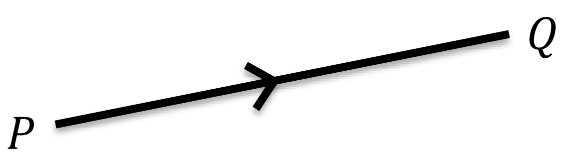
A

A **coordinate** represents a **position** in space, while a **vector** represents a **change** between two positions.

A vector has 2 properties:

- <u>Direction</u>
- <u>Magnitude</u> (i.e. length)

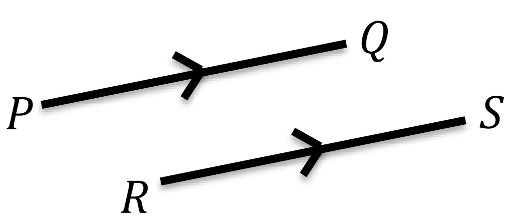
If P and Q are two points, then $\overrightarrow{PQ}$ is the vector between them.



B

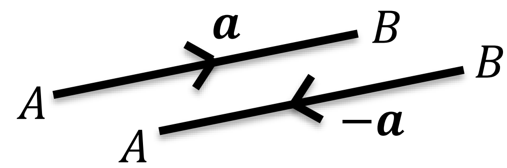
If two vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction, **they're the same vector** and are **parallel**.

This might seem obvious, but students sometimes think the vector is different because the movement occurred at a different point in space. Nope!



C

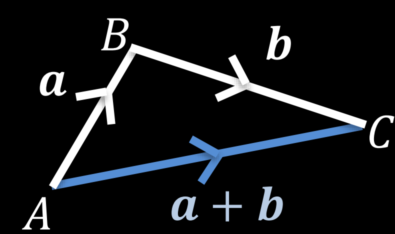
$\overrightarrow{AB} = -\overrightarrow{BA}$ and the two vectors are parallel, equal in magnitude but in **opposite directions**.



D

Triangle Law for vector addition:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



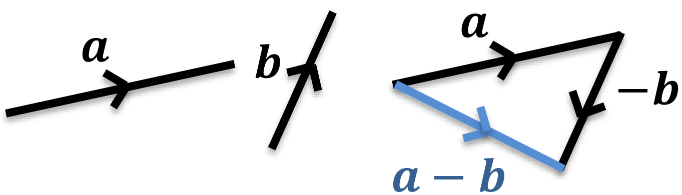
The vector of multiple vectors is known as the **resultant vector.**

(you will encounter this term in Mechanics)

Vector Basics

E Vector **subtraction** is defined using vector addition and negation:

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$



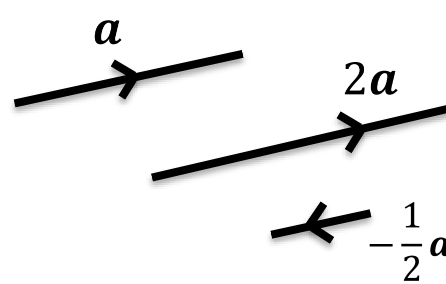
F The zero vector $\mathbf{0}$ (a bold 0), represents no movement.

$$\overrightarrow{PQ} + \overrightarrow{QP} = \mathbf{0}$$

In 2D: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

G A **scalar** is a numerical factor that doesn't depend on direction, hence can be used to 'scale' a vector.

- The **direction** will be the **same**.
- But the **magnitude** will be **different** (unless the scalar is 1).



If scalar is negative, opposite direction.

H Any vector parallel to the vector $\boldsymbol{a}$ can be written as $\lambda\boldsymbol{a}$, where λ is a scalar.

The implication is that if we can write one vector **as a multiple of** another, then we can show they are parallel.

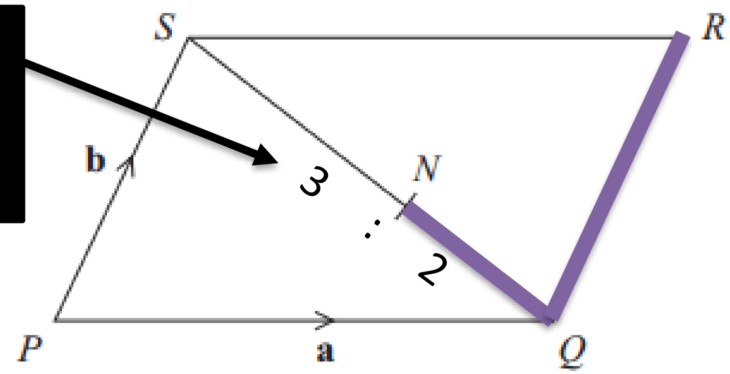
"Show $2\boldsymbol{a} + 4\boldsymbol{b}$ and $3\boldsymbol{a} + 6\boldsymbol{b}$ are parallel".

$3\boldsymbol{a} + 6\boldsymbol{b} = \frac{3}{2}(\boldsymbol{a} + 2\boldsymbol{b})$ $\therefore$ parallel

Example

Edexcel GCSE June 2013 1H Q27

Tip: This ratio wasn't in the original diagram. It was added as a visual aid.



$PQRS$ is a parallelogram.
N is the point on SQ such that $SN : NQ = 3 : 2$
$\overrightarrow{PQ} = \mathbf{a}$ $\quad \overrightarrow{PS} = \mathbf{b}$

(a) Write down, in terms of **a** and **b**, an expression for $\overrightarrow{SQ}$.

(b) Express $\overrightarrow{NR}$ in terms of **a** and **b**.

a $\quad \overrightarrow{SQ} =$?

For (b), there's two possible paths to get from N to R: via S or via Q. But which is best?

?

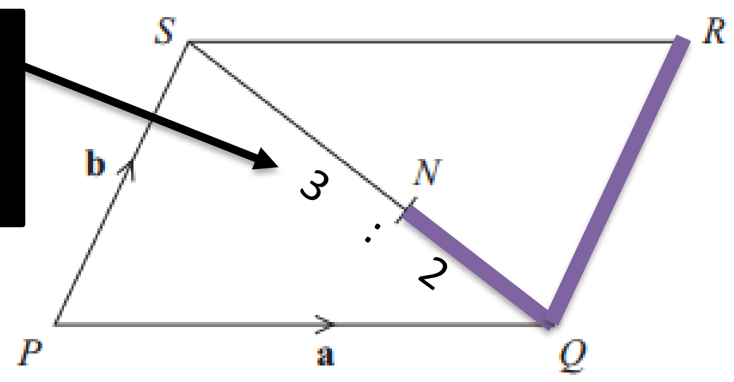
b ?

Fro Workings Tip: While you're welcome to start your working with the second line, I recommend the first line so that your chosen route is clearer.

Example

Edexcel GCSE June 2013 1H Q27

Tip: This ratio wasn't in the original diagram. It was added as a visual aid.



$PQRS$ is a parallelogram.
N is the point on SQ such that $SN : NQ = 3 : 2$
$\overrightarrow{PQ} = \mathbf{a}$ $\quad \overrightarrow{PS} = \mathbf{b}$

(a) Write down, in terms of **a** and **b**, an expression for $\overrightarrow{SQ}$.

(b) Express $\overrightarrow{NR}$ in terms of **a** and **b**.

a

$$\overrightarrow{SQ} = -\boldsymbol{b} + \boldsymbol{a}$$

For (b), there's two possible paths to get from N to R: via S or via Q. But which is best?
In (a) we found S to Q rather than Q to S, so it makes sense to go in this direction so that we can use our result in (a).

b

$$\overrightarrow{NR} = \frac{2}{5}\overrightarrow{SQ} + \boldsymbol{b}$$
$$= \frac{2}{5}(-\boldsymbol{b} + \boldsymbol{a}) + \boldsymbol{b}$$
$$= \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{b}$$

$\overrightarrow{QR}$ is also b because it is exactly the same movement as $\overrightarrow{PS}$.

Fro Workings Tip: While you're welcome to start your working with the second line, I recommend the first line so that your chosen route is clearer.

Test Your Understanding

Edexcel GCSE June 2012

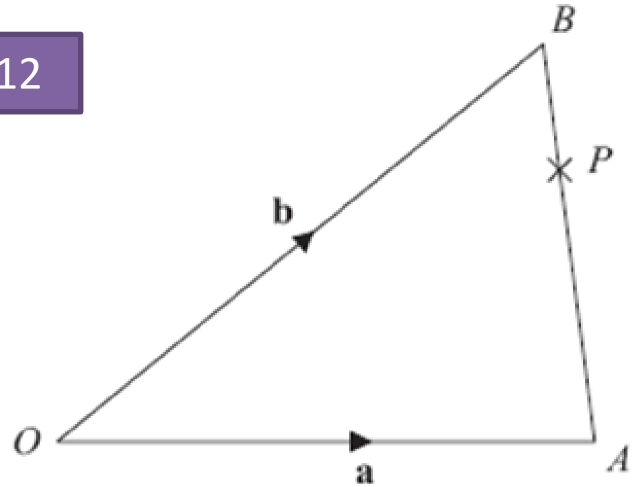


Diagram **NOT** accurately drawn

OAB is a triangle.

$\overrightarrow{OA} = \mathbf{a}$
$\overrightarrow{OB} = \mathbf{b}$

(a) Find $\overrightarrow{AB}$ in terms of **a** and **b**.

?

(1)

?

P is the point on AB such that $AP : PB = 3 : 1$

(b) Find $\overrightarrow{OP}$ in terms of **a** and **b**.
Give your answer in its simplest form.

Test Your Understanding

Edexcel GCSE June 2012

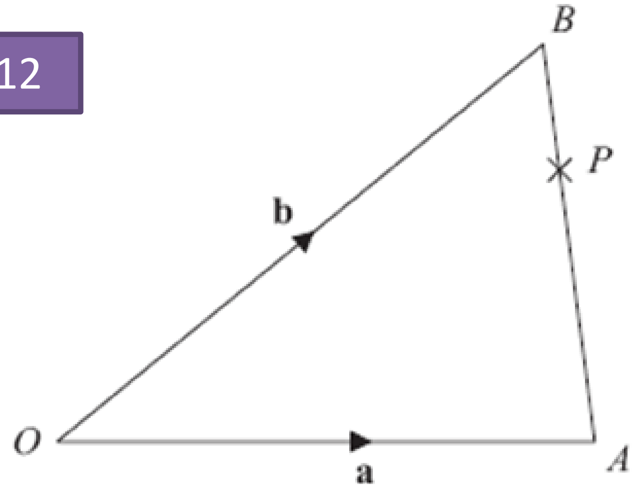


Diagram **NOT** accurately drawn

OAB is a triangle.

$\overrightarrow{OA} = \mathbf{a}$
$\overrightarrow{OB} = \mathbf{b}$

(a) Find $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\boldsymbol{a} + \boldsymbol{b}$$

(1)

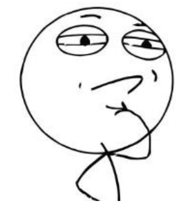
P is the point on AB such that $AP : PB = 3 : 1$

(b) Find $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
Give your answer in its simplest form.

$$\overrightarrow{OP} = \boldsymbol{a} + \frac{3}{4}\overrightarrow{AB}$$
$$= \boldsymbol{a} + \frac{3}{4}(-\boldsymbol{a} + \boldsymbol{b})$$
$$= \frac{1}{4}\boldsymbol{a} + \frac{3}{4}\boldsymbol{b}$$

Just for your interest...

Have you ever wondered what happens if you ‘multiply’ two vectors or two sets?



Erm...

In KS2/3 you probably only experienced variables holding numerical values. You since saw that variables can represent other mathematical types, such as sets or vectors:

$$x = 3$$
$$\boldsymbol{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$T = \{2,5,9,10\}$$

But when we do we need to define **explicitly** what operators like ‘+’ and ‘×’ mean.

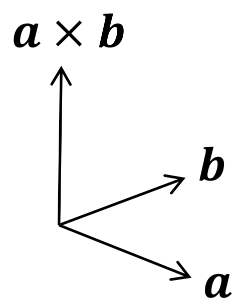
$$1 + 3 = 4 \qquad \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = ? \qquad \{a,b,c\} \times \{d,e\} = ?$$

Often these operators are defined to give it properties that are consistent with its usage elsewhere, e.g. ‘commutativity’: $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ for vector addition just as $2 + 4 = 4 + 2$ for numbers.

In FM, you will see that ‘multiplying’ two 3D vectors (known as the **cross product**) gives you a vector **perpendicular to the two**.
Vector multiplication is not commutative, so $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{b} \times \boldsymbol{a}$ (however it is ‘distributive’, so $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$)
The Casio Classwiz can calculate this in Vector mode!

$$\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$$



‘Multiplying’ **two sets** (known again as the cross product) finds each possible combination of members, one from each:

$$\{a,b,c\} \times \{d,e\} = \{\{a,d\},\{a,e\},\{b,d\},\{b,e\},\{c,d\},\{c,e\}\}$$

It has the nice property that $n(A \times B) = n(A) \times n(B)$, where $n(A)$ gives the size of the set A. Also, $A \times B = B \times A$.

ME-WOW!

Exercise 11.1

Pearson Pure Mathematics Year 1/AS

Pages 86-87

Homework Exercise

1 The diagram shows the vectors **a**, **b**, **c** and **d**.

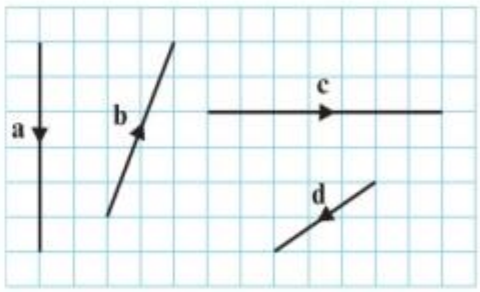
Draw a diagram to illustrate these vectors:

a $\mathbf{a} + \mathbf{c}$ **b** $-\mathbf{b}$

c $\mathbf{c} - \mathbf{d}$ **d** $\mathbf{b} + \mathbf{c} + \mathbf{d}$

e $\mathbf{a} - 2\mathbf{b}$ **f** $2\mathbf{c} + 3\mathbf{d}$

g $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$

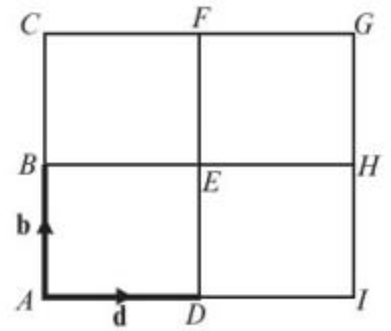


2 *ACGI* is a square, *B* is the midpoint of *AC*, *F* is the midpoint of *CG*, *H* is the midpoint of *GI*, *D* is the midpoint of *AI*. $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AD} = \mathbf{d}$. Find, in terms of **b** and **d**:

a $\overrightarrow{AC}$ **b** $\overrightarrow{BE}$ **c** $\overrightarrow{HG}$ **d** $\overrightarrow{DF}$

e $\overrightarrow{AE}$ **f** $\overrightarrow{DH}$ **g** $\overrightarrow{HB}$ **h** $\overrightarrow{FE}$

i $\overrightarrow{AH}$ **j** $\overrightarrow{BI}$ **k** $\overrightarrow{EI}$ **l** $\overrightarrow{FB}$



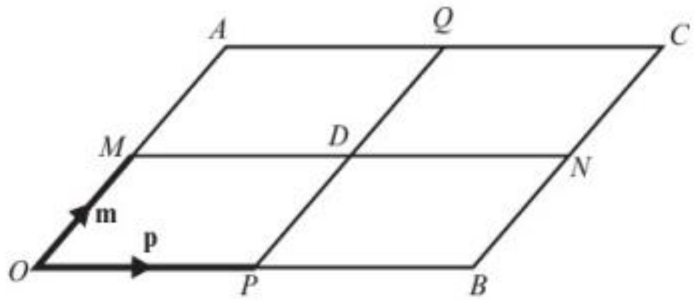
3 *OACB* is a parallelogram. *M*, *Q*, *N* and *P* are the midpoints of *OA*, *AC*, *BC* and *OB* respectively.

Vectors **p** and **m** are equal to $\overrightarrow{OP}$ and $\overrightarrow{OM}$ respectively. Express in terms of **p** and **m**.

a $\overrightarrow{OA}$ **b** $\overrightarrow{OB}$ **c** $\overrightarrow{BN}$ **d** $\overrightarrow{DQ}$

e $\overrightarrow{OD}$ **f** $\overrightarrow{MQ}$ **g** $\overrightarrow{OQ}$ **h** $\overrightarrow{AD}$

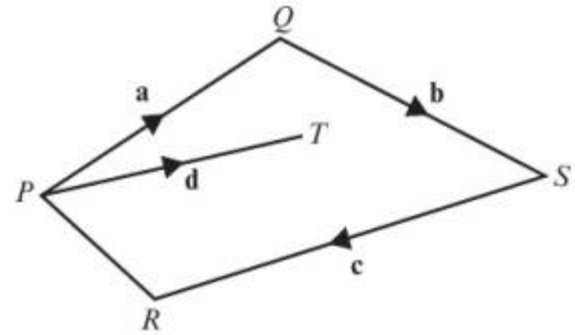
i $\overrightarrow{CD}$ **j** $\overrightarrow{AP}$ **k** $\overrightarrow{BM}$ **l** $\overrightarrow{NO}$



Homework Exercise

4 In the diagram, $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QS} = \mathbf{b}$, $\overrightarrow{SR} = \mathbf{c}$ and $\overrightarrow{PT} = \mathbf{d}$.
Find in terms of **a**, **b**, **c** and **d**:

a $\overrightarrow{QT}$ **b** $\overrightarrow{PR}$
c $\overrightarrow{TS}$ **d** $\overrightarrow{TR}$



5 In the triangle PQR, $PQ = 2\mathbf{a}$ and $QR = 2\mathbf{b}$.
The midpoint of PR is M. Find, in terms of **a** and **b**:

a $\overrightarrow{PR}$ **b** $\overrightarrow{PM}$ **c** $\overrightarrow{QM}$

6 $ABCD$ is a trapezium with AB parallel to DC and $DC = 3AB$.
M divides DC such that $DM : MC = 2 : 1$. $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$.
Find, in terms of **a** and **b**:

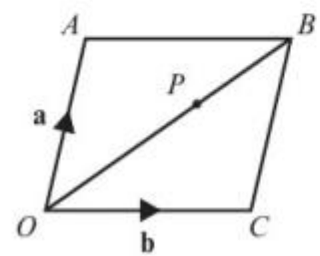
a $\overrightarrow{AM}$ **b** $\overrightarrow{BD}$ **c** $\overrightarrow{MB}$ **d** $\overrightarrow{DA}$

Problem-solving

Draw a sketch to show the information given in the question.

7 $OABC$ is a parallelogram. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{b}$.
The point P divides OB in the ratio 5:3.
Find, in terms of **a** and **b**:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OP}$ **c** $\overrightarrow{AP}$



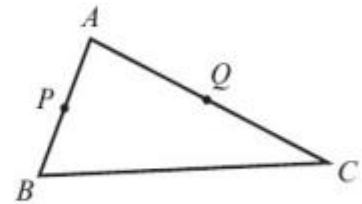
Homework Exercise

8 State with a reason whether each of these vectors is parallel to the vector $\mathbf{a} - 3\mathbf{b}$:

a $2\mathbf{a} - 6\mathbf{b}$ **b** $4\mathbf{a} - 12\mathbf{b}$ **c** $\mathbf{a} + 3\mathbf{b}$ **d** $3\mathbf{b} - \mathbf{a}$ **e** $9\mathbf{b} - 3\mathbf{a}$ **f** $\frac{1}{2}\mathbf{a} - \frac{2}{3}\mathbf{b}$

9 In triangle ABC, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$.

P is the midpoint of AB and Q is the midpoint of AC.

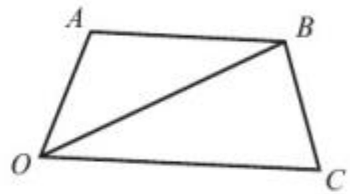


a Write in terms of **a** and **b**:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{AP}$ **iii** $\overrightarrow{AQ}$ **iv** $\overrightarrow{PQ}$

b Show that PQ is parallel to BC.

10 $OABC$ is a quadrilateral. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = 3\mathbf{b}$ and $\overrightarrow{OB} = \mathbf{a} + 2\mathbf{b}$.



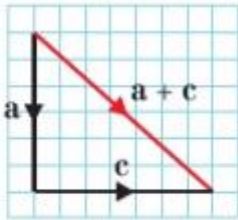
a Find, in terms of **a** and **b**:

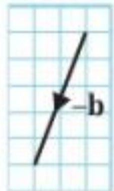
i $\overrightarrow{AB}$ **ii** $\overrightarrow{CB}$

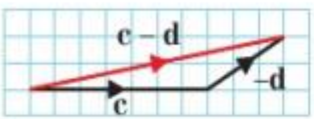
b Show that AB is parallel to OC.

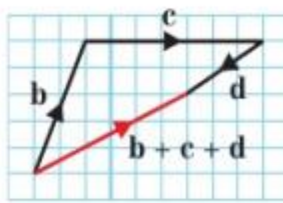
11 The vectors $2\mathbf{a} + k\mathbf{b}$ and $5\mathbf{a} + 3\mathbf{b}$ are parallel. Find the value of k.

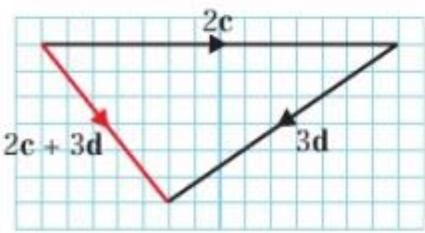
Homework Answers

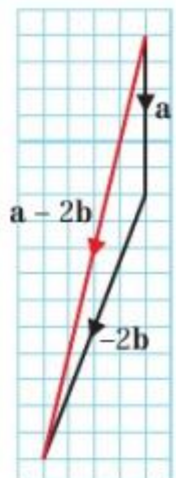
1 a


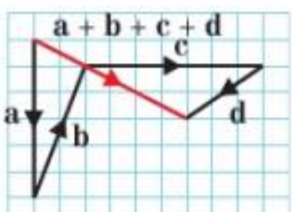
b


c


d


e


f


g


2 **a** 2**b** **b** **d** **c** **b**
d 2**b** **e** **d** + **b** **f** **d** + **b**
g –2**d** **h** –**b** **i** 2**d** + **b**
j –**b** + 2**d** **k** –**b** + **d** **l** –**d** – **b**

3 **a** 2**m** **b** 2**p** **c** **m**
d **m** **e** **p** + **m** **f** **p** + **m**
g **p** + 2**m** **h** **p** – **m** **i** –**m** –**p**
j –2**m** + **p** **k** –2**p** + **m** **l** –**m** – 2**p**

4 **a** **d** – **a** **b** **a** + **b** + **c**
c **a** + **b** – **d** **d** **a** + **b** + **c** – **d**

5 **a** 2**a** + 2**b** **b** **a** + **b** **c** **b** – **a**

6 **a** **b** **b** **b** – 3**a** **c** **a** – **b**
d 2**a** – **b**

7 **a** $\overrightarrow{OB} = \mathbf{a} + \mathbf{b}$ **b** $\overrightarrow{OP} = \frac{5}{8}(\mathbf{a} + \mathbf{b})$ **c** $\overrightarrow{AP} = \frac{5}{8}\mathbf{b} - \frac{3}{8}\mathbf{a}$

8 **a** Yes ($\lambda = 2$) **b** Yes ($\lambda = 4$) **c** No
d Yes ($\lambda = -1$) **e** Yes ($\lambda = -3$) **f** No

9 **a** **i** **b** – **a** **ii** $\frac{1}{2}\mathbf{a}$ **iii** $\frac{1}{2}\mathbf{b}$ **iv** $\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$
b $\overrightarrow{BC} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{PQ} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$ so *PQ* is parallel to *BC*.

10 **a** **i** 2**b** **ii** **a** – **b**
b $\overrightarrow{AB} = 2\mathbf{b}$, $\overrightarrow{OC} = 3\mathbf{b}$ so *AB* is parallel to *OC*.

11 1.2