P2 Chapter 3: Sequences and Series

Infinite Sums

Divergent vs Convergent

What can you say about the sum of each series up to infinity?

$$1 + 2 + 4 + 8 + 16 + \ldots$$

This is **divergent** – the sum of the values tends towards infinity.

$$1 - 2 + 3 - 4 + 5 - 6 + \ldots$$

This is **divergent** – the running total alternates either side of 0, but gradually gets further away from 0.

$$1 + 0.5 + 0.25 + 0.125 + \ldots$$

This is **convergent** – the sum of the values tends towards a fixed value, in this case 2.

Definitely NOT in the A Level syllabus, and just for fun...

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

This is **divergent** . This is known as the Harmonic Series

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

This is **convergent** . This is known as the Basel Problem, and the value is $\pi^2/6$.

Sum to Infinity

$1 + 0.5 + 0.25 + 0.125 + \ldots$ ← Why did this infinite sum converge (to 2)...

$1 + 2 + 4 + 8 + 16 + \ldots$ ← ...but this diverge to infinity?

- The infinite series will converge provided that $-1 < r < 1$ (which can be written as $|r| < 1$), because the terms will get smaller.

- Provided that $|r| < 1$, what happens to r^n as $n \to \infty$?
  For example $\left(\frac{1}{2}\right)^{100000}$ is very close to 0.
  We can see that as $n \to \infty, r^n \to 0$.

✎ A geometric series is convergent if $|r| < 1$.

- How therefore can we use the $S_n = \frac{a(1-r^n)}{1-r}$ formula to find the sum to infinity, i.e. S_∞?

✎ For a <u>convergent</u> geometric series,

$$S_\infty = \frac{a}{1-r}$$

Quickfire Examples

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

$$27, -9, 3, -1, \ldots$$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

$$p, p^2, p^3, p^4, \ldots$$

$where \;\; -1 < p < 1$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

$$p, 1, \frac{1}{p}, \frac{1}{p^2}, \ldots$$

$\boldsymbol{a} =$? $\quad \boldsymbol{r} =$? $\quad \boldsymbol{S_\infty} =$?

Quickfire Examples

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

$$\boldsymbol{a = 1,} \quad \boldsymbol{r = \frac{1}{2}} \quad \boldsymbol{S_\infty = 2}$$

$$27, -9, 3, -1, \ldots$$

$$\boldsymbol{a = 27,} \quad \boldsymbol{r = -\frac{1}{3}} \quad \boldsymbol{S_\infty = \frac{81}{4}}$$

$$p, p^2, p^3, p^4, \ldots$$

$$where \ -1 < p < 1$$

$$\boldsymbol{a = p,} \quad \boldsymbol{r = p} \quad \boldsymbol{S_\infty = \frac{p}{1-p}}$$

$$p, 1, \frac{1}{p}, \frac{1}{p^2}, \ldots$$

$$\boldsymbol{a = p,} \quad \boldsymbol{r = \frac{1}{p}} \quad \boldsymbol{S_\infty = \frac{p^2}{p-1}}$$

Further Examples

[Textbook] The fourth term of a geometric series is 1.08 and the seventh term is 0.23328.

a) Show that this series is convergent.
b) Find the sum to infinity of this series.

? a

? b

[Textbook] For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a) Find the possible values of r.
b) Given that all the terms in the series are positive, find the value of a.

Fro Warning: The power is n in the S_n formula but $n-1$ in the u_n formula.

? a

? b

Further Examples

[Textbook] The fourth term of a geometric series is 1.08 and the seventh term is 0.23328.

a) Show that this series is convergent.
b) Find the sum to infinity of this series.

$$u_4 = 1.08 \quad \rightarrow \quad ar^3 = 1.08$$
$$u_7 = 0.23328 \quad \rightarrow \quad ar^6 = 0.23328$$

Dividing:

$$r^3 = 0.216 \quad \therefore r = 0.6$$

The series converges because $|0.6| < 1$.

$$a = \frac{1.08}{r^3} = \frac{1.08}{0.216} = 5$$
$$\therefore S_\infty = \frac{5}{1 - 0.6} = 12.5$$

[Textbook] For a geometric series with first term a and common ratio r, $S_4 = 15$ and $S_\infty = 16$.

a) Find the possible values of r.
b) Given that all the terms in the series are positive, find the value of a.

Fro Warning: The power is n in the S_n formula but $n - 1$ in the u_n formula.

$$S_4 = 15 \quad \rightarrow \quad \frac{a(1 - r^4)}{1 - r} = 15 \quad (1)$$

$$S_\infty = 16 \quad \rightarrow \quad \frac{a}{1 - r} = 16 \quad (2)$$

Substituting $\frac{a}{1-r}$ for 16 in equation (1):

$16(1 - r^4) = 15$

Solving: $r = \pm\frac{1}{2}$

As terms positive, $r = \frac{1}{2}$, $\therefore a = 16\left(1 - \frac{1}{2}\right) = 8$

Test Your Understanding

Edexcel C2 May 2011 Q6

6. The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio, $r =$?
(2)

(*b*) the first term, $a =$?
(2)

(*c*) the sum to infinity, $S_\infty =$?
(2)

(*d*) the smallest value of n for which the sum of the first n terms of the series exceeds 1000.
(4)

?

Test Your Understanding

Edexcel C2 May 2011 Q6

6. The second and third terms of a geometric series are 192 and 144 respectively.

For this series, find

(*a*) the common ratio, $$r = \frac{3}{4}$$ **(2)**

(*b*) the first term, $$a = 256$$ **(2)**

(*c*) the sum to infinity, $$S_\infty = 1024$$ **(2)**

(*d*) the smallest value of *n* for which the sum of the first *n* terms of the series exceeds 1000. **(4)**

$$\frac{256\left(1-\left(\frac{3}{4}\right)^n\right)}{0.25} > 1000$$

$$n > \frac{\log\left(\frac{6}{256}\right)}{\log(0.75)} \quad \Rightarrow n = 14$$

Exercise 3.5

Pearson Pure Mathematics Year 2/AS
Page 22

Extension

1 [MAT 2006 1H] How many solutions does the equation
$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$$
have in the range $0 \leq x < 2\pi$

?

2 [MAT 2003 1F] Two players take turns to throw a fair six-sided die until one of them scores a six. What is the probability that the first player to throw the die is the first to score a six?

?

☠ [Frost] Determine the value of x where:
$$x = \frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \cdots$$
(Hint: Use an approach similar to proof of geometric S_n formula)

?

Exercise 3.5

Pearson Pure Mathematics Year 2/AS
Page 22

Extension

1 [MAT 2006 1H] How many solutions does the equation

$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$$

have in the range $0 \leq x < 2\pi$

RHS is geometric series. $\boldsymbol{a = \sin x, r = \sin x}$.

$$\mathbf{2 = \frac{\sin x}{1 - \sin x} \quad \rightarrow \quad 2 - 2\sin x = \sin x}$$

$$\mathbf{\sin x = \frac{2}{3}}$$

By considering the graph of sin for $\mathbf{0 \leq x < 2\pi}$ **, this has 2 solutions.**

[Frost] Determine the value of x where:

$$x = \frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \frac{6}{32} + \cdots$$

(Hint: Use an approach similar to proof of geometric S_n formula)

Doubling:

$$\mathbf{2x = 2 + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}$$

Subtracting the two equations:

$$\mathbf{x = 2 + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots}$$

$$\mathbf{= 2 + 2 = 4}$$

2 [MAT 2003 1F] Two players take turns to throw a fair six-sided die until one of them scores a six. What is the probability that the first player to throw the die is the first to score a six?

$$\boldsymbol{P(win\ 1st\ throw) = \frac{1}{6}}$$

$$\boldsymbol{P(win\ 3rd\ throw) = \left(\frac{5}{6}\right)^2 \times \left(\frac{1}{6}\right)}$$

$$\boldsymbol{P(win\ 5th\ throw) = \left(\frac{5}{6}\right)^4 \times \left(\frac{1}{6}\right)}$$

The total probability of the 1st player winning is the sum of these up to infinity.

$$\boldsymbol{a = \frac{1}{6}, r = \left(\frac{5}{6}\right)^2 = \frac{25}{36}}$$

$$\boldsymbol{S_\infty = \frac{1/6}{1 - 25/36} = \frac{6}{11}}$$

Homework Exercise

1 For each of the following series:

i state, with a reason, whether the series is convergent.

ii If the series is convergent, find the sum to infinity.

a $1 + 0.1 + 0.01 + 0.001 + \ldots$ **b** $1 + 2 + 4 + 8 + 16 + \ldots$ **c** $10 - 5 + 2.5 - 1.25 + \ldots$

d $2 + 6 + 10 + 14 + \ldots$ **e** $1 + 1 + 1 + 1 + 1 + \ldots$ **f** $3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$

g $0.4 + 0.8 + 1.2 + 1.6 + \ldots$ **h** $9 + 8.1 + 7.29 + 6.561 + \ldots$

2 A geometric series has first term 10 and sum to infinity 30. Find the common ratio.

3 A geometric series has first term -5 and sum to infinity -3. Find the common ratio.

4 A geometric series has sum to infinity 60 and common ratio $\frac{2}{3}$. Find the first term.

5 A geometric series has common ratio $-\frac{1}{3}$ and $S_\infty = 10$. Find the first term.

6 Find the fraction equal to the recurring decimal $0.\dot{2}\dot{3}$.

Hint $0.\dot{2}\dot{3} = \frac{23}{100} + \frac{23}{10\,000} + \frac{23}{1\,000\,000} + \ldots$

7 For a geometric series $a + ar + ar^2 + \ldots$, $S_3 = 9$ and $S_\infty = 8$, find the values of a and r.

8 Given that the geometric series $1 - 2x + 4x^2 - 8x^3 + \ldots$ is convergent,

a find the range of possible values of x **(3 marks)**

b find an expression for S_∞ in terms of x. **(1 mark)**

9 In a convergent geometric series the common ratio is r and the first term is 2.
Given that $S_\infty = 16 \times S_3$,

a find the value of the common ratio, giving your answer to 4 significant figures **(3 marks)**

b find the value of the fourth term. **(2 marks)**

Homework Exercise

10 The first term of a geometric series is 30. The sum to infinity of the series is 240.

a Show that the common ratio, r, is $\frac{7}{8}$ **(2 marks)**

b Find to 3 significant figures, the difference between the 4th and 5th terms. **(2 marks)**

c Calculate the sum of the first 4 terms, giving your answer to 3 significant figures. **(2 marks)**

The sum of the first n terms of the series is greater than 180.

d Calculate the smallest possible value of n. **(4 marks)**

11 A geometric series has first term a and common ratio r. The second term of the series is $\frac{15}{8}$ and the sum to infinity of the series is 8.

a Show that $64r^2 - 64r + 15 = 0$. **(4 marks)**

b Find the two possible values of r. **(2 marks)**

c Find the corresponding two possible values of a. **(2 marks)**

Given that r takes the smaller of its two possible values,

d find the smallest value of n for which S_n exceeds 7.99. **(2 marks)**

Challenge

The sum to infinity of a geometric series is 7. A second series is formed by squaring every term in the first geometric series.

a Show that the second series is also geometric.

b Given that the sum to infinity of the second series is 35, show that the common ratio of the original series is $\frac{1}{6}$

Homework Answers

1 **a** Yes as $|r| < 1$, $\frac{10}{9}$ **b** No as $|r| \geqslant 1$

c Yes as $|r| < 1$, $6\frac{2}{3}$

d No; arithmetic series does not converge.

e No as $|r| \geqslant 1$ **f** Yes as $|r| < 1$, $4\frac{1}{2}$

g No; arithmetic series does not converge.

h Yes as $|r| < 1$, 90

2 $\frac{2}{3}$ 3 $-\frac{2}{3}$ 4 20 5 $13\frac{1}{3}$

6 $\frac{23}{99}$ 7 $r = -\frac{1}{2}$, $a = 12$

8 **a** $-\frac{1}{2} < x < \frac{1}{2}$ **b** $S_\infty = \dfrac{1}{1 + 2x}$

9 **a** 0.9787 **b** 1.875

10 **a** $\dfrac{30}{1 - r} = 240 \Rightarrow 1 - r = \dfrac{1}{8} \Rightarrow r = \dfrac{7}{8}$

b 2.51 **c** 99.3 **d** 11

11 **a** $ar = \dfrac{15}{8} \Rightarrow a = \dfrac{15}{8r}$

$\dfrac{a}{1 - r} = 8 \Rightarrow a = 8(1 - r)$

$\dfrac{15}{8r} = 8(1 - r) \Rightarrow 15 = 64r - 64r^2$

$\Rightarrow 64r^2 - 64r + 15 = 0$

b $\frac{3}{8}, \frac{5}{8}$ **c** 5, 3 **d** 7

Challenge

a First series: $a + ar + ar^2 + ar^3 + \ldots$

Second series: $a^2 + a^2r^2 + a^2r^4 + a^2r^6 + \ldots$

Second series is geometric with common ratio is r^2 and first term a^2.

b $\dfrac{a}{1 - r} = 7 \Rightarrow a = 7(1 - r) \Rightarrow a^2 = 49(1 - r)(1 - r)$

$\dfrac{a^2}{1 - r^2} = 35 \Rightarrow \dfrac{49(1 - r)(1 - r)}{(1 - r)(1 + r)} = 35$

$49(1 - r) = 35(1 + r) \Rightarrow 49 - 49r = 35 + 35r \Rightarrow r = \dfrac{1}{6}$