P2 Chapter 1: Algebra Techniques

Chapter Practice

Key Points

1. To prove a statement by contradiction you start by assuming it is **not true**. You then use logical steps to show that this assumption leads to something impossible (either a contradiction of the assumption or a contradiction of a fact you know to be true). You can conclude that your assumption was incorrect, and the original statement **was true**.

2. A rational number can be written as $\frac{a}{b}$, where a and b are integers.

   An irrational number cannot be expressed in the form $\frac{a}{b}$, where a and b are integers.

3. To multiply fractions, cancel any common factors, then multiply the numerators and multiply the denominators.

4. To divide two fractions, multiply the first fraction by the reciprocal of the second fraction.

5. To add or subtract two fractions, find a common denominator.

6. A single fraction with two distinct linear factors in the denominator can be split into two separate fractions with linear denominators. This is called splitting it into **partial fractions**:

   $$\frac{5}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

Key Points

7 The method of partial fractions can also be used when there are more than two distinct linear factors in the denominator:

$$\frac{7}{(x-2)(x+6)(x+3)} = \frac{A}{x-2} + \frac{B}{x+6} + \frac{C}{x+3}$$

8 A single fraction with a repeated linear factor in the denominator can be split into two or more separate fractions:

$$\frac{2x+9}{(x-5)(x+3)^2} = \frac{A}{x-5} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

9 An improper algebraic fraction is one whose numerator has a degree equal to or larger than the denominator. An improper fraction must be converted to a mixed fraction before you can express it in partial fractions.

10 You can either use:

- algebraic division
- or the relationship $F(x) = Q(x) \times \text{divisor} + \text{remainder}$

to convert an improper fraction into a mixed fraction.

Chapter Exercises

1 Prove by contradiction that $\sqrt{\frac{1}{2}}$ is an irrational number. **(5 marks)**

2 Prove that if q^2 is an irrational number then q is an irrational number.

3 Simplify:

a $\dfrac{x-4}{6} \times \dfrac{2x+8}{x^2-16}$ **b** $\dfrac{x^2-3x-10}{3x^2-21} \times \dfrac{6x^2+24}{x^2+6x+8}$ **c** $\dfrac{4x^2+12x+9}{x^2+6x} \div \dfrac{4x^2-9}{2x^2+9x-18}$

4 **a** Simplify fully $\dfrac{4x^2-8x}{x^2-3x-4} \times \dfrac{x^2+6x+5}{2x^2+10x}$ **(3 marks)**

b Given that $\ln((4x^2-8x)(x^2+6x+5)) = 6 + \ln((x^2-3x-4)(2x^2+10x))$ find x in terms of e. **(4 marks)**

5 $g(x) = \dfrac{4x^3-9x^2-9x}{32x+24} \div \dfrac{x^2-3x}{6x^2-13x-5}$

a Show that $g(x)$ can be written in the form ax^2+bx+c, where a, b and c are constants to be found. **(4 marks)**

b Hence differentiate $g(x)$ and find $g'(-2)$. **(3 marks)**

6 Express $\dfrac{6x+1}{x-5} + \dfrac{5x+3}{x^2-3x-10}$ as a single fraction in its simplest form. **(4 marks)**

Chapter Exercises

7 $f(x) = x + \frac{3}{x-1} - \frac{12}{x^2+2x-3}, x \in \mathbb{R}, x > 1$

Show that $f(x) = \frac{x^2+3x+3}{x+3}$ **(4 marks)**

8 $f(x) = \frac{x-3}{x(x-1)}$

Show that $f(x)$ can be written in the form $\frac{A}{x} + \frac{B}{x-1}$ where A and B are constants to be found. **(3 marks)**

9 $\frac{-15x+21}{(x-2)(x+1)(x-5)} \equiv \frac{P}{x-2} + \frac{Q}{x+1} + \frac{R}{x-5}$

Find the values of the constants P, Q and R. **(4 marks)**

10 Show that $\frac{16x-1}{(3x+2)(2x-1)}$ can be written in the form $\frac{D}{3x+2} + \frac{E}{2x-1}$ and find the values of the constants D and E. **(4 marks)**

11 $\frac{7x^2+2x-2}{x^2(x+1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

Find the values of the constants A, B and C. **(4 marks)**

Chapter Exercises

12 $h(x) = \dfrac{21x^2 - 13}{(x+5)(3x-1)^2}$

Show that h(x) can be written in the form $\dfrac{D}{x+5} + \dfrac{E}{(3x-1)} + \dfrac{F}{(3x-1)^2}$ where D, E and F are constants to be found. **(5 marks)**

13 Find the values of the constants A, B, C and D in the following identity:

$$x^3 - 6x^2 + 11x + 2 \equiv (x-2)(Ax^2 + Bx + C) + D$$

(5 marks)

14 Show that $\dfrac{4x^3 - 6x^2 + 8x - 5}{2x+1}$ can be put in the form $Ax^2 + Bx + C + \dfrac{D}{2x+1}$

Find the values of the constants A, B, C and D. **(5 marks)**

15 Show that $\dfrac{x^4 + 2}{x^2 - 1} \equiv Ax^2 + Bx + C + \dfrac{D}{x^2-1}$ where A, B, C and D are constants to be found. **(5 marks)**

16 $\dfrac{x^4}{x^2 - 2x + 1} \equiv Ax^2 + Bx + C + \dfrac{D}{x-1} + \dfrac{E}{(x-1)^2}$

Find the values of the constants A, B, C, D and E. **(5 marks)**

17 $\mathrm{h}(x) = \dfrac{2x^2 + 2x - 3}{x^2 + 2x - 3}$

Show that h(x) can be written in the form $A + \dfrac{B}{x+3} + \dfrac{C}{x-1}$ where A, B and C are constants to be found. **(5 marks)**

18 Given that $\dfrac{x^2+1}{x(x-2)} \equiv P + \dfrac{Q}{x} + \dfrac{R}{x-2}$, find the values of the constants P, Q and R. **(5 marks)**

19 Given that $\mathrm{f}(x) = 2x^3 + 9x^2 + 10x + 3$:

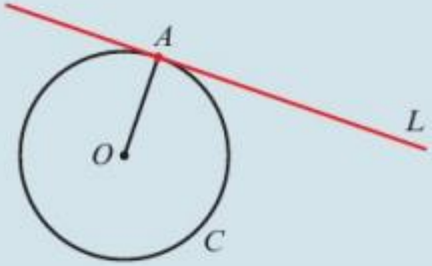
a show that -3 is a root of f(x)

b express $\dfrac{10}{\mathrm{f}(x)}$ as partial fractions.

Challenge

The line L meets the circle C with centre O at exactly one point, A.

Prove by contradiction that the line L is perpendicular to the radius OA.



Hint In a right-angled triangle, the side opposite the right-angle is always the longest side.

Chapter Answers

1 Assume $\sqrt{\frac{1}{2}}$ is a rational number.

Then $\sqrt{\frac{1}{2}} = \frac{a}{b}$ for some integers a and b.

Further assume that this fraction is in its simplest terms: there are no common factors between a and b.

So $0.5 = \frac{a^2}{b^2}$ or $2a^2 = b^2$.

Therefore b^2 must be a multiple of 2.
We know that this means b must also be a multiple of 2.
Write $b = 2c$, which means $b^2 = (2c)^2 = 4c^2$.
Now $4c^2 = 2a^2$, or $2c^2 = a^2$.
Therefore a^2 must be a multiple of 2, which implies a is also a multiple of 2.
If a and b are both multiples of 2, this contradicts the statement that there are no common factors between a and b.
Therefore, $\sqrt{\frac{1}{2}}$ is an irrational number.

2 Assume there exists a rational number q such that q^2 is irrational.

So write $q = \frac{a}{b}$ where a and b are integers.

$q^2 = \frac{a^2}{b^2}$

As a and b are integers a^2 and b^2 are integers.
So q^2 is rational.

This contradicts assumption that q^2 is irrational.
Therefore if q^2 is irrational then q is irrational.

3 **a** $\frac{1}{3}$ **b** $\frac{2(x^2+4)(x-5)}{(x^2-7)(x+4)}$ **c** $\frac{2x+3}{x}$

4 **a** $\frac{2x-4}{x-4}$ **b** $\frac{4(e^6-1)}{e^6-2}$

5 **a** $a = \frac{3}{4}, b = -\frac{13}{8}, c = -\frac{5}{8}$

b $g'(x) = \frac{3}{2}x - \frac{13}{8}, g'(-2) = -\frac{37}{8}$

6 $\frac{6x^2+18x+5}{x^2-3x-10}$

7 $x + \frac{3}{x-1} - \frac{12}{x^2+2x-3}$

$= \frac{x(x+3)(x-1)}{(x+3)(x-1)} + \frac{3(x+3)}{(x+3)(x-1)} - \frac{12}{(x+3)(x-1)}$

$= \frac{(x^2+3x+3)(x-1)}{(x+3)(x-1)} = \frac{x^2+3x+3}{x+3}$

8 $A = 3, B = -2$

9 $P = 1, Q = 2, R = -3$

10 $D = 5, E = 2$

11 $A = 4, B = -2, C = 3$
12 $D = 2, E = 1, F = -2$
13 $A = 1, B = -4, C = 3, D = 8$
14 $A = 2, B = -4, C = 6, D = -11$
15 $A = 1, B = 0, C = 1, D = 3$
16 $A = 1, B = 2, C = 3, D = 4, E = 1.$
17 $A = 2, B = -\frac{9}{4}, C = \frac{1}{4}$
18 $P = 1, Q = -\frac{1}{2}, R = \frac{5}{2}$
19 **a** $f(-3) = 0$ or $f(x) = (x + 3)(2x^2 + 3x + 1)$
b $\frac{1}{(x+3)} + \frac{8}{(2x+1)} - \frac{5}{(x+1)}$

Challenge
Assume L is not perpendicular to OA. Draw the line through O which is perpendicular to L. This line meets L at a point B, outside the circle. Triangle OBA is right-angled at B, so OA is the hypotenuse of this triangle, so $OA > OB$. This gives a contradiction, as B is outside the circle, so $OA < OB$. Therefore L is perpendicular to OA.