P2 Chapter 7: Trigonometric Equations

Chapter Practice

Key Points

Summary of key points

1 The **addition** (or compound-angle) formulae are:

- $\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$ $\quad\quad$ $\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$
- $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$ $\quad\quad$ $\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$
- $\tan(A + B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\quad\quad$ $\tan(A - B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

2 The **double-angle** formulae are:

- $\sin 2A \equiv 2 \sin A \cos A$
- $\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$

3 For positive values of a and b,

- $a \sin x \pm b \cos x$ can be expressed in the form $R \sin(x \pm \alpha)$
- $a \cos x \pm b \sin x$ can be expressed in the form $R \cos(x \mp \alpha)$

with $R > 0$ and $0 < \alpha < 90°$ $\left(\text{or } \frac{\pi}{2}\right)$

where $R \cos \alpha = a$ and $R \sin \alpha = b$ and $R = \sqrt{a^2 + b^2}$.

Chapter Exercises

1 Without using a calculator, find the value of:

a $\sin 40° \cos 10° - \cos 40° \sin 10°$ **b** $\frac{1}{\sqrt{2}}\cos 15° - \frac{1}{\sqrt{2}}\sin 15°$ **c** $\frac{1 - \tan 15°}{1 + \tan 15°}$

2 Given that $\sin x = \frac{1}{\sqrt{5}}$ where x is acute and that $\cos(x - y) = \sin y$, show that $\tan y = \frac{\sqrt{5} + 1}{2}$

3 The lines l_1 and l_2, with equations $y = 2x$ and $3y = x - 1$ respectively, are drawn on the same set of axes. Given that the scales are the same on both axes and that the angles l_1 and l_2 make with the positive x-axis are A and B respectively,

a write down the value of $\tan A$ and the value of $\tan B$;

b without using your calculator, work out the acute angle between l_1 and l_2.

4 In $\triangle ABC$, $AB = 5$ cm and $AC = 4$ cm, $\angle ABC = (\theta - 30°)$ and $\angle ACB = (\theta + 30°)$. Using the sine rule, show that $\tan \theta = 3\sqrt{3}$.

5 The first three terms of an arithmetic series are $\sqrt{3}\cos\theta$, $\sin(\theta - 30°)$ and $\sin\theta$, where θ is acute. Find the value of θ.

6 Two of the angles, A and B, in $\triangle ABC$ are such that $\tan A = \frac{3}{4}$, $\tan B = \frac{5}{12}$

a Find the exact value of: **i** $\sin(A + B)$ **ii** $\tan 2B$.

b By writing C as $180° - (A + B)$, show that $\cos C = -\frac{33}{65}$

Chapter Exercises

7 The angles x and y are acute angles such that $\sin x = \dfrac{2}{\sqrt{5}}$ and $\cos y = \dfrac{3}{\sqrt{10}}$

a Show that $\cos 2x = -\frac{3}{5}$

b Find the value of $\cos 2y$.

c Show without using your calculator, that:

i $\tan(x + y) = 7$ ii $x - y = \dfrac{\pi}{4}$

8 Given that $\sin x \cos y = \frac{1}{2}$ and $\cos x \sin y = \frac{1}{3}$,

a show that $\sin(x + y) = 5\sin(x - y)$.

Given also that $\tan y = k$, express in terms of k:

b $\tan x$

c $\tan 2x$

9 a Given that $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$, show that $\tan 2\theta = \dfrac{1}{\sqrt{3}}$ **(2 marks)**

b Hence solve, for $0 \leqslant \theta \leqslant \pi$, the equation $\sqrt{3}\sin 2\theta + 2\sin^2\theta = 1$. **(4 marks)**

10 a Show that $\cos 2\theta = 5\sin\theta$ may be written in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants to be found. **(3 marks)**

b Hence solve, for $-\pi \leqslant \theta \leqslant \pi$, the equation $\cos 2\theta = 5\sin\theta$. **(4 marks)**

11 **a** Given that $2\sin x = \cos(x - 60)°$, show that $\tan x = \dfrac{1}{4 - \sqrt{3}}$ **(4 marks)**

b Hence solve, for $0 \leqslant x \leqslant 360°$, $2\sin x = \cos(x - 60°)$, giving your answers to 1 decimal place. **(2 marks)**

12 **a** Given that $4\sin(x + 70°) = \cos(x + 20°)$, show that $\tan x = -\frac{3}{5}\tan 70°$. **(4 marks)**

b Hence solve, for $0 \leqslant x \leqslant 180°$, $4\sin(x + 70°) = \cos(x + 20°)$, giving your answers to 1 decimal place. **(3 marks)**

13 **a** Given that α is acute and $\tan\alpha = \frac{3}{4}$, prove that

$$3\sin(\theta + \alpha) + 4\cos(\theta + \alpha) \equiv 5\cos\theta$$

b Given that $\sin x = 0.6$ and $\cos x = -0.8$, evaluate $\cos(x + 270°)$ and $\cos(x + 540°)$.

14 **a** Prove, by counter-example, that the statement

$$\sec(A + B) \equiv \sec A + \sec B, \text{ for all } A \text{ and } B$$

is false. **(2 marks)**

b Prove that $\tan\theta + \cot\theta \equiv 2\operatorname{cosec} 2\theta$, $\theta \neq \dfrac{n\pi}{2}$, $n \in \mathbb{Z}$. **(4 marks)**

15 Using $\tan 2\theta \equiv \dfrac{2\tan\theta}{1-\tan^2\theta}$ with an appropriate value of θ,

a show that $\tan\dfrac{\pi}{8} = \sqrt{2} - 1$.

b Use the result in **a** to find the exact value of $\tan\dfrac{3\pi}{8}$

16 **a** Express $\sin x - \sqrt{3}\cos x$ in the form $R\sin(x-\alpha)$, with $R > 0$ and $0 < \alpha < 90°$. **(4 marks)**

b Hence sketch the graph of $y = \sin x - \sqrt{3}\cos x$, for $-360° \leqslant x \leqslant 360°$, giving the coordinates of all points of intersection with the axes. **(4 marks)**

17 Given that $7\cos 2\theta + 24\sin 2\theta \equiv R\cos(2\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$, find:

a the value of R and the value of α, to 2 decimal places **(4 marks)**

b the maximum value of $14\cos^2\theta + 48\sin\theta\cos\theta$. **(1 mark)**

c Solve the equation $7\cos 2\theta + 24\sin 2\theta = 12.5$, for $0 \leqslant \theta \leqslant \pi$, giving your answers to 2 decimal places. **(5 marks)**

Chapter Exercises

18 **a** Express $1.5\sin 2x + 2\cos 2x$ in the form $R\sin(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving your values of R and α to 3 decimal places where appropriate. **(4 marks)**

b Express $3\sin x\cos x + 4\cos^2 x$ in the form $a\sin 2x + b\cos 2x + c$, where a, b and c are constants to be found. **(3 marks)**

c Hence, using your answer to part **a**, deduce the maximum value of $3\sin x\cos x + 4\cos^2 x$. **(1 mark)**

19 **a** Given that $\sin^2\frac{\theta}{2} = 2\sin\theta$, show that $\sqrt{17}\sin(\theta + \alpha) = 1$ and state the value of α, where $0 \leqslant \alpha \leqslant 90°$. **(3 marks)**

b Hence, or otherwise, solve $\sin^2\frac{\theta}{2} = 2\sin\theta$ for $0 \leqslant \theta \leqslant 360°$. **(4 marks)**

20 **a** Given that $2\cos\theta = 1 + 3\sin\theta$, show that $R\cos(\theta + \alpha) = 1$, where R and α are constants to be found, and $0 \leqslant \alpha \leqslant 90°$. **(2 marks)**

b Hence, or otherwise, solve $2\cos\theta = 1 + 3\sin\theta$ for $0 \leqslant \theta \leqslant 360°$. **(4 marks)**

21 Using known trigonometric identities, prove the following:

a $\sec\theta\,\text{cosec}\,\theta \equiv 2\,\text{cosec}\,2\theta$

b $\tan\left(\frac{\pi}{4} + x\right) - \tan\left(\frac{\pi}{4} - x\right) \equiv 2\tan 2x$

c $\sin(x + y)\sin(x - y) \equiv \cos^2 y - \cos^2 x$

d $1 + 2\cos 2\theta + \cos 4\theta \equiv 4\cos^2\theta\cos 2\theta$

22 **a** Use the double-angle formulae to prove that $\dfrac{1-\cos 2x}{1+\cos 2x} \equiv \tan^2 x$. **(4 marks)**

b Hence find, for $-\pi \leqslant x \leqslant \pi$, all the solutions of $\dfrac{1-\cos 2x}{1+\cos 2x} = 3$, leaving your answers in terms of π. **(2 marks)**

23 **a** Prove that $\cos^4 2\theta - \sin^4 2\theta \equiv \cos 4\theta$. **(4 marks)**

b Hence find, for $0 \leqslant \theta \leqslant 180°$, all the solutions of $\cos^4 2\theta - \sin^4 2\theta = \frac{1}{2}$ **(2 marks)**

24 **a** Prove that $\dfrac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta$. **(4 marks)**

b Verify that $\theta = 180°$ is a solution of the equation $\sin 2\theta = 2 - 2\cos 2\theta$. **(1 mark)**

c Using the result in part **a**, or otherwise, find the two other solutions, $0 < \theta < 360°$, of the equation $\sin 2\theta = 2 - 2\cos 2\theta$. **(3 marks)**

25 The curve on an oscilloscope screen satisfies the equation $y = 2\cos x - \sqrt{5}\sin x$.

a Express the equation of the curve in the form $y = R\cos(x + \alpha)$, where R and α are constants and $R > 0$ and $0 \leqslant \alpha < \dfrac{\pi}{2}$ **(4 marks)**

b Find the values of x, $0 \leqslant x < 2\pi$, for which $y = -1$. **(3 marks)**

26 **a** Express $1.4\sin\theta - 5.6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Round R and α to 3 decimal places. **(4 marks)**

b Hence find the maximum value of $1.4\sin\theta - 5.6\cos\theta$ and the smallest positive value of θ for which this maximum occurs. **(3 marks)**

The length of daylight, $d(t)$ at a location in northern Scotland can be modelled using the equation

$$d(t) = 12 - 5.6\cos\left(\frac{360t}{365}\right)^{\circ} + 1.4\sin\left(\frac{360t}{365}\right)^{\circ}$$

where t is the numbers of days into the year.

c Calculate the minimum number of daylight hours in northern Scotland as given by this model. **(2 marks)**

d Find the value of t when this minimum number of daylight hours occurs. **(1 mark)**

27 **a** Express $12\sin x + 5\cos x$ in the form $R\sin(x + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Round α to 1 decimal place. **(4 marks)**

A runner's speed, v in m/s, in an endurance race can be modelled by the equation

$$v(x) = \frac{50}{12\sin\left(\frac{2x}{5}\right)^{\circ} + 5\cos\left(\frac{2x}{5}\right)^{\circ}}, \quad 0 \leqslant x \leqslant 300$$

where x is the time in minutes since the beginning of the race.

b Find the minimum value of v. **(2 marks)**

c Find the time into the race when this speed occurs. **(1 mark)**

Chapter Exercises

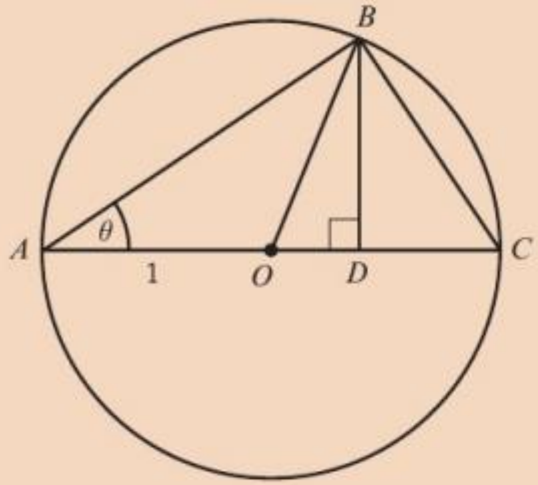
Challenge

1 Prove the identities:

a $\dfrac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv -\cot\theta$

b $\cos x + 2\cos 3x + \cos 5x \equiv 4\cos^2 x \cos 3x$

2 The points A, B and C lie on a circle with centre O and radius 1. AC is a diameter of the circle and point D lies on OC such that $\angle ODB = 90°$.



Use this construction to prove:

a $\sin 2\theta \equiv 2\sin\theta\cos\theta$ **b** $\cos 2\theta \equiv 2\cos^2\theta - 1$

Hint Find expressions for $\angle BOD$ and AB, then consider the lengths OD and DB.

Chapter Answers

1 a $\frac{1}{2}$ b $\frac{1}{2}$ c $\frac{\sqrt{3}}{3}$

2 $\sin x = \frac{1}{\sqrt{5}}$, so $\cos x = \frac{2}{\sqrt{5}}$

$\cos(x - y) = \sin y \Rightarrow \frac{2}{\sqrt{5}} \cos y + \frac{1}{\sqrt{5}} \sin y = \sin y$

$\Rightarrow (\sqrt{5} - 1) \sin y = 2 \cos y \Rightarrow \tan y = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2}$

3 a $\tan A = 2$, $\tan B = \frac{1}{3}$ b $45°$

4 Use the sine rule and addition formulae to get

$\frac{1}{20} \sin\theta \times \frac{\sqrt{3}}{2} = \frac{9}{20} \cos\theta \times \frac{1}{2}$

Then rearrage to get $\tan\theta = 3\sqrt{3}$.

5 $75°$

6 a i $\frac{56}{65}$ ii $\frac{120}{119}$

b Use $\cos\{180° - (A + B)\} \equiv -\cos(A + B)$ and expand. You can work out all the required trig. ratios (A and B are acute).

7 a Use $\cos 2x \equiv 1 - 2\sin^2 x$ b $\frac{4}{5}$

c i Use $\tan x = 2$, $\tan y = \frac{1}{3}$ in the expansion of $\tan(x + y)$.

ii Find $\tan(x - y) = 1$ and note that $x - y$ has to be acute.

8 a Show that both sides are equal to $\frac{5}{6}$.

b $\frac{3k}{2}$ c $\frac{12k}{4 - 9k^2}$

9 a $\sqrt{3} \sin 2\theta = 1 - 2\sin^2\theta = \cos 2\theta$

$\Rightarrow \sqrt{3} \tan 2\theta = 1 \Rightarrow \tan 2\theta = \frac{1}{\sqrt{3}}$

b $\frac{\pi}{12}, \frac{7\pi}{12}$

10 a $a = 2$, $b = 5$, $c = -1$ b 0.187, 2.95

11 a $\cos(x - 60°) = \cos x \cos 60° + \sin x \sin 60°$

$= \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x$

So $\left(2 - \frac{\sqrt{3}}{2}\right) \sin x = \frac{1}{2} \cos x \Rightarrow \tan x = \frac{\frac{1}{2}}{2 - \frac{\sqrt{3}}{2}} = \frac{1}{4 - \sqrt{3}}$

b $23.8°$, $203.8°$

12 a $\cos(x + 20°) = \sin(90° - 20° - x) = \sin(70° - x)$

Using addition formulae:

$\cos x \cos 20° - \sin x \sin 20°$

$= \sin 70° \cos x - \cos 70° \sin x$

Rearrange to get: $\sin x(5\cos 70°) + \cos x(3\sin 70°) = 0$

$\Rightarrow \tan x = \frac{\sin x}{\cos x} = -\frac{3\sin 70°}{5\cos 70°} = -\frac{3}{5} \tan 70°$

b $121.2°$

13 a Find $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$ and insert in expansions on L.H.S. Result follows.

b 0.6, 0.8

14 a Example: $A = 60°, B = 0°$; $\sec(A + B) = 2$,
$\sec A + \sec B = 2 + 1 = 3$

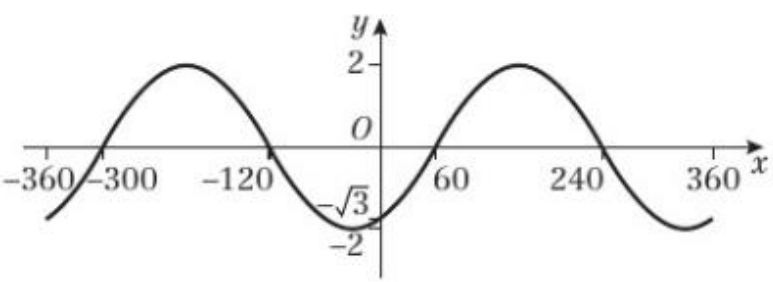
b L.H.S. $= \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} \equiv \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$
$\equiv \dfrac{1}{\frac{1}{2}\sin 2\theta} \equiv 2\operatorname{cosec} 2\theta =$ R.H.S.

15 a Setting $\theta = \frac{\pi}{8}$ gives resulting quadratic equation in t,
$t^2 + 2t - 1 = 0$, where $t = \tan\left(\frac{\pi}{8}\right)$.
Solving this and taking +ve value for t gives result.

b Expanding $\tan\left(\frac{\pi}{4} + \frac{\pi}{8}\right)$ gives answer: $\sqrt{2} + 1$

16 a $2\sin(x - 60)°$

b



Graph crosses y-axis at $(0, -\sqrt{3})$
Graph crosses x-axis at $(-300°, -0)$, $(-120°, 0)$, $(60°, 0)$, $(240°, 0)$

17 a $R = 25, \alpha = 1.29$ **b** 32 **c** $\theta = 0.12, 1.17$

18 a $2.5\sin(2x + 0.927)$ **b** $\frac{3}{2}\sin 2x + 2\cos 2x + 2$ **c** 4.5

19 a $\alpha = 14.0°$ **b** $0°, 151.9°, 360°$

20 a $R = \sqrt{13}$, $\alpha = 56.3°$ **b** $\theta = 17.6°, 229.8°$

21 a L.H.S. $= \dfrac{1}{\cos\theta} \cdot \dfrac{1}{\sin\theta} \equiv \dfrac{1}{\frac{1}{2}\sin 2\theta} \equiv 2\operatorname{cosec} 2\theta =$ R.H.S.

b L.H.S. $= \dfrac{1 + \tan x}{1 - \tan x} - \dfrac{1 - \tan x}{1 + \tan x}$
$\equiv \dfrac{(1 + \tan x)^2 - (1 - \tan x)^2}{(1 + \tan x)(1 - \tan x)}$
$\equiv \dfrac{(1 + 2\tan x + \tan^2 x) - (1 - 2\tan x + \tan^2 x)}{1 - \tan^2 x}$
$\equiv \dfrac{4\tan x}{1 - \tan^2 x} = \dfrac{2(2\tan x)}{1 - \tan^2 x} = 2\tan 2x =$ R.H.S.

c L.H.S. $= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)$
$= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$
$= (1 - \cos^2 x)\cos^2 y - \cos^2 x(1 - \sin^2 y) =$ R.H.S.

d L.H.S. $= 2\cos 2\theta + 1 + (2\cos^2 2\theta - 1)$
$\equiv 2\cos 2\theta(1 + \cos 2\theta) \equiv 2\cos 2\theta(2\cos^2\theta)$
$\equiv 4\cos^2\theta\cos 2\theta \equiv$ R.H.S.

22 a $\dfrac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)} \equiv \dfrac{2\sin^2 x}{2\cos^2 x}$
$\equiv \tan^2 x$

b $\pm\dfrac{\pi}{3}, \pm\dfrac{2\pi}{3}$

23 a L.H.S. $= \cos^4 2\theta - \sin^4 2\theta$
$\equiv (\cos^2 2\theta - \sin^2 2\theta)(\cos^2 2\theta + \sin^2 2\theta)$
$\equiv (\cos^2 2\theta - \sin^2 2\theta)(1)$
$\equiv \cos 4\theta =$ R.H.S.

b $15°, 75°, 105°, 165°$

24 **a** Use $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.
b $\sin 360° = 0$, $2 - 2\cos(360°) = 2 - 2 = 0$
c $26.6°$, $206.6°$
25 **a** $R = 3$, $\alpha = 0.841$ **b** $x = 1.07, 3.53$
26 **a** $R = 5.772$, $\alpha = 75.964°$ **b** 5.772 when $\theta = 166.0°$
c 6.228 hours **d** 350.8 days
27 **a** $13\sin(x + 22.6°)$ **b** 3.8 m/s
c 168.5 minutes

Challenge

1 **a** $\dfrac{\cos 2\theta + \cos 4\theta}{\sin 2\theta - \sin 4\theta} \equiv \dfrac{2\cos 3\theta\cos\theta}{-2\cos 3\theta\sin\theta} \equiv -\cot\theta$

b $\cos 5x + \cos x + 2\cos 3x$
$\equiv 2\cos 3x\cos 2x + 2\cos 3x$
$\equiv 2\cos 3x(\cos 2x + 1)$
$\equiv 2\cos 3x(2\cos^2 x)$
$\equiv 4\cos^2 x\cos 3x$