S1 Chapter 6: Statistical Distributions

Cumulative Probability

Cumulative Probabilities

Often we wish to find the probability of a range of values.
For a Binomial distribution, this was relatively easy if the range was narrow, e.g. $P(X \leq 1) = P(X = 0) + P(X = 1)$, but would be much more computationally expensive if we wanted say $P(X \leq 6)$.

If $X \sim B(10, 0.3)$, find $P(X \leq 6)$.

How to calculate on your ClassWiz:

Press Menu then 'Distributions'.
Choose "Binomial CD" (the C stands for 'Cumulative').
Choose 'Variable'.

$$x = 6$$
$$N = 10$$
$$p = 0.3$$

Pressing = gives the desired value.

Using tables (e.g. Page 204 of textbook)

Look up $n = 10$ and the column $p = 0.3$.
Then look up the row $x = 6$.
The value should be 0.9894.

Important Note: The tables only have limited values of p. You may have to use your calculator.
You will need to use your calculator in the exam anyway.

Cumulative Probabilities

The random variable $X \sim B(20,0.4)$. Find:

$P(X \leq 7) =$?

$P(X < 6) =$?

$P(X \geq 15) =$?

Look up $n = 20, p = 0.4, x = 7$

Note that the table requires $\leq$

To get this right, just say in your head *"What's the opposite of 'at least 15'?"*. Hopefully you can see it's *'at most 14'*.

Given that $X \sim B(25,0.25)$

$P(X = 6) =$?

$P(X > 20) =$?

$P(6 < X \leq 10) =$?

X can be 7 to 10. So we want up to 10, with everything up to 6 excluded.

Quickfire Questions

Write the following in terms of cumulative probabilities, e.g. $P(X < 7) = P(X \leq 6)$

$P(X < 5) =$?

$P(X \geq 7) =$?

$P(X > 7) =$?

$P(10 \leq X < 20) =$?

$P(10 \leq X \leq 20) =$?

$P(X = 100) =$?

$P(20 < X < 30) =$?

"at least 30" = ?

"greater than 30" = ?

Cumulative Probabilities

The random variable $X \sim B(20, 0.4)$. Find:

$P(X \leq 7) = 0.4159$

$P(X < 6) = P(X \leq 5) = 0.1256$

$P(X \geq 15) = 1 - P(X \leq 14) = 0.0016$

Look up $n = 20, p = 0.4, x = 7$

Note that the table requires $\leq$

To get this right, just say in your head *"What's the opposite of 'at least 15'?"*. Hopefully you can see it's *'at most 14'*.

Given that $X \sim B(25, 0.25)$

$P(X = 6) = \boldsymbol{P(X \leq 6) - P(X \leq 5)}$

$\quad = \mathbf{0.1828}$

$P(X > 20) = \boldsymbol{1 - P(X \leq 20)}$

$P(6 < X \leq 10) = \boldsymbol{P(X \leq 10) - P(X \leq 6)}$

X can be 7 to 10. So we want up to 10, with everything up to 6 excluded.

Quickfire Questions

Write the following in terms of cumulative probabilities, e.g. $P(X < 7) = P(X \leq 6)$

$P(X < 5) = \boldsymbol{P(X \leq 4)}$

$P(X \geq 7) = \boldsymbol{1 - P(X \leq 6)}$

$P(X > 7) = \boldsymbol{1 - P(X \leq 7)}$

$P(10 \leq X < 20) = \boldsymbol{P(X \leq 19) - P(X \leq 9)}$

$P(10 \leq X \leq 20) = \boldsymbol{P(X \leq 20) - P(X \leq 9)}$

$P(X = 100) = \boldsymbol{P(X \leq 100) - P(X \leq 99)}$

$P(20 < X < 30) = \boldsymbol{P(X \leq 29) - P(X \leq 20)}$

"at least 30" $= \boldsymbol{P(X \geq 30) = 1 - P(X \leq 29)}$

"greater than 30" $= \boldsymbol{P(X > 30) = 1 - P(X \leq 30)}$

More Challenging Example

Q

An awkward Tiffin boy ventures into Tiffin Girls. He asks 20 girls out on the date. The probability that each girl says yes is 0.3.
Determine the probability that he will end up with:
a) Less than 6 girls on his next date.
b) At least 9 girls on his next date.

The boy considers the evening a success if he dated at least 9 girls that evening. He repeats this process across 5 evenings.
c) Calculate the probability that he had at least 4 successful evenings.
(Note: You won't be able to use your table for (c) as p is not a nice round number – calculate prob directly)

a ?

b ?

c ?

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a

$X \sim B(20, 0.3)$ X is the number of girls dated in an evening.

$$P(X < 6) = P(X \leq 5)$$
$$= \mathbf{0.4164}$$

b

$$P(X \geq 9) = 1 - P(X \leq 8)$$
$$= \mathbf{1 - 0.8867}$$
$$= \mathbf{0.1133}$$

c

$Y \sim B(5, 0.1133)$ Y is the number of successful evenings.

$$\boldsymbol{P(X \geq 4) = P(X = 4) + P(X = 5)}$$
$$= \binom{5}{4} \mathbf{0.1133^4 \times 0.8867} \quad + \quad \mathbf{0.1133^5}$$
$$= \mathbf{0.000749}$$

This is an interesting problem because the probability from a Binomial distribution is then used as the p of a **second** separate Binomial distribution.

Dealing with Probability Ranges

Q

A spinner is designed so that probability it lands on red is 0.3. Jane has 12 spins.

a) Find the probability that Jane obtains at least 5 reds.

Jane decides to use this spinner for a class competition. She wants the probability of winning a prize to be < 0.05. Each member of the class will have 12 spins and the number of reds will be recorded.

b) Find how many reds are needed to win the prize.

?

Dealing with Probability Ranges

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$$P(X \geq 5) = 1 - P(X \leq 4) = 0.2763$$

$$P(X \geq r) < 0.05$$
$$1 - P(X \leq r - 1) < 0.05$$
$$P(X \leq r - 1) > 0.95$$
$$r - 1 = 6$$
$$\boldsymbol{r = 7}$$

STEP 1: Represent the sentence using probability.

STEP 2: Ensure LHS involves $\leq$ inside probability.

STEP 3: Rearrange.

STEP 4: Use table backwards to find value corresponding to closest probability.

Note that the textbook does this in a less methodical way: but the method above is what you would find in S2 exam mark schemes, so ignore at your peril.

Test Your Understanding

Q

At Camford University, students have 20 exams at the end of the year. All students pass each individual exam with probability 0.45. Students are only allowed to continue into the next year if they pass some minimum of exams out of the 20.
What do the university administrators set this minimum number such that the probability of continuing to next year is at least 90%?

?

Test Your Understanding

Q

At Camford University, students have 20 exams at the end of the year. All students pass each individual exam with probability 0.45. Students are only allowed to continue into the next year if they pass some minimum of exams out of the 20.
What do the university administrators set this minimum number such that the probability of continuing to next year is at least 90%?

$$X \sim B(20, 0.45)$$
$$P(X \geq k) \geq 0.9$$
$$1 - P(X \leq k-1) \geq 0.9$$
$$P(X \leq k-1) \leq 0.1$$
$$k - 1 = 5$$
$$\boldsymbol{k = 6}$$

This is **exactly** what you should write.

Exercise 6.3

Pearson Applied Year 1/AS
Pages 42-43

Homework Exercise

1 The random variable $X \sim B(9, 0.2)$. Find:

a $P(X \leqslant 4)$ **b** $P(X < 3)$ **c** $P(X \geqslant 2)$ **d** $P(X = 1)$

2 The random variable $X \sim B(20, 0.35)$. Find:

a $P(X \leqslant 10)$ **b** $P(X > 6)$

c $P(X = 5)$ **d** $P(2 \leqslant X \leqslant 7)$

Hint $P(2 \leqslant X \leqslant 7) = P(X \leqslant 7) - P(X \leqslant 1)$

3 The random variable $X \sim B(40, 0.47)$. Find:

a $P(X < 20)$ **b** $P(X > 16)$

c $P(11 \leqslant X \leqslant 15)$ **d** $P(10 < X < 17)$

Watch out For questions **3** and **4** the values of n and p aren't given in the tables. Use the binomial cumulative distribution function on your calculator.

4 The random variable $X \sim B(37, 0.65)$. Find:

a $P(X > 20)$ **b** $P(X \leqslant 26)$

c $P(15 \leqslant X < 20)$ **d** $P(X = 23)$

5 Eight fair coins are tossed and the total number of heads showing is recorded. Find the probability of:

a no heads **b** at least 2 heads **c** more heads than tails.

6 For a particular type of plant 25% have blue flowers. A garden centre sells these plants in trays of 15 plants of mixed colours. A tray is selected at random.
Find the probability that the number of plants with blue flowers in this tray is:

a exactly 4

b at most 3

c between 3 and 6 (inclusive).

Homework Exercise

7 The random variable $X \sim B(50, 0.40)$. Find:

a the largest value of k such that $P(X \leqslant k) < 0.05$ **(1 mark)**

b the smallest number r such that $P(X > r) < 0.01$. **(2 marks)**

8 The random variable $X \sim B(40, 0.10)$. Find:

a the largest value of k such that $P(X < k) < 0.02$ **(1 mark)**

b the smallest number r such that $P(X > r) < 0.01$ **(2 marks)**

c $P(k \leqslant X \leqslant r)$. **(2 marks)**

9 In a town, 30% of residents listen to the local radio.
Ten residents are chosen at random.
X = the number of these 10 residents that listen to the local radio.

a Suggest a suitable distribution for X and comment on any necessary assumptions. **(2 marks)**

b Find the probability that at least half of these 10 residents listen to local radio. **(2 marks)**

c Find the smallest value of s so that $P(X \geqslant s) < 0.01$. **(2 marks)**

10 A factory produces a component for the motor trade and 5% of the components are defective. A quality control officer regularly inspects a random sample of 50 components. Find the probability that the next sample contains:

a fewer than 2 defectives **(1 mark)**

b more than 5 defectives. **(2 marks)**

The officer will stop production if the number of defectives in the sample is greater than a certain value d. Given that the officer stops production less than 5% of the time:

c find the smallest value of d. **(2 marks)**

Homework Answers

1	**a**	0.9804	**b**	0.7382	**c**	0.5638	**d**	0.3020
2	**a**	0.9468	**b**	0.5834	**c**	0.1272	**d**	0.5989
3	**a**	0.5888	**b**	0.7662	**c**	0.1442	**d**	0.2302
4	**a**	0.8882	**b**	0.7992	**c**	0.0599	**d**	0.1258
5	**a**	0.0039	**b**	0.9648	**c**	0.3633		
6	**a**	0.2252	**b**	0.4613	**c**	0.7073		
7	**a**	$k = 13$	**b**	$r = 28$				
8	**a**	$k = 1$	**b**	$r = 9$	**c**	0.9801		

9 a $X \sim B(10, 0.30)$ Assumptions: The random variable can take two values (listen or don't listen), there are a fixed number of trials (10) and a fixed probability of success (0.3), each member in the sample is independent.

b 0.1503 **c** $s = 8$

10 a 0.2794 **b** 0.0378 **c** $d = 5$