P2 Chapter 1: Algebra Techniques

Repeated Partial Fractions

Repeated linear factors

Suppose we wished to express $\frac{2x+1}{(x+1)^2}$ as $\frac{A}{x+1}+\frac{B}{x+1}$. What's the problem?

Because the denominators are the same, we'd get $\frac{A+B}{x+1}$. There's no constant values of A and B we can choose such that $\frac{2x+1}{(x+1)^2} \equiv \frac{A+B}{x+1}$ because the denominators will still be different.

Q Split $\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$ into partial fractions.

The problem is resolved by having the factor **both squared and non-squared** (explanation of why we do this at the end of these slides).

$$\frac{11x^2+14x+5}{(x+1)^2(2x+1)} \equiv \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{2x+1}$$

$$11x^2+14x+5 \equiv A(x+1)(2x+1)+B(2x+1)+C(x+1)^2$$

When $x=-1$: $2=-B \quad \rightarrow \quad B=-2$

When $x=-\frac{1}{2}$: $\frac{3}{4}=\frac{1}{4}C \quad \rightarrow \quad C=3$

At this point we could substitute something else (e.g. $x=1$) but it's easier to equate x^2 terms.

$$11=2A+C$$
$$A=4$$

Test Your Understanding

C4 June 2011 Q1

$$\frac{9x^2}{(x-1)^2(2x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(2x+1)}.$$

Find the values of the constants A, B and C. **(4)**

?

Test Your Understanding

C4 June 2011 Q1

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Find the values of the constants A, B and C. **(4)**

	$9x^2 = A(x-1)(2x+1) + B(2x+1) + C(x-1)^2$		B1
$x \to 1$	$9 = 3B \Rightarrow B = 3$		M1
$x \to -\frac{1}{2}$	$\frac{9}{4} = \left(-\frac{3}{2}\right)^2 C \Rightarrow C = 1$	Any two of A, B, C	A1
x^2 terms	$9 = 2A + C \Rightarrow A = 4$	All three correct	A1

Informal proof of method for repeated factors

If a factor is repeated, why do have a partial fraction with the squared and one without?
When we split into partial fractions, we want each fraction to be non-top-heavy algebraic fractions – recall this means that the 'order' of the numerator has to be less than the order of the denominator. We assume the most generic non-top-heavy fraction possible, i.e. where the order of the numerator is one less than the denominator…

order 1

$$\frac{3x^2+1}{(x+1)x^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2}$$

order 2

Since the denominator is quadratic (order 2), we want the **most generic possible numerator whilst avoiding the fraction being top heavy**, i.e. linear (order 1). Just putting $\frac{B}{x^2}$ wouldn't be good enough because there *might* have been an x term in the numerator.

$$\frac{3x^2+1}{(x+1)x^2} = \frac{A}{x+1} + \frac{B\cancel{x}}{x^{\cancel{2}}} + \frac{C}{x^2}$$

Split the fraction.

$$\frac{3x^2+1}{(x+1)x^2} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2}$$

We can see therefore we have both the x with an without the squared in the denominator.

Informal proof of method for repeated factors

This is easy enough if the repeated factor is x^2, but what about more general repeated linear factors?

$$\frac{3x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x+1)^2}$$

We want something in the numerator where a $(x+1)$ will cancel, so a bit of clever manipulation is required.

$$\frac{3x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B(x+1)-B+C}{(x+1)^2}$$

Split the fraction as before.

$$\frac{3x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B(x+1)}{(x+1)^2} + \frac{C-B}{(x+1)^2}$$

$$\frac{3x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{D}{(x+1)^2}$$

$C-B$ is just a generic constant, so replace with a single constant.

Exercise 1.4

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Homework Exercise

1 $\mathrm{f}(x) = \dfrac{3x^2 + x + 1}{x^2(x+1)}, x \neq 0, x \neq -1$

Given that f(x) can be expressed in the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$, find the values of A, B and C. **(4 marks)**

2 $\mathrm{g}(x) = \dfrac{-x^2 - 10x - 5}{(x+1)^2(x-1)}, x \neq -1, x \neq 1$

Find the values of the constants D, E and F such that $\mathrm{g}(x) = \dfrac{D}{x+1} + \dfrac{E}{(x+1)^2} + \dfrac{F}{x-1}$ **(4 marks)**

3 Given that, for $x < 0$, $\dfrac{2x^2 + 2x - 18}{x(x-3)^2} \equiv \dfrac{P}{x} + \dfrac{Q}{x-3} + \dfrac{R}{(x-3)^2}$, where P, Q and R are constants, find the values of P, Q and R. **(4 marks)**

4 Show that $\dfrac{5x^2 - 2x - 1}{x^3 - x^2}$ can be written in the form $\dfrac{C}{x} + \dfrac{D}{x^2} + \dfrac{E}{x-1}$ where C, D and E are constants to be found. **(4 marks)**

5 $\mathrm{p}(x) = \dfrac{2x}{(x+2)^2}, x \neq -2.$

Find the values of the constants A and B such that $\mathrm{p}(x) = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$ **(4 marks)**

Homework Exercise

6 $\dfrac{10x^2 - 10x + 17}{(2x+1)(x-3)^2} \equiv \dfrac{A}{2x+1} + \dfrac{B}{x-3} + \dfrac{C}{(x-3)^2}, x > 3$

Find the values of the constants A, B and C. **(4 marks)**

7 Show that $\dfrac{39x^2 + 2x + 59}{(x+5)(3x-1)^2}$ can be written in the form $\dfrac{A}{x+5} + \dfrac{B}{3x-1} + \dfrac{C}{(3x-1)^2}$ where A, B and C are constants to be found. **(4 marks)**

8 Express the following as partial fractions:

a $\dfrac{4x+1}{x^2 + 10x + 25}$

b $\dfrac{6x^2 - x + 2}{4x^3 - 4x^2 + x}$

Homework Answers

1 $A = 0, B = 1, C = 3$

2 $D = 3, E = -2, F = -4$

3 $P = -2, Q = 4, R = 2$

4 $C = 3, D = 1, E = 2$

5 $A = 2, B = -4$

6 $A = 2, B = 4, C = 11$

7 $A = 4, B = 1$ and $C = 12$.

8 **a** $\dfrac{4}{x+5} - \dfrac{19}{(x+5)^2}$ **b** $\dfrac{2}{x} - \dfrac{1}{2x-1} + \dfrac{6}{(2x-1)^2}$