P1 Chapter 14: Logarithms

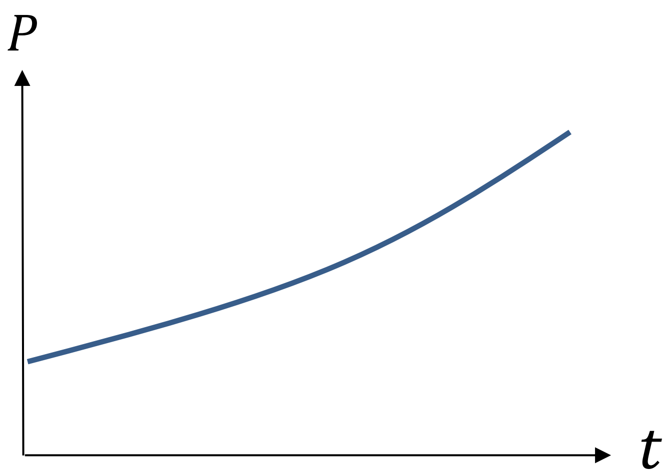
Exponential Modelling

Exponential Modelling

There are two key features of exponential functions which make them suitable for **population growth**:

1. **a^x gets a times bigger each time x increases by 1. (Because $a^{x+1} = a \times a^x$)**
   With population growth, we typically have a fixed percentage increase each year. So suppose the growth was 10% a year, and we used the equivalent decimal multiplier, 1.1, as a. Then 1.1^t, where t is the number of years, would get 1.1 times bigger each year.
2. **The rate of increase is proportional to the size of the population at a given moment.**
   This makes sense: The 10% increase of a population will be twice as large if the population itself is twice as large.

Suppose the population P in *The Republic of Dave*  is modelled by $P = 100e^{3t}$ where t is the numbers years since *The Republic* was established.



What is the initial population?

When $t = 0$,
$$P = 100e^0 = 100$$

What is the initial rate of population growth?

$$\frac{dP}{dt} = 300e^{3t}$$

When $t = 0$, $\frac{dP}{dt} = 300$

"Use of Technology" Monkey says:
When I'm not busy eating ticks off other monkeys, I use the $e^{\square}$ key. I can also use [ALPHA] [e] to get e without a power.

Another Example

[Textbook] The density of a pesticide in a given section of field, P mg/m^2, can be modelled by the equation $P = 160e^{-0.006t}$
where t is the time in days since the pesticide was first applied.

a. Use this model to estimate the density of pesticide after 15 days.

b. Interpret the meaning of the value 160 in this model.

c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.

d. Interpret the significance of the sign of your answer in part (c).

e. Sketch the graph of P against t.

a ?

In general, the 'initial value', when $t = 0$, is the coefficient of the exponential term.

b ?

c ?

d ?

?

ave
n
t.

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e. Sketch the graph of P against t.

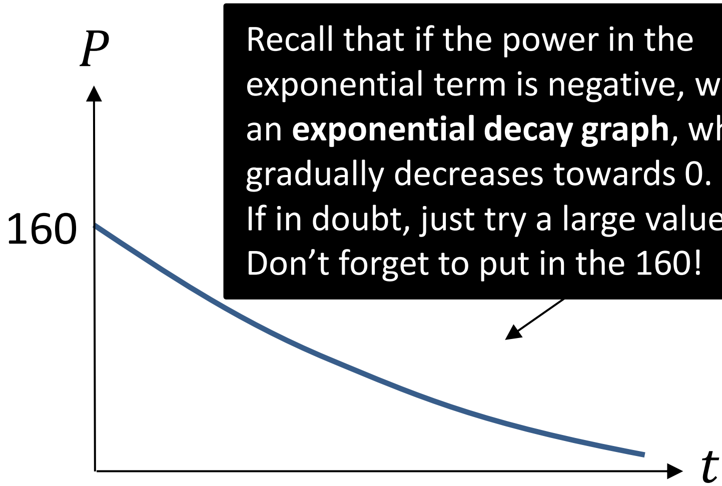
a When $t = 15$, $P = 160e^{-0.006(15)}$
$= 145.2\ mg/m^2$

In general, the 'initial value', when $t = 0$, is the coefficient of the exponential term.

b When $t = 0$, then $P = 160$. Thus 160 is the initial density of pesticide in the field.

c $\frac{dP}{dt} = (160 \times -0.006)e^{-0.006t}$
$= -0.96e^{-0.006t}$ $\quad \therefore k = -0.96$

d The rate is negative, thus the density of pesticide is decreasing.



Recall that if the power in the exponential term is negative, we have an **exponential decay graph**, which gradually decreases towards 0.
If in doubt, just try a large value of t.
Don't forget to put in the 160!

Exercise 14.3

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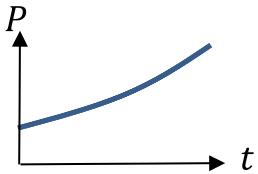
Application 1: Solutions to many 'differential equations'.

Frequently in physics/maths, **the rate of change of a variable is proportional to the value itself**. So with a population P behaving in this way, if the population doubled, the rate of increase would double.

$$P \propto \frac{dP}{dt} \quad \rightarrow \quad P = k\frac{dP}{dt}$$



This is known as a 'differential equation' because the equation involves both the variable and its derivative $\frac{dP}{dt}$.

The 'solution' to a differentiation equation means to have an equation relating P and t without the $\frac{dP}{dt}$. We end up with (using Year 2 techniques):

$$\boldsymbol{P = Ae^{kt}}$$

where A and k are constants. This is expected, because we know from experience that population growth is usually exponential.

Application 2: Russian Roulette

A scene from one of Dr Frost's favourite films, *The Deer Hunter*.

I once wondered (as you do), if I was playing Russian Roulette, where you randomly rotate the barrel of a gun each time with n chambers, but with one bullet, what's the probability I'm still alive after n shots?

The probability of surviving each time is $1 - \frac{1}{n}$, so the probability of surviving all n shots is $\left(1 - \frac{1}{n}\right)^n$. We might consider what happens when n becomes large, i.e. $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n$. In general, $e^k = \lim_{n\to\infty}\left(1 + \frac{k}{n}\right)^n$.

Thus $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = e^{-1} = \frac{1}{e}$, i.e. I have a 1 in e chance of surviving. Bad odds!

This is also applicable to the lottery. If there was a 1 in 20 million chance of winning the lottery, we might naturally wonder what happens if we bought 20 million (random) lottery tickets. There's a 1 in e (roughly a third) chance of winning no money at all!

Application 3: Secret Santa

ABC, ACB, BAC, BCA, CAB, CBA

You might have encountered $n! = n \times (n-1) \times \cdots \times 2 \times 1$, said "$n$ factorial" meaning "*the number of ways of arranging n objects in a line*". So if we had 3 letters ABC, we have $3! = 6$ ways of arranging them.

Meanwhile, $!n$ means the number of **derangements** of n, i.e. the arrangements where **no letter appears in its original place**.

For ABC, that only gives BCA or CAB, so $!3 = 2$. This is applicable to '**Secret Santa**' (where each person is given a name out a hat of whom to give their present to) because ideally we want the scenario where *no person gets their own name*.

Remarkably, a derangement occurs an e-th of the time. So if there are 5 people and hence $5! = 120$ possible allocations of recipient names, we only get the ideal Secret Santa situation just $\frac{120}{e} = 44.15 \rightarrow 44$ times. And so we get **my favourite result in the whole of mathematics**:

$$!n = \left[\frac{n!}{e}\right]$$

(where [...] means round)

Homework Exercise

1 The value of a car is modelled by the formula

$$V = 20\,000e^{-\frac{t}{12}}$$

where V is the value in £s and t is its age in years from new.

a State its value when new.

b Find its value (to the nearest £) after 4 years.

c Sketch the graph of V against t.

2 The population of a country is modelled using the formula

$$P = 20 + 10e^{\frac{t}{50}}$$

where P is the population in thousands and t is the time in years after the year 2000.

a State the population in the year 2000.

b Use the model to predict the population in the year 2030.

c Sketch the graph of P against t for the years 2000 to 2100.

d Do you think that it would be valid to use this model to predict the population in the year 2500? Explain your answer.

Homework Exercise

3 The number of people infected with a disease is modelled by the formula

$$N = 300 - 100e^{-0.5t}$$

where N is the number of people infected with the disease and t is the time in years after detection.

a How many people were first diagnosed with the disease?

b What is the long term prediction of how this disease will spread?

c Sketch the graph of N against t for $t > 0$.

4 The number of rabbits, R, in a population after m months is modelled by the formula

$$R = 12e^{0.2m}$$

a Use this model to estimate the number of rabbits after

i 1 month **ii** 1 year

b Interpret the meaning of the constant 12 in this model.

Problem-solving

Your answer to part **b** must refer to the context of the model.

c Show that after 6 months, the rabbit population is increasing by almost 8 rabbits per month.

d Suggest one reason why this model will stop giving valid results for large enough values of t.

Homework Exercise

5 On Earth, the atmospheric pressure, p, in bars can be modelled approximately by the formula $p = \mathrm{e}^{-0.13h}$ where h is the height above sea level in kilometres.

a Use this model to estimate the pressure at the top of Mount Rainier, which has an altitude of 4.394 km. **(1 mark)**

b Demonstrate that $\frac{\mathrm{d}p}{\mathrm{d}h} = kp$ where k is a constant to be found. **(2 marks)**

c Interpret the significance of the sign of k in part **b**. **(1 mark)**

d This model predicts that the atmospheric pressure will change by s% for every kilometre gained in height. Calculate the value of s. **(3 marks)**

6 Nigel has bought a tractor for £20 000. He wants to model the depreciation of the value of his tractor, £T, in t years. His friend suggests two models:

Model 1: $T = 20\,000\mathrm{e}^{-0.24t}$

Model 2: $T = 19\,000\mathrm{e}^{-0.255t} + 1000$

a Use both models to predict the value of the tractor after one year. Compare your results. **(2 marks)**

b Use both models to predict the value of the tractor after ten years. Compare your results. **(2 marks)**

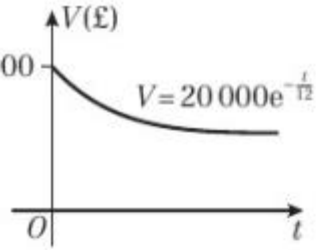
c Sketch a graph of T against t for both models. **(2 marks)**

d Interpret the meaning of the 1000 in model 2, and suggest why this might make model 2 more realistic. **(1 mark)**

Homework Answers

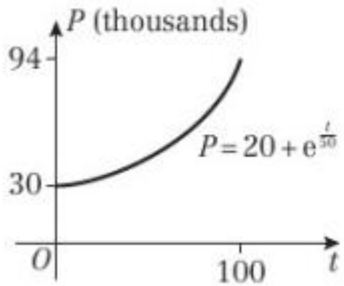
1 **a** £20 000 **b** £14 331

c



2 **a** 30 000 **b** 38 221

c

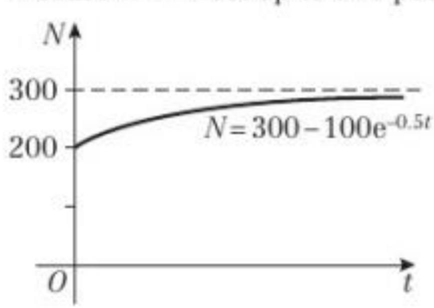


d Model predicts population of the country to be over 200 million, this is highly unlikely and by 2500 new factors are likely to affect population growth. Model not valid for predictions that far into the future.

3 **a** 200

b Disease will infect up to 300 people.

c



4 **a** **i** 15 rabbits **ii** 132 rabbits

b The initial number of rabbits

c $\frac{dR}{dm} = 2.4e^{0.2m}$

When $m = 6$, $\frac{dR}{dm} = 7.97 \approx 8$

d The rabbits may begin to run out of food or space

5 **a** 0.565 bars

b $\frac{dp}{dh} = -0.13e^{-0.13h} = -0.13p$, $k = -0.13$

c The atmospheric pressure decreases exponentially as the altitude increases

d 12%

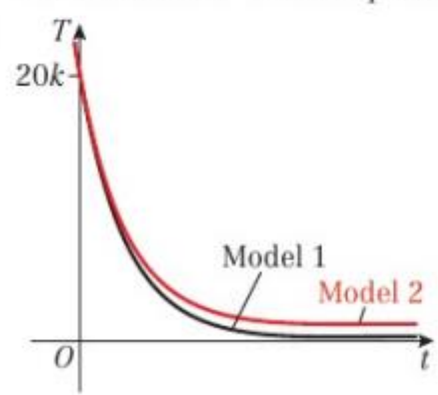
6 **a** Model 1: £15 733

Model 2: £15 723 Similar results

b Model 1: £1814

Model 2: £2484 Model 2 predicts a larger value

c



d In Model 2 the tractor will always be worth at least £1000. This could be the value of the tractor as scrap metal.