P2 Chapter 7: Trigonometric Equations

Addition Formulae

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Addition Formulae allow us to deal with a **sum or difference of angles**.

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin \mathrm{A}\cos \mathrm{B} - \cos \mathrm{A}\sin \mathrm{B}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(\mathrm{A}-\mathrm{B}) = \frac{\tan \mathrm{A} - \tan \mathrm{B}}{1 + \tan \mathrm{A}\tan \mathrm{B}}$$

Do I need to memorise these?
They're all technically in the formula booklet, but you REALLY want to eventually memorise these (particularly the *sin* and *cos* ones).

How to memorise:

First notice that for all of these the first thing on the RHS is the same as the first thing on the LHS!

- For sin, the operator in the middle is the same as on the LHS.
- For cos, it's the opposite.
- For tan, it's the same in the numerator, opposite in the denominator.

- For sin, we mix sin and cos.
- For cos, we keep the cos's and sin's together.

Common Student Error

Why is $\sin(A+B)$ not just $\sin(A)+\sin(B)$?

Because $\boldsymbol{sin}$ is a <u>function</u>, not a quantity that can be expanded out like this. It's a bit like how $(\boldsymbol{a}+\boldsymbol{b})^2 \not\equiv \boldsymbol{a}^2+\boldsymbol{b}^2$.
We can easily disprove it with a counterexample.

Addition Formulae

Now can you reproduce them without peeking at your notes?

$\sin(A+B) \equiv$?

$\sin(A-B) \equiv$?

$\cos(A+B) \equiv$?

$\cos(A-B) \equiv$?

$\tan(A+B) \equiv$?

$tan(A-B) \equiv$?

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Addition Formulae

Now can you reproduce them without peeking at your notes?

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

How to memorise:

First notice that for all of these the first thing on the RHS is the same as the first thing on the LHS!

- For sin, the operator in the middle is the same as on the LHS.
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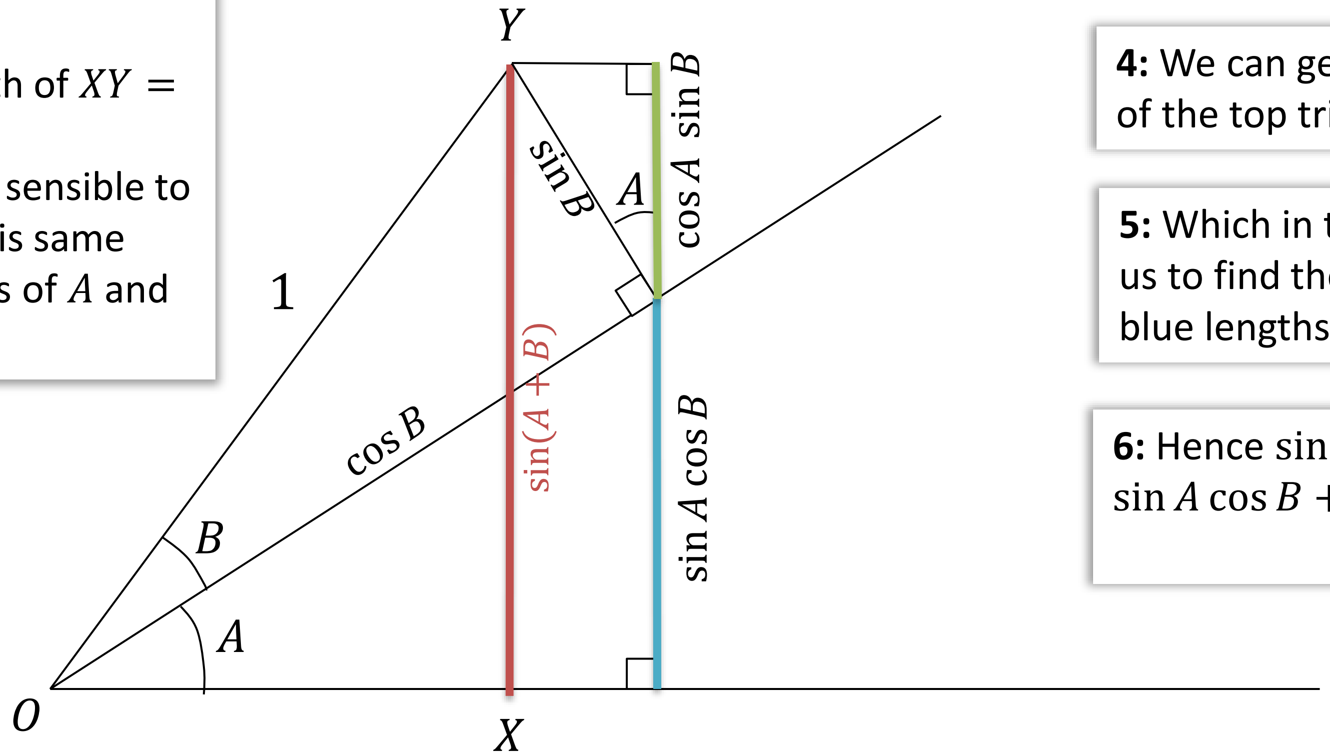
Proof of $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$

(Not needed for exam)

1: Suppose we had a line of length 1 projected an angle of $A+B$ above the horizontal.
Then the length of $XY = \sin(A+B)$
It would seem sensible to try and find this same length in terms of A and B individually.



2: We can achieve this by forming two right-angled triangles.

3: Then we're looking for the combined length of these two lines.

4: We can get the lengths of the top triangle...

5: Which in turn allows us to find the green and blue lengths.

6: Hence $\sin(A+B) = \sin A \cos B + \cos A \sin B$ $\square$

Proof of other identities

Can you think how to use our geometrically proven result $\sin(A+B) = \sin A \cos B + \cos A \sin B$ to prove the identity for $\sin(A-B)$?

$\boldsymbol{\sin(A-B)} =$?

$=$?

$=$

What about $\cos(A+B)$? (Hint: what links sin and cos?)

$\boldsymbol{\cos(A+B)} =$?

$=$?

$=$

Proof of other identities

Can you think how to use our geometrically proven result $\sin(A+B) = \sin A \cos B + \cos A \sin B$ to prove the identity for $\sin(A-B)$?

$$\begin{aligned}\boldsymbol{\sin(A-B)} &= \boldsymbol{\sin\big(A+(-B)\big)} \\ &= \boldsymbol{\sin(A)\cos(-B) + \cos(A)\sin(-B)} \\ &= \boldsymbol{\sin A \cos B - \cos A \sin B}\end{aligned}$$

What about $\cos(A+B)$? (Hint: what links sin and cos?)

$$\boldsymbol{\cos(A+B) = \sin\left(\frac{\pi}{2} - (A+B)\right) = \sin\left(\left(\frac{\pi}{2} - A\right) + (-B)\right)}$$

$$\begin{aligned}&= \boldsymbol{\sin\left(\frac{\pi}{2} - A\right)\cos(-B) + \cos\left(\frac{\pi}{2} - A\right)\sin(-B)} \\ &= \boldsymbol{\cos A \cos B - \sin A \sin B}\end{aligned}$$

Proof of other identities

Edexcel C3 Jan 2012 Q8

(a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(4)

?

Proof of other identities

Edexcel C3 Jan 2012 Q8

(a) Starting from the formulae for $\sin(A+B)$ and $\cos(A+B)$, prove that

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(4)

$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$	M1A1
$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A\sin B}{\cos A\cos B}}$ $\quad (\div \cos A\cos B)$	M1
$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$	A1 *

The important thing is that you explicit show the division of each term by $\cos A \cos B$.

Examples

Q Given that $2\sin(x+y) = 3\cos(x-y)$ express $\tan x$ in terms of $\tan y$.

?

Examples

Q Given that $2\sin(x+y) = 3\cos(x-y)$ express $\tan x$ in terms of $\tan y$.

Using your formulae:

$$2\sin x\cos y + 2\cos x\sin y = 3\cos x\cos y + 3\sin x\sin y$$

We need to get tan x and tan y in there. Dividing by cos x cos y would seem like a sensible step:

$$2\tan x + 2\tan y = 3 + 3\tan x\tan y$$

Rearranging:

$$\tan x = \frac{3 - 2\tan y}{2 - 3\tan y}$$

Exercise 7.1

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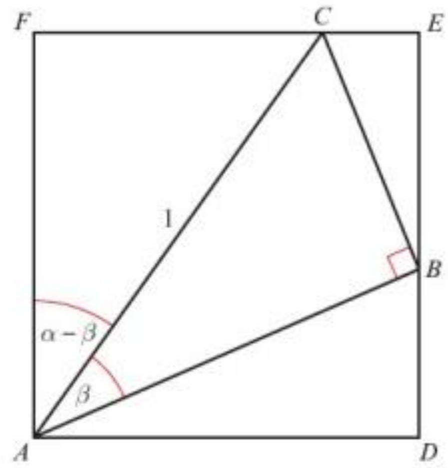
Homework Exercise

1 In the diagram $\angle BAC = \beta$, $\angle CAF = \alpha - \beta$ and $AC = 1$. Additionally lines AB and BC are perpendicular.



a Show each of the following:

i $\angle FAB = \alpha$ **ii** $\angle ABD = \alpha$ and $\angle ECB = \alpha$

iii $AB = \cos\beta$ **iv** $BC = \sin\beta$

b Use $\triangle ABD$ to write an expression for the lengths

i AD **ii** BD

c Use $\triangle BEC$ to write an expression for the lengths

i CE **ii** BE

d Use $\triangle FAC$ to write an expression for the lengths

i FC **ii** FA

e Use your completed diagram to show that:

i $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

ii $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

2 Use the formulae for $\sin(A - B)$ and $\cos(A - B)$ to show that

$$\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

3 By substituting $A = P$ and $B = -Q$ into the addition formula for $\sin(A + B)$, show that $\sin(P - Q) \equiv \sin P\cos Q - \cos P\sin Q$.

Homework Exercise

4 A student makes the mistake of thinking that $\sin(A + B) \equiv \sin A + \sin B$.
Choose non-zero values of A and B to show that this identity is not true.

Watch out This is a common mistake. One counter-example is sufficient to disprove the statement.

5 Using the expansion of $\cos(A - B)$ with $A = B = \theta$, show that $\sin^2\theta + \cos^2\theta \equiv 1$.

6 **a** Use the expansion of $\sin(A - B)$ to show that $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

b Use the expansion of $\cos(A - B)$ to show that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

7 Write $\sin\left(x + \frac{\pi}{6}\right)$ in the form $p\sin x + q\cos x$ where p and q are constants to be found.

8 Write $\cos\left(x + \frac{\pi}{3}\right)$ in the form $a\cos x + b\sin x$ where a and b are constants to be found.

Homework Exercise

9 Express the following as a single sine, cosine or tangent:

a $\sin 15° \cos 20° + \cos 15° \sin 20°$

b $\sin 58° \cos 23° - \cos 58° \sin 23°$

c $\cos 130° \cos 80° - \sin 130° \sin 80°$

d $\dfrac{\tan 76° - \tan 45°}{1 + \tan 76° \tan 45°}$

e $\cos 2\theta \cos \theta + \sin 2\theta \sin \theta$

f $\cos 4\theta \cos 3\theta - \sin 4\theta \sin 3\theta$

g $\sin \frac{1}{2}\theta \cos 2\frac{1}{2}\theta + \cos \frac{1}{2}\theta \sin 2\frac{1}{2}\theta$

h $\dfrac{\tan 2\theta + \tan 3\theta}{1 - \tan 2\theta \tan 3\theta}$

i $\sin (A + B) \cos B - \cos (A + B) \sin B$

j $\cos\left(\dfrac{3x + 2y}{2}\right) \cos\left(\dfrac{3x - 2y}{2}\right) - \sin\left(\dfrac{3x + 2y}{2}\right) \sin\left(\dfrac{3x - 2y}{2}\right)$

10 Use the addition formulae for sine or cosine to write each of the following as a single trigonometric function in the form $\sin (x \pm \theta)$ or $\cos (x \pm \theta)$, where $0 < \theta < \frac{\pi}{2}$

a $\dfrac{1}{\sqrt{2}}(\sin x + \cos x)$

b $\dfrac{1}{\sqrt{2}}(\cos x - \sin x)$

c $\dfrac{1}{2}(\sin x + \sqrt{3} \cos x)$

d $\dfrac{1}{\sqrt{2}}(\sin x - \cos x)$

11 Given that $\cos y = \sin (x + y)$, show that $\tan y = \sec x - \tan x$.

12 Given that $\tan (x - y) = 3$, express $\tan y$ in terms of $\tan x$.

Homework Exercise

13 Given that $\sin x(\cos y + 2\sin y) = \cos x(2\cos y - \sin y)$, find the value of $\tan(x + y)$.

Hint First multiply out the brackets.

14 In each of the following, calculate the exact value of $\tan x$.

a $\tan(x - 45°) = \frac{1}{4}$ **b** $\sin(x - 60°) = 3\cos(x + 30°)$ **c** $\tan(x - 60°) = 2$

15 Given that $\tan\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$, show that $\tan x = 8 - 5\sqrt{3}$. **(3 marks)**

16 Prove that

$$\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right) = 0$$

You must show each stage of your working. **(4 marks)**

Challenge

This triangle is constructed from two right-angled triangles T_1 and T_2.

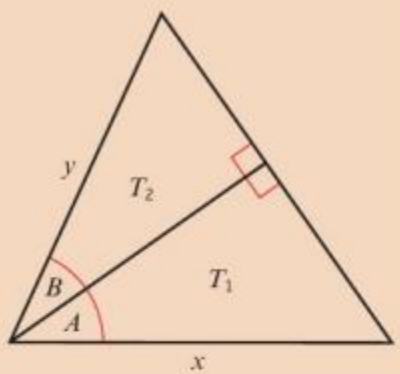
a Find expressions involving x, y, A and B for:

i the area of T_1

ii the area of T_2

iii the area of the large triangle.

b Hence prove that $\sin(A + B) = \sin A\cos B + \cos A\sin B$



Hint For part **a** your expressions should all involve **all four** variables. You will need to use the formula Area $= \frac{1}{2}ab\sin\theta$ in each case.

Homework Answers

1 **a** **i** $(\alpha - \beta) + \beta = \alpha$. So $\angle FAB = \alpha$.

ii $\angle FAB = \angle ABD$ (alternate angles)

$\angle CBE = 90 - \alpha$, so $\angle BCE = 90 - (90 - \alpha) = \alpha$.

iii $\cos\beta = \dfrac{AB}{1} \Rightarrow AB = \cos\beta$

iv $\sin\beta = \dfrac{BC}{1} \Rightarrow BC = \sin\beta$

b **i** $\sin\alpha = \dfrac{AD}{\cos\beta} \Rightarrow AD = \sin\alpha\cos\beta$

ii $\cos\alpha = \dfrac{BD}{\cos\beta} \Rightarrow BD = \cos\alpha\cos\beta$

c **i** $\cos\alpha = \dfrac{CE}{\sin\beta} \Rightarrow CE = \cos\alpha\sin\beta$

ii $\sin\alpha = \dfrac{BE}{\sin\beta} \Rightarrow BE = \sin\alpha\sin\beta$

d **i** $\sin(\alpha - \beta) = \dfrac{FC}{1} \Rightarrow FC = \sin(\alpha - \beta)$

ii $\cos(\alpha - \beta) = \dfrac{FA}{1} \Rightarrow FA = \cos(\alpha - \beta)$

e **i** $FC + CE = AD$, so $FC = AD - CE$

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

ii $AF = DB + BE$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

2 $\tan(A - B) = \dfrac{\sin(A - B)}{\cos(A - B)} = \dfrac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} - \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} + \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

3 $\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(P + (-Q)) = \sin P\cos(-Q) + \cos P\sin(-Q)$

$\sin(P - Q) = \sin P\cos Q - \cos P\sin Q$

4 Example: with $A = 60°$, $B = 30°$,

$\sin(A + B) = \sin 90° = 1$; $\sin A + \sin B = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2} \neq 1$

[You can find examples of A and B for which the statement is true, e.g. $A = 30°$, $B = -30°$, but one counter-example shows that it is not an identity.]

5 $\cos(\theta - \theta) \equiv \cos\theta\cos\theta + \sin\theta\sin\theta$

$\Rightarrow \sin^2\theta + \cos^2\theta \equiv 1$ as $\cos 0 = 1$

6 **a** $\sin\left(\dfrac{\pi}{2} - \theta\right) \equiv \sin\dfrac{\pi}{2}\cos\theta - \cos\dfrac{\pi}{2}\sin\theta$

$\equiv (1)\cos\theta - (0)\sin\theta = \cos\theta$

b $\cos\left(\dfrac{\pi}{2} - \theta\right) \equiv \cos\dfrac{\pi}{2}\cos\theta - \sin\dfrac{\pi}{2}\sin\theta$

$\equiv (0)\cos\theta - (1)\sin\theta = \sin\theta$

7 $\sin\left(x + \dfrac{\pi}{6}\right) = \sin x\cos\dfrac{\pi}{6} + \cos x\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}\sin x + \dfrac{1}{2}\cos x$

8 $\cos\left(x + \dfrac{\pi}{3}\right) = \cos x\cos\dfrac{\pi}{3} - \sin x\sin\dfrac{\pi}{3} = \dfrac{1}{2}\cos x - \dfrac{\sqrt{3}}{2}\sin x$

9 **a** $\sin 35°$ **b** $\sin 35°$ **c** $\cos 210°$ **d** $\tan 31°$

e $\cos\theta$ **f** $\cos 7\theta$ **g** $\sin 3\theta$ **h** $\tan 5\theta$

i $\sin A$ **j** $\cos 3x$

10 **a** $\sin\left(x + \dfrac{\pi}{4}\right)$ or $\cos\left(x - \dfrac{\pi}{4}\right)$ **b** $\cos\left(x + \dfrac{\pi}{4}\right)$

c $\sin\left(x + \dfrac{\pi}{3}\right)$ or $\cos\left(x - \dfrac{\pi}{6}\right)$ **d** $\sin\left(x - \dfrac{\pi}{4}\right)$

Homework Answers

11 $\cos y = \sin x \cos y + \sin y \cos x$
Divide by $\cos x \cos y \Rightarrow \sec x = \tan x + \tan y$,
so $\tan y = \sec x - \tan x$

12 $\dfrac{\tan x - 3}{3\tan x + 1}$ **13** 2

14 a $\frac{5}{3}$ **b** $\sqrt{3}$ **c** $-\left(\dfrac{8 + 5\sqrt{3}}{11}\right)$

15 $\dfrac{\tan x + \sqrt{3}}{1 - \sqrt{3}\tan x} = \dfrac{1}{2} \Rightarrow (2 + \sqrt{3})\tan x = 1 - 2\sqrt{3}$, so

$$\tan x = \frac{1 - 2\sqrt{3}}{2 + \sqrt{3}} = \frac{(1 - 2\sqrt{3})(2 - \sqrt{3})}{1} = 8 - 5\sqrt{3}$$

16 Write θ as $\left(\theta + \dfrac{2\pi}{3}\right) - \dfrac{2\pi}{3}$ and $\theta + \dfrac{4\pi}{3}$ as $\left(\theta + \dfrac{2\pi}{3}\right) + \dfrac{2\pi}{3}$.
Use the addition formulae for cos and simplify.

Challenge

a i Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}x(y\cos B)(\sin A) = \frac{1}{2}xy\sin A\cos B$

ii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}y(x\cos A)(\sin B) = \frac{1}{2}xy\cos A\sin B$

iii Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}xy\sin(A + B)$

b Area of large triangle = area T_1 + area T_2

$$\tfrac{1}{2}xy\sin(A + B) = \tfrac{1}{2}xy\sin A\cos B + \tfrac{1}{2}xy\cos A\sin B$$

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$