P1 Chapter 8: Binomial Expansion

Chapter Practice

Key Points

1. Pascal's triangle is formed by adding adjacent pairs of numbers to find the numbers on the next row.

2. The $(n + 1)$th row of Pascal's triangle gives the coefficients in the expansion of $(a + b)^n$.

3. $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$.

4. You can use factorial notation and your calculator to find entries in Pascal's triangle quickly.
   - The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
   - The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$.

5. The binomial expansion is:

   $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \ (n \in \mathbb{N})$

   where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

6. In the expansion of $(a + b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

7. The first few terms in the binomial expansion can be used to find an approximate value for a complicated expression.

Chapter Exercises

1 The 16th row of Pascal's triangle is shown below.

1 15 105 … …

a Find the next two values in the row.

b Hence find the coefficient of x^3 in the expansion of $(1 + 2x)^{15}$.

2 Given that $\binom{45}{17} = \frac{45!}{17!a!}$, write down the value of a. **(1 mark)**

3 20 people play a game at a school fete.
The probability that exactly n people win a prize is modelled as $\binom{20}{n}p^n(1-p)^{20-n}$, where p is the probability of any one person winning.

Calculate the probability of:

a 5 people winning when $p = \frac{1}{2}$

b nobody winning when $p = 0.7$

c 13 people winning when $p = 0.6$

Give your answers to 3 significant figures.

4 When $(1 - \frac{3}{2}x)^p$ is expanded in ascending powers of x, the coefficient of x is -24.

a Find the value of p. **(2 marks)**

b Find the coefficient of x^2 in the expansion. **(3 marks)**

c Find the coefficient of x^3 in the expansion. **(1 mark)**

5 Given that:

$$(2 - x)^{13} \equiv A + Bx + Cx^2 + \dots$$

find the values of the integers A, B and C. **(4 marks)**

Chapter Exercises

6 **a** Expand $(1 - 2x)^{10}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient in the expansion. **(4 marks)**

b Use your expansion to find an approximation of 0.98^{10}, stating clearly the substitution which you have used for x. **(3 marks)**

7 **a** Use the binomial series to expand $(2 - 3x)^{10}$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an integer. **(4 marks)**

b Use your series expansion, with a suitable value for x, to obtain an estimate for 1.97^{10}, giving your answer to 2 decimal places. **(3 marks)**

8 **a** Expand $(3 + 2x)^4$ in ascending powers of x, giving each coefficient as an integer. **(4 marks)**

b Hence, or otherwise, write down the expansion of $(3 - 2x)^4$ in ascending powers of x. **(2 marks)**

c Hence by choosing a suitable value for x show that $(3 + 2\sqrt{2})^4 + (3 - 2\sqrt{2})^4$ is an integer and state its value. **(2 marks)**

9 The coefficient of x^2 in the binomial expansion of $\left(1 + \frac{x}{2}\right)^n$, where n is a positive integer, is 7.

a Find the value of n. **(2 marks)**

b Using the value of n found in part **a**, find the coefficient of x^4. **(4 marks)**

10 **a** Use the binomial theorem to expand $(3 + 10x)^4$ giving each coefficient as an integer. **(4 marks)**

b Use your expansion, with an appropriate value for x, to find the exact value of 1003^4. State the value of x which you have used. **(3 marks)**

Chapter Exercises

11 **a** Expand $(1 + 2x)^{12}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient. **(4 marks)**

b By substituting a suitable value for x, which must be stated, into your answer to part **a**, calculate an approximate value of 1.02^{12}. **(3 marks)**

c Use your calculator, writing down all the digits in your display, to find a more exact value of 1.02^{12}. **(1 mark)**

d Calculate, to 3 significant figures, the percentage error of the approximation found in part **b**. **(1 mark)**

12 Expand $\left(x - \frac{1}{x}\right)^5$, simplifying the coefficients. **(4 marks)**

13 In the binomial expansion of $(2k + x)^n$, where k is a constant and n is a positive integer, the coefficient of x^2 is equal to the coefficient of x^3.

a Prove that $n = 6k + 2$. **(3 marks)**

b Given also that $k = \frac{2}{3}$, expand $(2k + x)^n$ in ascending powers of x up to and including the term in x^3, giving each coefficient as an exact fraction in its simplest form. **(4 marks)**

14 **a** Expand $(2 + x)^6$ as a binomial series in ascending powers of x, giving each coefficient as an integer. **(4 marks)**

b By making suitable substitutions for x in your answer to part **a**, show that $(2 + \sqrt{3})^6 - (2 - \sqrt{3})^6$ can be simplified to the form $k\sqrt{3}$, stating the value of the integer k. **(3 marks)**

Chapter Exercises

15 The coefficient of x^2 in the binomial expansion of $(2 + kx)^8$, where k is a positive constant, is 2800.

a Use algebra to calculate the value of k. **(2 marks)**

b Use your value of k to find the coefficient of x^3 in the expansion. **(4 marks)**

16 a Given that

$$(2 + x)^5 + (2 - x)^5 \equiv A + Bx^2 + Cx^4,$$

find the value of the constants A, B and C. **(4 marks)**

b Using the substitution $y = x^2$ and your answers to part **a**, solve

$$(2 + x)^5 + (2 - x)^5 = 349.$$ **(3 marks)**

17 In the binomial expansion of $(2 + px)^5$, where p is a constant, the coefficient of x^3 is 135. Calculate:

a the value of p, **(4 marks)**

b the value of the coefficient of x^4 in the expansion. **(2 marks)**

18 Find the constant term in the expansion of $\left(\frac{x^2}{2} - \frac{2}{x}\right)^9$.

19 a Find the first three terms, in ascending powers of x of the binomial expansion of $(2 + px)^7$, where p is a constant. **(2 marks)**

The first 3 terms are 128, $2240x$ and qx^2, where q is a constant.

b Find the value of p and the value of q. **(4 marks)**

Chapter Exercises

20 **a** Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 - px)^{12}$, where p is a non-zero constant. **(2 marks)**

b Given that, in the expansion of $(1 - px)^{12}$, the coefficient of x is q and the coefficient of x^2 is $6q$, find the value of p and the value of q. **(4 marks)**

21 **a** Find the first 3 terms, in ascending powers of x, of the binomial expansion of $\left(2 + \frac{x}{2}\right)^7$, giving each term in its simplest form. **(4 marks)**

b Explain how you would use your expansion to give an estimate for the value of 2.05^7. **(1 mark)**

22 $g(x) = (4 + kx)^5$, where k is a constant.

Given that the coefficient of x^3 in the binomial expansion of $g(x)$ is 20, find the value of k. **(3 marks)**

Challenge

1 $f(x) = (2 - px)(3 + x)^5$ where p is a constant.

There is no x^2 term in the expansion of $f(x)$.

Show that $p = \frac{4}{3}$

2 Find the coefficient of x^2 in the expansion of $(1 + 2x)^8(2 - 5x)^7$.

Chapter Answers

1 **a** 455, 1365 **b** 3640
2 $a = 28$
3 **a** 0.0148 **b** 0.000 000 000 034 9 **c** 0.166
4 **a** $p = 16$ **b** 270 **c** -1890
5 $A = 8192$, $B = -53\,248$, $C = 159\,744$
6 **a** $1 - 20x + 180x^2 - 960x^3$
b 0.817 04, $x = 0.01$
7 **a** $1024 - 153\,60x + 103\,680x^2 - 414\,720x^3$
b 880.35
8 **a** $81 + 216x + 216x^2 + 96x^3 + 16x^4$
b $81 - 216x + 216x^2 - 96x^3 + 16x^4$
c 1154
9 **a** $n = 8$ **b** $\frac{35}{8}$
10 **a** $81 + 1080x + 5400x^2 + 12\,000x^3 + 10\,000x^4$
b 1 012 054 108 081, $x = 100$
11 **a** $1 + 24x + 264x^2 + 1760x^3$ **b** 1.268 16
c 1.268 241 795 **d** 0.006 45% (3 sf)
12 $x^5 - 5x^3 + 10x - \dfrac{10}{x} + \dfrac{5}{x^3} - \dfrac{1}{x^5}$
13 **a** $\binom{n}{2}(2k)^{n-2} = \binom{n}{3}(2k)^{n-3}$

$$\frac{n!(2k)^{n-2}}{2!(n-2)!} = \frac{n!(2k)^{n-3}}{3!(n-3)!}$$

$$\frac{2k}{n-2} = \frac{1}{3}$$

So $n = 6k + 2$

13 **b** $\dfrac{4096}{729} + \dfrac{2048}{81}x + \dfrac{1280}{27}x^2 + \dfrac{1280}{27}x^3$
14 **a** $64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$
b $k = 1560$
15 **a** $k = 1.25$ **b** 3500
16 **a** $A = 64$, $B = 160$, $C = 20$ **b** $x = \pm\sqrt{\frac{3}{2}}$
17 **a** $p = 1.5$ **b** 50.625
18 672
19 **a** $128 + 448px + 672p^2x^2$
b $p = 5$, $q = 16\,800$
20 **a** $1 - 12px + 66p^2x^2$
b $p = -1\frac{1}{11}$, $q = 13\frac{1}{11}$
21 **a** $128 + 224x + 168x^2$
b Substitute $x = 0.1$ into the expansion.
22 $k = \frac{1}{2}$

Challenge

1 $540 - 405p = 0$, $p = \frac{4}{3}$
2 -4704