M2 Chapter 6: Projectiles

Projectile Motion Formulae

Projectile Formulae

There's nothing new here, but you may be asked to prove more general results regarding projectile motion.

[Textbook] A particle is projected from a point on a horizontal plane with an initial velocity U at an angle α above the horizontal and moves freely under gravity until it hits the plane at point B. Given that that acceleration due to gravity is g, find expressions for:
(a) the time of flight, T
(b) the range, R, on the horizontal plane.

a

$$R(\uparrow): \quad s = 0, u = U\sin\alpha, \cancel{v = \quad}, a = -g, t = ?$$

$$s = ut + \frac{1}{2}at^2$$

$$0 = (U\sin\alpha)t - \frac{1}{2}gt^2 = t\left(U\sin\alpha - \frac{1}{2}gt\right)$$

$$t = 0 \text{ (at } A) \; or \; t = \frac{2U\sin\alpha}{g}$$

b

$$R(\rightarrow): \quad R = U\cos\alpha \times \frac{2U\sin\alpha}{g} = \frac{2U^2}{g}\sin\alpha\cos\alpha$$

$$R = \frac{U^2\sin 2\alpha}{g}$$

Using double-angle formula for sin

Projectile Formulae

[Textbook] A particle is projected from a point with speed U at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x, its height above the point of projection is y.

(a) Show that $y = x\tan\alpha - \frac{gx^2}{2u^2}(1 + \tan^2\alpha)$

A particle is projected from a point O on a horizontal plane, with speed 28 ms^{-1} at an angle of elevation α. The particle passes through a point B, which is at a horizontal distance of 32m from O and at a height of 8m above the plane.

(b) Find the two possible values of α, giving your answers to the nearest degree.

a

?

Don't be intimidated by the lack of numerical values. Just do what you'd usually do and resolve both vertically and horizontally!

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a

$$R(\rightarrow): \quad x = U\cos\alpha \times t \qquad (1)$$

$$R(\uparrow): \quad s = ut + \frac{1}{2}at^2$$

$$y = U\sin\alpha \times t - \frac{1}{2}gt^2 \qquad (2)$$

Don't be intimidated by the lack of numerical values. Just do what you'd usually do and resolve both vertically and horizontally!

The target equation doesn't contain t, so much t the subject of (1) and substitute into (2):

$$t = \frac{x}{U\cos\alpha} \quad \Rightarrow \quad y = U\sin\alpha\left(\frac{x}{U\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{gx^2}{2U^2}\sec^2\alpha$$

$$= x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$$

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b

?

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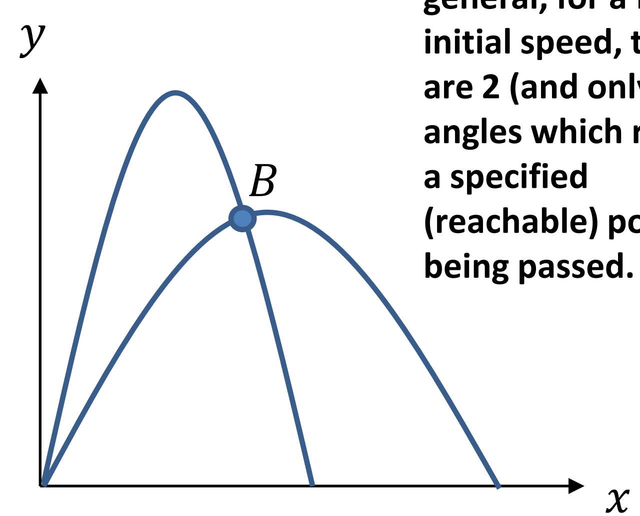
b

$$y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$$
$$8 = 32\tan\alpha - 6.4(1+\tan^2\alpha)$$

This is quadratic in $\tan\alpha$:

$$6.4\tan^2\alpha - 32\tan\alpha + 14.4 = 0$$
$$4\tan^2\alpha - 20\tan\alpha + 9 = 0$$
$$(2\tan\alpha - 1)(2\tan\alpha - 9) = 0$$
$$\tan\alpha = \frac{1}{2}, \frac{9}{2}$$
$$\alpha = 27°, 77°$$



This is an interesting result, because in general, for a fixed initial speed, there are 2 (and only 2) angles which result in a specified (reachable) point being passed.

General Results

Exam Note: You may be asked to derive these. But don't attempt to memorise them or even actually use them to solve exam problems:
– instead, use the techniques used earlier in the chapter.

✎ For a particle projected with initial velocity U at angle α above horizontal and moving freely under gravity:

- Time of flight $= \frac{2U \sin \alpha}{g}$
- Time to reach greatest height $= \frac{U \sin \alpha}{g}$
- Range on horizontal plane $= \frac{U^2 \sin 2\alpha}{g}$
- Equation of trajectory: $y = x \tan \alpha - \frac{gx^2}{2U^2}(1 + \tan^2 \alpha)$
  where y is vertical height of particle and x horizontal distance.

Exercise 6D

Pearson Stats/Mechanics Year 2
Pages 52-53

Homework Exercise

Whenever a numerical value of g is required, take $g = 9.8\,\text{m s}^{-2}$ unless otherwise stated.

1 A particle is launched from a point on a horizontal plane with initial velocity $U\,\text{m s}^{-1}$ at an angle of elevation α. The particle moves freely under gravity until it strikes the plane.
The greatest height of the particle is h m.

Show that $h = \dfrac{U^2 \sin^2 \alpha}{2g}$

2 A particle is projected from a point with speed $21\,\text{m s}^{-1}$ at an angle of elevation α and moves freely under gravity. When the particle has moved a horizontal distance x m, its height above the point of projection is y m.

a Show that $y = x \tan \alpha - \dfrac{x^2}{90 \cos^2 \alpha}$

b Given that $y = 8.1$ when $x = 36$, find the value of $\tan \alpha$.

3 A projectile is launched from a point on a horizontal plane with initial speed $U\,\text{m s}^{-1}$ at an angle of elevation α. The particle moves freely under gravity until it strikes the plane. The range of the projectile is R m.

a Show that the time of flight of the particle is $\dfrac{2U \sin \alpha}{g}$ seconds.

b Show that $R = \dfrac{U^2 \sin 2\alpha}{g}$.

c Deduce that, for a fixed u, the greatest possible range is when $\alpha = 45°$.

d Given that $R = \dfrac{2U^2}{5g}$, find the two possible values of the angle of elevation at which the projectile could have been launched.

Homework Exercise

4 A firework is launched vertically with a speed of $v\,\text{m s}^{-1}$. When it reaches its maximum height, the firework explodes into two parts, which are projected horizontally in opposite directions, each with speed $2v\,\text{m s}^{-1}$.
Show that the two parts of the firework land a distance $\frac{4v^2}{g}$ m apart.

5 In this question use $g = 10\,\text{m s}^{-2}$.
A particle is projected from a point O with speed U at an angle of elevation α above the horizontal and moves freely under gravity. When the particle has moved a horizontal distance x, its height above O is y.

a Show that $y = x\tan\alpha - \dfrac{gx^2}{2U^2\cos^2\alpha}$ **(4 marks)**

A boy throws a stone from a point P at the end of a pier. The point P is 15 m above sea level. The stone is projected with a speed of $8\,\text{m s}^{-1}$ at an angle of elevation of 40°. By modelling the ball as a particle moving freely under gravity,

b find the horizontal distance of the stone from P when the ball is 2 m above sea level. **(5 marks)**

6 A particle is projected from a point with speed U at an angle of elevation α above the horizontal and moves freely under gravity. When it has moved a horizontal distance x, its height above the point of projection is y.

a Show that $y = x\tan\alpha - \dfrac{gx^2}{2U^2}(1 + \tan^2\alpha)$ **(5 marks)**

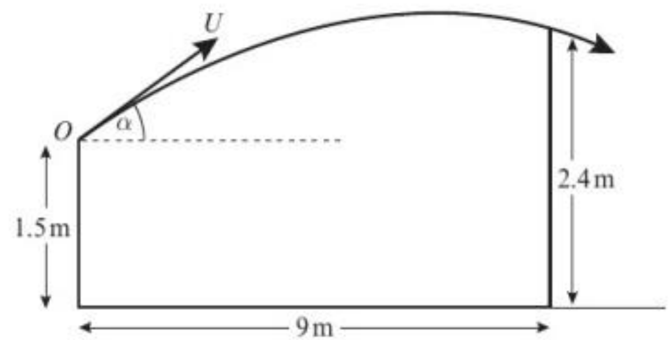
An athlete throws a javelin from a point P at a height of 2 m above horizontal ground. The javelin is projected at an angle of elevation of 45° with a speed of $30\,\text{m s}^{-1}$. By modelling the javelin as a particle moving freely under gravity,

b find, to 3 significant figures, the horizontal distance of the javelin from P when it hits the ground **(5 marks)**

c find, to 2 significant figures, the time elapsed from the point the javelin is thrown to the point it hits the ground. **(2 marks)**

Homework Exercise

7 A girl playing volleyball on horizontal ground hits the ball towards the net 9 m away from a point 1.5 m above the ground. The ball moves in a vertical plane which is perpendicular to the net. The ball just passes over the top of the net, which is 2.4 m above the ground, as shown in the diagram.
The ball is modelled as a particle projected with initial speed U m s^{-1} from point O, 1.5 m above the ground at an angle α to the horizontal.



a By writing down expressions for the horizontal and vertical distances from O to the ball, t seconds after it was hit, show that when the ball passes over the net

$$0.9 = 9\tan\alpha - \frac{81g}{2U^2\cos^2\alpha}$$ **(6 marks)**

Given that $\alpha = 30°$,

b find the speed of the ball as it passes over the net. **(6 marks)**

8 In this question **i** and **j** are unit vectors in a horizontal and upward vertical direction respectively. An object is projected from a fixed point A on horizontal ground with velocity $(k\mathbf{i} + 2k\mathbf{j})$ m s^{-1}, where k is a positive constant. The object moves freely under gravity until it strikes the ground at B, where it immediately comes to rest. Relative to O, the position vector of a point on the path of the object is $(x\mathbf{i} + y\mathbf{j})$ m.

a Show that $y = 2x - \dfrac{gx^2}{2k^2}$ **(5 marks)**

Given that $AB = R$ m and the maximum vertical height of the object above the ground is H m,

b using the result in part **a**, or otherwise, find, in terms of k and g,

i R **ii** H **(6 marks)**

Homework Exercise

Challenge

A stone is projected from a point on a straight sloping hill. Given that the hill slopes downwards at an angle of 45°, and that the stone is projected at an angle of 45° above the horizontal with speed U m s^{-1}.

Show that the stone lands a distance $\frac{2\sqrt{2}U^2}{g}$ m down the hill.

Homework Answers

1 R(↑): $v^2 = U^2 \sin^2 \alpha - 2gh$

At maximum height, $v = 0$ so $0 = U^2 \sin^2 \alpha - 2gh$

Rearrange to give $h = \dfrac{U^2 \sin^2 \alpha}{2g}$

2 **a** R(→): $x = 21 \cos\alpha \times t$, so $t = \dfrac{x}{21 \cos\alpha}$

R(↑): $y = 21 \sin\alpha \times \dfrac{x}{21\cos\alpha} - \dfrac{1}{2}g\left(\dfrac{x}{21\cos\alpha}\right)^2$

$y = x \tan \alpha - \dfrac{x^2}{90\cos^2\alpha}$

b $\tan\alpha = 1.25$

3 **a** R(↑): $s = U\sin\alpha t - \dfrac{g}{2}t^2$

When particle strikes plane, $s = 0 = t(U\sin\alpha - \dfrac{g}{2}t)$

So $t = 0$ or $t = \dfrac{2U\sin\alpha}{g}$

b R(→): $s = ut = U\cos\alpha\left(\dfrac{2U\sin\alpha}{g}\right) = \dfrac{U^2\sin 2\alpha}{g}$

c Range $s = \dfrac{U^2 \sin 2\alpha}{g}$ is greatest when $\sin 2\alpha = 1$

Occurs when $2\alpha = 90° \Rightarrow \alpha = 45°$

d 12° and 78°

4 Using $v = u + at$, at max height $t = \dfrac{v}{g}$

So time taken to return to the ground $= \dfrac{v}{g}$

Using $s = ut + \frac{1}{2}at^2$, distance travelled by one part =

$2v\left(\dfrac{v}{g}\right) = \dfrac{2v^2}{g}$

So two parts of firework are $\dfrac{2v^2}{g} + \dfrac{2v^2}{g} = \dfrac{4v^2}{g}$ apart.

Homework Answers

5 **a** R(→): $x = U\cos\alpha \times t$, so $t = \frac{x}{U\cos\alpha}$

R(↑): $y = U\sin\alpha \times t - \frac{1}{2}gt^2$

Substitute for $t \Rightarrow y = U\sin\alpha\left(\frac{x}{U\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$

Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and rearrange to give

$y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$.

b 13.7 m

6 **a** R(→): $x = U\cos\alpha \times t$, so $t = \frac{x}{U\cos\alpha}$

R(↑): $y = U\sin\alpha \times t - \frac{1}{2}gt^2$

Substitute for $t \Rightarrow y = U\sin\alpha\left(\frac{x}{U\cos\alpha}\right) - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$

Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and $\frac{1}{\cos\alpha} = \sec\alpha$, and rearrange to give

$y = x\tan\alpha - \frac{gx^2}{2U^2}\sec^2\alpha$.

Use $\sec^2\alpha = 1 + \tan^2\alpha$, and rearrange to give

$y = x\tan\alpha - \frac{gx^2}{2U^2}(1 + \tan^2\alpha)$.

b 93.8 m

c 4.4 s

7 **a** R(→): $x = 9 = U\cos\alpha \times t$, so $t = \frac{9}{U\cos\alpha}$

R(↑): $y = U\sin\alpha \times t - \frac{1}{2}gt^2$

Substitute for $t \Rightarrow y = U\sin\alpha\left(\frac{9}{U\cos\alpha}\right) - \frac{1}{2}g\left(\frac{9}{U\cos\alpha}\right)^2$

Use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ and $y = 0.9$. Rearrange to give

$0.9 = 9\tan\alpha - \frac{81g}{2U^2\cos^2\alpha}$.

b $10.3\,\text{m s}^{-1}$

Homework Answers

8 **a** R(→): $x = kt$, so $t = \frac{x}{k}$

R(↑): $y = 2kt - \frac{gt^2}{2}$

Substitute for $t \Rightarrow y = 2x - \frac{gx^2}{2k^2}$

b **i** $\frac{4k^2}{g}$ m **ii** $\frac{2k^2}{g}$ m

Challenge

For the projectile: $y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$ so for $\alpha = 45°$

$y = x - \frac{gx^2}{U^2}$

For the slope: $y = -x$

Projectile intersects the slope when $-x = x - \frac{gx^2}{U^2} \Rightarrow x = \frac{2U^2}{g}$,

$y = -\frac{2U^2}{g}$

Distance $= \sqrt{\left(\frac{2U^2}{g}\right)^2 + \left(\frac{2U^2}{g}\right)^2} = \sqrt{8\left(\frac{U^2}{g}\right)^2} = \frac{2\sqrt{2}\,U^2}{g}$