P1 Chapter 14: Logarithms

Logarithm Laws

Laws of Logs

Three main laws:

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

$$\log_a (x^k) = k \log_a x$$

Special cases:

$$\log_a a = 1 \quad (a > 0, a \neq 1)$$

$$\log_a 1 = 0 \quad (a > 0, a \neq 1)$$

$$\log\left(\frac{1}{x}\right) = \log(x^{-1}) = -\log(x)$$

Not in syllabus (but in MAT/PAT):

$$\log_a b = \frac{\log_c b}{\log_c a}$$

The logs must have a consistent base.

i.e. You can move the power to the front.

We often try to avoid leaving fractions inside logs. So if the answer was:

$$\log_2\left(\frac{1}{3}\right)$$

You should write your answer as: $-\log_2 3$

Reciprocating the input negates the output.

This is known as **changing the base**. So to get $\log_2 9$ in terms of log base 3:

$$log_2 9 = \frac{log_3 9}{log_3 2} = \frac{2}{log_3 2}$$

Examples

Write as a single logarithm:

a. $\log_3 6 + \log_3 7$

b. $\log_2 15 - \log_2 3$

c. $2\log_5 3 + 3\log_5 2$

d. $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$

a. $\log_a(x^2yz^3)$

b. $\log_a\left(\frac{x}{y^3}\right)$

c. $\log_a\left(\frac{x\sqrt{y}}{z}\right)$

d. $\log_a\left(\frac{x}{a^4}\right)$

a ?

b ?

c ?

d ?

a ?

b ?

c ?

d ?

Examples

Write as a single logarithm:

a. $\log_3 6 + \log_3 7$

b. $\log_2 15 - \log_2 3$

c. $2\log_5 3 + 3\log_5 2$

d. $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$

Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$

a. $\log_a(x^2yz^3)$

b. $\log_a\left(\frac{x}{y^3}\right)$

c. $\log_a\left(\frac{x\sqrt{y}}{z}\right)$

d. $\log_a\left(\frac{x}{a^4}\right)$

a $\log_3 42$

b $\log_2 5$

c $\log_5 9 + \log_2 8 = \log^5 72$

d $\log_{10} 3 - \log_{10}\left(\frac{1}{16}\right) = \log_{10} 48$

a $\log_a(x^2yz^3) = \log_a(x^2) + \log_a y + \log_a z^3$
$= 2\log_a x + \log_a y + 3\log_a z$

b $\log_a\left(\frac{x}{y^3}\right) = \log_a(x) - \log_a(y^3)$
$= \log_a x - 3\log_a y$

c $\log_a\left(xy^{\frac{1}{2}}\right) - \log_a z = \log_a x + \log_a\left(y^{\frac{1}{2}}\right) - \log_a z$
$= \log_a x + \frac{1}{2}\log_a\left(\frac{1}{2}\right) - \log_a z$

d $\log_a\left(\frac{x}{a^4}\right) = \log_a x - 4\log_a a = \log_a x - 4$

Anti Laws

These are **NOT LAWS OF LOGS**, but are mistakes students often make:

$$\log_a(b+c) = \log_a b + \log_a c$$

?

✘ FAIL

$$(\log_2 x)^3 = 3\log_2 x$$

?

✘ FAIL

Anti Laws

These are **NOT LAWS OF LOGS**, but are mistakes students often make:

$$\log_a(b+c) = \log_a b + \log_a c$$

There is **no method** to simplify the **log of a sum**, only the sum of two logs!

✘ FAIL

$$(\log_2 x)^3 = 3\log_2 x$$

The power must be on the input (here the x), but here the power is around the entire log.

✘ FAIL

Solving Equations with Logs

Solve the equation $\log_{10} 4 + 2\log_{10} x = 2$

This is a very common type of exam question.
The strategy is to **combine the logs into one** and isolate on one side.

?

Solving Equations with Logs

Solve the equation $\log_{10} 4 + 2\log_{10} x = 2$

This is a very common type of exam question.
The strategy is to **combine the logs into one** and isolate on one side.

$$\log_{10} 4 + \log_{10} x^2 = 2$$
$$\log_{10}(4x^2) = 2$$
$$4x^2 = 10^2$$
$$x^2 = 25$$
$$x = 5$$

We've used the laws of logs to combine them into one.

Use your favourite method of rearranging. Either do "10 the power of each side" to "undo" the log, or the "insert the 2 between the 10 and $4x^2$" method.

The subtle bit: You must check each value in the original equation. If $x = -5$, then we'd have $\log_{10}(-5)$ but we're not allowed to log a negative number.

Test Your Understanding

Edexcel C2 Jan 2013 Q6

Given that $2\log_2(x+15) - \log_2 x = 6$,

(*a*) show that $x^2 - 34x + 225 = 0$.

(5)

(*b*) Hence, or otherwise, solve the equation $2\log_2(x+15) - \log_2 x = 6$.

(2)

a

?

b

?

Test Your Understanding

Edexcel C2 Jan 2013 Q6

Given that $2\log_2(x+15) - \log_2 x = 6$,

(*a*) show that $x^2 - 34x + 225 = 0$.

(5)

(*b*) Hence, or otherwise, solve the equation $2\log_2(x+15) - \log_2 x = 6$.

(2)

a

$$\log_2(x+15)^2 - \log_2 x = 6$$

$$\log_2\left(\frac{(x+15)^2}{x}\right) = 6$$

Those who feel confident with their laws could always skip straight to this line.

$$2^6 = \frac{(x+15)^2}{x}$$

$$64x = (x+15)^2$$

$$64x = x^2 + 30x + 225$$

$$x^2 - 34x + 225 = 0$$

b

$$(x-25)(x-9) = 0$$

$$x = 25\ or\ x = 9$$

These are both valid solutions when substituted into the original equation.

Exercise 14.5

Pearson Pure Mathematics Year 1/AS
Page 117

Extension

1 *[AEA 2010 Q1b]* Solve the equation

$$\log_3(x-7) - \frac{1}{2}\log_3 x = 1 - \log_3 2$$

2 *[AEA 2008 Q5i]* Anna, who is confused about the rules of logarithms, states that

$$(\log_3 p)^2 = \log_3(p^2)$$
$$\log_3(p+q) = \log_3 p + \log_3 q$$

However, there is a value for p and a value for q for which both statements are correct. Find their values.

3 *[MAT 2007 1I]* Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$$

what is the greatest possible value of a?

4 *[MAT 2002 1F]* Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$. From these facts, we can deduce that $\log_2 3$ is:

A) between $1\frac{1}{3}$ and $1\frac{1}{2}$

B) between $1\frac{1}{2}$ and $1\frac{2}{3}$

C) between $1\frac{2}{3}$ and 2

D) none of the above

These are all strictly **non-calculator**!

Solutions on next slide.

Solutions to Extension Exercises

1 [AEA 2010 Q1b] Solve the equation

$$\log_3(x-7) - \frac{1}{2}\log_3 x = 1 - \log_3 2$$

Solution: $\boldsymbol{x = \frac{49}{4}}$

2 [AEA 2008 Q5i] Anna, who is confused about the rules of logarithms, states that

$$(\log_3 p)^2 = \log_3(p^2)$$
$$\log_3(p+q) = \log_3 p + \log_3 q$$

However, there is a value for p and a value for q for which both statements are correct. Find their values.

First equation:

$$\boldsymbol{(\log_2 p)^2 = 2\log_3 p}$$
$$\boldsymbol{\therefore \log_3 p\,(\log_3 p - 2) = 0}$$
$$\boldsymbol{\log_3 p \Rightarrow p = 1}$$
$$\boldsymbol{\log_3 p = 2 \quad \Rightarrow \quad p = 9}$$

Second equation:

$$\boldsymbol{\log_3(p+q) = \log_3(pq)}$$
$$\boldsymbol{\therefore p + q = pq \quad \Rightarrow \quad q = \frac{p}{p-1} \quad \therefore p \neq 1}$$
$$\boldsymbol{p = 9, q = \frac{9}{8}}$$

3 [MAT 2007 1I] Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$$

what is the greatest possible value of a?

To make $\boldsymbol{a}$ as large as possible we make $\boldsymbol{(\log_{10} b)^2}$ as small as possible. Anything squared is at least 0: $\boldsymbol{b = 1}$ will achieve this.

$$\boldsymbol{\therefore 4(\log_{10} a)^2 = 1}$$
$$\boldsymbol{\log_{10} a = \pm\frac{1}{2}}$$
$$\boldsymbol{a = \sqrt{10}}$$

4 [MAT 2002 1F] Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$. From these facts, we can deduce that $\log_2 3$ is:

A) between $1\frac{1}{3}$ and $1\frac{1}{2}$

B) between $1\frac{1}{2}$ and $1\frac{2}{3}$

C) between $1\frac{2}{3}$ and 2

D) none of the above

Suppose that $\boldsymbol{\log_2 3 > \frac{3}{2}}$. Taking 2 to the power of each side:

$$\boldsymbol{3 > 2^{\frac{3}{2}} \quad \rightarrow \quad 3^2 > 2^3 \quad \rightarrow \quad 9 > 8}$$

This is true, so answer is not (A).

Next try $\boldsymbol{\log_2 3 > \frac{5}{3}}$

$$\boldsymbol{3 > 2^{\frac{5}{3}} \quad \rightarrow \quad 3^3 > 2^5 \quad \rightarrow \quad 27 > 32}$$

This is not true, so answer is (B).

Homework Exercise

1 Write as a single logarithm.

a $\log_2 7 + \log_2 3$ **b** $\log_2 36 - \log_2 4$ **c** $3\log_5 2 + \log_5 10$

d $2\log_6 8 - 4\log_6 3$ **e** $\log_{10} 5 + \log_{10} 6 - \log_{10}\left(\frac{1}{4}\right)$

2 Write as a single logarithm, then simplify your answer.

a $\log_2 40 - \log_2 5$ **b** $\log_6 4 + \log_6 9$ **c** $2\log_{12} 3 + 4\log_{12} 2$

d $\log_8 25 + \log_8 10 - 3\log_8 5$ **e** $2\log_{10} 2 - (\log_{10} 5 + \log_{10} 8)$

3 Write in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a(x^3y^4z)$ **b** $\log_a\left(\frac{x^5}{y^2}\right)$ **c** $\log_a(a^2x^2)$

d $\log_a\left(\frac{x}{\sqrt{y}z}\right)$ **e** $\log_a\sqrt{ax}$

4 Solve the following equations:

a $\log_2 3 + \log_2 x = 2$ **b** $\log_6 12 - \log_6 x = 3$

c $2\log_5 x = 1 + \log_5 6$ **d** $2\log_9(x+1) = 2\log_9(2x-3) + 1$

Hint Move the logarithms onto the same side if necessary and use the division law.

Homework Exercise

5 **a** Given that $\log_3(x+1) = 1 + 2\log_3(x-1)$, show that $3x^2 - 7x + 2 = 0$. **(5 marks)**

b Hence, or otherwise, solve $\log_3(x+1) = 1 + 2\log_3(x-1)$. **(2 marks)**

6 Given that a and b are positive constants, and that $a > b$, solve the simultaneous equations

$$a + b = 13$$
$$\log_6 a + \log_6 b = 2$$

Problem-solving

Pay careful attention to the conditions on a and b given in the question.

Challenge

By writing $\log_a x = m$ and $\log_a y = n$, prove that $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$.

Homework Answers

1 **a** $\log_2 21$ **b** $\log_2 9$ **c** $\log_5 80$
d $\log_6 \left(\frac{64}{81}\right)$ **e** $\log_{10} 120$

2 **a** $\log_2 8 = 3$ **b** $\log_6 36 = 2$ **c** $\log_{12} 144 = 2$
d $\log_8 2 = \frac{1}{3}$ **e** $\log_{10} 10 = 1$

3 **a** $3\log_a x + 4\log_a y + \log_a z$
b $5\log_a x - 2\log_a y$
c $2 + 2\log_a x$
d $\log_a x - \frac{1}{2}\log_a y - \log_a z$
e $\frac{1}{2} + \frac{1}{2}\log_a x$

4 **a** $\frac{4}{3}$ **b** $\frac{1}{18}$ **c** $\sqrt{30}$ **d** 2

5 **a** $\log_3 (x + 1) - 2\log_3 (x - 1) = 1$

$$\log_3\left(\frac{x+1}{(x-1)^2}\right) = 1$$

$$\frac{x+1}{(x-1)^2} = 3$$

$x + 1 = 3(x-1)^2$
$x + 1 = 3(x^2 - 2x + 1)$
$3x^2 - 7x + 2 = 0$

b $x = 2$

6 $a = 9, b = 4$

Challenge

$\log_a x = m$ and $\log_a y = n$
$x = a^m$ and $y = a^n$
$x \div y = a^m \div a^n = a^{m-n}$

$$\log_a\left(\frac{x}{y}\right) = m - n = \log_a x - \log_a y$$