Pure2 Chapter 11: Vectors

Chapter Practice

Key Points

1 The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$

2 The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

3 The unit vectors along the x-, y- and z-axes are denoted by **i**, **j** and **k** respectively.

$$\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Any 3D vector can be written in column form as $p\mathbf{i} + q\mathbf{j} + r\mathbf{k} = \begin{pmatrix}p\\q\\r\end{pmatrix}$

4 If the vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ makes an angle θ_x with the positive x-axis then $\cos\theta_x = \frac{x}{|\mathbf{a}|}$ and similarly for the angles θ_y and θ_z.

5 If **a**, **b** and **c** are vectors in three dimensions which do not all lie in the same plane then you can compare their coefficients on both sides of an equation.

Chapter Exercises

1 The points $A(2, 7, 3)$ and $B(4, 3, 5)$ are joined to form the line segment AB. The point M is the midpoint of AB. Find the distance from M to the point $C(5, 8, 7)$.

2 The coordinates of P and Q are $(2, 3, a)$ and $(a - 2, 6, 7)$. Given that the distance from P to Q is $\sqrt{14}$, find the possible values of a.

3 $\overrightarrow{AB}$ is the vector $-3\mathbf{i} + t\mathbf{j} + 5\mathbf{k}$, where $t > 0$. Given that $|\overrightarrow{AB}| = 5\sqrt{2}$, show that $\overrightarrow{AB}$ is parallel to $6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k}$.

4 P is the point $(5, 6, -2)$, Q is the point $(2, -2, 1)$ and R is the point $(2, -3, 6)$.

a Find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$.

b Hence, or otherwise, find the area of triangle PQR.

5 The points D, E and F have position vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix}$ respectively.

a Find the vectors $\overrightarrow{DE}$, $\overrightarrow{EF}$ and $\overrightarrow{FD}$. **(3 marks)**

b Find $|\overrightarrow{DE}|$, $|\overrightarrow{EF}|$ and $|\overrightarrow{FD}|$ giving your answers in exact form. **(6 marks)**

c Describe triangle DEF. **(1 mark)**

6 P is the point $(-6, 2, 1)$, Q is the point $(3, -2, 1)$ and R is the point $(1, 3, -2)$.

a Find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$. **(3 marks)**

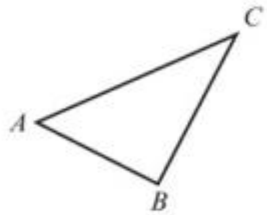
b Hence find the lengths of the sides of triangle PQR. **(6 marks)**

c Given that angle $QRP = 90°$ find the size of angle PQR. **(2 marks)**

Chapter Exercises

7 The diagram shows the triangle ABC.

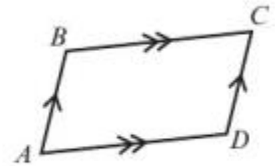
Given that $\overrightarrow{AB} = -\mathbf{i} + \mathbf{j}$ and $\overrightarrow{BC} = \mathbf{i} - 3\mathbf{j} + \mathbf{k}$, find $\angle ABC$ to 1 d.p.



(5 marks)

8 The diagram shows the quadrilateral $ABCD$.

Given that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ -2 \\ 11 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 15 \\ 8 \\ 5 \end{pmatrix}$, find the area of the quadrilateral. **(7 marks)**



9 A is the point $(2, 3, -2)$, B is the point $(0, -2, 1)$ and C is the point $(4, -2, -5)$. When A is reflected in the line BC it is mapped to the point D.

a Work out the coordinates of the point D.

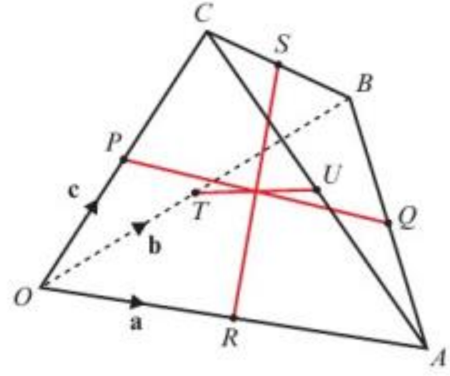
b Give the mathematical name for the shape $ABCD$.

c Work out the area of $ABCD$.

10 The diagram shows a tetrahedron $OABC$. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of A, B and C respectively.

P, Q, R, S, T and U are the midpoints of OC, AB, OA, BC, OB and AC respectively.

Prove that the line segments PQ, RS and TU meet at a point and bisect each other.

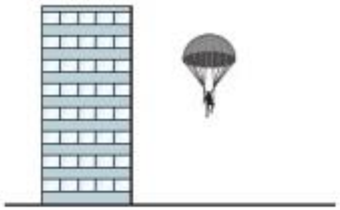


11 A particle of mass 2 kg is acted upon by three forces:

$\mathbf{F}_1 = (b\mathbf{i} + 2\mathbf{j} + \mathbf{k})\,\text{N}$
$\mathbf{F}_2 = (3\mathbf{i} - b\mathbf{j} + 2\mathbf{k})\,\text{N}$
$\mathbf{F}_3 = (-2\mathbf{i} + 2\mathbf{j} + (4 - b)\mathbf{k})\,\text{N}$

Given that the particle accelerates at $3.5\,\text{m}\,\text{s}^{-2}$, work out the possible values of b.

12 In this question **i** and **j** are the unit vectors due east and due north respectively, and **k** is the unit vector acting vertically upwards.



A BASE jumper descending with a parachute is modelled as a particle of mass 50 kg subject to forces describing the wind, **W**, and air resistance, **F**, where:

$\mathbf{W} = (20\mathbf{i} + 16\mathbf{j})\,\text{N}$
$\mathbf{F} = (-4\mathbf{i} - 3\mathbf{j} + 450\mathbf{k})\,\text{N}$

a With reference to the model, suggest a reason why the **k** component of **F** is greater than the other components.

b Taking $g = 9.8\,\text{m}\,\text{s}^{-2}$, find the resultant force acting on the BASE jumper.

c Given that the BASE jumper starts from rest and travels a distance of 180 m before landing, find the total time of the descent.

Challenge

A student writes the following hypothesis:

If **a**, **b** and **c** are three non-parallel vectors in three dimensions, then
$p\mathbf{a} + q\mathbf{b} + r\mathbf{c} = s\mathbf{a} + t\mathbf{b} + u\mathbf{c} \Rightarrow p = s,\ q = t \text{ and } r = u$

Show, by means of a counter-example, that this hypothesis is not true.

1 $\sqrt{22}$ 2 $a = 5$ or $a = 6$

3 $|\overrightarrow{AB}| = 5\sqrt{2} \Rightarrow 9 + t^2 + 25 = 50 \Rightarrow t^2 = 16 \Rightarrow t = 4$

$6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k} = 6\mathbf{i} - 8\mathbf{j} - 10\mathbf{k} = -2(-3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) = -2\overrightarrow{AB}$

So $\overrightarrow{AB}$ is parallel to $6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k}$

4 a $\overrightarrow{PQ} = -3\mathbf{i} - 8\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{PR} = -3\mathbf{i} - 9\mathbf{j} + 8\mathbf{k}$, $\overrightarrow{QR} = -\mathbf{j} + 5\mathbf{k}$
b 20.0

5 a $\overrightarrow{DE} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, $\overrightarrow{EF} = -3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, $\overrightarrow{FD} = -\mathbf{i} + \mathbf{j} - 8\mathbf{k}$
b $|\overrightarrow{DE}| = \sqrt{41}$, $|\overrightarrow{EF}| = \sqrt{41}$, $|\overrightarrow{FD}| = \sqrt{66}$ c isosceles

6 a $\overrightarrow{PQ} = 9\mathbf{i} - 4\mathbf{j}$, $\overrightarrow{PR} = 7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\overrightarrow{QR} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$
b $|\overrightarrow{PQ}| = \sqrt{97}$, $|\overrightarrow{PR}| = \sqrt{59}$, $|\overrightarrow{QR}| = \sqrt{38}$ c 51.3°

7 31.5°

8 184 (3 s.f.)

9 a (2, −7, −2) b rhombus c 36.1

Challenge

If $\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\mathbf{c} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, then $\mathbf{a} + \mathbf{b} + \mathbf{c} = 2\mathbf{a} + 2\mathbf{b} + 0\mathbf{c}$.

10 $\overrightarrow{PQ} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$, $\overrightarrow{RS} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$, $\overrightarrow{TU} = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Let $\overrightarrow{PQ}$, $\overrightarrow{RS}$ and $\overrightarrow{TU}$ intersect at X: $\overrightarrow{PX} = r\overrightarrow{PQ} = \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{RX} = s\overrightarrow{RS} = \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{TX} = t\overrightarrow{TU} = \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$ for scalars r, s and t

$\overrightarrow{RX} = \overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

Comparing coefficients in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ gives $r = s = \frac{1}{2}$

$\overrightarrow{TX} = \overrightarrow{TO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{b} + \mathbf{c}) + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c}) = \frac{1}{4}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Comparing coefficients in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ gives $t = \frac{1}{2}$

So the line segments PQ, RS and TU meet at a point and bisect each other.

11 $b = 1$ or $\frac{17}{3}$

12 a Air resistance acts in opposition to the motion of the BASE jumper. The motion downwards will be greater than the motion in the other directions.
b $(16\mathbf{i} + 13\mathbf{j} - 40\mathbf{k})$ N c 20 seconds