
P2 Chapter 1: CoAngle Trigonometry

Chapter Practice

Key Points

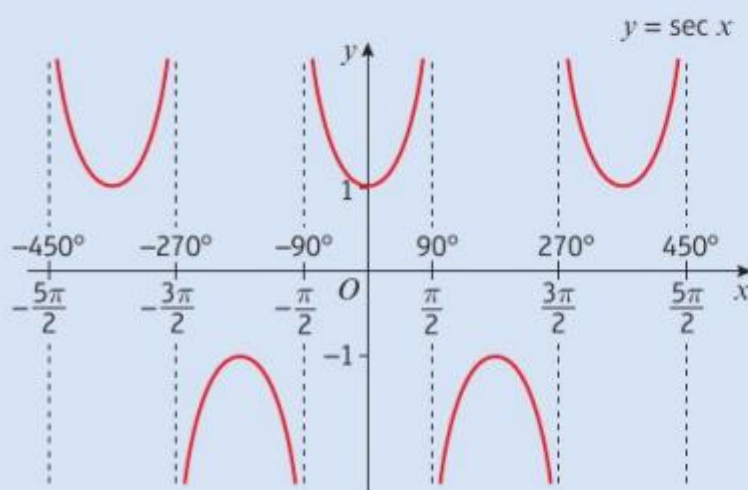
1 • $\sec x = \frac{1}{\cos x}$ (undefined for values of x for which $\cos x = 0$)

• $\operatorname{cosec} x = \frac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)

• $\cot x = \frac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)

• $\cot x = \frac{\cos x}{\sin x}$

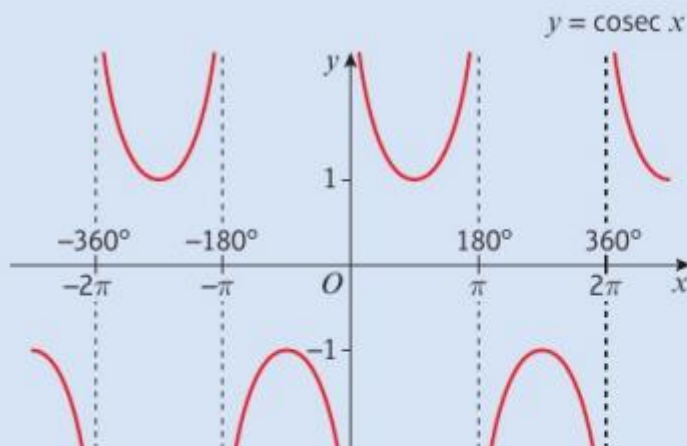
- 2 The graph of $y = \sec x$, $x \in \mathbb{R}$, has symmetry in the y -axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$.



- The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90^\circ, 270^\circ, 450^\circ, \dots$ or any odd multiple of 90° .
- In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ or any odd multiple of $\frac{\pi}{2}$.
- The range of $y = \sec x$ is $y \leq -1$ or $y \geq 1$.

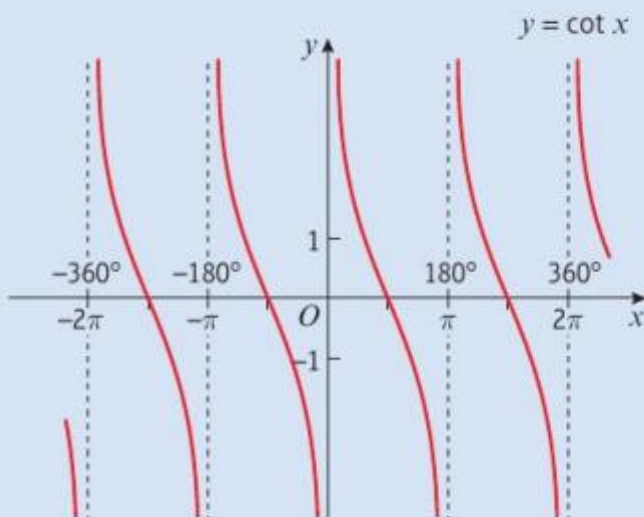
Key Points

- 3 The graph of $y = \operatorname{cosec} x$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$.



- The domain of $y = \operatorname{cosec} x$ is $x \in \mathbb{R}$, $x \neq 0^\circ, 180^\circ, 360^\circ, \dots$ or any multiple of 180° .
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \dots$ or any multiple of π
- The range of $y = \operatorname{cosec} x$ is $y \leq -1$ or $y \geq 1$.

- 4 The graph of $y = \cot x$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$.



- The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0^\circ, 180^\circ, 360^\circ, \dots$ or any multiple of 180° .
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \dots$ or any multiple of π .
- The range of $y = \cot x$ is $y \in \mathbb{R}$.

- 5 $\sec x = k$ and $\operatorname{cosec} x = k$ have no solutions for $-1 < k < 1$.

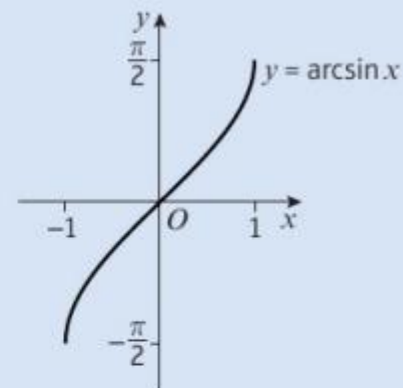
Key Points

6 You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities:

- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$

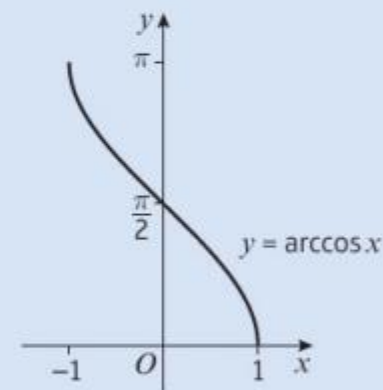
7 The inverse function of $\sin x$ is called **arcsin x** .

- The domain of $y = \arcsin x$ is $-1 \leq x \leq 1$
- The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$ or $-90^\circ \leq \arcsin x \leq 90^\circ$



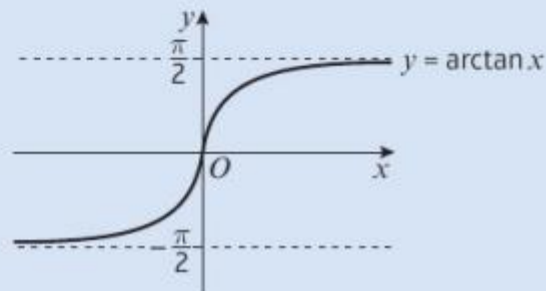
8 The inverse function of $\cos x$ is called **arccos x** .

- The domain of $y = \arccos x$ is $-1 \leq x \leq 1$
- The range of $y = \arccos x$ is $0 \leq \arccos x \leq \pi$ or $0^\circ \leq \arccos x \leq 180^\circ$



9 The inverse function of $\tan x$ is called **arctan x** .

- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$ or $-90^\circ < \arctan x < 90^\circ$



Chapter Exercises

Give any non-exact answers to equations to 1 decimal place.

1 Solve $\tan x = 2 \cot x$, in the interval $-180^\circ \leq x \leq 90^\circ$. (4 marks)

2 Given that $p = 2 \sec \theta$ and $q = 4 \cos \theta$, express p in terms of q . (4 marks)

3 Given that $p = \sin \theta$ and $q = 4 \cot \theta$, show that $p^2 q^2 = 16(1 - p^2)$. (4 marks)

4 a Solve, in the interval $0 < \theta < 180^\circ$,

i $\operatorname{cosec} \theta = 2 \cot \theta$

ii $2 \cot^2 \theta = 7 \operatorname{cosec} \theta - 8$

b Solve, in the interval $0 \leq \theta \leq 360^\circ$,

i $\sec(2\theta - 15^\circ) = \operatorname{cosec} 135^\circ$

ii $\sec^2 \theta + \tan \theta = 3$

c Solve, in the interval $0 \leq x \leq 2\pi$,

i $\operatorname{cosec}\left(x + \frac{\pi}{15}\right) = -\sqrt{2}$

ii $\sec^2 x = \frac{4}{3}$

5 Given that $5 \sin x \cos y + 4 \cos x \sin y = 0$, and that $\cot x = 2$, find the value of $\cot y$. (5 marks)

6 Prove that:

a $(\tan \theta + \cot \theta)(\sin \theta + \cos \theta) \equiv \sec \theta + \operatorname{cosec} \theta$

b $\frac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x} \equiv \sec^2 x$

c $(1 - \sin x)(1 + \operatorname{cosec} x) \equiv \cos x \cot x$

d $\frac{\cot x}{\operatorname{cosec} x - 1} - \frac{\cos x}{1 + \sin x} \equiv 2 \tan x$

e $\frac{1}{\operatorname{cosec} \theta - 1} + \frac{1}{\operatorname{cosec} \theta + 1} \equiv 2 \sec \theta \tan \theta$

f $\frac{(\sec \theta - \tan \theta)(\sec \theta + \tan \theta)}{1 + \tan^2 \theta} \equiv \cos^2 \theta$

Chapter Exercises

7 a Prove that $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} \equiv 2 \operatorname{cosec} x$. (4 marks)

b Hence solve, in the interval $-2\pi \leq x \leq 2\pi$, $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = -\frac{4}{\sqrt{3}}$ (4 marks)

8 Prove that $\frac{1 + \cos \theta}{1 - \cos \theta} \equiv (\operatorname{cosec} \theta + \cot \theta)^2$ (4 marks)

9 Given that $\sec A = -3$, where $\frac{\pi}{2} < A < \pi$,

a calculate the exact value of $\tan A$ (3 marks)

b show that $\operatorname{cosec} A = \frac{3\sqrt{2}}{4}$ (3 marks)

10 Given that $\sec \theta = k$, $|k| \geq 1$, and that θ is obtuse, express in terms of k :

a $\cos \theta$

b $\tan^2 \theta$

c $\cot \theta$

d $\operatorname{cosec} \theta$

11 Solve, in the interval $0 \leq x \leq 2\pi$, the equation $\sec\left(x + \frac{\pi}{4}\right) = 2$, giving your answers in terms of π . (5 marks)

12 Find, in terms of π , the value of $\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$. (4 marks)

Chapter Exercises

- 13 Solve, in the interval $0 \leq x \leq 2\pi$, the equation $\sec^2 x - \frac{2\sqrt{3}}{3} \tan x - 2 = 0$, giving your answers in terms of π . (5 marks)
- 14 a Factorise $\sec x \operatorname{cosec} x - 2 \sec x - \operatorname{cosec} x + 2$. (2 marks)
b Hence solve $\sec x \operatorname{cosec} x - 2 \sec x - \operatorname{cosec} x + 2 = 0$, in the interval $0 \leq x \leq 360^\circ$. (4 marks)
- 15 Given that $\arctan(x - 2) = -\frac{\pi}{3}$, find the value of x . (3 marks)
- 16 On the same set of axes sketch the graphs of $y = \cos x$, $0 \leq x \leq \pi$, and $y = \arccos x$, $-1 \leq x \leq 1$, showing the coordinates of points at which the curves meet the axes. (4 marks)
- 17 a Given that $\sec x + \tan x = -3$, use the identity $1 + \tan^2 x \equiv \sec^2 x$ to find the value of $\sec x - \tan x$. (3 marks)
b Deduce the values of:
i $\sec x$ ii $\tan x$ (3 marks)
c Hence solve, in the interval $-180^\circ \leq x \leq 180^\circ$, $\sec x + \tan x = -3$. (3 marks)
- 18 Given that $p = \sec \theta - \tan \theta$ and $q = \sec \theta + \tan \theta$, show that $p = \frac{1}{q}$ (4 marks)
- 19 a Prove that $\sec^4 \theta - \tan^4 \theta = \sec^2 \theta + \tan^2 \theta$. (3 marks)
b Hence solve, in the interval $-180^\circ \leq \theta \leq 180^\circ$, $\sec^4 \theta = \tan^4 \theta + 3 \tan \theta$. (4 marks)

Chapter Exercises

- 20 a** Sketch the graph of $y = \sin x$ and shade in the area representing $\int_0^{\frac{\pi}{2}} \sin x \, dx$.
- b** Sketch the graph of $y = \arcsin x$ and shade in the area representing $\int_0^1 \arcsin x \, dx$.
- c** By considering the shaded areas explain why $\int_0^{\frac{\pi}{2}} \sin x \, dx + \int_0^1 \arcsin x \, dx = \frac{\pi}{2}$
- 21** Show that $\cot 60^\circ \sec 60^\circ = \frac{2\sqrt{3}}{3}$
- 22 a** Sketch, in the interval $-2\pi \leq x \leq 2\pi$, the graph of $y = 2 - 3 \sec x$. **(3 marks)**
- b** Hence deduce the range of values of k for which the equation $2 - 3 \sec x = k$ has no solutions. **(2 marks)**
- 23 a** Sketch the graph of $y = 3 \arcsin x - \frac{\pi}{2}$, showing clearly the exact coordinates of the end-points of the curve. **(4 marks)**
- b** Find the exact coordinates of the point where the curve crosses the x -axis. **(3 marks)**
- 24 a** Prove that for $0 < x \leq 1$, $\arccos x = \arctan \frac{\sqrt{1-x^2}}{x}$
- b** Prove that for $-1 \leq x < 0$, $\arccos x = k + \arctan \frac{\sqrt{1-x^2}}{x}$, where k is a constant to be found.

Chapter Answers

1 $-125.3^\circ, \pm 54.7^\circ$

2 $p = \frac{8}{q}$

3 $p^2 q^2 = \sin^2 \theta \times 4^2 \cot^2 \theta = 16 \sin^2 \theta \times \frac{\cos^2 \theta}{\sin^2 \theta}$
 $= 16 \cos^2 \theta = 16(1 - \sin^2 \theta) = 16(1 - p^2)$

4 a i 60°

ii $30^\circ, 41.8^\circ, 138.2^\circ, 150^\circ$

b i $30^\circ, 165^\circ, 210^\circ, 345^\circ$

ii $45^\circ, 116.6^\circ, 225^\circ, 296.6^\circ$

c i $\frac{71\pi}{60}, \frac{101\pi}{60}$

ii $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

5 $-\frac{8}{5}$

Chapter Answers

$$\begin{aligned} 6 \quad \mathbf{a} \quad \text{L.H.S.} &\equiv \left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \right) (\sin \theta + \cos \theta) \\ &\equiv \frac{(\sin^2 \theta + \cos^2 \theta)}{\cos \theta \sin \theta} (\sin \theta + \cos \theta) \\ &\equiv \frac{\sin \theta}{\sin \theta \cos \theta} + \frac{\cos \theta}{\cos \theta \sin \theta} \\ &\equiv \sec \theta + \operatorname{cosec} \theta \equiv \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{L.H.S.} &\equiv \frac{\frac{1}{\sin x}}{\frac{1}{\sin x} - \sin x} \\ &\equiv \frac{\frac{1}{\sin x}}{\frac{1 - \sin^2 x}{\sin x}} \equiv \frac{1}{\sin x} \times \frac{\sin x}{\cos^2 x} \equiv \frac{1}{\cos^2 x} \equiv \sec^2 x \equiv \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{L.H.S.} &\equiv 1 - \sin x + \operatorname{cosec} x - 1 \equiv \frac{1}{\sin x} - \sin x \\ &\equiv \frac{1 - \sin^2 x}{\sin x} \equiv \frac{\cos^2 x}{\sin x} \equiv \cos x \frac{\cos x}{\sin x} \equiv \cos x \cot x \\ &\equiv \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \text{L.H.S.} &\equiv \frac{\cot x(1 + \sin x) - \cos x(\operatorname{cosec} x - 1)}{(\operatorname{cosec} x - 1)(1 + \sin x)} \\ &\equiv \frac{\cot x + \cos x - \cot x + \cos x}{\operatorname{cosec} x - 1 + 1 - \sin x} \equiv \frac{2 \cos x}{\operatorname{cosec} x - \sin x} \\ &\equiv \frac{2 \cos x}{\frac{1}{\sin x} - \sin x} \equiv \frac{2 \cos x}{\left(\frac{1 - \sin^2 x}{\sin x} \right)} \equiv \frac{2 \cos x \sin x}{\cos^2 x} \\ &\equiv 2 \tan x \equiv \text{R.H.S.} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \text{L.H.S.} &\equiv \frac{\operatorname{cosec} \theta + 1 + \operatorname{cosec} \theta - 1}{(\operatorname{cosec}^2 \theta - 1)} \equiv \frac{2 \operatorname{cosec} \theta}{\cot^2 \theta} \\ &\equiv \frac{2}{\sin \theta} \cdot \frac{\sin^2 \theta}{\cos^2 \theta} \equiv \frac{2 \sin \theta}{\cos^2 \theta} \equiv \frac{2}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta} \\ &\equiv 2 \sec \theta \tan \theta \equiv \text{R.H.S.} \end{aligned}$$

$$\mathbf{f} \quad \text{L.H.S.} \equiv \frac{\sec^2 \theta - \tan^2 \theta}{\sec^2 \theta} \equiv \frac{1}{\sec^2 \theta} \equiv \cos^2 \theta \equiv \text{R.H.S.}$$

Chapter Answers

$$7 \quad \mathbf{a} \quad \text{L.H.S.} \equiv \frac{\sin^2 x + (1 + \cos x)^2}{(1 + \cos x) \sin x}$$

$$\equiv \frac{\sin^2 x + 1 + 2 \cos x + \cos^2 x}{(1 + \cos x) \sin x} \equiv \frac{2 + 2 \cos x}{(1 + \cos x) \sin x}$$

$$\equiv \frac{2(1 + \cos x)}{(1 + \cos x) \sin x} \equiv \frac{2}{\sin x} \equiv 2 \operatorname{cosec} x$$

$$\mathbf{b} \quad -\frac{\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

$$8 \quad \text{R.H.S.} \equiv \left(\frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} \right)^2 \equiv \frac{(1 + \cos \theta)^2}{\sin^2 \theta} \equiv \frac{(1 + \cos \theta)^2}{1 - \cos^2 \theta}$$

$$\equiv \frac{(1 + \cos \theta)^2}{(1 - \cos \theta)(1 + \cos \theta)} \equiv \frac{1 + \cos \theta}{1 - \cos \theta} \equiv \text{L.H.S.}$$

$$9 \quad \mathbf{a} \quad -2\sqrt{2}$$

$$\mathbf{b} \quad \operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{1}{8} = \frac{9}{8}$$

$$\Rightarrow \operatorname{cosec} A = \pm \frac{3}{2\sqrt{2}} = \pm \frac{3\sqrt{2}}{4}$$

$$\text{As } A \text{ is obtuse, } \operatorname{cosec} A \text{ is +ve, } \Rightarrow \operatorname{cosec} A = \frac{3\sqrt{2}}{4}$$

$$10 \quad \mathbf{a} \quad \frac{1}{k} \quad \mathbf{b} \quad k^2 - 1 \quad \mathbf{c} \quad -\frac{1}{\sqrt{k^2 - 1}} \quad \mathbf{d} \quad -\frac{k}{\sqrt{k^2 - 1}}$$

$$11 \quad \frac{\pi}{12}, \frac{17\pi}{12}$$

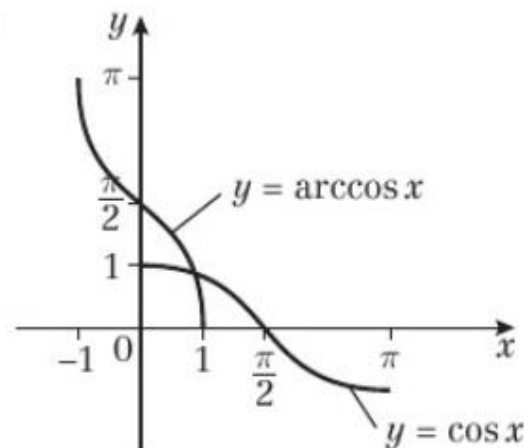
$$12 \quad \frac{\pi}{3}$$

$$13 \quad \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$$

$$14 \quad \mathbf{a} \quad (\sec x - 1)(\operatorname{cosec} x - 2) \quad \mathbf{b} \quad 30^\circ, 150^\circ$$

$$15 \quad 2 - \sqrt{3}$$

16



$$17 \quad \mathbf{a} \quad -\frac{1}{3} \quad \mathbf{b} \quad \mathbf{i} \quad -\frac{5}{3}, \quad \mathbf{ii} \quad -\frac{4}{3} \quad \mathbf{c} \quad 126.9^\circ$$

$$18 \quad pq = (\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = \sec^2 \theta - \tan^2 \theta$$

$$= 1 \Rightarrow p = \frac{1}{q}$$

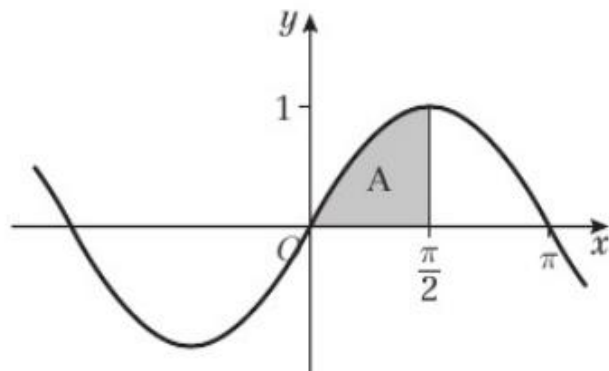
$$19 \quad \mathbf{a} \quad \text{L.H.S.} = (\sec^2 \theta - \tan^2 \theta)(\sec^2 \theta + \tan^2 \theta)$$

$$= 1 \times (\sec^2 \theta + \tan^2 \theta) = \sec^2 \theta + \tan^2 \theta = \text{R.H.S.}$$

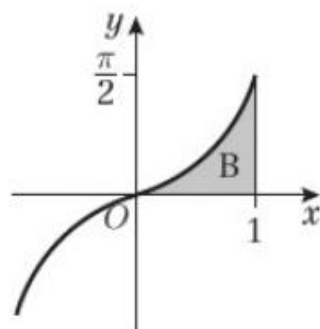
$$\mathbf{b} \quad -153.4^\circ, -135^\circ, 26.6^\circ, 45^\circ$$

Chapter Answers

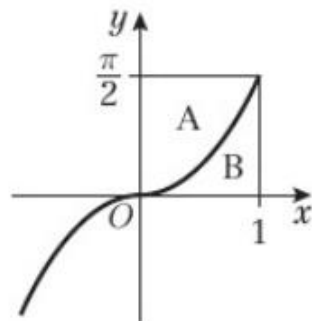
20 a



b



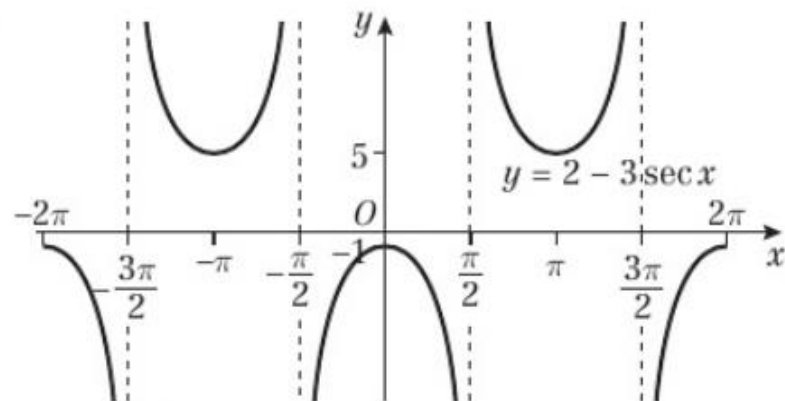
c The regions A and B fit together to make a rectangle.



$$\text{Area} = 1 \times \frac{\pi}{2} = \frac{\pi}{2}$$

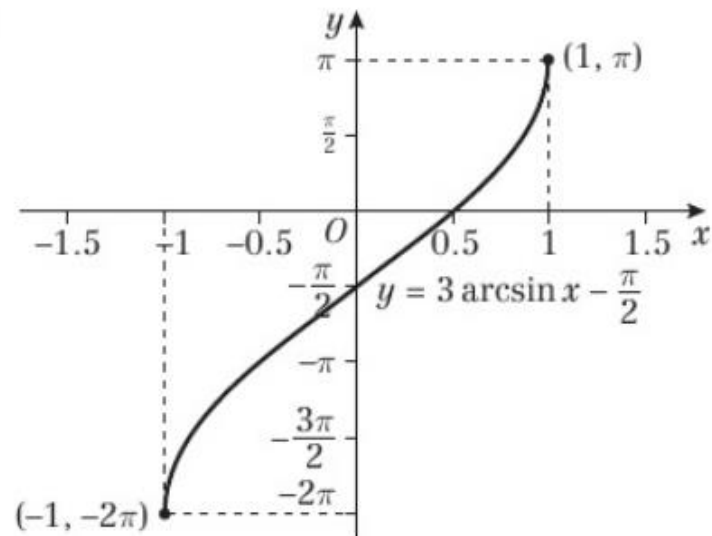
$$21 \quad \cot 60^\circ \sec 60^\circ = \frac{1}{\tan 60^\circ} \times \frac{1}{\cos 60^\circ} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

22 a



b $-1 < k < 5$

23 a



b $(\frac{1}{2}, 0)$

Chapter Answers

24 a Let $y = \arccos x$. So $\cos y = x$, $\sin y = \sqrt{1 - x^2}$.

Thus $\tan y = \frac{\sqrt{1 - x^2}}{x}$, which is valid for $x \in (0, 1]$.

Therefore $\arccos x = \arctan \frac{\sqrt{1 - x^2}}{x}$ for $0 < x \leq 1$.

b Letting $y = \arccos x$, $x \in [-1, 0) \Rightarrow y \in \left(\frac{\pi}{2}, \pi\right]$

$$\tan y = \frac{\sin y}{\cos y} = \frac{\sqrt{1 - x^2}}{x}$$

$\arctan \frac{\sqrt{1 - x^2}}{x}$ gives values in the range $\left(-\frac{\pi}{2}, 0\right]$,

so for $y \in \left(\frac{\pi}{2}, \pi\right]$ you need to add π :

$$y = \pi + \arctan \frac{\sqrt{1 - x^2}}{x}$$

Therefore $\arccos x = \pi + \arctan \frac{\sqrt{1 - x^2}}{x}$