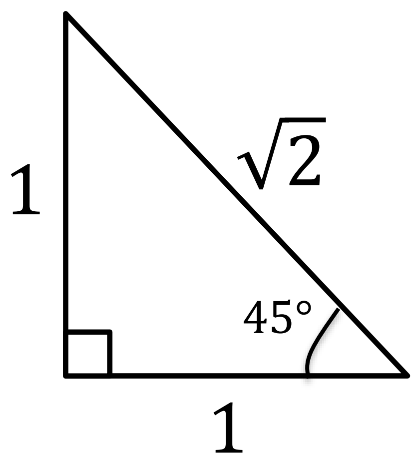
P1 Chapter 10: Trigonometry Equations

The Unit Circle

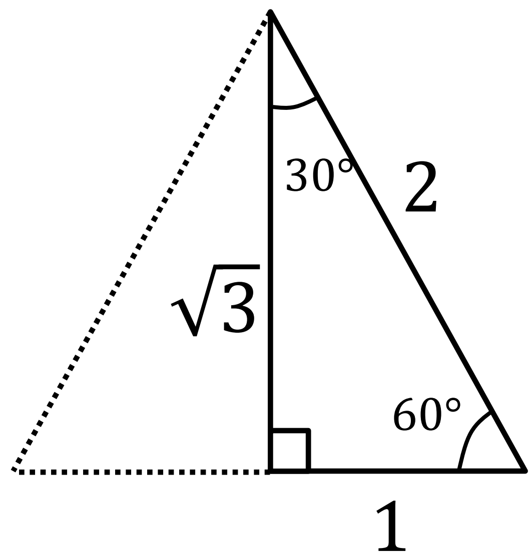
sin/cos/tan of 30°, 45°, 60°

You will frequently encounter angles of $30°, 60°, 45°$ in geometric problems. Why?
We see these angles in equilateral triangles and half squares.

Although you will always have a calculator, you need to know how to derive these.
All you need to remember:
✎ **Draw half a unit square and half an equilateral triangle of side 2.**



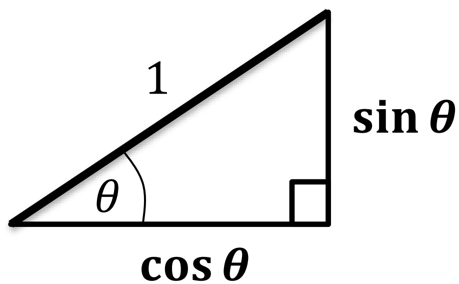
$$\sin(45°) = \frac{1}{\sqrt{2}}$$
$$\cos(45°) = \frac{1}{\sqrt{2}}$$
$$\tan(45°) = 1$$



$$\sin(30°) = \frac{1}{2}$$
$$\cos(30°) = \frac{\sqrt{3}}{2}$$
$$\tan(30°) = \frac{1}{\sqrt{3}}$$
$$\sin(60°) = \frac{\sqrt{3}}{2}$$
$$\cos(60°) = \frac{1}{2}$$
$$\tan(60°) = \sqrt{3}$$

The Unit Circle and Trigonometry

For values of θ in the range $0 < \theta < 90°$, you know that $\sin\theta$ and $\cos\theta$ are lengths on a right-angled triangle:



And what would be the **gradient** of the bold line?

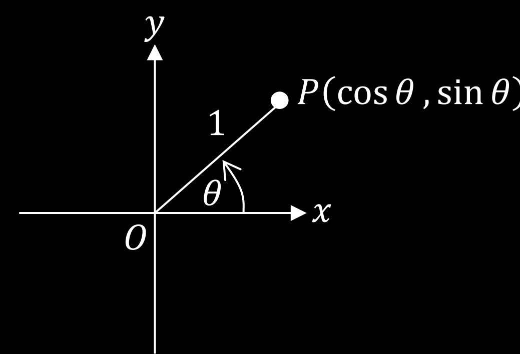
$$\boldsymbol{m = \frac{\Delta y}{\Delta x} = \frac{\sin\theta}{\cos\theta}}$$

But also: $\boldsymbol{\tan\theta = \frac{opp}{adj} = \frac{\sin\theta}{\cos\theta}}$

$$\boldsymbol{\therefore m = \tan\theta}$$

But how do we get the rest of the graph for sin, cos and tan when $90° \leq \theta \leq 360°$?

✏ The point P on a unit circle, such that OP makes an angle θ with the positive x-axis, has coordinates $(\cos\theta, \sin\theta)$.
OP has gradient $\tan\theta$.



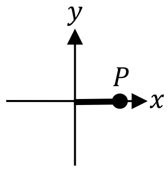
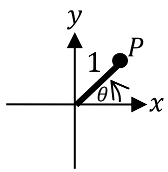
Angles are always measured **anticlockwise**.
(Further Mathematicians will encounter the same when they get to Complex Numbers)

We can consider the coordinate $(\cos\theta, \sin\theta)$ as θ increases from 0 to 360°…

Mini-Exercise

Use the unit circle to determine each value in the table, **using either “0”, “+ve”, “-ve”, “1”, “-1” or “undefined”**. Recall that the point on the unit circle has coordinate $(\cos\theta, \sin\theta)$ and OP has gradient $\tan\theta$.

x-value → $\cos\theta$; y-value → $\sin\theta$; Gradient of OP. → $\tan\theta$

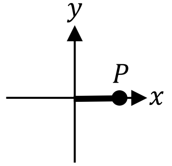
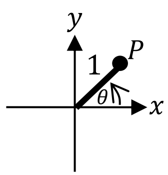
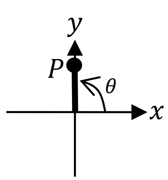
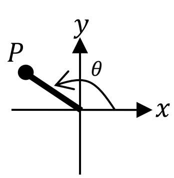
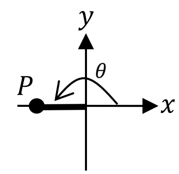
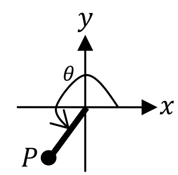
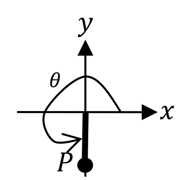
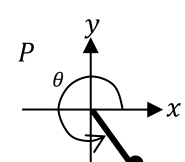
	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta = 0$ 	1	0	0
$0 < \theta < 90°$ 	?		
$\theta = 90°$?	?		
$90° < \theta < 180°$?	?		

	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta = 180°$?	?		
$180° < \theta < 270°$?	?		
$\theta = 270°$?	?		
$270° < \theta < 360°$?	?		

Mini-Exercise

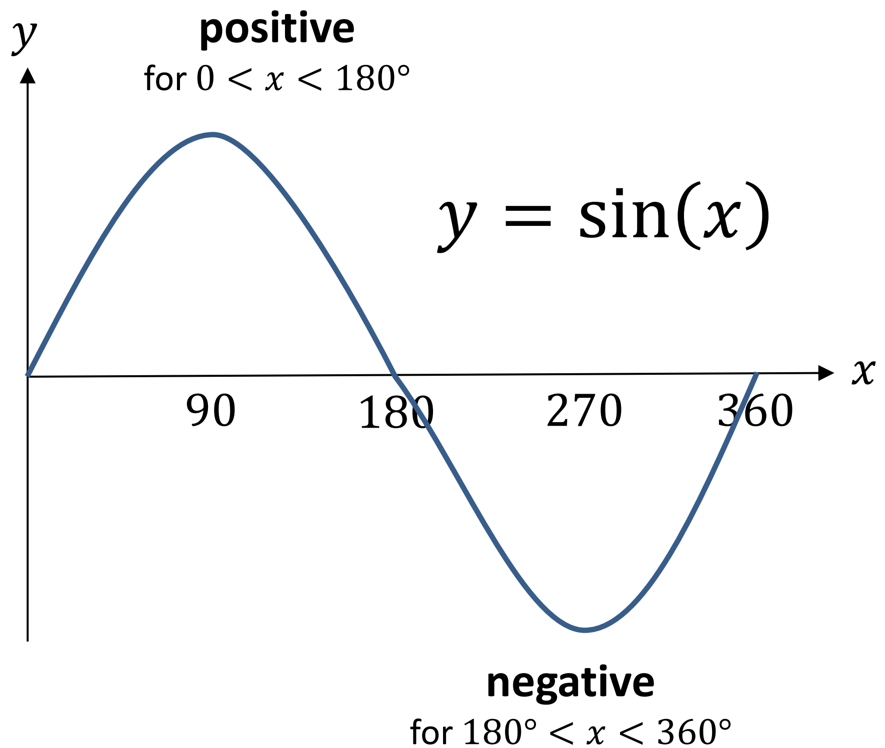
Use the unit circle to determine each value in the table, **using either "0", "+ve", "-ve", "1", "-1" or "undefined"**. Recall that the point on the unit circle has coordinate $(\cos\theta, \sin\theta)$ and OP has gradient $\tan\theta$.

x-value | y-value | Gradient of OP.

	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta = 0$ 	1	0	0
$0 < \theta < 90°$ 	+ve	+ve	+ve
$\theta = 90°$ 	0	1	Undefined (vertical lines don't have a well-defined gradient)
$90° < \theta < 180°$ 	-ve	+ve	-ve
$\theta = 180°$ 	-1	0	0
$180° < \theta < 270°$ 	-ve	-ve	+ve
$\theta = 270°$ 	0	-1	Undefined
$270° < \theta < 360°$ 	+ve	-ve	-ve

The Unit Circle and Trigonometry

The unit circles explains the behaviour of these trigonometric graphs beyond 90°. However, the easiest way to remember whether $\sin(x)$, $\cos(x)$, $\tan(x)$ are positive or negative is to just do a **very quick sketch (preferably mentally!)** of the corresponding graph.



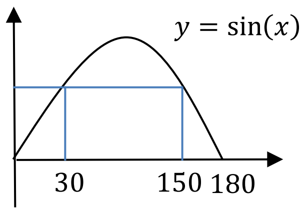
Note: The textbook uses something called '*CAST diagrams*'. I will not be using them in these slides, but you may wish to look at these technique as an alternative approach to various problems in the chapter.

A Few Trigonometric Angle Laws

The following are all easily derivable using a quick sketch of a trigonometric graph, and are merely a <u>convenience</u> so you don't always have to draw out a graph every time.
You are highly encouraged to **memorise these** so that you can do exam questions faster.

1 $\sin(x) = \sin(180° - x)$

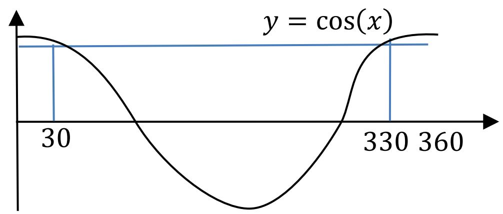
e.g. $\sin(150°) = \sin(30°)$



We saw this in the previous chapter when covering the 'ambiguous case' when using the sine rule.

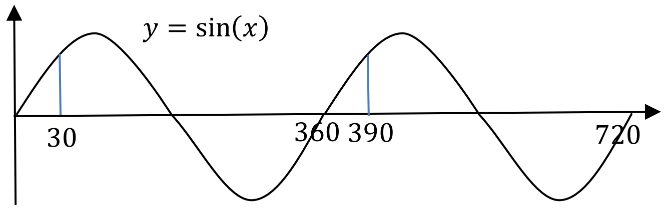
2 $\cos(x) = \cos(360° - x)$

e.g. $\cos(330°) = \cos(30°)$



3 *sin* and *cos* repeat every 360° but *tan* every 180°

e.g. $\sin(390°) = \sin(30°)$



4 $\sin(x) = \cos(90° - x)$

e.g. $\sin(50°) = \cos(40°)$

Remember from the previous chapter that "cosine" by definition is the sine of the "complementary" angle. This was/is never covered in the textbook but caught everyone by surprise when it came up in a C3 exam.

Exercise 10.1

Pearson Pure Mathematics Year 1/AS
Page 78

Homework Exercise

1 Draw diagrams to show the following angles. Mark in the acute angle that OP makes with the x-axis.

a $-80°$ **b** $100°$ **c** $200°$ **d** $165°$ **e** $-145°$
f $225°$ **g** $280°$ **h** $330°$ **i** $-160°$ **j** $-280°$

2 State the quadrant that OP lies in when the angle that OP makes with the positive x-axis is:

a $400°$ **b** $115°$ **c** $-210°$ **d** $255°$ **e** $-100°$

3 Without using a calculator, write down the values of:

a $\sin(-90°)$ **b** $\sin 450°$ **c** $\sin 540°$ **d** $\sin(-450°)$ **e** $\cos(-180°)$
f $\cos(-270°)$ **g** $\cos 270°$ **h** $\cos 810°$ **i** $\tan 360°$ **j** $\tan(-180°)$

4 Express the following in terms of trigonometric ratios of acute angles:

a $\sin 240°$ **b** $\sin(-80°)$ **c** $\sin(-200°)$ **d** $\sin 300°$ **e** $\sin 460°$
f $\cos 110°$ **g** $\cos 260°$ **h** $\cos(-50°)$ **i** $\cos(-200°)$ **j** $\cos 545°$
k $\tan 100°$ **l** $\tan 325°$ **m** $\tan(-30°)$ **n** $\tan(-175°)$ **o** $\tan 600°$

5 Given that θ is an acute angle, express in terms of $\sin\theta$:

a $\sin(-\theta)$ **b** $\sin(180° + \theta)$ **c** $\sin(360° - \theta)$
d $\sin(-(180° + \theta))$ **e** $\sin(-180° + \theta)$ **f** $\sin(-360° + \theta)$
g $\sin(540° + \theta)$ **h** $\sin(720° - \theta)$ **i** $\sin(\theta + 720°)$

Hint The results obtained in questions **5** and **6** are true for all values of θ.

Homework Exercise

6 Given that θ is an acute angle, express in terms of $\cos\theta$ or $\tan\theta$:

a $\cos(180° - \theta)$	**b** $\cos(180° + \theta)$	**c** $\cos(-\theta)$	**d** $\cos(-(180° - \theta))$
e $\cos(\theta - 360°)$	**f** $\cos(\theta - 540°)$	**g** $\tan(-\theta)$	**h** $\tan(180° - \theta)$
i $\tan(180° + \theta)$	**j** $\tan(-180° + \theta)$	**k** $\tan(540° - \theta)$	**l** $\tan(\theta - 360°)$

Challenge

a Prove that $\sin(180° - \theta) = \sin\theta$

b Prove that $\cos(-\theta) = \cos\theta$

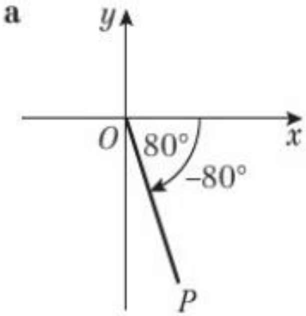
c Prove that $\tan(180° - \theta) = -\tan\theta$

Problem-solving

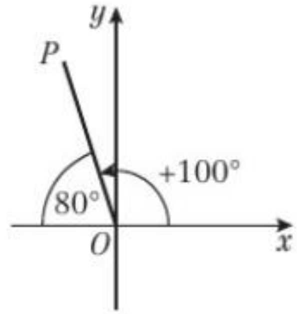
Draw a diagram showing the positions of θ and $180° - \theta$ on the unit circle.

Homework Answers

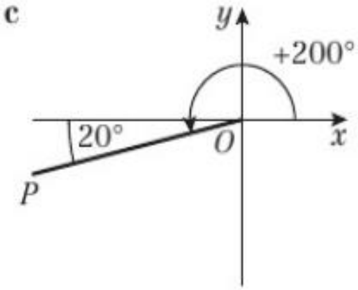
1 **a**



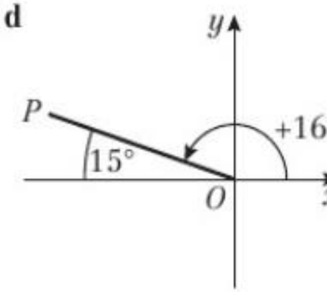
b



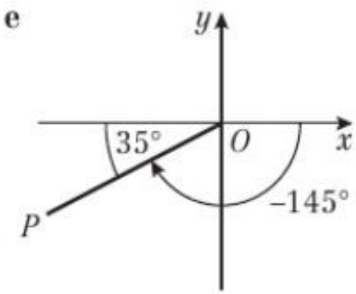
c



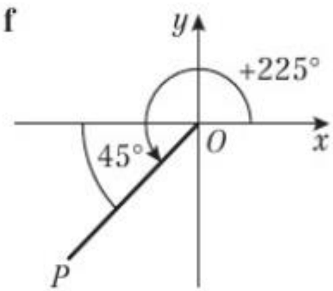
d



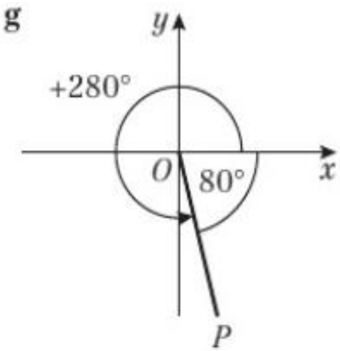
e



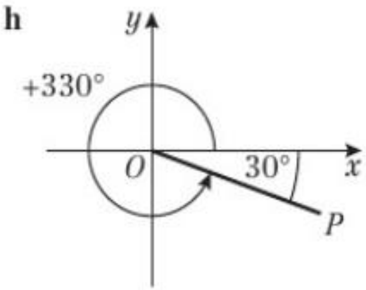
f



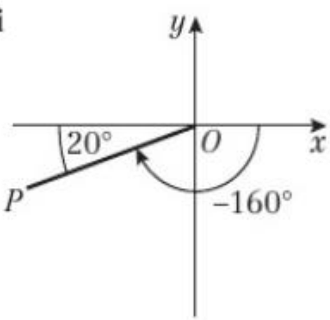
g



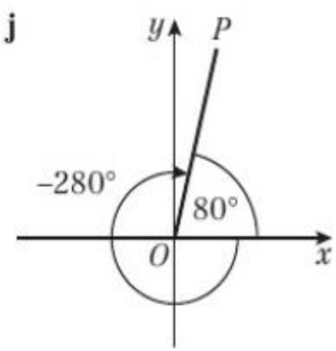
h



i



j



Homework Answers

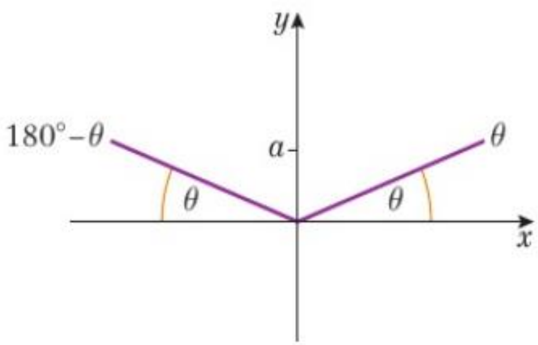
2	**a**	First			**b**	Second		**c**	Second					
	d	Third			**e**	Third								
3	**a**	-1	**b**	1			**c**	0		**d**	-1		**e**	-1
	f	0	**g**	0			**h**	0		**i**	0		**j**	0

4	**a**	$-\sin 60°$	**b**	$-\sin 80°$	**c**	$\sin 20°$
	d	$-\sin 60°$	**e**	$\sin 80$	**f**	$-\cos 70°$
	g	$-\cos 80°$	**h**	$\cos 50°$	**i**	$-\cos 20°$
	j	$-\cos 5°$	**k**	$-\tan 80°$	**l**	$-\tan 35°$
	m	$-\tan 30°$	**n**	$\tan 5°$	**o**	$\tan 60°$
5	**a**	$-\sin\theta$	**b**	$-\sin\theta$	**c**	$-\sin\theta$
	d	$\sin\theta$	**e**	$-\sin\theta$	**f**	$\sin\theta$
	g	$-\sin\theta$	**h**	$-\sin\theta$	**i**	$\sin\theta$
6	**a**	$-\cos\theta$	**b**	$-\cos\theta$	**c**	$\cos\theta$
	d	$-\cos\theta$	**e**	$\cos\theta$	**f**	$-\cos\theta$
	g	$-\tan\theta$	**h**	$-\tan\theta$	**i**	$\tan\theta$
	j	$\tan\theta$	**k**	$-\tan\theta$	**l**	$\tan\theta$

Homework Answers

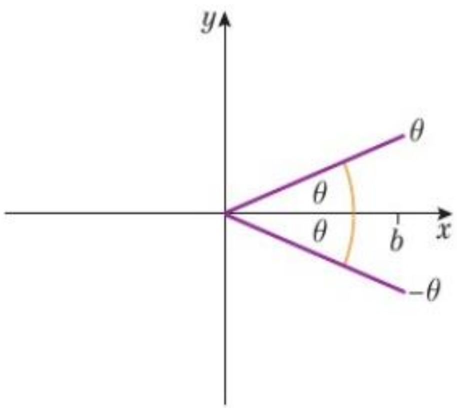
Challenge

a



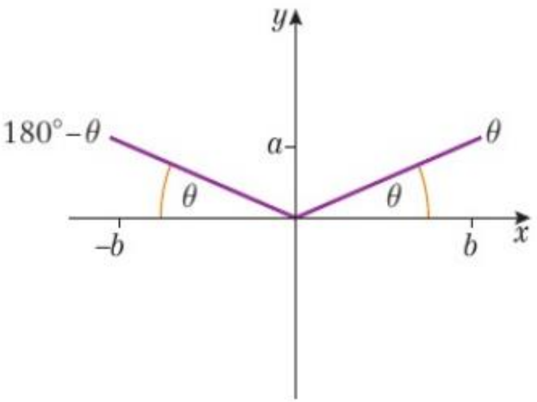
$\sin\theta = \sin(180° - \theta) = a$

b



$\cos\theta = \cos(-\theta) = b$

c



$\tan\theta = \frac{a}{b}$; $\tan(180° - \theta) = \frac{a}{-b} = -\tan\theta$

For $\tan\theta = \frac{x}{y}$