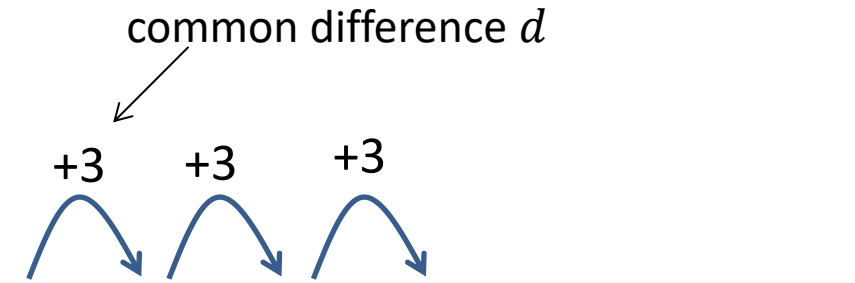
P2 Chapter 3: Sequences and Series

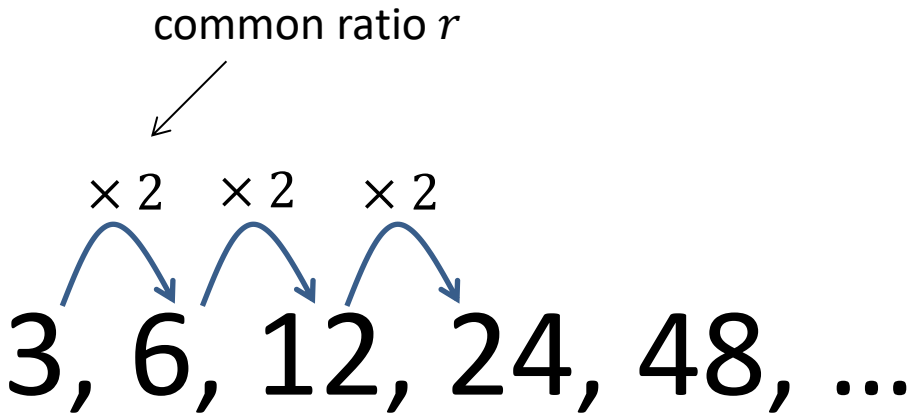
Geometric Sequences

Recap of Arithmetic vs Geometric Sequences



This is an:

Arithmetic Sequence



Geometric Sequence

A geometric sequence is one in which there is a **common ratio** between terms.

Quickfire Common Ratio

Identify the common ratio r:

1. $1, 2, 4, 8, 16, 32, \ldots$ $r =$?
2. $27, 18, 12, 8, \ldots$ $r =$?
3. $10, 5, 2.5, 1.25, \ldots$ $r =$?
4. $5, -5, 5, -5, 5, -5, \ldots$ $r =$?
5. $x, -2x^2, 4x^3$ $r =$?
6. $1, p, p^2, p^3, \ldots$ $r =$?
7. $4, -1, 0.25, -0.0625, \ldots$ $r =$?

An **alternating sequence** is one which oscillates between positive and negative.

Quickfire Common Ratio

Identify the common ratio r:

1. $1, 2, 4, 8, 16, 32, \ldots$ $\quad r = 2$
2. $27, 18, 12, 8, \ldots$ $\quad r = 2/3$
3. $10, 5, 2.5, 1.25, \ldots$ $\quad r = 1/2$
4. $5, -5, 5, -5, 5, -5, \ldots$ $\quad r = -1$
5. $x, -2x^2, 4x^3$ $\quad r = -2x$
6. $1, p, p^2, p^3, \ldots$ $\quad r = p$
7. $4, -1, 0.25, -0.0625, \ldots$ $\quad r = -0.25$

An **alternating sequence** is one which oscillates between positive and negative.

n^{th} term

Arithmetic Sequence

$u_n =$?

Geometric Sequence

$u_n =$?

Determine the 10th and n^{th} terms of the following:

3, 6, 12, 24, …

?

40, -20, 10, -5, …

?

Fro Tip: As before, write out $a =$ and $r =$ first before substituting.

n^{th} term

Arithmetic Sequence	Geometric Sequence
$u_n = a + (n-1)d$	$u_n = ar^{n-1}$

Determine the 10th and n^{th} terms of the following:

3, 6, 12, 24, …

$a = 3, r = 2$

$u_{10} = 3 \times 2^9 = 1536$

$u_n = 3(2^{n-1})$

40, -20, 10, -5, …

$a = 40, r = -\frac{1}{2}$

$u_{10} = 40 \times \left(-\frac{1}{2}\right)^9 = -\frac{5}{64}$

$u_n = (-1)^{n-1} \times \frac{5}{2^{n-4}}$

Fro Tip: As before, write out $a =$ and $r =$ first before substituting.

Further Example

[Textbook] The second term of a geometric sequence is 4 and the 4th term is 8. The common ratio is positive. Find the exact values of:

a) The common ratio.
b) The first term.
c) The 10th term.

?

Fro Tip: Explicitly writing $u_2 = 4$ first helps you avoid confusing the n^{th} term with the 'sum of the first n terms' (the latter of which we'll get onto).

Further Example

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$$u_2 = 4 \quad \rightarrow \quad ar = 4 \ (1)$$
$$u_4 = 8 \quad \rightarrow \quad ar^3 = 8 \ (2)$$

a) Dividing (2) by (1) gives $r^2 = 2, so\ r = \sqrt{2}$

b) Substituting, $a = \frac{4}{r} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

c) $u_{10} = ar^9 = 64$

Fro Tip: Explicitly writing $u_2 = 4$ first helps you avoid confusing the n^{th} term with the 'sum of the first n terms' (the latter of which we'll get onto).

Further Example

[Textbook] The numbers $3, x$ and $x + 6$ form the first three terms of a positive geometric sequence. Find:

a) The value of x.
b) The 10^{th} term in the sequence.

Hint: You're told it's a geometric sequence, which means the ratio between successive terms must be the same. Consequently $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

?

Exam Note: This kind of question has appeared in the exam multiple times.

Further Example

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$$\frac{x}{3} = \frac{x+6}{x} \quad \rightarrow \quad x^2 = 3x + 18$$
$$x^2 - 3x - 18 = 0$$
$$(x+3)(-6) = 0$$
$$x = 6 \; or \; -3$$

But there are no negative terms so $x = 6$

$$r = \frac{x}{3} = \frac{6}{3} = 2 \qquad a = 3 \qquad n = 10$$
$$u_{10} = 3 \times 2^9 = 1536$$

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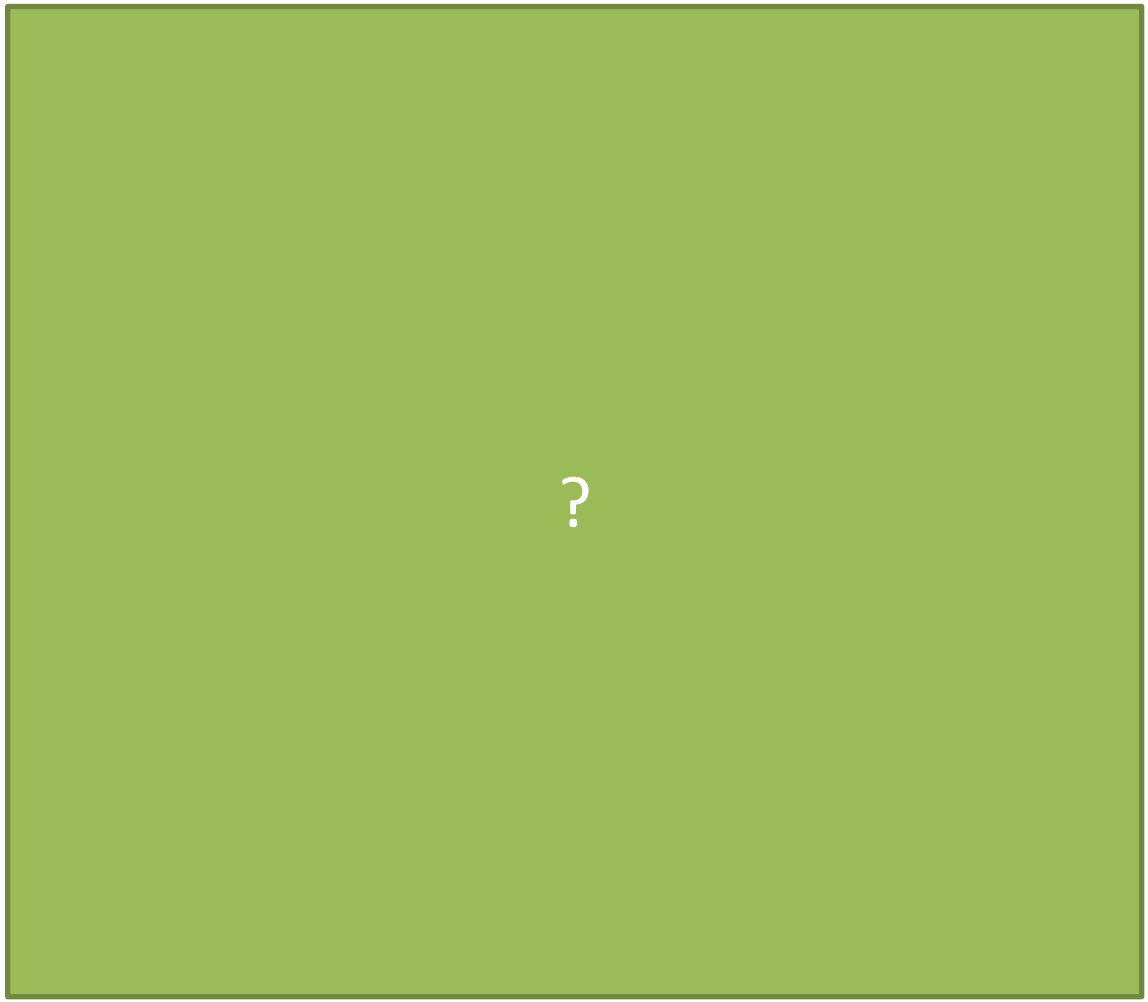
n^{th} term with inequalities

[Textbook] What is the first term in the geometric progression $3, 6, 12, 24, \ldots$ to exceed 1 million?



n^{th} term with inequalities

[Textbook] What is the first term in the geometric progression $3, 6, 12, 24, \ldots$ to exceed 1 million?

$$u_n > 1000000 \qquad a = 3, r = 2$$
$$\therefore\ 3 \times 2^{n-1} > 1000000$$
$$2^{n-1} > \frac{1000000}{3}$$
$$\log 2^{n-1} > \log \frac{1000000}{3}$$
$$n - 1 > \frac{\log\left(\frac{1000000}{3}\right)}{log2}$$
$$n - 1 > 18.35$$
$$n > 19.35$$
$$n = 20$$

Test Your Understanding

All the terms in a geometric sequence are positive.
The third term of the sequence is 20 and the fifth term 80. What is the 20^{th} term?

?

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x+6, \qquad 5x-6$$

a) Determine the possible values of x.
b) Given the common ratio is positive, find the common ratio.
c) Hence determine the possible values for the first term of the sequence.

?

Test Your Understanding

All the terms in a geometric sequence are positive.
The third term of the sequence is 20 and the fifth term 80. What is the 20th term?

$$u_3 = 20 \quad \rightarrow \quad ar^2 = 20$$
$$u_5 = 80 \quad \rightarrow \quad ar^4 = 80$$

Dividing: $r^2 = 4, \ \therefore r = 2$

$$a = \frac{20}{r^2} = \frac{20}{4} = 5$$
$$u^{20} = 5 \times 2^{19} = 2621440$$

The second, third and fourth term of a geometric sequence are the following:

$$x, \qquad x + 6, \qquad 5x - 6$$

a) Determine the possible values of x.
b) Given the common ratio is positive, find the common ratio.
c) Hence determine the possible values for the first term of the sequence.

$$\frac{x+6}{x} = \frac{5x-6}{x+6}$$
$$(x+6)^2 = x(5x-6)$$
$$x^2 + 12x + 36 = 5x^2 - 6x$$
$$4x^2 - 18x - 36 = 0$$
$$2x^2 - 9x - 18 = 0$$
$$(2x+3)(x-6)$$

$$x = -\frac{3}{2} \ or \ x = 6$$
$$r = \frac{x+6}{x} = \frac{6+6}{6} = 2$$
$$u_2 = x = 6$$
$$ar = 6$$
$$a = \frac{6}{2} = 3$$

Exercise 3.3

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Homework Exercise

1 Which of the following are geometric sequences? For the ones that are, give the value of the common ratio, r.

a 1, 2, 4, 8, 16, 32, …

b 2, 5, 8, 11, 14, …

c 40, 36, 32, 28, …

d 2, 6, 18, 54, 162, …

e 10, 5, 2.5, 1.25, …

f 5, −5, 5, −5, 5, …

g 3, 3, 3, 3, 3, 3, 3, …

h 4, −1, 0.25, −0.0625, …

2 Continue the following geometric sequences for three more terms.

a 5, 15, 45, …

b 4, −8, 16, …

c 60, 30, 15, …

d $1, \frac{1}{4}, \frac{1}{16}, \ldots$

e $1, p, p^2, \ldots$

f $x, -2x^2, 4x^3, \ldots$

3 If 3, x and 9 are the first three terms of a geometric sequence, find:

a the exact value of x,

b the exact value of the 4th term.

Problem-solving

In a geometric sequence the common ratio can be calculated by $\frac{u_2}{u_1}$ or $\frac{u_3}{u_2}$

4 Find the sixth and nth terms of the following geometric sequences.

a 2, 6, 18, 54, …

b 100, 50, 25, 12.5, …

c 1, −2, 4, −8, …

d 1, 1.1, 1.21, 1.331, …

5 The nth term of a geometric sequence is 2×5^n. Find the first and 5th terms.

Homework Exercise

6 The sixth term of a geometric sequence is 32 and the 3rd term is 4. Find the first term and the common ratio.

7 A geometric sequence has first term 4 and third term 1. Find the two possible values of the 6th term.

8 The first three terms of a geometric sequence are given by $8 - x$, $2x$, and x^2 respectively where $x > 0$.

a Show that $x^3 - 4x^2 = 0$. **(2 marks)**

b Find the value of the 20th term. **(3 marks)**

c State, with a reason, whether 4096 is a term in the sequence. **(1 mark)**

9 A geometric sequence has first term 200 and a common ratio p where $p > 0$.
The 6th term of the sequence is 40.

a Show that p satisfies the equation $5 \log p + \log 5 = 0$. **(3 marks)**

b Hence or otherwise, find the value of p correct to 3 significant figures. **(1 mark)**

Homework Exercise

10 A geometric sequence has first term 4 and fourth term 108. Find the smallest value of k for which the kth term in this sequence exceeds 500 000.

11 The first three terms of a geometric sequence are 9, 36, 144. State, with a reason, whether 383 616 is a term in the sequence.

Problem-solving

Determine the values of a and r and find the general term of the sequence. Set the number given equal to the general term and solve to find n. If n is an integer, then the number is in the sequence.

12 The first three terms of a geometric sequence are 3, −12, 48. State, with a reason, whether 49 152 is a term in the sequence.

13 Find which term in the geometric progression 3, 12, 48, …is the first to exceed 1 000 000.

Homework Answers

1 **a** Geometric, $r = 2$ **b** Not geometric
c Not geometric **d** Geometric, $r = 3$
e Geometric, $r = \frac{1}{2}$ **f** Geometric, $r = -1$
g Geometric, $r = 1$ **h** Geometric, $r = -\frac{1}{4}$

2 **a** 135, 405, 1215 **b** $-32, 64, -128$
c 7.5, 3.75, 1.875 **d** $\frac{1}{64}, \frac{1}{256}, \frac{1}{1024}$
e p^3, p^4, p^5 **f** $-8x^4, 16x^5, -32x^6$

3 **a** $x = 3\sqrt{3}$ **b** $9\sqrt{3}$

4 **a** $486, 2 \times 3^{n-1}$ **b** $\frac{25}{8}, 100 \times \left(\frac{1}{2}\right)^{n-1}$
c $-32, (-2)^{n-1}$ **d** $1.61051, (1.1)^{n-1}$

5 10, 6250 **6** $a = 1, r = 2$ **7** $\frac{1}{8}, -\frac{1}{8}$

8 **a** $\frac{x^2}{2x} = \frac{2x}{8-x} \Rightarrow x^2(8-x) = 4x^2 \Rightarrow x^3 - 4x^2 = 0$
b 2 097 152
c Yes, 4096 is in sequence as n is integer, $n = 11$

9 **a** $ar^5 = 40 \Rightarrow 200p^5 = 40$
$\Rightarrow p^5 = \frac{1}{5} \Rightarrow \log p^5 = \log\left(\frac{1}{5}\right)$
$\Rightarrow 5\log p = \log 1 - \log 5 \Rightarrow 5\log p + \log 5 = 0$
b $p = 0.725$

10 $k = 12$
11 $n = 8.69$, so not a sequence as n not an integer
12 No, -49152 is in sequence
13 $n = 11$, 3 145 728