# P1 Chapter 12: Differentiation

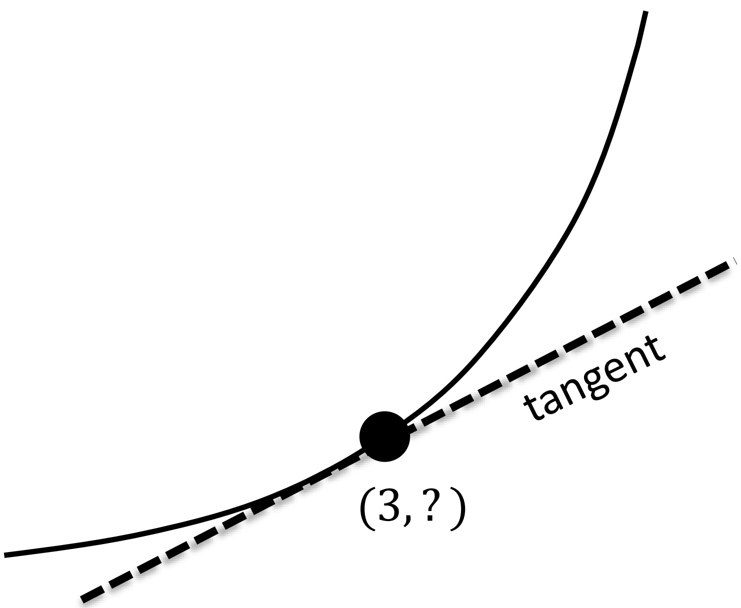
## Tangents and Normals

# Finding equations of tangents

Find the equation of the **tangent** to the curve $y = x^2$ when $x = 3$.

We want to use $y - y_1 = m(x - x_1)$ for the tangent (as it is a straight line!).
Therefore we need:

(a) A point $(x_1, y_1)$
(b) The gradient $m$.




Gradient function: $\boldsymbol{\frac{dy}{dx} = 2x}$

Gradient when $x = 3$: $\boldsymbol{m = 6}$

$y$-value when $x = 3$: $\boldsymbol{y = 9}$

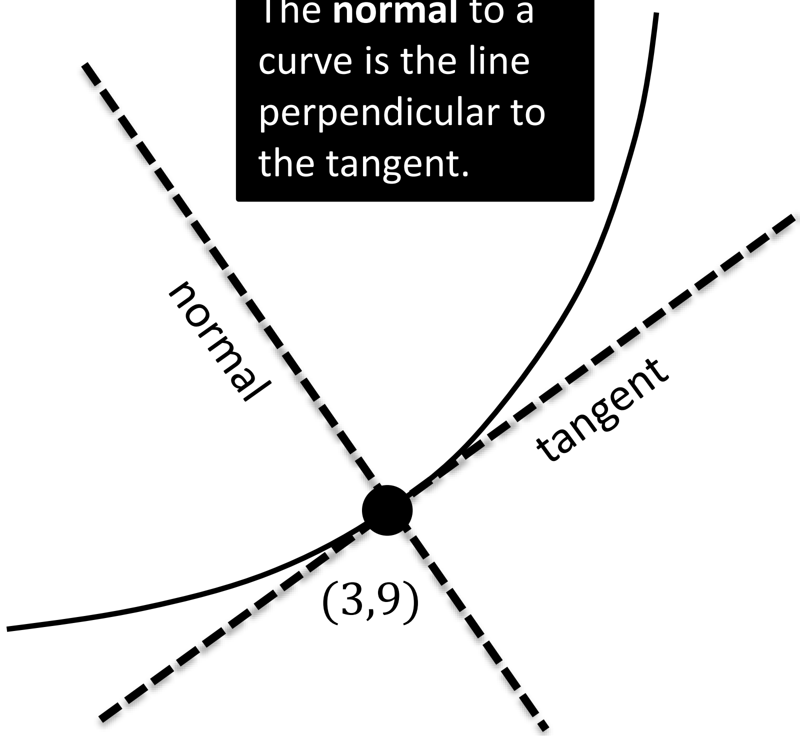
So, equation of tangent:

$$\boldsymbol{y - 9 = 6(x - 3)}$$

# Finding equations of normals

Find the equation of the **normal** to the curve $y = x^2$ when $x = 3$.

The **normal** to a curve is the line perpendicular to the tangent.




Equation of tangent (from earlier):

$$\boldsymbol{y - 9 = 6(x - 3)}$$

Therefore equation of normal:

$$\boldsymbol{y - 9 = -\frac{1}{6}(x - 3)}$$

**Fro Exam Tip**: A **very common error** is for students to accidentally forget whether the question is asking for the tangent or for the normal.

# Test Your Understanding

Find the equation of the **normal** to the curve $y = x + 3\sqrt{x}$ when $x = 9$.

When $x = 9$, $y =$ ? y

? gradient

**Fro Tip:** I like to use $m_T$ and $m_N$ to make clear to the examiner (and myself) what gradient I've found.

? Final equation

# Test Your Understanding

Find the equation of the **normal** to the curve $y = x + 3\sqrt{x}$ when $x = 9$.

When $x = 9$, $y = 9 + 3\sqrt{9} = 18$

$$y = x + 3x^{\frac{1}{2}}$$
$$\therefore \frac{dy}{dx} = 1 + \frac{3}{2}x^{-\frac{1}{2}}$$
$$m_T = 1 + \frac{3}{2}\left(9^{-\frac{1}{2}}\right) = \frac{3}{2}$$
$$\therefore m_N = -\frac{2}{3}$$

**Fro Tip:** I like to use $m_T$ and $m_N$ to make clear to the examiner (and myself) what gradient I've found.

Equation of normal: $y - 18 = -\frac{2}{3}(x - 9)$

# Exercise 12.6

## Pearson Pure Mathematics Year 1/AS
## Page 96

The 'difference of two cubes':
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
will help for both of these.

**Extension**

1 *[STEP I 2005 Q2]*
The point $P$ has coordinates $(p^2, 2p)$ and the point $Q$ has coordinates $(q^2, 2q)$, where $p$ and $q$ are non-zero and $p \neq q$. The curve $C$ is given by $y^2 = 4x$. The point $R$ is the intersection of the tangent to $C$ at $P$ and the tangent to $C$ at $Q$. Show that $R$ has coordinates $(pq, p+q)$.

The point $S$ is the intersection of the normal to $C$ at $P$ and the normal to $C$ at $Q$. If $p$ and $q$ are such that $(1,0)$ lies on the line $PQ$, show that $S$ has coordinates $(p^2 + q^2 + 1, p+q)$, and that the quadrilateral $PSQR$ is a rectangle.

**Solutions on next slide.**

2 *[STEP I 2012 Q4]*
The curve $C$ has equation $xy = \frac{1}{2}$.
The tangents to $C$ at the distinct points $P\left(p, \frac{1}{2p}\right)$ and $Q\left(q, \frac{1}{2q}\right)$, where $p$ and $q$ are positive, intersect at $T$ and the normal to $C$ at these points intersect at $N$. Show that $T$ is the point

$$\left(\frac{2pq}{p+q}, \frac{1}{p+q}\right)$$

In the case $pq = \frac{1}{2}$, find the coordinates of $N$. Show (in this case) that $T$ and $N$ lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

# Solutions to Extension Question 1

$y^2 = 4x \Rightarrow 2yy' = 4 \Rightarrow y' = \dfrac{2}{y}$

This is using something called 'implicit differentiation' (Year 2), but you could easily do:

$$y^2 = 4x \;\rightarrow y = 2x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} = \frac{2}{y}$$

Equation of tangent at $P$: $y - 2p = \dfrac{1}{p}\left(x - p^2\right) \Rightarrow py = x + p^2$.

Equation of tangent at $Q$: $qy = x + q^2$

Intersect where $qy - q^2 = py - p^2$

$\Rightarrow (q-p)\,y = q^2 - p^2$

$\Rightarrow y = p + q \Rightarrow x = pq$ by substitution. Hence $R\,(pq\,,\ p+q)$.

Equation of normal at $P$: $y - 2p = -p\left(x - p^2\right) \Rightarrow y + px = 2p + p^3$.

Equation of normal at $Q$: $y + qx = 2q + q^3$

Intersect where $2p + p^3 - px = 2q + q^3 - qx$

$\Rightarrow x\,(p-q) = 2\,(p-q) + (p-q)\left(p^2 + pq + q^2\right)$ using the identity $p^3 - q^3 \equiv (p-q)\left(p^2 + pq + q^2\right)$

$\Rightarrow x = p^2 + pq + q^2 + 2 \Rightarrow y = -pq\,(p+q)$ by substitution.

But $(1\,,\ 0)$ lies on $PQ$ so the gradient of the line segement from $P$ to $(1\,,\ 0)$ equals the gradient of the line segment from $Q$ to $(1\,,\ 0)$.

$\Rightarrow \dfrac{2p}{p^2-1} = \dfrac{2q}{q^2-1}$

$\Rightarrow 2pq^2 - 2p = 2qp^2 - 2q \Rightarrow pq\,(q-p) = p - q \Rightarrow pq = -1$

$\Rightarrow S$, where the normals intersect, has coordinates $\left(p^2 + q^2 + 1\,,\ p+q\right)$.

Obviously, $SP$ is perpendicular to $PR$ and $QS$ is perpendicular to $QR$, because each of these is a tangent-normal pair.

Furthermore, the gradient of $PR$ $\times$ the gradient of $QR$ $= \frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = -1$, so $PR$ is perpendicular to $QR$.

Also, the gradient of $PS$ $\times$ the gradient of $QS = -p \times -q = pq = -1$, so $PS$ is perpendicular to $QS$.

Therefore all four angles are right angles, proving that PSQR is a rectangle. It is not sufficient to consider only the lengths of the sides, since the quadrilateral could be a parallelogram.

# Homework Exercise

1 Find the equation of the tangent to the curve:

a $y = x^2 - 7x + 10$ at the point $(2, 0)$

b $y = x + \frac{1}{x}$ at the point $\left(2, 2\frac{1}{2}\right)$

c $y = 4\sqrt{x}$ at the point $(9, 12)$

d $y = \frac{2x - 1}{x}$ at the point $(1, 1)$

e $y = 2x^3 + 6x + 10$ at the point $(-1, 2)$

f $y = x^2 - \frac{7}{x^2}$ at the point $(1, -6)$

2 Find the equation of the normal to the curve:

a $y = x^2 - 5x$ at the point $(6, 6)$

b $y = x^2 - \frac{8}{\sqrt{x}}$ at the point $(4, 12)$

3 Find the coordinates of the point where the tangent to the curve $y = x^2 + 1$ at the point $(2, 5)$ meets the normal to the same curve at the point $(1, 2)$.

4 Find the equations of the normals to the curve $y = x + x^3$ at the points $(0, 0)$ and $(1, 2)$, and find the coordinates of the point where these normals meet.

5 For $f(x) = 12 - 4x + 2x^2$, find the equations of the tangent and the normal at the point where $x = -1$ on the curve with equation $y = f(x)$.

6 The point $P$ with $x$-coordinate $\frac{1}{2}$ lies on the curve with equation $y = 2x^2$.
The normal to the curve at $P$ intersects the curve at points $P$ and $Q$.
Find the coordinates of $Q$. **(6 marks)**

**Problem-solving**

Draw a sketch showing the curve, the point $P$ and the normal. This will help you check that your answer makes sense.

**Challenge**

The line $L$ is a tangent to the curve with equation $y = 4x^2 + 1$. $L$ cuts the $y$-axis at $(0, -8)$ and has a positive gradient. Find the equation of $L$ in the form $y = mx + c$.

**Hint** Use the discriminant to find the value of $m$ when the line just touches the curve.

# Homework Answers

**1** **a** $y + 3x - 6 = 0$ **b** $4y - 3x - 4 = 0$
**c** $3y - 2x - 18 = 0$ **d** $y = x$
**e** $y = 12x + 14$ **f** $y = 16x - 22$

**2** **a** $7y + x - 48 = 0$ **b** $17y + 2x - 212 = 0$

**3** $(1\frac{2}{9}, 1\frac{8}{9})$

**4** $y = -x$, $4y + x - 9 = 0$; $(-3, 3)$

**5** $y = -8x + 10$, $8y - x - 145 = 0$

**6** $\left(-\frac{3}{4}, \frac{9}{8}\right)$

**Challenge**
$L$ has equation $y = 12x - 8$.