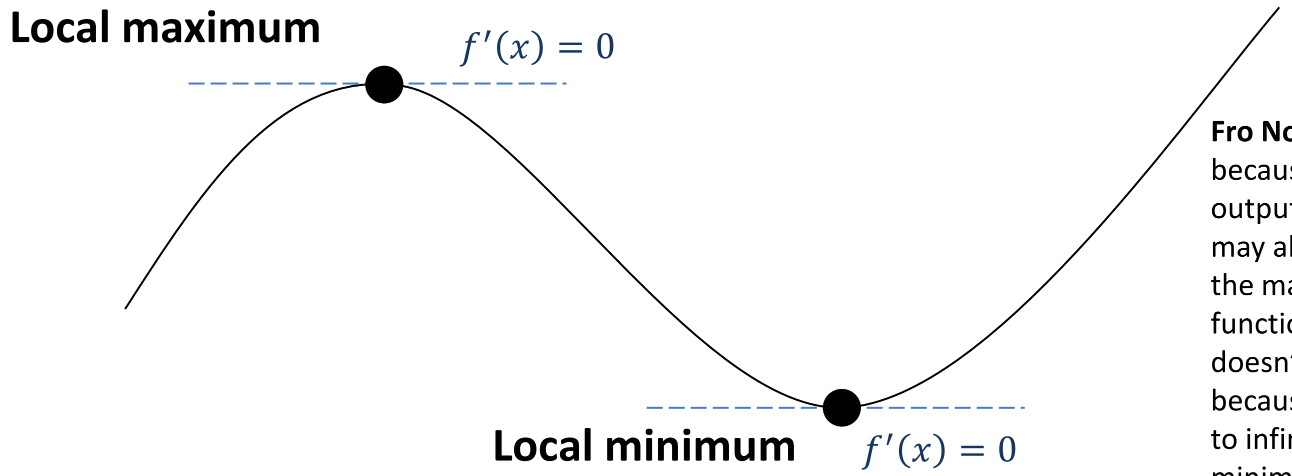
P1 Chapter 12: Differentiation

Stationary Points

Stationary/Turning Points

A stationary point is where the gradient is 0, i.e. $f'(x) = 0$.



Fro Note: It's called a '**local**' maximum because it's the function's largest output within the vicinity. Functions may also have a '**global**' maximum, i.e. the maximum output across the entire function. This particular function doesn't have a global maximum because the output keeps increasing up to infinity. It similarly has no global minimum, as with all cubics.

Find the coordinates of the turning points of $y = x^3 + 6x^2 - 135x$

?

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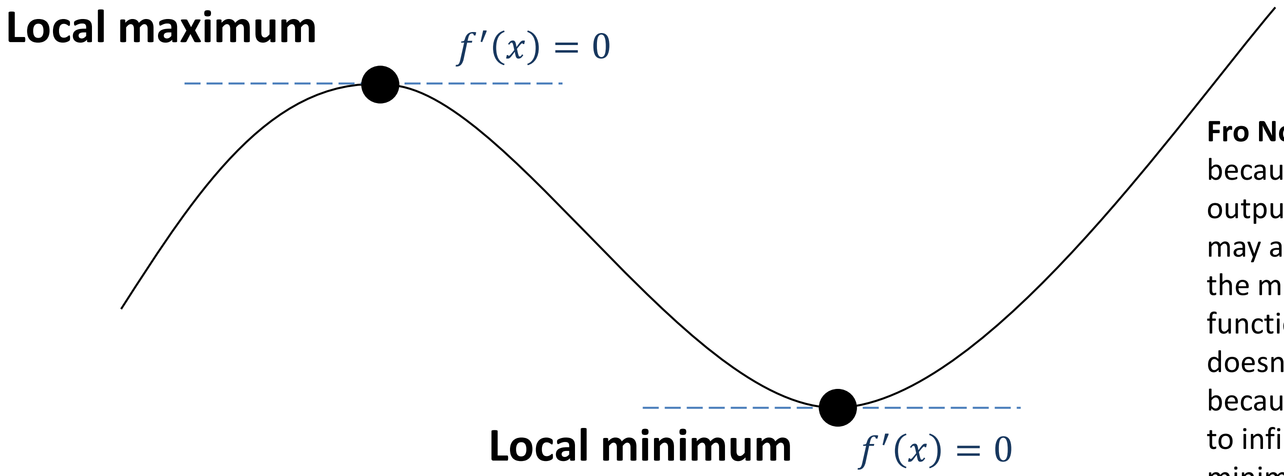

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Find the coordinates of the turning points of $y = x^3 + 6x^2 - 135x$

$$\frac{dy}{dx} = 3x^2 + 12x - 135 = 0$$
$$x^2 + 4x - 45 = 0$$
$$(x+9)(x-5) = 0$$
$$x = -9 \text{ or } x = 5$$

When $x = 5, y = 5^3 + 6(5^2) - 135(5) = -400 \quad \rightarrow \quad (5, -400)$

When $x = -9, y = (-9)^3 + 6(-9)^2 - 135(-9) = 972 \quad \rightarrow \quad (-9, 972)$

More Examples

Find the least value of
$$f(x) = x^2 - 4x + 9$$

?

Method 1:
Differentiation

?

Method 2:
Completing the Square

Fro Note: Method 2 is only applicable for quadratic functions. For others, differentiation must be used.

Find the turning point of
$$y = \sqrt{x} - x$$

?

More Examples

Find the least value of

$$f(x) = x^2 - 4x + 9$$

Find the turning point of

$$y = \sqrt{x} - x$$

Method 1: Differentiation

$$f'(x) = 2x - 4 = 0$$
$$x = 2$$
$$f(2) = 2^2 - 4(2) + 9 = 5$$

So 5 is the minimum value.

Method 2: Completing the square

$$f(x) = (x-2)^2 + 5$$

Therefore the minimum value of $f(x)$ is 5, and this occurs when $x = 2$.

Fro Note: Method 2 is only applicable for quadratic functions. For others, differentiation must be used.

$$y = x^{\frac{1}{2}} - x$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 1 = 0$$
$$\frac{1}{2}x^{-\frac{1}{2}} = 1$$
$$x^{-\frac{1}{2}} = 2 \quad \rightarrow \quad \frac{1}{\sqrt{x}} = 2$$
$$\sqrt{x} = \frac{1}{2} \quad \rightarrow \quad x = \frac{1}{4}$$

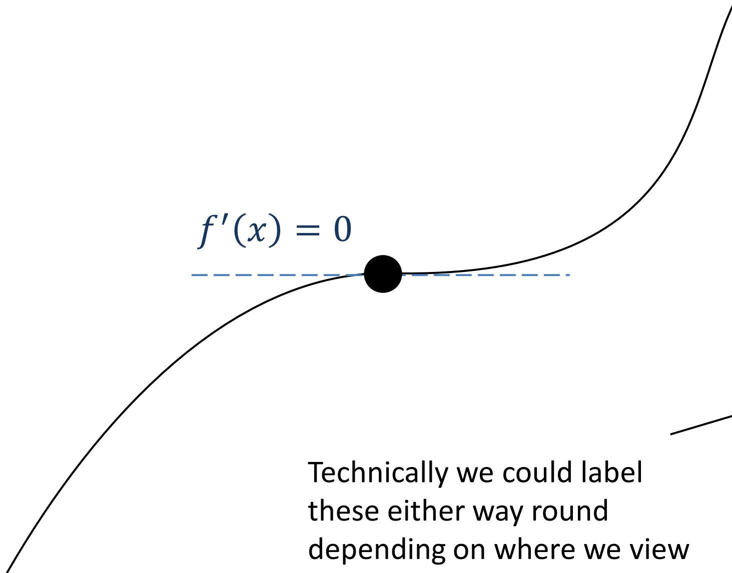
When $x = \frac{1}{4}$,

$$y = \sqrt{\frac{1}{4}} - \frac{1}{4} = \frac{1}{4}$$

So turning point is $\left(\frac{1}{4}, \frac{1}{4}\right)$

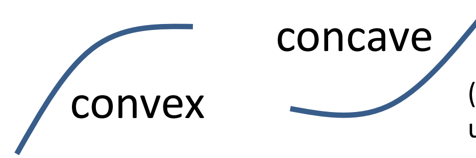
Points of Inflection

There’s a third type of stationary point (that we’ve encountered previously):



Technically we could label these either way round depending on where we view the curve from. What’s important is that the **concavity changes**.

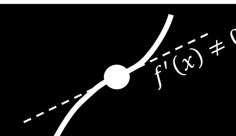
A point of inflection is where the curve changes from convex concave (or vice versa).



(the same terms used in optics!)

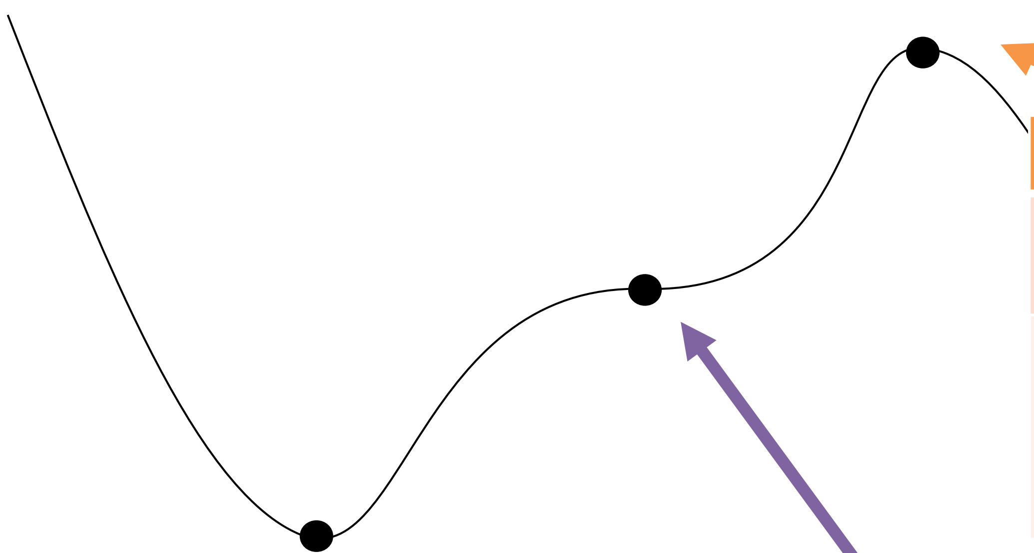
i.e. the line curves in one direction before the point of inflection, then curves in the other direction after.

Fro Side Note: Not all points of inflection are stationary points, as can be seen in the example on the right.
A point of inflection which **is** a stationary point is known as a *saddle point*.



How do we tell what type of stationary point?

Method 1: Look at gradient just before and just after point.



Local Maximum		
Gradient just before	Gradient at maximum	Gradient just after
+ve	0	+ve

Point of Inflection		
Gradient just before	Gradient at p.o.i	Gradient just after
+ve	0	+ve

Local Minimum		
Gradient just before	Gradient at minimum	Gradient just after
-ve	0	+ve

How do we tell what type of stationary point?

Method 1: Look at gradient just before and just after point.

Find the stationary point on the curve with equation $y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

? Turning Point

Strategy: ?

? Determine point type

How do we tell what type of stationary point?

Method 1: Look at gradient just before and just after point.

Find the stationary point on the curve with equation $y = x^4 - 32x$, and determine whether it is a local maximum, a local minimum or a point of inflection.

$$\frac{dy}{dx} = 4x^3 - 32 = 0$$

$x = 2 \quad \therefore \quad y = -48$

Stationary point is $(2, -48)$

Strategy: Find the gradient for values just before and after $x = 2$. Let's try $x = 1.9$ and $x = 2.1$.

	$x = 1.9$	$x = 2$	$x = 2.1$
Gradient	-4.56	0	5.04
Shape			

Looking at the shape, we can see that $(2, -48)$ is a minimum.

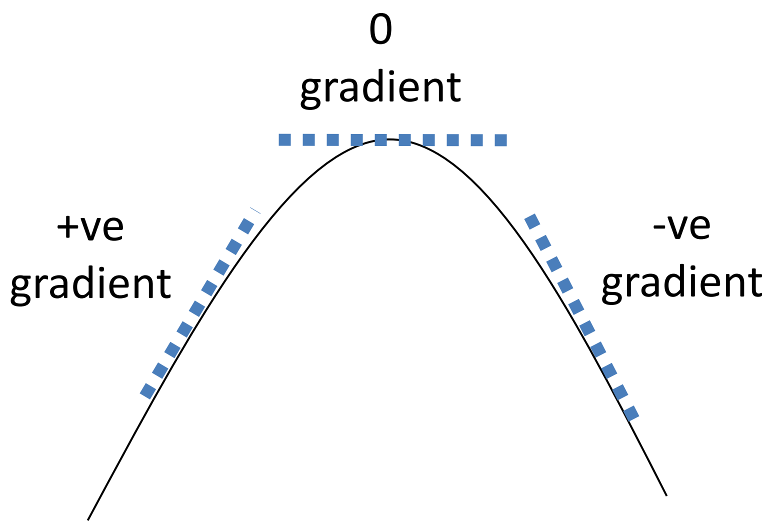
Method 2: Using the second derivative

The method of substituting values of x just before and after is a bit cumbersome.
It also has the potential for problems: what if two different types of stationary points are really close together?

Recall the gradient gives a measure of the **rate of change** of y, i.e. how much the y value changes as x changes.

Thus by differentiating the gradient function, the **second derivative tells us the rate at what the gradient is changing**.

Thus if the second derivative is positive, the gradient is increasing.
If the second derivative is negative, the gradient is decreasing.



At a maximum point, we can see that as x increases, the gradient is decreasing from a positive value to a negative value.

$$\therefore \frac{d^2y}{dx^2} < 0$$

Method 2: Using the second derivative

✎ At a stationary point $x = a$:

- If $f''(a) > 0$ the point is a local minimum.
- If $f''(a) < 0$ the point is a local maximum.
- If $f''(a) = 0$ it could be any type of point, so resort to Method 1.

I will eventually do a 'Just for your Interest…' thingy on why we can't classify the point when $f''(x) = 0$, and how we could use the **third derivative**!

The stationary point of $y = x^4 - 32x$ is $(2, -48)$. Use the second derivative to classify this stationary point.

?

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The stationary point of $y = x^4 - 32x$ is $(2, -48)$. Use the second derivative to classify this stationary point.

$$\frac{dy}{dx} = 4x^3 - 32$$

$$\frac{d^2y}{dx^2} = 12x^2$$

When $x = 2$, $\frac{d^2y}{dx^2} = 12(2)^2 > 0$

Therefore the stationary point is a minimum point.

Find $\frac{d^2y}{dx^2}$ for the x at the stationary point.

Test Your Understanding

Edexcel C2 May 2013 Q9

The curve with equation

$$y = x^2 - 32\sqrt{x} + 20, \qquad x > 0,$$

has a stationary point P.

Use calculus

(a) to find the coordinates of P,

(6)

(b) to determine the nature of the stationary point P.

(3)

(a)

?

(b)

?

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(3)

(a)	$\left\{\frac{\mathrm{d}y}{\mathrm{d}x} = \right\}\ 2x - 16x^{-\frac{1}{2}}$	M1 A1
	$2x - 16x^{-\frac{1}{2}} = 0 \Rightarrow\ x^{\frac{3}{2}} = ,\ x^{-\frac{3}{2}} =$,or $2x - = 16x^{-\frac{1}{2}}$ then squared then obtain $x^3 =$	M1
	[or $2x - 16x^{-\frac{1}{2}} = 0 \Rightarrow x = 4$ (no wrong work seen)]	
	$(x^{\frac{3}{2}} = 8 \Rightarrow) x = 4$	A1
	$x = 4,\ y = 4^2 - 32\sqrt{4} + 20 = -28$ (ignore $y = 100$ as second answer)	M1 A1
		(6)
(b)	$\left\{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \right\}\ 2 + 8x^{-\frac{3}{2}}$	M1 A1
	$(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0 \Rightarrow) y$ is a minimum (there should be no wrong reasoning)	A1
		(3)

Exercise 12.9

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Homework Exercise

1 Find the least value of the following functions:

a $f(x) = x^2 - 12x + 8$ **b** $f(x) = x^2 - 8x - 1$ **c** $f(x) = 5x^2 + 2x$

2 Find the greatest value of the following functions:

a $f(x) = 10 - 5x^2$ **b** $f(x) = 3 + 2x - x^2$ **c** $f(x) = (6 + x)(1 - x)$

Hint For each part of questions **1** and **2**:
- Find $f'(x)$.
- Set $f'(x) = 0$ and solve to find the value of x at the stationary point.
- Find the corresponding value of $f(x)$.

3 Find the coordinates of the points where the gradient is zero on the curves with the given equations. Establish whether these points are local maximum points, local minimum points or points of inflection in each case.

a $y = 4x^2 + 6x$ **b** $y = 9 + x - x^2$ **c** $y = x^3 - x^2 - x + 1$

d $y = x(x^2 - 4x - 3)$ **e** $y = x + \dfrac{1}{x}$ **f** $y = x^2 + \dfrac{54}{x}$

g $y = x - 3\sqrt{x}$ **h** $y = x^{\frac{1}{2}}(x - 6)$ **i** $y = x^4 - 12x^2$

4 Sketch the curves with equations given in question **3** parts **a**, **b**, **c** and **d**, labelling any stationary points with their coordinates.

5 By considering the gradient on either side of the stationary point on the curve $y = x^3 - 3x^2 + 3x$, show that this point is a point of inflection.
Sketch the curve $y = x^3 - 3x^2 + 3x$.

6 Find the maximum value and hence the range of values for the function $f(x) = 27 - 2x^4$.

7 $f(x) = x^4 + 3x^3 - 5x^2 - 3x + 1$

a Find the coordinates of the stationary points of $f(x)$, and determine the nature of each.

b Sketch the graph of $y = f(x)$.

Hint Use the **factor theorem** with small positive integer values of x to find one factor of $f'(x)$.

Homework Answers

1 **a** -28 **b** -17 **c** $-\frac{1}{5}$

2 **a** 10 **b** 4 **c** 12.25

3 **a** $\left(-\frac{3}{4}, -\frac{9}{4}\right)$ minimum

b $\left(\frac{1}{2}, 9\frac{1}{4}\right)$ maximum

c $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$ maximum, $(1, 0)$ minimum

d $(3,-18)$ minimum, $\left(-\frac{1}{3}, \frac{14}{27}\right)$ maximum

e $(1, 2)$ minimum, $(-1, -2)$ maximum

f $(3, 27)$ minimum

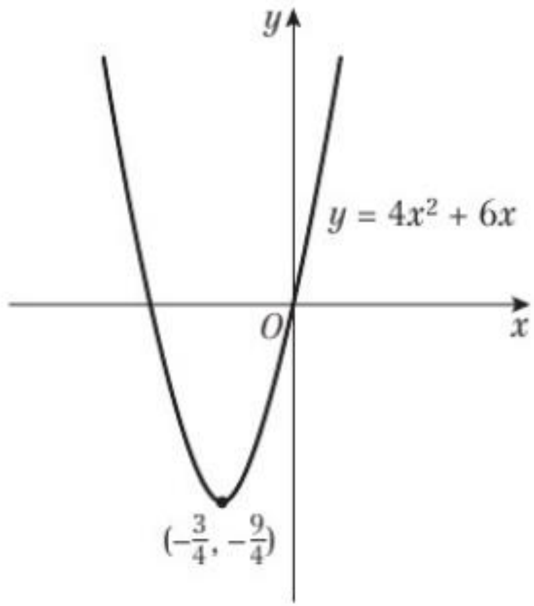
g $\left(\frac{9}{4}, -\frac{9}{4}\right)$ minimum

h $(2, -4\sqrt{2})$ minimum

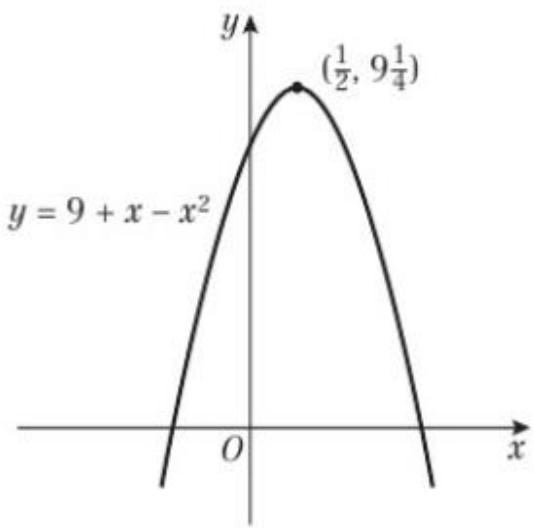
i $(\sqrt{6}, -36)$ minimum, $(-\sqrt{6}, -36)$ minimum, $(0,0)$ maximum

Homework Answers

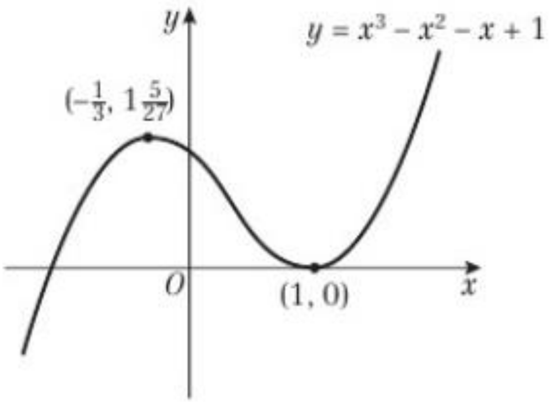
4 a



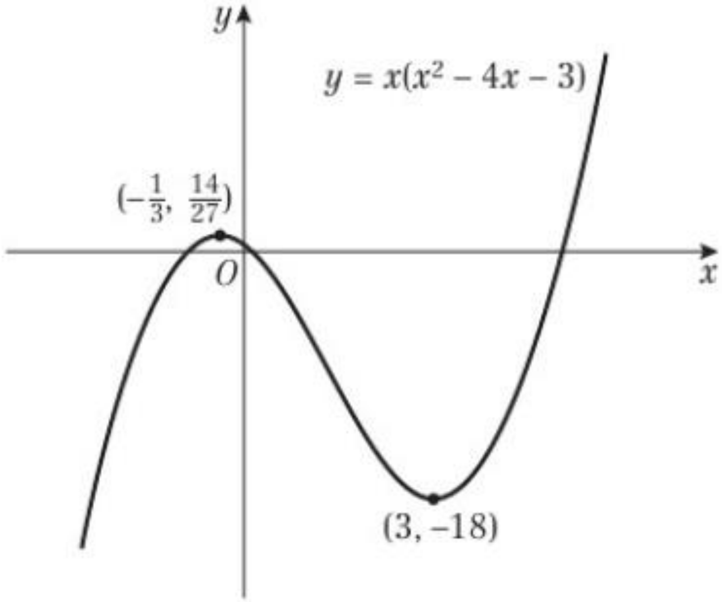
b



c

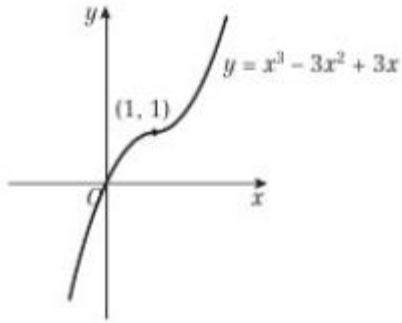


d



Homework Answers

5 (1, 1) inflection (gradient is positive either side of point)



6 Maximum value is 27; $f(x) \leqslant 27$

7 **a** (1, −3): minimum, (−3, −35): minimum, $\left(-\frac{1}{4}, \frac{357}{256}\right)$: maximum

b