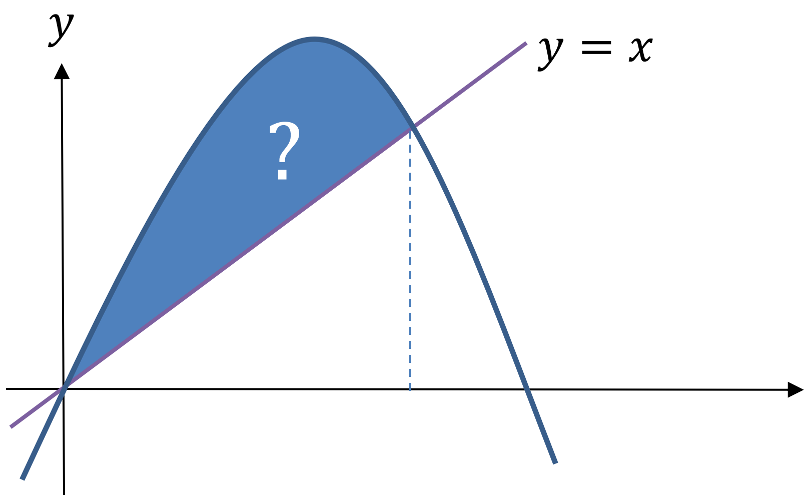
P1 Chapter 13: Integration

Bounded Areas

Areas between curves and lines



How could we find the area between the line and the curve?

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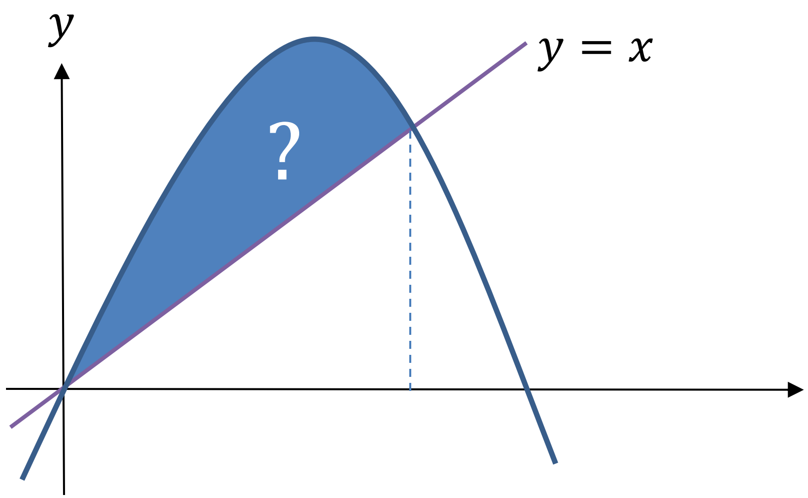
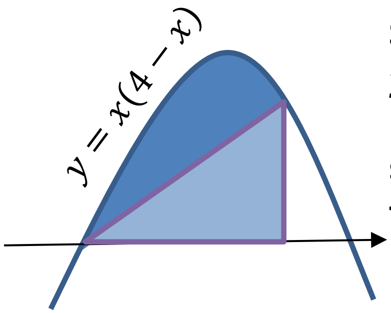
Determine the area between the lines with equations $y = x(4 - x)$ and $y = x$

?

Areas between curves and lines



How could we find the area between the line and the curve?



Start with the area under $y = x(4 - x)$ up to the point of intersection, then subtract the area of the triangle to 'cut it out'.

Determine the area between the lines with equations $y = x(4 - x)$ and $y = x$

Find point of intersection:

$$x(4 - x) = x$$
$$\therefore x = 0 \text{ } or \text{ } x = 3$$

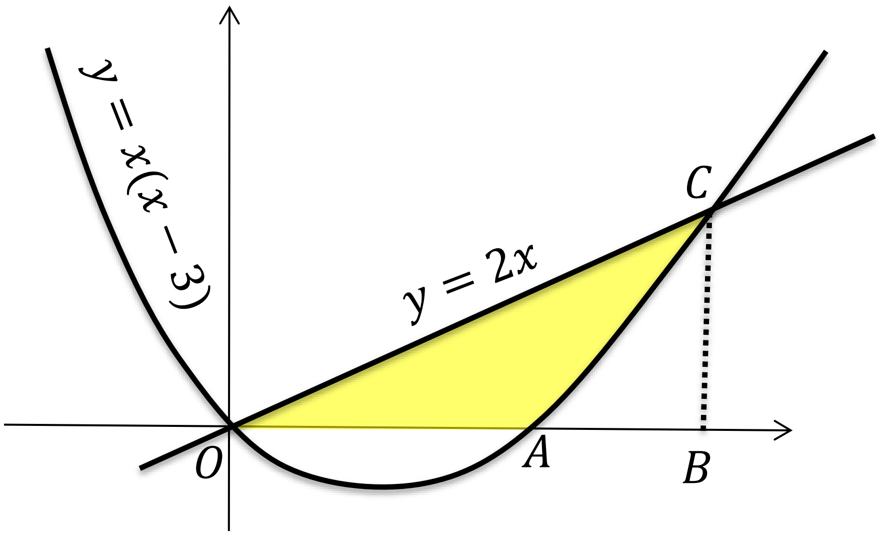
Area under curve:

$$\int_0^3 x(4 - x)\, dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^3 = 9$$

Area of triangle $= \frac{1}{2} \times 3 \times 3 = \frac{9}{2}$

$\therefore$ Shaded area $= 9 - \frac{9}{2} = \frac{9}{2}$

A Harder One



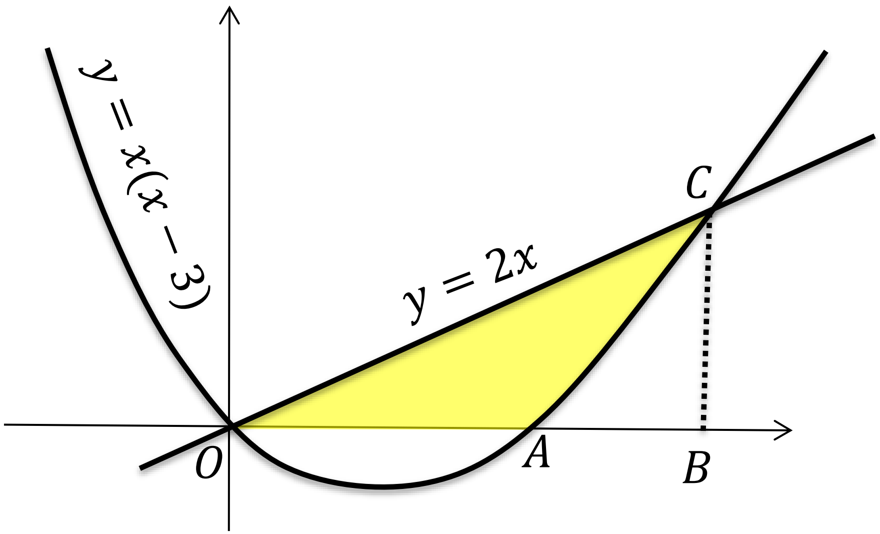
[Textbook] The diagram shows a sketch of the curve with equation $y = x(x-3)$ and the line with equation $y = 2x$.
Find the area of the shaded region OAC.

What areas should we subtract this time?

?

?

A Harder One



[Textbook] The diagram shows a sketch of the curve with equation $y = x(x-3)$ and the line with equation $y = 2x$.
Find the area of the shaded region OAC.

What areas should we subtract this time?
Start with triangle OBC and subtract the area under the curve AC.

First find points of intersection:

$$x(x-3) = 2x \quad \rightarrow \quad x = 0 \text{ or } x = 5$$

When $x = 5,\ y = 10 \quad \rightarrow \quad C(5, 10)$

Also need to find the point A:

$$x(x-3) = 0 \quad \rightarrow \quad A(3, 0)$$

$\therefore$ **Area of triangle** $OBC = \frac{1}{2} \times 5 \times 10 = 25$

Area under AC: $\int_3^5 x(x-3)dx = \ \dots = \frac{26}{3}$

$\therefore$ **Shaded area** $= 25 - \frac{26}{3} = \frac{49}{3}$

Test Your Understanding

Edexcel C2 May 2012 Q5

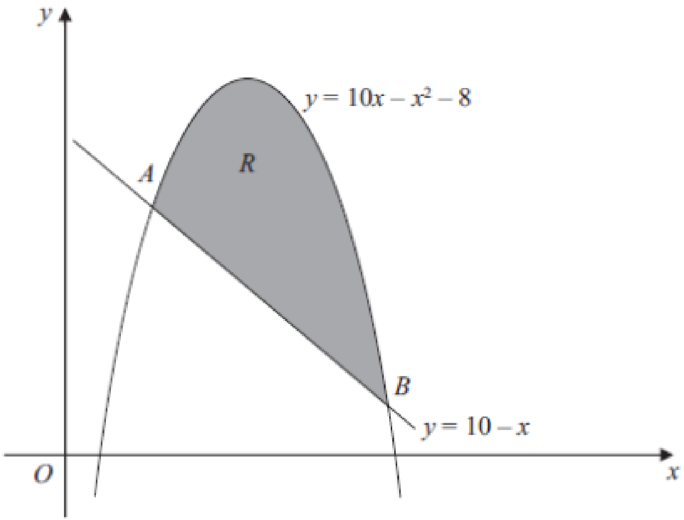


Figure 2 shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$.

The line and the curve intersect at the points A and B, and O is the origin.

(a) Calculate the coordinates of A and the coordinates of B.

(5)

The shaded area R is bounded by the line and the curve, as shown in Figure 2.

(b) Calculate the exact area of R.

(7)

a

? a

b

? b

Alternative Method:

If the top curve has equation $y = f(x)$ and the bottom curve $y = g(x)$, the area between them is:

$$\int_b^a \big(f(x) - g(x)\big)\, dx$$

This means you can integrate a single expression to get the final area, without any adjustment required after.

Test Your Understanding

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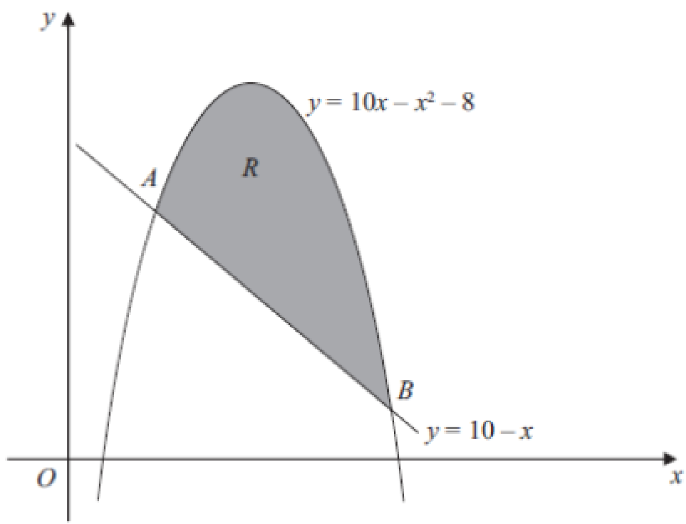

Figure 2 shows the line with equation $y = 10 - x$ and the curve with equation $y = 10x - x^2 - 8$.

The line and the curve intersect at the points A and B, and O is the origin.

(*a*) Calculate the coordinates of A and the coordinates of B. **(5)**

The shaded area R is bounded by the line and the curve, as shown in Figure 2.

(*b*) Calculate the exact area of R. **(7)**

a $A(2,8), \qquad B(9,1)$

b $57\frac{1}{6}$ or $\frac{343}{6}$

Alternative Method:
If the top curve has equation $y = f(x)$ and the bottom curve $y = g(x)$, the area between them is:

$$\int_b^a \left(f(x) - g(x)\right)\,dx$$

This means you can integrate a single expression to get the final area, without any adjustment required after.

Exercise 13.7

Pearson Pure Mathematics Year 1/AS
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Extension

1 *[MAT 2005 1A]* What is the area of the region bounded by the curves $y = x^2$ and $y = x + 2$?

? 1

2 *[MAT 2016 1H]* Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?
(Your answer should be an inequality in terms of a)

? 2

Exercise 13.7

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Extension

1 *[MAT 2005 1A]* What is the area of the region bounded by the curves $y = x^2$ and $y = x + 2$?

$$\mathbf{\frac{9}{2}}$$

2 *[MAT 2016 1H]* Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$? *(Your answer should be an inequality in terms of a)*

(Official solution)
The area bounded by the x-axis and the curve $y = f(x)$, A_1 is equal to

$$A_1 = \int_{\sqrt{a}}^{\sqrt{a}} f(x)\ dx = \frac{4}{3}a^{\frac{3}{2}}$$

whilst the area bounded by the x-axis and the curve $y = g(x)$, A_2 is equal to

$$A_2 = \left| \int_{-\sqrt[4]{a}}^{\sqrt[4]{a}} g(x)dx \right| = \frac{8}{5}a^{\frac{5}{4}}$$

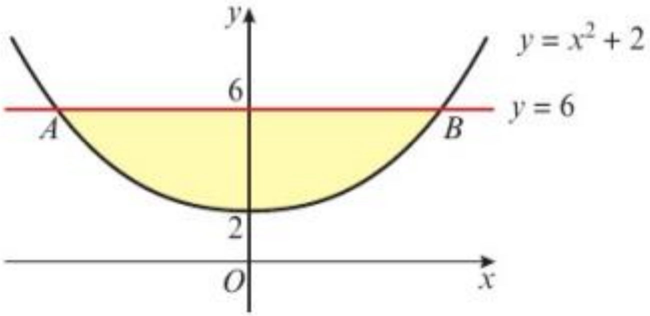
We require an a such that $A_1 > A_2$ so

$$\frac{4}{3}a^{\frac{3}{2}} > \frac{8}{5}a^{\frac{5}{4}}$$
$$20a^{\frac{6}{4}} > 24a^{\frac{5}{4}}$$
$$a^{\frac{1}{4}} > \frac{6}{5}$$

and so the answer is (e).

Homework Exercise

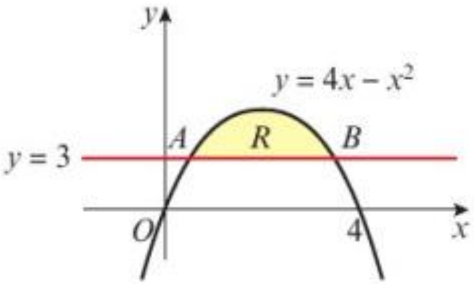
1 The diagram shows part of the curve with equation $y = x^2 + 2$ and the line with equation $y = 6$. The line cuts the curve at the points A and B.



a Find the coordinates of the points A and B.

b Find the area of the finite region bounded by line AB and the curve.

2 The diagram shows the finite region, R, bounded by the curve with equation $y = 4x - x^2$ and the line $y = 3$. The line cuts the curve at the points A and B.

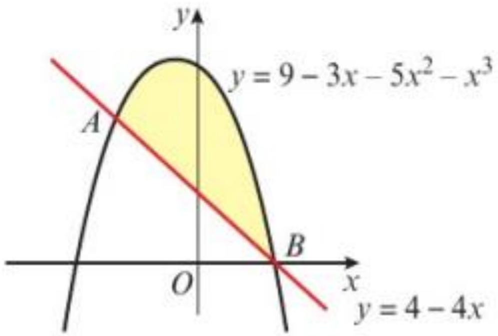


a Find the coordinates of the points A and B.

b Find the area of R.

3 The diagram shows a sketch of part of the curve with equation $y = 9 - 3x - 5x^2 - x^3$ and the line with equation $y = 4 - 4x$. The line cuts the curve at the points $A\ (-1, 8)$ and $B(1, 0)$.

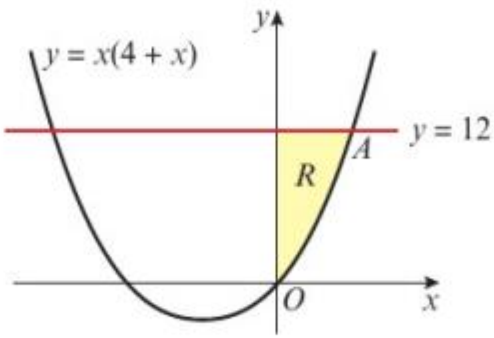
Find the area of the shaded region between AB and the curve.



Homework Exercise

4 Find the area of the finite region bounded by the curve with equation $y = (1 - x)(x + 3)$ and the line $y = x + 3$.

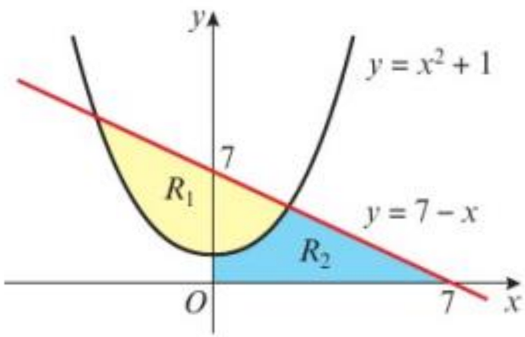
5 The diagram shows the finite region, R, bounded by the curve with equation $y = x(4 + x)$, the line with equation $y = 12$ and the y-axis.



a Find the coordinates of the point A where the line meets the curve.

b Find the area of R.

6 The diagram shows a sketch of part of the curve with equation $y = x^2 + 1$ and the line with equation $y = 7 - x$. The finite region, R_1 is bounded by the line and the curve. The finite region, R_2 is below the curve and the line and is bounded by the positive x- and y-axes as shown in the diagram.



a Find the area of R_1.

b Find the area of R_2.

Homework Exercise

7 The curve C has equation $y = x^{\frac{2}{3}} - \dfrac{2}{x^{\frac{1}{3}}} + 1$.

a Verify that C crosses the x-axis at the point $(1, 0)$.

b Show that the point $A(8, 4)$ also lies on C.

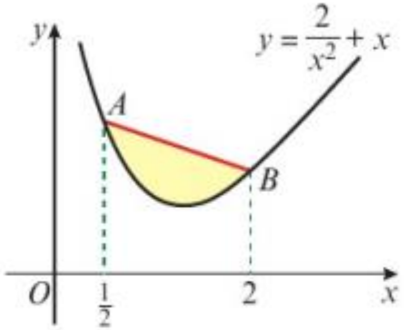
c The point B is $(4, 0)$. Find the equation of the line through AB.

The finite region R is bounded by C, AB and the positive x-axis.

d Find the area of R.

8 The diagram shows part of a sketch of the curve with equation $y = \dfrac{2}{x^2} + x$. The points A and B have x-coordinates $\frac{1}{2}$ and 2 respectively.

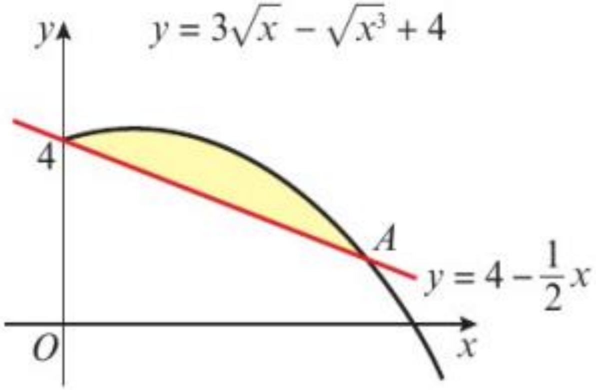
Find the area of the finite region between AB and the curve.



9 The diagram shows part of the curve with equation $y = 3\sqrt{x} - \sqrt{x^3} + 4$ and the line with equation $y = 4 - \frac{1}{2}x$.

a Verify that the line and the curve cross at the point $A(4, 2)$.

b Find the area of the finite region bounded by the curve and the line.



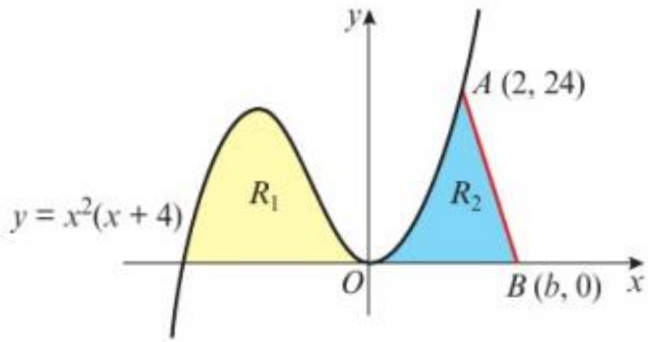
Homework Exercise

10 The sketch shows part of the curve with equation $y = x^2(x + 4)$. The finite region R_1 is bounded by the curve and the negative x-axis. The finite region R_2 is bounded by the curve, the positive x-axis and AB, where $A(2, 24)$ and $B(b, 0)$.

The area of R_1 = the area of R_2.

a Find the area of R_1.

b Find the value of b.



Problem-solving

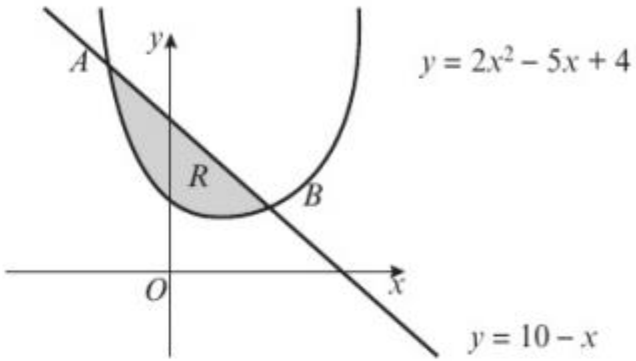
Split R_2 into two areas by drawing a vertical line at $x = 2$.

11 The line with equation $y = 10 - x$ cuts the curve with equation $y = 2x^2 - 5x + 4$ at the points A and B, as shown.

a Find the coordinates of A and the coordinates of B. **(5 marks)**

The shaded region R is bounded by the line and the curve as shown.

b Find the exact area of R. **(6 marks)**



Homework Answers

1 **a** $A(-2, 6), B(2, 6)$ **b** $10\frac{2}{3}$

2 **a** $A(1, 3), B(3, 3)$ **b** $1\frac{1}{3}$

3 $6\frac{2}{3}$

4 4.5

5 **a** (2, 12) **b** $13\frac{1}{3}$

6 **a** $20\frac{5}{6}$ **b** $17\frac{1}{6}$

7 **a, b** Substitute into equation for y

c $y = x - 4$ **d** $8\frac{3}{5}$

8 $3\frac{3}{8}$

9 **a** Substitute $x = 4$ into both equations

b 7.2

10 **a** $21\frac{1}{3}$ **b** $2\frac{5}{9}$

11 **a** (–1, 11) and (3, 7) **b** $21\frac{1}{3}$