M1 Chapter 11: Variable Acceleration

Calculus Proof of Suvat

Constant acceleration formulae

For constant acceleration we worked out the various $suvat$ formulae by using a velocity-time graph. It's also possible to derive all of these using integration, using the fact that **acceleration is constant**.

Given a body has constant acceleration a, at t=0 the initial velocity is u and its initial displacement is 0 m, prove that:

(a) Final velocity: $v = u + at$

(b) Displacement: $s = ut + \frac{1}{2}at^2$

a

$v = \int a\, dt = at + c$

When $t = 0, v = u$, therefore

$$u = 0 + c \quad \rightarrow \quad c = u$$

$$\therefore v = u + at$$

Note that a is a **constant** not a variable, so integrates as such.
Just as $\int 3\, dx = 3x + c$, we find $\int a\, dt = at + c$

b

Integrating again:

$$s = \int v\, dt = \int (u + at) dt$$

$$s = ut + \frac{1}{2}at^2 + c$$

Again, because u is fixed, we can treat it as a constant.

Since $s = 0$ when $t = 0$,

$0 = 0 + 0 + c \quad \rightarrow \quad c = 0$

$$s = ut + \frac{1}{2}at^2$$

Exercise 11.5 Constant Acceleration Formulae

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Homework Exercise

1 A particle moves on the x-axis with constant acceleration $a\,\text{m s}^{-2}$. The particle has initial velocity 0 and initial displacement x m. After time t seconds the particle has velocity $v\,\text{m s}^{-1}$ and displacement s m.

Prove that $s = \frac{1}{2}at^2 + x$.

2 A particle moves in a straight line with constant acceleration $5\,\text{m s}^{-2}$.

a Given that its initial velocity is $12\,\text{m s}^{-1}$, use calculus to show that its velocity at time t s is given by $v = 12 + 5t$.

b Given that the initial displacement of the particle is 7 m, show that $s = 12t + 2.5t^2 + 7$.

3 A particle moves in a straight line from a point O. At time t seconds, its displacement, s m, from P is given by $s = ut + \frac{1}{2}at^2$ where u and a are constants. Prove that the particle moves with constant acceleration a.

4 Which of these equations for displacement describe constant acceleration? Explain your answers.

A $s = 2t^2 - t^3$ B $s = 4t + 7$ C $s = \frac{t^2}{4}$ D $s = 3t - \frac{2}{t^2}$ E $s = 6$

Homework Exercise

5 A particle moves in a straight line with constant acceleration. The initial velocity of the particle is $5\,\mathrm{m\,s^{-1}}$ and after 2 seconds it is moving with velocity $13\,\mathrm{m\,s^{-1}}$.

a Find the acceleration of the particle. **(3 marks)**

b Without making use of the kinematics formulae, show that the displacement, s m, of the particle from its starting position is given by:

$$s = pt^2 + qt + r,\ t \geqslant 0$$

where p, q and r are constants to be found. **(5 marks)**

Watch out An exam question might specify that you cannot use certain formulae or techniques. In this case you need to use calculus to find the answer to part **b**.

6 A train travels along a straight track, passing point A at time $t = 0$ and passing point B 40 seconds later. Its distance from A at time t seconds is given by:

$$s = 25t - 0.2t^2,\ 0 \leqslant t \leqslant 40$$

a Find the distance AB. **(1 mark)**

b Show that the train travels with constant acceleration. **(3 marks)**

A bird passes point B at time $t = 0$ at an initial velocity towards A of $7\,\mathrm{m\,s^{-1}}$. It flies in a straight line towards point A with constant acceleration $0.6\,\mathrm{m\,s^{-2}}$.

c Find the distance from A at which the bird is directly above the train. **(6 marks)**

Homework Answers

1 $v = \int a\mathrm{d}t = at + c$

$a \times 0 + c = 0 \Rightarrow c = 0 \Rightarrow v = at$

$s = \int v\mathrm{d}t = \int at\mathrm{d}t = \frac{1}{2}at^2 + k$

$\frac{1}{2}a \times 0^2 + k = x \Rightarrow k = x$

so $s = \frac{1}{2}at^2 + x$

2 **a** $a = 5$, $v = \int 5\,\mathrm{d}t = 5t + c$; when $t = 0$, $u = 12$ so $c = 12$, $v = 12 + 5t$

b $s = \int 12 + 5t\,\mathrm{d}t = 12t + \frac{5t^2}{2} + d$, when $t = 0$,

$s = 7$ so $d = 7$, $s = 12t + 2.5t^2 + 7$

3 $v = \frac{\mathrm{d}s}{\mathrm{d}t} = u + at$; $\frac{\mathrm{d}v}{\mathrm{d}t} = a$ so constant acceleration a

4 **A** $a = 4 - 6t$, not constant

B $a = 0$. no acceleration

C $a = \frac{1}{2}$, constant

D $a = -\frac{12}{t^4}$ not constant

E $v = 0$, particle stationary

5 **a** $4\,\mathrm{m\,s^{-2}}$ **b** $p = 2$, $q = 5$, $r = 0$

6 **a** 680 m

b $\frac{\mathrm{d}s}{\mathrm{d}t} = 25 - 0.4t \Rightarrow \frac{\mathrm{d}^2s}{\mathrm{d}t^2} = -0.4 \therefore a$ is constant

c 420 m from A