P1 Chapter 11: 3D Vectors

Gibbs Notation in 3D

$\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ notation

In 2D you were previously introduced to $\boldsymbol{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\boldsymbol{j} = \begin{pmatrix}0\\1\end{pmatrix}$ as unit vectors in each of the x and y directions.

It meant for example that $\begin{pmatrix}8\\-2\end{pmatrix}$ could be written as $8\boldsymbol{i} - 2\boldsymbol{j}$ since $8\begin{pmatrix}1\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}8\\-2\end{pmatrix}$

Unsurprisingly, in 3D:

$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Quickfire Questions

1 Put in i, j, k notation:

$\begin{pmatrix}1\\2\\3\end{pmatrix} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$

$\begin{pmatrix}-7\\3\\0\end{pmatrix} = -7\boldsymbol{i} + 3\boldsymbol{j}$

$\begin{pmatrix}3\\0\\-1\end{pmatrix} = 3\boldsymbol{i} - \boldsymbol{k}$

2 Write as a column vector:

$4\boldsymbol{j} + \boldsymbol{k} = \begin{pmatrix}0\\4\\1\end{pmatrix}$ $\quad$ $\boldsymbol{i} - \boldsymbol{j} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

3 If $A(1,2,3), B(4,0,-1)$ then

$$\overrightarrow{AB} = \begin{pmatrix}3\\-2\\-4\end{pmatrix}$$

4 If $\boldsymbol{a} = \begin{pmatrix}2\\3\\4\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}0\\-1\\3\end{pmatrix}$ then $3\boldsymbol{a} + 2\boldsymbol{b} = \begin{pmatrix}6\\7\\18\end{pmatrix}$

Examples

Find the magnitude of $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

?

If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$ is $2\boldsymbol{a} - 3\boldsymbol{b}$ parallel to $4\boldsymbol{i} - 5\boldsymbol{k}$.

?

Examples

Find the magnitude of $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and hence find $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

Magnitude of a is $|a| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{1}{\sqrt{21}}(2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$$

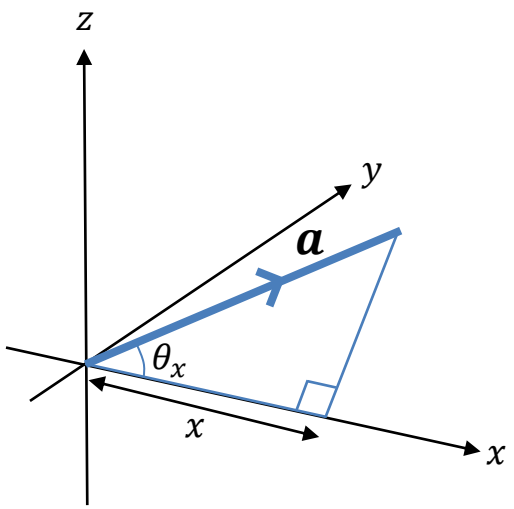
Recall from Year 1 that if the length/magnitude of the vector is $\sqrt{21}$, then clearly dividing this vector by $\sqrt{21}$ will give a length of 1. It has unit length and therefore is known as a unit vector.

If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$ is $2\boldsymbol{a} - 3\boldsymbol{b}$ parallel to $4\boldsymbol{i} - 5\boldsymbol{k}$.

$$2\boldsymbol{a} - 3\boldsymbol{b} = \begin{pmatrix} 8 \\ 0 \\ -10 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} \text{ which is a multiple of } \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$$

$\therefore$ parallel.

Angles between vectors and an axis



How could you work out the angle between a vector and the x-axis?

Just form a right-angle triangle!

The angle between $\boldsymbol{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and the x-axis is:

$$\cos\theta_x = \frac{x}{|\boldsymbol{a}|}$$

and similarly for the y and z axes.

[Textbook] Find the angles that the vector $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k}$ makes with each of the positive coordinate axis.

$$|a| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{14}$$

$$\cos\theta_x = \frac{2}{\sqrt{14}} \quad \rightarrow \quad \theta_x = 57.7°$$

$$\cos\theta_y = \frac{-3}{\sqrt{14}} \quad \rightarrow \quad \theta_y = 143.3°$$

$$\cos\theta_z = \frac{-1}{\sqrt{14}} \quad \rightarrow \quad \theta_z = 105.5°$$

Test Your Understanding

[Textbook] The points A and B have position vectors $4\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ relative to a fixed origin, O. Find $\overrightarrow{AB}$ and show that ΔOAB is isosceles.

?

(a) Find the angle that the vector $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ makes with the x-axis.
(b) Find the angle that $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ makes with the x-axis.
(c) Explain why the difference between these angles does not give you the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

a

?

b

?

c

?

Test Your Understanding

[Textbook] The points A and B have position vectors $4\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$ relative to a fixed origin, O. Find $\overrightarrow{AB}$ and show that ΔOAB is isosceles.

$$\overrightarrow{\boldsymbol{AB}} = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix} \quad \therefore \left|\overrightarrow{\boldsymbol{AB}}\right| = \sqrt{(-1)^2 + 2^2 + (-8)^2} = \sqrt{69}$$

$$\left|\overrightarrow{\boldsymbol{OA}}\right| = \sqrt{4^2 + 2^2 + 7^2} = \sqrt{69}$$

$$\left|\overrightarrow{\boldsymbol{OB}}\right| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$$

$\left|\overrightarrow{\boldsymbol{AB}}\right| = \left|\overrightarrow{\boldsymbol{OA}}\right|$ **(but** $\neq \left|\overrightarrow{\boldsymbol{OB}}\right|$**) therefore** $\boldsymbol{OAB}$ **is isosceles.**

We can find $\overrightarrow{AB}$ by simply looking at the change from 4 to 3, and so on. The textbook has a longwinded way of writing $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$, then finding $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. Overkill!

(a) Find the angle that the vector $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ makes with the x-axis.
(b) Find the angle that $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ makes with the x-axis.
(c) Explain why the difference between these angles does not give you the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

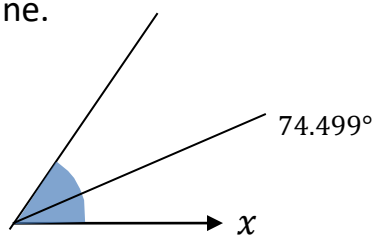
a $|\boldsymbol{a}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

$$\cos\theta_x = \frac{2}{\sqrt{6}} \qquad \theta_x = 35.264°$$

b $|\boldsymbol{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

$$\cos\theta_x = \frac{1}{\sqrt{14}} \qquad \theta_x = 74.499°$$

c The difference ($74.5 - 35.3 = 39.2°$) would only be the angle between a and b if a, b and i were coplanar, i.e. lied on the same plane.



Exercise 12.2

Pearson Pure Mathematics Year 2/AS
Pages 106-107

Homework Exercise

1 The vectors $\mathbf{a}$ and $\mathbf{b}$ are defined by $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$.

a Find:

i $\mathbf{a} - \mathbf{b}$ **ii** $-\mathbf{a} + 3\mathbf{b}$

b State with a reason whether each of these vectors is parallel to $6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}$.

2 The vectors $\mathbf{a}$ and $\mathbf{b}$ are defined by $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix}$.

Show that the vector $3\mathbf{a} + 2\mathbf{b}$ is parallel to $6\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$.

3 The vectors $\mathbf{a}$ and $\mathbf{b}$ are defined by $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$.

Given that $\mathbf{a} + 2\mathbf{b} = 5\mathbf{i} + 4\mathbf{j}$, find the values of p, q and r.

4 Find the magnitude of:

a $3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$ **b** $4\mathbf{i} - 2\mathbf{k}$ **c** $\mathbf{i} + \mathbf{j} - \mathbf{k}$

d $5\mathbf{i} - 9\mathbf{j} - 8\mathbf{k}$ **e** $\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$

5 Given that $\mathbf{p} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$, find in column vector form:

a $\mathbf{p} + \mathbf{q}$ **b** $\mathbf{q} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} + \mathbf{r}$

d $3\mathbf{p} - \mathbf{r}$ **e** $\mathbf{p} - 2\mathbf{q} + \mathbf{r}$

6 The position vector of the point A is $2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{AB} = 5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.
Find the position vector of the point B.

Homework Exercise

7 Given that $\mathbf{a} = t\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, and that $|\mathbf{a}| = 7$, find the possible values of t.

8 Given that $\mathbf{a} = 5t\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$, and that $|\mathbf{a}| = 3\sqrt{10}$, find the possible values of t.

9 The points A, B and C have coordinates (2, 1, 4), (3, −2, 4) and (−1, 2, 2).

a Find, in terms of **i**, **j** and **k**:

i the position vectors of A, B and C

ii $\overrightarrow{AC}$

b Find the exact value of:

i $|\overrightarrow{AC}|$

ii $|\overrightarrow{OC}|$

10 P is the point (3, 0, 7) and Q is the point (−1, 3, −5). Find:

a the vector $\overrightarrow{PQ}$

b the distance between P and Q

c the unit vector in the direction of $\overrightarrow{PQ}$.

11 $\overrightarrow{OA}$ is the vector $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB}$ is the vector $-2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Find:

a the vector $\overrightarrow{AB}$

b the distance between A and B

c the unit vector in the direction of $\overrightarrow{AB}$.

12 Find the unit vector in the direction of each of the following vectors.

a $\mathbf{p} = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$ **b** $\mathbf{q} = \begin{pmatrix} \sqrt{2} \\ -4 \\ -\sqrt{7} \end{pmatrix}$ **c** $\mathbf{r} = \begin{pmatrix} \sqrt{5} \\ -2\sqrt{2} \\ -\sqrt{3} \end{pmatrix}$

Homework Exercise

13 The points A, B and C have position vectors $\begin{pmatrix} 8 \\ -7 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 8 \\ -3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ -6 \\ 3 \end{pmatrix}$ respectively.

a Find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. **(3 marks)**

b Find $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$ and $|\overrightarrow{BC}|$ giving your answers in exact form. **(6 marks)**

c Describe triangle ABC. **(1 mark)**

14 A is the point $(3, 4, 8)$, B is the point $(1, -2, 5)$ and C is the point $(7, -5, 7)$.

a Find the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. **(3 marks)**

b Hence find the lengths of the sides of triangle ABC. **(6 marks)**

c Given that angle $ABC = 90°$ find the size of angle BAC. **(2 marks)**

15 For each of the given vectors,

a $-\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ **b** $\begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$

find the angle made by the vector with:

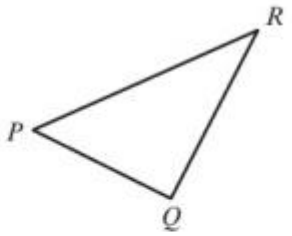
i the positive x-axis **ii** the positive y-axis **iii** the positive z-axis

16 A scalene triangle has the coordinates $(2, 0, 0)$, $(5, 0, 0)$ and $(4, 2, 3)$.
Work out the area of the triangle.

Homework Exercise

17 The diagram shows the triangle PQR.

Given that $\overrightarrow{PQ} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{QR} = -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, show that $\angle PQR = 78.5°$ to 1 d.p.



(5 marks)

Challenge

Find the acute angle that the vector $\mathbf{a} = -2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ makes with the xy-plane. Give your answer to 1 d.p.

Homework Answers

1 **a** **i** $\begin{pmatrix}-3\\5\\-9\end{pmatrix}$ **ii** $\begin{pmatrix}11\\-11\\19\end{pmatrix}$

b $\mathbf{a}-\mathbf{b}$ is parallel as $-2(\mathbf{a}-\mathbf{b}) = 6\mathbf{i}-10\mathbf{j}+18\mathbf{k}$
$-\mathbf{a}+3\mathbf{b}$ is not parallel as it is not a multiple of $6\mathbf{i}-10\mathbf{j}+18\mathbf{k}$.

2 $3\mathbf{a}+2\mathbf{b} = 3\begin{pmatrix}3\\2\\-1\end{pmatrix}+2\begin{pmatrix}-3\\-2\\4\end{pmatrix} = 3\mathbf{i}+2\mathbf{j}+5\mathbf{k} = \frac{1}{2}(6\mathbf{i}+4\mathbf{j}+10\mathbf{k})$

3 $p = 2, q = 1, r = 2$

4 **a** $\sqrt{35}$ **b** $2\sqrt{5}$ **c** $\sqrt{3}$ **d** $\sqrt{170}$ **e** $5\sqrt{3}$

5 **a** $\begin{pmatrix}7\\1\\-1\end{pmatrix}$ **b** $\begin{pmatrix}-5\\5\\-5\end{pmatrix}$ **c** $\begin{pmatrix}14\\-3\\1\end{pmatrix}$ **d** $\begin{pmatrix}8\\4\\4\end{pmatrix}$ **e** $\begin{pmatrix}8\\-6\\10\end{pmatrix}$

6 $7\mathbf{i}-3\mathbf{j}+2\mathbf{k}$ **7** 6 or −6 **8** $\sqrt{3}$ or $-\sqrt{3}$

9 **a** **i** A: $2\mathbf{i}+\mathbf{j}+4\mathbf{k}$, B: $3\mathbf{i}-2\mathbf{j}+4\mathbf{k}$, C: $-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$
ii $-3\mathbf{i}+\mathbf{j}-2\mathbf{k}$

b **i** $\sqrt{14}$ **ii** 3

10 **a** $-4\mathbf{i}+3\mathbf{j}-12\mathbf{k}$ **b** 13 **c** $-\frac{4}{13}\mathbf{i}+\frac{3}{13}\mathbf{j}-\frac{12}{13}\mathbf{k}$

11 **a** $-6\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ **b** $\sqrt{61}$ **c** $-\frac{6}{\sqrt{61}}\mathbf{i}+\frac{4}{\sqrt{61}}\mathbf{j}+\frac{3}{\sqrt{61}}\mathbf{k}$

12 **a** $\frac{3}{\sqrt{29}}\mathbf{i}-\frac{4}{\sqrt{29}}\mathbf{j}-\frac{2}{\sqrt{29}}\mathbf{k}$ **b** $\frac{\sqrt{2}}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}-\frac{\sqrt{7}}{5}\mathbf{k}$

c $\frac{\sqrt{5}}{4}\mathbf{i}-\frac{2\sqrt{2}}{4}\mathbf{j}-\frac{\sqrt{3}}{4}\mathbf{k}$

13 **a** $\overrightarrow{AB} = 4\mathbf{j}-\mathbf{k}$, $\overrightarrow{AC} = 4\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\overrightarrow{BC} = 4\mathbf{i}-3\mathbf{j}$

b $|\overrightarrow{AB}| = \sqrt{17}$, $|\overrightarrow{AC}| = 3\sqrt{2}$, $|\overrightarrow{BC}| = 5$

c scalene

14 **a** $\overrightarrow{AB} = -2\mathbf{i}-6\mathbf{j}-3\mathbf{k}$, $\overrightarrow{AC} = 4\mathbf{i}-9\mathbf{j}-\mathbf{k}$,
$\overrightarrow{BC} = 6\mathbf{i}-3\mathbf{j}+2\mathbf{k}$

b $|\overrightarrow{AB}| = 7$, $|\overrightarrow{AC}| = 7\sqrt{2}$, $|\overrightarrow{BC}| = 7$ **c** 45°

15 **a** **i** 98.0° **ii** 11.4° **iii** 82.0°
b **i** 69.6° **ii** 62.3° **iii** 35.5°
c **i** 56.3° **ii** 90° **iii** 146.3°

16 5.41

17 $|\overrightarrow{PQ}| = \sqrt{14}$, $|\overrightarrow{QR}| = \sqrt{29}$, $|\overrightarrow{PR}| = \sqrt{35}$
Let $\theta = \angle PQR$. $14 + 29 - 2\sqrt{406}\cos\theta = 35$
$\Rightarrow \cos\theta = 0.198\ldots \Rightarrow \theta = 78.5°$ (1 d.p.)

Challenge
25.4°