P1 Chapter 4: Transforming Graphs

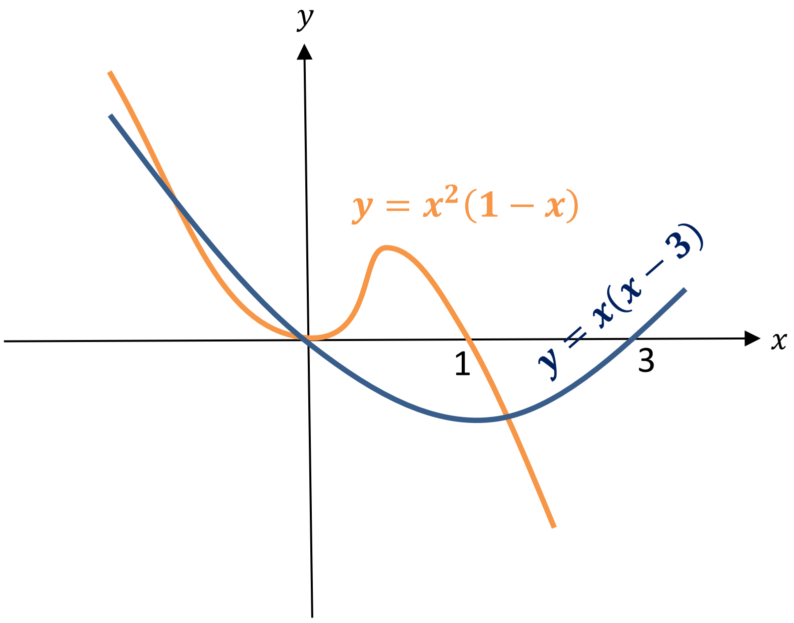
Points of Intersection

Points of Intersection

In the previous chapter we saw why the points of intersection of two graphs gave the solutions to the simultaneous equations corresponding to these graphs.

If $y = f(x)$ and $y = g(x)$, then the x values of the points of intersection can be found when $f(x) = g(x)$.

Example: On the same diagram sketch the curves with equations $y = x(x-3)$ and $y = x^2(1-x)$. Find the coordinates of their points of intersection.



$$x(x-3) = x^2(1-x)$$
$$x^2 - 3x = x^2 - x^3$$
$$x^3 - 3x = 0$$
$$x(x^2 - 3) = 0$$

$x = 0$ or $x = -\sqrt{3}$ or $x = +\sqrt{3}$

Substituting these values back into either equation, we obtain points:

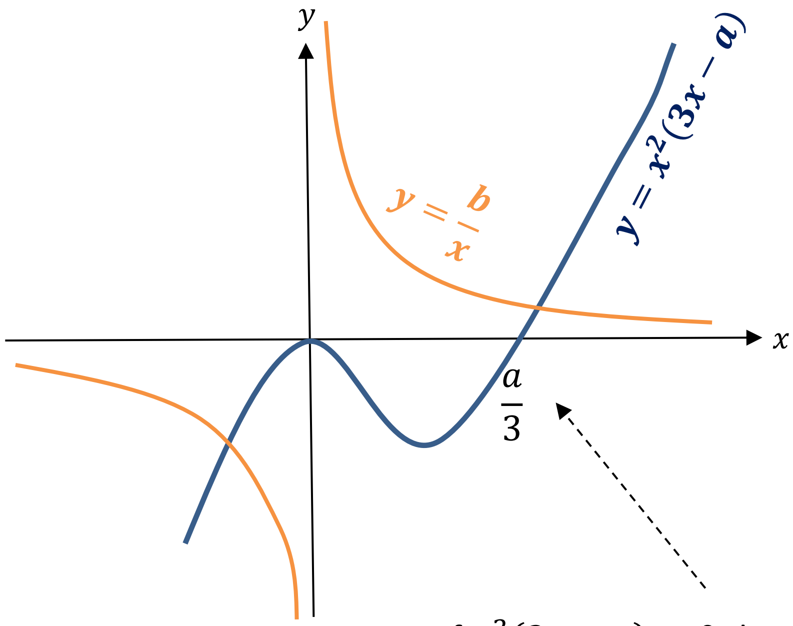
$$(-\sqrt{3}, 3 + 3\sqrt{3}),$$
$$(0,0),$$
$$(\sqrt{3}, 3 - 3\sqrt{3})$$

Note: Cubics generally have 3 solutions. And this seems good news as we have 3 points of intersection.

Tip: A classic mistake is to divide by x to get $x^2 - 3 = 0$. NEVER divide an equation by a variable, because you lose a solution. Always factorise.

Further example involving unknown constants

On the same diagram sketch the curves with equations $y = x^2(3x - a)$ and $y = \frac{b}{x}$, where a, b are positive constants. State, giving a reason, the number of real solutions to the equation $x^2(3x - a) - \frac{b}{x} = 0$



If $x^2(3x - a) = 0$ then $x = 0$ or $x = \frac{a}{3}$
We were told that a is positive, thus this latter root is positive.

If the points of intersection are given by:

$$x^2(3x - a) = \frac{b}{x}$$

then clearly:

$$x^2(3x - a) - \frac{b}{x} = 0$$

There are 2 points of intersection, thus **2 solutions to this equation**.

Note: Note that the question is asking for the number of solutions, not the solutions themselves. We'd have to solve a quartic, with roots in terms of a and b. While there is a 'quartic formula' (like the quadratic formula), it is absolutely horrific.

Test Your Understanding

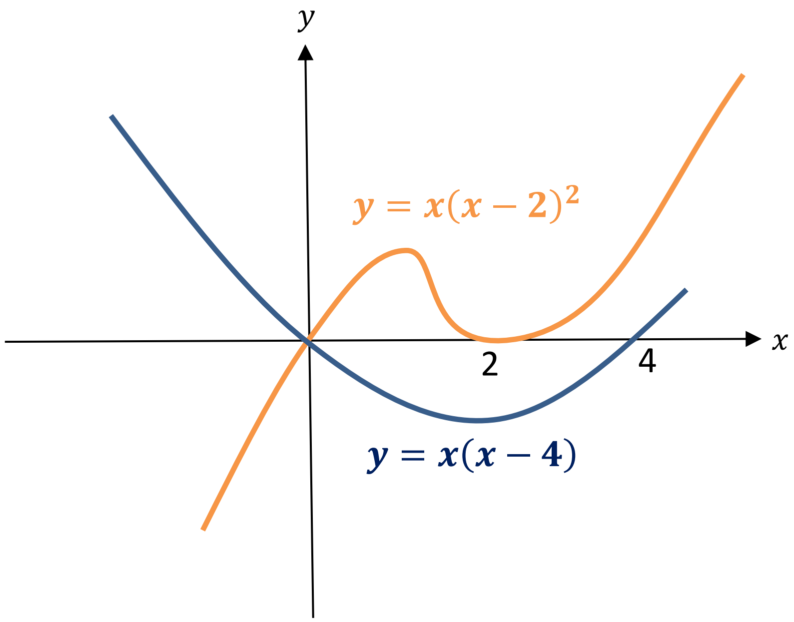
On the same diagram sketch the curves with equations $y = x(x-4)$ and $y = x(x-2)^2$, and hence find the coordinates of any points of intersection.

?

Hint: Remember you can use the discriminant to reason about the number of solutions of a quadratic.

Test Your Understanding

On the same diagram sketch the curves with equations $y = x(x-4)$ and $y = x(x-2)^2$, and hence find the coordinates of any points of intersection.



Looking at the diagram we expect that $(0,0)$ will be the only point of intersection (as the cubic will rise more rapidly than the quadratic). But we need to show this algebraically.

$$x(x-2)^2 = x(x-4)$$
$$x(x^2-4x+4) = x^2-4x$$
$$x^3-4x^2+4x = x^2-4x$$
$$x^3-5x^2+8x = 0$$
$$x(x^2-5x+8) = 0$$

Thus $x = 0$ giving $(0,0)$.

But the discriminant of x^2-5x+8 is -7, thus there are no further solutions to this equation.

Hint: Remember you can use the discriminant to reason about the number of solutions of a quadratic.

Exercise 4.4

Pearson Pure Mathematics Year 1/AS
Pages 1

Extension

1 [MAT 2010 1A] The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely

A) $k \leq 0$ B) $k \geq -4$

C) $k \geq 0$ or $k \leq -4$ D) $-4 \leq k \leq 0$

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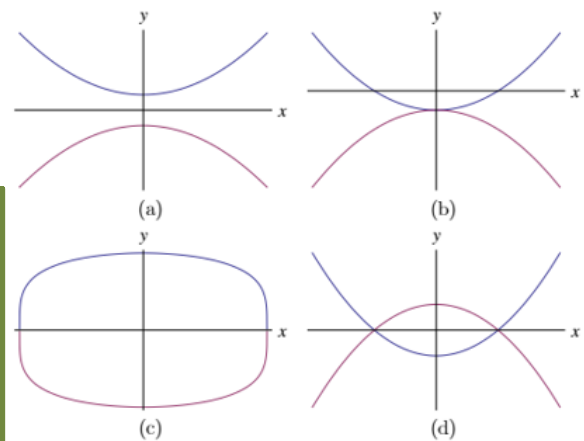
2 [MAT 2013 1D]
Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



?

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1 [MAT 2010 1A] The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely

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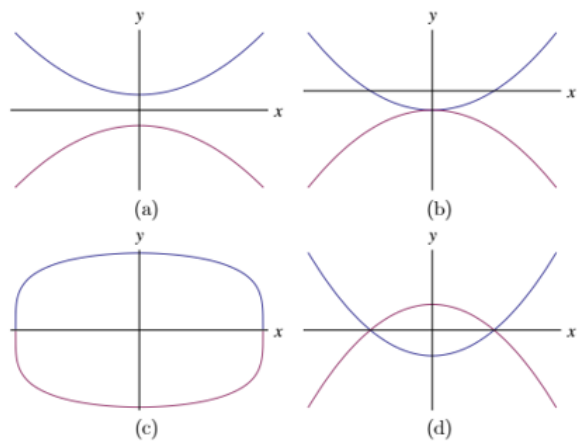
Equating:

$$\boldsymbol{(x-1)^2 = kx}$$
$$\boldsymbol{x^2 - 2x + 1 = kx}$$
$$\boldsymbol{x^2 + (-2-k)x + 1 = 0}$$

Discriminant:

$\boldsymbol{a = 1, b = -2-k, c = 1} \rightarrow \boldsymbol{k^2 + 4k + 4 - 4 \geq 0}$
$\boldsymbol{k(k+4) \geq 0}$
$\boldsymbol{k \leq -4 \text{ or } k \geq 0}$

2 [MAT 2013 1D]
Which of the following sketches is a graph of $x^4 - y^2 = 2y + 1$?



$$x^4 = y^2 + 2y + 1 = (y+1)^2$$
$$\therefore x^2 = \pm(y+1)$$

i.e. $y = x^2 - 1$
$or\ y = -x^2 - 1$

Answer is (b).

Homework Exercise

1 In each case:

i sketch the two curves on the same axes

ii state the number of points of intersection

iii write down a suitable equation which would give the x-coordinates of these points. (You are not required to solve this equation.)

a $y = x^2,\ y = x(x^2 - 1)$

b $y = x(x + 2),\ y = -\frac{3}{x}$

c $y = x^2,\ y = (x + 1)(x - 1)^2$

d $y = x^2(1 - x),\ y = -\frac{2}{x}$

e $y = x(x - 4),\ y = \frac{1}{x}$

f $y = x(x - 4),\ y = -\frac{1}{x}$

g $y = x(x - 4),\ y = (x - 2)^3$

h $y = -x^3,\ y = -\frac{2}{x}$

i $y = -x^3,\ y = x^2$

j $y = -x^3,\ y = -x(x + 2)$

k $y = 4,\ y = x(x - 1)(x + 2)^2$

l $y = x^3,\ y = x^2(x + 1)^2$

2 **a** On the same axes sketch the curves given by $y = x^2(x - 3)$ and $y = \frac{2}{x}$

b Explain how your sketch shows that there are only two real solutions to the equation $x^3(x - 3) = 2$.

3 **a** On the same axes sketch the curves given by $y = (x + 1)^3$ and $y = 3x(x - 1)$.

b Explain how your sketch shows that there is only one real solution to the equation $x^3 + 6x + 1 = 0$.

4 **a** On the same axes sketch the curves given by $y = \frac{1}{x}$ and $y = -x(x - 1)^2$.

b Explain how your sketch shows that there are no real solutions to the equation $1 + x^2(x - 1)^2 = 0$.

Homework Exercise

Problem-solving

Even though you don't know the values of a and b, you know they are positive, so you know the shapes of the graphs. You can label the point a on the x-axis on your sketch of $y = x^2(x + a)$.

5 **a** On the same axes sketch the curves given by $y = x^2(x + a)$ and $y = \frac{b}{x}$ where a and b are both positive constants. **(5 marks)**

b Using your sketch, state, giving a reason, the number of real solutions to the equation $x^4 + ax^3 - b = 0$. **(1 mark)**

6 **a** On the same set of axes sketch the graphs of $y = \frac{4}{x^2}$ and $y = 3x + 7$. **(3 marks)**

b Write down the number of real solutions to the equation $\frac{4}{x^2} = 3x + 7$. **(1 mark)**

c Show that you can rearrange the equation to give $(x + 1)(x + 2)(3x - 2) = 0$. **(2 marks)**

d Hence determine the exact coordinates of the points of intersection. **(3 marks)**

7 **a** On the same axes sketch the curve $y = x^3 - 3x^2 - 4x$ and the line $y = 6x$.

b Find the coordinates of the points of intersection.

8 **a** On the same axes sketch the curve $y = (x^2 - 1)(x - 2)$ and the line $y = 14x + 2$.

b Find the coordinates of the points of intersection.

9 **a** On the same axes sketch the curves with equations $y = (x - 2)(x + 2)^2$ and $y = -x^2 - 8$.

b Find the coordinates of the points of intersection.

Homework Exercise

10 **a** Sketch the graphs of $y = x^2 + 1$ and $2y = x - 1$. **(3 marks)**

b Explain why there are no real solutions to the equation $2x^2 - x + 3 = 0$. **(2 marks)**

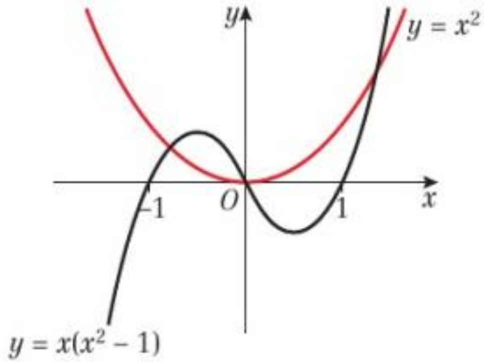
c Work out the range of values of a such that the graphs of $y = x^2 + a$ and $2y = x - 1$ have two points of intersection. **(5 marks)**

11 **a** Sketch the graphs of $y = x^2(x - 1)(x + 1)$ and $y = \frac{1}{3}x^3 + 1$. **(5 marks)**

b Find the number of real solutions to the equation $3x^2(x - 1)(x + 1) = x^3 + 3$. **(1 mark)**

Homework Answers

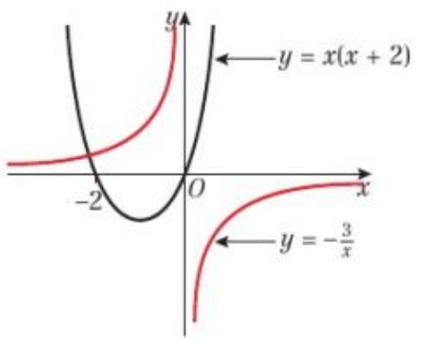
1 a i



ii 3

iii $x^2 = x(x^2 - 1)$

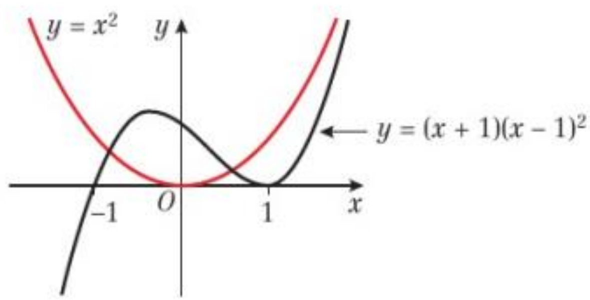
b i



ii 1

iii $x(x + 2) = -\dfrac{3}{x}$

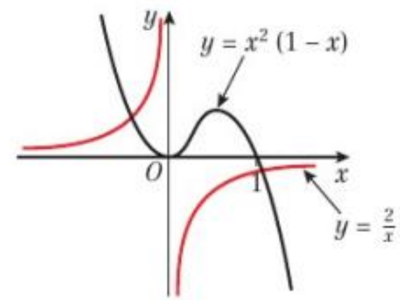
c i



ii 3

iii $x^2 = (x + 1)(x - 1)^2$

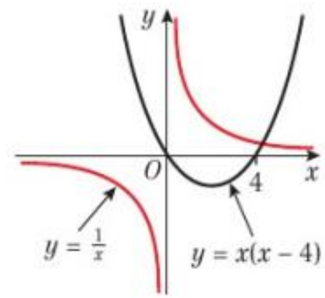
d i



ii 2

iii $x^2(1 - x) = -\dfrac{2}{x}$

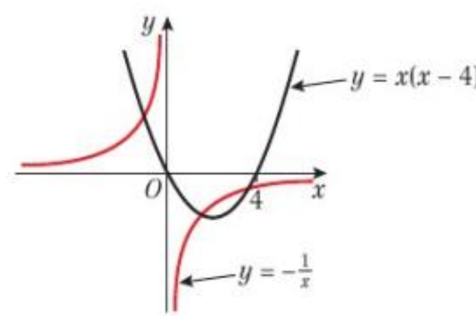
e i



ii 1

iii $x(x - 4) = \dfrac{1}{x}$

f i

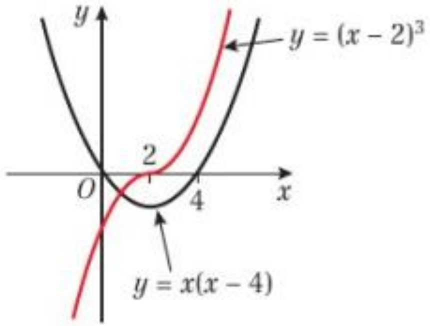


ii 3

iii $x(x - 4) = -\dfrac{1}{x}$

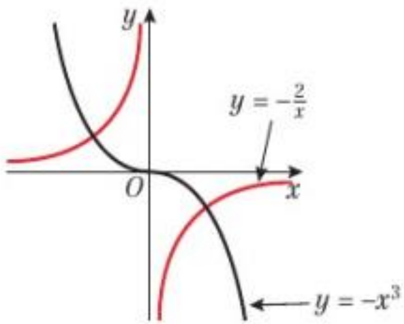
Homework Answers

1 **g** **i**



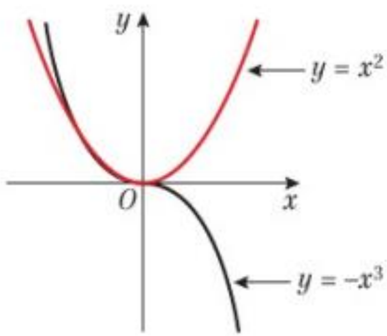
ii 1 **iii** $x(x-4) = (x-2)^3$

h **i**



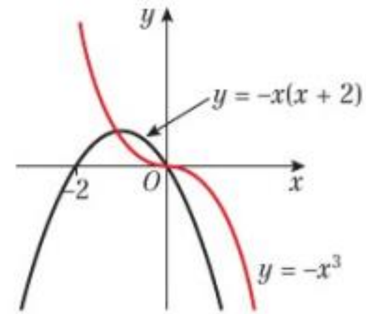
ii 2 **iii** $-x^3 = -\frac{2}{x}$

i **i**



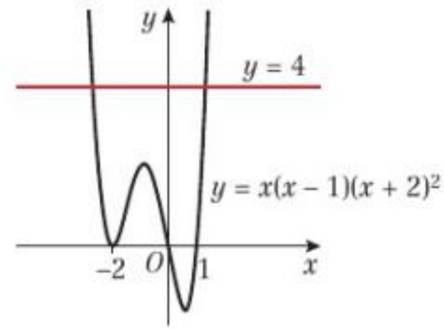
ii 2 **iii** $-x^3 = x^2$

j **i**



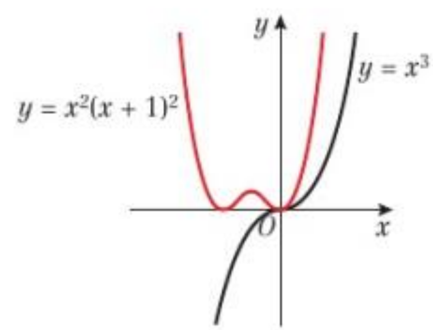
ii 3 **iii** $-x^3 = -x(x+2)$

k **i**



ii 2 **iii** $x(x-1)(x+2)^2 = 4$

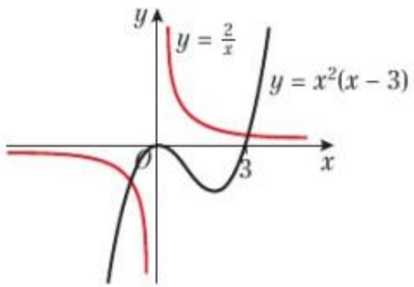
l **i**



ii 1 **iii** $x^3 = x^2(x+1)^2$

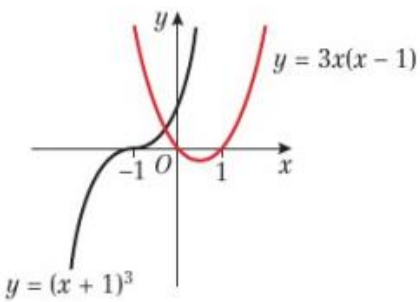
Homework Answers

2 **a**



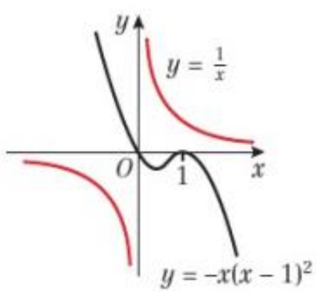
b Only 2 intersections

3 **a**



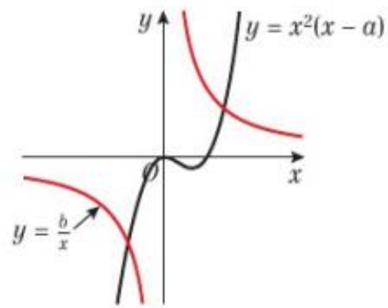
b Only 1 intersection

4 **a**



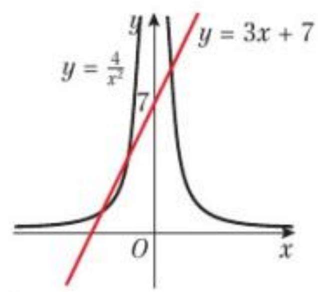
b Graphs do not intersect

5 **a**



b 2; the graphs cross in two places so there are two solutions.

6 **a**

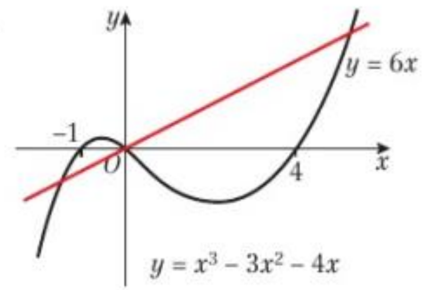


b 3

c Expand brackets and rearrange.

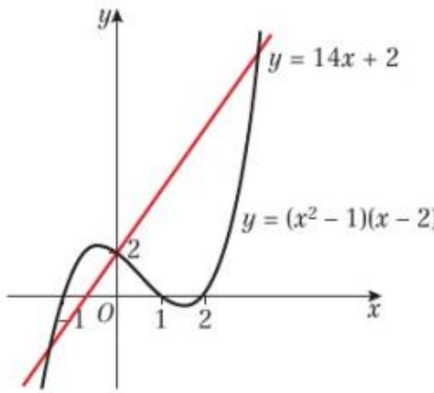
d $(-2, 1)$, $(-1, 4)$, $(\frac{2}{3}, 9)$

7 **a**



b $(0, 0)$; $(-2, -12)$; $(5, 30)$

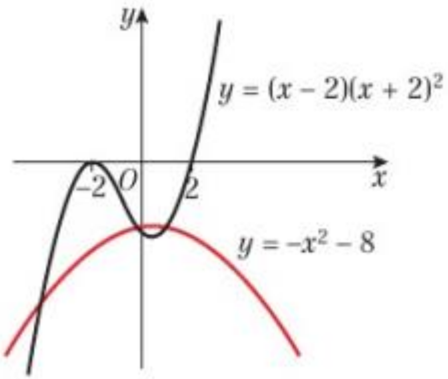
8 **a**



b $(0, 2)$; $(-3, -40)$; $(5, 72)$

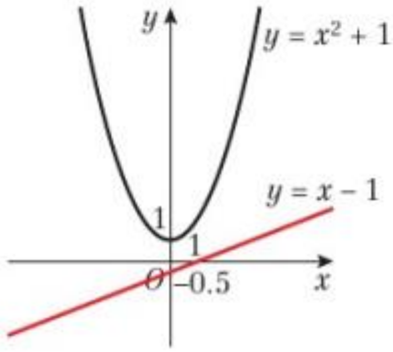
Homework Answers

9 **a**



b $(0, -8)$; $(1, -9)$; $(-4, -24)$

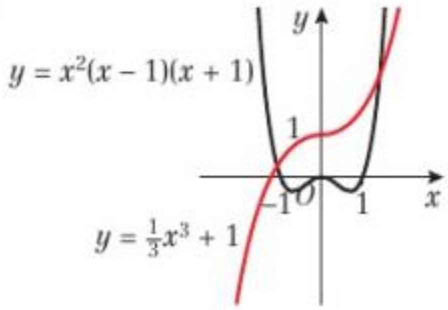
10 **a**



b Graphs do not intersect.

c $a < -\frac{7}{16}$

11 **a**



b 2