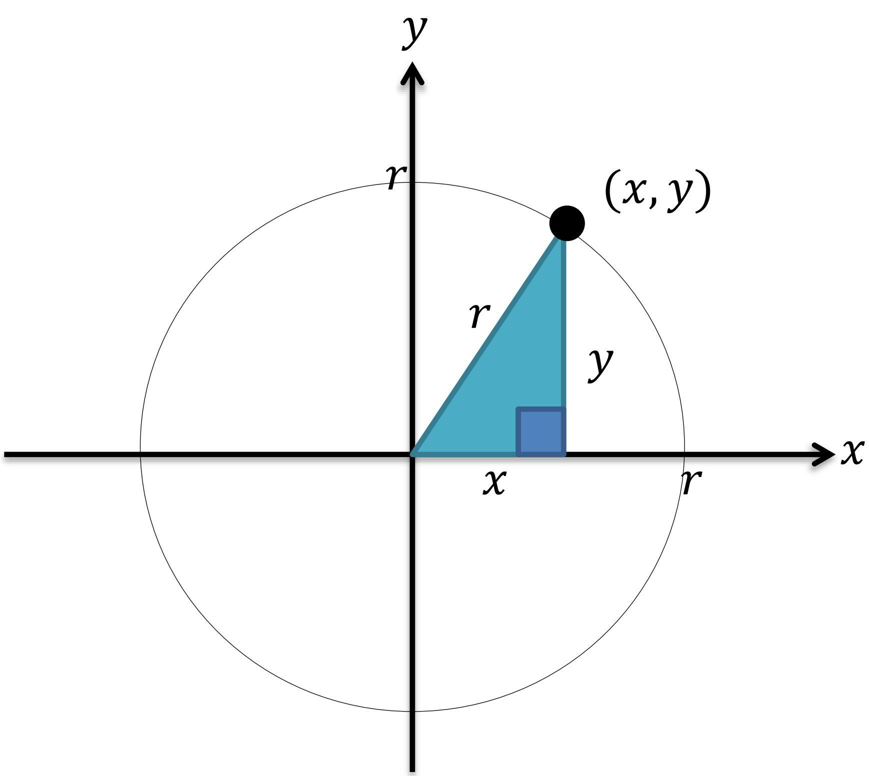
P1 Chapter 6: Circles

The Circle Equation

Equation of a circle

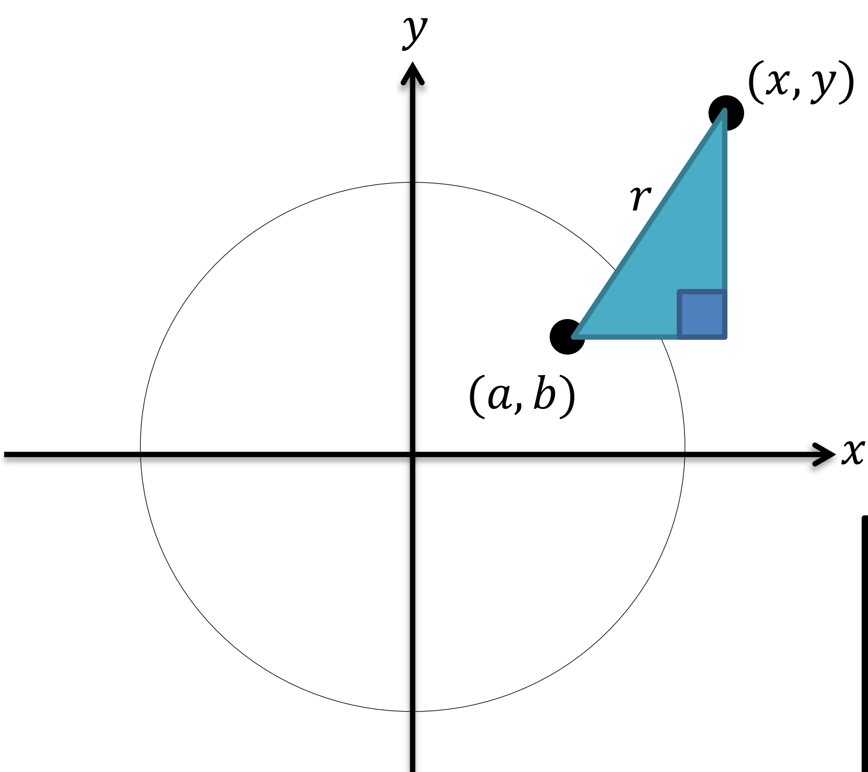


Recall that a line can be a set of points (x, y) that satisfy some equation. Suppose we have a point (x, y) on a circle centred at the origin, with radius r. What equation must (x, y) satisfy?

(Hint: draw a right-angled triangle inside your circle, with one vertex at the origin and another at the circumference)

$$x^2 + y^2 = r^2$$

Equation of a circle



Now suppose we shift the circle so it's now centred at (a, b). What's the equation now?

(Hint: What would the sides of this right-angled triangle be now?)

The equation of a circle with centre (a, b) and radius r is:

$$(x-a)^2 + (y-b)^2 = r^2$$

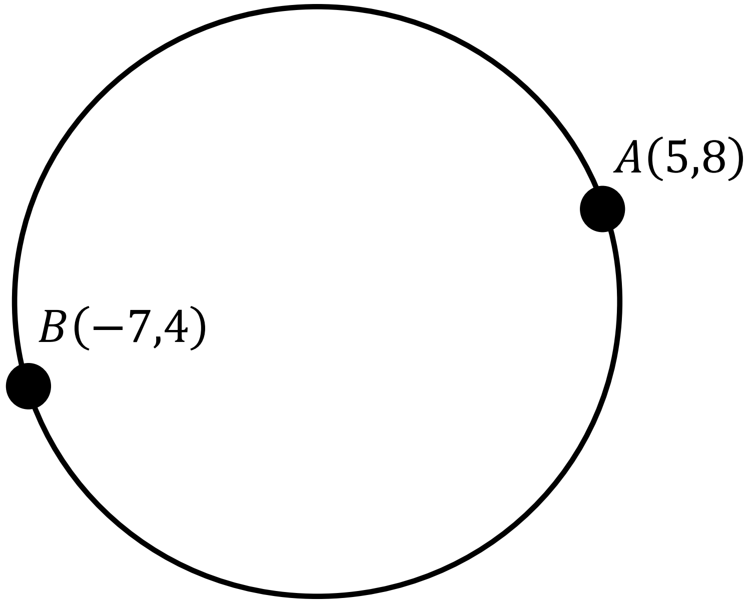
Quickfire Questions

Centre	Radius	Equation
$(0,0)$	5	?
$(1,2)$	6	?
?	?	$(x+3)^2+(y-5)^2=1$
?	?	$(x+5)^2+(y-2)^2=49$
?	?	$(x+6)^2+y^2=16$
?	?	$(x-1)^2+(y+1)^2=3$
?	?	$(x+2)^2+(y-3)^2=8$

Quickfire Questions

Centre	Radius	Equation
$(0,0)$	5	$x^2 + y^2 = 25$
$(1,2)$	6	$(x-1)^2 + (y-2)^2 = 36$
$(-3,5)$	1	$(x+3)^2 + (y-5)^2 = 1$
$(-5,2)$	7	$(x+5)^2 + (y-2)^2 = 49$
$(-6,0)$	4	$(x+6)^2 + y^2 = 16$
$(1,-1)$	$\sqrt{3}$	$(x-1)^2 + (y+1)^2 = 3$
$(-2,3)$	$2\sqrt{2}$	$(x+2)^2 + (y-3)^2 = 8$

Finding the equation using points

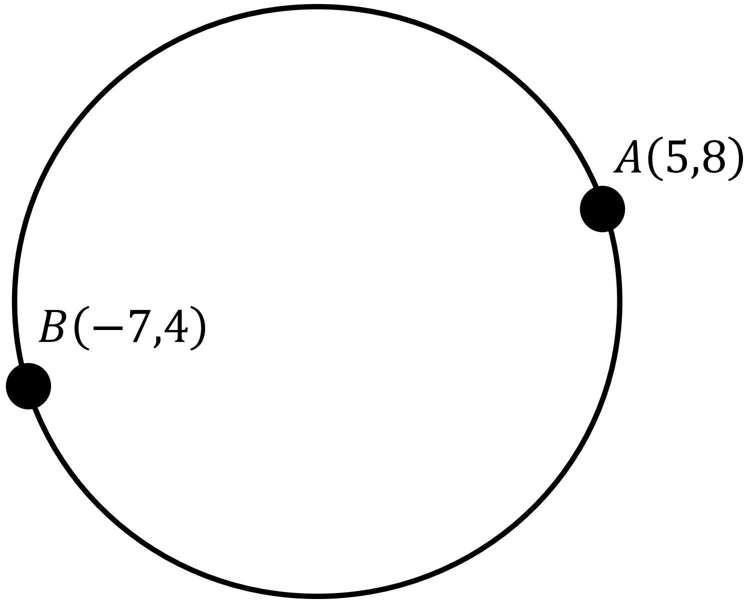


A line segment AB is the diameter of a circle, where A and B have coordinates $(5,8)$ and $(-7,4)$ respectively. Determine the equation of the circle.

Hint: What two things do we need to use the circle formula?

?

Finding the equation using points



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Hint: What two things do we need to use the circle formula?

Centre: $C(-1, 6)$

We can use the distance BC or AC as the radius. Using $B(-7, 4)$ and $M(-1, 6)$

$$r = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$$

$$\therefore \quad (x+1)^2 + (y-6)^2 = 40$$

Test Your Understanding

Edexcel C2 Jan 2005 Q2

The points A and B have coordinates $(5, -1)$ and $(13, 11)$ respectively.

(a) Find the coordinates of the mid-point of AB.

(2)

Given that AB is a diameter of the circle C,

(b) find an equation for C.

(4)

a) ?

b) ?

Test Your Understanding

Edexcel C2 Jan 2005 Q2

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(4)

a) $(9, 5)$

b) $r = \sqrt{4^2 + 6^2} = \sqrt{52}$

$$(x-9)^2 + (y-5)^2 = 52$$

Completing the square

When the equation of a circle is in the form $(x-a)^2+(y-b)^2=r^2$, we can instantly read off the centre (a,b) and the radius r.
But what if the equation wasn't in this form?

Find the centre and radius of the circle with equation $x^2+y^2-6x+2y-6=0$

Hint: Have we seen a method in a previous chapter that allows us to turn a x^2 term and a x term into a single expression involving x?

?

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$$x^2-6x+y^2+2y-6=0$$
$$(x-3)^2-9+(y+1)^2-1-6=0$$
$$(x-3)^2+(y+1)^2=16$$

Rearrange terms so that x terms are together and y terms are together.

Complete the square!

Further Example

Edexcel C2 June 2012 Q3a,b

The circle C with centre T and radius r has equation

$$x^2 + y^2 - 20x - 16y + 139 = 0$$

(a) Find the coordinates of the centre of C.

(3)

(b) Show that $r = 5$

(2)

?

Further Example

Edexcel C2 June 2012 Q3a,b

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(a) Find the coordinates of the centre of C.

(3)

(b) Show that $r = 5$

(2)

$$x^2 - 20x + y^2 - 16y + 139 = 0$$
$$(x - 10)^2 - 100 + (y - 8)^2 - 64 + 139 = 0$$
$$(x - 10)^2 + (y - 8)^2 = 25$$

$$C(10,8), \qquad r = \sqrt{25} = 5$$

Exercise 6.2

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Extension:

1 [MAT 2009 1B] The point on the circle $x^2 + y^2 + 6x + 8y = 75$ which is closest to the origin, is at what distance from the origin?

?

2 [MAT 2007 1D]
The point on the circle
$(x-5)^2 + (y-4)^2 = 4$
which is closest to the circle
$(x-1)^2 + (y-1)^2 = 1$
has what coordinates?

?

3 [MAT 2016 1I] Let a and b be positive real numbers. If $x^2 + y^2 \leq 1$ then the largest that $ax + by$ can equal is what? Give your expression in terms of a and b.

?

Exercise 6.2

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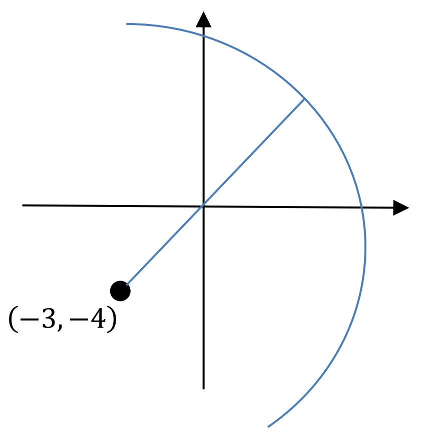
Extension:

1 [MAT 2009 1B] The point on the circle $x^2 + y^2 + 6x + 8y = 75$ which is closest to the origin, is at what distance from the origin?

$$(x+3)^2 + (y+4)^2 = 100$$
$$\therefore C(-3,-4), \qquad r = 10$$

The closest point P lies on the line between the circle centre and the origin. Since $(-3,-4)$ is 5 away from the origin, the distance between the origin and P must be $10 - 5 = 5$

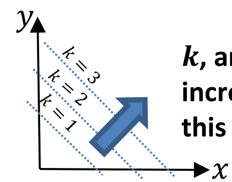


2 [MAT 2007 1D] The point on the circle $(x-5)^2 + (y-4)^2 = 4$ which is closest to the circle $(x-1)^2 + (y-1)^2 = 1$ has what coordinates?

Drawing the circles on the same axes, and drawing a straight line connecting their centres, the point is where the straight line intersects the first circle. The circle centres are 5 apart, so we need to go $\frac{2}{5}$ of the way across this line, giving $(3.4, 2.8)$

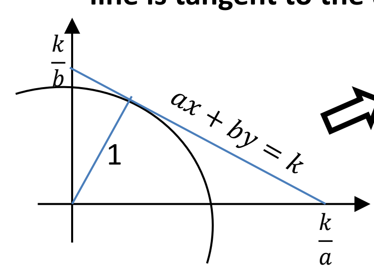
3 [MAT 2016 1I] Let a and b be positive real numbers. If $x^2 + y^2 \leq 1$ then the largest that $ax + by$ can equal is what? Give your expression in terms of a and b.

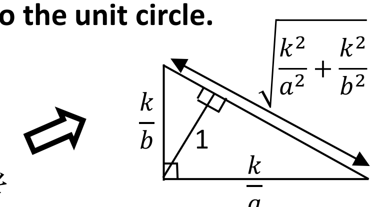
Many MAT questions consider maximising an expression in terms of x and y. Consider for example the simple case $x + y = k$. As we increase k, the line stays in the same direction but 'sweeps' across:



k, and therefore $x + y$, increase as we move in this direction.

If we similarly consider the line $ax + by = k$, $ax + by$ is therefore maximised when the line is tangent to the unit circle.





Using similar triangles, we can obtain $k = \sqrt{a^2 + b^2}$

Homework Exercise

1 Write down the equation of each circle:

a Centre $(3, 2)$, radius 4 **b** Centre $(-4, 5)$, radius 6 **c** Centre $(5, -6)$, radius $2\sqrt{3}$

d Centre $(2a, 7a)$, radius $5a$ **e** Centre $(-2\sqrt{2}, -3\sqrt{2})$, radius 1

2 Write down the coordinates of the centre and the radius of each circle:

a $(x + 5)^2 + (y - 4)^2 = 9^2$ **b** $(x - 7)^2 + (y - 1)^2 = 16$ **c** $(x + 4)^2 + y^2 = 25$

d $(x + 4a)^2 + (y + a)^2 = 144a^2$ **e** $(x - 3\sqrt{5})^2 + (y + \sqrt{5})^2 = 27$

3 In each case, show that the circle passes through the given point:

a $(x - 2)^2 + (y - 5)^2 = 13$, point $(4, 8)$ **b** $(x + 7)^2 + (y - 2)^2 = 65$, point $(0, -2)$

c $x^2 + y^2 = 25^2$, point $(7, -24)$ **d** $(x - 2a)^2 + (y + 5a)^2 = 20a^2$, point $(6a, -3a)$

e $(x - 3\sqrt{5})^2 + (y - \sqrt{5})^2 = (2\sqrt{10})^2$ point, $(\sqrt{5}, -\sqrt{5})$

4 The point $(4, -2)$ lies on the circle centre $(8, 1)$.
Find the equation of the circle.

Hint First find the radius of the circle.

5 The line PQ is the diameter of the circle, where P and Q are $(5, 6)$ and $(-2, 2)$ respectively.
Find the equation of the circle. **(5 marks)**

6 The point $(1, -3)$ lies on the circle $(x - 3)^2 + (y + 4)^2 = r^2$. Find the value of r. **(3 marks)**

7 The points $P(2, 2)$, $Q(2 + \sqrt{3}, 5)$ and $R(2 - \sqrt{3}, 5)$ lie on the circle $(x - 2)^2 + (y - 4)^2 = r^2$.

a Find the value of r. **(2 marks)**

b Show that $\triangle PQR$ is equilateral. **(3 marks)**

Homework Exercise

8 **a** Show that $x^2 + y^2 - 4x - 11 = 0$ can be written in the form $(x - a)^2 + y^2 = r^2$, where a and r are numbers to be found. **(2 marks)**

b Hence write down the centre and radius of the circle with equation $x^2 + y^2 - 4x - 11 = 0$ **(2 marks)**

Problem-solving

Start by writing $(x^2 - 4x)$ in the form $(x - a)^2 - b$.

9 **a** Show that $x^2 + y^2 - 10x + 4y - 20 = 0$ can be written in the form $(x - a)^2 + (y - b)^2 = r^2$, where a, b and r are numbers to be found. **(2 marks)**

b Hence write down the centre and radius of the circle with equation $x^2 + y^2 - 10x + 4y - 20 = 0$. **(2 marks)**

10 Find the centre and radius of the circle with each of the following equations.

a $x^2 + y^2 - 2x + 8y - 8 = 0$

b $x^2 + y^2 + 12x - 4y = 9$

c $x^2 + y^2 - 6y = 22x - 40$

d $x^2 + y^2 + 5x - y + 4 = 2y + 8$

e $2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$

Hint Start by writing the equation in one of the following forms:
$(x - a)^2 + (y - b)^2 = r^2$
$x^2 + y^2 + 2fx + 2gy + c = 0$

11 A circle C has equation $x^2 + y^2 + 12x + 2y = k$, where k is a constant.

a Find the coordinates of the centre of C. **(2 marks)**

b State the range of possible values of k. **(2 marks)**

Problem-solving

A circle must have a positive radius.

Homework Exercise

12 The point $P(7, -14)$ lies on the circle with equation $x^2 + y^2 + 6x - 14y = 17$.
The point Q also lies on the circle such that PQ is a diameter.
Find the coordinates of point Q. **(4 marks)**

13 The circle with equation $(x - k)^2 + y^2 = 41$ passes through the point $(3, 4)$.
Find the two possible values of k. **(5 marks)**

Challenge

1 A circle with equation $(x - k)^2 + (y - 2)^2 = 50$ passes through the point $(4, -5)$.
Find the possible values of k and the equation of each circle.

2 By completing the square for x and y, show that the equation $x^2 + y^2 + 2fx + 2gy + c = 0$ describes a circle with centre $(-f, -g)$ and radius $\sqrt{f^2 + g^2 - c}$.

Homework Answers

1 a $(x-3)^2+(y-2)^2=16$
b $(x+4)^2+(y-5)^2=36$
c $(x-5)^2+(y+6)^2=12$
d $(x-2a)^2+(y-7a)^2=25a^2$
e $(x+2\sqrt{2})^2+(y+3\sqrt{2})^2=1$

2 a $(-5, 4)$, 9 b $(7, 1)$, 4
c $(-4, 0)$, 5 d $(-4a, -a)$, $12a$
e $(3\sqrt{5}, -\sqrt{5})$, $3\sqrt{3}$

3 a $(4-2)^2+(8-5)^2=4+9=13$
b $(0+7)^2+(-2-2)^2=49+16=65$
c $7^2+(-24)^2=49+576=625=25^2$
d $(6a-2a)^2+(-3a+5a)^2=16a^2+4a^2=20a^2$
e $(\sqrt{5}-3\sqrt{5})^2+(-\sqrt{5}-\sqrt{5})^2=(-2\sqrt{5})^2+(-2\sqrt{5})^2$
$=20+20=40=(2\sqrt{10})^2$

4 $(x-8)^2+(y-1)^2=25$

5 $(x-\frac{3}{2})^2+(y-4)^2=\frac{65}{4}$

6 $\sqrt{5}$

7 a $r=2$
b Distance $PQ=PR=RQ=2\sqrt{3}$, three equal length sides triangle is equilateral.

8 a $(x-2)^2+y^2=15$
b Centre (2, 0) and radius $=\sqrt{15}$

9 a $(x-5)^2+(y+2)^2=49$
b Centre (5, −2) and radius = 7

10 a Centre (1, −4), radius 5
b Centre (−6, 2), radius 7
c Centre (11, −3), radius $3\sqrt{10}$
d 10 Centre (−2.5, 1.5), radius $\frac{5\sqrt{2}}{2}$
e Centre (2 ,−2), radius

11 a Centre (−6, −1)
b $k>-37$

12 $Q(-13, 28)$

13 $k=-2$ and $k=8$

Challenge

1 $k=3$, $(x-3)^2+(y-2)^2=50$
$k=5$, $(x-5)^2+(y-2)^2=50$

2 $(x+f)^2-f^2+(y+g)^2-g^2+c=0$
So $(x+f)^2+(y+g)^2=f^2+g^2-c$
Circle with centre $(-f, -g)$ and radius $\sqrt{f^2+g^2-c}$.