P2 Chapter 3: Sequences and Series

Recurrence Relations

Recurrence Relations

$$u_n = 2n^2 + 3$$

This is an example of a position-to-term sequence, because each term is based on the position n.

$$u_{n+1} = 2u_n + 4$$
$$u_1 = 3$$

We need the first term because the recurrence relation alone is not enough to know what number the sequence starts at.

But a term might be defined based on previous terms.
If $\boldsymbol{u_n}$ refers to the current term, $\boldsymbol{u_{n+1}}$ refers to the next term.
So, the example in words says "the next term is twice the previous term + 4"

This is known as a term-to-term sequence, or more formally as a **recurrence relation**, as the sequence 'recursively' refers to itself.

Example

Important Note: With recurrence relation questions, the **the sequence will likely not be arithmetic nor geometric**. So your previous u_n and S_n formulae do not apply.

Edexcel C1 May 2013 (R)

6. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$
$$x_{n+1} = (x_n)^2 - kx_n, \qquad n \geq 1,$$

where k is a constant.

(*a*) Find an expression for x_2 in terms of k. **(1)**

$$x_2 = (x_1)^2 - kx_1 = 1 - k$$

(*b*) Show that $x_3 = 1 - 3k + 2k^2$. **(2)**

$$\begin{aligned} x_3 &= (x_2)^2 - kx_2 \\ &= (1-k)^2 - k(1-k) \\ &= 1 - 3k + 2k^2 \end{aligned}$$

Given also that $x_3 = 1$,

(*c*) calculate the value of k. **(3)**

$$k = \frac{3}{2}$$

(*d*) Hence find the value of $\sum_{n=1}^{100} x_n$. **(3)**

$$= 1 + \left(-\frac{1}{2}\right) + 1 + \left(-\frac{1}{2}\right) + \cdots$$
$$= 50 \times \left(1 - \frac{1}{2}\right) = 25$$

Tip: When a Σ question comes up for recurrence relations, it will most likely be some kind of repeating sequence. Just work out the sum of terms in each repeating bit, and how many times it repeats.

Test Your Understanding

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4. A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = a x_n + 5, \qquad n \geq 1,$$

where a is a constant.

(*a*) Write down an expression for x_2 in terms of a. ? **(1)**

(*b*) Show that $x_3 = a^2 + 5a + 5$. ? **(2)**

Given that $x_3 = 41$

(*c*) find the possible values of a. ? **(3)**

Test Your Understanding

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4. A sequence $x_1, x_2, x_3, \ldots$ is defined by

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where a is a constant.

(*a*) Write down an expression for x_2 in terms of a.

$$x_2 = a + 5$$

(1)

(*b*) Show that $x_3 = a^2 + 5a + 5$.

$$x_3 = a(a+5) + 5 = \cdots$$

(2)

Given that $x_3 = 41$

(*c*) find the possible values of a.

$$a^2 + 5a + 5 = 41$$
$$a^2 + 5a - 36 = 0$$
$$(a+9)(a-4) = 0$$
$$a = -9 \text{ or } 4$$

(3)

Exercise 3.7

Pearson Pure Mathematics Year 2/AS
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1 *[AEA 2011 Q3]* A sequence $\{u_n\}$ is given by

$$u_1 = k$$
$$u_{2n} = u_{2n-1} \times p, \qquad n \geq 1$$
$$u_{2n+1} = u_{2n} \times q \qquad n \geq 1$$

(a) Write down the first 6 terms in the sequence.

(b) Show that $\sum_{r=1}^{2n} u_r = \frac{k(1+p)(1-(pq)^n)}{1-pq}$

(c) $[x]$ means the integer part of x, for example $[2.73] = 2, [4] = 4.$

Find $\sum_{r=1}^{\infty} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]}$

?

2 [MAT 2014 1H] The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geq 2$,

$F(n) = F(n-1) + 2$ if 2 divides n but 3 does not divide ,
$F(n) = F(n-1) + 3$ if 3 divides n but 2 does not divide n,
$F(n) = F(n-1) + 4$ if 2 and 3 both divide n
$F(n) = F(n-1)$ if neither 2 nor 3 divides n.

Then the value of $F(6000)$ equals what?

?

3 [MAT 2016 1G] The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and

$x_n = \sum_{k=0}^{n-1}(x_k)$ for $n \geq 1$

Determine the value of the sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

Solution on next slide.

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Pearson Pure Mathematics Year 2/AS
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(a)	$k,\ kp,\ kpq;\ \ kp^2q,\ kp^2q^2,\ kp^3q^2$	M1 A2/1/0 (3)	M1 for 1st 3 terms A2/1/0 (-1 eeoo) for next 3
(b)	**[Need one line clearly showing factorisation or split]** Identify: $k + kpq + kp^2q^2 \ldots$ is GP with $a = k,\ r = pq$	M1A1	M1 for splitting into 2 series A1 for 1st a and r
	Identify: $kp + kp^2q + kp(pq)^2 \ldots$ is GP with $a = kp,\ r = pq$	M1A1	M1 for identifying 2nd GP A1 for 2nd a and r
	$S_{2n} = \frac{k(1-(pq)^n)}{1-pq} + \frac{kp(1-(pq)^n)}{1-pq}$	M1	Use of Sn formula twice. One correct ft their a & r
	$= \frac{k(1+p)(1-(pq)^n)}{1-pq}$	A1cso (6)	
(c)	$\sum_1^{\infty} = 6 + 6\times\left(\frac{4}{3}\right) + 6\times\left(\frac{4}{3}\right)\times\left(\frac{3}{5}\right) + \ldots$ i.e. $k = 6,\ p = \frac{4}{3}, q = \frac{3}{5}$	B1	Identify link with above and values for k, p and q
	$r = pq = \frac{4}{5}$ ($r < 1\ \therefore\ S_\infty$ formula can be used)	M1	Attempt to find r. (S+ for noting $r < 1$ etc)
	$S_\infty = \frac{k(1+p)}{1-pq} = \frac{6\times\frac{7}{3}}{1-\frac{4}{5}},\quad = \frac{210}{3} = \underline{70}$	A1,A1 (4) **(13)**	A1 for an expression can be in k, p or q. ft their values A1 for 70

2 [MAT 2014 1H] The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geq 2$,

$F(n) = F(n-1) + 2$ if 2 divides n but 3 does not divide ,
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$F(n) = F(n-1) + 4$ if 2 and 3 both divide n
$F(n) = F(n-1)$ if neither 2 nor 3 divides n.

Then the value of $F(6000)$ equals what?

Solution: 11000

3 [MAT 2016 1G] The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and $x_n = \sum_{k=0}^{n-1}(x_k)$ for $n \geq 1$

Determine the value of the sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

Solution on next slide.

Solution to Extension Question 3

[MAT 2016 1G] The sequence (x_n), where $n \geq 0$, is defined by $x_0 = 1$ and
$x_n = \sum_{k=0}^{n-1}(x_k)$ for $n \geq 1$
Determine the value of the sum

$$\sum_{k=0}^{\infty} \frac{1}{x_k}$$

Solution:

$$\begin{aligned}
\boldsymbol{x_0} &= \mathbf{1} \\
\boldsymbol{x_1} &= \mathbf{1} \\
\boldsymbol{x_2} &= \mathbf{1 + 1 = 2} \\
\boldsymbol{x_3} &= \mathbf{1 + 1 + 2 = 4}
\end{aligned}$$

This is a geometric sequence from $\boldsymbol{x_1}$ onwards.

Therefore

$$\boldsymbol{\sum_{k=0}^{\infty} \frac{1}{x_k} = \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}$$
$$\boldsymbol{= 1 + 2 = 3}$$

Increasing, decreasing and periodic sequences

A sequence is **strictly increasing** if the terms are always increasing, i.e. $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.
e.g. $1, 2, 4, 8, 16, \ldots$

Textbook Error: It uses the term 'increasing' when it means 'strictly increasing'.

Similarly a sequence is **strictly decreasing** if $u_{n+1} < u_n$ for all $n \in \mathbb{N}$

A sequence is **periodic** if the terms repeat in a cycle. The **order** k of a sequence is **how often it repeats**, i.e. $u_{n+k} = u_n$ for all n.
e.g. 2, 3, 0, 2, 3, 0, 2, 3, 0, 2, … is periodic and has order 3.

[Textbook] For each sequence:
i) State whether the sequence is increasing, decreasing or periodic.
ii) If the sequence is periodic, write down its order.

a) $u_{n+1} = u_n + 3,\ u_1 = 7$?
b) $u_{n+1} = (u_n)^2, \quad u_1 = \frac{1}{2}$?
c) $u_{n+1} = \sin(90n°)$?

Homework Exercise

1 Find the first four terms of the following recurrence relationships.

a $u_{n+1} = u_n + 3$, $u_1 = 1$

b $u_{n+1} = u_n - 5$, $u_1 = 9$

c $u_{n+1} = 2u_n$, $u_1 = 3$

d $u_{n+1} = 2u_n + 1$, $u_1 = 2$

e $u_{n+1} = \dfrac{u_n}{2}$, $u_1 = 10$

f $u_{n+1} = (u_n)^2 - 1$, $u_1 = 2$

2 Suggest possible recurrence relationships for the following sequences. (Remember to state the first term.)

a 3, 5, 7, 9, …

b 20, 17, 14, 11, …

c 1, 2, 4, 8, …

d 100, 25, 6.25, 1.5625, …

e 1, −1, 1, −1, 1, …

f 3, 7, 15, 31, …

g 0, 1, 2, 5, 26, …

h 26, 14, 8, 5, 3.5, …

3 By writing down the first four terms or otherwise, find the recurrence formula that defines the following sequences:

a $u_n = 2n - 1$

b $u_n = 3n + 2$

c $u_n = n + 2$

d $u_n = \dfrac{n+1}{2}$

e $u_n = n^2$

f $u_n = 3^n - 1$

4 A sequence of terms is defined for $n \geqslant 1$ by the recurrence relation $u_{n+1} = ku_n + 2$, where k is a constant. Given that $u_1 = 3$,

a find an expression in terms of k for u_2

b hence find an expression for u_3

Given that $u_3 = 42$:

c find the possible values of k.

Homework Exercise

5 A sequence is defined for $n \geqslant 1$ by the recurrence relation

$$u_{n+1} = pu_n + q,\ u_1 = 2$$

Given that $u_2 = -1$ and $u_3 = 11$, find the values of p and q. **(4 marks)**

6 A sequence is given by

$$x_1 = 2$$

$$x_{n+1} = x_n(p - 3x_n)$$

where p is an integer.

a Show that $x_3 = -10p^2 + 132p - 432$. **(2 marks)**

b Given that $x_3 = -288$ find the value of p. **(1 mark)**

c Hence find the value of x_4. **(1 mark)**

7 A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$

$$a_{n+1} = 4a_n + 5$$

a Find a_3 in terms of k. **(2 marks)**

b Show that $\sum_{r=1}^{4} a_r$ is a multiple of 5. **(3 marks)**

Homework Exercise

1 For each sequence:

i state whether the sequence is increasing, decreasing, or periodic.

ii If the sequence is periodic, write down its order.

a 2, 5, 8, 11, 14 **b** $3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ **c** 5, 9, 15, 23, 33 **d** 3, −3, 3, −3, 3

2 For each sequence:

i write down the first 5 terms of the sequence

ii state whether the sequence is increasing, decreasing, or periodic.

iii If the sequence is periodic, write down its order.

a $u_n = 20 - 3n$ **b** $u_n = 2^{n-1}$ **c** $u_n = \cos(180n°)$

d $u_n = (-1)^n$ **e** $u_{n+1} = u_n - 5$, $u_1 = 20$ **f** $u_{n+1} = 5 - u_n$, $u_1 = 20$

g $u_{n+1} = \frac{2}{3}u_n$, $u_1 = k$

3 The sequence of numbers $u_1, u_2, u_3, \ldots$ is given by $u_{n+1} = ku_n$, $u_1 = 5$.
Find the range of values of k for which the sequence is strictly decreasing.

Homework Exercise

4 The sequence with recurrence relation $u_{k+1} = pu_k + q$, $u_1 = 5$, where p is a constant and $q = 13$, is periodic with order 2.
Find the value of p. **(5 marks)**

5 A sequence has nth term $a_n = \cos(90n°)$, $n \geqslant 1$.

a Find the order of the sequence.

b Find $\sum_{r=1}^{444} a_r$

Challenge

The sequence of numbers $u_1, u_2, u_3, \ldots$ is given by $u_{n+2} = \dfrac{1 + u_{n+1}}{u_n}$, $u_1 = a$, $u_2 = b$, where a and b are positive integers.

a Show that the sequence is periodic for all positive a and b.

b State the order of the sequence.

Hint Each term in this sequence is defined in terms of the **previous two** terms.

Homework Answers

1 **a** 1, 4, 7, 10 **b** 9, 4, −1, −6
c 3, 6, 12, 24 **d** 2, 5, 11, 23
e 10, 5, 2.5, 1.25 **f** 2, 3, 8, 63

2 **a** $u_{n+1} = u_n + 2,\ u_1 = 3$ **b** $u_{n+1} = u_n - 3,\ u_1 = 20$
c $u_{n+1} = 2u_n,\ u_1 = 1$ **d** $u_{n+1} = \dfrac{u_n}{4},\ u_1 = 100$
e $u_{n+1} = -1 \times u_n,\ u_1 = 1$ **f** $u_{n+1} = 2u_n + 1,\ u_1 = 3$
g $u_{n+1} = (u_n)^2 + 1,\ u_1 = 0$ **h** $u_{n+1} = \dfrac{u_n + 2}{2},\ u_1 = 26$

3 **a** $u_{n+1} = u_n + 2,\ u_1 = 1$ **b** $u_{n+1} = u_n + 3,\ u_1 = 5$
c $u_{n+1} = u_n + 1,\ u_1 = 3$ **d** $u_{n+1} = u_n + \frac{1}{2},\ u_1 = 1$
e $u_{n+1} = u_n + 2n + 1,\ u_1 = 1$ **f** $u_{n+1} = 3u_n + 2,\ u_1 = 2$

4 **a** $3k + 2$ **b** $3k^2 + 2k + 2$ **c** $\frac{10}{3}, -4$

5 $p = -4,\ q = 7$

6 **a** $x_2 = x_1(p - 3x_1) = 2(p - 3(2)) = 2p - 12$
$x_3 = (2p - 12)(p - 3(2p - 12)) = (2p - 12)(-5p + 36)$
$= -10p^2 + 132p - 432$
b 12 **c** −252288

7 **a** $16k + 25$
b $a_4 = 4(16k + 25) + 5 = 64k + 105$
$\sum_{r=1}^{4} a_r = k + 4k + 5 + 16k + 25 + 64k + 105$
$= 85k + 135 = 5(17k + 27)$

Homework Answers

1 **a** **i** increasing
b **i** decreasing
c **i** increasing
d **i** periodic **ii** 2

2 **a** **i** 17, 14, 11, 8, 5 **ii** decreasing
b **i** 1, 2, 4, 8, 16 **ii** increasing
c **i** −1, 1, −1, 1, −1 **ii** periodic
iii 2
d **i** −1, 1, −1, 1, −1 **ii** periodic
iii 2
e **i** 20, 15, 10, 5, 0 **ii** decreasing
f **i** 20, −15, 20, −15, 20 **ii** periodic
iii 2
g **i** $k, \frac{2k}{3}, \frac{4k}{9}, \frac{8k}{27}, \frac{16k}{81}$
ii dependent on value of k

3 $0 < k < 1$ **4** $p = -1$

5 **a** 4 **b** 0

Challenge

$u_3 = \frac{1+b}{a}$, $u_4 = \frac{a+b+1}{ab}$, $u_5 = \frac{a+1}{b}$, $u_6 = a$, $u_7 = b$

Order is 5 as $u_6 = u_1$ and $u_7 = u_2$