P2 Chapter 7: Trigonometric Equations

Using Addition Formulae

Uses of Addition Formulae

Q [Textbook] Using a suitable angle formulae, show that $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$.

?

Q [Textbook] Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$, find the value of: (a) $\cos(A - B)$ (b) $\tan(A + B)$

?

Fro Tip: You can get cos in terms of sin and vice versa by using a rearrangement of $\sin^2 x + \cos^2 x \equiv 1$.
So $\cos A = \sqrt{1 - \sin^2 A}$

Uses of Addition Formulae

Q [Textbook] Using a suitable angle formulae, show that $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$.

$$\sin 15 = \sin(45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Q [Textbook] Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$, find the value of: (a) $\cos(A - B)$ (b) $\tan(A + B)$

$\boldsymbol{cos(A-B) \equiv cos(A)cos(B) + sin(A)sin(B)}$

$\boldsymbol{cos\ A \equiv \sqrt{1 - sin^2 A} = \pm\frac{4}{5}}$. **But we know from the graph of $\boldsymbol{cos}$ that it is negative between $\boldsymbol{180° < A < 270°}$, so $\boldsymbol{cos\ A = -\frac{4}{5}}$.**

Similarly $\boldsymbol{sin\ B = \pm\frac{5}{13}}$, and similarly considering the graph for $\boldsymbol{sin}$, $\boldsymbol{sin\ B = +\frac{5}{13}}$.

$$\therefore cos(A - B) \equiv \left(-\frac{4}{5}\right) \times \left(-\frac{12}{13}\right) + \left(-\frac{3}{5}\right) \times \left(\frac{5}{13}\right) = \frac{33}{65}$$

Continued…

Fro Tip: You can get cos in terms of sin and vice versa by using a rearrangement of $\sin^2 x + \cos^2 x \equiv 1$.
So $\cos A = \sqrt{1 - \sin^2 A}$

Continued…

Q

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$, find the value of: (b) $\tan(A + B)$

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Continued…

Q

Given that $\sin A = -\frac{3}{5}$ and $180° < A < 270°$, and that $\cos B = -\frac{12}{13}$, find the value of: (b) $\tan(A + B)$

$$\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan A \tan B}$$

We earlier found $\sin A = -\frac{3}{5}, \sin B = \frac{5}{13}, \cos A = -\frac{4}{5}, \cos B = -\frac{12}{13}$

$\therefore \tan A = \left(-\frac{3}{5}\right) \div \left(-\frac{4}{5}\right) = \frac{3}{4}$ **and** $\tan B = \frac{5}{13} \div \left(-\frac{12}{13}\right) = -\frac{5}{12}$

By substitution:

$$\tan(A+B) = \frac{16}{63}$$

Test Your Understanding

Without using a calculator, determine the exact value of:

a) $\cos(75°)$

b) $\tan(75°)$

?

?

Test Your Understanding

Without using a calculator, determine the exact value of:

a) $\cos(75°)$

b) $\tan(75°)$

$$\cos(75°) = \cos(45° + 30°) = \cos(45°)\cos(30°) - \sin(45°)\sin(30°)$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\tan(75°) = \frac{\sin(75°)}{\cos(75°)}$$

$$\sin(75°) = \sin(45° + 30°) = \sin(45°)\cos(30°) + \cos(45°)\sin(30°)$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\therefore \tan(75°) = \left(\frac{\sqrt{6} + \sqrt{2}}{4}\right) \div \left(\frac{\sqrt{6} - \sqrt{2}}{4}\right) = \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} = 2 + \sqrt{3}$$

Challenging question

Edexcel June 2013 Q3

Given that

$$2\cos(x+50)^\circ = \sin(x+40)^\circ.$$

(*a*) Show, without using a calculator, that

$$\tan x^\circ = \frac{1}{3}\tan 40^\circ.$$

(*b*) Hence solve, for $0 \leq \theta < 360$,

$$2\cos(2\theta+50)^\circ = \sin(2\theta+40)^\circ,$$

giving your answers to 1 decimal place.

a

?

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a

'Expanding' both sides:

$$2\cos x\cos 50 - 2\sin x\sin 50$$
$$= \sin x\cos 40 + \cos x\sin 40$$

Since thing to prove only has 40 in it, use $\cos 50 = \sin 40$ and $\sin 50 = \cos 40$.

$$2\cos x\sin 40 - 2\sin x\cos 40$$
$$= \sin x\cos 40 + \cos x\sin 40$$

As per usual, when we want tans, divide by $\cos x\cos 40$:

$$2\tan 40 - 2\tan x = \tan x + \tan 40$$
$$\tan 40 = 3\tan x$$
$$\frac{1}{3}\tan 40 = \tan x$$

Exercise 7.2

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Extension

[STEP I 2010 Q3] Show that

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y$$

and deduce that

$$\sin A-\sin B=2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

Show also that

$$\cos A-\cos B=-2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$ and $(a\cos s, b\sin s)$ respectively, where $0\le p<q<r<s<2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r+s-p-q=2\pi$$

Homework Exercise

1 Without using your calculator, find the exact value of:

a $\cos 15°$ **b** $\sin 75°$ **c** $\sin(120° + 45°)$ **d** $\tan 165°$

2 Without using your calculator, find the exact value of:

a $\sin 30° \cos 60° + \cos 30° \sin 60°$

b $\cos 110° \cos 20° + \sin 110° \sin 20°$

c $\sin 33° \cos 27° + \cos 33° \sin 27°$

d $\cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$

e $\sin 60° \cos 15° - \cos 60° \sin 15°$

f $\cos 70° (\cos 50° - \tan 70° \sin 50°)$

g $\dfrac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°}$

h $\dfrac{1 - \tan 15°}{1 + \tan 15°}$

i $\dfrac{\tan\frac{7\pi}{12} - \tan\frac{\pi}{3}}{1 + \tan\frac{7\pi}{12}\tan\frac{\pi}{3}}$

j $\sqrt{3}\cos 15° - \sin 15°$

3 **a** Express $\tan(45° + 30°)$ in terms of $\tan 45°$ and $\tan 30°$. **(2 marks)**

b Hence show that $\tan 75° = 2 + \sqrt{3}$. **(2 marks)**

4 Given that $\cot A = \frac{1}{4}$ and $\cot(A + B) = 2$, find the value of $\cot B$.

5 **a** Using $\cos(\theta + \alpha) \equiv \cos\theta\cos\alpha - \sin\theta\sin\alpha$, or otherwise, show that $\cos 105° = \dfrac{\sqrt{2} - \sqrt{6}}{4}$ **(4 marks)**

b Hence, or otherwise, show that $\sec 105° = -\sqrt{a}(1 + \sqrt{b})$, where a and b are constants to be found. **(3 marks)**

Homework Exercise

6 Given that $\sin A = \frac{4}{5}$ and $\sin B = \frac{1}{2}$, where A and B are both acute angles, calculate the exact value of:

a $\sin(A + B)$ **b** $\cos(A - B)$ **c** $\sec(A - B)$

7 Given that $\cos A = -\frac{4}{5}$, and A is an obtuse angle measured in radians, find the exact value of:

a $\sin A$ **b** $\cos(\pi + A)$ **c** $\sin\left(\frac{\pi}{3} + A\right)$ **d** $\tan\left(\frac{\pi}{4} + A\right)$

8 Given that $\sin A = \frac{8}{17}$, where A is acute, and $\cos B = -\frac{4}{5}$, where B is obtuse, calculate the exact value of:

a $\sin(A - B)$ **b** $\cos(A - B)$ **c** $\cot(A - B)$

9 Given that $\tan A = \frac{7}{24}$, where A is reflex, and $\sin B = \frac{5}{13}$, where B is obtuse, calculate the exact value of:

a $\sin(A + B)$ **b** $\tan(A - B)$ **c** $\operatorname{cosec}(A + B)$

10 Given that $\tan A = \frac{1}{5}$ and $\tan B = \frac{2}{3}$, calculate, without using your calculator, the value of $A + B$ in degrees, where:

a A and B are both acute,

b A is reflex and B is acute.

Homework Answers

1 **a** $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$ **b** $\frac{\sqrt{2}(\sqrt{3}+1)}{4}$ **c** $\frac{\sqrt{2}(\sqrt{3}-1)}{4}$ **d** $\sqrt{3}-2$

2 **a** 1 **b** 0 **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{\sqrt{2}}{2}$ **e** $\frac{\sqrt{2}}{2}$

f $-\frac{1}{2}$ **g** $\sqrt{3}$ **h** $\frac{\sqrt{3}}{3}$ **i** 1 **j** $\sqrt{2}$

3 **a** $\tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

b $$\tan 75° = \frac{1 + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}} = \frac{3+\sqrt{3}}{3-\sqrt{3}} = \frac{(3+\sqrt{3})(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})}$$

$$= \frac{12+6\sqrt{3}}{9-3} = 2+\sqrt{3}$$

4 $-\frac{6}{7}$

5 **a** $$\cos 105° = \cos(45° + 60°)$$
$$= \cos 45° \cos 60° - \sin 45° \sin 60°$$
$$= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

b $a = 2, b = 3$

6 **a** $\frac{3+4\sqrt{3}}{10}$ **b** $\frac{4+3\sqrt{3}}{10}$ **c** $\frac{10(3\sqrt{3}-4)}{11}$

7 **a** $\frac{3}{5}$ **b** $\frac{4}{5}$ **c** $\frac{3-4\sqrt{3}}{10}$ **d** $\frac{1}{7}$

8 **a** $-\frac{77}{85}$ **b** $-\frac{36}{85}$ **c** $\frac{36}{77}$

9 **a** $-\frac{36}{325}$ **b** $\frac{204}{253}$ **c** $-\frac{325}{36}$

10 **a** $45°$ **b** $225°$