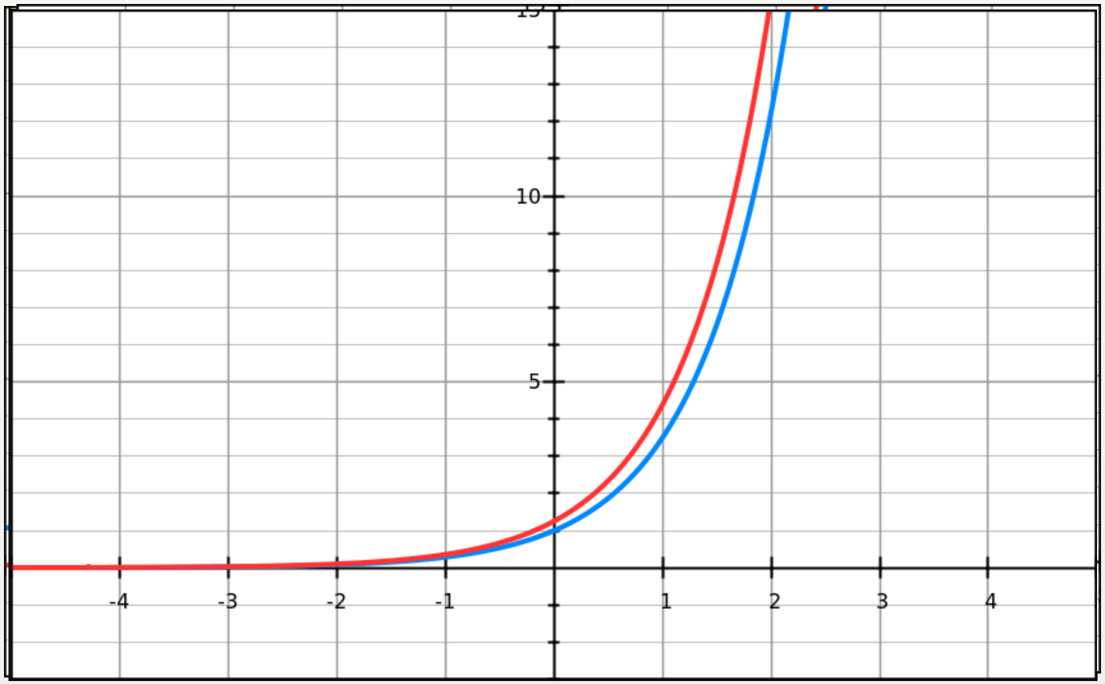
P1 Chapter 14: Logarithms

The Natural Base

$y = e^x$

Click

Function	Gradient
$y = 1^x$	$\frac{dy}{dx} = 0$
$y = 1.5^x$	$\frac{dy}{dx} = 0.41 \times 1.5^x$
$y = 2^x$	$\frac{dy}{dx} = 0.69 \times 2^x$
$y = 2.5^x$	$\frac{dy}{dx} = 0.92 \times 2.5^x$
$y = 3^x$	$\frac{dy}{dx} = 1.10 \times 3^x$
$y = 3.5^x$	$\frac{dy}{dx} = 1.25 \times 3.5^x$



Compare each exponential function against its respective gradient function. What do you notice?

$y = e^x$

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$y = 2.5^x$ and $y = 3^x$ seem to be similar to their respective gradient functions. So is there a base between 2.5 and 3 where the **function is equal to its gradient function**?

$e = 2.71828\ldots$ is known as **Euler's Number.**

It is one of the five most fundamental constants in mathematics ($0, 1, i, e, \pi$).

It has the property that:

$$y = e^x \quad \rightarrow \quad \frac{dy}{dx} = e^x$$

Although any function of the form $y = a^x$ is known as **an** exponential function, e^x is known as "**the**" exponential function.

You can find the exponential function on your calculator, to the right (above the "ln" key)

Differentiating $y = ae^{kx}$

✎ If $y = e^{kx}$, where k is a constant, then $\frac{dy}{dx} = ke^{kx}$

Note: This is not a standalone rule but an application of something called the '*chain rule*', which you will encounter in Year 2.

Different e^{5x} with respect to x.

?

Different e^{-x} with respect to x.

?

Different $4e^{3x}$ with respect to x.

?

Note: In general, when you scale the function, you scale the derivative/integral.

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Different e^{5x} with respect to x.

$$\frac{dy}{dx} = 5e^{5x}$$

Different e^{-x} with respect to x.

$$y = e^{-1x} \quad \therefore \quad \frac{dy}{dx} = -e^{-x}$$

Different $4e^{3x}$ with respect to x.

$$\frac{dy}{dx} = 12e^{3x}$$

Note: In general, when you scale the function, you scale the derivative/integral.

More Graph Transformations

Sketch $y = e^{3x}$

?

Sketch $y = 5e^{-x}$

?

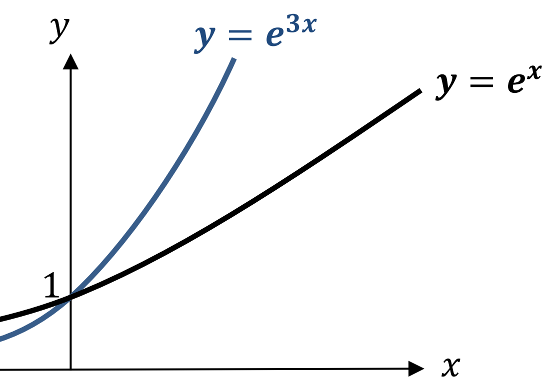
Sketch $y = 2 + e^{\frac{1}{3}x}$

?

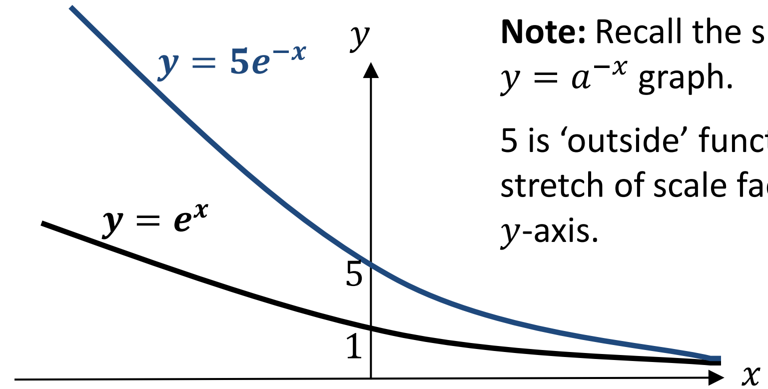
More Graph Transformations

Sketch $y = e^{3x}$

Input x is being replaced with $3x$, so stretch of scale factor $\frac{1}{3}$ on x-axis.



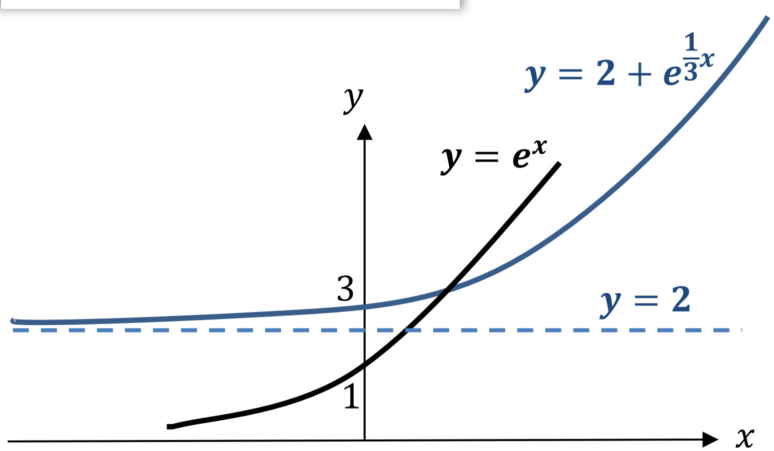
Sketch $y = 5e^{-x}$



Note: Recall the shape of a $y = a^{-x}$ graph.

5 is 'outside' function, so stretch of scale factor 5 on y-axis.

Sketch $y = 2 + e^{\frac{1}{3}x}$



We have a stretch on x-axis by scale factor 3, and a translation up by 2.

Important Note: Because the original **asymptote** was $y = 0$, it is now $y = 2$ and you must indicate this along with its equation.

Test Your Understanding

Sketch $y = e^{-2x} - 1$

?

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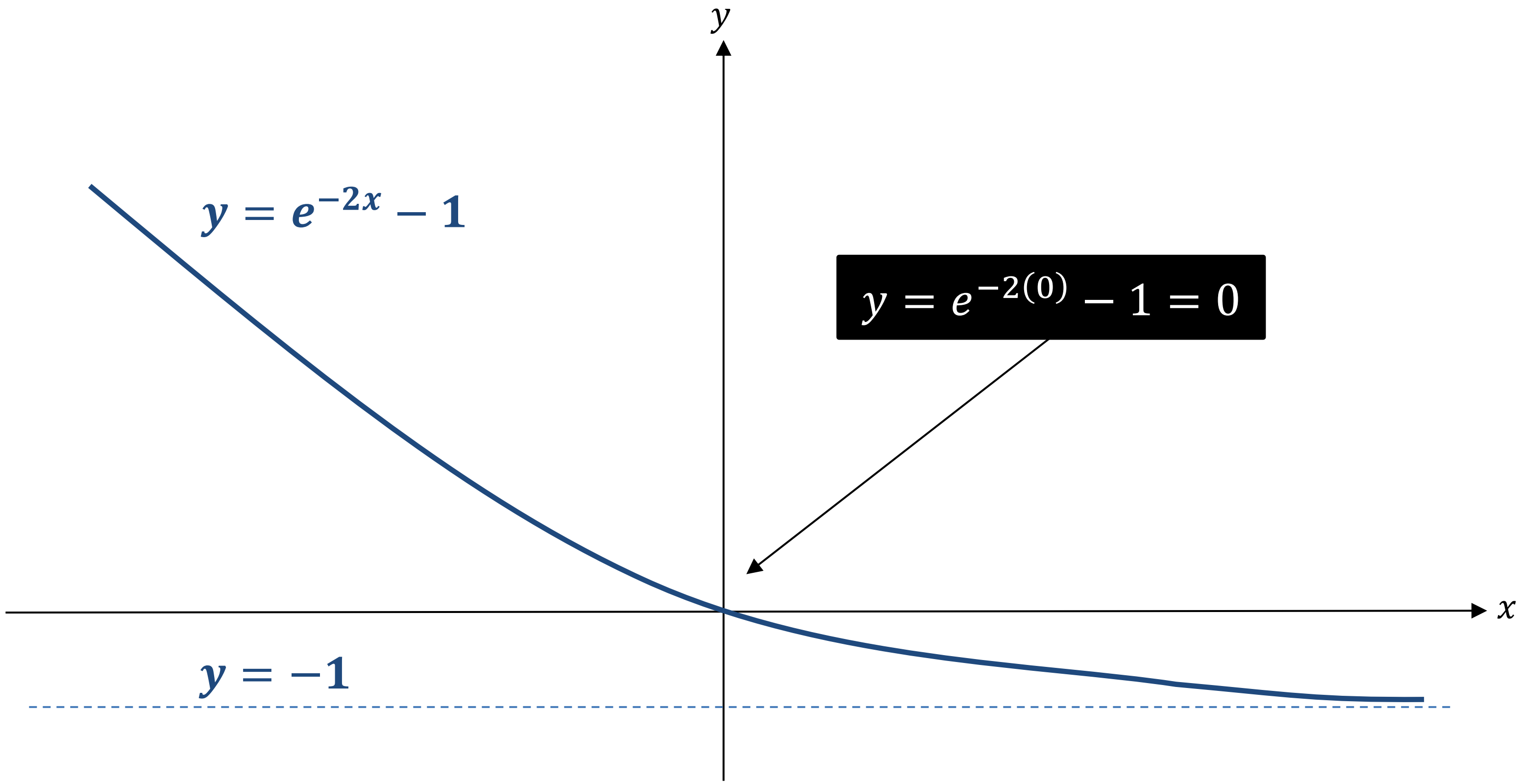
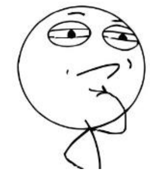
Sketch $y = e^{-2x} - 1$



Just for your interest...

Where does e come from, and why is it so important?



$e = 2.71828\ldots$
is known as **Euler's Number**, and is considered one of the five fundamental constants in maths:
$0,\ 1,\ \pi,\ e,\ i$

Its value was originally encountered by Bernoulli who was solving the following problem:
You have £1. If you put it in a bank account with 100% interest, how much do you have a year later? If the interest is split into 2 instalments of 50% interest, how much will I have? What about 3 instalments of 33.3%? And so on...

No. Instalments	Money after a year
1	$1 \times 2^1 = £2$
2	$1 \times 1.5^2 = £2.25$
3	$1 \times 1.\dot{3}^3 = £2.37$
4	$1 \times 1.25^4 = £2.44$
n	$\left(1 + \frac{1}{n}\right)^n$

As n becomes larger, the amount after a year approaches £2.71..., i.e. e!

Thus:
$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

But we have seen that differentiation by first principles uses 'limits'. It is therefore possible to prove from the definition above that $\frac{d}{dx}(e^x) = e^x$, and **these two definitions of e are considered to be equivalent***.

e therefore tends to arise in problems involving limits, and also therefore crops up all the time in anything involving differentiation and integration. Let's see some applications...

*You can find a full proof here in my Graph Sketching/Limits slides:
http://www.drfrostmaths.com/resources/resource.php?rid=163

Exercise 14.2

Pearson Pure Mathematics Year 1/AS
Page 114

Homework Exercise

1 Use a calculator to find the value of e^x to 4 decimal places when

a $x = 1$ **b** $x = 4$ **c** $x = -10$ **d** $x = 0.2$

2 **a** Draw an accurate graph of $y = e^x$ for $-4 \leqslant x \leqslant 4$.

b By drawing appropriate tangent lines, estimate the gradient at $x = 1$ and $x = 3$.

c Compare your answers to the actual values of e and e^3.

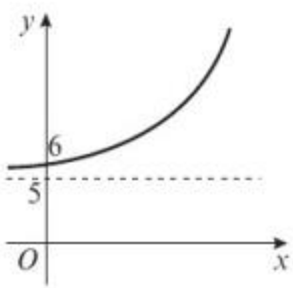
3 Sketch the graphs of:

a $y = e^{x+1}$ **b** $y = 4e^{-2x}$ **c** $y = 2e^x - 3$

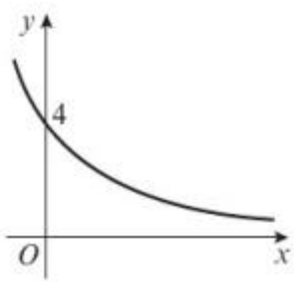
d $y = 4 - e^x$ **e** $y = 6 + 10e^{\frac{1}{2}x}$ **f** $y = 100e^{-x} + 10$

4 Each of the sketch graphs below is of the form $y = Ae^{bx} + C$, where A, b and C are constants. Find the values of A and C for each graph, and state whether b is positive or negative.

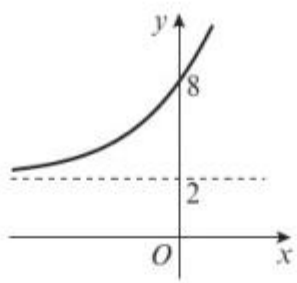
a



b



c



Hint You do not have enough information to work out the value of b, so simply state whether it is positive or negative.

5 Rearrange $f(x) = e^{3x+2}$ into the form $f(x) = Ae^{bx}$, where A and b are constants whose values are to be found. Hence, or otherwise, sketch the graph of $y = f(x)$.

Hint $e^{m+n} = e^m \times e^n$

Homework Exercise

6 Differentiate the following with respect to x.

a e^{6x} **b** $e^{-\frac{1}{3}x}$ **c** $7e^{2x}$

d $5e^{0.4x}$ **e** $e^{3x} + 2e^{x}$ **f** $e^{x}(e^{x} + 1)$

Hint For part **f**, start by expanding the bracket.

7 Find the gradient of the curve with equation $y = e^{3x}$ at the point where

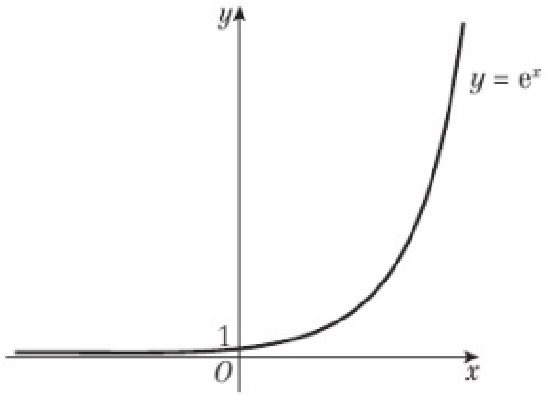
a $x = 2$ **b** $x = 0$ **c** $x = -0.5$

8 The function f is defined as $f(x) = e^{0.2x}$, $x \in \mathbb{R}$. Show that the tangent to the curve at the point $(5, e)$ goes through the origin.

Homework Answers

1 **a** 2.718 28 **b** 54.598 15 **c** 0.000 04 **d** 1.221 40

2 **a**

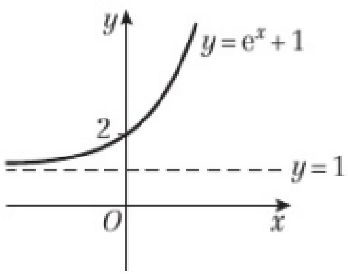


b Student's own answers

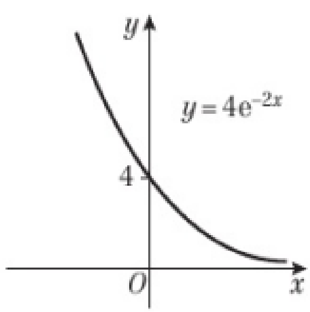
c e = 2.71828…

$e^3 = 20.08553…$

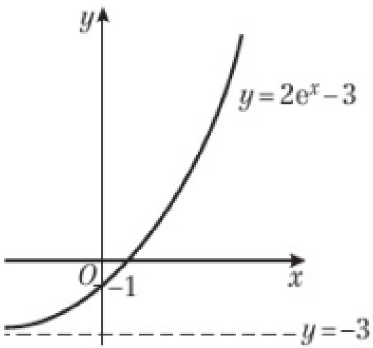
3 **a**



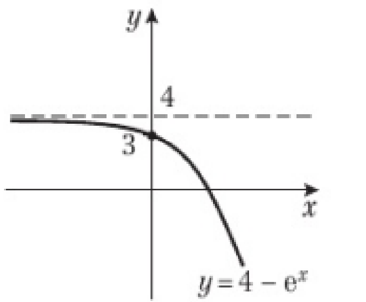
b



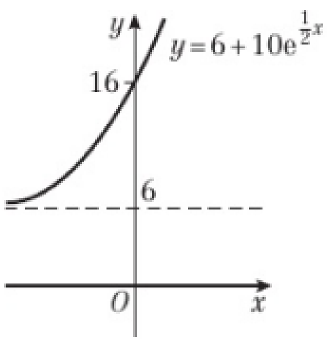
3 **c**



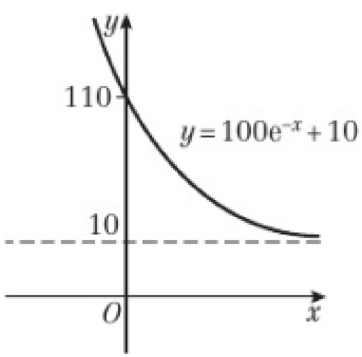
d



e



f



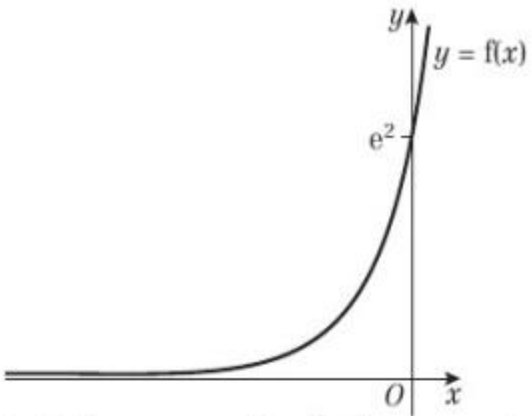
4 **a** $A = 1$, $C = 5$, b is positive

b $A = 4$, $C = 0$, b is negative

c $A = 6$, $C = 2$, b is positive

Homework Answers

5 $A = e^2, b = 3$



6 **a** $6e^{6x}$ **b** $-\frac{1}{3}e^{-\frac{1}{3}x}$ **c** $14e^{2x}$

d $2e^{0.4x}$ **e** $3e^{3x} + 2e^{x}$ **f** $2e^{2x} + e^{x}$

7 **a** $3e^{6}$ **b** 3 **c** $3e^{-1.5}$

8 $f'(x) = 0.2e^{0.2x}$

The gradient of the tangent when $x = 5$ is

$f'(5) = 0.2e^{1} = 0.2e$.

The equation of the tangent is therefore $y = (0.2e)x + c$.

At $(5, e)$, $e = 0.2e \times 5 + c$, so $c = 0$ and when $x = 0$, $y = 0$.