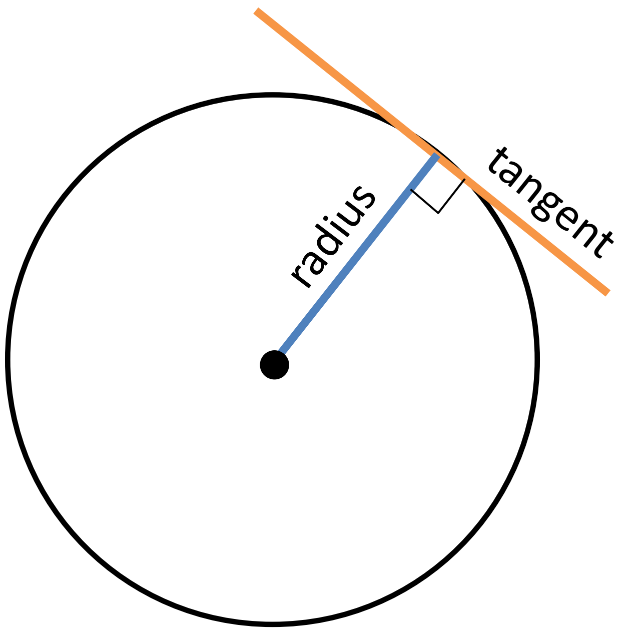
P1 Chapter 6: Circles

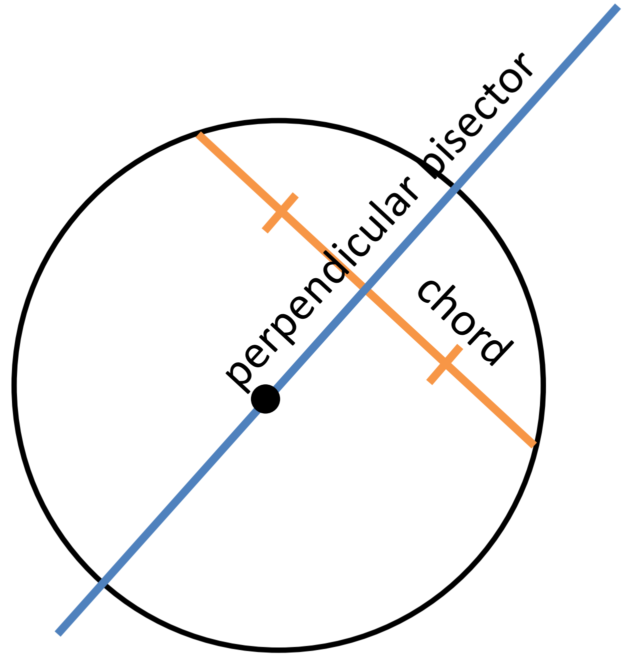
Tangents and Chords

Tangents, Chords, Perpendicular Bisectors

There are two circle theorems that are of particular relevance to problems in this chapter, the latter you might be less familiar with:



The tangent is perpendicular to the radius (at the point of intersection).



The perpendicular bisector of any chord passes through the centre of the circle.

Why this will help:
If we knew the centre of the circle and the point of intersection, we can easily find the gradient of the radius, and thus the gradient and hence equation of the tangent.

Why this will help:
The first thing we did in this chapter is find the equation of the perpendicular bisector. If we had two chords, and hence found two bisectors, we could find the point of intersection, which would be the centre of the circle.

Examples

Note that the GCSE 2015+ syllabus had questions like this, except with circles centred at the origin only.

The circle C has equation
$(x-3)^2+(y-7)^2=100$.

a) Verify the point $P(11,1)$ lies on C.
b) Find an equation of the tangent to C at the point P, giving your answer in the form $ax+by+c=0$

?

A circle C has equation

$$(x-4)^2+(y+4)^2=10$$

The line l is a tangent to the circle and has gradient -3. Find two possible equations for l, giving your answers in the form $y=mx+c$.

?

Examples

Note that the GCSE 2015+ syllabus had questions like this, except with circles centred at the origin only.

The circle C has equation
$(x-3)^2 + (y-7)^2 = 100$.

a) Verify the point $P(11,1)$ lies on C.

b) Find an equation of the tangent to C at the point P, giving your answer in the form $ax + by + c = 0$

a) $(11-3)^2 + (1-7)^2$
$= 8^2 + 6^2 = 100$

b) Circle centre: $(3,7)$

Gradient of radius:

$$m_r = \frac{1-7}{11-3} = -\frac{3}{4}$$

$$\therefore m_t = \frac{4}{3}$$

$$y - 1 = \frac{4}{3}(x - 11)$$

$$3y - 3 = 4(x - 11)$$

$$3y - 3 = 4x - 44$$

$$4x - 3y - 41 = 0$$

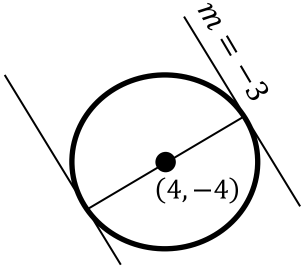
Tip: Use 'subscripting' of variables to make clear to the examiner (and yourself!) what you're calculating.

A circle C has equation

$$(x-4)^2 + (y+4)^2 = 10$$

The line l is a tangent to the circle and has gradient -3. Find two possible equations for l, giving your answers in the form $y = mx + c$.

This time we have the gradient, but don't have the points where the tangent(s) intersect the radius.



Equation of radius/diameter:

$$y + 4 = \frac{1}{3}(x - 4)$$

$$y = \frac{1}{3}x - \frac{16}{3}$$

Intersecting with circle:

$$(x-4)^2 + \left(\frac{1}{3}x - \frac{16}{3} + 4\right)^2 = 10$$

Solving results in the points $(1,-5), (7,-3)$.

Then equations of tangents:

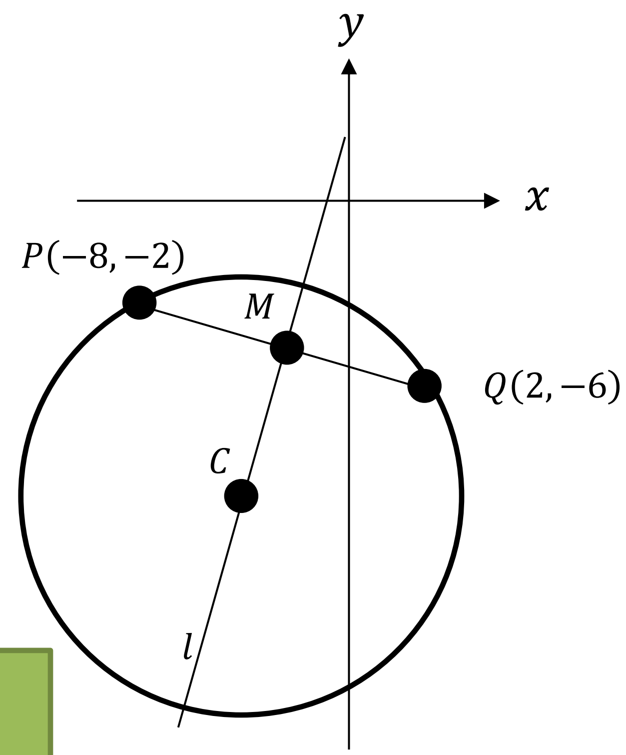
$$y + 5 = -3(x - 1) \rightarrow \boldsymbol{y = -3x - 2}$$

$$y + 3 = -3(x - 7) \rightarrow \boldsymbol{y = -3x + 18}$$

Determining the Circle Centre

The points P and Q lie on a circle with centre C, as shown in the diagram. The point P has coordinates $(-8, -2)$ and the point Q has coordinates $(2, -6)$. M is the midpoint of the line segment PQ.
The line l passes through the points M and C.

a) Find an equation for l.
b) Given that the y-coordinate of C is -9:
   i) show that the x-coordinate of C is -5.
   ii) find an equation of the circle.



a
?

b
?

c
?

Determining the Circle Centre

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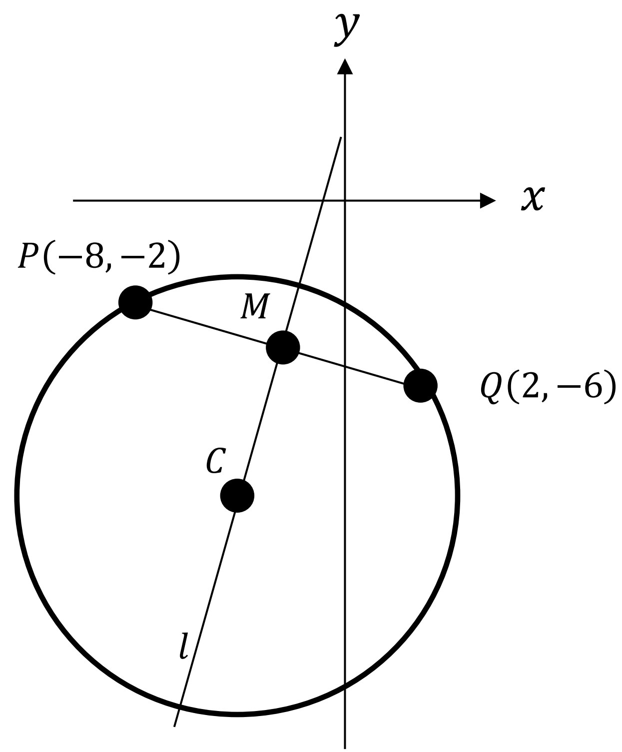

a We know the line going through the midpoint of a chord and a centre of the circle is the perpendicular bisector of the chord.
$M(-3,-4)$
$m_{PQ} = -\frac{4}{10} = -\frac{2}{5} \quad \therefore \quad m_{MC} = \frac{5}{2}$
Equation of l: $\quad y + 4 = \frac{5}{2}(x+3)$

Exam Tip: If you're not asked for the equation in a particular form, just leave it as it is.

b When $y=-9$: $\quad -9+4 = \frac{5}{2}(x+3) \quad \ldots \quad x=-5 \quad \rightarrow \quad C(-5,-9)$

c Use either PC or CQ for the radius.
$r = PC = \sqrt{3^2+7^2} = \sqrt{58}$
$\therefore \quad (x+5)^2 + (y+9)^2 = 58$

Test Your Understanding

A circle has centre $C(3,5)$, and goes through the point $P(6,9)$. Find the equation of the tangent of the circle at the point P, giving your equation in the form $ax + by + c = 0$ where a, b, c are integers..

?

A circle passes through the points $A(0,0)$ and $B(4,2)$. The centre of the circle has x value -1. Determine the equation of the circle.

?

Test Your Understanding

A circle has centre $C(3,5)$, and goes through the point $P(6,9)$. Find the equation of the tangent of the circle at the point P, giving your equation in the form $ax + by + c = 0$ where a, b, c are integers..

$$m_r = \frac{4}{3} \therefore m_t = -\frac{3}{4}$$
$$y - 9 = -\frac{3}{4}(x - 6)$$
$$4y - 36 = -3(x - 6)$$
$$4y - 36 = -3x + 18$$
$$3x + 4y - 54 = 0$$

A circle passes through the points $A(0,0)$ and $B(4,2)$. The centre of the circle has x value -1. Determine the equation of the circle.

$$m_{AB} = \frac{2}{4} = \frac{1}{2} \quad m_{\perp} = -2$$
$$M(2,1)$$

Equation of perpendicular bisector of chord:

$$y - 1 = -2(x - 2)$$

When $x = -1$:

$$y - 1 = -2(-1 - 2)$$
$$y - 1 = 6$$
$$y = 7$$

Centre: $C(-1,7)$

Radius (using length AC): $\sqrt{1^2 + 7^2} = \sqrt{50}$

$$(x + 1)^2 + (y - 7)^2 = 50$$

Exercise 6.4

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Extension:

1 *[MAT 2012 1A]* Which of the following lines is a tangent to the circle with equation
$x^2 + y^2 = 4$?
A) $x + y = 2$
B) $y = x - 2\sqrt{2}$
C) $x = \sqrt{2}$
D) $y = \sqrt{2} - x$

?

Mark scheme on next slide.

2 *[AEA 2006 Q4]* The line with equation $y = mx$ is a tangent to the circle C_1 with equation $(x+4)^2 + (y-7)^2 = 13$
(a) Show that m satisfies the equation $3m^2 + 56m + 36 = 0$
The tangents from the origin O to C_1 touch C_1 at the points A and B.
(b) Find the coordinates of the points A and B.
Another circle C_2 has equation $x^2 + y^2 = 13$. The tangents from the point $(4, -7)$ to C_2 touch it at the points P and Q.
(c) Find the coordinates of either the point P or the point Q.
Mark scheme on next slide.

(This is not a tangent/chord question but is worthwhile regardless!)

3 *[STEP 2005 Q6]*
(i) The point A has coordinates $(5,16)$ and the point B has coordinates $(4, -4)$. The variable P has coordinates (x, y) and moves on a path such that $AP = 2BP$. Show that the Cartesian equation of the path of P is $(x+7)^2 + y^2 = 100$.
(ii) The point C has coordinates $(a, 0)$ and the point D has coordinates $(b, 0)$. The variable point Q moves on a path such that $QC = k \times QD$, where $k > 1$.
Given that the path of Q is the same as the path of P, show that
$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}$$
Show further that $(a+7)(b+7) = 100$, in the case $a \neq b$.

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B) $y = x - 2\sqrt{2}$

C) $x = \sqrt{2}$

D) $y = \sqrt{2} - x$

Solution: B. Note that we could avoid using algebra by using a suitable diagram and simple use of Pythagoras to get the y-intercept.

Mark scheme on next slide.

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Mark Scheme for Extension Question 2

(a) $(x+4)^2+(y-7)^2=13$ and $y=mx$

$\therefore (x^2+8x+16)+(m^2x^2-14mx+49)=13$ M1

$(1+m^2)x^2+(8-14m)x+52=0$ (3 t-quad) A1

Touches, so "$b^2=4ac$" $\quad (8-14m)^2=4\times52\times(1+m^2)$ M1 ($b^2=4ac$)

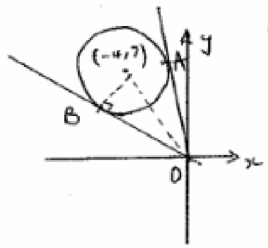
$(4-7m)^2=52+52m^2$

$\therefore \underline{3m^2+56m+36=0}$ * A1 cso

(4)

(b) $(3m+2)(m+18)=0$ M1

$\therefore m=-2/3$ or -18 (both m) A1



Let A or B be (X,Y)

then $(X^2+Y^2)+13=4^2+7^2=65$

$X^2+Y^2=52$ M1, A1

$Y=-\frac{2}{3}X \Rightarrow \frac{13}{9}X^2=52 \Rightarrow X=\pm6$

From the configuration $X_B=-6 \therefore Y_B=+4$: $\underline{B\text{ is }(-6,4)}$ M1, A1

$Y=-18X \Rightarrow 325X^2=52 \Rightarrow X^2=\frac{4}{25}$

Again $X<0$ for A $\therefore X_A=-\frac{2}{5}$; $Y_A=\frac{36}{5}$

$\underline{A=\left(-\frac{2}{5},\frac{36}{5}\right)}$ M1, A1

(8)

(c) Situation is a translation of problem in (b) by $\begin{pmatrix}4\\-7\end{pmatrix}$ M1

So P, Q are $\begin{pmatrix}-6\\4\end{pmatrix}+\begin{pmatrix}4\\-7\end{pmatrix}$ and $\left(-\frac{2}{5},\frac{36}{5}\right)+(4,-7)$

$=\underline{(-2,-3)}$ and $\underline{\left(\frac{18}{5},\frac{1}{5}\right)}$ A1 (either)

(2)

Mark Scheme for Extension Question 3

(i) $PA = 2PB \Rightarrow (x-5)^2 + (y-16)^2 = 4\left((x+4)^2 + (y-4)^2\right)$

$\Rightarrow x^2 + y^2 - 10x - 32y + 281 = 4x^2 + 4y^2 + 32x - 32y + 128$

$\Rightarrow 3x^2 + 3y^2 + 42x - 153 = 0$

$\Rightarrow x^2 + y^2 + 14x - 51 = 0$

$\Rightarrow (x+7)^2 - 49 + y^2 - 51 = 0$

$\Rightarrow (x+7)^2 + y^2 = 100$

which is a circle centre $(-7, 0)$ with radius 10.

(ii) $QC = k \times QD \Rightarrow (x-a)^2 + y^2 = k^2(x-b)^2 + k^2y^2$

$\Rightarrow x^2\left(k^2-1\right) + y^2\left(k^2-1\right) + x\left(2a - 2k^2b\right) + \left(k^2b^2 - a^2\right) = 0$

If this locus is the same as the locus of P, then the ratios of the coefficients must be the same.

$\Rightarrow \dfrac{2a - 2k^2b}{k^2-1} = 14$ and $\dfrac{k^2b^2 - a^2}{k^2-1} = -51.$

Notice that you **cannot** conclude that $k^2 - 1 = 1$.

$\Rightarrow k^2 = \dfrac{a+7}{b+7}$ and $k^2 = \dfrac{a^2+51}{b^2+51}$

$\Rightarrow \dfrac{a+7}{b+7} = \dfrac{a^2+51}{b^2+51}$

$\Rightarrow (a+7)\left(b^2+51\right) = (b+7)\left(a^2+51\right)$

$\Rightarrow ab^2 - a^2b = 7\left(a^2 - b^2\right) + 51(b-a)$

$\Rightarrow ab(b-a) = 7(a-b)(a+b) + 51(b-a)$

$\Rightarrow ab = 51 - 7(a+b)$ since $a \neq b \Rightarrow a - b \neq 0$

$\Rightarrow ab + 7(a+b) = 51$

$\Rightarrow ab + 7(a+b) + 49 = 51 + 49$

$\Rightarrow (a+7)(b+7) = 100$

Homework Exercise

1 The line $x + 3y - 11 = 0$ touches the circle $(x + 1)^2 + (y + 6)^2 = r^2$ at (2, 3).

a Find the radius of the circle.

b Show that the radius at (2, 3) is perpendicular to the line.

2 The point $P(1, -2)$ lies on the circle centre (4, 6).

a Find the equation of the circle.

b Find the equation of the tangent to the circle at P.

3 The points A and B with coordinates $(-1, -9)$ and $(7, -5)$ lie on the circle C with equation $(x - 1)^2 + (y + 3)^2 = 40$.

a Find the equation of the perpendicular bisector of the line segment AB.

b Show that the perpendicular of bisector AB passes through the centre of the circle C.

4 The points P and Q with coordinates (3, 1) and $(5, -3)$ lie on the circle C with equation $x^2 - 4x + y^2 + 4y = 2$.

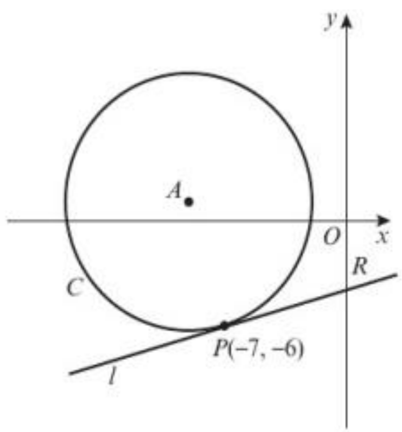
a Find the equation of the perpendicular bisector of the line segment PQ.

b Show that the perpendicular bisector of PQ passes through the centre of the circle C.

5 The circle C has equation $x^2 + 18x + y^2 - 2y + 29 = 0$.

a Verify the point $P(-7, -6)$ lies on C. **(2 marks)**

b Find an equation for the tangent to C at the point P, giving your answer in the form $y = mx + b$. **(4 marks)**



c Find the coordinates of R, the point of intersection of the tangent and the y-axis. **(2 marks)**

d Find the area of the triangle APR. **(2 marks)**

Homework Exercise

6 The tangent to the circle $(x + 4)^2 + (y - 1)^2 = 242$ at $(7, -10)$ meets the y-axis at S and the x-axis at T.

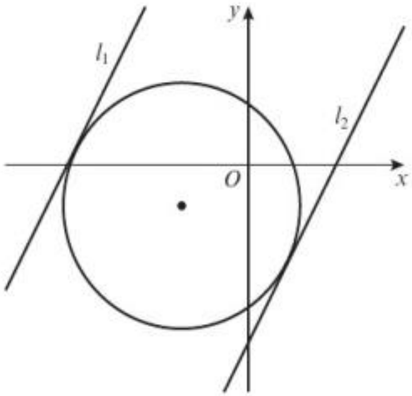
a Find the coordinates of S and T. **(5 marks)**

b Hence, find the area of $\triangle OST$, where O is the origin. **(3 marks)**

7 The circle C has equation $(x + 5)^2 + (y + 3)^2 = 80$.

The line l is a tangent to the circle and has gradient 2.

Find two possible equations for l giving your answers in the form $y = mx + c$. **(8 marks)**



8 The line with equation $2x + y - 5 = 0$ is a tangent to the circle with equation

$(x - 3)^2 + (y - p)^2 = 5$

a Find the two possible values of p. **(8 marks)**

b Write down the coordinates of the centre of the circle in each case. **(2 marks)**

Problem-solving

The line is a tangent to the circle so it must intersect at exactly one point. You can use the discriminant to determine the values of p for which this occurs.

9 The circle C has centre $P(11, -5)$ and passes through the point $Q(5, 3)$.

a Find an equation for C. **(3 marks)**

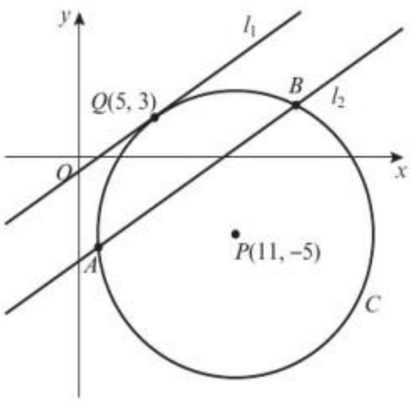
The line l_1 is a tangent to C at the point Q.

b Find an equation for l_1. **(4 marks)**

The line l_2 is parallel to l_1 and passes through the midpoint of PQ. Given that l_2 intersects C at A and B

c find the coordinates of points A and B **(4 marks)**

d find the length of the line segment AB, leaving your answer in its simplest surd form. **(3 marks)**



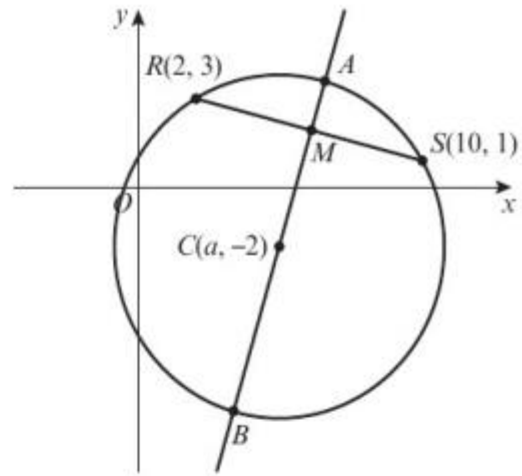
Homework Exercise

10 The points R and S lie on a circle with centre $C(a, -2)$, as shown in the diagram.

The point R has coordinates $(2, 3)$ and the point S has coordinates $(10, 1)$.

M is the midpoint of the line segment RS.

The line l passes through M and C.



a Find an equation for l. **(4 marks)**

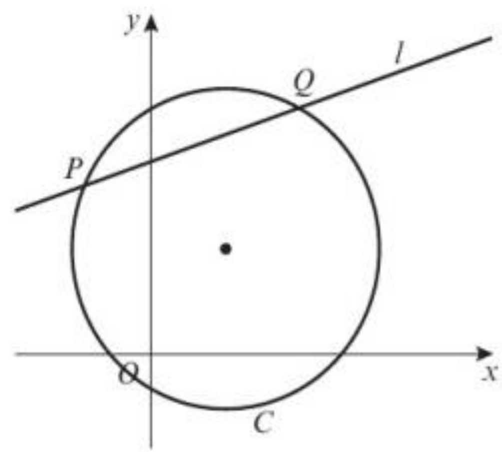
b Find the value of a. **(2 marks)**

c Find the equation of the circle. **(3 marks)**

d Find the points of intersection, A and B, of the line l and the circle. **(5 marks)**

11 The circle C has equation $x^2 - 4x + y^2 - 6y = 7$.

The line l with equation $x - 3y + 17 = 0$ intersects the circle at the points P and Q.



a Find the coordinates of the point P and the point Q. **(4 marks)**

b Find the equation of the tangent at the point P and the point Q. **(4 marks)**

c Find the equation of the perpendicular bisector of the chord PQ. **(3 marks)**

d Show that the two tangents and the perpendicular bisector intersect at a single point and find the coordinates of the point of intersection. **(2 marks)**

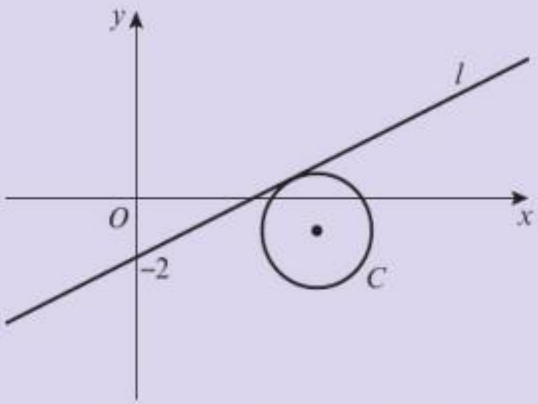
Homework Exercise

Challenge

1 The circle C has equation $(x - 7)^2 + (y + 1)^2 = 5$.

The line l with positive gradient passes through $(0, -2)$ and is a tangent to the circle.

Find an equation of l, giving your answer in the form $y = mx + c$.



2 The circle with centre C has equation $(x - 2)^2 + (y - 1)^2 = 10$.

The tangents to the circle at points P and Q meet at the point R with coordinates $(6, -1)$.

a Show that $CPRQ$ is a square.

b Hence find the equations of both tangents.

Problem-solving

Use the point $(0, -2)$ to write an equation for the tangent in terms of m. Substitute this equation into the equation for the circle.

Homework Answers

1 **a** $3\sqrt{10}$

b Gradient of radius = $3x$, gradient of line = $-\frac{1}{3}$, gradients are negative reciprocals and therefore perpendicular.

2 **a** $(x-4)^2+(y-6)^2=73$ **b** $3x+8y+13=0$

3 **a** $y=-2x-1$

b Centre of circle $(1,-3)$ satisfies $y=-2x-1$.

4 **a** $y=\frac{1}{2}x-3$

b Centre of circle $(2,-2)$ satisfies $y=\frac{1}{2}x-3$

5 **a** $(-7,-6)$ satisfies $x^2+18x+y^2-2y+29=0$

b $y=\frac{2}{7}x-4$ **c** $R(0,-4)$ **d** $\frac{53}{2}$

6 **a** $(0,-17)$, $(17,0)$

b 144.5

7 $y=2x+27$ and $y=2x-13$

8 **a** $p=4$, $p=-6$

b $(3,4)$ and $(3,-6)$

9 **a** $(x-11)^2+(y+5)^2=100$

b $y=\frac{3}{4}x-\frac{3}{4}$

c $A(8-4\sqrt{3},-1-3\sqrt{3})$ and $B(8+4\sqrt{3},-1+3\sqrt{3})$

d $10\sqrt{3}$

10 **a** $y=4x-22$

b $a=5$

c $(x-5)^2+(y+2)^2=34$

d $A(5+\sqrt{2},-2+4\sqrt{2})$ and $B(5-\sqrt{2},-2-4\sqrt{2})$

11 **a** $P(-2,5)$ and $Q(4,7)$

b $y=2x+9$ and $y=-\frac{1}{2}x+9$

c $y=-3x+9$

d $(0,9)$

Challenge

1 $y=\frac{1}{2}x-2$

2 **a** $\angle CPR=\angle CQR=90°$ (Angle between tangent and radius)

$CP=CQ=\sqrt{10}$ (Radii of circle)

$CR=\sqrt{(6-2)^2+(-1-1)^2}=\sqrt{20}$

So using Pythagoras' Theorem,

$PR=QR=\sqrt{20-10}=\sqrt{10}$

4 equal sides and two opposite right–angles, so $CPRQ$ is a square

b $y=\frac{1}{3}x-3$ and $y=-3x+17$