M1 Chapter 8: Modelling

Units and Vectors

SI units

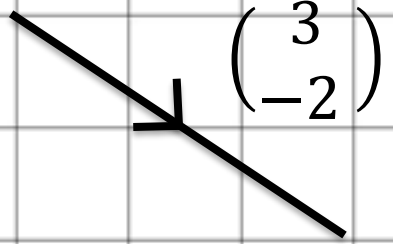
The SI units are a standard system of units, used internationally ("Système International"). These are the ones you will use:

Quantity	Unit	Symbol
Mass	kilogram	kg
Length/Displacement	metre	m
Time	seconds	s
Speed/Velocity	metres per second	$m\ s^{-1}$
Acceleration	metres per second per second	$m\ s^{-2}$
Force/Weight	newton	N (= $kg\ m\ s^{-2}$)

This unit is consistent with force being mass × acceleration

- The MKS units {metres, kilogram, seconds} are the ***mechanical base*** units defined in terms of the speed of light constant and Planck's quantum constant.
- All other units in mechanics are ***derived*** units that can be written as a combination of base units.

Representing Vectors

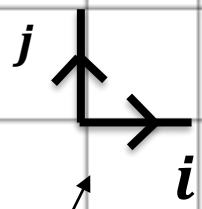


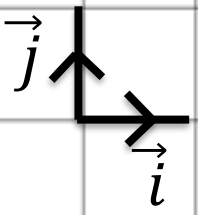
You should already be familiar that the value of a vector is the **displacement** in the x and y direction (if in 2D).

$$\vec{a} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad \vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{a} + \vec{b} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$$

$$2\vec{a} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$





✎ **Bold notation** is used in textbooks and exam papers but is awkward for handwriting.

✎ A **unit vector** is a vector of magnitude 1. $\vec{i}$ and $\vec{j}$ are unit vectors in the x-axis and y-axis respectively.

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

e.g. $\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4\vec{i} + 3\vec{j}$

Examples

If $\boldsymbol{a} = 3\boldsymbol{i}, \quad \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$ then:

1) Write $\boldsymbol{a}$ in column vector form.
2) Find $\boldsymbol{b} + 2\boldsymbol{c}$ in $\boldsymbol{i}, \boldsymbol{j}$ form.

1 ?

2 ?

Examples

If $\boldsymbol{a} = 3\boldsymbol{i}, \quad \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}, \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$ then:

1) Write $\boldsymbol{a}$ in column vector form.
2) Find $\boldsymbol{b} + 2\boldsymbol{c}$ in $\boldsymbol{i}, \boldsymbol{j}$ form.

1 $\quad \boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

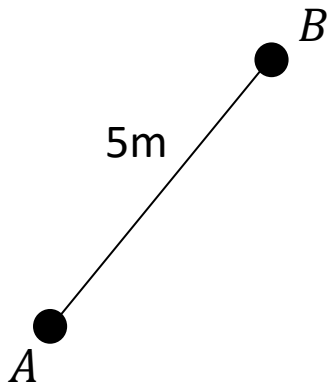
2 $\quad \boldsymbol{b} + 2\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} + 2(\boldsymbol{i} - 2\boldsymbol{j})$

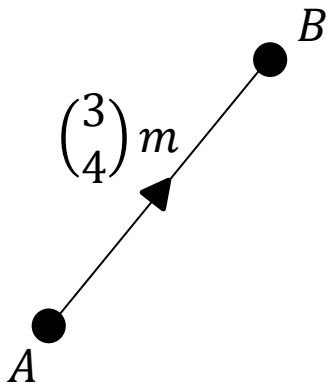
$= \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{i} - 4\boldsymbol{j}$

$= 3\boldsymbol{i} - 3\boldsymbol{j}$

Vectors ↔ Scalars

In Mechanics you will often need to convert between the scalar magnitude of a quantity and the vector version.





Scalar Magnitude	Vector Form
Distance	Displacement
Speed	Velocity

A scalar quantity has magnitude (i.e. size) only.
The 5m is a distance.
The value is always positive.

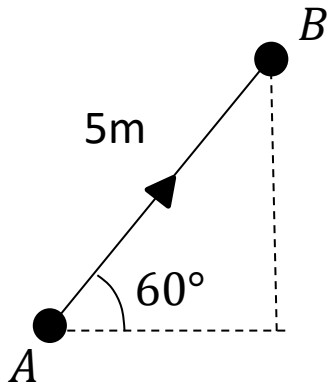
A vector quantity also has direction.
The vector equivalent of distance is displacement.

Quantities which are vectors *with* a scalar magnitude:
force, acceleration

Quantities which are only scalars:
time, mass

Representing Vectors

Geometric Form	Vector Form



$$\begin{pmatrix} 5\cos 60° \\ 5\sin 60° \end{pmatrix} = \begin{pmatrix} 2.5 \\ 4.33 \end{pmatrix} m$$

To convert to vector form, just use basic trigonometry to find the x-change and y-change.

Speed Tip: If x is the magnitude, use $x\cos\theta$ for the side adjacent to the angle and $x\sin\theta$ for the side opposite it.

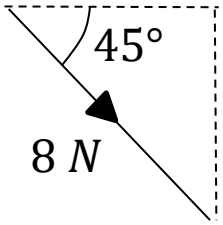
Speed:

$$\sqrt{5^2 + (-12)^2} = 13\, ms^{-1}$$

Velocity:

$$\begin{pmatrix} 5 \\ -12 \end{pmatrix} ms^{-1}$$

To convert scalar form, just find the **magnitude** of the vector using Pythagoras.



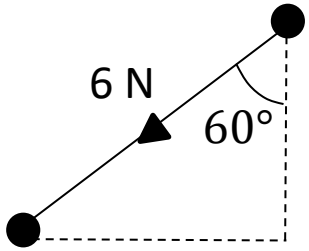
Force vector:

$$\begin{pmatrix} 8cos45° \\ -8\sin 45° \end{pmatrix} = \begin{pmatrix} 4\sqrt{2} \\ -4\sqrt{2} \end{pmatrix} N$$

In the y-direction the force is acting downwards.

Further Examples

Geometric Form	Vector Form

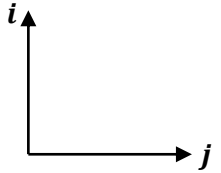


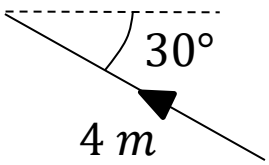
$$\begin{pmatrix} -6\sin 60° \\ -6\cos 60° \end{pmatrix} = \begin{pmatrix} -3\sqrt{3} \\ -3 \end{pmatrix} N$$

$$\sqrt{6^2 + (-8)^2} = 10\ ms^{-2}$$

$$(6\boldsymbol{i} - 8\boldsymbol{j})\ ms^{-2}$$

Recall from Pure Year 1 that $6\boldsymbol{i} - 8\boldsymbol{j}$ is another way of writing $\begin{pmatrix} 6 \\ -8 \end{pmatrix}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the positive x and y directions.





Displacement:

$$(-4\cos 30°)\boldsymbol{i} + (2\sin 30°)\boldsymbol{j}$$
$$= \left(-2\sqrt{3}\boldsymbol{i} + 2\boldsymbol{j}\right) m$$

Test Your Understanding

[Textbook] A man walks from A to B and then from B to C.
His displacement from A to B is $6\boldsymbol{i} + 4\boldsymbol{j}$ m.
His displacement from B to C is $5\boldsymbol{i} - 12\boldsymbol{j}$ m.
(a) What is the magnitude of the displacement from A to C?
(b) What is the total distance the man has walked in getting from A to C.

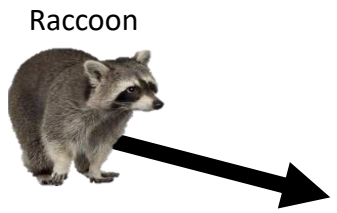
? Suitable Diagram

a ?

b ?

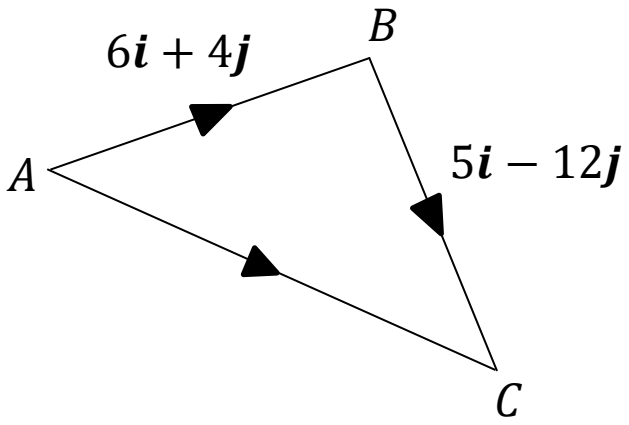
A raccoon has a velocity of $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ ms^{-1}.
Determine the angle the trajectory of the raccoon makes with the unit vector $\boldsymbol{i}$.

?



Test Your Understanding

[Textbook] A man walks from A to B and then from B to C.
His displacement from A to B is $6\boldsymbol{i} + 4\boldsymbol{j}$ m.
His displacement from B to C is $5\boldsymbol{i} - 12\boldsymbol{j}$ m.
(a) What is the magnitude of the displacement from A to C?
(b) What is the total distance the man has walked in getting from A to C.



a

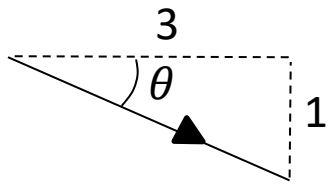
$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
$= 11\boldsymbol{i} - 8\boldsymbol{j}$
$\left|\overrightarrow{AC}\right| = \sqrt{11^2 + 8^2} = 13.6\ km$

b

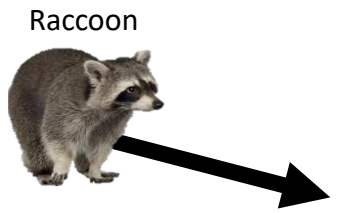
$\left|\overrightarrow{AB}\right| = \sqrt{6^2 + 4^2} = 7.21\ km$
$\left|\overrightarrow{BC}\right| = \sqrt{5^2 + 12^2} = 13\ km$

Total distance:
$7.21 + 13 = 20.21\ km$

A raccoon has a velocity of $\begin{pmatrix} 3 \\ -1 \end{pmatrix}\ ms^{-1}$.
Determine the angle the trajectory of the raccoon makes with the unit vector $\boldsymbol{i}$.



$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.4°\ (3sf)$



Exercise 8C, 8D

Pearson Stats/Mechanics Year 1

1) Exercise 8.3 page 53
2) Exercise 8.4 pages 54-55

Homework Exercise

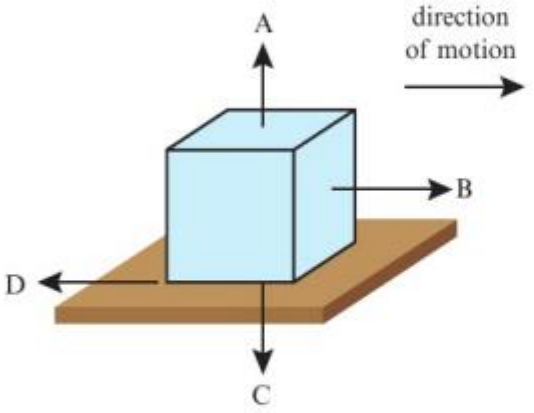
1 Convert to SI units:

a $65\,\text{km}\,\text{h}^{-1}$ **b** $15\,\text{g}\,\text{cm}^{-2}$ **c** 30 cm per minute

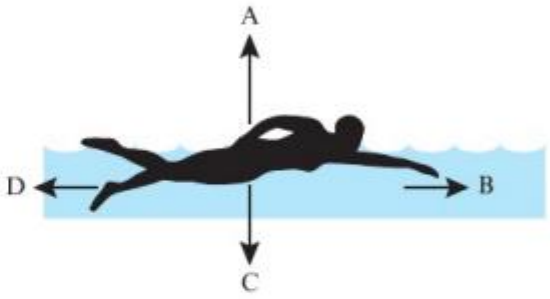
d $24\,\text{g}\,\text{m}^{-3}$ **e** $4.5 \times 10^{-2}\,\text{g}\,\text{cm}^{-3}$ **f** $6.3 \times 10^{-3}\,\text{kg}\,\text{cm}^{-2}$

2 Write down the names of the forces shown in each of these diagrams.

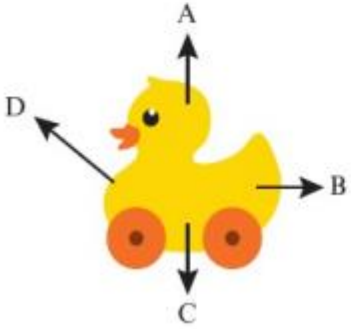
a A box being pushed along rough ground



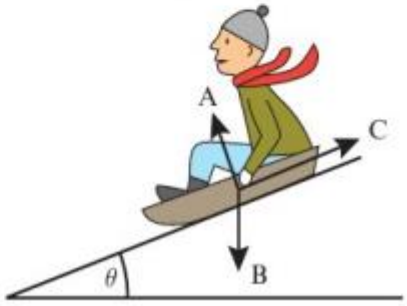
b A man swimming through the water



c A toy duck being pulled along by a string



d A man sliding down a hill on a sledge

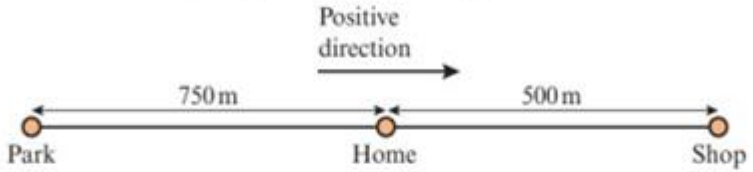


Homework Exercise

3 A man walks from his home along a straight road to a shop with a speed of $2.1\,\text{m s}^{-1}$ and walks home again at a speed of $1.8\,\text{m s}^{-1}$.

He then jogs along a straight road from his home to the park with a speed of $2.7\,\text{m s}^{-1}$ and returns home at a speed of $2.5\,\text{m s}^{-1}$.

The park, the man's home and the shop all lie on a straight line, as shown in the diagram.



Taking the positive direction as shown in the diagram, state the man's:

a velocity on the journey from his home to the shop

b displacement from his home when he reaches the shop

c velocity on the journey from the shop to his home

d velocity on the journey from his home to the park

e displacement from his home when he reaches the park

f velocity on the journey from the park to his home.

4 The velocity of a car is given by $\mathbf{v} = 12\mathbf{i} - 10\mathbf{j}\,\text{m s}^{-1}$. Find:

a the speed of the car

b the angle the direction of motion of the car makes with the unit vector $\mathbf{i}$.

Problem-solving

Draw a sketch to help you find the direction. $\mathbf{j}$ acts in the positive y-direction, so the angle between $\mathbf{j}$ and the vector $3\mathbf{i} - 4\mathbf{j}$ will be obtuse.

Homework Exercise

5 The acceleration of a motorbike is given by $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}\,\text{m s}^{-2}$. Find:

a the magnitude of the acceleration

b the angle the direction of the acceleration vector makes with the unit vector $\mathbf{j}$.

6 A girl cycles from A to B and then from B to C.

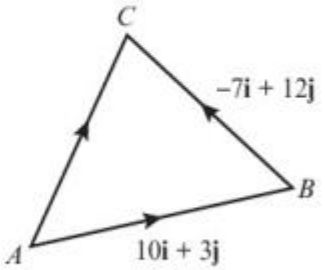
The displacement from A to B is $10\mathbf{i} + 3\mathbf{j}$ km.

The displacement from B to C is $-7\mathbf{i} + 12\mathbf{j}$ km.

a Find the magnitude of the displacement from A to C.

b Find the total distance the girl has cycled in getting from A to C.

c Work out the angle $\overrightarrow{AC}$ makes with the unit vector $\mathbf{i}$.



Homework Answers

1 **a** $18.1\,m\,s^{-1}$ **b** $150\,kg\,m^{-2}$ **c** $5 \times 10^{-3}\,m\,s^{-1}$
d $0.024\,kg\,m^{-3}$ **e** $45\,kg\,m^{-3}$ **f** $63\,kg\,m^{-2}$

2 **a** A: Normal reaction, B: Forward thrust, C: Weight, D: Friction.
b A: Buoyancy, B: Forward thrust, C: Weight, D: Water resistance or drag.
c A: Normal reaction, B: Friction, C: Weight, D: Tension.
d A: Normal reaction, B: Weight, C: Friction.

3 **a** $2.1\,m\,s^{-1}$ **b** 500 m **c** $-1.8\,m\,s^{-1}$

4 **d** $-2.7\,m\,s^{-1}$ **e** −750 m **f** $2.5\,m\,s^{-1}$

5 **a** $15.6\,m\,s^{-1}$ **b** 39.8°

6 **a** $5\,m\,s^{-2}$ **b** 143°

a 15.3 m **b** 24.3 m **c** 78.7°