P2 Chapter 3: Sequences and Series

Geometric Series

Sum of the first n terms of a geometric **series**

Arithmetic Series

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Geometric Series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Proof:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}$$

Mutiplying by r:

$$rS_n = \quad ar + ar^2 + \cdots \quad + ar^{n-1} + ar^n$$

Subtracting:

$$S_n - rS_n = a - ar^n$$
$$S_n(1 - r) = a(1 - r^n)$$
$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Exam Note: This once came up in an exam. And again is a university interview favourite!

Geometric Series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Find the sum of the first 10 terms.

$3, 6, 12, 24, 48, \ldots$

$a = 3, r = 2, n = 10$

$$S_n = \frac{3(1 - 2^{10})}{1 - 2} = 3069$$

$4, 2, 1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \ldots$

$a = 4, r = \dfrac{1}{2}, n = 10$

$$S_n = \frac{4\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}} = \frac{1023}{128}$$

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \cdots$ to n terms would exceed 2 000 000.

?

?

Harder Example

Find the least value of n such that the sum of $1 + 2 + 4 + 8 + \cdots$ to n terms would exceed 2 000 000.

$$a = 1, r = 2, n = ?$$
$$S_n > 2\ 000\ 000$$

$$\frac{1(1 - 2^n)}{1 - 2} > 2\ 000\ 000$$
$$2^n - 1 > 2\ 000\ 000$$
$$2^n > 2\ 000\ 001$$
$$n > \log_2 2000001$$
$$n > 20.9$$

So 21 terms needed.

log_2 both sides to cancel out "2 to the power of".

Test Your Understanding

The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(a) the common ratio,

(2)

(b) the first term,

(2)

(d) the smallest value of n for which the sum of the first n terms of the series exceeds 1000.

(4)

(a) ?

(b) ?

(d) ?

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(4)

(a)	$\{ ar = 192 \quad \text{and} \quad ar^2 = 144 \}$		
	$r = \dfrac{144}{192}$	Attempt to eliminate a.	M1
	$r = \frac{3}{4}$ or 0.75	$\frac{3}{4}$ or 0.75	A1
			[2]
(b)	$a(0.75) = 192$		M1
	$a \left\{ = \dfrac{192}{0.75} \right\} = 256$	256	A1
			[2]
(d)	$\dfrac{256(1 - (0.75)^n)}{1 - 0.75} > 1000$	Applies S_n with their a and r and "uses" 1000 at any point in their working. (Allow with = or <).	M1
	$(0.75)^n < 1 - \dfrac{1000(0.25)}{256} \left\{ = \dfrac{6}{256} \right\}$	Attempt to isolate $+(r)^n$ from S_n formula. (Allow with = or >).	M1
	$n \log(0.75) < \log\left(\dfrac{6}{256} \right)$	Uses the power law of logarithms correctly. (Allow with = or >).	M1
	$n > \dfrac{\log\left(\frac{6}{256} \right)}{\log(0.75)} = 13.0471042... \Rightarrow n = 14$	$n = 14$	A1 cso
			[4]

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Extension

1 [MAT 2010 1B]
The sum of the first $2n$ terms of
$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \ldots$$
is

A) $2^n + 1 - 2^{1-n}$
B) $2^n + 2^{-n}$
C) $2^{2n} - 2^{3-2n}$
D) $\frac{2^n - 2^{-n}}{3}$

?

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There are interleaved sequences. If we want $2n$ terms, we want n terms of $1, 2, 4, 8, \dots$ and n terms of $1, \frac{1}{2}, \frac{1}{4}, \dots$

For first sequence:

$$S_n = \frac{1(1 - 2^n)}{1 - 2} = 2^n - 1$$

For second sequence:

$$S_n = \frac{1\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = \frac{1 - 2^{-n}}{1/2} = 2 - 2^{1-n}$$

Therefore the sum of both is (A).

1 Find the sum of the following geometric series (to 3 d.p. if necessary).

a $1 + 2 + 4 + 8 + \dots$ (8 terms)

b $32 + 16 + 8 + \dots$ (10 terms)

c $\dfrac{2}{3} + \dfrac{4}{15} + \dfrac{8}{75} + \dots + \dfrac{256}{234\,375}$

d $4 - 12 + 36 - 108 + \dots$ (6 terms)

e $729 - 243 + 81 - \dots - \dfrac{1}{3}$

f $-\dfrac{5}{2} + \dfrac{5}{4} - \dfrac{5}{8} \dots - \dfrac{5}{32\,768}$

2 A geometric series has first three terms $3 + 1.2 + 0.48\dots$ Evaluate S_{10} giving your answer to 4 d.p.

3 A geometric series has first term 5 and common ratio $\dfrac{2}{3}$. Find the value of S_8.

4 The sum of the first three terms of a geometric series is 30.5. If the first term is 8, find possible values of r.

5 Find the least value of n such that the sum $3 + 6 + 12 + 24 + \dots$ to n terms exceeds 1.5 million.

6 Find the least value of n such that the sum $5 + 4.5 + 4.05 + \dots$ to n terms exceeds 45.

7 A geometric series has first term 25 and common ratio $\dfrac{3}{5}$

Given that the sum to k terms of the series is greater than 61,

a show that $k > \dfrac{\log(0.024)}{\log(0.6)}$

(4 marks)

b find the smallest possible value of k.

(1 mark)

8 A geometric series has first term a and common ratio r.
The sum of the first two terms of the series is 4.48.
The sum of the first four terms is 5.1968. Find the two
possible values of r. **(4 marks)**

9 The first term of a geometric series is a and the common ratio is $\sqrt{3}$.
Show that $S_{10} = 121a(\sqrt{3} + 1)$. **(4 marks)**

10 A geometric series has first term a and common ratio 2. A different geometric series has
first term b and common ratio 3. Given that the sum of the first 4 terms of both series

is the same, show that $a = \dfrac{8}{3}b$. **(4 marks)**

11 The first three terms of a geometric series are $(k - 6)$, k, $(2k + 5)$, where k is a positive constant.
   a Show that $k^2 - 7k - 30 = 0$. **(4 marks)**
   b Hence find the value of k. **(2 marks)**
   c Find the common ratio of this series. **(1 mark)**
   d Find the sum of the first 10 terms of this series, giving your answer to the nearest
      whole number. **(2 marks)**

1 **a** 255 **b** 63.938 **c** 1.110

  d −728 **e** $546\frac{2}{3}$ **f** −1.667

2 4.9995

3 14.4147

4 $\frac{5}{4}, -\frac{9}{4}$

5 19 terms

6 22 terms

7 **a** $\dfrac{25\left(1 - \left(\frac{3}{5}\right)^k\right)}{\left(1 - \frac{3}{5}\right)} > 61 \Rightarrow 1 - \left(\frac{3}{5}\right)^k > \dfrac{122}{125} \Rightarrow \left(\frac{3}{5}\right)^k < \dfrac{3}{125}$

  $\Rightarrow k\log\left(\frac{3}{5}\right) < \log\left(\frac{3}{125}\right) \Rightarrow k > \dfrac{\log(0.024)}{\log(0.6)}$

  b $k = 8$

8 $r = \pm 0.4$

9 $S_{10} = \dfrac{a\left[\left(\sqrt{3}\right)^{10} - 1\right]}{\sqrt{3} - 1} = \dfrac{a(243 - 1)}{\sqrt{3} - 1}$

  $= \dfrac{242a(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = 121a(\sqrt{3} + 1)$

10 $\dfrac{a(2^4 - 1)}{1} = \dfrac{b(3^4 - 1)}{2}$

  $15a = 40b \Rightarrow a = \dfrac{8}{3}b$

11 **a** $\dfrac{2k + 5}{k} = \dfrac{k}{k - 6} \Rightarrow (k - 6)(2k + 5) = k^2$

  $k^2 - 7k - 30 = 0$

  b $k = 10$

  c 2.5

  d 25429