Stats Yr2 Chapter 3: Distribution-N

The Standard Normal

Standard Normal Distribution

✎ Z is the number of standard deviations above the mean.

If again we use IQ distributed as $X \sim N(100, 15^2)$ then: (in your head!)

IQ	Z
100	0
130	2
85	-1
165	4.333
62.5	-2.5

This formula makes sense if you think about the definition above. For an IQ of 130: $Z = \frac{130-100}{15} = 2$ as expected.

✎ Z represents the coding:

$$Z = \frac{X - \mu}{\sigma}$$

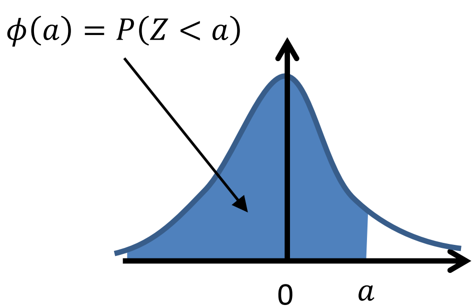
and $Z \sim N(0, 1^2)$. Z is known as a **standard** normal distribution.

The 0 and 1 of $Z \sim N(0, 1^2)$ are the result of the coding. If we've subtracted μ from each value the mean of the normal distribution is now 0. If we've divided all the values by σ the standard deviation is now $\frac{\sigma}{\sigma} = 1$

Standard Normal Distribution

The point of coding in this context is that all different possible normal distributions become a single unified distribution where we no longer have to worry about the mean and standard deviation.
It means for example when we calculate $P(Z < 3)$, this will always give the same probability regardless of the original distribution.

It also means we can look up probabilities in a ***z*-table**:



$\Phi(a) = P(Z < a)$ is the cumulative distribution for the standard normal distribution. The values of $\Phi(a)$ can be found in a z-table.

This is a traditional z-table in the old A Level syllabus (but also found elsewhere). You no longer get given this and are expected to use your calculator.

This is from the new formula booklet. This is sometimes known as a 'reverse z-table', because you're looking up the z-value for a probability. Beware: p here it the probability of **<u>exceeding</u>** z rather than being up to z. Let's use it...

THE NORMAL DISTRIBUTION FUNCTION

The function tabulated below is $\Phi(z)$, defined as $\Phi(z) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{z} e^{-\frac{1}{2}t^2}\, dt$.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.9772
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.02	0.9783
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.04	0.9793
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.06	0.9803
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.08	0.9812
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.10	0.9821
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.12	0.9830
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.14	0.9838
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.16	0.9846
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.18	0.9854
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.20	0.9861

Percentage Points of The Normal Distribution

The values z in the table are those which a random variable $Z \sim N(0, 1)$ exceeds with probability p; that is, $P(Z > z) = 1 - \Phi(z) = p$.

p	z	p	z
0.5000	0.0000	0.0500	1.6449
0.4000	0.2533	0.0250	1.9600
0.3000	0.5244	0.0100	2.3263
0.2000	0.8416	0.0050	2.5758
0.1500	1.0364	0.0010	3.0902
0.1000	1.2816	0.0005	3.2905

Examples

[Textbook] The random variable $X \sim N(50, 4^2)$. Write in terms of $\Phi(z)$ for some value of z.

(a) $P(X < 53)$ (b) $P(X \geq 55)$

a

?

b

?

[Textbook] The systolic blood pressure of an adult population, S mmHg, is modelled as a normal distribution with mean 127 and standard deviation 16. A medical research wants to study adults with blood pressures higher than the 95th percentile. Find the minimum blood pressure for an adult included in her study.

?

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Examples

[Textbook] The random variable $X \sim N(50, 4^2)$. Write in terms of $\Phi(z)$ for some value of z.
(a) $P(X < 53)$ (b) $P(X \geq 55)$

a

$$P(X < 53)$$
$$= P\left(Z < \frac{53-50}{4}\right) = P(Z < 0.75)$$
$$= \Phi(0.75)$$

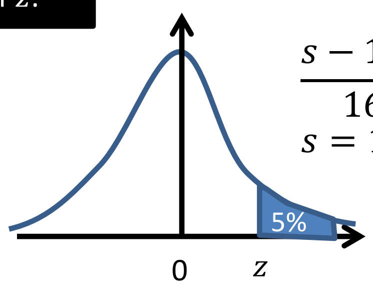
'Standardise' to turn X into Z.

b

$$P(X \geq 55)$$
$$= P\left(Z \geq \frac{55-50}{4}\right) = P(Z \geq 1.25)$$
$$= 1 - P(Z < 1.25)$$
$$= 1 - \Phi(1.25)$$

[Textbook] The systolic blood pressure of an adult population, S mmHg, is modelled as a normal distribution with mean 127 and standard deviation 16. A medical research wants to study adults with blood pressures higher than the 95th percentile. Find the minimum blood pressure for an adult included in her study.

A sketch will help you visualise whether you expect a positive or negative value of z.



$S \sim N(127, 16^2)$
Looking at the table, we see that the top 5% corresponds to a z-value of 1.6449.

$$\frac{s-127}{16} = 1.6449$$
$$s = 153\ (3sf)$$

Use $Z = \frac{X-\mu}{\sigma}$ formula as soon as you determine z.

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Further Examples

(a) Determine $P(Z > -1.3)$
(b) Determine $P(-2 < Z < 1)$
(c) Determine the a such that $P(Z > a) = 0.7$
(d) Determine the a such that $P(-a < Z < a) = 0.6$

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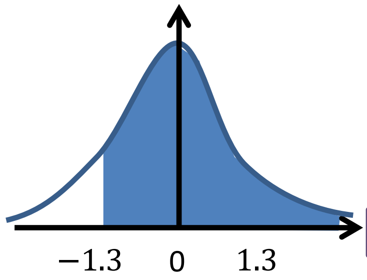
a



?

Fro Tip: Either changing < to/from > or changing the sign (+ to/from -) has the effect of "1 –". However, if you change both, the "1 –"s cancel out!

b

?

$$P(a < Z < b) = P(Z < b) - P(Z < a)$$

c

?

d

?

Further Examples

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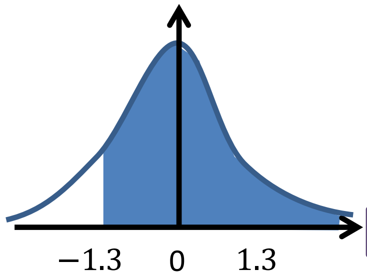
a



$$P(Z > -1.3) = P(Z < 1.3)$$
$$= 0.9032$$

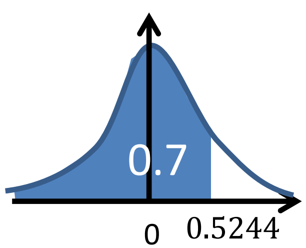
Fro Tip: Either changing < to/from > or changing the sign (+ to/from -) has the effect of "1 –". However, if you change both, the "1 –"s cancel out!

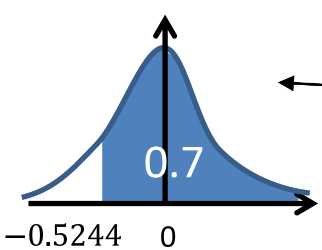
b

$$P(-2 < Z < 1)$$
$$= P(Z < 1) - P(Z < -2)$$
$$= P(Z < 1) - \big(1 - P(Z < 2)\big)$$
$$= 0.8413 - (1 - 0.9772)$$
$$= 0.8185\ (4dp)$$

$$P(a < Z < b)$$
$$= P(Z < b) - P(Z < a)$$

c

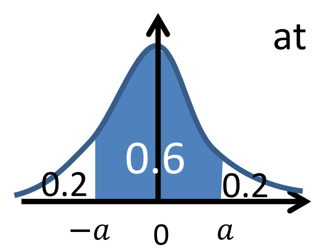




From the second sketch we can see that our z value is in the bottom half, so it is negative. However, our table only gives positive values of z.
From the graphs on the left, we can see by symmetry that the z value for the top 30% must be the negation of the z-value for the bottom 30% (i.e. top 70%).

$$P(Z > a) = 0.7$$
$$a = -0.5244$$

d



By symmetry, if 0.6 at the centre, must be 0.2 at each tail.

$$P(Z > a) = 0.2$$
$$a = 0.8416$$

Test Your Understanding

1 IQ is distributed with mean 100 and standard deviation 15. Using an appropriate table, determine the IQ corresponding to the
(a) top 10% of people.
(b) bottom 20% of people.

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0.5000	0.0000	0.0500	1.6449
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a ?

b ?

2 If $X \sim N(100, 15^2)$, determine, in terms of Φ:
(a) $P(X > 115)$
(b) $P(77.5 < X < 112)$

a ?

b ?

3 Find the a such that:
(a) $P(-a < Z < a) = 0.2$
(b) $P(0 < Z < a) = 0.35$

a ?

b ?

Test Your Understanding

1 IQ is distributed with mean 100 and standard deviation 15. Using an appropriate table, determine the IQ corresponding to the
(a) top 10% of people.
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a
$$X \sim N(100, 15^2)$$
$$1.2816 = \frac{x - 100}{15}$$
$$x = 119.24$$

b
$$-0.8416 = \frac{x - 100}{15}$$
$$x = 87.376$$

2 If $X \sim N(100, 15^2)$, determine, in terms of Φ:
(a) $P(X > 115)$
(b) $P(77.5 < X < 112)$

a
$$P(X > 115) = P\left(Z > \frac{115 - 100}{15}\right)$$
$$= P(Z > 1) = 1 - P(Z < 1) = 1 - \Phi(1)$$

b
$$P(77.5 < X < 112)$$
$$= P(X < 112) - P(X < 77.5)$$
$$= P(Z < 0.8) - P(Z < -1.5)$$
$$= \Phi(0.8) - \big(1 - \Phi(1.5)\big)$$
$$= \Phi(0.8) + \Phi(1.5) - 1$$

3 Find the a such that:
(a) $P(-a < Z < a) = 0.2$
(b) $P(0 < Z < a) = 0.35$

a Using appropriate sketch:
$$P(Z > a) = 0.4$$
$$a = 0.2533$$

b
$$P(Z > a) = 0.15$$
$$a = 1.0364$$

Exercise 3.4

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Homework Exercise

1 For the standard normal distribution $Z \sim N(0, 1^2)$, find:

a $P(Z < 2.12)$ **b** $P(Z < 1.36)$ **c** $P(Z > 0.84)$ **d** $P(Z < -0.38)$

e $P(-2.30 < Z < 0)$ **f** $P(Z < -1.63)$ **g** $P(-2.16 < Z < -0.85)$ **h** $P(-1.57 < Z < 1.57)$

2 For the standard normal distribution $Z \sim N(0, 1^2)$, find values of a such that:

a $P(Z < a) = 0.9082$ **b** $P(Z > a) = 0.0314$

c $P(Z > a) = 0.1500$ **d** $P(Z > a) = 0.9500$

e $P(0 < Z < a) = 0.3554$ **f** $P(0 < Z < a) = 0.4946$

g $P(-a < Z < a) = 0.80$ **h** $P(-a < Z < a) = 0.40$

Hint For parts **g** and **h** you will need to use the symmetry properties of the distribution.

3 The random variable $X \sim N(0.8, 0.05^2)$. For each of the following values of X, write down the corresponding value of the standardised normal distribution, $Z \sim N(0, 1^2)$.

a $x = 0.8$ **b** $x = 0.792$ **c** $x = 0.81$ **d** $x = 0.837$

4 The normal distribution $X \sim N(154, 12^2)$. Write in terms of $\Phi(z)$:

a $P(X < 154)$ **b** $P(X < 160)$

c $P(X > 151)$ **d** $P(140 < X < 155)$

Hint Write your answer to part **d** in the form $\Phi(z_1) - \Phi(z_2)$.

5 **a** Use the percentage points table to find a value of z such that $P(Z > z) = 0.025$. **(1 mark)**

b A fighter jet training programme takes only the top 2.5% of candidates on a test. Given that the scores can be modelled using a normal distribution with mean 80 and standard deviation 4, use your answer to part **a** to find the score necessary to get on the programme. **(2 marks)**

Homework Exercise

6 **a** Use the percentage points table to find a value of z such that $P(Z < z) = 0.15$. **(1 mark)**

b A hat manufacturer makes a special 'petite' hat which should fit 15% of its customers. Given that hat sizes can be modelled using a normal distribution with mean 57 cm and standard deviation 2 cm, use your answer to part **a** to find the size of a 'petite' hat. **(2 marks)**

7 **a** Use the percentage points table to find the values of z that correspond to the 10% to 90% interpercentile range. **(2 marks)**

A particular brand of light bulb has a life modelled as a normal distribution with mean 1175 hours and standard deviation 56 hours. The bulb life is considered 'standard' if its life falls into the 10% to 90% interpercentile range.

b Use your answer to part **a** to find the range of life to the nearest hour for a 'standard' bulb. **(2 marks)**

Homework Answers

For Chapter 3, student answers may differ slightly from those shown here when calculators are used rather than table values.

1	**a** 0.9830		**b** 0.9131		**c** 0.2005		**d** 0.3520	
	e 0.4893		**f** 0.0516		**g** 0.1823		**h** 0.8836	
2	**a** 1.33		**b** 1.86		**c** 1.0364		**d** −1.6449	
	e 1.06		**f** 2.55		**g** 1.2816		**h** 0.5244	
3	**a** 0		**b** −0.16		**c** 0.2		**d** 0.74	
4	**a** $\Phi(0)$		**b** $\Phi(0.5)$		**c** $1 - \Phi(-0.25)$			
	d $\Phi(0.0833) - \Phi(-1.17)$							
5	**a** 1.96		**b** 87.8 (3 s.f.)					
6	**a** −1.0364		**b** 54.9 cm					
7	**a** $-1.2816 < z < 1.2816$				**b** 1103–1247 hours			