P2 Chapter 7: Trigonometric Equations

Double Angle Formulae

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Double-angle formula allow you to halve the angle within a trig function.

$$\sin(2A) \equiv 2\sin A\cos A$$
$$\cos(2A) \equiv \cos^2 A - \sin^2 A$$
$$\equiv 2\cos^2 A - 1$$
$$\equiv 1 - 2\sin^2 A$$

This first form is relatively rare.

Tip: Way to remember which way round these go is that the cos on the RHS is 'attracted' to the cos on the LHS, whereas the sin is pushed away.

$$\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$$

These are all easily derivable by just setting $A = B$ in the compound angle formulae. e.g.

$$\sin(2A) = \sin(A + A)$$
$$= \sin A\cos A + \cos A\sin A$$
$$= 2\sin A\cos A$$

Examples

[Textbook] Use the double-angle formulae to write each of the following as a single trigonometric ratio.

a) $\cos^2 50° - \sin^2 50°$

b) $\dfrac{2\tan\left(\frac{\pi}{6}\right)}{1-\tan^2\left(\frac{\pi}{6}\right)}$

c) $\dfrac{4\sin 70°}{\sec 70°}$

$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

a ?

b ?

c ?

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$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

a

This matches $\cos(2A) = \cos^2 A - \sin^2 A$:

$$cos^2\, 50° - sin^2\, 50° = \cos(100°)$$

b

This matches $\tan(2A) = \frac{2\tan A}{1-\tan^2 A}$

$$\frac{2\, tan\left(\frac{\pi}{6}\right)}{1 - tan^2\left(\frac{\pi}{6}\right)} = \tan\left(\frac{\pi}{3}\right)$$

c

$$\frac{4\sin 70°}{\sec 70°} = \frac{4\sin 70°}{1 \div \cos 70°} = 4\sin 70° \cos 70°$$

This matches $\sin(2A) = 2\sin A\cos A$

$$4\sin 70°\ cos 70° = 2\sin 140°$$

Examples

[Textbook] Given that $x = 3\sin\theta$ and $y = 3 - 4cos2\theta$, eliminate θ and express y in terms of x.

?

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Note: This question is an example of turning a set of **parametric equations** into a single **Cartesian** one. You will cover this in the next chapter.

Given that $\cos x = \frac{3}{4}$ and x is acute, find the exact value of

(a) $\sin 2x$ (b) $\tan 2x$

?

Examples

[Textbook] Given that $x = 3\sin\theta$ and $y = 3 - 4cos2\theta$, eliminate θ and express y in terms of x.

$$\begin{aligned} y &= 3 - 4(1 - 2\sin^2\theta) \\ &= 8\sin^2\theta - 1 \end{aligned}$$

$$\sin\theta = \frac{x}{3}$$

$$\therefore y = 8\left(\frac{x}{3}\right)^2 - 1$$

$$\begin{aligned} \sin 2x &= 2\sin x\cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \\ \tan 2x &= \frac{2\tan x}{1 - \tan^2 x} \end{aligned}$$

Note: This question is an example of turning a set of **parametric equations** into a single **Cartesian** one. You will cover this in the next chapter.

Given that $\cos x = \frac{3}{4}$ and x is acute, find the exact value of

(a) $\sin 2x$ (b) $\tan 2x$

Since $sin^2 x + \cos^2 x = 1$, $sin\ x = \pm\sqrt{1 - cos^2 x}$

$sin\ x = \sqrt{1 - \left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}$ **(and is positive given** $0 < x < 90°$**)**

$$\therefore sin\ 2x = 2sin\ x\ cos\ x = 2 \times \frac{\sqrt{7}}{4} \times \frac{3}{4} = \frac{3\sqrt{7}}{8}$$

$$tan\ x = \frac{sin\ x}{cos} = \frac{3\sqrt{7}/8}{3/4} = \frac{\sqrt{7}}{3} \qquad \therefore tan2x = \frac{2 \times \frac{\sqrt{7}}{3}}{1 - \left(\frac{\sqrt{7}}{3}\right)^2} = 3\sqrt{7}$$

Exercise 7.3

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Homework Exercise

1 Use the expansion of $\sin(A + B)$ to show that $\sin 2A \equiv 2\sin A \cos A$.

Hint Set $B = A$.

2 **a** Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, show that $\cos 2A \equiv \cos^2 A - \sin^2 A$.

b Hence show that:

i $\cos 2A \equiv 2\cos^2 A - 1$

ii $\cos 2A \equiv 1 - 2\sin^2 A$

Problem-solving

Use $\sin^2 A + \cos^2 A \equiv 1$

3 Use the expansion of $\tan(A + B)$ to express $\tan 2A$ in terms of $\tan A$.

4 Write each of the following expressions as a single trigonometric ratio.

a $2\sin 10° \cos 10°$

b $1 - 2\sin^2 25°$

c $\cos^2 40° - \sin^2 40°$

d $\dfrac{2\tan 5°}{1 - \tan^2 5°}$

e $\dfrac{1}{2\sin(24.5)° \cos(24.5)°}$

f $6\cos^2 30° - 3$

g $\dfrac{\sin 8°}{\sec 8°}$

h $\cos^2\dfrac{\pi}{16} - \sin^2\dfrac{\pi}{16}$

5 Without using your calculator find the exact values of:

a $2\sin 22.5° \cos 22.5°$

b $2\cos^2 15° - 1$

c $(\sin 75° - \cos 75°)^2$

d $\dfrac{2\tan\frac{\pi}{8}}{1 - \tan^2\frac{\pi}{8}}$

Homework Exercise

6 **a** Show that $(\sin A + \cos A)^2 \equiv 1 + \sin 2A$. **(3 marks)**

b Hence find the exact value of $\left(\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right)^2$. **(2 marks)**

7 Write the following in their simplest form, involving only one trigonometric function:

a $\cos^2 3\theta - \sin^2 3\theta$

b $6\sin 2\theta\cos 2\theta$

c $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

d $2 - 4\sin^2\frac{\theta}{2}$

e $\sqrt{1+\cos 2\theta}$

f $\sin^2\theta\cos^2\theta$

g $4\sin\theta\cos\theta\cos 2\theta$

h $\dfrac{\tan\theta}{\sec^2\theta - 2}$

i $\sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta$

8 Given that $p = 2\cos\theta$ and $q = \cos 2\theta$, express q in terms of p.

9 Eliminate θ from the following pairs of equations:

a $x = \cos^2\theta$, $y = 1 - \cos 2\theta$

b $x = \tan\theta$, $y = \cot 2\theta$

c $x = \sin\theta$, $y = \sin 2\theta$

d $x = 3\cos 2\theta + 1$, $y = 2\sin\theta$

10 Given that $\cos x = \frac{1}{4}$, find the exact value of $\cos 2x$.

11 Find the possible values of $\sin\theta$ when $\cos 2\theta = \frac{23}{25}$

Homework Exercise

12 Given that $\tan\theta = \frac{3}{4}$, and that θ is acute,

a find the exact value of: **i** $\tan 2\theta$ **ii** $\sin 2\theta$ **iii** $\cos 2\theta$

b deduce the value of $\sin 4\theta$.

13 Given that $\cos A = -\frac{1}{3}$, and that A is obtuse,

a find the exact value of: **i** $\cos 2A$ **ii** $\sin A$ **iii** $\operatorname{cosec} 2A$

b show that $\tan 2A = \dfrac{4\sqrt{2}}{7}$

14 Given that $\pi < \theta < \dfrac{3\pi}{2}$, find the value of $\tan\dfrac{\theta}{2}$ when $\tan\theta = \frac{3}{4}$ **(4 marks)**

15 Given that $\cos x + \sin x = m$ and $\cos x - \sin x = n$, where m and n are constants, write down, in terms of m and n, the value of $\cos 2x$. **(4 marks)**

16 In $\triangle PQR$, $PQ = 3\,\text{cm}$, $PR = 6\,\text{cm}$, $QR = 5\,\text{cm}$ and $\angle QPR = 2\theta$.

a Use the cosine rule to show that $\cos 2\theta = \frac{5}{9}$ **(3 marks)**

b Hence find the exact value of $\sin\theta$. **(2 marks)**

17 The line l, with equation $y = \frac{3}{4}x$, bisects the angle between the x-axis and the line $y = mx$, $m > 0$. Given that the scales on each axis are the same, and that l makes an angle θ with the x-axis,

a write down the value of $\tan\theta$ **(1 mark)**

b show that $m = \frac{24}{7}$ **(3 marks)**

Homework Exercise

18 **a** Use the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, to show that $\cos 2A \equiv 2\cos^2 A - 1$.
(2 marks)

The curves C_1 and C_2 have equations

C_1: $y = 4\cos 2x$
C_2: $y = 6\cos^2 x - 3\sin 2x$

b Show that the x-coordinates of the points where C_1 and C_2 intersect satisfy the equation

$\cos 2x + 3\sin 2x - 3 = 0$ **(3 marks)**

19 Use the fact that $\tan 2A \equiv \dfrac{\sin 2A}{\cos 2A}$ to derive the formula for $\tan 2A$ in terms of $\tan A$.

Hint Use the identities for $\sin 2A$ and $\cos 2A$ and then divide both the numerator and denominator by $\cos^2 A$.

Homework Answers

1 $\sin 2A = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$

2 **a** $\cos 2A = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$

b **i** $\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A)$
$= 2\cos^2 A - 1$

ii $\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A$

3 $\tan 2A = \dfrac{\tan A + \tan A}{1 - \tan A\tan A} = \dfrac{2\tan A}{1 - \tan^2 A}$

4 **a** $\sin 20°$ **b** $\cos 50°$ **c** $\cos 80°$
d $\tan 10°$ **e** $\operatorname{cosec} 49°$ **f** $3\cos 60°$
g $\frac{1}{2}\sin 16°$ **h** $\cos\left(\frac{\pi}{8}\right)$

5 **a** $\dfrac{\sqrt{2}}{2}$ **b** $\dfrac{\sqrt{3}}{2}$ **c** $\frac{1}{2}$ **d** 1

6 **a** $(\sin A + \cos A)^2 = \sin^2 A + 2\sin A\cos A + \cos^2 A$
$= 1 + \sin 2A$

b $\left(\sin\frac{\pi}{8} + \cos\frac{\pi}{8}\right)^2 = 1 + \sin\frac{\pi}{4} = 1 + \dfrac{\sqrt{2}}{2} = \dfrac{2 + \sqrt{2}}{2}$

7 **a** $\cos 6\theta$ **b** $3\sin 4\theta$ **c** $\tan\theta$
d $2\cos\theta$ **e** $\sqrt{2}\cos\theta$ **f** $\frac{1}{4}\sin^2 2\theta$
g $\sin 4\theta$ **h** $-\frac{1}{2}\tan 2\theta$ **i** $\cos^2 2\theta$

8 $q = \dfrac{p^2}{2} - 1$

9 **a** $y = 2(1 - x)$ **b** $2xy = 1 - x^2$
c $y^2 = 4x^2(1 - x^2)$ **d** $y^2 = \dfrac{2(4 - x)}{3}$

10 $-\frac{7}{8}$ 11 $\pm\frac{1}{5}$

12 **a** **i** $\frac{24}{7}$ **ii** $\frac{24}{25}$ **iii** $\frac{7}{25}$ **b** $\frac{336}{625}$

13 **a** **i** $-\frac{7}{9}$ **ii** $\dfrac{2\sqrt{2}}{3}$ **iii** $-\dfrac{9\sqrt{2}}{8}$

b $\tan 2A = \dfrac{\sin 2A}{\cos 2A} = -\dfrac{4\sqrt{2}}{9} \times -\dfrac{9}{7} = \dfrac{4\sqrt{2}}{7}$

14 -3 15 mn

16 **a** $\cos 2\theta = \dfrac{3^2 + 6^2 - 5^2}{2 \times 3 \times 6} = \dfrac{20}{36} = \dfrac{5}{9}$ **b** $\dfrac{\sqrt{2}}{3}$

17 **a** $\frac{3}{4}$ **b** $m = \tan 2\theta = \dfrac{2\left(\frac{3}{4}\right)}{1 - \left(\frac{3}{4}\right)^2} = \dfrac{3}{2} \times \dfrac{16}{7} = \dfrac{24}{7}$

18 **a** $\cos 2A = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$
$= \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$

b $4\cos 2x = 6\cos^2 x - 3\sin 2x$
$\cos 2x + 3\cos 2x - 6\cos^2 x + 3\sin 2x = 0$
$\cos 2x + 3(2\cos^2 x - 1) - 6\cos^2 x + 3\sin 2x = 0$
$\cos 2x - 3 + 3\sin 2x = 0$
$\cos 2x + 3\sin 2x - 3 = 0$

19 $\tan 2A \equiv \dfrac{\sin 2A}{\cos 2A} \equiv \dfrac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$

$$\equiv \frac{\dfrac{2\sin A\cos A}{\cos^2 A}}{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}} \equiv \frac{2\tan A}{1 - \tan^2 A}$$