# P2 Chapter 5: Radians

## Small Angle Approximations

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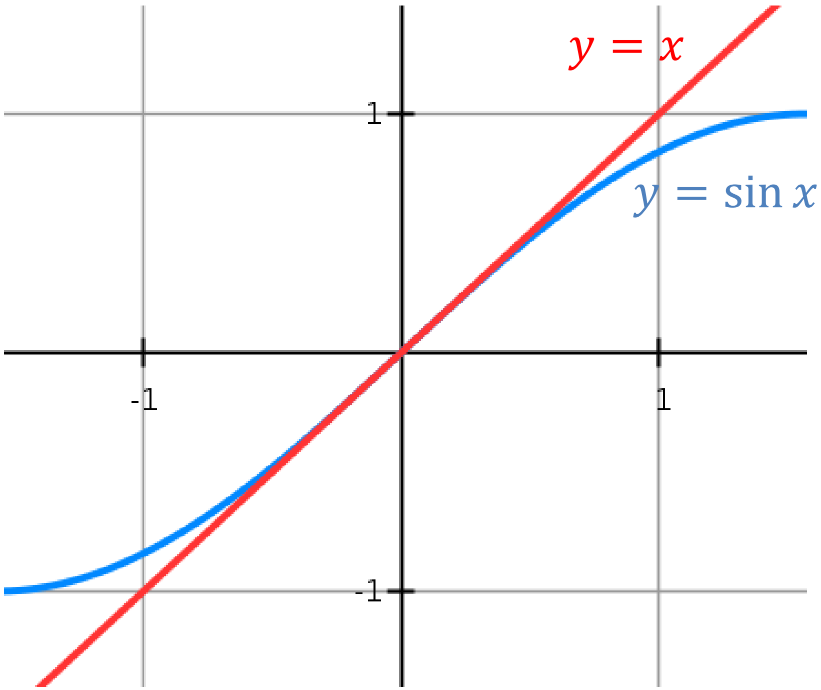
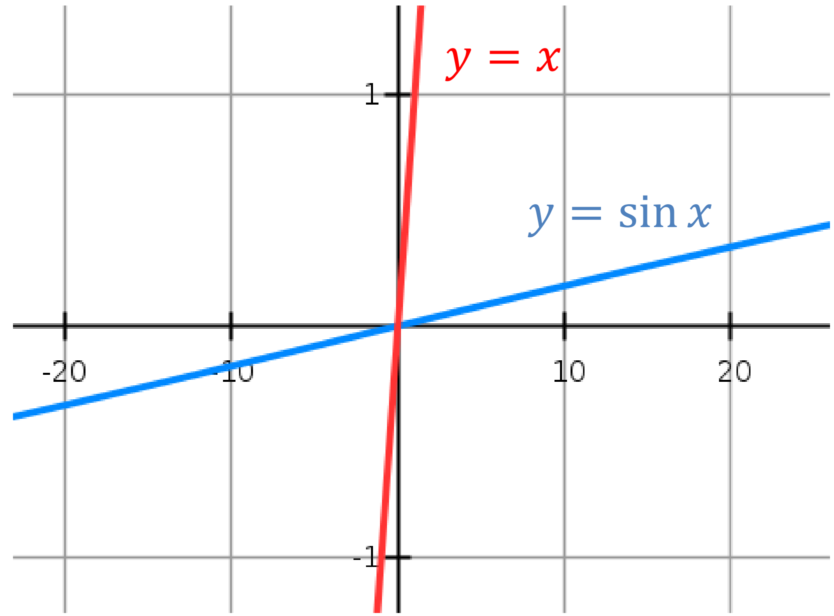


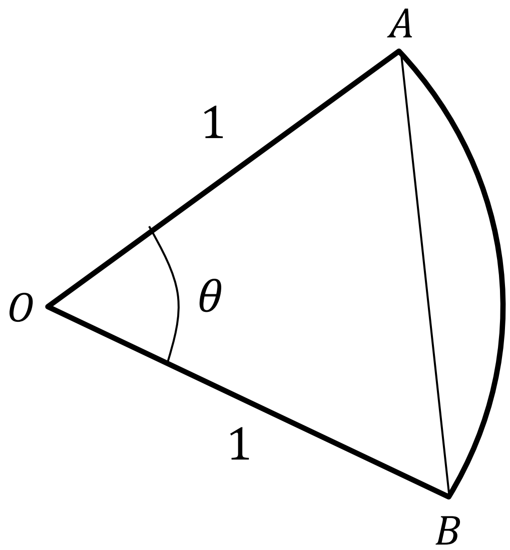

If $x$ is in radians, we can see from the graph that as $x$ approaches 0, the two graphs are approximately the same, i.e. $\sin x \approx x$

If $x$ was in degrees however, then we can see this is not the case.

✎ When $\theta$ is small and measured in radians:

- $\sin \theta \approx \theta$
- $\tan \theta \approx \theta$
- $\cos \theta \approx 1 - \frac{\theta^2}{2}$

# Small Angle Approximations



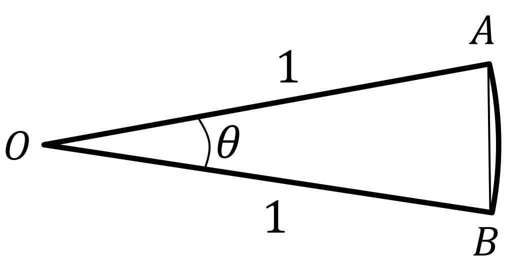

**Geometric Proof that $\boldsymbol{sin\ \theta \approx \theta}$:**

The area of sector $OAB$ is:

$$\frac{1}{2} \times 1^2 \times \theta = \frac{1}{2}\theta$$

The area of triangle $OAB$ is:

$$\frac{1}{2} \times 1^2 \times sin\theta = \frac{1}{2}\sin\theta$$




As $\theta$ becomes small, the area of the triangle is approximately equal to that of the sector, so:

$$\frac{1}{2}\sin\theta \approx \frac{1}{2}\theta$$
$$\sin\theta \approx \theta$$

Note that this only works for radians, because we used the sector area formula for radians. The fact that $\sin\theta \approx \theta$ is enormously important when we come to differentiation, because we can use it to prove that $\frac{d}{dx}(\sin x) = \cos x$.

# Examples

When $\theta$ is small and measured in radians:

- $\sin\theta \approx \theta$
- $\tan\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$

[Textbook] When $\theta$ is small, find the approximate value of:

a) $\frac{\sin 2\theta + \tan\theta}{2\theta}$

b) $\frac{\cos 4\theta - 1}{\theta \sin 2\theta}$

[Textbook] a) Show that, when $\theta$ is small,

$$\sin 5\theta + \tan 2\theta - \cos 2\theta \approx 2\theta^2 + 7\theta - 1$$

b) Hence state the approximate value of $\sin 5\theta + \tan 2\theta - \cos 2\theta$ for small values of $\theta$.

**a)** ?

**b)** ?

?

?

# Examples

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b) Hence state the approximate value of $\sin 5\theta + \tan 2\theta - \cos 2\theta$ for small values of $\theta$.

**a)** $\frac{\sin 2\theta + \tan\theta}{2\theta} \approx \frac{2\theta + \theta}{2\theta} = \frac{3}{2}$

**b)** $\frac{\cos 4\theta - 1}{\theta \sin 2\theta} \approx \frac{1 - \frac{(4\theta)^2}{2} - 1}{\theta \times 2\theta}$

$$= \frac{1 - \frac{16\theta^2}{2} - 1}{2\theta^2}$$

$$= \frac{-8\theta^2}{2\theta^2} = -4$$

$$\sin 5\theta + \tan 2\theta - \cos 2\theta$$

$$\approx 5\theta + 2\theta - \left(1 - \frac{(2\theta)^2}{2}\right)$$

$$= 7\theta - 1 + 2\theta^2$$

**b) The $7\theta$ and $2\theta^2$ terms tend towards 0, thus the approximate value is -1.**

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# Homework Exercise

**1** When $\theta$ is small, find the approximate values of:

**a** $\dfrac{\sin 4\theta - \tan 2\theta}{3\theta}$ **b** $\dfrac{1 - \cos 2\theta}{\tan 2\theta \sin\theta}$ **c** $\dfrac{3\tan\theta - \theta}{\sin 2\theta}$

**2** When $\theta$ is small, show that:

**a** $\dfrac{\sin 3\theta}{\theta \sin 4\theta} = \dfrac{3}{4\theta}$ **b** $\dfrac{\cos\theta - 1}{\tan 2\theta} = -\dfrac{\theta}{4}$ **c** $\dfrac{\tan 4\theta + \theta^2}{3\theta - \sin 2\theta} = 4 + \theta$

**3** **a** Find $\cos(0.244 \text{ rad})$ correct to 6 decimal places.

**b** Use the approximation for $\cos\theta$ to find an approximate value for $\cos(0.244 \text{ rad})$.

**c** Calculate the percentage error in your approximation.

**d** Calculate the percentage error in the approximation for $\cos 0.75$ rad.

**e** Explain the difference between your answers to parts **c** and **d**.

**4** The percentage error for $\sin\theta$ for a given value of $\theta$ is 1%. Show that $100\theta = 101\sin\theta$.

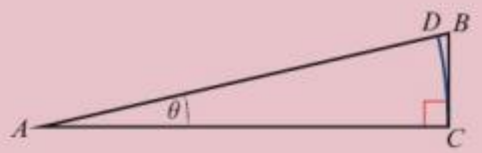
**5** **a** When $\theta$ is small, show that the expression $\dfrac{4\cos 3\theta - 2 + 5\sin\theta}{1 - \sin 2\theta}$ can be written as $9\theta + 2$. **(3 marks)**

**b** Hence write down the value of $\dfrac{4\cos 3\theta - 2 + 5\sin\theta}{1 - \sin 2\theta}$ when $\theta$ is small. **(1 mark)**

# Homework Exercise

**Challenge**

1 The diagram shows a right-angled triangle $ABC$. $\angle BAC = \theta$. An arc, $CD$, of the circle with centre $A$ and radius $AC$ has been drawn on the diagram in blue.




**a** Write an expression for the arc length $CD$ in terms of $AC$ and $\theta$.

Given that $\theta$ is small so that, $AC = AD \approx AB$ and $CD \approx BC$,

**b** deduce that $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$.

2 **a** Using the binomial expansion and ignoring terms in $x^4$ and higher powers of $x$, find an approximation for $\sqrt{1-x^2}$, $|x| < 1$.

**b** Hence show that for small $\theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$. You may assume that $\sin\theta \approx \theta$.

# Homework Answers

1 **a** 0.795, 5.49 **b** 3.34, 6.08
**c** 1.37, 4.51 **d** $\pi$

2 **a** 0.848, 2.29 **b** 0.142, 3.28
**c** 1.08, 4.22 **d** 0.886, 5.40

3 **a** 1.16, 5.12 **b** 3.61, 5.82
**c** 0.896, 4.04 **d** 0.421, 5.86

4 **a** $-\frac{5\pi}{6}, \frac{\pi}{6}$ **b** 0.201, 2.94
**c** −5.39, −0.896, 0.896, 5.39
**d** −1.22, 1.22, 5.06, 7.51
**e** 1.77, 4.91, 8.05, 11.2
**f** 4.89

5 **a** 0.322, 2.82, 3.46, 5.96
**b** 1.18, 1.96, 3.28, 4.05, 5.37, 6.15
**c** $\frac{\pi}{24}, \frac{7\pi}{24}, \frac{13\pi}{24}, \frac{19\pi}{24}, \frac{25\pi}{24}, \frac{31\pi}{24}, \frac{37\pi}{24}, \frac{43\pi}{24}$
**d** 0.232, 2.91, 3.37, 6.05

6 **a** $-\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$
**b** $-\frac{5\pi}{6}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$
**c** −5.92, −4.35, −2.78, −1.21, 0.359, 1.93, 3.50, 5.07
**d** −2.46, −0.685, 0.685, 2.46, 3.83, 5.60, 6.97, 8.74

7 **a** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **b** 0, 2.82, $\pi$, 5.96, $2\pi$
**c** $\pi$ **d** 0.440, 2.70, 3.58, 5.84

8 **a** $\pi$ **b** 0.501, 2.64, 3.64, 5.78
**c** No solutions **d** 1.10, 5.18

9 **a** $\frac{\pi}{3}, \frac{11\pi}{6}$ **b** $\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}$
**c** −0.986, 0.786 **d** $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

10 **a** $-\frac{\pi}{4}, \frac{3\pi}{4}$, 0.412, 2.73 **b** 0, 0.644, $\pi$, 5.64

11 0.3, 0.5, 2.6, 2.9

12 0.7, 2.4, 3.9, 5.6

13 $8\sin^2 x + 4\sin x - 20 = 4$
$8\sin^2 x + 4\sin x - 24 = 0$
$2\sin^2 x + \sin x - 6 = 0$
Let $Y = \sin x \Rightarrow 2Y^2 + Y - 6 = 0$
$\Rightarrow (2Y - 3)(Y + 2) = 0 \Rightarrow$ So $Y = 1.5$ or $Y = -2$
Since $Y = \sin x$, $\sin x = 1.5 \rightarrow$ No Solutions,
$\sin x = -2 \rightarrow$ No Solutions

14 **a** Using the quadratic formula with $a = 1$, $b = -2$ and $c = -6$ (can complete the square as well)

$$\tan x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-6)}}{2 \times 1}$$

$$\tan x = \frac{2 \pm \sqrt{4 + 24}}{2} = \frac{2 \pm \sqrt{28}}{2} = \frac{2 \pm 2\sqrt{7}}{2} = 1 \pm \sqrt{7}$$

**b** 1.3, 2.1, 4.4, 5.3, 7.6, 8.4

15 **a** $\sin x = 0.599$ (3 d.p.)
**b** 0.64, 2.50