P1 Chapter 12: Differentiation

Sketching Gradient Functions

Sketching Graphs

All the way back in Chapter 4, we used features such as intercepts with the axes, and behaviour when $x \to \infty$ and $x \to -\infty$ in order to sketch graphs.
Now we can also find stationary/turning points!

[Textbook] By first finding the stationary points, sketch the graph of $y = \frac{1}{x} + 27x^3$

? Turning Points

? As $x \to \infty$, $x \to -\infty$

? Vertical Asymptotes

? Graph

Sketching Graphs

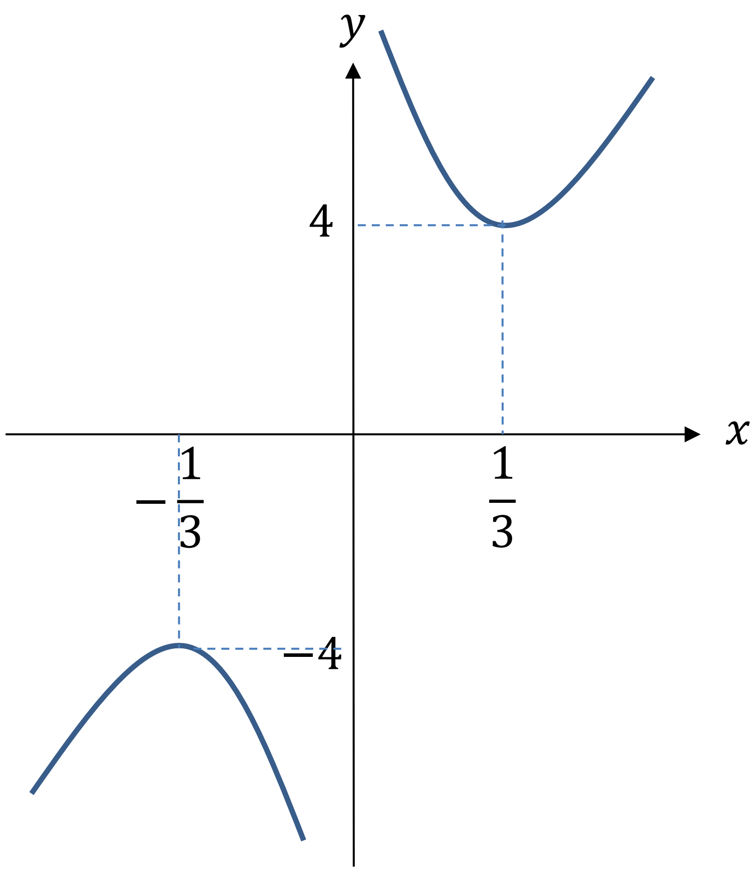
All the way back in Chapter 4, we used features such as intercepts with the axes, and behaviour when $x \to \infty$ and $x \to -\infty$ in order to sketch graphs.
Now we can also find stationary/turning points!

[Textbook] By first finding the stationary points, sketch the graph of $y = \frac{1}{x} + 27x^3$

$$y = x^{-1} + 27x^3$$
$$\frac{dy}{dx} = -x^{-2} + 81x^2 = 0$$
$$-\frac{1}{x^2} + 81x^2 = 0 \quad \to \quad 81x^2 = \frac{1}{x^2}$$
$$81x^4 = 1 \quad \to \quad x = \frac{1}{3}\ or\ -\frac{1}{3}$$
$$\left(\mathbf{\frac{1}{3}}, \mathbf{4}\right), \left(-\mathbf{\frac{1}{3}}, \mathbf{-4}\right)$$

As $x \to \infty$, $\boldsymbol{y} \to \infty$
As $x \to -\infty$, $\boldsymbol{y} \to -\infty$

x not defined at **0** (due to $\frac{1}{x}$ term)



Exercise

Extension

1 [MAT 2014 1C] The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

A) $k > 0$
B) $0 < k < 1$
C) $k > \frac{1}{2}$
D) $k < 3$
E) all values of k

2 *[MAT 2004 1B]* The smallest value of the function:

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

In the range $0 \leq x \leq 2$ is what?

3 *[MAT 2001 1E]* The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \leq x \leq 2\frac{1}{5}$ occurs when:

A) $x = 0$
B) $x = 1 - \frac{1}{\sqrt{3}}$
C) $x = 1 + \frac{1}{\sqrt{3}}$
D) $x = 2\frac{1}{5}$

Hint: When two curves touch, their y values must match, but what else must also match?

4 [STEP I 2007 Q8] A curve is given by:

$$y = ax^3 - 6ax^2 + (12a + 12)x - (8a + 16)$$

where a is a real number. Show that this curve touches the curve with equation $y = x^3$ at $(2,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes these two curves when $a = 2$.

(ii) … when $a = 1$ (iii) when $a = -2$

Solutions to Extension Questions

1 [MAT 2014 1C] The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

A) $k > 0$

B) $0 < k < 1$

C) $k > \frac{1}{2}$

D) $k < 3$

E) all values of k

$$\boldsymbol{\frac{dy}{dx} = 3kx^2 - 2(k+1)x + (2-k)}$$

When $\boldsymbol{x = 1, \frac{dy}{dx} = 3k - 2k - 2 + 2 - k \equiv 0}$

Therefore there is a turning point for all values of $\boldsymbol{k}$. However, this must be a minimum.

$$\boldsymbol{\frac{d^2y}{dx^2} = 6kx - 2(k+1)}$$

When $\boldsymbol{x = 1, \frac{d^2y}{dx^2} = 4k - 2}$

If minimum: $\boldsymbol{4k - 2 > 0 \quad \rightarrow \quad k > \frac{1}{2}}$

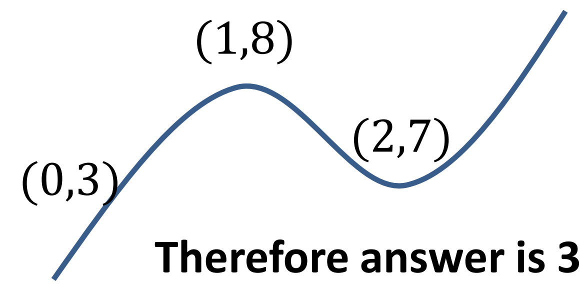
2 *[MAT 2004 1B]* The smallest value of the function:

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

In the range $0 \leq x \leq 2$ is what?

$$\boldsymbol{f'(x) = 6x^2 - 18x + 12 = 0}$$
$$\boldsymbol{x^2 - 3x + 2 = 0}$$
$$\boldsymbol{(x-1)(x-2) = 0}$$
$$\boldsymbol{x = 1 \quad \rightarrow \quad f(1) = 8}$$
$$\boldsymbol{x = 2 \quad \rightarrow \quad f(2) = 7}$$

At start of range: $\boldsymbol{f(0) = 3}$



Therefore answer is 3.

Solutions to Extension Questions

3 [MAT 2001 1E] The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \leq x \leq 2\frac{1}{5}$ occurs when:

A) $x = 0$

B) $x = 1 - \frac{1}{\sqrt{3}}$

C) $x = 1 + \frac{1}{\sqrt{3}}$

D) $x = 2\frac{1}{5}$

Gradient: $\frac{dy}{dx} = 4x^3 - 12x^2 + 8x$

So find max value of $4x^3 - 12x^2 + 8x$

$$\frac{d^2y}{dx^2} = 12x^2 - 24x + 8 = 0$$

$$3x^2 - 6x + 2 = 0$$

$$x = 1 \pm \frac{1}{\sqrt{3}}$$

Due to the shape of a cubic, we have the a local maximum gradient at $x = 1 - \frac{1}{\sqrt{3}}$

and a local minimum at $x = 1 + \frac{1}{\sqrt{3}}$

In range $0 \leq x \leq 2\frac{1}{5}$**, answer must either be** $x = 1 + \frac{1}{\sqrt{3}}$ **or** $x = 2\frac{1}{5}$**. Substituting these into** y **yields a higher value for the latter, so answer is (D).**

Solutions to Extension Questions

4 [STEP I 2007 Q8] A curve is given by:

$y = ax^3 - 6ax^2$
$+(12a + 12)x - (8a + 16)$

where a is a real number. Show that this curve touches the curve with equation $y = x^3$ at (2,8). Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes these two curves when $a = 2$.

(ii) … when $a = 1$

(iii) … when $a = -2$

Consider $\mathrm{f}(x) = ax^3 - 6ax^2 + (12a + 12)x - (8a + 16)$.

Since $\mathrm{f}(2) = 8$ and $\mathrm{f}'(2) = 12$, the curve $y = \mathrm{f}(x)$ touches $y = x^3$ at $(2, 8)$: notice that both calculations are necessary to prove that the curves **touch**.

To find the other intersection point, let $\mathrm{f}(x) = x^3$

$\Rightarrow (a-1)x^3 - 6ax^2 + (12a + 12)x - (8a + 16) = 0$

Substituting $x = 2$: $8(a-1) - 24a + 2(12a + 12) - (8a + 16) \equiv 0$

$\Rightarrow (x-2)\left[(a-1)x^2 - (4a+2)x + (4a+8)\right] = 0$ (notice that factorising by inspection is much easier than using a method such as long division)

Substituting $x = 2$ into the quadratic factor: $4(a-1) - 2(4a+2) + (4a+8) \equiv 0$.

$\Rightarrow (x-2)(x-2)\left[(a-1)x - (2a+4)\right] = 0$

So the other intersection point has coordinates $\left(\dfrac{2a+4}{a-1}, \left[\dfrac{2a+4}{a-1}\right]^3\right)$

(i) When $a = 2$, $\dfrac{2a+4}{a-1} = 8$.

Hence the two graphs touch at $(2, 8)$ and intersect at $(8, 512)$.

$y = 2x^3 - 12x^2 + 36x - 32$ has no turning points: consider the derivative $6x^2 - 24x + 36 = 0$.

(ii) When $a = 1$, $\dfrac{2a+4}{a-1}$ is undefined.

Hence the two graphs touch at $(2, 8)$, and do not intersect elsewhere.

$y = x^3 - 6x^2 + 24x - 24$ has no turning points: consider the derivative $3x^2 - 12x + 24 = 0$.

(iii) When $a = -2$, $\dfrac{2a+4}{a-1} = 0$.

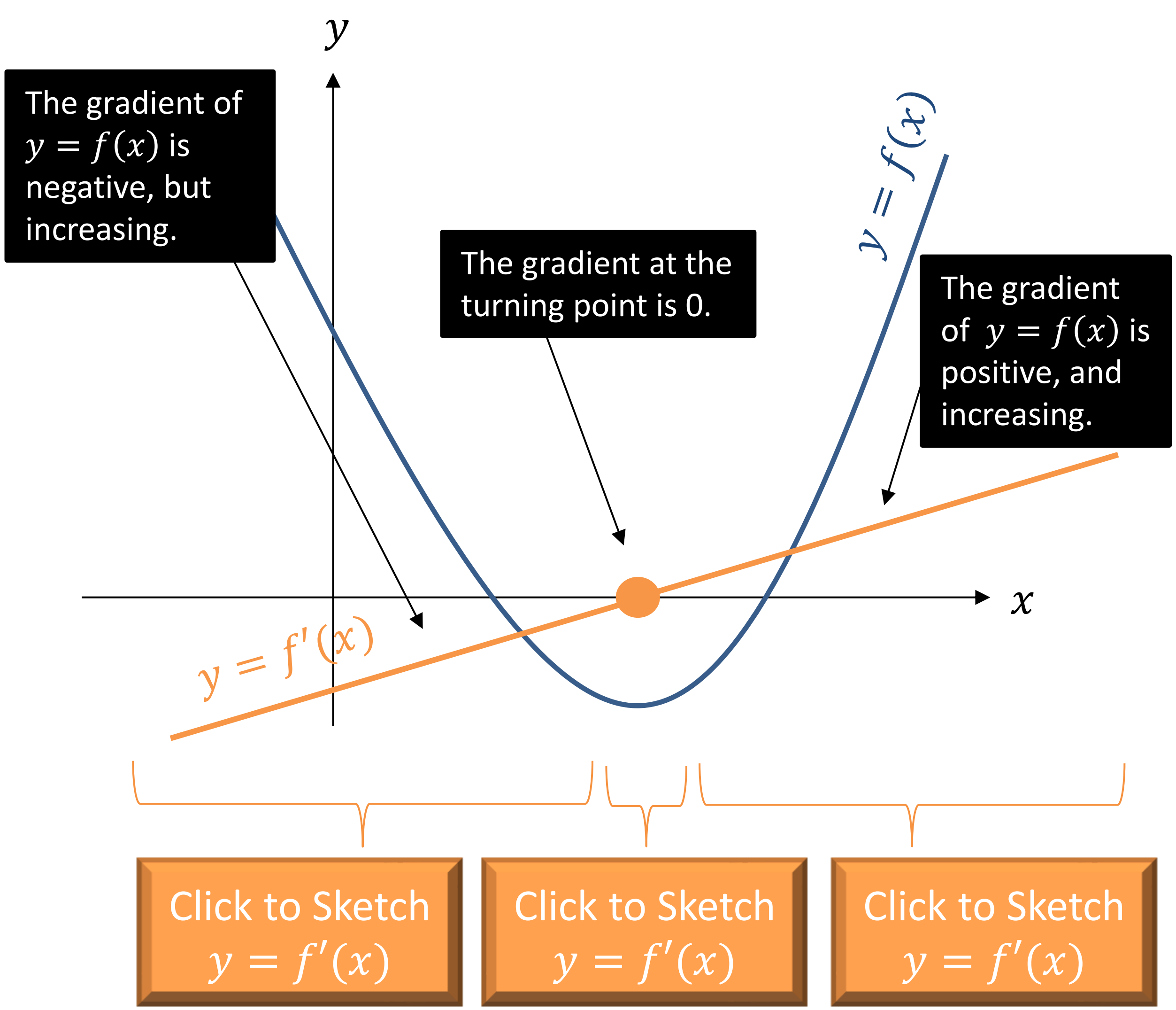
Hence the two graphs touch at $(2, 8)$ and intersect at $(0, 0)$.

$y = -2x^3 + 12x^2 - 12x$ turns when $x = 2 \pm \sqrt{2} \approx 3.4$ and 0.6: consider the derivative $-6x^2 + 24x - 12 = 0$.

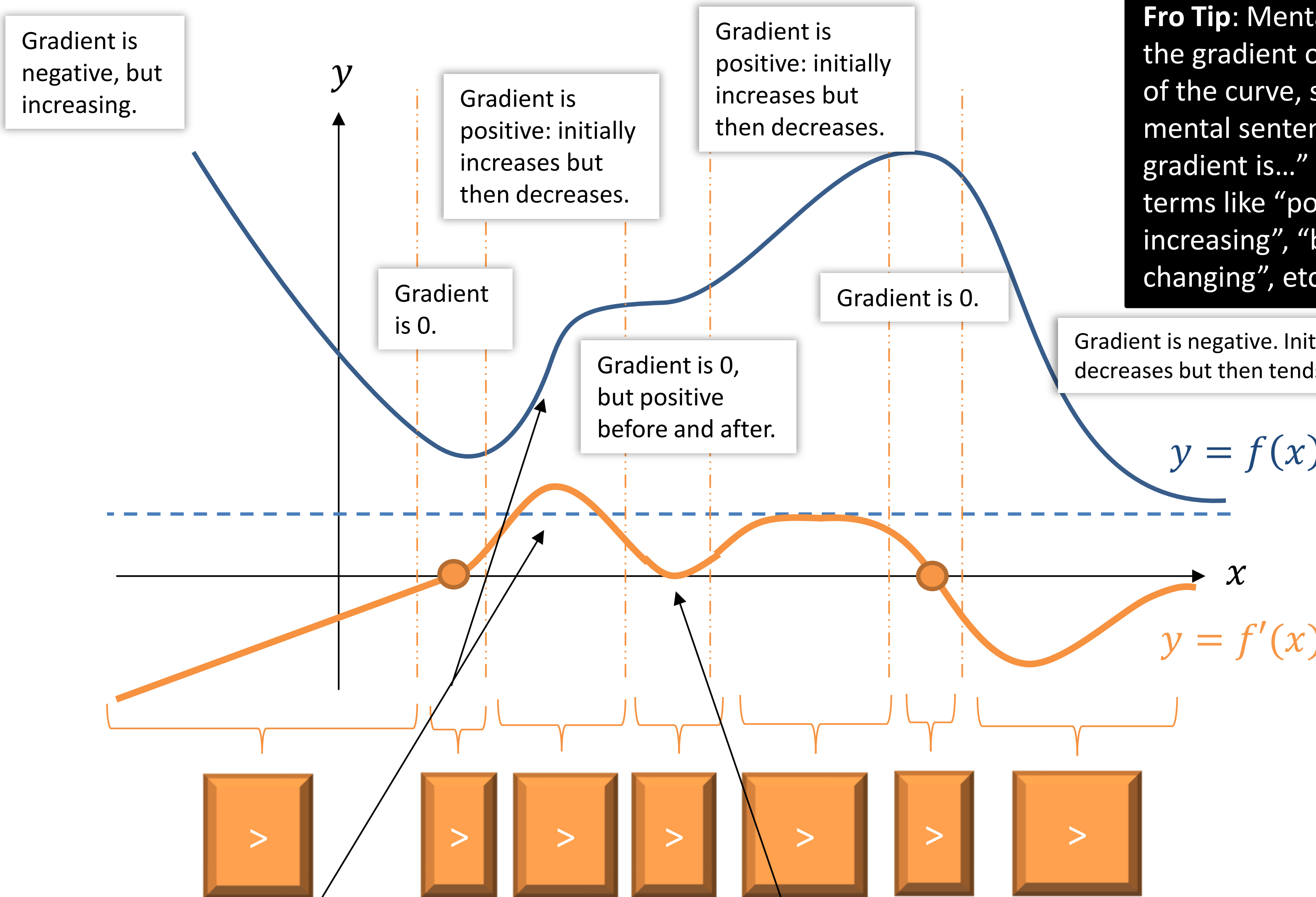
Sketching Gradient Functions

The new A Level specification specifically mentions being able to sketch $y = f'(x)$.

If you know the function $f'(x)$ explicitly (e.g. because you differentiated $y = f(x)$), you can use your knowledge of sketching straight line/quadratic/cubic graphs.
But in other cases **you won't be given the function explicitly**, but **just the sketch**.



A Harder One

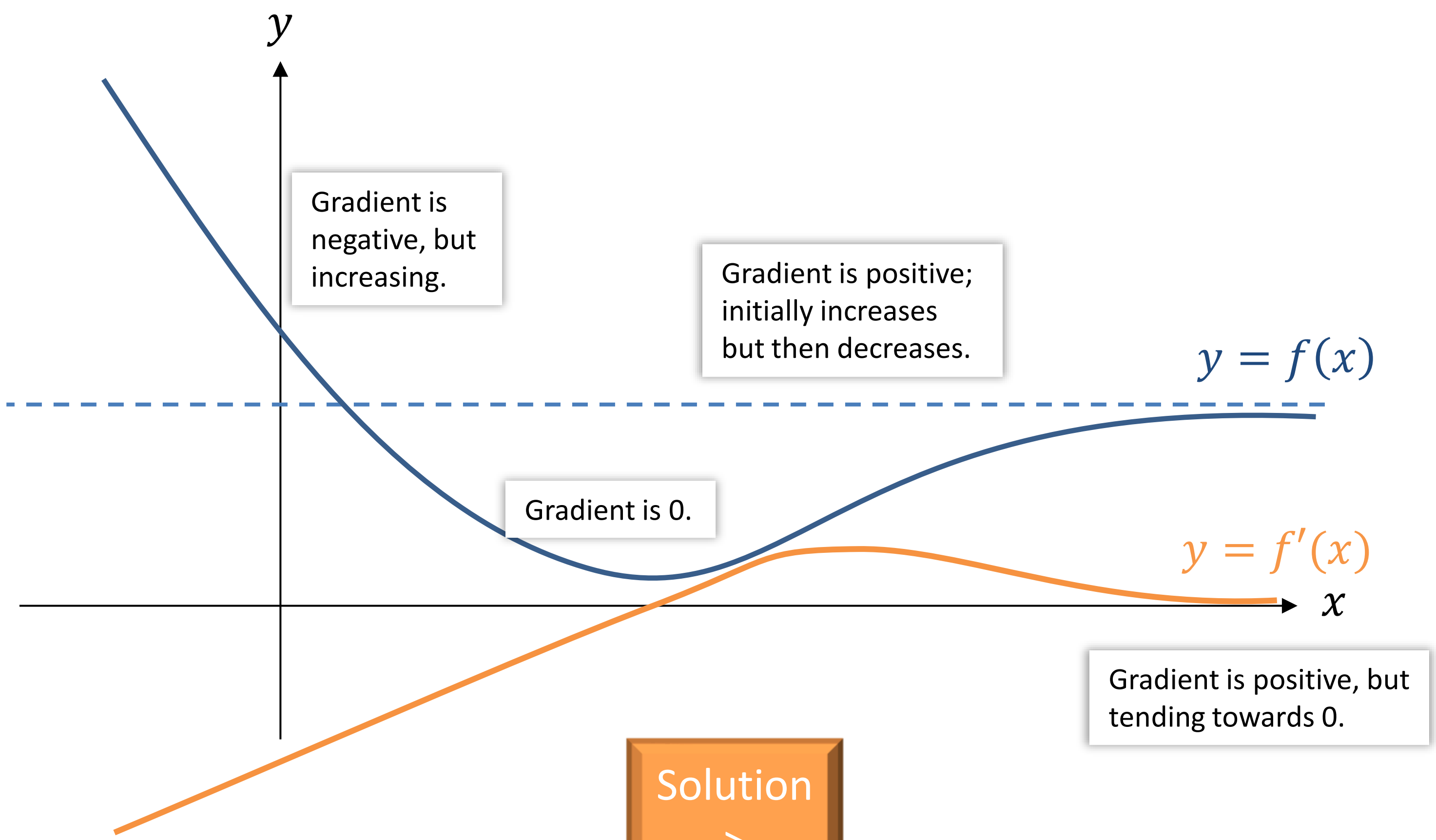


Fro Tip: Mentally describe the gradient of each section of the curve, starting your mental sentence with "The gradient is…" and using terms like "positive", "but increasing", "but not changing", etc.

Fro Pro Tip: The gradient is momentarily not changing so the gradient of the gradient is 0. We get a **turning point** in $y = f'(x)$ whenever the **curve has a point of inflection**.

Again, on $y = f(x)$ this is a point of inflection so the gradient function has a turning point. It is also a stationary point, so $f'(x) = 0$. i.e. The $f'(x)$ curve touches the x-axis.

Test Your Understanding



Solution >

Exercise 12.10

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Extension

1 [MAT 2015 1B]

$$f(x) = (x + a)^n$$

where a is a real number and n is a positive whole number, and $n \geq 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will:

A) always be odd
B) always be even
C) depend on a but not n
D) depend on n but not a
E) depend on both a and n.

?

Exercise 12.10

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Extension

1 [MAT 2015 1B]

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where a is a real number and n is a positive whole number, and $n \geq 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will:

A) always be odd
B) always be even
C) depend on a but not n
D) depend on n but not a
E) depend on both a and n.

(Official solution)
If $\boldsymbol{f(x)}$ is an even function then $\boldsymbol{f'(x)}$ will be an odd function, and vice versa. $\boldsymbol{f(x)}$ and $\boldsymbol{f'(x)}$ will cross the $\boldsymbol{x}$-axis at $\boldsymbol{(-a, 0)}$ and will have one further intersection when $\boldsymbol{y}$ and $\boldsymbol{x}$ are greater than 0.

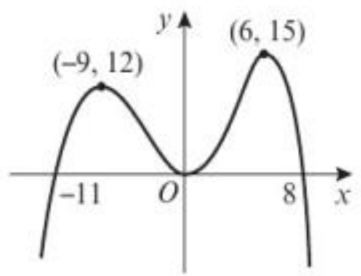
The answer is (b).

Recall that:
Odd function: $f(-x) = -f(x)$
Even function: $f(-x) = f(x)$

Homework Exercise

1 For each graph given, sketch the graph of the corresponding gradient function on a separate set of axes. Show the coordinates of any points where the curve cuts or meets the x-axis, and give the equations of any asymptotes.

a



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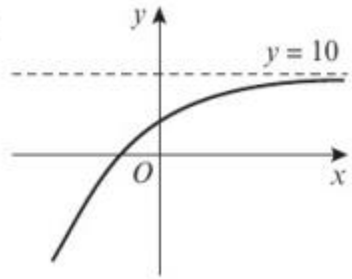

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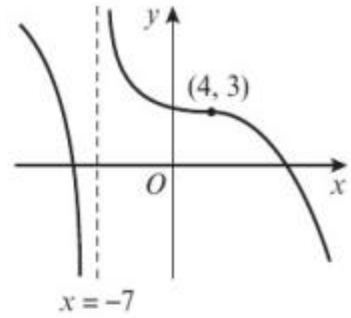

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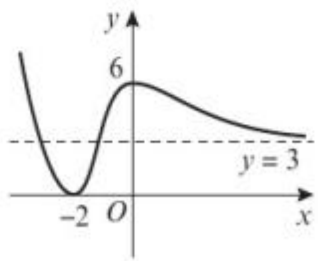

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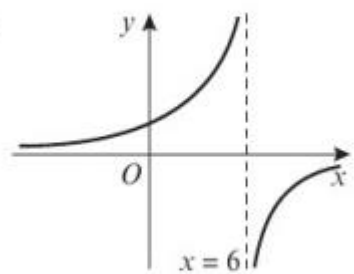

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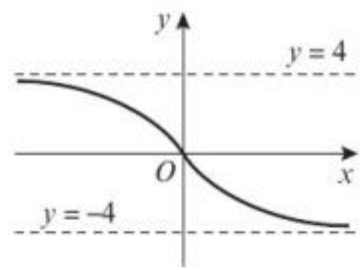


2 $f(x) = (x + 1)(x - 4)^2$

a Sketch the graph of $y = f(x)$.

b On a separate set of axes, sketch the graph of $y = f'(x)$.

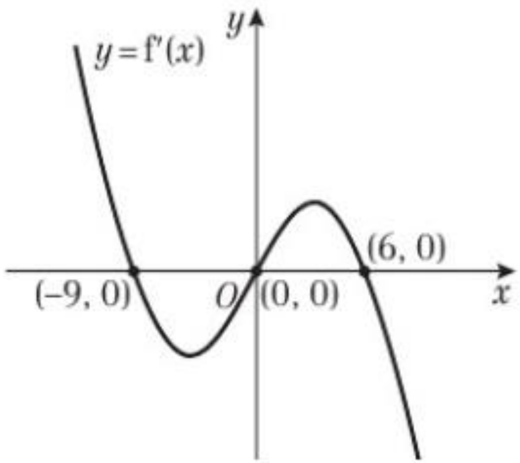
c Show that $f'(x) = (x - 4)(3x - 2)$.

d Use the derivative to determine the exact coordinates of the points where the gradient function cuts the coordinate axes.

Hint This is an x^3 graph with a positive coefficient of x^3. ← Section 4.1

Homework Answers

1 a



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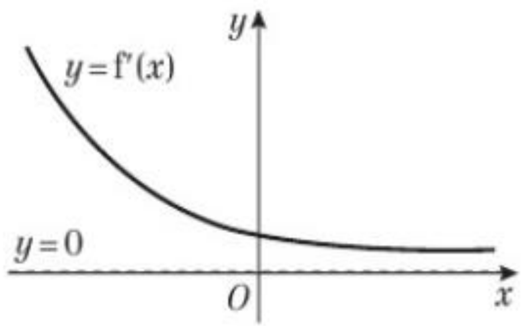

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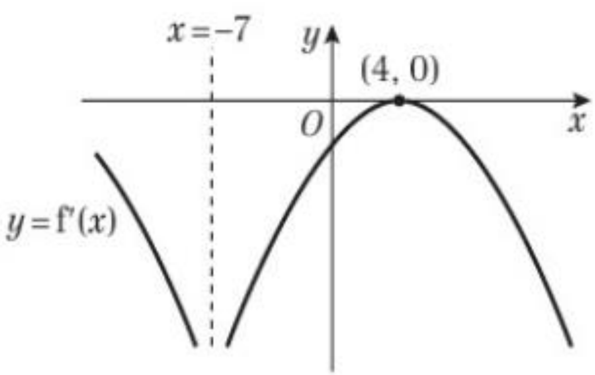

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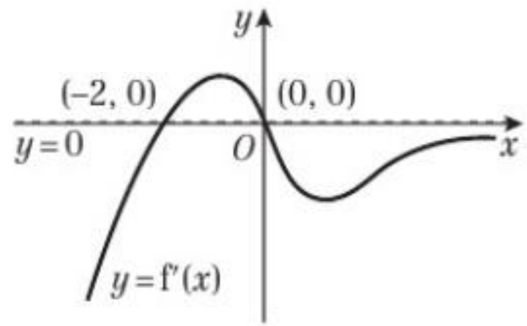

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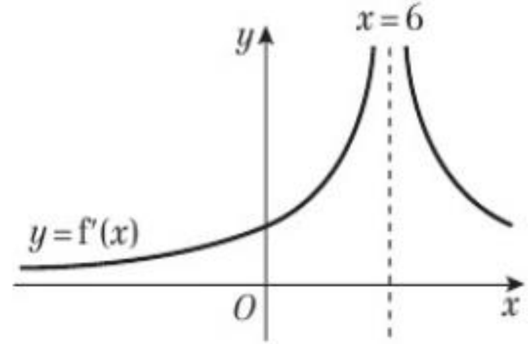

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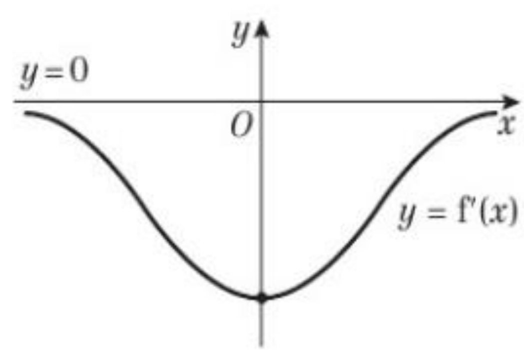
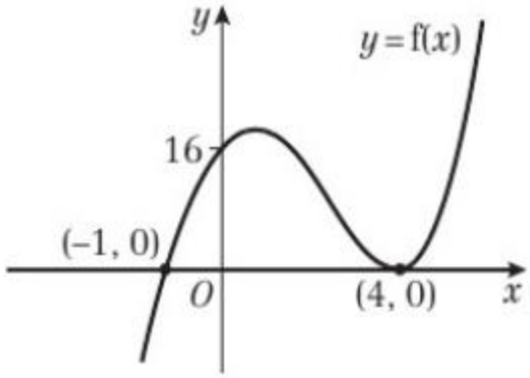


Homework Answers

2 **a**



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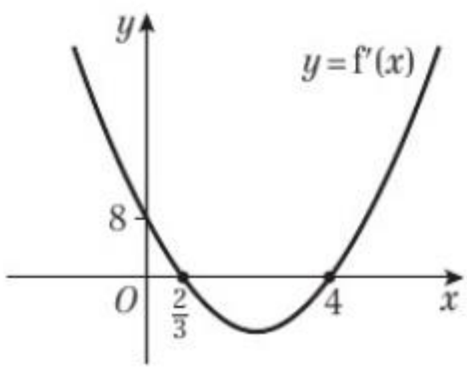

c $f(x) = x^3 - 7x^2 + 8x + 16$

$f'(x) = 3x^2 - 14x + 8 = (3x - 2)(x - 4)$

d $(4, 0)$, $\left(\frac{2}{3}, 0\right)$ and $(0, 8)$