P1 Chapter 14: Logarithms

Chapter Practice

Key Points

1 For all real values of x:

- If $f(x) = e^x$ then $f'(x) = e^x$
- If $y = e^x$ then $\frac{dy}{dx} = e^x$

2 For all real values of x and for any constant k:

- If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$
- If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

3 $\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$

4 **The laws of logarithms:**

- $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$ (the division law)
- $\log_a (x^k) = k \log_a x$ (the power law)

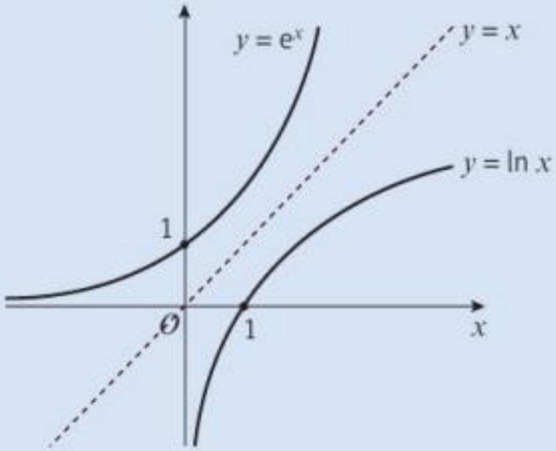
5 You should also learn to recognise the following special cases:

- $\log_a \left(\frac{1}{x}\right) = \log_a (x^{-1}) = -\log_a x$ (the power law when $k = -1$)
- $\log_a a = 1$ $(a > 0, a \neq 1)$
- $\log_a 1 = 0$ $(a > 0, a \neq 1)$

6 Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$

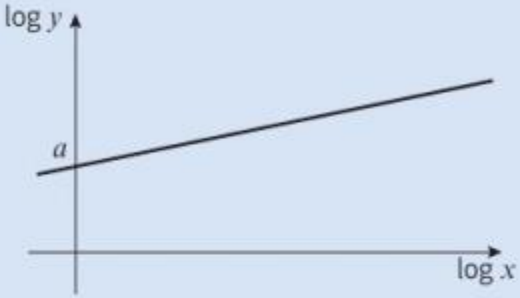
Key Points

7 The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.



8 $e^{\ln x} = \ln(e^x) = x$

9 If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.



10 If $y = ab^x$ then the graph of $\log y$ against x will be a straight line with gradient $\log b$ and vertical intercept $\log a$.

Chapter Exercises

1 Sketch each of the following graphs, labelling all intersections and asymptotes.

a $y = 2^{-x}$ **b** $y = 5e^x - 1$ **c** $y = \ln x$

Hint Recall that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

2 **a** Express $\log_a(p^2q)$ in terms of $\log_a p$ and $\log_a q$.

b Given that $\log_a(pq) = 5$ and $\log_a(p^2q) = 9$, find the values of $\log_a p$ and $\log_a q$.

3 Given that $p = \log_q 16$, express in terms of p,

a $\log_q 2$

b $\log_q(8q)$

4 Solve these equations, giving your answers to 3 significant figures.

a $4^x = 23$ **b** $7^{2x+1} = 1000$ **c** $10^x = 6^{x+2}$

5 **a** Using the substitution $u = 2^x$, show that the equation $4^x - 2^{x+1} - 15 = 0$ can be written in the form $u^2 - 2u - 15 = 0$. **(2 marks)**

b Hence solve the equation $4^x - 2^{x+1} - 15 = 0$, giving your answer to 2 decimal places. **(3 marks)**

6 Solve the equation $\log_2(x + 10) - \log_2(x - 5) = 4$. **(4 marks)**

7 Differentiate each of the following expressions with respect to x.

a e^{-x} **b** e^{11x} **c** $6e^{5x}$

8 Solve the following equations, giving exact solutions.

a $\ln(2x - 5) = 8$ **b** $e^{4x} = 5$ **c** $24 - e^{-2x} = 10$

d $\ln x + \ln(x - 3) = 0$ **e** $e^x + e^{-x} = 2$ **f** $\ln 2 + \ln x = 4$

9 The price of a computer system can be modelled by the formula

$$P = 100 + 850\,e^{-\frac{t}{2}}$$

where P is the price of the system in £s and t is the age of the computer in years after being purchased.

a Calculate the new price of the system.

b Calculate its price after 3 years.

c When will it be worth less than £200?

d Find its price as $t \to \infty$.

e Sketch the graph showing P against t.

f Comment on the appropriateness of this model.

10 The points P and Q lie on the curve with equation $y = e^{\frac{1}{2}x}$.
The x-coordinates of P and Q are $\ln 4$ and $\ln 16$ respectively.

a Find an equation for the line PQ.

b Show that this line passes through the origin O.

c Calculate the length, to 3 significant figures, of the line segment PQ.

Chapter Exercises

11 The temperature, T °C, of a cup of tea is given by $T = 55e^{-\frac{t}{8}} + 20 \qquad t \geqslant 0$
where t is the time in minutes since measurements began.

a Briefly explain why $t \geqslant 0$. **(1 mark)**

b State the starting temperature of the cup of tea. **(1 mark)**

c Find the time at which the temperature of the tea is 50 °C, giving your answer to the nearest minute. **(3 marks)**

d By sketching a graph or otherwise, explain why the temperature of the tea will never fall below 20 °C. **(2 marks)**

12 The table below gives the surface area, S, and the volume, V of five different spheres, rounded to 1 decimal place.

S	18.1	50.3	113.1	221.7	314.2
V	7.2	33.5	113.1	310.3	523.6

Given that $S = aV^b$, where a and b are constants,

a show that $\log S = \log a + b \log V$. **(2 marks)**

b copy and complete the table of values of $\log S$ and $\log V$, giving your answers to 2 decimal places. **(1 mark)**

log S					
log V	0.86				

c plot a graph of $\log V$ against $\log S$ and draw in a line of best fit. **(2 marks)**

d use your graph to confirm that $b = 1.5$ and estimate the value of a to one significant figure. **(4 marks)**

13 The radioactive decay of a substance is modelled by the formula $R = 140e^{kt}$ $\quad t \geqslant 0$
where R is a measure of radioactivity (in counts per minute) at time t days, and k is a constant.

a Explain briefly why k must be negative. **(1 mark)**

b Sketch the graph of R against t. **(2 marks)**

After 30 days the radiation is measured at 70 counts per minute.

c Show that $k = c \ln 2$, stating the value of the constant c. **(3 marks)**

14 The total number of views (in millions) V of a viral video in x days is modelled by

$$V = e^{0.4x} - 1$$

a Find the total number of views after 5 days.

b Find $\frac{dV}{dx}$.

c Find the rate of increase of the number of views after 100 days, stating the units of your answer.

d Use your answer to part **c** to comment on the validity of the model after 100 days.

15 The moment magnitude scale is used by seismologists to express the sizes of earthquakes.
The scale is calculated using the formula

$$M = \tfrac{2}{3}\log_{10}(S) - 10.7$$

where S is the seismic moment in dyne cm.

a Find the magnitude of an earthquake with a seismic moment of 2.24×10^{22} dyne cm.

b Find the seismic moment of an earthquake with

i magnitude 6 **ii** magnitude 7

c Using your answers to part **b** or otherwise, show that an earthquake of magnitude 7 is approximately 32 times as powerful as an earthquake of magnitude 6.

16 A student is asked to solve the equation

$$\log_2 x - \tfrac{1}{2}\log_2(x + 1) = 1$$

The student's attempt is shown

$$\begin{aligned}
\log_2 x - \log_2\sqrt{x+1} &= 1 \\
x - \sqrt{x+1} &= 2^1 \\
x - 2 &= \sqrt{x+1} \\
(x-2)^2 &= x + 1 \\
x^2 - 5x + 3 &= 0 \\
x = \frac{5+\sqrt{13}}{2} \quad & x = \frac{5-\sqrt{13}}{2}
\end{aligned}$$

a Identify the error made by the student. **(1 mark)**

b Solve the equation correctly. **(3 marks)**

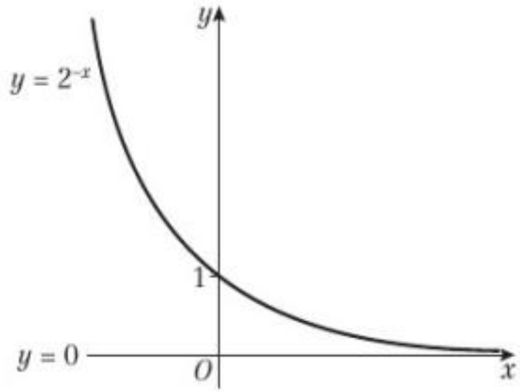
Challenge

a Given that $y = 9^x$, show that $\log_3 y = 2x$.

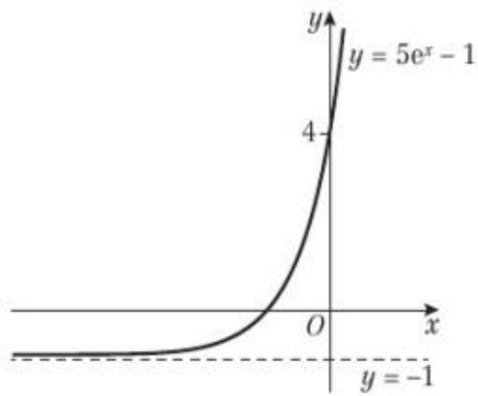
b Hence deduce that $\log_3 y = \log_9 y^2$.

c Use your answer to part **b** to solve the equation $\log_3(2 - 3x) = \log_9(6x^2 - 19x + 2)$

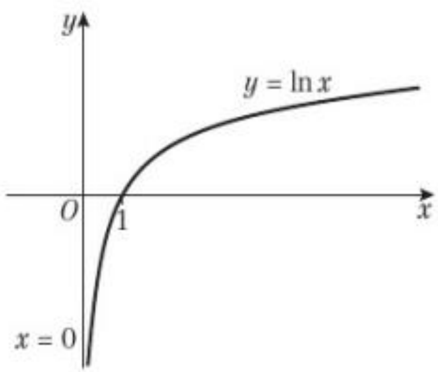
1 **a**



b



c



2 **a** $2\log_a p + \log_a q$ **b** $\log_a p = 4$, $\log_a q = 1$

3 **a** $\frac{1}{4}p$ **b** $\frac{3}{4}p + 1$

4 **a** 2.26 **b** 1.27 **c** 7.02

5 **a** $4^x - 2^{x+1} - 15 = 0$

$2^{2x} - 2 \times 2^x - 15 = 0$

$(2^x)^2 - 2 \times 2^x - 15 = 0$

$u^2 - 2u - 15 = 0$

b 2.32

6 $x = 6$

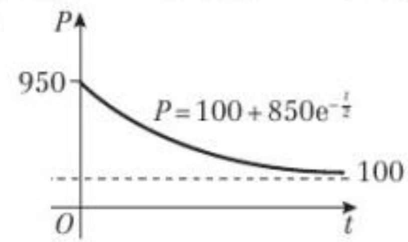
7 **a** $-e^{-x}$ **b** $11e^{11x}$ **c** $30e^{5x}$

8 **a** $\dfrac{e^8 + 5}{2}$ **b** $\dfrac{\ln 5}{4}$ **c** $-\frac{1}{2}\ln 14$ **d** $\dfrac{3 + \sqrt{13}}{2}$

e 0 **f** $\dfrac{e^4}{2}$

9 **a** £950 **b** £290 **c** 4.28 years **d** £100

e



f A good model. The computer will always be worth something

10 **a** $y = \left(\dfrac{2}{\ln 4}\right)x$

b (0, 0) satisfies the equation of the line.

c 2.43

11 **a** We cannot go backwards in time

b 75°C

c 5 minutes

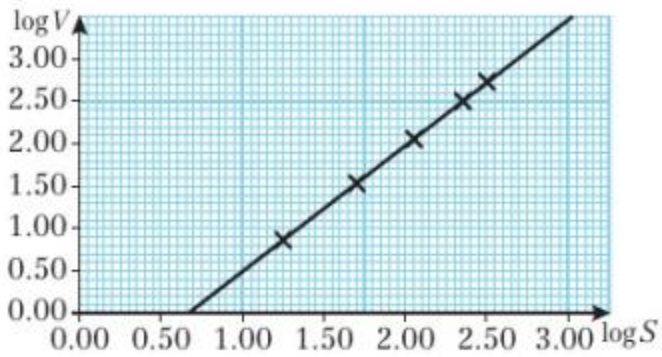
d The exponential term will always be positive, so the overall temperature will be greater than 20°C.

12 a $S = aV^b$
$\log S = \log(aV^b)$
$\log S = \log a + \log(V^b)$
$\log S = \log a + b\log V$

b

$\log S$	1.26	1.70	2.05	2.35	2.50
$\log V$	0.86	1.53	2.05	2.49	2.72

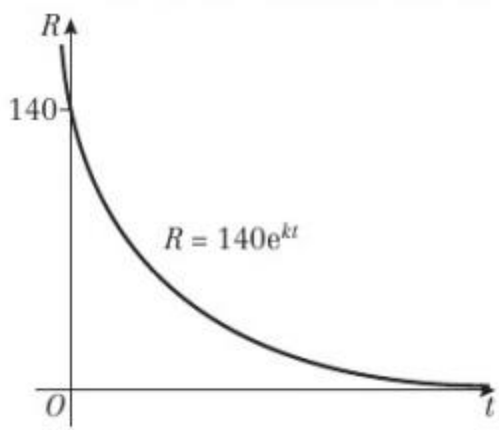
c



d The gradient is approximately 1.5; $a = 0.09$

13 a The model concerns decay, not growth

b



13 c $70 = 140e^{30k}$
$\frac{1}{2} = e^{30k}$
$\ln\left(\frac{1}{2}\right) = 30k$
$k = \frac{1}{30}\ln\left(\frac{1}{2}\right)$
$k = -\frac{1}{30}\ln(2)$, so $c = -\frac{1}{30}$

14 a 6.3 million views

b $\frac{dV}{dx} = 0.4e^{0.4x}$

c 9.42×10^{16} new views per day

d This is too big, so the model is not valid after 100 days

15 a 4.2

b i 1.12×10^{25} dyne cm

ii 3.55×10^{26} dyne cm

c divide **b ii** by **b i**

16 a They exponentiated the two terms on RHS separately rather than combining them first.

b $x = 2 \pm \sqrt{5}$

Challenge

a $y = 9^x = 3^{2x}$, $\log_3(y) = 2x$

b $y^2 = (9^x)^2 = 9^{2x}$, $\log_9(y^2) = 2x$

c $x = -\frac{1}{3}$ or $x = -2$