P1 Chapter 8: Binomial Expansion

Binomial Coefficients

Using Binomial Coefficients to Expand

In the previous section we learnt about the 'choose' function and how this related to Pascal's Triangle.

1 5 10 10 5 1

$$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$$

Why do rows of Pascal's Triangle give us the coefficients in a Binomial Expansion?

One possible selection of terms from each bracket.

Consider: $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$

Each term of the expansion involves picking one term from each bracket.

How many times will a^3b^2 appear in the expansion?

To get $\boldsymbol{a^3b^2}$ we must have chosen 3 $\boldsymbol{a}$'s from the 5 brackets (the rest $\boldsymbol{b}$'s).

That's $\binom{5}{3}$ ways, giving us $\binom{5}{3}a^3b^2$ in the expansion of $(a+b)^5$.

Using Binomial Coefficients to Expand

$\mathbb{N}$ is the set of natural numbers, i.e. positive integers. This formula is only valid for positive integers n. In Year 2 you will see how to deal with fractional/negative n.

✎ The binomial expansion, when $n \in \mathbb{N}$:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + b^n$$

Find the first 4 terms in the expansion of $(3x+1)^{10}$, in ascending powers of x.

?

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Find the first 4 terms in the expansion of $(3x+1)^{10}$, in ascending powers of x.

$$(3x+1)^{10} = \binom{10}{0}(1^{10}) + \binom{10}{1}(1^9)(3x)^1 + \binom{10}{2}(1^8)(3x)^2 + \binom{10}{3}(1^7)(3x)^3 + \cdots$$

$$= 1 + 30x + 405x^2 + 3240x^3 + \cdots$$

This is exactly the same method as before, except we've just had to calculate the Binomial coefficients ourselves rather than read them off Pascal's Triangle.

Test Your Understanding

Find the first 3 terms in the expansion of $\left(2-\frac{1}{3}x\right)^7$, in ascending powers of x.

?

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Find the first 3 terms in the expansion of $\left(2-\frac{1}{3}x\right)^7$, in ascending powers of x.

$$\left(2-\frac{1}{3}x\right)^7 = \binom{7}{0}(2^7) + \binom{7}{1}(2^6)\left(-\frac{1}{3}\right) + \binom{7}{2}(2^5)\left(-\frac{1}{3}\right)^2 + \cdots$$

$$= 128 - \frac{448}{3}x + \frac{224}{3}x^2 + \cdots$$

Note: The "$+\cdots$" indicates that there would have been other terms in the expansion.

Exercise 8.3

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Extension

1 [AEA 2013 Q1a] In the binomial expansion of $\left(1+\frac{12n}{5}x\right)^n$ the coefficients of x^2 and x^3 are equal and non-zero. Find the possible values of n.

?

Hint: Remember that

$$\binom{n}{2}=\frac{n(n-1)}{2!}$$

Can you similarly simplify $\binom{n}{3}$ using $\binom{n}{r}=\frac{n!}{r!(n-r)!}$?

2 [STEP I 2010 Q5a] By considering the expansion of $(1+x)^n$, where n is a positive integer, or otherwise, show that:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

?

Note: This means that the sum of each row in Pascal's Triangle gives successive powers of 2. S

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1 [AEA 2013 Q1a] In the binomial expansion of $\left(1+\frac{12n}{5}x\right)^n$ the coefficients of x^2 and x^3 are equal and non-zero. Find the possible values of n.

$$\frac{n(n-1)}{2!}\left(\frac{12n}{5}\right)^2 = \frac{n(n-1)(n-2)}{3!}\left(\frac{12n}{5}\right)^3$$

$3\times 5 = n(n-2)\times 12$ or $4n^2-8n-5=0$ (o.e.)

$(2n+1)(2n-5)=0$

$$\underline{n=-\tfrac{1}{2},\tfrac{5}{2}}$$

Hint: Remember that

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$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

Note: This means that the sum of each row in Pascal's Triangle gives successive powers of 2.

$$\boldsymbol{(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n}$$

Letting $x=1$ gives the desired result.

Homework Exercise

1 Write down the expansion of the following:

a $(1+x)^4$ **b** $(3+x)^4$ **c** $(4-x)^4$ **d** $(x+2)^6$ **e** $(1+2x)^4$ **f** $\left(1-\frac{1}{2}x\right)^4$

2 Use the binomial theorem to find the first four terms in the expansion of:

a $(1+x)^{10}$ **b** $(1-2x)^5$ **c** $(1+3x)^6$ **d** $(2-x)^8$ **e** $\left(2-\frac{1}{2}x\right)^{10}$ **f** $(3-x)^7$

3 Use the binomial theorem to find the first four terms in the expansion of:

a $(2x+y)^6$ **b** $(2x+3y)^5$ **c** $(p-q)^8$ **d** $(3x-y)^6$ **e** $(x+2y)^8$ **f** $(2x-3y)^9$

4 Use the binomial expansion to find the first four terms, in ascending powers of x, of:

a $(1+x)^8$ **b** $(1-2x)^6$ **c** $\left(1+\frac{x}{2}\right)^{10}$

d $(1-3x)^5$ **e** $(2+x)^7$ **f** $(3-2x)^3$

g $(2-3x)^6$ **h** $(4+x)^4$ **i** $(2+5x)^7$

Hint Your answers should be in the form $a+bx+cx^2+dx^3$ where a, b, c and d are numbers.

5 Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(2-x)^6$ and simplify each term. **(4 marks)**

6 Find the first 3 terms, in ascending powers of x, of the binomial expansion of $(3-2x)^5$ giving each term in its simplest form. **(4 marks)**

7 Find the binomial expansion of $\left(x+\frac{1}{x}\right)^5$ giving each term in its simplest form. **(4 marks)**

Challenge

a Show that $(a+b)^4-(a-b)^4=8ab(a^2+b^2)$.

b Given that $82\,896=17^4-5^4$, write $82\,896$ as a product of its prime factors.

Homework Answers

1 **a** $1 + 4x + 6x^2 + 4x^3 + x^4$
b $81 + 108x + 54x^2 + 12x^3 + x^4$
c $256 - 256x + 96x^2 - 16x^3 + x^4$
d $x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$
e $1 + 8x + 24x^2 + 32x^3 + 16x^4$
f $1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

2 **a** $1 + 10x + 45x^2 + 120x^3$
b $1 - 10x + 40x^2 - 80x^3$
c $1 + 18x + 135x^2 + 540x^3$
d $256 - 1024x + 1792x^2 - 1792x^3$
e $1024 - 2560x + 2880x^2 - 1920x^3$
f $2187 - 5103x + 5103x^2 - 2835x^3$

3 **a** $64x^6 + 192x^5y + 240x^4y^2 + 160x^3y^3$
b $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3$
c $p^8 - 8p^7q + 28p^6q^2 - 56p^5q^3$
d $729x^6 - 1458x^5y + 1215x^4y^2 - 540x^3y^3$
e $x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3$
f $512x^9 - 6912x^8y + 41\,472x^7y^2 - 145\,152x^6y^3$

4 **a** $1 + 8x + 28x^2 + 56x^3$
b $1 - 12x + 60x^2 - 160x^3$
c $1 + 5x + \frac{45}{4}x^2 + 15x^3$
d $1 - 15x + 90x^2 - 270x^3$
e $128 + 448x + 672x^2 + 560x^3$
f $27 - 54x + + 36x^2 - 8x^3$
g $64 - 576x + 2160x^2 - 4320x^3$
h $256 + 256x + 96x^2 + 16x^3$
i $128 + 2240x + 16\,800x^2 + 70\,000x^3$

5 $64 - 192x + 240x^2$

6 $243 - 810x + 1080x^2$

7 $x^5 + 5x^3 + 10x + \dfrac{10}{x} + \dfrac{5}{x^3} + \dfrac{1}{x^5}$

Challenge

a $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$
$(a + b)^4 - (a - b)^4 = 8a^3b + 8ab^3 = 8ab(a^2 + b^2)$

b $82\,896 = 2^4 \times 3 \times 11 \times 157$