P1 Chapter 13: Integration

Negative Areas

'Negative Areas'

Sketch the curve $y = x(x-1)(x-2)$ (which expands to give $y = x^3 - 3x^2 + 2x$).
Now calculate $\int_0^2 x(x-1)(x-2)\ dx$. Why is this result surprising?

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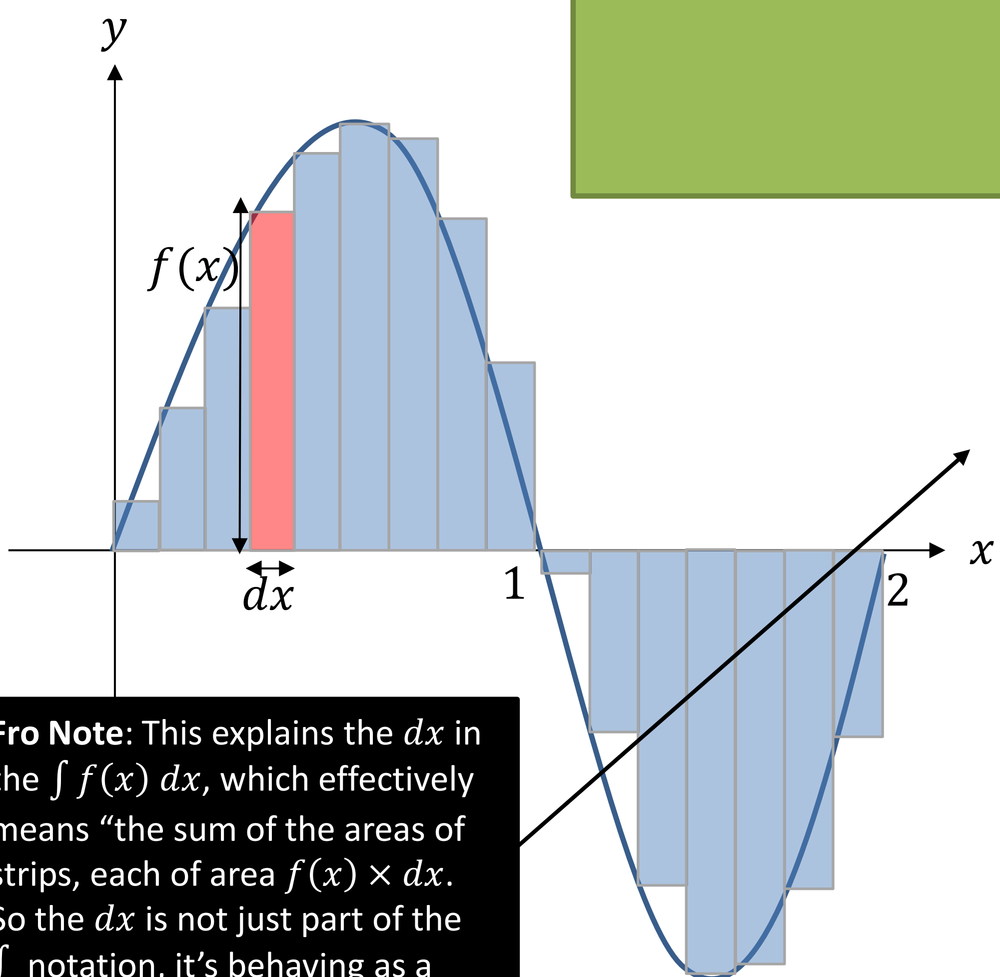

Integration $\int f(x)\, dx$ is just the sum of areas of infinitely thin rectangles, where the current y value (i.e. $f(x)$) is each height, and the widths are dx.
i.e. The area of each is $f(x) \times dx$

The problem is, when $f(x)$ is negative, then $f(x) \times dx$ is negative, i.e. a negative area!
The result is that the 'positive area' from 0 to 1 is cancelled out by the 'negative area' from 1 to 2, giving an overall 'area' of 0.
So how do we resolve this?

Fro Note: This explains the dx in the $\int f(x)\, dx$, which effectively means "the sum of the areas of strips, each of area $f(x) \times dx$. So the dx is not just part of the $\int$ notation, it's behaving as a physical quantity! (i.e. length)

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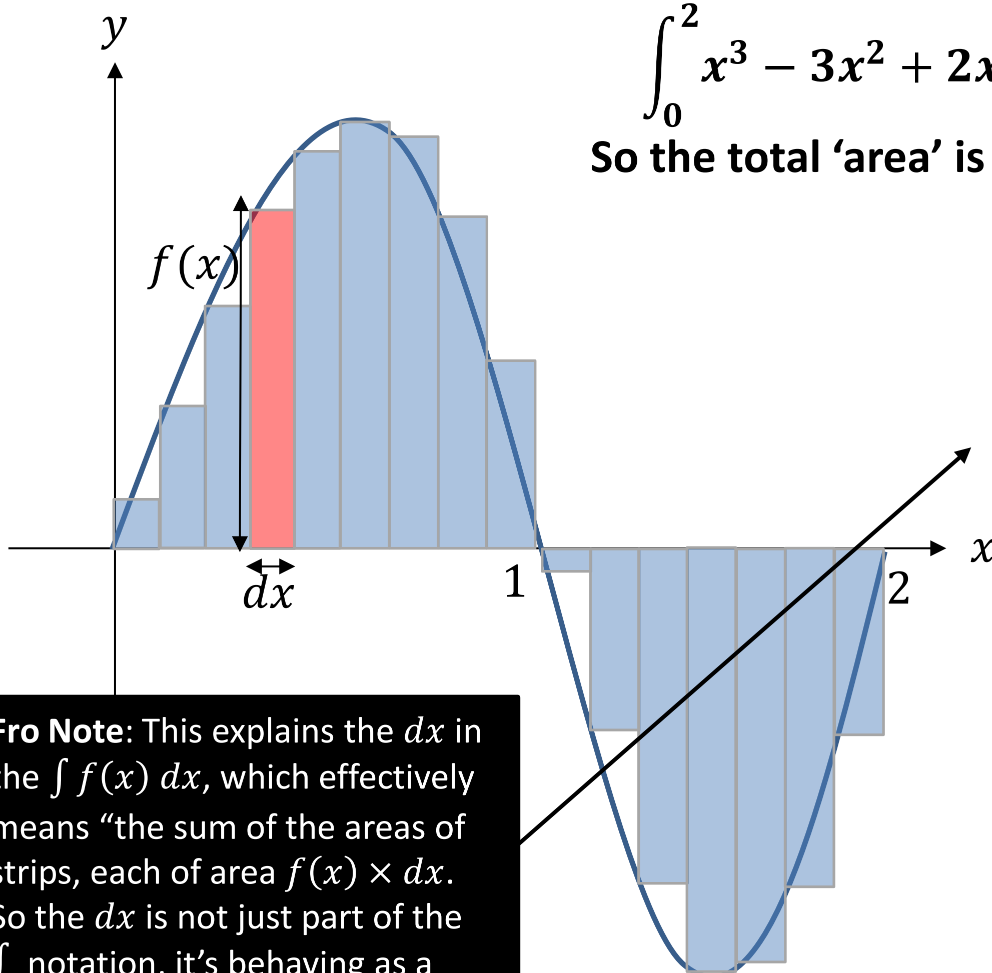
$$\int_0^2 x^3 - 3x^2 + 2x\ dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^2 = 0$$

So the total ‘area’ is 0!



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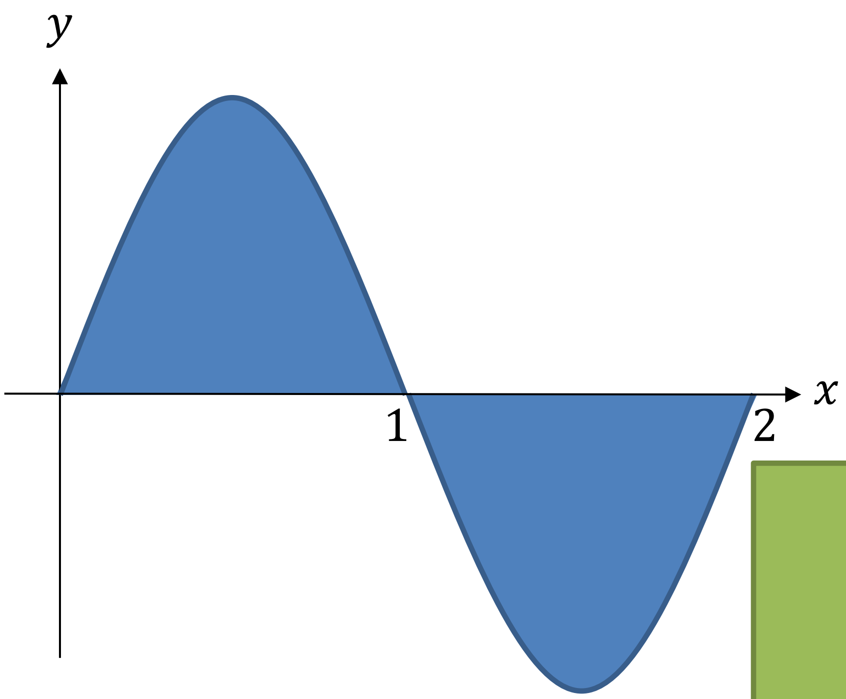
Example

Find the total area bound between the curve $y = x(x-1)(x-2)$ and the x-axis.



Strategy:

?

? Solution

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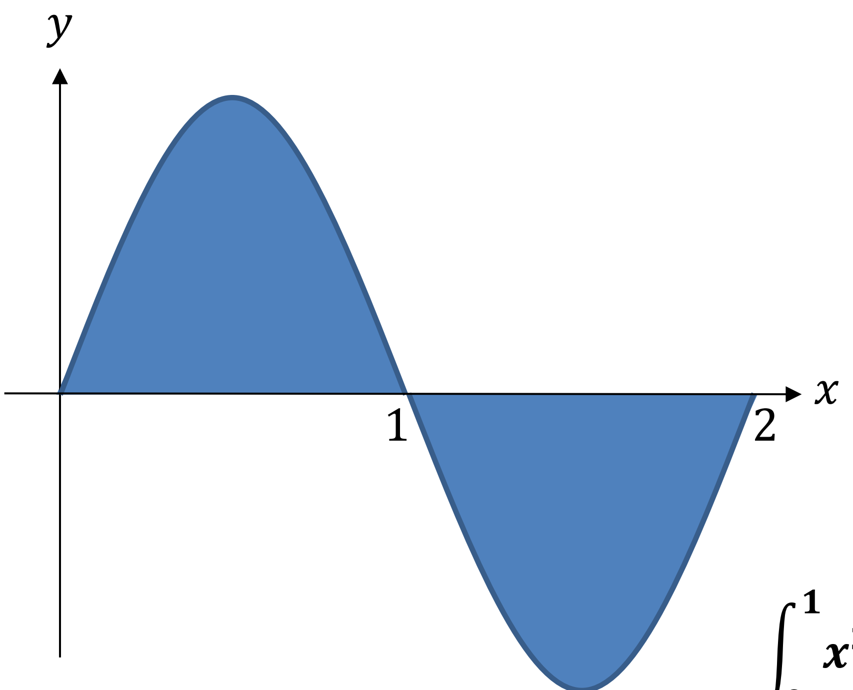

Strategy:
Separately find the area between $x = 0$ and 1, and between 1 and 2. Treat any negative areas as positive.

$$\boldsymbol{x(x-1)(x-2) = x^3 - 3x^2 + 2x}$$

$$\boldsymbol{\int_0^1 x^3 - 3x^2 + 2x \; dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^1 = \frac{1}{4}}$$

$$\boldsymbol{\int_1^2 x^3 - 3x^2 + 2x \; dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^2 = -\frac{1}{4}}$$

Treating both as positive:

$$\boldsymbol{Area = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}}$$

Test Your Understanding

Edexcel C2 May 2013 Q6

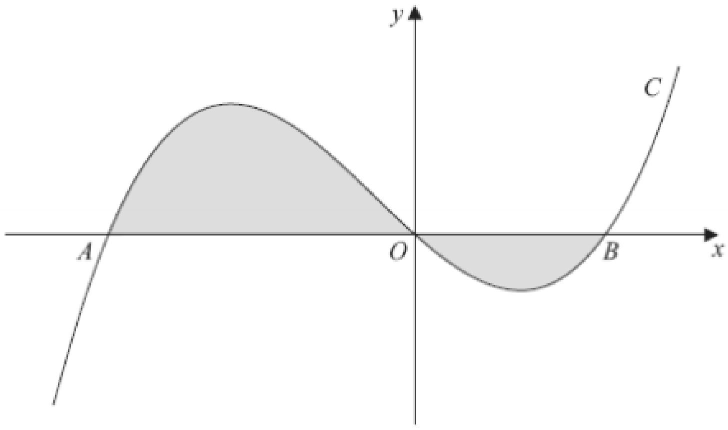

Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = x(x + 4)(x - 2).$$

The curve C crosses the x-axis at the origin O and at the points A and B.

(a) Write down the x-coordinates of the points A and B.

(1)

The finite region, shown shaded in Figure 3, is bounded by the curve C and the x-axis.

(b) Use integration to find the total area of the finite region shown shaded in Figure 3.

(7)

(a) ?

(b) ?

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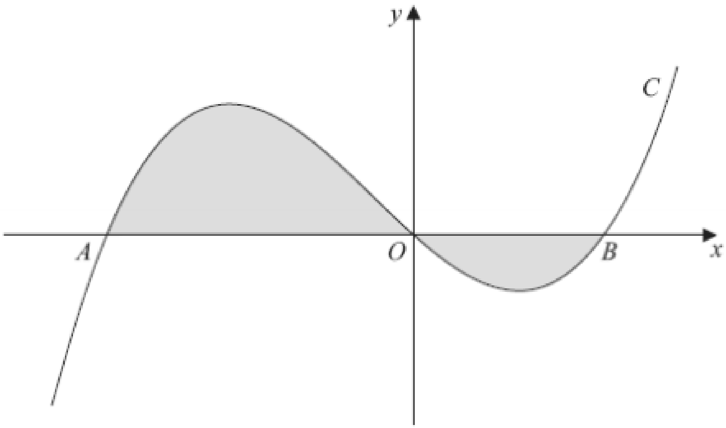

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(*b*) Use integration to find the total area of the finite region shown shaded in Figure 3. **(7)**

(a)	Seeing -4 and 2.	B1 **(1)**
(b)	$x(x+4)(x-2) = \underline{x^3 + 2x^2 - 8x}$ or $\underline{x^3 - 2x^2 + 4x^2 - 8x}$ (without simplifying)	B1
	$\int (x^3 + 2x^2 - 8x)\,dx = \frac{x^4}{4} + \frac{2x^3}{3} - \frac{8x^2}{2} \{+c\}$ or $\frac{x^4}{4} - \frac{2x^3}{3} + \frac{4x^3}{3} - \frac{8x^2}{2} \{+c\}$	M1A1ft
	$\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{-4}^{0} = (0) - \left(64 - \frac{128}{3} - 64\right)$ **or** $\left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{8x^2}{2}\right]_{0}^{2} = \left(4 + \frac{16}{3} - 16\right) - (0)$	dM1
	One integral $= \pm 42\frac{2}{3}$ (42.6 or awrt 42.7) **or** other integral $= \pm 6\frac{2}{3}$ (6.6 or awrt 6.7)	A1
	Hence Area $=$ "$their 42\frac{2}{3}$" $+$ "$their 6\frac{2}{3}$" or Area $=$ "$their 42\frac{2}{3}$" $-$ "$- their 6\frac{2}{3}$"	dM1
	$= 49\frac{1}{3}$ or 49.3 or $\frac{148}{3}$ (NOT $-\frac{148}{3}$)	A1
	(An answer of $= 49\frac{1}{3}$ may not get the final two marks – check solution carefully)	**(7)**

Exercise 13.6

Pearson Pure Mathematics Year 1/AS
Page 108

Hint: It's not actually even possible to integrate 2^{x^2}, but we can still sketch the graph. Reflect on what $\frac{dI}{da}$ actually means.

Extension

1 *[MAT 2010 1I]* For a positive number a, let

$$I(a) = \int_0^a \left(4 - 2^{x^2}\right) dx$$

When $\frac{dI}{da} = 0$ then a is what value?

?

2 *[STEP I 2014 Q3]*
The numbers a and b, where $b > a \geq 0$, are such that

$$\int_a^b x^2\, dx = \left(\int_a^b x\, dx\right)^2$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies the equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q.
Deduce that $1 < b - a \leq \frac{4}{3}$

Guidance for this problem on next slide.

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Pearson Pure Mathematics Year 1/AS
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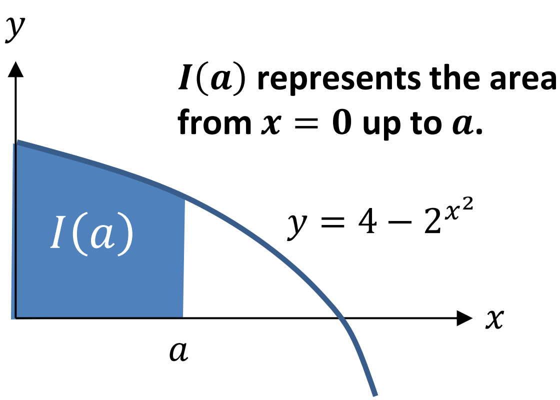

$\frac{dI}{da}$ represents the rate of change of area as a increases. Thus if $\frac{dI}{da} = 0$, the area is not changing. This must happen at the x-intercept of the graph, because once the curve goes negative, the total area will start to decrease.

$$4 - 2^{x^2} = 0 \quad \rightarrow \quad x = \sqrt{2}$$

The answer is $a = \sqrt{2}$.

2 *[STEP I 2014 Q3]*
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Guidance for this problem on next slide.

Guidance for Extension Problem 2

[STEP I 2014 Q3] The numbers a and b, where $b > a \geq 0$, are such that $\int_a^b x^2\,dx = \left(\int_a^b x\,dx\right)^2$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies the equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leq \frac{4}{3}$

This question was actually devised to address what happens when students misunderstand or mis-apply a "rule" of mathematics and *it turns out to give the right answer*. Part (i) starts you off gently: integrating both terms, squaring the RHS and solving very quickly gives $b = \frac{4}{3}$. Part (ii) develops in much the same way, but with a non-zero lower limit to the integrals, and we immediately see that the algebra gets much more involved. Importantly, it should be very clear that whatever expression materialises must have $(b - 1)$ as a factor (since setting $b = a$ would definitely give a zero area, thus trivially satisfying the given integral statement). This leads to the required cubic equation.

The final part of (ii) requires a mixture of different ideas (and can be done in a number of different ways). The most basic approach to demonstrating that a cubic curve has only one zero is to illustrate that both of its TPs lie on the same side of the x-axis (or to show there are no TPs). The popular *Change-of-Sign Rule* for continuous functions can be used to identify the position of this zero.

Having got you started with some simple lower limits, part (iii) develops matters more generally, and derives the (perhaps) surprising result that the exploration of this initial "stupid idea" requires b and a to be "not too far apart" to an extent that is easily identifiable.

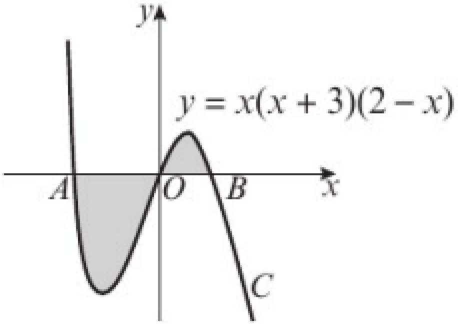
Homework Exercise

1 Sketch the following and find the total area of the finite region or regions bounded by the curves and the x-axis:

a $y = x(x + 2)$ **b** $y = (x + 1)(x - 4)$ **c** $y = (x + 3)x(x - 3)$

d $y = x^2(x - 2)$ **e** $y = x(x - 2)(x - 5)$

2 The graph shows a sketch of part of the curve C with equation $y = x(x + 3)(2 - x)$.



The curve C crosses the x-axis at the origin O and at points A and B.

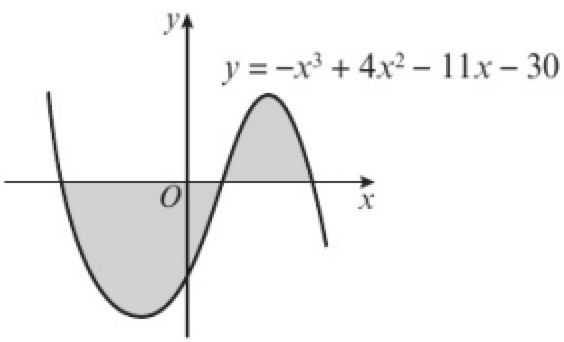
a Write down the x-coordinates of A and B. **(1 mark)**

The finite region, shown shaded, is bounded by the curve C and the x-axis.

b Use integration to find the total area of the finite shaded region. **(7 marks)**

3 $f(x) = -x^3 + 4x^2 + 11x - 30$

The graph shows a sketch of part of the curve with equation $y = -x^3 + 4x^2 + 11x - 30$.



a Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

b Write $f(x)$ in the form $(x + 3)(Ax^2 + Bx + C)$.

c Hence, factorise $f(x)$ completely.

d Hence, determine the x-coordinates where the curve intersects the x-axis.

e Hence, determine the total shaded area shown on the sketch.

Homework Exercise

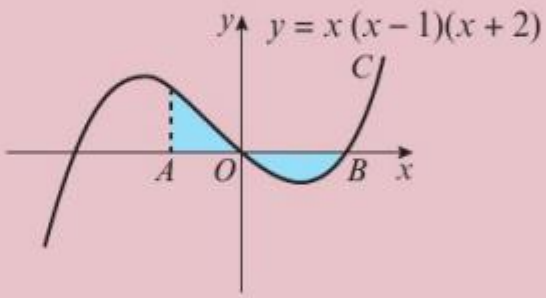
Challenge

1 Given that $f(x) = x(3 - x)$, find the area of the finite region bounded by the x-axis and the curve with equation

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = af(x)$

d $y = f(x + a)$ **e** $y = f(ax)$.

2 The graph shows a sketch of part of the curve C with equation $y = x(x - 1)(x + 2)$. The curve C crosses the x-axis at the origin O and at point B. The shaded areas above and below the x-axis are equal.



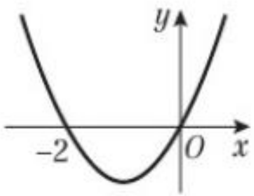
a Show that the x-coordinate of A satisfies the equation

$$(x - 1)^2(3x^2 + 10x + 5) = 0$$

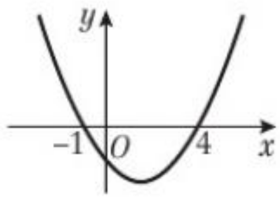
b Hence find the exact coordinates of A, and interpret geometrically the other roots of this equation.

Homework Answers

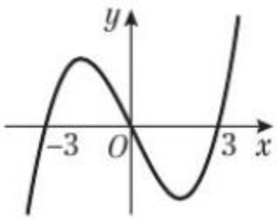
1 **a** $1\frac{1}{3}$



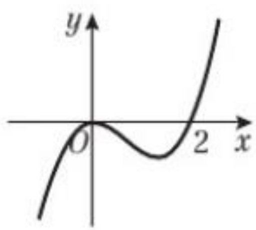
b $20\frac{5}{6}$



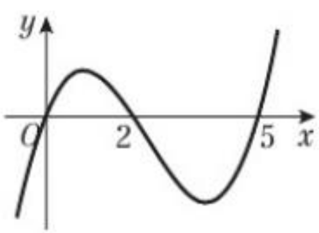
c $40\frac{1}{2}$



d $1\frac{1}{3}$



e $21\frac{1}{12}$



2 **a** $(-3, 0)$ and $(2, 0)$ **b** $21\frac{1}{12}$

3 **a** $f(-3) = 0$

b $f(x) = (x + 3)(-x^2 + 7x - 10)$

c $f(x) = (x + 3)(x - 5)(2 - x)$

d $(-3, 0)$, $(2, 0)$ and $(5, 0)$

e $143\frac{5}{6}$

Challenge

1 **a** $4\frac{1}{2}$ **b** 9 **c** $\dfrac{9a}{2}$ **d** $4\frac{1}{2}$ **e** $\dfrac{9}{2a}$

2 **a** B has x-coordinate 1

$\int_0^1 (x^3 + x^2 - 2x)\mathrm{d}x = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right]_0^1$

$= \frac{1}{4} + \frac{1}{3} - 1 = -\frac{5}{12}$

So area under x-axis is $\frac{5}{12}$

Area above x-axis is

$\left(\frac{1}{4}0^4 + \frac{1}{3}0^3 - 0^2\right) - \left(\frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2\right) = \frac{5}{12}$

So the x-coordinate of a satisfies

$3x^4 + 4x^3 - 12x^2 + 5 = 0$

Then use the factor theorem twice to get

$(x - 1)^2(3x^2 + 10x + 5) = 0$

b A has coordinates $\left(\dfrac{-5 + \sqrt{10}}{3}, \dfrac{-80 + 37\sqrt{10}}{27}\right)$

The roots at 1 correspond to point B.

The root $\dfrac{-5 - \sqrt{10}}{3}$ gives a point on the curve to the left of -2 below the x-axis, so cannot be A.