P2 Chapter 7: Trigonometric Equations

Trigonometric Solutions

Solving Trigonometric Equations

This is effectively the same type of question you encountered in Chapter 6 and in Year 1, except you may need to use either the **addition formulae or double angle formulae.**

[Textbook] Solve $3\cos 2x - \cos x + 2 = 0$ for $0 \leq x \leq 360°$.

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[Textbook] Solve $3\cos 2x - \cos x + 2 = 0$ for $0 \leq x \leq 360°$.

$$3(2\cos^2 x - 1) - \cos x + 2 = 0$$
$$6\cos^2 x - 3 - \cos x + 2 = 0$$
$$6\cos^2 x - \cos x - 1 = 0$$
$$(3\cos x + 1)(2\cos x - 1) = 0$$
$$\cos x = -\frac{1}{3} \; or \; \cos x = \frac{1}{2}$$
$$x = 109.5°, 250.5° \qquad x = 60°, 300°$$

Recall that we have a choice of double angle identities for $\cos 2x$. This is clearly the most sensible choice as we end up with an equation just in terms of cos.

Further Examples

[Textbook] By noting that $3A = 2A + A$, :

a) Show that $\sin(3A) = 3\sin A - 4\sin^3 A$.

b) Hence or otherwise, solve, for $0 < \theta < 2\pi$, the equation $16\sin^3\theta - 12\sin\theta - 2\sqrt{3} = 0$

Exam Note: A question pretty much just like this came up in an exam once.

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[Textbook] Solve $4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$ in the range $0 \leq \theta < 360°$.

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$$\sin(2A + A)$$
$$= \sin 2A \cos A + \cos 2A \sin A$$
$$= 2\sin A \cos A \cos A + (1 - 2\sin^2 A)\sin A$$
$$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$$
$$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$$
$$= 3\sin A - 4\sin^3 A$$

Note that $16\,sin^3\theta - 12\,sin\,\theta$
$$= -4(3\sin A - 4\sin^3 A) = -4\sin 3\theta.$$
$$\therefore -4\sin 3\theta = 2\sqrt{3}$$
$$\sin 3\theta = -\frac{\sqrt{3}}{2}$$
$$3\theta = \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}, \frac{16\pi}{3}, \frac{17\pi}{3}$$
$$\theta = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$$

[Textbook] Solve $4\cos(\theta - 30°) = 8\sqrt{2}\sin\theta$ in the range $0 \leq \theta < 360°$.

Your instinct should be to use the addition formulae first to break up the $4\,cos(\theta - 30°)$.

$$4\cos\theta\cos 30° + 4\sin\theta\sin 30° = 8\sqrt{2}\sin\theta$$
$$2\sqrt{3}\cos\theta + 2\sin\theta = 8\sqrt{2}\sin\theta$$
$$2\sqrt{3}\cos\theta = 8\sqrt{2}\sin\theta - 2\sin\theta$$
$$2\sqrt{3}\cos\theta = (8\sqrt{2} - 2)\sin\theta$$
$$\frac{2\sqrt{3}}{8\sqrt{2} - 2} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$
$$\theta = 20.4°, 200.4°$$

Test Your Understanding

Edexcel C3 Jan 2013 Q6

6. (i) Without using a calculator, find the exact value of

$$(\sin 22.5° + \cos 22.5°)^2.$$

You must show each stage of your working.

(5)

(ii) (*a*) Show that $\cos 2\theta + \sin \theta = 1$ may be written in the form

$$k \sin^2 \theta - \sin \theta = 0, \text{ stating the value of } k.$$

(2)

(*b*) Hence solve, for $0 \leq \theta < 360°$, the equation

$$\cos 2\theta + \sin \theta = 1.$$

(4)

?

If you finish that quickly…

[Textbook] Solve $2 \tan 2y \tan y = 3$ for $0 \leq y < 2\pi$, giving your answer to 2dp.

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(*b*) Hence solve, for $0 \leq \theta < 360°$, the equation

$$\cos 2\theta + \sin\theta = 1.$$ **(4)**

6.	(i) $(\sin 22.5 + \cos 22.5)^2 = \sin^2 22.5 + \cos^2 22.5 + \ldots\ldots$	M1
	$= \sin^2 22.5 + \cos^2 22.5 + 2\sin 22.5\cos 22.5$	
	States or uses $\sin^2 22.5 + \cos^2 22.5 = 1$	B1
	Uses $2\sin x\cos x = \sin 2x \Rightarrow 2\sin 22.5\cos 22.5 = \sin 45$	M1
	$(\sin 22.5 + \cos 22.5)^2 = 1 + \sin 45$	A1
	$= 1 + \frac{\sqrt{2}}{2}$ or $1 + \frac{1}{\sqrt{2}}$ cso	A1
	(ii) (a) $\cos 2\theta + \sin\theta = 1 \Rightarrow 1 - 2\sin^2\theta + \sin\theta = 1$	M1
	$\sin\theta - 2\sin^2\theta = 0$	
	$2\sin^2\theta - \sin\theta = 0$ or $k = 2$	A1*
	(b) $\sin\theta(2\sin\theta - 1) = 0$	M1
	$\sin\theta = 0,\ \ \sin\theta = \frac{1}{2}$	A1
	Any two of 0,30,150,180	B1
	All four answers 0,30,150,180	A1

If you finish that quickly…

[Textbook] Solve $2\tan 2y\tan y = 3$ for $0 \leq y < 2\pi$, giving your answer to 2dp.

$$2\tan 2y\tan y = 3$$

$$2\left(\frac{2\tan y}{1 - \tan^2 y}\right)\tan y = 3$$

…

$$7\tan^2 y = 3$$

$$\tan y = \pm\sqrt{\frac{3}{7}}$$

$$y = 0.58, 2.56, 3.72, 5.70$$

Forgetting the $\pm$ is an incredibly common source of lost marks.

Exercise 7.4

Pearson Pure Mathematics Year 2/AS
Page 51

Extension

[AEA 2013 Q2]

1 (a) Use the formula for $\sin(A-B)$ to show that $\sin(90° - x) = \cos x$ **(1)**

(b) Solve for $0 < \theta < 360°$

$$2\sin(\theta + 17°) = \frac{\cos(\theta + 8°)}{\cos(\theta + 17°)}$$

(7)

?

2 [AEA 2009 Q3]

(a) Solve, for $0 \leqslant \theta < 2\pi$,

$$\sin\left(\frac{\pi}{3} - \theta\right) = \frac{1}{\sqrt{3}}\cos\theta .$$

(5)

(b) Find the value of x for which

$$\arcsin(1 - 2x) = \frac{\pi}{3} - \arcsin x, \qquad 0 < x < 0.5$$

[$\arcsin x$ is an alternative notation for $\sin^{-1} x$]

(7)

(Solution on next slide)

Exercise 7.4

Pearson Pure Mathematics Year 2/AS
Page 51

Extension

[AEA 2013 Q2]

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$$2\sin(\theta + 17°) = \frac{\cos(\theta + 8°)}{\cos(\theta + 17°)}$$

(7)

Question	Scheme	Marks	Notes
(a)	$\sin(90 - x) = \sin 90 \cos x - \cos 90 \sin x = 1.\cos x - 0.\sin x = \cos x$	B1 (1)	One intermediate line
(b)	$2\sin(\theta+17)\cos(\theta+17) = \cos(\theta+8) \Rightarrow \sin[2(\theta+17)] = \cos(\theta+8)$	M1	Use of $\sin 2A = \ldots$
	$2\theta + 34 = 90 - (\theta + 8)$	dM1	Use of (a) – not trigθ
	$3\theta = 82 - 34 = 48$ so $\underline{\theta = 16}$	A1	
	$2\theta + 34 = 180 - [90 - (\theta+8)]$ or $2\theta + 34 = [90 - (\theta + 8)] + 360$	M1	2nd eqn for θ
	$\theta = 98 - 34$ or $\underline{\theta = 64}$	A1	
	$3\theta = 48 + 460$ $\underline{\theta = 136}$	A1	
	$\underline{\theta = 256}$	A1 (7)	
NB	$\sin(2\theta+34) - \sin(82 - \theta)$ gives $2\cos[(\theta+116)/2]\sin[(3\theta - 48)/2]$ Then: $\theta/2 + 58 = 90$ gets M1 and e.g. $3\theta/2 - 24 = 0$ gets M1	**(8)**	

2 [AEA 2009 Q3]

(a) Solve, for $0 \leqslant \theta < 2\pi$,

$$\sin\left(\frac{\pi}{3} - \theta\right) = \frac{1}{\sqrt{3}}\cos\theta .$$

(5)

(b) Find the value of x for which

$$\arcsin(1 - 2x) = \frac{\pi}{3} - \arcsin x, \qquad 0 < x < 0.5$$

[$\arcsin x$ is an alternative notation for $\sin^{-1} x$]

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(Solution on next slide)

Solution to Extension Question 2

(a) Solve, for $0 \leqslant \theta < 2\pi$,

$$\sin\left(\frac{\pi}{3}-\theta\right)=\frac{1}{\sqrt{3}}\cos\theta .$$

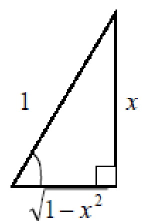
(5)

(b) Find the value of x for which

$$\arcsin(1-2x)=\frac{\pi}{3}-\arcsin x, \qquad 0<x<0.5$$

[$\arcsin x$ is an alternative notation for $\sin^{-1}x$]

(7)

(a)	$\sin\frac{\pi}{3}\cos\theta-\cos\frac{\pi}{3}\sin\theta \quad =\frac{1}{\sqrt{3}}\cos\theta$	M1	Use of $\sin(A-B)$
	$\frac{1}{\sqrt{3}}\cos\theta \quad =\sin\theta$ (o.e)	M1	Use of $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$ and collect terms
	$\Rightarrow \quad \tan\theta=\frac{1}{\sqrt{3}}$	A1	$\tan\theta=\frac{1}{\sqrt{3}}$ oe.
	$\theta=\frac{\pi}{6},\frac{7\pi}{6}$	A1, B1$\sqrt{}$	
		(5)	
(b)	$\sin[\arcsin(1-2x)]=\sin\left[\frac{\pi}{3}-\arcsin x\right]$		
	$\sin[\arcsin(1-2x)]=\sin\frac{\pi}{3}\cos[\arcsin x]-\cos\frac{\pi}{3}\sin(\arcsin x)$	M1	Use of $\sin(A\pm B)$
	$1-2x=\frac{\sqrt{3}}{2}\sqrt{1-x^2}-\frac{1}{2}x$	M1,B1	B1 for $\cos[\arcsin x]=\sqrt{1-x^2}$
	 $[2-3x=\sqrt{3}\sqrt{1-x^2}]$		
	$4-12x+9x^2=3-3x^2$	M1	Simplify to quadratic in x
	$12x^2-12x+1$ (=0)	A1	correct 3TQ
	$x=\frac{12\pm\sqrt{144-48}}{24}$	M1	Attempt to solve if at least one previous M scored in (b)
	$x=\frac{3\pm\sqrt{6}}{6}$		
	$\because \quad 0<x<0.5 \quad x=\frac{3-\sqrt{6}}{6}$ (o.e.)	A1	Must choose '_'
		(7)	

Homework Exercise

1 Solve, in the interval $0 \leqslant \theta < 360°$, the following equations. Give your answers to 1 d.p.

a $3\cos\theta = 2\sin(\theta + 60°)$

b $\sin(\theta + 30°) + 2\sin\theta = 0$

c $\cos(\theta + 25°) + \sin(\theta + 65°) = 1$

d $\cos\theta = \cos(\theta + 60°)$

2 **a** Show that $\sin\left(\theta + \frac{\pi}{4}\right) \equiv \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$ **(2 marks)**

b Hence, or otherwise, solve the equation $\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta) = \frac{1}{\sqrt{2}}$, $0 \leqslant \theta \leqslant 2\pi$. **(4 marks)**

c Use your answer to part **b** to write down the solutions to $\sin\theta + \cos\theta = 1$ over the same interval. **(2 marks)**

3 **a** Solve the equation $\cos\theta\cos 30° - \sin\theta\sin 30° = 0.5$, for $0 \leqslant \theta \leqslant 360°$.

b Hence write down, in the same interval, the solutions of $\sqrt{3}\cos\theta - \sin\theta = 1$.

4 **a** Given that $3\sin(x - y) - \sin(x + y) = 0$, show that $\tan x = 2\tan y$.

b Solve $3\sin(x - 45°) - \sin(x + 45°) = 0$, for $0 \leqslant x \leqslant 360°$.

5 Solve the following equations, in the intervals given.

a $\sin 2\theta = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$

b $\cos 2\theta = 1 - \cos\theta$, $-180° < \theta \leqslant 180°$

c $3\cos 2\theta = 2\cos^2\theta$, $0 \leqslant \theta < 360°$

d $\sin 4\theta = \cos 2\theta$, $0 \leqslant \theta \leqslant \pi$

e $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0$, $0 \leqslant \theta < 720°$

f $\cos^2\theta - \sin 2\theta = \sin^2\theta$, $0 \leqslant \theta \leqslant \pi$

g $2\sin\theta = \sec\theta$, $0 \leqslant \theta \leqslant 2\pi$

h $2\sin 2\theta = 3\tan\theta$, $0 \leqslant \theta < 360°$

i $2\tan\theta = \sqrt{3}(1 - \tan\theta)(1 + \tan\theta)$, $0 \leqslant \theta \leqslant 2\pi$

j $\sin^2\theta = 2\sin 2\theta$, $-180° < \theta < 180°$

k $4\tan\theta = \tan 2\theta$, $0 \leqslant \theta \leqslant 360°$

Homework Exercise

6 In $\triangle ABC$, $AB = 4$ cm, $AC = 5$ cm, $\angle ABC = 2\theta$ and $\angle ACB = \theta$. Find the value of θ, giving your answer, in degrees, to 1 decimal place. **(4 marks)**

7 a Show that $5\sin 2\theta + 4\sin\theta = 0$ can be written in the form $a\sin\theta\,(b\cos\theta + c) = 0$, stating the values of a, b and c. **(2 marks)**

b Hence solve, for $0 \leqslant \theta < 360°$, the equation $5\sin 2\theta + 4\sin\theta = 0$. **(4 marks)**

8 a Given that $\sin 2\theta + \cos 2\theta = 1$, show that $2\sin\theta\,(\cos\theta - \sin\theta) = 0$. **(2 marks)**

b Hence, or otherwise, solve the equation $\sin 2\theta + \cos 2\theta = 1$ for $0 \leqslant \theta < 360°$. **(4 marks)**

9 a Prove that $(\cos 2\theta - \sin 2\theta)^2 \equiv 1 - \sin 4\theta$. **(4 marks)**

b Use the result to solve, for $0 \leqslant \theta < \pi$, the equation $\cos 2\theta - \sin 2\theta = \frac{1}{\sqrt{2}}$

Give your answers in terms of π. **(3 marks)**

10 a Show that:

i $\sin\theta \equiv \dfrac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$ **ii** $\cos\theta \equiv \dfrac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$

b By writing the following equations as quadratics in $\tan\frac{\theta}{2}$, solve, in the interval $0 \leqslant \theta \leqslant 360°$:

i $\sin\theta + 2\cos\theta = 1$ **ii** $3\cos\theta - 4\sin\theta = 2$

Homework Exercise

11 **a** Show that $3\cos^2 x - \sin^2 x \equiv 1 + 2\cos 2x$. **(3 marks)**

b Hence sketch, for $-\pi \leqslant x \leqslant \pi$, the graph of $y = 3\cos^2 x - \sin^2 x$, showing the coordinates of points where the curve meets the axes. **(3 marks)**

12 **a** Express $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2}$ in the form $a\cos\theta + b$, where a and b are constants. **(4 marks)**

b Hence solve $2\cos^2\frac{\theta}{2} - 4\sin^2\frac{\theta}{2} = -3$, in the interval $0 \leqslant \theta < 360°$. **(3 marks)**

13 **a** Use the identity $\sin^2 A + \cos^2 A \equiv 1$ to show that $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$. **(5 marks)**

b Deduce that $\sin^4 A + \cos^4 A \equiv \frac{1}{4}(3 + \cos 4A)$. **(3 marks)**

c Hence solve $8\sin^4\theta + 8\cos^4\theta = 7$, for $0 < \theta < \pi$. **(3 marks)**

Hint Start by squaring $(\sin^2 A + \cos^2 A)$.

14 **a** By writing 3θ as $2\theta + \theta$, show that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. **(4 marks)**

b Hence, or otherwise, for $0 < \theta < \pi$, solve $6\cos\theta - 8\cos^3\theta + 1 = 0$ giving your answer in terms of π. **(5 marks)**

Homework Answers

1 **a** 51.7°, 231.7° **b** 170.1°, 350.1°
c 56.5°, 303.5° **d** 150°, 330°

2 **a** $\sin\left(\theta + \frac{\pi}{4}\right) \equiv \sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}$

$$\equiv \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta \equiv \frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)$$

b $0, \frac{\pi}{2}, 2\pi$ **c** $0, \frac{\pi}{2}, 2\pi$

3 **a** 30°, 270° **b** 30°, 270°

4 **a** $3(\sin x\cos y - \cos x\sin y)$
$-(\sin x\cos y + \cos x\sin y) = 0$
$\Rightarrow 2\sin x\cos y - 4\cos x\sin y = 0$
Divide throughout by $2\cos x\cos y$
$\Rightarrow \tan x - 2\tan y = 0$, so $\tan x = 2\tan y$

b Using **a** $\tan x = 2\tan y = 2\tan 45° = 2$
so $x = 63.4°, 243.4°$

5 **a** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ **b** ±38.7°
c 30°, 150°, 210°, 330° **d** $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}$
e 60°, 300°, 443.6°, 636.4° **f** $\frac{\pi}{8}, \frac{5\pi}{8}$
g $\frac{\pi}{4}, \frac{5\pi}{4}$
h 0°, 30°, 150°, 180°, 210°, 330° **i** $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$
j −104.0°, 0°, 76.0°
k 0°, 35.3°, 144.7°, 180°, 215.3°, 324.7°, 360°

6 51.3°

7 **a** $5\sin 2\theta = 10\sin\theta\cos\theta$, so equation becomes
$10\sin\theta\cos\theta + 4\sin\theta = 0$, or $2\sin\theta(5\cos\theta + 2) = 0$

b 0°, 180°, 113.6°, 246.4°

8 **a** $2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta = 1$
$\Rightarrow 2\sin\theta\cos\theta - 2\sin^2\theta = 0$
$\Rightarrow 2\sin\theta(\cos\theta - \sin\theta) = 0$

b 0°, 180°, 45°, 225°

9 **a** L.H.S. $= \cos^2 2\theta + \sin^2 2\theta - 2\sin 2\theta\cos 2\theta$
$= 1 - \sin 4\theta =$ R.H.S.

b $\frac{\pi}{24}, \frac{17\pi}{24}$

10 **a** **i** R.H.S. $= \dfrac{2\tan\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)} = 2\dfrac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} \times \dfrac{\cos^2\left(\frac{\theta}{2}\right)}{1}$

$= 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \sin\theta$

ii R.H.S. $= \dfrac{1 - \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1 - \tan^2\left(\frac{\theta}{2}\right)}{\sec^2\left(\frac{\theta}{2}\right)}$

$= \cos^2\left(\frac{\theta}{2}\right)\left\{1 - \tan^2\left(\frac{\theta}{2}\right)\right\} = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$

$= \cos\theta =$ L.H.S.

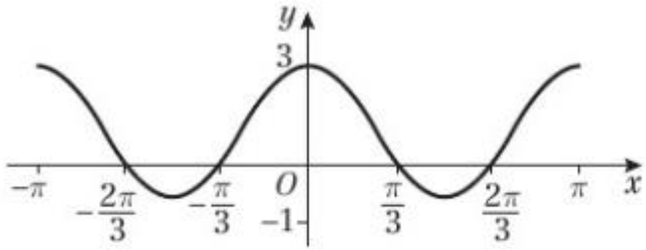
b **i** 90°, 323.1° **ii** 13.3°, 240.4°

Homework Answers

11 a L.H.S. $\equiv \dfrac{3(1+\cos 2x)}{2} - \dfrac{(1-\cos 2x)}{2}$

$\equiv 1 + 2\cos 2x$

b



Crosses y-axis at $(0, 3)$

Crosses x-axis at $\left(-\frac{2\pi}{3}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{3}, 0\right), \left(\frac{2\pi}{3}, 0\right)$

12 a $2\cos^2\left(\frac{\theta}{2}\right) - 4\sin^2\left(\frac{\theta}{2}\right) = 2\left(\frac{1+\cos\theta}{2}\right) - 4\left(\frac{1-\cos\theta}{2}\right)$

$= 1 + \cos\theta - 2 + 2\cos\theta = 3\cos\theta - 1$

b $131.8°, 228.2°$

13 a $(\sin^2 A + \cos^2 A)^2 \equiv \sin^4 A + \cos^4 A + 2\sin^2 A\cos^2 A$

So $\quad 1 \equiv \sin^4 A + \cos^4 A + \dfrac{(2\sin A\cos A)^2}{2}$

$\Rightarrow \quad 2 \equiv 2(\sin^4 A + \cos^4 A) + \sin^2 2A$

$\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$

b Using **a**: $\sin^4 A + \cos^4 A \equiv \frac{1}{2}(2 - \sin^2 2A)$

$\equiv \frac{1}{2}\left\{2 - \dfrac{(1-\cos 4A)}{2}\right\} \equiv \dfrac{(4 - 1 + \cos 4A)}{4} \equiv \dfrac{3 + \cos 4A}{4}$

c $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$

14 a $\cos 3\theta \equiv \cos(2\theta + \theta) \equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1 - \cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta$

b $\frac{\pi}{9}, \frac{5\pi}{9}$ and $\frac{7\pi}{9}$