P2 Chapter 6: CoAngle Trigonometry

Using Cotangents

Using $sec, cosec, cot$

Questions in the exam usually come in two flavours: (a) 'provey' questions requiring to prove some identity and (b) 'solvey' questions.

[Textbook]

(a) Simplify $\sin\theta \cot\theta \sec\theta$

(b) Simplify $\sin\theta \cos\theta \ (\sec\theta + cosec\ \theta)$

(c) Prove that $\frac{\cot\theta\ cosec\ \theta}{\sec^2\theta + cosec^2\ \theta} \equiv \cos^3\theta$

Tip 1: Get everything in terms of sin and cos first (using $\cot x = \frac{\cos x}{\sin x}$ rather than $\cot x = \frac{1}{\tan x}$)

Tip 2: Whenever you have algebraic fractions being added/subtracted, whether $\frac{a}{b} + \frac{c}{d}$ or $\frac{a}{b} + c$, combine them into one (as we can typically then use $\sin^2 x + \cos^2 x = 1$)

a

?

b

?

c

?

Using *sec*, *cosec*, *cot*

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(b) Simplify $\sin\theta\cos\theta\ (\sec\theta + cosec\ \theta)$
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a

$$\sin\theta\cot\theta\sec\theta \equiv \sin\theta \times \frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos\theta}$$
$$\equiv 1$$

b

$$\sin\theta\cos\theta\ (\sec\theta + cosec\ \theta)$$
$$= \sin\theta\cos\theta\left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)$$
$$= \sin\theta\cos\theta\left(\frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}\right)$$
$$= \sin\theta + \cos\theta$$

c

$$\frac{\cot\theta\ cosec\ \theta}{\sec^2\theta + cosec^2\ \theta} \equiv \frac{\frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta}}{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}}$$
$$\equiv \frac{\left(\frac{\cos\theta}{\sin^2\theta}\right)}{\left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}\right)} \equiv \frac{\left(\frac{\cos\theta}{\sin^2\theta}\right)}{\left(\frac{1}{\sin^2\theta\cos^2\theta}\right)}$$
$$\equiv \frac{\cos^3\theta}{1}$$
$$\equiv \cos^3\theta$$

Multiply top and bottom by $\sin^2\theta\cos^2\theta$

Test Your Understanding

1 $\sec x - \cos x \equiv \sin x \tan x$

?

2 $(1 + \cos x)(cosec\, x - \cot x) \equiv \sin x$

?

Test Your Understanding

1

$\sec x - \cos x \equiv \sin x \tan x$

$$\textbf{LHS} = \frac{1}{\cos x} - \cos x$$

$$= \frac{1 - \cos^2 x}{\cos x}$$

$$= \frac{\sin^2 x}{\cos x}$$

$$= \sin x \tan x$$

2

$(1 + \cos x)(cosec\, x - \cot x) \equiv \sin x$

$$\textbf{LHS} = \operatorname{cosec} x - \cot x + \cos x \operatorname{cosec} x - \cos x \cot x$$

$$= \frac{1}{\sin x} - \cot x + \cot x - \frac{\cos^2 x}{\sin x}$$

$$= \frac{1 - \cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x$$

Solvey Questions

[Textbook] Solve the following equations in the interval $0 \leq \theta \leq 360°$:

a) $\sec \theta = -2.5$

b) $\cot 2\theta = 0.6$

a

?

b

?

Solve $\cot \theta = 0$ in the interval $0 \leq \theta \leq 2\pi$.

?

Solvey Questions

[Textbook] Solve the following equations in the interval $0 \leq \theta \leq 360°$:

a) $\sec\theta = -2.5$

b) $\cot 2\theta = 0.6$

a

$\sec\theta = -2.5$

$\cos\theta = -0.4$

$\theta = 113.6°, 246.5°$

Reciprocate both sides.

b

$\cot 2\theta = 0.6 \qquad 0 \leq 2\theta \leq 720°$

$\tan 2\theta = \frac{5}{3}$

$2\theta = 59.0°, 239.0°, 419.0°, 599.0°$

$\theta = 29.5°, 120°, 210°, 300°\ (3sf)$

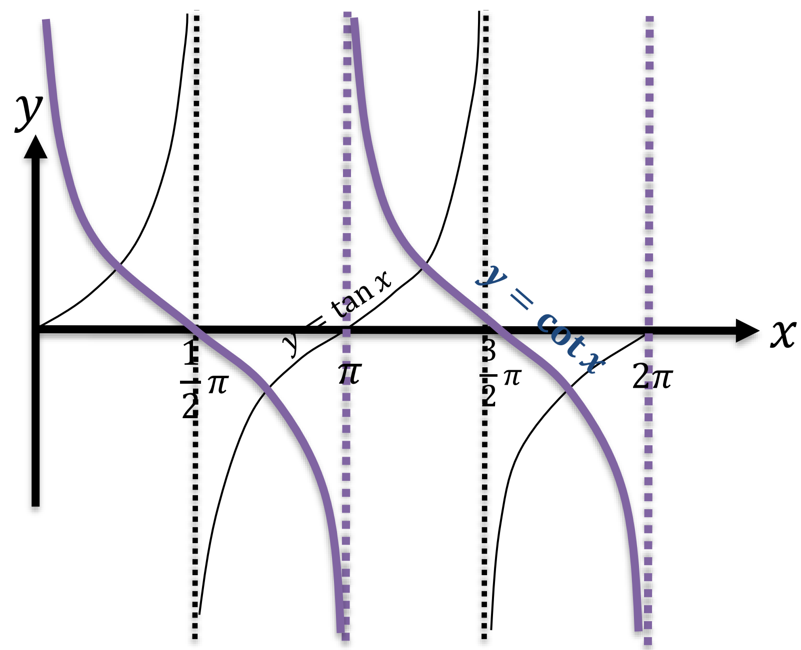
Solve $\cot\theta = 0$ in the interval $0 \leq \theta \leq 2\pi$.

$\cot\theta = 0$

$\therefore \tan\theta \to \infty$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

You can't reciprocate 0. However, the value tends towards $\pm\infty$, which coincides with the asymptotes of tan.



Test Your Understanding

Solve in the interval $0 \leq \theta < 360°$:

$$cosec\ 3\theta = 2$$

?

Test Your Understanding

Solve in the interval $0 \leq \theta < 360°$:

$$cosec\ 3\theta = 2$$

$0 \leq 3\theta < 1080°$

$\sin 3\theta = \frac{1}{2}$

$3\theta = 30°, 150°, 390°, 510°, 750°, 870°$

$\theta = 10°, 50°, 130°, 170°, 250°, 290°$

Exercise 6.3

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Homework Exercise

1 Rewrite the following as powers of $\sec\theta$, $\operatorname{cosec}\theta$ or $\cot\theta$.

a $\dfrac{1}{\sin^3\theta}$ **b** $\dfrac{4}{\tan^6\theta}$ **c** $\dfrac{1}{2\cos^2\theta}$ **d** $\dfrac{1-\sin^2\theta}{\sin^2\theta}$

e $\dfrac{\sec\theta}{\cos^4\theta}$ **f** $\sqrt{\operatorname{cosec}^3\theta\cot\theta\sec\theta}$ **g** $\dfrac{2}{\sqrt{\tan\theta}}$ **h** $\dfrac{\operatorname{cosec}^2\theta\tan^2\theta}{\cos\theta}$

2 Write down the value(s) of $\cot x$ in each of the following equations.

a $5\sin x = 4\cos x$ **b** $\tan x = -2$ **c** $3\dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\sin x}$

3 Using the definitions of **sec**, **cosec**, **cot** and **tan** simplify the following expressions.

a $\sin\theta\cot\theta$ **b** $\tan\theta\cot\theta$

c $\tan 2\theta \operatorname{cosec} 2\theta$ **d** $\cos\theta\sin\theta(\cot\theta + \tan\theta)$

e $\sin^3 x \operatorname{cosec} x + \cos^3 x \sec x$ **f** $\sec A - \sec A \sin^2 A$

g $\sec^2 x \cos^5 x + \cot x \operatorname{cosec} x \sin^4 x$

4 Prove that:

a $\cos\theta + \sin\theta\tan\theta \equiv \sec\theta$ **b** $\cot\theta + \tan\theta \equiv \operatorname{cosec}\theta\sec\theta$

c $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$ **d** $(1-\cos x)(1+\sec x) \equiv \sin x\tan x$

e $\dfrac{\cos x}{1-\sin x} + \dfrac{1-\sin x}{\cos x} \equiv 2\sec x$ **f** $\dfrac{\cos\theta}{1+\cot\theta} \equiv \dfrac{\sin\theta}{1+\tan\theta}$

Homework Exercise

5 Solve, for values of θ in the interval $0 \leqslant \theta \leqslant 360°$, the following equations. Give your answers to 3 significant figures where necessary.

a $\sec\theta = \sqrt{2}$ **b** $\operatorname{cosec}\theta = -3$ **c** $5\cot\theta = -2$ **d** $\operatorname{cosec}\theta = 2$

e $3\sec^2\theta - 4 = 0$ **f** $5\cos\theta = 3\cot\theta$ **g** $\cot^2\theta - 8\tan\theta = 0$ **h** $2\sin\theta = \operatorname{cosec}\theta$

6 Solve, for values of θ in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations:

a $\operatorname{cosec}\theta = 1$ **b** $\sec\theta = -3$ **c** $\cot\theta = 3.45$

d $2\operatorname{cosec}^2\theta - 3\operatorname{cosec}\theta = 0$ **e** $\sec\theta = 2\cos\theta$ **f** $3\cot\theta = 2\sin\theta$

g $\operatorname{cosec} 2\theta = 4$ **h** $2\cot^2\theta - \cot\theta - 5 = 0$

7 Solve the following equations for values of θ in the interval $0 \leqslant \theta \leqslant 2\pi$. Give your answers in terms of π.

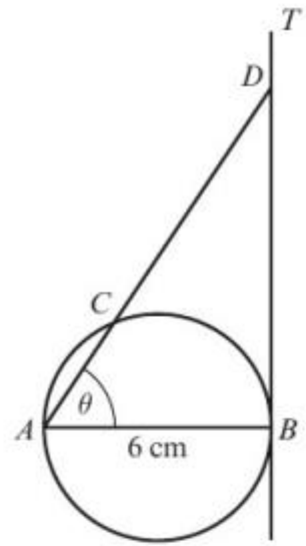
a $\sec\theta = -1$ **b** $\cot\theta = -\sqrt{3}$

c $\operatorname{cosec}\frac{1}{2}\theta = \frac{2\sqrt{3}}{3}$ **d** $\sec\theta = \sqrt{2}\tan\theta \left(\theta \neq \frac{\pi}{2}, \theta \neq \frac{3\pi}{2}\right)$

8 In the diagram $AB = 6$ cm is the diameter of the circle and BT is the tangent to the circle at B. The chord AC is extended to meet this tangent at D and $\angle DAB = \theta$.

a Show that $CD = 6(\sec\theta - \cos\theta)$ cm. **(4 marks)**

b Given that $CD = 16$ cm, calculate the length of the chord AC. **(3 marks)**



Problem-solving

AB is the diameter of the circle, so $\angle ACB = 90°$.

9 **a** Prove that $\dfrac{\operatorname{cosec} x - \cot x}{1 - \cos x} \equiv \operatorname{cosec} x$. **(4 marks)**

b Hence solve, in the interval $-\pi \leqslant x \leqslant \pi$, the equation $\dfrac{\operatorname{cosec} x - \cot x}{1 - \cos x} = 2$. **(3 marks)**

10 **a** Prove that $\dfrac{\sin x \tan x}{1 - \cos x} - 1 \equiv \sec x$. **(4 marks)**

b Hence explain why the equation $\dfrac{\sin x \tan x}{1 - \cos x} - 1 = -\dfrac{1}{2}$ has no solutions. **(1 mark)**

11 Solve, in the interval $0 \leqslant x \leqslant 360°$, the equation $\dfrac{1 + \cot x}{1 + \tan x} = 5$. **(8 marks)**

Homework Answers

1 **a** $\operatorname{cosec}^3\theta$ **b** $4\cot^6\theta$ **c** $\frac{1}{2}\sec^2\theta$
d $\cot^2\theta$ **e** $\sec^5\theta$ **f** $\operatorname{cosec}^2\theta$
g $2\cot^{\frac{1}{2}}\theta$ **h** $\sec^3\theta$

2 **a** $\frac{5}{4}$ **b** $-\frac{1}{2}$ **c** $\pm\sqrt{3}$

3 **a** $\cos\theta$ **b** 1 **c** $\sec 2\theta$
d 1 **e** 1 **f** $\cos A$
g $\cos x$

4 **a** $$\text{L.H.S.} = \cos\theta + \sin\theta\frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta = \text{R.H.S.}$$

b $$\text{L.H.S.} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \equiv \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} \equiv \frac{1}{\sin\theta\cos\theta} = \frac{1}{\sin\theta}\times\frac{1}{\cos\theta} \equiv \operatorname{cosec}\theta\sec\theta = \text{R.H.S.}$$

c $$\text{L.H.S.} = \frac{1}{\sin\theta} - \sin\theta \equiv \frac{1-\sin^2\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta} \equiv \cos\theta\times\frac{\cos\theta}{\sin\theta} \equiv \cos\theta\cot\theta = \text{R.H.S.}$$

4 **d** $$\text{L.H.S.} = (1-\cos x)\left(1+\frac{1}{\cos x}\right) \equiv 1 - \cos x + \frac{1}{\cos x} - 1 \equiv \frac{1}{\cos x} - \cos x \equiv \frac{1-\cos^2 x}{\cos x} \equiv \frac{\sin^2 x}{\cos x} \equiv \sin x\times\frac{\sin x}{\cos x} \equiv \sin x\tan x = \text{R.H.S.}$$

e $$\text{L.H.S.} = \frac{\cos^2 x + (1-\sin x)^2}{(1-\sin x)\cos x} \equiv \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{(1-\sin x)\cos x} \equiv \frac{2-2\sin x}{(1-\sin x)\cos x} \equiv \frac{2(1-\sin x)}{(1-\sin x)\cos x} \equiv 2\sec x = \text{R.H.S.}$$

f $$\text{L.H.S.} = \frac{\cos\theta}{1+\frac{1}{\tan\theta}} \equiv \frac{\cos\theta}{\left(\frac{\tan\theta+1}{\tan\theta}\right)} \equiv \frac{\cos\theta\tan\theta}{\tan\theta+1} \equiv \frac{\sin\theta}{1+\tan\theta} = \text{R.H.S.}$$

Homework Answers

5 **a** 45°, 315° **b** 199°, 341°
c 112°, 292° **d** 30°, 150°
e 30°, 150°, 210°, 330° **f** 36.9°, 90°, 143°, 270°
g 26.6°, 207° **h** 45°, 135°, 225°, 315°

6 **a** 90° **b** ±109°
c −164°, 16.2° **d** 41.8°, 138°
e ±45°, ±135° **f** ±60°
g −173°, −97.2°, 7.24°, 82.8°
h −152°, −36.5°, 28.4°, 143°

7 **a** π **b** $\frac{5\pi}{6}, \frac{11\pi}{6}$

c $\frac{2\pi}{3}, \frac{4\pi}{3}$ **d** $\frac{\pi}{4}, \frac{3\pi}{4}$

8 **a** $\frac{AB}{AD} = \cos\theta \Rightarrow AD = 6\sec\theta$

$\frac{AC}{AB} = \cos\theta \Rightarrow AC = 6\cos\theta$

$CD = AD - AC \Rightarrow CD = 6\sec\theta - 6\cos\theta$
$= 6(\sec\theta - \cos\theta)$

b 2 cm

9 **a** $$\frac{\operatorname{cosec} x - \cot x}{1 - \cos x} \equiv \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{1 - \cos x} \equiv \frac{1}{\sin x} \times \frac{1 - \cos x}{1 - \cos x}$$
$$\equiv \operatorname{cosec} x$$

b $x = \frac{\pi}{6}, \frac{5\pi}{6}$

10 **a** $$\frac{\sin x \tan x}{1 - \cos x} - 1 \equiv \frac{\sin^2 x}{\cos x(1 - \cos x)} - 1$$
$$\equiv \frac{\sin^2 x - \cos x + \cos^2 x}{\cos x(1 - \cos x)} \equiv \frac{1 - \cos x}{\cos x(1 - \cos x)}$$
$$\equiv \frac{1}{\cos x} \equiv \sec x$$

b Would need to solve $\sec x = -\frac{1}{2}$, which is equivalent to $\cos x = -2$, which has no solutions.

11 $x = 11.3°, 191.3°$ (1 d.p.)