
P1 Chapter 12: Differentiation

Second Order Derivatives

Second Order Derivatives

When you differentiate once, the expression you get is known as the **first derivative**. Unsurprisingly, when we differentiate a second time, the resulting expression is known as the **second derivative**. And so on...

Original Function

First Derivative

Second Derivative

The diagram illustrates three methods of differentiation for the function $y = x^4$. On the left, the function is written as $y = x^4$. Three arrows point from this function to the first derivative column:

- An arrow labeled "Leibniz's" points to $\frac{dy}{dx} = 4x^3$.
- A straight arrow points to $y' = 4x^3$.
- An arrow labeled "Newton's" points to $\dot{y} = 4x^3$.

From each first derivative expression, a straight arrow points to the second derivative column:

- From $\frac{dy}{dx} = 4x^3$, an arrow points to $\frac{d^2y}{dx^2} = 12x^2$.
- From $y' = 4x^3$, an arrow points to $y'' = 12x^2$.
- From $\dot{y} = 4x^3$, an arrow points to $\ddot{y} = 12x^2$.

Lagrange's

$$f(x) = x^4 \longrightarrow f'(x) = 4x^3 \longrightarrow f''(x) = 12x^2$$

You can similarly have the third derivative ($\frac{d^3y}{dx^3}$), although this is no longer in the A Level syllabus. We'll see why might use the second derivative soon...

Examples

If $y = 3x^5 + \frac{4}{x^2}$, find $\frac{d^2y}{dx^2}$.

?

If $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find $f''(x)$.

?

Examples

$$\text{If } y = 3x^5 + \frac{4}{x^2}, \text{ find } \frac{d^2y}{dx^2}.$$

$$y = 3x^5 + 4x^{-2}$$

$$\frac{dy}{dx} = 15x^4 - 8x^{-3}$$

$$\frac{d^2y}{dx^2} = 60x^3 + 24x^{-4}$$

This could also be written as:

$$60x^3 + \frac{24}{x^4}$$

$$\text{If } f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}, \text{ find } f''(x).$$

$$f(x) = 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$$

$$f'(x) = \frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{4}x^{-\frac{3}{2}}$$

$$f''(x) = -\frac{3}{4}x^{-\frac{3}{2}} + \frac{3}{8}x^{-\frac{5}{2}}$$

Test Your Understanding

If $y = 5x^3 - \frac{x}{3\sqrt{x}}$, find $\frac{d^2y}{dx^2}$.

?

Test Your Understanding

If $y = 5x^3 - \frac{x}{3\sqrt{x}}$, find $\frac{d^2y}{dx^2}$.

$$y = 5x^3 - \frac{1}{3}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 15x^2 - \frac{1}{6}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 30x + \frac{1}{12}x^{-\frac{3}{2}}$$

Exercise 12.8

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Just for your interest...

How does the notation $\frac{d^2y}{dx^2}$ work?
Why are the squareds where they are?



Suppose that $y = x^3 + 1$.
Then when we write $\frac{dy}{dx}$, we're
effectively doing $\frac{d(x^3+1)}{dx}$ (by
substitution), although this would
typically be written:

$$\frac{d}{dx}(x^3 + 1)$$



The $\frac{d}{dx}(\dots)$ notation is quite handy,
because it behaves as a function and
allows us to write the original
expression and the derivative within
a single equation:

$$\frac{d}{dx}(x^3 + 1) = 3x^2$$



Therefore, if we wanted to differentiate y twice, we'd do:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{d}{dx}(y) \right) \\ &= \frac{d^2}{dx^2}(y) \\ &= \frac{d^2y}{dx^2} \end{aligned}$$

Homework Exercise

- 1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when y equals:

a $12x^2 + 3x + 8$

b $15x + 6 + \frac{3}{x}$

c $9\sqrt{x} - \frac{3}{x^2}$

d $(5x + 4)(3x - 2)$

e $\frac{3x + 8}{x^2}$

- 2 The displacement of a particle in metres at time t seconds is modelled by the function

$$f(t) = \frac{t^2 + 2}{\sqrt{t}}$$

Links

The velocity of the particle will be $f'(t)$ and its acceleration will be $f''(t)$.

→ Statistics and Mechanics Year 2, Section 6.2

The acceleration of the particle in m s^{-2} is the second derivative of this function.

Find an expression for the acceleration of the particle at time t seconds.

- 3 Given that $y = (2x - 3)^3$, find the value of x when $\frac{d^2y}{dx^2} = 0$.

4 $f(x) = px^3 - 3px^2 + x^2 - 4$

When $x = 2$, $f''(x) = -1$. Find the value of p .

Problem-solving

When you differentiate with respect to x , you treat any other letters as constants.

Homework Answers

- 1 a $24x + 3, 24$
 b $15 - 3x^{-2}, 6x^{-3}$
 c $\frac{9}{2}x^{-\frac{1}{2}} + 6x^{-3}, -\frac{9}{4}x^{-\frac{3}{2}} - 18x^{-4}$
 d $30x + 2, 30$
 e $-3x^{-2} - 16x^{-3}, 6x^{-3} + 48x^{-4}$
- 2 Acceleration $= \frac{3}{4}t^{-\frac{1}{2}} + \frac{3}{2}t^{-\frac{5}{2}}$
- 3 $\frac{3}{2}$
- 4 $-\frac{1}{2}$