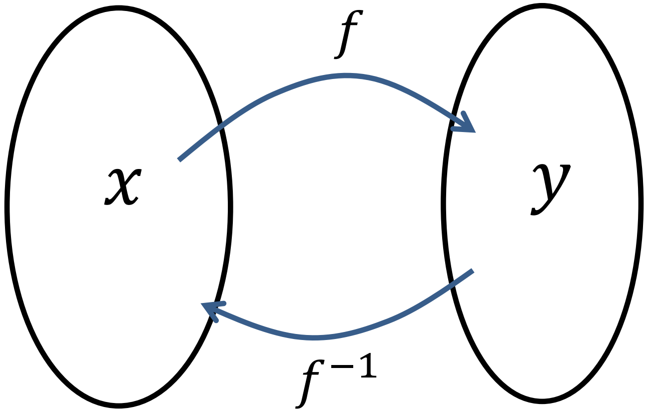
P2 Chapter 2: Graphing Functions

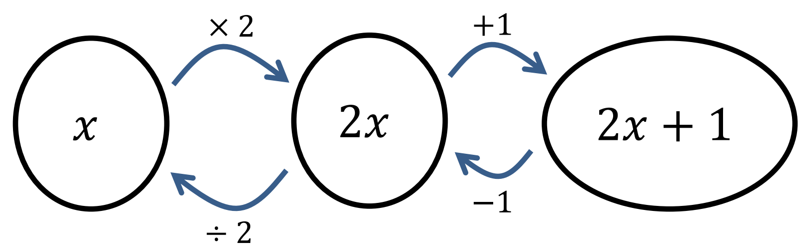
Inverse Functions

Inverse Functions



An inverse function f^{-1} **does the opposite of the original function**. For example, if $f(4) = 2$, then $f^{-1}(2) = 4$.

If $f(x) = 2x + 1$, we could do the opposite operations within the function in reverse order to get back to the original input:



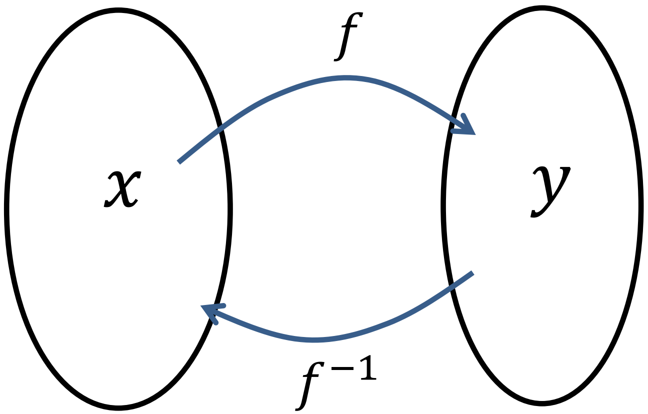
Thus $f^{-1}(x) = \frac{x-1}{2}$

Notation: Just like f^2 means "apply f twice", f^{-1} means "apply f -1 times", i.e. once backwards!
This is why we write $\sin^{-1}(x)$ to mean "inverse sin".

This has appeared in exams before.

Explain why the function must be one-to-one for an inverse function to exist:
If the mapping was many-to-one, then the inverse mapping would be one-to-many. But this is not a function!

More on Inverse Functions



In the original function, we have the **output y in terms of the input x**, e.g. $y = 2x + 1$

Therefore if we **change the subject to get x in terms of y**, then we have the input in terms of the output, i.e. the inverse function!

$$x = \frac{y-1}{2}$$

However, we tend to write a function in terms of x, so would write;

$$f^{-1}(x) = \frac{x-1}{2}$$

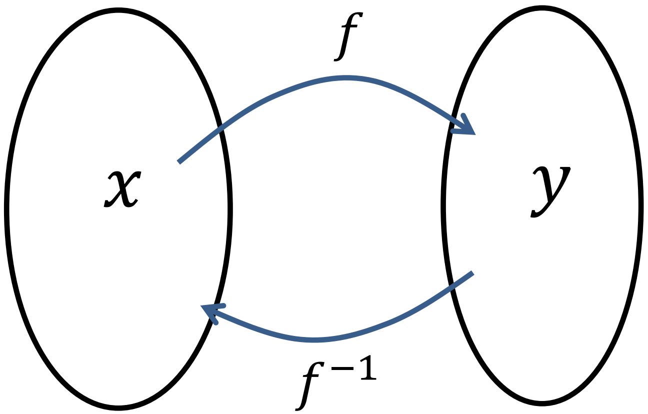
If $f(x) = 3 - 4x$, find $f^{-1}(x)$

?

If $f(x) = \frac{x+2}{2x-1}$, $x \neq \frac{1}{2}$, determine $f^{-1}(x)$

?

More on Inverse Functions



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However, we tend to write a function in terms of x, so would write;

$$f^{-1}(x) = \frac{x-1}{2}$$

If $f(x) = 3 - 4x$, find $f^{-1}(x)$

$$y = 3 - 4x$$

Use y instead of $f(x)$ for convenience.

$$4x = 3 - y$$

$$x = \frac{3-y}{4}$$

Make x the subject.

$$f^{-1}(x) = \frac{3-x}{4}$$

Replace y's with x at end.

If $f(x) = \frac{x+2}{2x-1}$, $x \neq \frac{1}{2}$, determine $f^{-1}(x)$

$$y = \frac{x+2}{2x-1}$$

$$y(2x-1) = x+2$$

$$2xy - y = x + 2 \qquad 2xy - x = y + 2$$

$$x(2y-1) = y+2$$

$$x = \frac{y+2}{2y-1}$$

$$f^{-1}(x) = \frac{x+2}{2x-1}$$

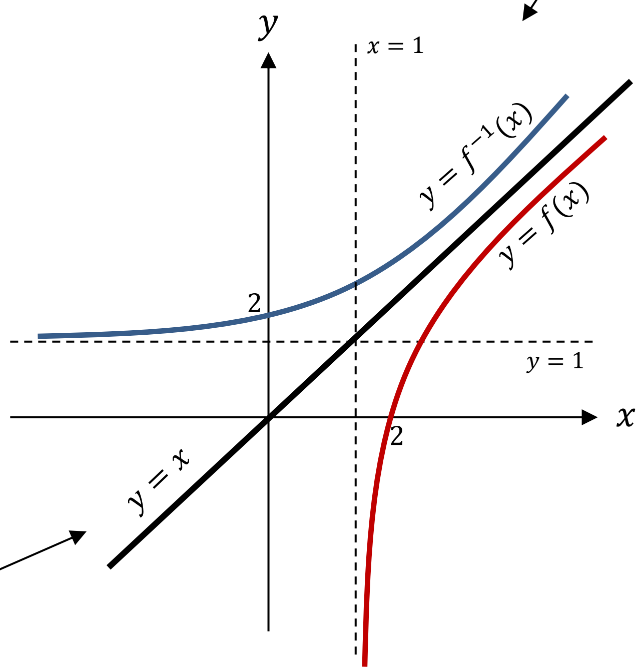
Graphing an Inverse Function

We saw that the inverse function effectively swaps the input x and output y. Thus the **x and y axis are swapped** when sketching the original function and its inverse.

And since the set of inputs and set of outputs is swapped…

✎ The domain of $f(x)$ is the range of $f^{-1}(x)$ and vice versa.

Notice that x-intercepts become y-intercepts, and vertical asymptotes become horizontal ones.



$y = f(x)$ and $y = f^{-1}(x)$ have the line $y = x$ as a line of symmetry.

Domain of f: $\boldsymbol{x > 1}$

Range of f^{-1}: $\boldsymbol{f^{-1}(x) > 1}$

The domain of the function is the same as the range of the inverse, but remember that we write a domain in terms of x, but a range in terms of $f(x)$ or $f^{-1}(x)$.

Example

[Textbook] If $g(x)$ is defined as $g(x) = \sqrt{x-2}\ \{x \in \mathbb{R}, x \geq 2\}$

a) Find the range of $g(x)$.

b) Calculate $g^{-1}(x)$

c) Sketch the graphs of both functions.

d) State the domain and range of $g^{-1}(x)$.

a ?

b ?

c ?

d ?

Example

[Textbook] If $g(x)$ is defined as $g(x) = \sqrt{x-2}\ \{x \in \mathbb{R}, x \geq 2\}$

a) Find the range of $g(x)$.

b) Calculate $g^{-1}(x)$

c) Sketch the graphs of both functions.

d) State the domain and range of $g^{-1}(x)$.

a

$f(x) \geq 0$

b

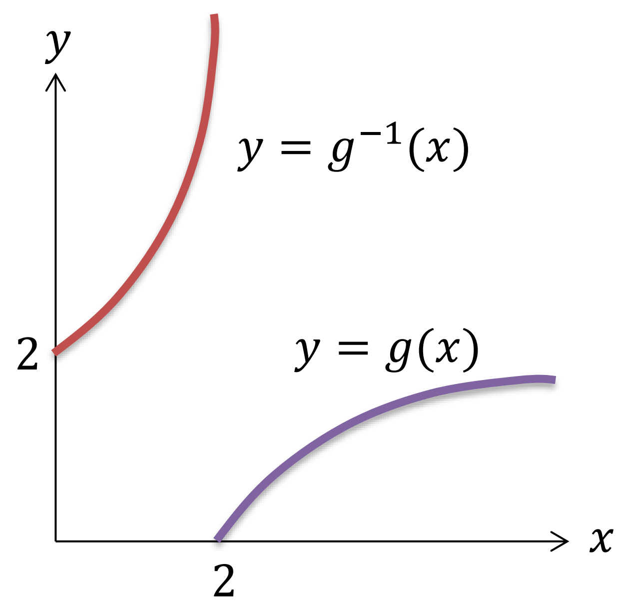
Start with $y = \sqrt{x-2}$ and make x the subject, before swapping y and x.

$y = \sqrt{x-2}$

$x = y^2 + 2$

$g^{-1}(x) = x^2 + 2$

c



d

Domain of $g^{-1}(x)$ is range of $g(x)$ and range of $g^{-1}(x)$ the domain of $g(x)$.

Domain:

$x \geq 0$

Range:

$f^{-1}(x) \geq 2$

Further Example

[Textbook] The function is defined by $f(x) = x^2 - 3,\ x \in \mathbb{R},\ x \geq 0$.

a) Find $f^{-1}(x)$

b) Sketch $y = f(x)$ and $y = f^{-1}(x)$ and state the domain of f^{-1}.

c) Solve the equation $f(x) = f^{-1}(x)$.

a

?

b

?

c

?

Further Example

[Textbook] The function is defined by $f(x) = x^2 - 3, x \in \mathbb{R}, x \geq 0$.

a) Find $f^{-1}(x)$

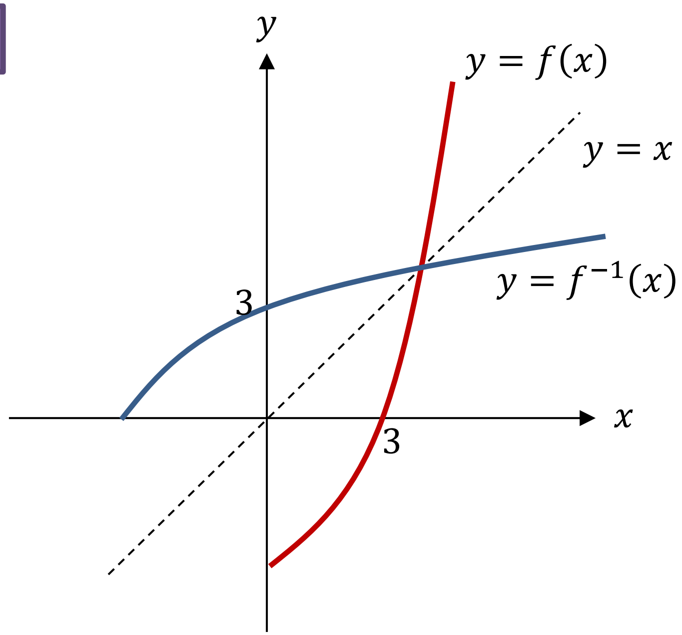
b) Sketch $y = f(x)$ and $y = f^{-1}(x)$ and state the domain of f^{-1}.

c) Solve the equation $f(x) = f^{-1}(x)$.

a

$$y = x^2 - 3$$
$$x = \sqrt{y+3}$$
$$f^{-1}(x) = \sqrt{x+3}$$

b



c

If the function is equal to its inverse, it must lie on the line $y = x$, i.e. $f(x) = x$:

$$x^2 - x = 3$$
$$x^2 - x - 3 = 0$$
$$x = \frac{1 \pm \sqrt{1+12}}{2}$$
$$= \frac{1 + \sqrt{13}}{2}$$

From graph, we can see we only want positive solution.

Note: There was once an exam question based on this principle.

Test Your Understanding

Edexcel C4 June 2012 Q6

The function f is defined by

$$f: x \to e^x + 2, \quad x \in \mathbb{R}$$

(d) Find f^{-1}, the inverse function of f, stating its domain.

(e) On the same axe sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes.

(d)

?

(e)

?

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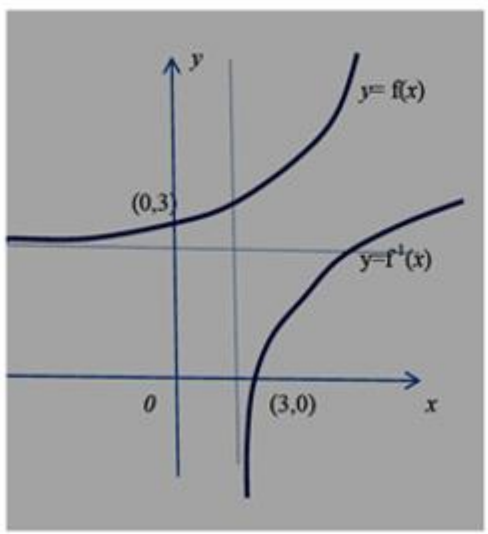
(e) On the same axe sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes.

(d)	Let $y = e^x + 2 \Rightarrow y - 2 = e^x \Rightarrow \ln(y-2) = x$	M1
	$f^{-1}(x) = \ln(x-2), \quad x > 2.$	A1 , B1ft
(e)		
	Shape for f(x)	B1
	(0, 3)	B1
	Shape for f^{-1}(x)	B1
	(3, 0)	B1

Exercise 2.4

Pearson Pure Mathematics Year 2/AS

Pages 9

Homework Exercise

1 For each of the following functions $f(x)$:

i state the range of $f(x)$

ii determine the equation of the inverse function $f^{-1}(x)$

iii state the domain and range of $f^{-1}(x)$

iv sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

a $f: x \mapsto 2x + 3, x \in \mathbb{R}$

b $f: x \mapsto \dfrac{x + 5}{2}, x \in \mathbb{R}$

c $f: x \mapsto 4 - 3x, x \in \mathbb{R}$

d $f: x \mapsto x^3 - 7, x \in \mathbb{R}$

2 Find the inverse of each function:

a $f(x) = 10 - x, x \in \mathbb{R}$

b $g(x) = \dfrac{x}{5}, x \in \mathbb{R}$

c $h(x) = \dfrac{3}{x}, x \neq 0, x \in \mathbb{R}$

d $k(x) = x - 8, x \in \mathbb{R}$

Notation Two of these functions are **self-inverse**. A function is self-inverse if $f^{-1}(x) = f(x)$. In this case $ff(x) = x$.

3 Explain why the function $g: x \mapsto 4 - x, \{x \in \mathbb{R}, x > 0\}$ is not identical to its inverse.

Homework Exercise

4 For each of the following functions $g(x)$ with a restricted domain:

i state the range of $g(x)$

ii determine the equation of the inverse function $g^{-1}(x)$

iii state the domain and range of $g^{-1}(x)$

iv sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same set of axes.

a $g(x) = \frac{1}{x}, \{x \in \mathbb{R}, x \geqslant 3\}$

b $g(x) = 2x - 1, \{x \in \mathbb{R}, x \geqslant 0\}$

c $g(x) = \frac{3}{x-2}, \{x \in \mathbb{R}, x > 2\}$

d $g(x) = \sqrt{x-3}, \{x \in \mathbb{R}, x \geqslant 7\}$

e $g(x) = x^2 + 2, \{x \in \mathbb{R}, x > 2\}$

f $g(x) = x^3 - 8, \{x \in \mathbb{R}, x \geqslant 2\}$

5 The function $t(x)$ is defined by
$t(x) = x^2 - 6x + 5, x \in \mathbb{R}, x \geqslant 5$
Find $t^{-1}(x)$. **(5 marks)**

Hint First complete the square for the function $t(x)$.

6 The function $m(x)$ is defined by $m(x) = x^2 + 4x + 9, x \in \mathbb{R}, x > a$, for some constant a.

a State the least value of a for which $m^{-1}(x)$ exists. **(4 marks)**

b Determine the equation of $m^{-1}(x)$. **(3 marks)**

c State the domain of $m^{-1}(x)$. **(1 mark)**

Homework Exercise

7 The function h(x) is defined by $h(x) = \frac{2x+1}{x-2}$, $\{x \in \mathbb{R}, x \neq 2\}$.

a What happens to the function as x approaches 2?

b Find $h^{-1}(3)$.

c Find $h^{-1}(x)$, stating clearly its domain.

d Find the elements of the domain that get mapped to themselves by the function.

8 The functions m and n are defined by

$m: x \mapsto 2x + 3, x \in \mathbb{R}$

$n: x \mapsto \frac{x-3}{2}, x \in \mathbb{R}$

a Find nm(x)

b What can you say about the functions m and n?

9 The functions s and t are defined by

$s(x) = \frac{3}{x+1}, x \neq -1$

$t(x) = \frac{3-x}{x}, x \neq 0$

Show that the functions are inverses of each other.

Homework Exercise

10 The function $f(x)$ is defined by $f(x) = 2x^2 - 3$, $\{x \in \mathbb{R}, x < 0\}$.
Determine:

a $f^{-1}(x)$ clearly stating its domain **(4 marks)**

b the value(s) of a for which $f(a) = f^{-1}(a)$. **(4 marks)**

11 The functions f and g are defined by

$f: x \mapsto e^x - 5, x \in \mathbb{R}$

$g: x \mapsto \ln(x - 4), x > 4$

a State the range of f. **(1 mark)**

b Find f^{-1}, the inverse function of f, stating its domain. **(3 marks)**

c On the same axes, sketch the curves with equation $y = f(x)$ and $y = f^{-1}(x)$, giving the coordinates of all the points where the curves cross the axes. **(4 marks)**

d Find g^{-1}, the inverse function of g, stating its domain. **(3 marks)**

e Solve the equation $g^{-1}(x) = 11$, giving your answer to 2 decimal places. **(3 marks)**

12 The function f is defined by

$$f: x \mapsto \frac{3(x+2)}{x^2 + x - 20} - \frac{2}{x-4}, \; x > 4$$

a Show that $f: x \mapsto \frac{1}{x+5}, x > 4$. **(4 marks)**

b Find the range of f. **(2 marks)**

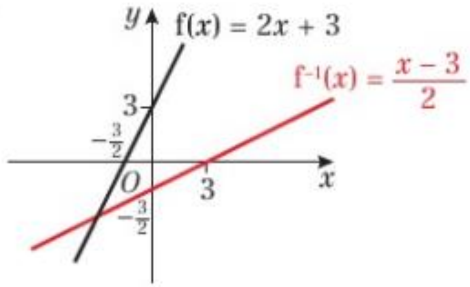
c Find $f^{-1}(x)$. State the domain of this inverse function. **(4 marks)**

Homework Answers

1 **a** **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = \dfrac{x-3}{2}$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

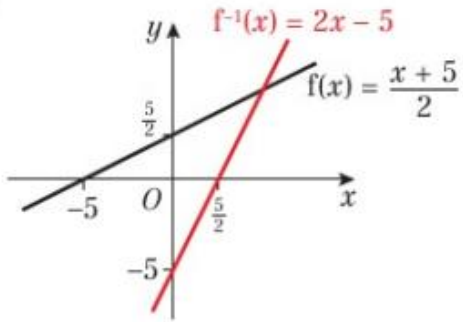
iv



b **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = 2x - 5$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

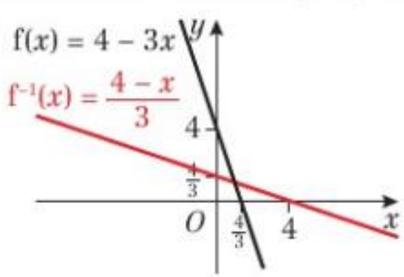
iv



c **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = \dfrac{4-x}{3}$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

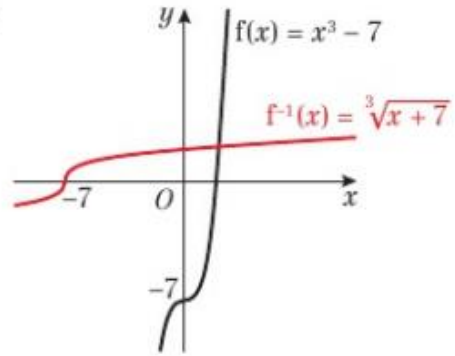
iv



d **i** $\{y \in \mathbb{R}\}$ **ii** $f^{-1}(x) = \sqrt[3]{x+7}$

iii Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}\}$

iv



2 **a** $f^{-1}(x) = 10 - x$, $\{x \in \mathbb{R}\}$ **b** $g^{-1}(x) = 5x$, $\{x \in \mathbb{R}\}$

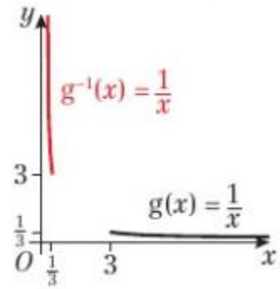
c $h^{-1}(x) = \dfrac{3}{x}$, $\{x \neq 0\}$ **d** $k^{-1}(x) = x + 8$, $\{x \in \mathbb{R}\}$

3 Domain becomes $x < 4$

4 **a** **i** $0 < g(x) \leqslant \frac{1}{3}$ **ii** $g^{-1}(x) = \frac{1}{x}$

iii $\{x \in \mathbb{R}, 0 < x \leqslant \frac{1}{3}\}$, $g^{-1}(x) \geqslant 3$

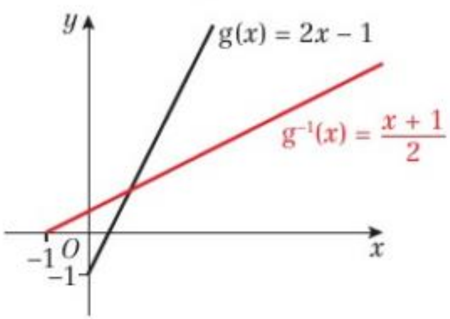
iv



b **i** $g(x) \geqslant -1$ **ii** $g^{-1}(x) = \frac{x+1}{2}$

iii $\{x \in \mathbb{R}, x \geqslant -1\}$, $g^{-1}(x) \geqslant 0$

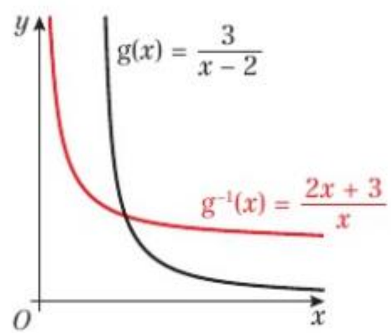
iv



c **i** $g(x) > 0$ **ii** $g^{-1}(x) = \frac{2x+3}{x}$

iii $\{x \in \mathbb{R}, x > 0\}$, $g^{-1}(x) > 2$

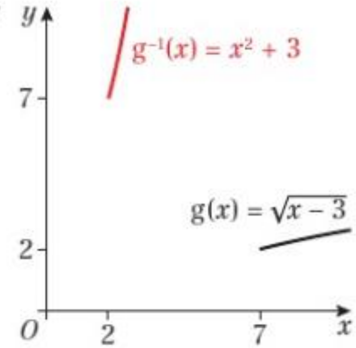
iv



d **i** $g(x) \geqslant 2$ **ii** $g^{-1}(x) = x^2 + 3$

iii $\{x \in \mathbb{R}, x \geqslant 2\}$, $g^{-1}(x) \geqslant 7$

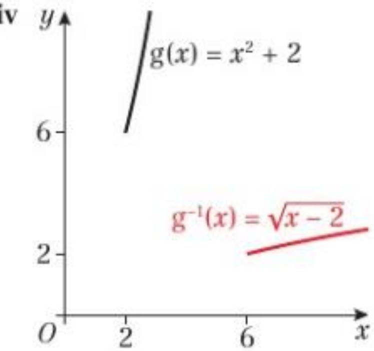
iv



e **i** $g(x) > 6$ **ii** $g^{-1}(x) = \sqrt{x-2}$

iii $\{x \in \mathbb{R}, x > 6\}$, $g^{-1}(x) > 2$

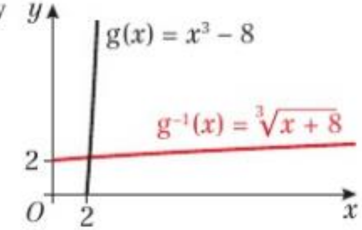
iv



f **i** $g(x) \geqslant 0$ **ii** $g^{-1}(x) = \sqrt[3]{x+8}$

iii $\{x \in \mathbb{R}, x \geqslant 0\}$, $g^{-1}(x) \geqslant 2$

iv



Homework Answers

5 $t^{-1}(x) = \sqrt{x+4} + 3$, $\{x \in \mathbb{R}, x \geqslant 0\}$

6 a -2 b $m^{-1}(x) = \sqrt{x-5} - 2$ c $x > 5$

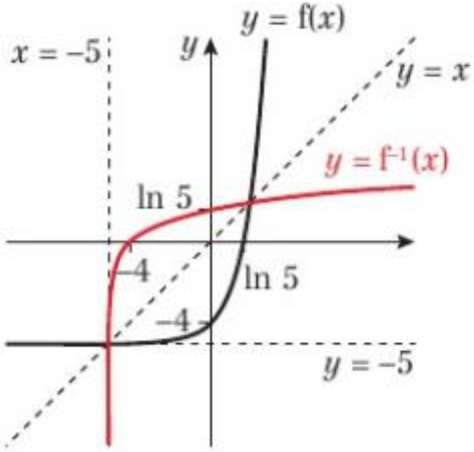
7 a tends to $\pm\infty$
b 7
c $h^{-1}(x) = \dfrac{2x+1}{x-2}$ $\{x \in \mathbb{R}, x \neq 2\}$
d $2 + \sqrt{5}$, $2 - \sqrt{5}$

8 a $nm(x) = x$
b The functions m and n are inverse of one another as $mn(x) = nm(x) = x$.

9 $st(x) = \dfrac{3}{\frac{3-x}{x} + 1} = x$, $ts(x) = \dfrac{3 - \frac{3}{x-1}}{\frac{3}{x+1}} = x$

10 a $f^{-1}(x) = -\sqrt{\dfrac{x+3}{2}}$ $\{x \in \mathbb{R}, x > -3\}$
b $a = -1$

11 a $f(x) > -5$ b $f^{-1}(x) = \ln(x+5)$ $\{x \in \mathbb{R}, x > -5\}$
c



d $g^{-1}(x) = e^x + 4$, $x \in \mathbb{R}$ e $x = 1.95$

12 a $f(x) = \dfrac{3(x+2)}{x^2 + x - 20} - \dfrac{2}{x-4}$

$$= \frac{3(x+2)}{(x+5)(x-4)} - \frac{2(x+5)}{(x+5)(x-4)} = \frac{x-4}{(x+5)(x-4)}$$

$$= \frac{1}{x+5}$$

b $\left\{y \in \mathbb{R}, 0 < y < \frac{1}{9}\right\}$
c $f^{-1}: x \to \dfrac{1}{x} - 5$. Domain is $\left\{x \in \mathbb{R}, 0 < x < \frac{1}{9} \text{ and } x \neq 0\right\}$