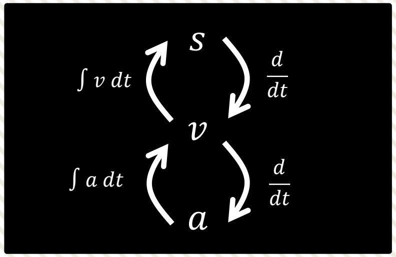
M1 Chapter 11: Variable Acceleration

Integrating Variable Motion

Using Integration



Differentiating (with respect to time) gets us from displacement to velocity, and from velocity to acceleration.
So naturally, integrating (with respect to time) gets us from acceleration to velocity, and from velocity to displacement. As mentioned earlier, it's helpful to picture the flowchart on the left: we move down to differentiate and up to integrate.

[Textbook] A particle is moving on the x-axis. At time $t = 0$, the particle is at the point where $x = 5$. The velocity of the particle at time t seconds (where $t \geq 0$) is $(6t - t^2)$ ms^{-1}. Find:
(a) An expression for the displacement of the particle from O at time t seconds.
(b) The distance of the particle from its starting point when $t = 6$.

a

$$x = \int v\,dt = 3t^2 - \frac{1}{3}t^3 + c$$

When $t = 0, x = 5, \therefore$

$$0 - 0 + c = 5 \quad \rightarrow \quad c = 5$$

$$x = 3t^2 - \frac{1}{3}t^3 + 5$$

Recall in Pure Year 1 that we can find the constant of integration by using known values.

b

When $t = 6$,

$$x = 3(6^2) - \frac{1}{3}(6^3) + 5 = 41$$

Distance is $41 - 5 = 36$ m

Further Example

[Textbook] A particle travels in a straight line. After t seconds its velocity, v ms^{-1}, is given by $v = 5 - 3t^2$, $t \geq 0$. Find the distance travelled by the particle in the third second of its motion.

?

Further Example

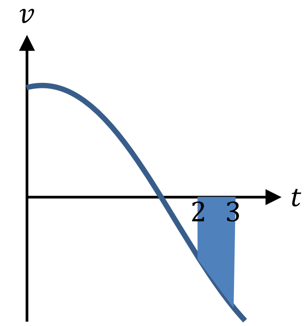
[Textbook] A particle travels in a straight line. After t seconds its velocity, v ms^{-1}, is given by $v = 5 - 3t^2$, $t \geq 0$. Find the distance travelled by the particle in the third second of its motion.

$$s = \int_2^3 (5 - 3t^2)\, dt$$

$$= [5t - t^3]_2^3 = (15 - 27) - (10 - 8)$$

$$= -12 - 2 = -14$$

Distance travelled is 14 m.



Further Example

Edexcel M2 June 2015 Q6

A particle P moves on the positive x-axis. The velocity of P at time t seconds is $(2t^2 - 9t + 4)$ m s^{-1}. When $t = 0$, P is 15 m from the origin O.

Find

(*a*) the values of t when P is instantaneously at rest, **(3)**

(*b*) the acceleration of P when $t = 5$, **(3)**

(*c*) the total distance travelled by P in the interval $0 \leq t \leq 5$. **(5)**

Hint: recall that if the curve goes above and below the x-axis, we need to find each **distance-area** separately.

a

?

b

?

c

?

Further Example

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a

At rest when $v = 0$: $(2t^2 - 9t + 4) = 0$
$= (2t-1)(t-4)$,
$t = \frac{1}{2}$, 4

b

$a = \frac{dv}{dt} = 4t - 9$
$t = 5$, $a = 11$ (m s^{-2})

c

$s = \int v\,dt = \frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t\,(+C)$	M1	Integrate v to obtain s (at least one power of t going up)
	A1	
Use of $t=0$, $t=\frac{1}{2}$, $t=4$, $t=5$ (and $t=0, s=15$) as limits in integrals	DM1	Correct strategy for their limits - requires subtraction of the negative distance. Dependent on the previous M1 and at least one positive solution for t in (0,5) from (a)
$\left[\frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t(+15)\right]_0^{\frac{1}{2}} - \left[\frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t(+15)\right]_{\frac{1}{2}}^4$ $+\left[\frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t(+15)\right]_4^5$	A1	NB: $\int_0^5 v\,dt$ scores M0A0A0
$(0, \frac{23}{24}, -\frac{40}{3}, \frac{-55}{6})$ $\left(15, 15\frac{23}{24}\left(\frac{383}{24}\right), \frac{5}{3}, 5.8\dot{3}\left(\frac{35}{6}\right)\right)$ $= \frac{23}{24} + \frac{343}{24} + \frac{100}{24} = 19.4$ (m)	A1 [5]	$19\frac{5}{12}$ $\left(\frac{233}{12}\right)$ or better
	(11)	

Exercise 11.4 Using Integration

Pearson Stats/Mechanics Year 1
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Homework Exercise

1 A particle is moving in a straight line. Given that $s = 0$ when $t = 0$, find an expression for the displacement of the particle if the velocity is given by:

a $v = 3t^2 - 1$ **b** $v = 2t^3 - \frac{3t^2}{2}$ **c** $v = 2\sqrt{t} + 4t^2$

2 A particle is moving in a straight line. Given that $v = 0$ when $t = 0$, find an expression for the velocity of the particle if the acceleration is given by:

a $a = 8t - 2t^2$ **b** $a = 6 + \frac{t^2}{3}$

3 A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(8 + 2t - 3t^2)\,\text{m s}^{-1}$ in the direction of x increasing. At time $t = 0$, P is at the point where $x = 4$. Find the distance of P from O when $t = 1$.

4 A particle P is moving on the x-axis. At time t seconds, the acceleration of P is $(16 - 2t)\,\text{m s}^{-2}$ in the direction of x increasing. The velocity of P at time t seconds is $v\,\text{m s}^{-1}$. When $t = 0$, $v = 6$ and when $t = 3$, $x = 75$. Find:

a v in terms of t **b** the value of x when $t = 0$.

5 A particle is moving in a straight line. At time t seconds, its velocity, $v\,\text{m s}^{-1}$, is given by $v = 6t^2 - 51t + 90$. When $t = 0$ the displacement is 0. Find the distance between the two points where P is instantaneously at rest.

Homework Exercise

6 At time t seconds, where $t \geqslant 0$, the velocity $v\,\text{m s}^{-1}$ of a particle moving in a straight line is given by $v = 12 + t - 6t^2$. When $t = 0$, P is at a point O on the line. Find the distance of P from O when $v = 0$.

7 A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(4t - t^2)\,\text{m s}^{-1}$ in the direction of x increasing. At time $t = 0$, P is at the origin O. Find:

a the value of x at the instant when $t > 0$ and P is at rest

b the total distance moved by P in the interval $0 \leqslant t \leqslant 5$.

Problem-solving

You will need to consider the motion when v is positive and negative separately.

8 A particle P is moving on the x-axis. At time t seconds, the velocity of P is $(6t^2 - 26t + 15)\,\text{m s}^{-1}$ in the direction of x increasing. At time $t = 0$, P is at the origin O. In the subsequent motion P passes through O twice. Find the two non-zero values of t when P passes through O.

9 A particle P moves along the x-axis. At time t seconds (where $t \geqslant 0$) the velocity of P is $(3t^2 - 12t + 5)\,\text{m s}^{-1}$ in the direction of x increasing. When $t = 0$, P is at the origin O. Find:

a the values of t when P is again at O

b the distance travelled by P in the interval $2 \leqslant t \leqslant 3$.

10 A particle P moves on the x-axis. The acceleration of P at time t seconds, $t \geqslant 0$, is $(4t - 3)\,\text{m s}^{-2}$ in the positive x-direction. When $t = 0$, the velocity of P is $4\,\text{m s}^{-1}$ in the positive x-direction. When $t = T$ $(T \neq 0)$, the velocity of P is $4\,\text{m s}^{-1}$ in the positive x-direction. Find the value of T. **(6 marks)**

Homework Exercise

11 A particle P travels in a straight line such that its acceleration at time t seconds is $(t - 3)$ m s^{-2}. The velocity of P at time t seconds is v m s^{-1}. When $t = 0$, $v = 4$. Find:

a v in terms of t **(4 marks)**

b the values of t when P is instantaneously at rest **(3 marks)**

c the distance between the two points at which P is instantaneously at rest. **(4 marks)**

12 A particle travels in a straight line such that its acceleration, a m s^{-2}, at time t seconds is given by $a = 6t + 2$. When $t = 2$ seconds, the displacement, s, is 10 metres and when $t = 3$ seconds the displacement is 38 metres. Find:

a the displacement when $t = 4$ seconds **(6 marks)**

b the velocity when $t = 4$ seconds. **(2 marks)**

Problem-solving

You need to use integration to find expressions for the velocity and displacement then substitute in the given values. Use simultaneous equations to find the values of the constants of integration.

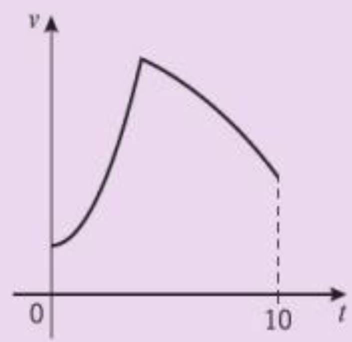
Homework Exercise

Challenge

The motion of a robotic arm moving along a straight track is modelled using the equations:

$$v = \frac{t^2}{2} + 2,\ 0 \leqslant t \leqslant k \text{ and } v = 10 + \frac{t}{3} - \frac{t^2}{12},\ k \leqslant t \leqslant 10$$

The diagram shows a sketch of the velocity–time graph of the motion of the arm.



Work out the total distance travelled by the robotic arm.

Homework Answers

1 **a** $s = t^3 - t$ **b** $s = \frac{t^4}{2} - \frac{t^3}{2}$ **c** $s = \frac{4}{3}t^{\frac{3}{2}} + \frac{4t^3}{3}$

2 **a** $v = 4t^2 - \frac{2t^3}{3}$ **b** $v = 6t + \frac{t^3}{9}$

3 12 m

4 **a** $v = 6 + 16t - t^2$ **b** -6

5 42.9 m (3 s.f.)

6 12.375 m

7 **a** $10\frac{2}{3}$ **b** 13 m

8 $t = \frac{3}{2}$ and $t = 5$

9 **a** $t = 1$ and $t = 5$ **b** 6 m

10 $T = 1.5$ s

11 **a** $v = \frac{t^2}{2} - 3t + 4$ **b** $t = 2$ and $t = 4$ **c** $\frac{2}{3}$ m

12 **a** 86 m **b** 60 m s^{-1}

Challenge

$\frac{200}{3}$ m