P2 Chapter 3: Sequences and Series

Sigma Notation

Sigma Notation

What does each bit of this expression mean?

The Greek letter, capital sigma, means 'sum'.

$$\sum_{r=1}^{5}(2r+1)$$

The numbers top and bottom tells us what r varies between. It goes up by 1 each time.

We work out this expression for each value of r (between 1 and 5), and add them together.

$$= \underset{r=1}{3} \; \underset{r=2}{+5} \; \underset{r=3}{+7} \; \underset{r=4}{+9} \; \underset{r=5}{+11} = 35$$

If the expression being summed (in this case $2r+1$) is **linear**, we get an **arithmetic series**. We can therefore apply our usual approach of establishing a, d and n before applying the S_n formula.

Determining the value

	First few terms?	Values of a, n, d or r?	Final result?
$\sum_{n=1}^{7} 3n$	?	?	?
$\sum_{k=5}^{15} (10-2k)$	?	?	?
$\sum_{k=1}^{12} 5 \times 3^{k-1}$	?	?	?
$\sum_{k=5}^{12} 5 \times 3^{k-1}$		?	

Determining the value

	First few terms?	Values of $a, n, d\ or\ r$?	Final result?
$\sum_{n=1}^{7} 3n$	$= 3 + 6 + 9 + \cdots$	$a = 3, d = 3, n = 7$	$S_7 = \frac{7}{2}(6 + 6 \times 3)$ $= 84$
$\sum_{k=5}^{15} (10 - 2k)$	$= 0 + (-2) + (-4) + \cdots$	$a = 0, d = -2, n = 11$ Be careful, there are 11 numbers between 5 and 15 inclusive. Subtract and +1.	$S_{11} = \frac{11}{2}(0 + 10 \times -2)$ $= -110$
$\sum_{k=1}^{12} 5 \times 3^{k-1}$	$= 5 + 15 + 45 + \cdots$	$a = 5, r = 3, n = 12$	$S_{12} = \frac{5(1 - 3^{12})}{1 - 3}$ $= 1328600$
$\sum_{k=5}^{12} 5 \times 3^{k-1}$			

Note: You can either find from scratch by finding the first few terms (the first when $k = 5$), or by calculating:

$$\sum_{k=1}^{12} 5 \times 3^{k-1} - \sum_{k=1}^{4} 5 \times 3^{k-1}$$

i.e. We start with the first 12 terms, and subtract the first 4 terms.

Testing Your Understanding

Solomon Paper A

Evaluate

$$\sum_{r=10}^{30} (7 + 2r). \qquad \textbf{(4)}$$

Method 1: Direct

?

Method 2: Subtraction

?

Testing Your Understanding

Solomon Paper A

Evaluate

$$\sum_{r=10}^{30} (7 + 2r). \qquad \textbf{(4)}$$

Method 1: Direct

$$\sum_{r=10}^{30} (7 + 2r) = 27 + 29 + 31 + \cdots$$

$$a = 27, d = 2, n = 21$$

$$\therefore S_{21} = \frac{21}{2}(54 + 20 \times 2)$$

$$= 987$$

Method 2: Subtraction

$$\sum_{r=10}^{30} (7 + 2r) = \sum_{r=1}^{30} (7 + 2r) - \sum_{r=1}^{9} (7 + 2r)$$

$$\sum_{r=1}^{30} (7 + 2r) = 9 + 11 + 13 + \cdots$$

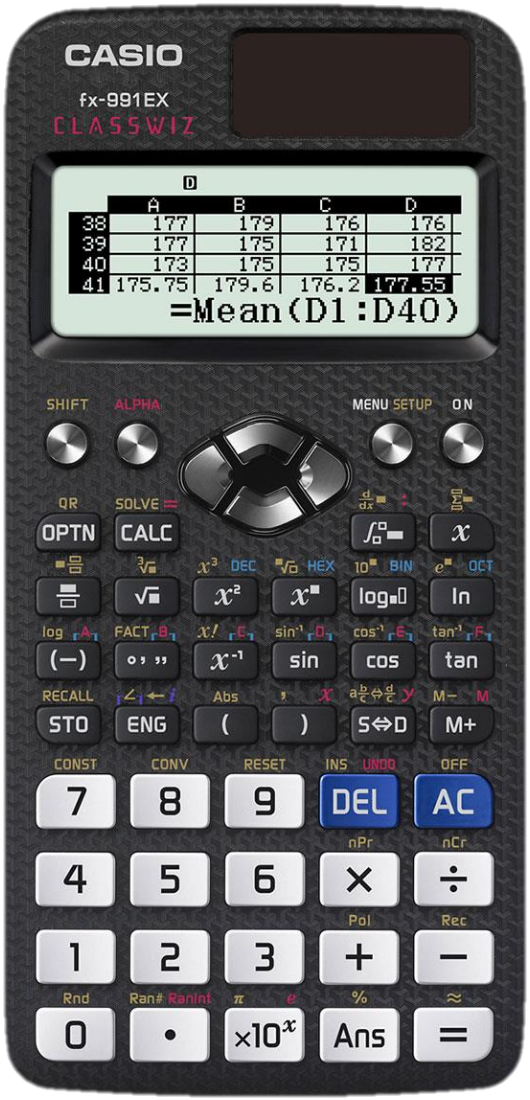
$$a = 9, d = 2, n = 30$$

$$S_{30} = \frac{30}{2}(18 + 29 \times 2) = 1140$$

$$S_9 = \frac{9}{2}(18 + 8 \times 2) = 153$$

$$1140 - 153 = 987$$

On your calculator



The Casio calculator has a Σ button.

Use it to find:

$$\sum_{k=5}^{12} 2 \times 3^k$$

Exercise 3.6

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Homework Exercise

1 For each series:

i write out every term in the series

ii hence find the value of the sum.

a $\sum_{r=1}^{5}(3r+1)$ **b** $\sum_{r=1}^{6}3r^2$ **c** $\sum_{r=1}^{5}\sin(90r°)$ **d** $\sum_{r=5}^{8}2\left(-\frac{1}{3}\right)^r$

2 For each series:

i write the series using sigma notation

ii evaluate the sum.

a $2+4+6+8$ **b** $2+6+18+54+162$ **c** $6+4.5+3+1.5+0-1.5$

3 For each series:

i find the number of terms in the series

ii write the series using sigma notation.

a $7+13+19+\ldots+157$ **b** $\frac{1}{3}+\frac{2}{15}+\frac{4}{75}+\ldots+\frac{64}{46875}$ **c** $8-1-10-19\ldots-127$

4 Evaluate:

a $\sum_{r=1}^{20}(7-2r)$ **b** $\sum_{r=1}^{10}3\times 4^r$ **c** $\sum_{r=1}^{100}(2r-8)$ **d** $\sum_{r=1}^{\infty}7\left(-\frac{1}{3}\right)^r$

5 Evaluate:

a $\sum_{r=9}^{30}\left(5r-\frac{1}{2}\right)$ **b** $\sum_{r=100}^{200}(3r+4)$ **c** $\sum_{r=5}^{100}3\times 0.5^r$ **d** $\sum_{i=5}^{100}1$

Problem-solving

$$\sum_{r=k}^{n}u_r=\sum_{r=1}^{n}u_r-\sum_{r=1}^{k-1}u_r$$

Homework Exercise

6 Show that $\sum_{r=1}^{n} 2r = n + n^2$.

7 Show that $\sum_{r=1}^{n} 2r - \sum_{r=1}^{n} (2r - 1) = n$.

8 Find in terms of k:

a $\sum_{r=1}^{k} 4(-2)^r$ **b** $\sum_{r=1}^{k} (100 - 2r)$ **c** $\sum_{r=10}^{k} (7 - 2r)$

9 Find the value of $\sum_{r=10}^{\infty} 200 \times \left(\frac{1}{4}\right)^r$

10 Given that $\sum_{r=1}^{k} (8 + 3r) = 377$,

a show that $(3k + 58)(k - 13) = 0$ **(3 marks)**

b hence find the value of k. **(1 mark)**

11 Given that $\sum_{r=1}^{k} 2 \times 3^r = 59\,046$,

a show that $k = \dfrac{\log 19\,683}{\log 3}$ **(4 marks)**

b For this value of k, calculate $\sum_{r=k+1}^{13} 2 \times 3^r$. **(3 marks)**

Homework Exercise

12 A geometric series is given by $1 + 3x + 9x^2 + \ldots$

The series is convergent.

a Write down the range of possible values of x. **(3 marks)**

Given that $\sum_{r=1}^{\infty}(3x)^{r-1} = 2$

b calculate the value of x. **(3 marks)**

Challenge

Given that $\sum_{r=1}^{10}(a + (r-1)d) = \sum_{r=11}^{14}(a + (r-1)d)$, show that $d = 6a$.

Homework Answers

1 **a** **i** $4 + 7 + 10 + 13 + 16$ **ii** 50
b **i** $3 + 12 + 27 + 48 + 75 + 108$ **ii** 273
c **i** $1 + 0 + (-1) + 0 + 1$ **ii** 1
d **i** $-\frac{2}{243} + \frac{2}{729} - \frac{2}{2187} + \frac{2}{6561}$ **ii** $-\frac{40}{6561}$

2 **a** **i** $\sum_{r=1}^{4} 2r$ **ii** 20
b **i** $\sum_{r=1}^{5} (2 \times 3^{r-1})$ **ii** 242
c **i** $\sum_{r=1}^{6} \left(-\frac{3}{2}r + \frac{15}{2}\right)$ **ii** 13.5

3 **a** **i** 26 **ii** $\sum_{r=1}^{26} (6r + 1)$
b **i** 7 **ii** $\sum_{r=1}^{7} \left(\frac{1}{3} \times \left(\frac{2}{5}\right)^{r-1}\right)$
c **i** 16 **ii** $\sum_{r=1}^{16} (17 - 9r)$

4 **a** -280 **b** 4 194 300
c 9300 **d** $-\frac{7}{4}$

5 **a** 2134 **b** 45854 **c** $\frac{3}{16}$ **d** 96

6 $\sum_{r=1}^{n} 2r = 2 + 4 + 6 + \ldots + 2n$; $a = 2$, $d = 2$

$S_n = \frac{n}{2}(4 + (n - 1)2) = \frac{n}{2}(2 + 2n) = n + n^2$

7 $\sum_{r=1}^{n} 2r = n + n^2$

$\sum_{r=1}^{n} (2r - 1) = \frac{n}{2}(2 + (n - 1)2) = \frac{n}{2}(2n) = n^2$

$\sum_{r=1}^{n} 2r - \sum_{r=1}^{n} (2r - 1) = n + n^2 - n^2 = n$

8 **a** $\frac{8}{3}((-2)^k - 1)$
b $99k - k^2$
c $6k - k^2 + 27$

9 $\frac{25}{98304}$

10 **a** $a = 11$, $d = 3$

$377 = \frac{k}{2}(2(11) + (k - 1)(3)) = \frac{k}{2}(19 + 3k)$

$3k^2 + 19k - 754 = 0 \Rightarrow (3k + 58)(k - 13) = 0$

b $k = 13$

Homework Answers

11 a $a = 6, d = 3; S_k = \dfrac{6(3^k - 1)}{3 - 1} = 3(3^k - 1)$

$\Rightarrow 3(3^k - 1) = 59046 \Rightarrow 3^k = 19683$

$\Rightarrow k \log 3 = \log 19683 \Rightarrow k = \dfrac{\log 19683}{\log 3}$

b 4723920

12 a $|x| < \frac{1}{3}$ **b** $\frac{1}{6}$

Challenge

$\sum_{r=1}^{10}[a + (r - 1)d]$

$S_{10} = 5(2a + 9d)$

$\sum_{r=11}^{14}[a + (r - 1)d] = \sum_{r=1}^{14}[a + (r - 1)d] - \sum_{r=1}^{10}[a + (r - 1)d]$

$= [7(2a + 13d) - 5(2a + 9d)] = 4a + 46d$

$4a + 46d = 10a + 45d \Rightarrow 6a = d$