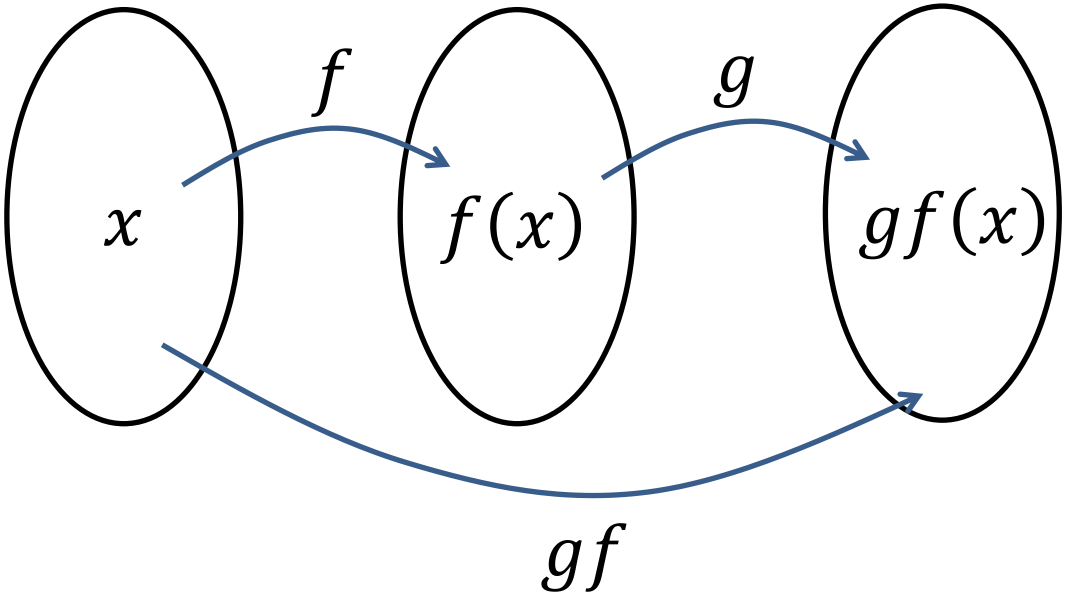
P2 Chapter 2: Graphing Functions

Composite Functions

Composite Functions

Sometimes we may apply multiple functions in succession to an input. These combined functions are known as a **composite function**.



✏ $gf(x)$ means $g(f(x))$, i.e. f is applied first, then g.

Examples

Let $f(x) = x^2 + 1$, and $g(x) = 4x - 2$.
What is...

$fg(2)$? $=$?

$fg(x)$? ?

$gf(x)$? ?

$f^2(x)$? ?

$f^2(x)$ means $ff(x)$

Solve $gf(x) = 38$?

Examples

Let $f(x) = x^2 + 1$, and $g(x) = 4x - 2$.
What is…

$fg(2)$? $= f\big(g(2)\big) = f(6) = 37$

$fg(x)$?

$$f\big(g(x)\big) = f(4x-2)$$
$$= (4x-2)^2 + 1$$
$$= 16x^2 - 16x + 5$$

Replace any instance of x in the outer function with the inner function.

$gf(x)$?

$$g\big(f(x)\big) = g(x^2+1)$$
$$= 4(x^2+1) - 2$$
$$= 4x^2 + 2$$

$f^2(x)$? $= f\big(f(x)\big) = (x^2+1)^2 + 1$

$f^2(x)$ means $ff(x)$

Solve $gf(x) = 38$

$$4x^2 + 2 = 38$$
$$x = \pm 3$$

Further Examples

The functions f and g are defined by

$$f: x \to |2x - 8|$$

$$g: x \to \frac{x+1}{2}$$

a) Find $fg(3)$ b) Solve $fg(x) = x$

a

?

b

?

Further Examples

The functions f and g are defined by

$$f: x \to |2x - 8|$$

$$g: x \to \frac{x+1}{2}$$

a) Find $fg(3)$ b) Solve $fg(x) = x$

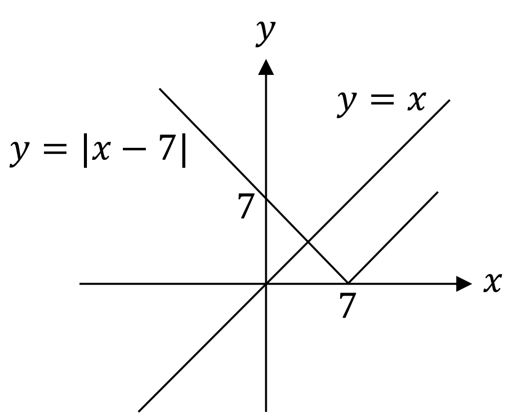
a

$$fg(3) = f\big(g(3)\big) = f(2)$$
$$= |4 - 8| = |-4| = 4$$

b

$$fg(x) = f\left(\frac{x+1}{2}\right) = \left|2\left(\frac{x+1}{2}\right) - 8\right|$$
$$= |x - 7|$$

$$|x - 7| = x$$



$$-x + 7 = x$$
$$x = 3.5$$

It's the reflected (negated) part of $|x - 7|$ that is intersecting with $y = x$

Test Your Understanding

Edexcel C4 June 2013(R) Q4

The functions and f and g are defined by

$$f: x \to 2|x| + 3, \qquad x \in \mathbb{R}$$
$$g: x \to 3 - 4x, \qquad x \in \mathbb{R}$$

b) Find $fg(1)$

d) Solve the equation

$$gg(x) + [g(x)]^2 = 0$$

Edexcel C4 June 2012 Q6

The functions f and g are defined by

$$f: x \to e^x + 2, \qquad x \in \mathbb{R}$$
$$g: x \to \ln x, \qquad x > 0$$

b) Find $fg(x)$, giving your answer in its simplest form.

? b

? d

?

Test Your Understanding

Edexcel C4 June 2013(R) Q4

The functions and f and g are defined by

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b) Find $fg(1)$

d) Solve the equation

$$gg(x) + [g(x)]^2 = 0$$

$$fg(1) = f(-1)$$
$$= 2|-1| + 3 = 5$$

$$3 - 4(3 - 4x) + (3 - 4x)^2 = 0$$
$$3 - 12 + 16x + 9 - 24x + 16x^2 = 0$$
$$16x^2 - 8x = 0$$
$$2x^2 - x = 0$$
$$x(2x - 1) = 0$$
$$x = 0, 0.5$$

Edexcel C4 June 2012 Q6

The functions f and g are defined by

$$f: x \to e^x + 2, \qquad x \in \mathbb{R}$$
$$g: x \to \ln x, \qquad x > 0$$

b) Find $fg(x)$, giving your answer in its simplest form.

$$fg(x) = f(\ln x)$$
$$= e^{\ln x} + 2$$
$$= x + 2$$

"e to the power of" and "ln of" are inverse functions so cancel each other out.

Exercise 2.3

Pearson Pure Mathematics Year 2/AS
Pages 8

Extension

1 [MAT 2014 1F]
The functions S and T are defined for real numbers by $S(x) = x + 1$ and $T(x) = -x$.
The function S is applied s times and the function T is applied t times, in some order, to produce the function

$$F(x) = 8 - x$$

It is possible to deduce that:

i) $s = 8$ and $t = 1$
ii) s is odd and t is even.
iii) s is even and t is odd.
iv) s and t are powers of 2.
v) none of the above.

?

2 [MAT 2012 Q2]
Let $f(x) = x + 1$ and $g(x) = 2x$.
i) Show that $f^2g(x) = gf(x)$
ii) Note that $gf^2g(x) = 4x + 4$
Find all the other ways of combining f and g that result in the function $4x + 4$.
iii) Let $i, j, k \geq 0$ be integers. Determine the function

$$f^i g f^j g f^k(x)$$

iv) Let $m \geq 0$ be an integer. How many different ways of combining the functions f and g are there that result in the function $4x + 4m$?

?

Exercise 2.3

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1 [MAT 2014 1F]
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ii) s is odd and t is even.
iii) s is even and t is odd.
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v) none of the above.

Each application of T will oscillate the sign of x, so clearly t is odd, eliminating (ii) and (iv). Each application of T doesn't change the magnitude of the constant term.
If $t = 0$ we'd need 8 applications of S to get to 8. An application of T might get us from say 3 to -3. We'd then require and even number of applications of S to get back up to 3 (in this case 6). So s must be even. This leaves (i) and (iii), but there is more than one way, so the answer is (iii).

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$$f^i g f^j g f^k(x)$$

iv) Let $m \geq 0$ be an integer. How many different ways of combining the functions f and g are there that result in the function $4x + 4m$?

i) $\boldsymbol{f^2g(x) = 2x + 1 + 1 = 2x + 2}$
$\boldsymbol{gf(x) = 2(x+1) = 2x + 2}$

ii) $\boldsymbol{f^4g^2,\ f^2gfg,\ gf^2g,\ g^2f}$

iii) $\boldsymbol{= 4x + 4k + 2j + i}$

iv) $\boldsymbol{4x + 4k + 2j + i = 4x + 4m}$
$\boldsymbol{\therefore 4k + 2j + i = 4m}$

If $k = 0$, j can vary between 0 and $2m$, which is $2m + 1$ possibilities. If $k = 1$, j can vary between 0 and $2m - 2$ which is $2m + 1$ possibilities. Continuing in this way we get $1 + 3 + 5 + \cdots + (2m + 1)$. This can be simplified to $(m + 1)^2$ (see Arithmetic Series).

Homework Exercise

1 Given the functions $p(x) = 1 - 3x$, $q(x) = \frac{x}{4}$ and $r(x) = (x - 2)^2$, find:

a $pq(-8)$ **b** $qr(5)$ **c** $rq(6)$ **d** $p^2(-5)$ **e** $pqr(8)$

2 Given the functions $f(x) = 4x + 1$, $g(x) = x^2 - 4$ and $h(x) = \frac{1}{x}$, find expressions for the functions:

a $fg(x)$ **b** $gf(x)$ **c** $gh(x)$ **d** $fh(x)$ **e** $f^2(x)$

3 The functions f and g are defined by

$f(x) = 3x - 2, x \in \mathbb{R}$

$g(x) = x^2, x \in \mathbb{R}$

a Find an expression for $fg(x)$. **(2 marks)**

b Solve $fg(x) = gf(x)$. **(4 marks)**

4 The functions p and q are defined by

$p(x) = \frac{1}{x - 2}, x \in \mathbb{R}, x \neq 2$

$q(x) = 3x + 4, x \in \mathbb{R}$

a Find an expression for $qp(x)$ in the form $\frac{ax + b}{cx + d}$ **(3 marks)**

b Solve $qp(x) = 16$. **(3 marks)**

Homework Exercise

5 The functions f and g are defined by:

f: $x \mapsto |9 - 4x|$

g: $x \mapsto \dfrac{3x - 2}{2}$

a Find fg(6). **(2 marks)**

b Solve $\mathrm{fg}(x) = x$. **(5 marks)**

6 Given $\mathrm{f}(x) = \dfrac{1}{x + 1}$, $x \neq -1$

a Prove that $\mathrm{f}^2(x) = \dfrac{x + 1}{x + 2}$

b Find an expression for $\mathrm{f}^3(x)$.

7 The functions s and t are defined by

$\mathrm{s}(x) = 2^x$, $x \in \mathbb{R}$

$\mathrm{t}(x) = x + 3$, $x \in \mathbb{R}$

a Find an expression for $\mathrm{st}(x)$.

b Find an expression for $\mathrm{ts}(x)$.

c Solve $\mathrm{st}(x) = \mathrm{ts}(x)$, leaving your answer in the form $\dfrac{\ln a}{\ln b}$

Hint Rearrange the equation in part **c** into the form $2^x = k$ where k is a real number, then take natural logs of both sides. ← **Year 1, Section 14.5**

8 Given $\mathrm{f}(x) = \mathrm{e}^{5x}$ and $\mathrm{g}(x) = 4\ln x$, find in its simplest form:

a $\mathrm{gf}(x)$ **(2 marks)**

b $\mathrm{fg}(x)$ **(2 marks)**

Homework Exercise

9 The functions p and q are defined by

p: $x \mapsto \ln(x + 3)$, $x \in \mathbb{R}$, $x > -3$

q: $x \mapsto e^{3x} - 1$, $x \in \mathbb{R}$

Hint The range of p will be the set of possible inputs for q in the function qp.

a Find qp(x) and state its range. **(3 marks)**

b Find the value of qp(7). **(1 mark)**

c Solve qp(x) = 124. **(3 marks)**

10 The function t is defined by

t: $x \mapsto 5 - 2x$

Solve the equation $t^2(x) - (t(x))^2 = 0$. **(5 marks)**

Problem-solving

You need to work out the intermediate steps for this problem yourself, so plan your answer before you start. You could start by finding an expression for tt(x).

11 The function g has domain $-5 \leqslant x \leqslant 14$ and is linear from $(-5, -8)$ to $(0, 12)$ and from $(0, 12)$ to $(14, 5)$.

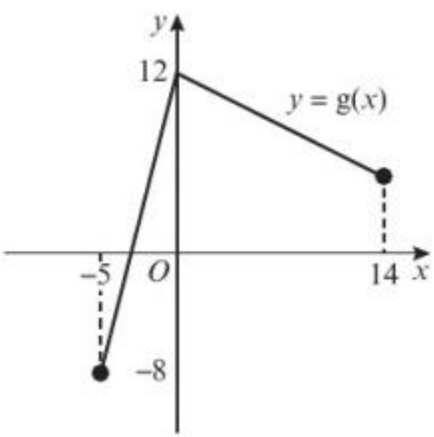
A sketch of the graph of $y = g(x)$ is shown in the diagram.

a Write down the range of g. **(1 mark)**

b Find gg(0). **(2 marks)**

The function h is defined by h: $x \mapsto \dfrac{2x - 5}{10 - x}$

c Find gh(7). **(2 marks)**



Homework Answers

1 a 7 b $\frac{9}{4}$ or 2.25 c 0.25 d -47 e -26

2 a $4x^2 - 15$ b $16x^2 + 8x - 3$ c $\dfrac{1}{x^2} - 4$

d $\dfrac{4}{x} + 1$ e $16x + 5$

3 a $fg(x) = 3x^2 - 2$ b $x = 1$

4 a $qp(x) = \dfrac{4x - 5}{x - 2}$ b $x = \frac{9}{4}$

5 a 23 b $x = \frac{13}{7}$ and $x = \frac{13}{5}$

6 a $f^2(x) = f\left(\dfrac{1}{x+1}\right) = \dfrac{1}{\left(\dfrac{1}{x+1}\right) + 1} = \dfrac{x+1}{x+2}$

b $f^3(x) = \dfrac{x+2}{2x+3}$

7 a 2^{x+3} b $2^x + 3$ c $\dfrac{\ln\left(\frac{3}{7}\right)}{\ln(2)}$

8 a $20x$ b x^{20}

9 a $(x+3)^3 - 1$, $qp(x) > -1$

b 999 c $x = 2$

10 $3 \pm \dfrac{\sqrt{6}}{2}$

11 a $-8 \leqslant g(x) \leqslant 12$ b 6 c 10.5