Stats1 Chapter 7: Hypothesis Testing

Critical Values

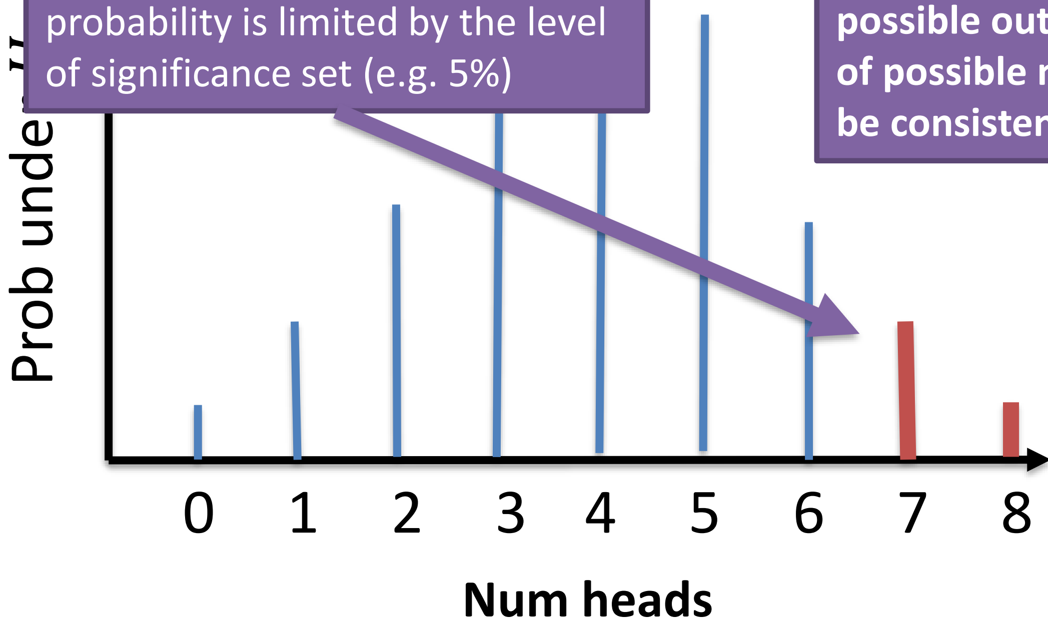
Critical Regions and Values

John wants to see whether a coin is unbiased or whether **it is biased towards coming down heads**. He tosses the coin 8 times and counts the number of times X, it lands head uppermost. **What values would lead to John's hypothesis being rejected?**

As before, we're interested how likely a given outcome is likely to happen 'just by chance' under the null hypothesis (i.e. when the coin is not biased).

However, there are values which collectively form a range of 'extreme values' where it would be unlikely that the coin would be unbiased. Their combined probability is limited by the level of significance set (e.g. 5%)

The probability of getting exactly 5 heads is only 22%, which is more likely to not happen than to happen. If we saw this number of heads, why would it not be sensible to think the coin is biased?
The probability is only low because there's lots of possible outcomes. But 5 heads forms part of a range of possible number of heads that collectively would be consistent with a coin not biased towards heads.



Critical Regions and Values

John wants to see whether a coin is unbiased or whether **it is biased towards coming down heads**. He tosses the coin 8 times and counts the number of times X, it lands head uppermost. **What values would lead to John's hypothesis being rejected**, if the significance level was 5%?

What's the probability that we would see **6 heads**, or an **even more extreme value**? Is this sufficiently unlikely to support John's claim that the coin is biased?

$$\boldsymbol{P(X \geq 6) = 1 - P(X \leq 5)}$$
$$\boldsymbol{= 0.1445}$$

Insufficient evidence to reject null hypothesis (since $\boldsymbol{0.1445 > 0.05}$).

What's the probability that we would see **7 heads**, or an **even more extreme value**?

$$\boldsymbol{P(X \geq 7) = 1 - P(X \leq 6)}$$
$$\boldsymbol{= 0.0352}$$

Since 0.0352 < 0.05, this is very unlikely, so we reject the null hypothesis and accept the alternative hypothesis that the coin is biased.

C.D.F. Binomial table: $p = 0.5, n = 8$

x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
3	0.3633
4	0.6367
5	0.8555
6	0.9648
7	0.9961

Critical Regions and Values

John wants to see whether a coin is unbiased or whether **it is biased towards coming down heads**. He tosses the coin 8 times and counts the number of times X, it lands head uppermost. **What values would lead to John's hypothesis being rejected**, if the significance level was 5%?

✎ The **critical region** is the range of values of the test statistic that would lead to you rejecting H_0

If level of significance 5%, critical region?

We saw that 95% is exceeded when $X = 6$. This means $P(X \geq 7) = 1 - P(X \leq 6)$
$= \mathbf{0.0352} < \mathbf{5\%}$
$\therefore \mathbf{7} \leq \boldsymbol{X} \leq \mathbf{8}$

Fro Tip: Use the first value AFTER the one in the table that exceeds 95%.

✎ The value(s) on the boundary of the critical region are called **critical value(s)**.

Critical value:

7

C.D.F. Binomial table:
$p = 0.5, n = 8$

x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
3	0.3633
4	0.6367
5	0.8555
6	0.9648
7	0.9961
8	1

Quickfire Critical Regions

Determine the critical region when we throw a coin where we're trying to establish if there's the specified bias, given the specified number of throws, when the level of significance is 5%.

Coin thrown 5 times. Trying to establish if biased towards heads.

$$p = 0.5, n = 5$$

x	$P(X \leq x)$
0	0.0312
1	0.1875
2	0.5000
3	0.8125
4	0.9688

Critical region:

$$X = 5$$

Coin thrown 10 times. Trying to establish if biased towards heads.

$$p = 0.5, n = 10$$

x	$P(X \leq x)$
0	0.0010
1	0.0107
2	0.0547
…	…
7	0.9453
8	0.9893
9	0.9990

Critical region:

$$9 \leq X \leq 10$$

Coin thrown 10 times. Trying to establish if biased towards tails.

$$p = 0.5, n = 10$$

x	$P(X \leq x)$
0	0.0010
1	0.0107
2	0.0547
…	…
7	0.9453
8	0.9893
9	0.9990

Critical region:

$$0 \leq X \leq 1$$

Fro Reminder: At the positive tail, use the value AFTER the first that exceeds 95% (100 - 5).

At the negative tail, we just use the first value that goes under the significance level.

Actual Significance Level

John wants to see whether a coin is unbiased or whether **it is biased towards coming down heads**. He tosses the coin 8 times and counts the number of times X, it lands head uppermost. **What values would lead to John's hypothesis being rejected**, if the significance level was 5%?

We saw earlier that the critical region was $X \geq 7$, i.e. the region in which John would reject the null hypothesis (and conclude the coin was biased).

We ensured that $P(X \geq 7)$ was less than the significance level of 5%.
But what actually is $P(X \geq 7)$?

$$\boldsymbol{P(X \geq 7) = 1 - P(X \leq 6) = 0.0352}$$

This is known as the actual significance level, i.e. the probability that we're in the critical region. We expected this to be less than, but close to, 5%.

✎ The **actual significance level** is the actual probability of being in the critical region.

C.D.F. Binomial table: $p = 0.5, n = 8$

x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
3	0.3633
4	0.6367
5	0.8555
6	0.9648
7	0.9961
8	1

Two-tailed test

Suppose I threw a coin 8 times and was now interested in how may heads would suggest it was a **biased coin** (i.e. either way!). How do we work out the critical values now, with 5% significance?

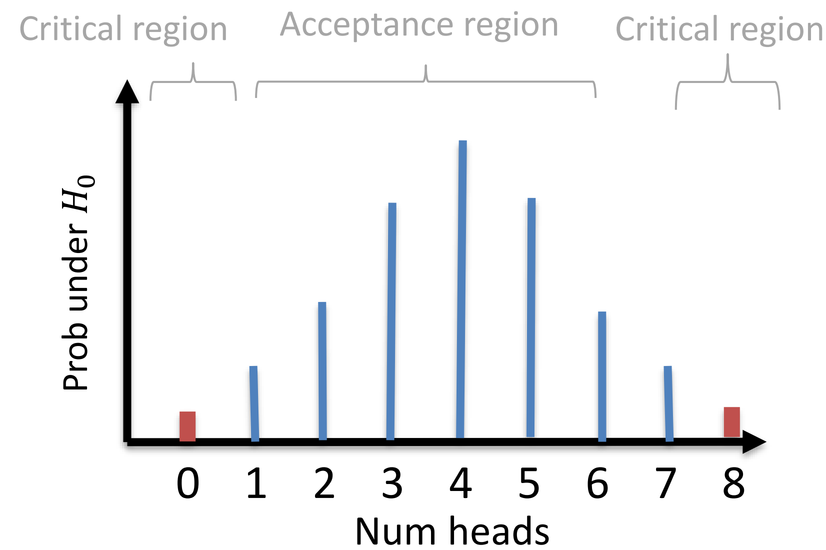
We split the 5% so there's 2.5% at either tail, then proceed as normal:

Critical region at positive tail:
Look at closest value above 0.975 (then go one above):

$$\boldsymbol{X = 8}$$

Critical region at negative tail:
Look at closest value below 0.025.

$$\boldsymbol{X = 0}$$



C.D.F. Binomial table:
$p = 0.5, n = 8$

x	$P(X \leq x)$
0	0.0039
1	0.0352
2	0.1445
…	…
6	0.9648
7	0.9961
8	1

Test Your Understanding

A random variable X has binomial distribution $B(40, p)$. A single observation is used to test $H_0: p = 0.25$ against $H_1: p \neq 0.25$.

a) Using the 2% level of significance, find the critical region of this test. The probability in each tail should be as close as possible to 0.01.

b) Write down the actual significance level of the test.

The $\neq$ indicates bias either way, i.e. two-tailed.

This means you find the closest to 0.01 (even if slightly above) rather than the closest under 0.01

a

?

b

?

C.D.F. Binomial table: $p = 0.25, n = 40$

x	$P(X \leq x)$
2	0.0010
3	0.0047
4	0.0160
5	0.0433
16	0.9884
17	0.9953
18	0.9983
19	0.9994

Warning: Textbook has several typos in this example.

Test Your Understanding

A random variable X has binomial distribution $B(40, p)$. A single observation is used to test $H_0: p = 0.25$ against $H_1: p \neq 0.25$.

a) Using the 2% level of significance, find the critical region of this test. The probability in each tail should be as close as possible to 0.01.

b) Write down the actual significance level of the test.

The $\neq$ indicates bias either way, i.e. two-tailed.

This means you find the closest to 0.01 (even if slightly above) rather than the closest under 0.01

a

(Half of 0.02 is 0.01)
$P(X \leq 3) = 0.0047$
$0 \leq X \leq 3$

To ensure all method marks always show the probability of being in the critical region (even if you don't subsequently need the value!)

$P(X \geq 17) = 1 - P(X \leq 16) = 0.0116$
$17 \leq X \leq 40$
Critical region is $0 \leq X \leq 3$ or $17 \leq X \leq 40$

Note that X can't go below 0 or exceed 40.

b

$0.0047 + 0.0116 = 0.0163 = 1.63\%$

C.D.F. Binomial table: $p = 0.25, n = 40$

x	$P(X \leq x)$
2	0.0010
3	0.0047
4	0.0160
5	0.0433
16	0.9884
17	0.9953
18	0.9983
19	0.9994

Warning: Textbook has several typos in this example.

Exercise 7.2

Pearson Applied Year 1/AS

Pages 46-47

Homework Exercise

1 Explain what you understand by the following terms:

a critical value **b** critical region **c** acceptance region.

2 A test statistic has a distribution B(10, p). Given that H_0: $p = 0.2$, H_1: $p > 0.2$, find the critical region for the test using a 5% significance level.

3 A random variable has a distribution B(20, p). A single observation is used to test H_0: $p = 0.15$ against H_1: $p < 0.15$. Using a 5% level of significance, find the critical region of this test.

4 A random variable has distribution B(20, p). A single observation is used to test H_0: $p = 0.4$ against H_1: $p \neq 0.4$.

a Using the 5% level of significance, find the critical region of this test. **(3 marks)**

b Write down the actual significance level of the test. **(1 mark)**

5 A test statistic has a distribution B(20, p). Given that H_0: $p = 0.18$, H_1: $p < 0.18$, find the critical region for the test using a 1% level of significance.

Watch out These probabilities are not found in statistical tables. You can use your calculator to find cumulative probabilities for B(n, p) with any values of n and p.

6 A random variable has distribution B(10, p). A single observation is used to test H_0: $p = 0.22$ against H_1: $p \neq 0.22$.

a Using a 1% level of significance, find the critical region of this test. The probability in each tail should be as close as possible to 0.005. **(3 marks)**

b Write down the actual significance level of the test. **(2 marks)**

Homework Exercise

7 A mechanical component fails, on average, 3 times out of every 10. An engineer designs a new system of manufacture that he believes reduces the likelihood of failure. He tests a sample of 20 components made using his new system.

a Describe the test statistic. **(1 mark)**

b State suitable null and alternative hypotheses. **(2 marks)**

c Using a 5% level of significance, find the critical region for a test to check his belief, ensuring the probability is as close as possible to 0.05. **(3 marks)**

d Write down the actual significance level of the test. **(1 mark)**

8 Seedlings come in trays of 36. On average, 12 seedlings survive to be planted on. A gardener decides to use a new fertiliser on the seedlings which she believes will improve the number that survive.

a Describe the test statistic and state suitable null and alternative hypotheses. **(3 marks)**

b Using a 10% level of significance, find the critical region for a test to check her belief. **(3 marks)**

c State the probability of incorrectly rejecting H_0 using this critical region. **(1 mark)**

9 A restaurant owner notices that her customers typically choose lasagne one fifth of the time. She changes the recipe and believes this will change the proportion of customers choosing lasagne.

a Suggest a model and state suitable null and alternative hypotheses. **(3 marks)**

She takes a random sample of 25 customers.

b Find, at the 5% level of significance, the critical region for a test to check her belief. **(4 marks)**

c State the probability of incorrectly rejecting H_0. **(1 mark)**

Homework Answers

1 **a** The critical value is the first value to fall inside of the critical region.
b A critical region is a region of the probability distribution which, if the test statistic falls within it, would cause you to reject the null hypothesis.
c The acceptance region is the area in which we accept the null hypothesis.

2 The critical value is $x = 5$ and the critical region is $X \geqslant 5$ since $P(X \geqslant 5) = 0.0328 < 0.05$.

3 The critical value is $x = 0$ and the critical region is $X = 0$.

4 **a** The critical region is $X \geqslant 13$ and $X \leqslant 3$.
b $0.037 = 3.7\%$

5 The critical value is $x = 0$. The critical region is $X = 0$.

6 **a** The critical region is $X = 0$ and $7 \leqslant X \leqslant 10$.
b 0.085

7 **a** The number of times the sample fails.
b H_0: $p = 0.3$, H_1: $p < 0.3$
c The critical value is $x = 10$ and the critical region is $X \geqslant 10$
d 4.8%

8 **a** The number of seedlings that survive.
H_0: $p = \frac{1}{3}$, H_1: $p > \frac{1}{3}$
b The critical value is $x = 17$ and the critical region is $X \geqslant 17$
c 5.84%

9 **a** H_0: $p = 0.2$, H_1: $p \neq 0.2$
b The critical region is $X \leqslant 1$ and $X \geqslant 10$
c 4.47%