P1 Chapter 7: Algebraic Methods

Mathematical Proof

Proof

Terminology

A **conjecture** is a mathematical statement that has yet to be proven.

One famous conjecture is **Goldbach's Conjecture**.
It states "*Every even integer greater than 2 can be expressed as the sum of two primes*."
It has been verified up to 4×10^{18} (that's big!); this provides evidence that it is true, but does not prove it is true!

A **theorem** is a mathematical statement that has been proven.

One famous misnomer was **Fermat's Last Theorem**, which states "*If* n is an integer where $n > 2$, *then* $a^n + b^n = c^n$ *has no non-zero integer solutions for* a, b, c." It was 300 years until this was proven in 1995. Only then was the 'Theorem' in the name then correct!

Types of Proof

A proof must show all **assumptions** you are using, have a clear **sequential list of steps** that logically follow, and must cover **all possible cases**.
You should usually make a **concluding statement**, e.g. restating the original conjecture that you have proven.

a. Proof by Deduction

This is the simplest type, where you start from known facts and reach the desired conclusion via deductive steps.

An **identity** is an equation that is true for **all values** of the variables. e.g. $x^2 = 4$ is true only for $x = \pm 2$, but $x(x-1) \equiv x^2 - x$ is true for all x.

"Prove that the product of two odd numbers is odd."

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"Prove that $(3x+2)(x-5)(x+7) \equiv 3x^3 + 8x^2 - 101x - 70$"

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a. Proof by Deduction

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"Prove that the product of two odd numbers is odd."
Let p, q be integers, then $2p + 1$ and $2q + 1$ are odd numbers.
$(2p + 1)(2q + 1) = 4pq + 2p + 2q + 1$
$= 2(2pq + p + q) + 1$
This is one more than a multiple of 2, and is therefore odd.

An **identity** is an equation that is true for **all values** of the variables. e.g. $x^2 = 4$ is true only for $x = \pm 2$, but $x(x - 1) \equiv x^2 - x$ is true for all x.

"Prove that $\boldsymbol{(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70}$"
$(3x + 2)(x - 5)(x + 7)$
$= (3x + 2)(x^2 + 2x - 35)$
$= 3x^3 + 6x^2 - 105x + 2x^2 + 4x - 70$
$= 3x^2 + 8x^2 - 101x - 70$

$\therefore (3x + 2)(x - 5)(x + 7)$
$\equiv 3x^3 + 8x^2 - 101x - 70$

Be Warned…

Proof by Deduction requires you to **start from known facts** and end up at the conclusion. It is **not** acceptable to start with to the conclusion, and verify it works, **because you are assuming the thing you are trying to prove**.

Example: Prove that if three consecutive integers are the sides of a right-angled triangle, they must be 3, 4 and 5.

Incorrect Proof:

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We are assuming the thing we are trying to prove!

The underlying problem is that this technique doesn't prove there can't be **other** consecutive integers that work – we have only verified 3,4,5 is one such solution.

Correct Proof:

We are only assuming things in the 'if' bit. This is fine!

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Be Warned…

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Example: Prove that if three consecutive integers are the sides of a right-angled triangle, they must be 3, 4 and 5.

Incorrect Proof:

Let the lengths be 3,4,5.
Therefore:

$$3^2 + 4^2 = 5^2$$
$$25 = 25$$

This satisfies Pythagoras' Theorem, and the numbers are consecutive.

We are assuming the thing we are trying to prove!

The underlying problem is that this technique doesn't prove there can't be **<u>other</u>** consecutive integers that work – we have only verified 3,4,5 is one such solution.

We are only assuming things in the 'if' bit. This is fine!

Correct Proof:

If the sides are consecutive, then let the sides be:

$$x, x+1, x+2$$

Then by Pythagoras' Theorem:

$$x^2 + (x+1)^2 = (x+2)^2$$
$$\ldots$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$

$x = 3$ (as x can't be negative)
Thus the sides are 3, 4, 5.

Types of Proof

a. Proof by Deduction

Exam Tip: This is quite a common last parter.

Prove that $x^2 + 4x + 5$ is positive for all values of x.

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Anything squared is at least 0. This is formally known as the '*trivial inequality*'.

Test Your Understanding

Prove that the sum of the squares of two consecutive odd numbers is 2 more than a multiple of 8.

?

Types of Proof

a. Proof by Deduction

Exam Tip: This is quite a common last parter.

Prove that $x^2 + 4x + 5$ is positive for all values of x.

$$x^2 + 4x + 5 = (x+2)^2 + 1$$

$(x+2)^2 \geq 0$ for all x

$$\therefore (x+2)^2 + 1 \geq 1 > 0$$

Anything squared is at least 0. This is formally known as the '*trivial inequality*'.

Test Your Understanding

Prove that the sum of the squares of two consecutive odd numbers is 2 more than a multiple of 8.

Let $2n-1$ and $2n+1$ be any two consecutive odd numbers, where n is an integer.

$(2n-1)^2 + (2n+1)^2 = 4n^2 - 4n + 1 + 4n^2 + 4n + 1$

$= 8n^2 + 2$ which is 2 more than a multiple of 8.

Exercise 7.4

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Extension

[STEP I 2005 Q1] 47231 is a five-digit number whose digits sum to $4 + 7 + 2 + 3 + 1 = 17$.

(i) Prove that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

? i

? ii

Exercise 7.4

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Extension

[STEP I 2005 Q1] 47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

(i) Prove that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

i) Since $5 \times 9 = 45$ then the digits drop by 2 in total from the maximum of 9, 9, 9, 9, 9. This can either be on one number (9,9,9,9,7) or spread across two numbers (9,9,9,8,8).
If 9, 9, 9, 9, 7 are used, the 7 can go in 5 positions, giving 5 numbers.
If 9, 9, 9, 8, 8 is used, the two 8s can go in $\frac{5\times4}{2}=10$ positions (as there are 5 choices for the first 8 and 4 for the second, but they can go either way round). Thus there are $5+10=15$ possibilities.

ii) This time we must drop the digit sum by 6 from the maximum of 9,9,9,9,9. This gives the possibilities: (9,9,9,9,3), (9,9,9,8,4), (9,9,9,7,5), (9,9,8,8,5), (9,9,9,6,6), (9,9,8,7,6), (9,8,8,8,6), (9,9,7,7,7), (9,8,8,7,7), (8,8,8,8,7). This give 5, 20, 20, 30, 10, 60, 20, 10, 30, 5 possibilities respectively, giving 210 possibilities in total. (I have omitted the calculation for each for brevity)

Homework Exercise

1 Prove that $n^2 - n$ is an even number for all values of n.

2 Prove that $\dfrac{x}{1+\sqrt{2}} \equiv x\sqrt{2} - x$.

3 Prove that $(x + \sqrt{y})(x - \sqrt{y}) \equiv x^2 - y$.

4 Prove that $(2x - 1)(x + 6)(x - 5) \equiv 2x^3 + x^2 - 61x + 30$.

5 Prove that $x^2 + bx \equiv \left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

6 Prove that the solutions of $x^2 + 2bx + c = 0$ are $x = -b \pm \sqrt{b^2 - c}$.

7 Prove that $\left(x - \dfrac{2}{x}\right)^3 \equiv x^3 - 6x + \dfrac{12}{x} - \dfrac{8}{x^3}$

8 Prove that $\left(x^3 - \dfrac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right) \equiv x^{\frac{1}{2}}\left(x^4 - \dfrac{1}{x^4}\right)$

9 Use completing the square to prove that $3n^2 - 4n + 10$ is positive for all values of n.

10 Use completing the square to prove that $-n^2 - 2n - 3$ is negative for all values of n.

Hint The proofs in this exercise are all proofs by deduction.

Problem-solving

Any expression that is squared must be $\geqslant 0$.

Homework Exercise

11 Prove that $x^2 + 8x + 20 \geqslant 4$ for all values of x. **(3 marks)**

12 The equation $kx^2 + 5kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leqslant k < \frac{12}{25}$ **(4 marks)**

13 The equation $px^2 - 5x - 6 = 0$, where p is a constant, has two distinct real roots. Prove that p satisfies the inequality $p > -\frac{25}{24}$ **(4 marks)**

14 Prove that $A(3, 1)$, $B(1, 2)$ and $C(2, 4)$ are the vertices of a right-angled triangle.

15 Prove that quadrilateral $A(1, 1)$, $B(2, 4)$, $C(6, 5)$ and $D(5, 2)$ is a parallelogram.

16 Prove that quadrilateral $A(2, 1)$, $B(5, 2)$, $C(4, -1)$ and $D(1, -2)$ is a rhombus.

17 Prove that $A(-5, 2)$, $B(-3, -4)$ and $C(3, -2)$ are the vertices of an isosceles right-angled triangle.

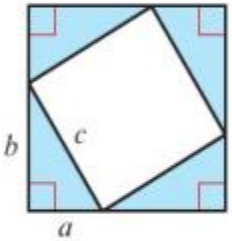
18 A circle has equation $(x - 1)^2 + y^2 = k$, where $k > 0$.
The straight line L with equation $y = ax$ cuts the circle at two distinct points.
Prove that $k > \frac{a^2}{1 + a^2}$ **(6 marks)**

Homework Exercise

19 Prove that the line $4y - 3x + 26 = 0$ is a tangent to the circle $(x + 4)^2 + (y - 3)^2 = 100$. **(5 marks)**

20 The diagram shows a square and four congruent right-angled triangles.

Use the diagram to prove that $a^2 + b^2 = c^2$.



Problem-solving

Find an expression for the area of the large square in terms of a and b.

Challenge

1 Prove that $A(7, 8)$, $B(-1, 8)$, $C(6, 1)$ and $D(0, 9)$ are points on the same circle.

2 Prove that any odd prime number can be written as the difference of two squares.

Homework Answers

1 $n^2 - n = n(n-1)$
If n is even, $n - 1$ is odd and even × odd = even
If n is odd, $n - 1$ is even and odd × even = even

2 $\dfrac{x}{(1+\sqrt{2})} \times \dfrac{(1-\sqrt{2})}{(1-\sqrt{2})} = \dfrac{x(1-\sqrt{2})}{(1-2)} = \dfrac{x - x\sqrt{2}}{-1} = x\sqrt{2} - x$

3 $(x + \sqrt{y})(x - \sqrt{y}) = x^2 - x\sqrt{y} + x\sqrt{y} - y = x^2 - y$

4 $(2x-1)(x+6)(x-5) = (2x-1)(x^2+x-30)$
$= 2x^3 + x^2 - 61x + 30$

5 LHS $= x^2 + bx$, using completing the square,
$\left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

6 $x^2 + 2bx + c = 0$, using completing the square
$(x+b)^2 + c - b^2 = 0$
$(x+b)^2 = b^2 - c$
$x + b = \pm\sqrt{b^2 - c}$
$x = -b \pm \sqrt{b^2 - c}$

7 $\left(x - \dfrac{2}{x}\right)^3 = \left(x - \dfrac{2}{x}\right)\left(x^2 - 4 + \dfrac{4}{x^2}\right) = x^3 - 6x + \dfrac{12}{x} - \dfrac{8}{x^3}$

8 $\left(x^3 - \dfrac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{\frac{-3}{2}}\right) = x^{\frac{9}{2}} + x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{-\frac{7}{2}} = x^{\frac{9}{2}} - x^{-\frac{7}{2}}$
$= x^{\frac{1}{2}}\left(x^4 - \dfrac{1}{x^4}\right)$

9 $3n^2 - 4n + 10 = 3\left[n^2 - \frac{4}{3}n + \frac{10}{3}\right] = 3\left[\left(n - \frac{2}{3}\right)^2 + \frac{10}{3} - \frac{4}{9}\right]$
$= 3\left(n - \frac{2}{3}\right)^2 + \frac{26}{3}$
The minimum value is $\frac{26}{3}$ so $3n^2 - 4n + 10$ is always positive.

10 $-n^2 - 2n - 3 = -[n^2 + 2n + 3] = -\left[(n+1)^2 + 3 - 1\right]$
$= -(n+1)^2 - 2$
The maximum value is -2 so $-n^2 - 2n - 3$ is always negative.

11 $x^2 + 8x + 20 = (x+4)^2 + 4$
The minimum value is 4 so $x^2 + 8x + 20$ is always greater than or equal to 4.

12 $kx^2 + 5kx + 3 = 0$, $b^2 - 4ac < 0$, $25k^2 - 12k < 0$,
$k(25k - 12) < 0$, $0 < k < \frac{12}{25}$.
When $k = 0$ there are no real roots, so $0 \leqslant k < \frac{12}{25}$

13 $px^2 - 5x - 6 = 0$, $b^2 - 4ac > 0$, $25 + 24p > 0$, $p > -\frac{25}{24}$

14 Gradient $AB = -\frac{1}{2}$, gradient $BC = 2$,
Gradient AB × gradient $BC = -\frac{1}{2} \times 2 = -1$,
so AB and BC are perpendicular.

Homework Answers

15 Gradient $AB = 3$, gradient $BC = \frac{1}{4}$, gradient $CD = 3$, gradient $AD = \frac{1}{4}$
Gradient AB = gradient CD so AB and CD are parallel.
Gradient BC = gradient AD so BC and AD are parallel.

16 Gradient $AB = \frac{1}{3}$, gradient $BC = 3$, gradient $CD = \frac{1}{3}$, gradient $AD = 3$
Gradient AB = gradient CD so AB and CD are parallel.
Gradient BC = gradient AD so BC and AD are parallel.
Length $AB = \sqrt{10}$, $BC = \sqrt{10}$, $CD = \sqrt{10}$ and $AD = \sqrt{10}$, all four sides are equal

17 Gradient $AB = -3$, gradient $BC = \frac{1}{3}$,
Gradient AB × gradient $BC = -3 \times \frac{1}{3} = -1$, so AB and BC are perpendicular
Length $AB = \sqrt{40}$, $BC = \sqrt{40}$, $AB = BC$

18 $(x-1)^2 + y^2 = k$, $y = ax$, $(x-1)^2 + a^2x^2 = k$,
$x^2(1 + a^2) - 2x + 1 - k = 0$
$b^2 - 4ac > 0$, $k > \dfrac{a^2}{1 + a^2}$.

19 $x = 2$. There is only one solution so the line $4y - 3x + 26 = 0$ only touches the circle in one place so is the tangent to the circle.

20 Area of square $= (a + b)^2 = a^2 + 2ab + b^2$
Shaded area $= 4\left(\frac{1}{2}ab\right)$
Area of smaller square: $a^2 + 2ab + b^2 - 2ab$
$= a^2 + b^2 = c^2$

Challenge

1 The equation of the circle is $(x - 3)^2 + (y - 5)^2 = 25$ and all four points satisfy this equation.

2 $\left(\frac{1}{2}(p + 1)\right)^2 - \left(\frac{1}{2}(p - 1)\right)^2 = \frac{1}{4}\left((p + 1)^2 - (p - 1)^2\right) = \frac{1}{4}(4p) = p$