P1 Chapter 14: Logarithms

Logarithms

Logarithms

You know the inverse of many mathematical operations; we can undo an addition by 2 for example by subtracting 2. But is there an inverse function for an exponential function?

Function **Inverse**

$$4 \longrightarrow \boxed{x \times 3} \xrightarrow{12} \boxed{x \div 3} \longrightarrow 4$$

$$4 \longrightarrow \boxed{x + 3} \xrightarrow{7} \boxed{x - 3} \longrightarrow 4$$

$$4 \longrightarrow \boxed{x^2} \xrightarrow{16} \boxed{\sqrt{x}} \longrightarrow 4$$

$$4 \longrightarrow \boxed{x^5} \xrightarrow{1024} \boxed{\sqrt[5]{x}} \longrightarrow 4$$

$$4 \longrightarrow \boxed{3^x} \xrightarrow{81} \boxed{\log_3 x} \longrightarrow 4$$

Such functions are known as **logarithms** and exist to provide an inverse to exponential functions.

Interchanging between exponential and log form

✎ $\log_a n$ ("said log base a of n") is equivalent to $a^x = n$.
The log function outputs the **missing power**.

$$3^2 = 9 \quad \Longleftrightarrow \quad \log_3 9 = 2$$

Here are two methods of interchanging between these forms.
Pick your favourite!

Method 1: 'Missing Power'

- Note first the base of the log must match the base of the exponential.
- $\log_2 8$ for example asks the question "2 to **what power** gives 8?"

We can imagine inserting the output of the log just after the base. Click the button!

$$log_2 \;\; 8 = 3$$

Method 2: Do same operation to each side of equation.

Since KS3 you're used to the idea of doing the same thing to each side of the equation that 'undoes' whatever you want to get rid of.

$$3x + 2 = 11$$
$$(-2) \qquad (-2)$$
$$3x = 9$$

"log base a" undoes "a to the power of" and vice versa, as they are inverse functions.

$$\log_2 8 = 3$$
$$(2^{\square}) \qquad (2^{\square})$$
$$8 = 2^3$$

Examples

$\log_5 25 = $?

Think: "5 to the power of what gives you 25?"

$\log_3 81 = $?

$\log_2 32 = $?

$\log_{10} 1000 = $?

$\log_4 1 = $?

✎ $\log_a 1 = 0$ for all a.

$\log_4 4 = $?

$\log_2 \left(\frac{1}{2}\right) = $?

$\log_3 \left(\frac{1}{27}\right) = $?

$\log_2 \left(\frac{1}{16}\right) = $?

$\log_a (a^3) = $?

$\log_4 (-1) = $?!!??

While a log can output a negative number, we **can't log negative numbers**.

Strictly Just For Your Interest: However, if we were to expand the range (i.e. output) of the log function to allow *complex numbers* (known as the '*complex logarithm*'), then we in fact get $\log_4(-1) = \frac{i\pi}{\ln(4)}$. It's probably better if you purge these last few sentences from your memory and move along…

Examples

$$\log_5 25 = \mathbf{2}$$

Think: "5 to the power of what gives you 25?"

$$\log_3 81 = \mathbf{4}$$

$$\log_2 32 = \mathbf{5}$$

$$\log_{10} 1000 = \mathbf{3}$$

$$\log_4 1 = \mathbf{0}$$

✎ $\log_a 1 = 0$ for all a.

$$\log_4 4 = \mathbf{1}$$

$$\log_2\left(\frac{1}{2}\right) = \mathbf{-1}$$

$$\log_3\left(\frac{1}{27}\right) = \mathbf{-3}$$

$$\log_2\left(\frac{1}{16}\right) = \mathbf{-4}$$

$$\log_a(a^3) = \mathbf{3}$$

$\log_4(-1) =$ No real answer

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With Your Calculator

There are three buttons on your calculator for computing logs:

$\log_{\square} \square$

$\log_3 7 =$?

$\log_5 0.3 =$?

I couldn't think of a word that rhymed with 'ln' so I recorded it for you.

ln of x

ln

$\ln 10 =$?

$\ln e =$?

ln is the "**natural log of x**", meaning "log to the base e", i.e. it the inverse of e^x.

$$\ln(x) = \log_e(x)$$

We will use it more extensively later this chapter.

log

$\log 100 =$?

Just like the $\sqrt{}$ symbol without a number is $\sqrt[2]{\square}$ by default, log without a base is **base 10** by default when used on your calculator (although confusingly "log" can mean "ln" in mathematical papers)

With Your Calculator

There are three buttons on your calculator for computing logs:

$\log_{\square} \square$

$$\log_3 7 = \mathbf{1.77124}\ldots$$
$$\log_5 0.3 = -\mathbf{0.74807}\ldots$$

I couldn't think of a word that rhymed with 'ln' so I recorded it for you.

ln of x

ln

$$\ln 10 = \mathbf{2.30258}\ldots$$
$$\ln e = \mathbf{1}$$

ln is the "**natural log of x**", meaning "log to the base e", i.e. it the inverse of e^x.

$$\ln(x) = \log_e(x)$$

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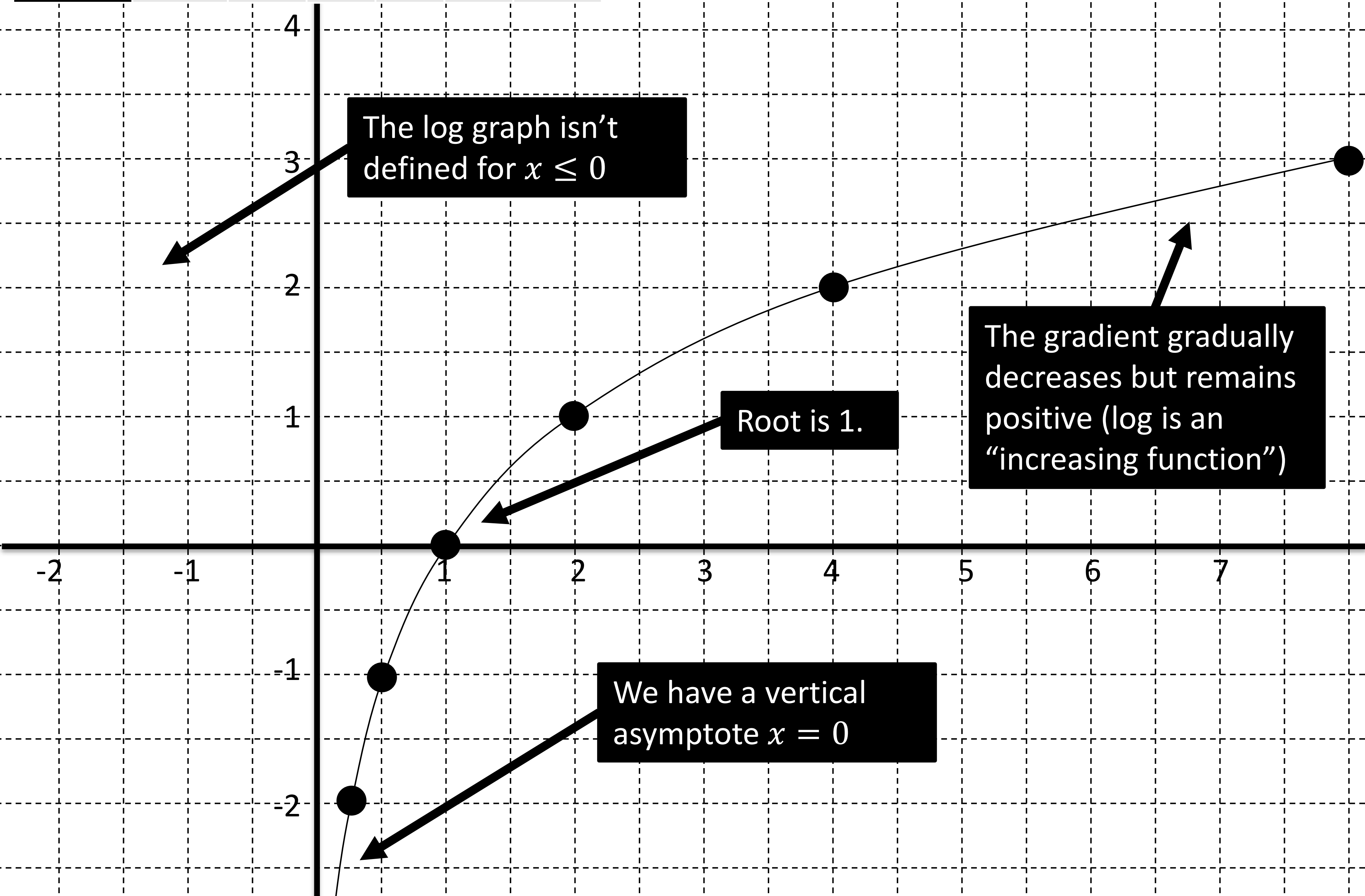
log

$$\log 100 = \mathbf{2}$$

Just like the $\sqrt{}$ symbol without a number is $\sqrt[2]{\square}$ by default, log without a base is **base 10** by default when used on your calculator (although confusingly "log" can mean "ln" in mathematical papers)

x	0.25	0.5	1	2	4	8
y	-2	-1	0	1	2	3

$y = \log_2 x$



Exercise 14.4

Pearson Pure Mathematics Year 1/AS
Page 116

Extension

1 [MAT 2015 1J] Which is the largest of the following numbers?

A) $\frac{\sqrt{7}}{2}$ B) $\frac{5}{4}$ C) $\frac{\sqrt{10!}}{3(6!)}$

D) $\frac{\log_2 30}{\log_3 85}$ E) $\frac{1+\sqrt{6}}{3}$

Non-calculator!

?

2 *[MAT 2013 1F]* Three *positive* numbers a, b, c satisfy

$$\log_b a = 2$$
$$\log_b(c-3) = 3$$
$$\log_a(c+5) = 2$$

This information:

A) specifies a uniquely;

B) is satisfied by two values of a;

C) is satisfied by infinitely many values of a;

D) is contradictory

?

Exercise 14.4

Pearson Pure Mathematics Year 1/AS
Page 116

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Non-calculator!

(Official solution) Squaring all answers results in (A) $\frac{7}{4}$ which is larger than (B) $\frac{25}{16}$. After squaring (C) it simplifies to $\frac{10\times9\times8\times7}{9\times6!}$ which further simplifies to $\frac{7}{9}$ which is smaller than (A).
$\log_2 30 \approx 5$ and $\log_3(85) \approx 4$, hence (D) is smaller than (A) after squaring. Comparing (A) with (E) after squaring results in a comparison of $\frac{7}{4}$ and $\frac{7+2\sqrt{6}}{9}$. As $2 < \sqrt{6} < 3$, (E) squared must be less than $\frac{13}{9}$ and hence less than $\frac{7}{4}$. The answer is (A).

2 *[MAT 2013 1F]* Three *positive* numbers a, b, c satisfy

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This information:

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If we take exponents of the three equations:

$$a = b^2, \qquad c-3 = b^3, \qquad c+5 = a^2$$

Hence eliminating a and c we get

$$b^3 + 3 = b^4 - 5 \Rightarrow b^3(b-1) = 8$$

We are only interested in positive solutions to b. Note that $b^3(b-1)$ is negative for $0 < b < 1$ and then both b^3 and $b-1$ are positive and increasing for $b > 1$. So there is only one positive solution to the equation ($b = 2$, so that $a = 4, c = 11$). Answer is (a).

Homework Exercise

1 Rewrite using a logarithm.

a $4^4 = 256$ **b** $3^{-2} = \frac{1}{9}$ **c** $10^6 = 1\,000\,000$

d $11^1 = 11$ **e** $(0.2)^3 = 0.008$

2 Rewrite using a power.

a $\log_2 16 = 4$ **b** $\log_5 25 = 2$ **c** $\log_9 3 = \frac{1}{2}$

d $\log_5 0.2 = -1$ **e** $\log_{10} 100\,000 = 5$

3 Without using a calculator, find the value of

a $\log_2 8$ **b** $\log_5 25$ **c** $\log_{10} 10\,000\,000$ **d** $\log_{12} 12$

e $\log_3 729$ **f** $\log_{10} \sqrt{10}$ **g** $\log_4 (0.25)$ **h** $\log_{0.25} 16$

i $\log_a (a^{10})$ **j** $\log_{\frac{2}{3}} \left(\frac{9}{4}\right)$

4 Without using a calculator, find the value of x for which

a $\log_5 x = 4$ **b** $\log_x 81 = 2$ **c** $\log_7 x = 1$

d $\log_2 (x - 1) = 3$ **e** $\log_3 (4x + 1) = 4$ **f** $\log_x (2x) = 2$

5 Use your calculator to evaluate these logarithms to three decimal places.

a $\log_9 230$ **b** $\log_5 33$ **c** $\log_{10} 1020$ **d** $\log_e 3$

Homework Exercise

Hint Use corresponding statements involving powers of 2.

6 **a** Without using a calculator, justify why the value of $\log_2 50$ must be between 5 and 6.

b Use a calculator to find the exact value of $\log_2 50$ to 4 significant figures.

7 **a** Find the values of:

i $\log_2 2$ **ii** $\log_3 3$ **iii** $\log_{17} 17$

b Explain why $\log_a a$ has the same value for all positive values of a ($a \neq 1$).

8 **a** Find the values of:

i $\log_2 1$ **ii** $\log_3 1$ **iii** $\log_{17} 1$

b Explain why $\log_a 1$ has the same value for all positive values of a ($a \neq 1$).

Homework Answers

1	a	$\log_4 256 = 4$			b	$\log_3 \frac{1}{9} = -2$			
	c	$\log_{10} 1\,000\,000 = 6$			d	$\log_{11} 11 = 1$			
	e	$\log_{0.2} 0.008 = 3$							
2	a	$2^4 = 16$	b	$5^2 = 25$					
	c	$9^{\frac{1}{2}} = 3$	d	$5^{-1} = 0.2$					
	e	$10^5 = 100\,000$							
3	a	3	b	2	c	7	d	1	
	e	6	f	$\frac{1}{2}$	g	-1	h	-2	
	i	10	j	-2					
4	a	625	b	9	c	7	d	9	
	e	20	f	2					
5	a	2.475	b	2.173	c	3.009	d	1.099	
6	a	$5 = \log_2 32 < \log_2 50 < \log_2 64 = 6$							
	b	5.644							
7	a	i 1	ii 1	iii 1	b	$a^1 \equiv a$			
8	a	i 0	ii 0	iii 0	b	$a^0 \equiv 1$			