P1 Chapter 12: Differentiation

Chapter Practice

Key Points

1 The **gradient** of a **curve** at a given point is defined as the gradient of the **tangent** to the curve at that point.

2 The **gradient function**, or **derivative**, of the curve $y = \text{f}(x)$ is written as $\text{f}'(x)$ or $\frac{dy}{dx}$

$$\text{f}'(x) = \lim_{h \to 0} \frac{\text{f}(x+h) - \text{f}(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

3 For all real values of n, and for a constant a:

- If $\text{f}(x) = x^n$ then $\text{f}'(x) = nx^{n-1}$
- If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$
- If $\text{f}(x) = ax^n$ then $\text{f}'(x) = anx^{n-1}$
- If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

4 For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by

$$\frac{dy}{dx} = 2ax + b$$

5 If $y = \text{f}(x) \pm \text{g}(x)$, then $\frac{dy}{dx} = \text{f}'(x) \pm \text{g}'(x)$.

6 The tangent to the curve $y = \text{f}(x)$ at the point with coordinates $(a, \text{f}(a))$ has equation

$$y - \text{f}(a) = \text{f}'(a)(x - a)$$

7 The normal to the curve $y = \text{f}(x)$ at the point with coordinates $(a, \text{f}(a))$ has equation

$$y - \text{f}(a) = -\frac{1}{\text{f}'(a)}(x - a)$$

Key Points

8 • The function $f(x)$ is **increasing** on the interval $[a, b]$ if $f'(x) \geqslant 0$ for all values of x such that $a < x < b$.

• The function $f(x)$ is **decreasing** on the interval $[a, b]$ if $f'(x) \leqslant 0$ for all values of x such that $a < x < b$.

9 Differentiating a function $y = f(x)$ twice gives you the second order derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$

10 Any point on the curve $y = f(x)$ where $f'(x) = 0$ is called a **stationary point**. For a small positive value h:

Type of stationary point	$f'(x-h)$	$f'(x)$	$f'(x+h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
Point of inflection	Negative	0	Negative
	Positive	0	Positive

11 If a function $f(x)$ has a stationary point when $x = a$, then:

• if $f''(a) > 0$, the point is a local minimum

• if $f''(a) < 0$, the point is a local maximum.

If $f''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to look at points on either side to determine its nature.

Chapter Exercises

1 Prove, from first principles, that the derivative of $10x^2$ is $20x$. **(4 marks)**

2 The point A with coordinates $(1, 4)$ lies on the curve with equation $y = x^3 + 3x$.
The point B also lies on the curve and has x-coordinate $(1 + \delta x)$.

a Show that the gradient of the line segment AB is given by $(\delta x)^2 + 3\delta x + 6$.

b Deduce the gradient of the curve at point A.

3 A curve is given by the equation $y = 3x^2 + 3 + \frac{1}{x^2}$, where $x > 0$. At the points A, B and C on the curve, $x = 1$, 2 and 3 respectively. Find the gradient of the curve at A, B and C.

4 Calculate the x-coordinates of the points on the curve with equation $y = 7x^2 - x^3$ at which the gradient is equal to 16. **(4 marks)**

5 Find the x-coordinates of the two points on the curve with equation $y = x^3 - 11x + 1$ where the gradient is 1. Find the corresponding y-coordinates.

6 The function f is defined by $f(x) = x + \frac{9}{x}, x \in \mathbb{R}, x \neq 0$.

a Find $f'(x)$. **(2 marks)**

b Solve $f'(x) = 0$. **(2 marks)**

7 Given that

$$y = 3\sqrt{x} - \frac{4}{\sqrt{x}}, \quad x > 0,$$

find $\frac{dy}{dx}$ **(3 marks)**

Chapter Exercises

8 A curve has equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$.

a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$. **(2 marks)**

b Find the coordinates of the point on the curve where the gradient is zero. **(2 marks)**

9 **a** Expand $(x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$. **(2 marks)**

b A curve has equation $y = (x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$, $x > 0$. Find $\frac{dy}{dx}$ **(2 marks)**

c Use your answer to part **b** to calculate the gradient of the curve at the point where $x = 4$. **(1 mark)**

10 Differentiate with respect to x:

$$2x^3 + \sqrt{x} + \frac{x^2 + 2x}{x^2}$$ **(3 marks)**

11 The curve with equation $y = ax^2 + bx + c$ passes through the point (1, 2). The gradient of the curve is zero at the point (2, 1). Find the values of a, b and c. **(5 marks)**

12 A curve C has equation $y = x^3 - 5x^2 + 5x + 2$.

a Find $\frac{dy}{dx}$ in terms of x. **(2 marks)**

b The points P and Q lie on C. The gradient of C at both P and Q is 2. The x-coordinate of P is 3.

i Find the x-coordinate of Q. **(3 marks)**

ii Find an equation for the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants. **(3 marks)**

iii If this tangent intersects the coordinate axes at the points R and S, find the length of RS, giving your answer as a surd. **(3 marks)**

13 A curve has equation $y = \frac{8}{x} - x + 3x^2$, $x > 0$. Find the equations of the tangent and the normal to the curve at the point where $x = 2$.

14 The normals to the curve $2y = 3x^3 - 7x^2 + 4x$, at the points $O(0, 0)$ and $A(1, 0)$, meet at the point N.

a Find the coordinates of N. **(7 marks)**

b Calculate the area of triangle OAN. **(3 marks)**

15 A curve C has equation $y = x^3 - 2x^2 - 4x - 1$ and cuts the y-axis at a point P.
The line L is a tangent to the curve at P, and cuts the curve at the point Q.
Show that the distance PQ is $2\sqrt{17}$. **(7 marks)**

16 Given that $y = x^{\frac{3}{2}} + \frac{48}{x}$, $x > 0$

a find the value of x and the value of y when $\frac{dy}{dx} = 0$. **(5 marks)**

b show that the value of y which you found in part **a** is a minimum. **(2 marks)**

17 A curve has equation $y = x^3 - 5x^2 + 7x - 14$. Determine, by calculation, the coordinates of the stationary points of the curve.

18 The function f, defined for $x \in \mathbb{R}$, $x > 0$, is such that:

$$f'(x) = x^2 - 2 + \frac{1}{x^2}$$

a Find the value of $f''(x)$ at $x = 4$. **(4 marks)**

b Prove that f is an increasing function. **(3 marks)**

19 A curve has equation $y = x^3 - 6x^2 + 9x$. Find the coordinates of its local maximum. **(4 marks)**

20 $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 20$

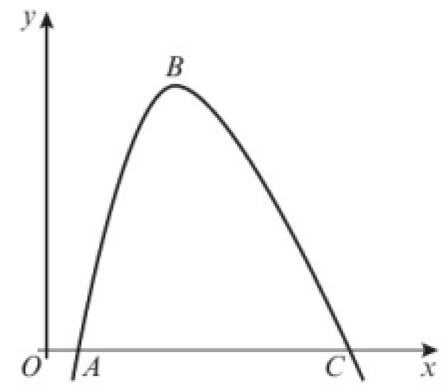
a Find the coordinates of the stationary points of $f(x)$, and determine the nature of each of them.

b Sketch the graph of $y = f(x)$.

21 The diagram shows part of the curve with equation $y = f(x)$, where:

$$f(x) = 200 - \frac{250}{x} - x^2, x > 0$$

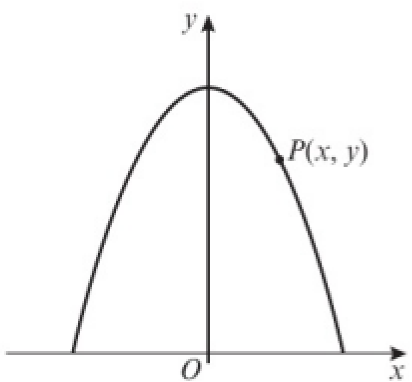
The curve cuts the x-axis at the points A and C.
The point B is the maximum point of the curve.



a Find $f'(x)$. **(3 marks)**

b Use your answer to part **a** to calculate the coordinates of B. **(4 marks)**

22 The diagram shows the part of the curve with equation $y = 5 - \frac{1}{2}x^2$ for which $y > 0$.
The point $P(x, y)$ lies on the curve and O is the origin.



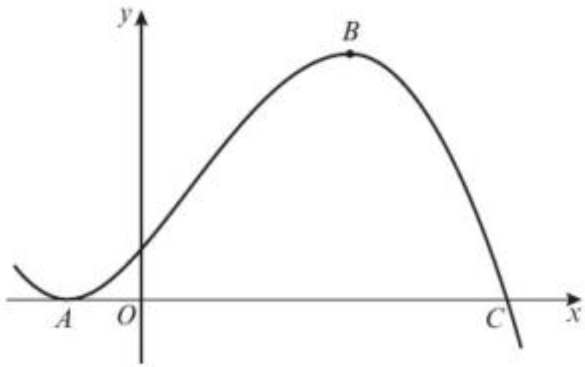
a Show that $OP^2 = \frac{1}{4}x^4 - 4x^2 + 25$. **(3 marks)**

Taking $f(x) = \frac{1}{4}x^4 - 4x^2 + 25$:

b Find the values of x for which $f'(x) = 0$. **(4 marks)**

c Hence, or otherwise, find the minimum distance from O to the curve, showing that your answer is a minimum. **(4 marks)**

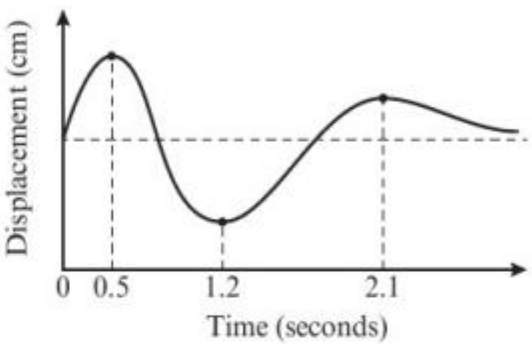
23 The diagram shows part of the curve with equation $y = 3 + 5x + x^2 - x^3$. The curve touches the x-axis at A and crosses the x-axis at C. The points A and B are stationary points on the curve.



a Show that C has coordinates (3, 0). **(1 mark)**

b Using calculus and showing all your working, find the coordinates of A and B. **(5 marks)**

24 The motion of a damped spring is modelled using this graph.

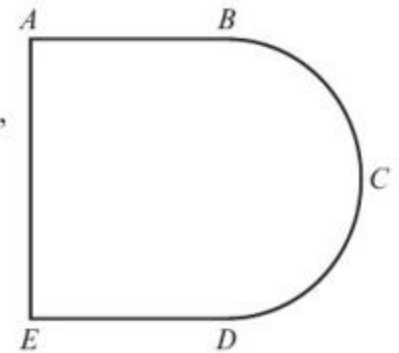


On a separate graph, sketch the gradient function for this model. Choose suitable labels and units for each axis, and indicate the coordinates of any points where the gradient function crosses the horizontal axis.

25 The volume, V cm^3, of a tin of radius r cm is given by the formula $V = \pi(40r - r^2 - r^3)$.
Find the positive value of r for which $\frac{dV}{dr} = 0$, and find the value of V which corresponds to this value of r.

26 The total surface area, A cm^2, of a cylinder with a fixed volume of 1000 cm^3 is given by the formula $A = 2\pi x^2 + \frac{2000}{x}$, where x cm is the radius. Show that when the rate of change of the area with respect to the radius is zero, $x^3 = \frac{500}{\pi}$

27 A wire is bent into the plane shape $ABCDE$ as shown. Shape $ABDE$ is a rectangle and BCD is a semicircle with diameter BD. The area of the region enclosed by the wire is $R\,\text{m}^2$, $AE = x$ metres, and $AB = ED = y$ metres. The total length of the wire is 2 m.



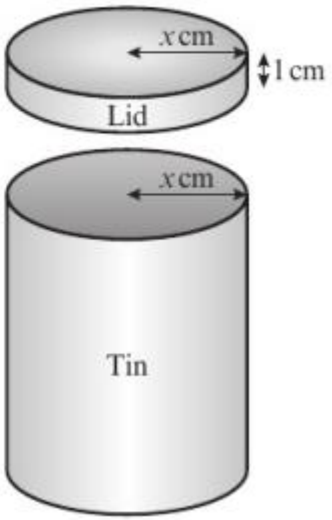
a Find an expression for y in terms of x. **(3 marks)**

b Prove that $R = \frac{x}{8}(8 - 4x - \pi x)$. **(4 marks)**

Given that x can vary, using calculus and showing your working:

c find the maximum value of R. (You do not have to prove that the value you obtain is a maximum.) **(5 marks)**

28 A cylindrical biscuit tin has a close-fitting lid which overlaps the tin by 1 cm, as shown. The radii of the tin and the lid are both x cm. The tin and the lid are made from a thin sheet of metal of area $80\pi\,\text{cm}^2$ and there is no wastage. The volume of the tin is $V\,\text{cm}^3$.



a Show that $V = \pi(40x - x^2 - x^3)$. **(5 marks)**

Given that x can vary:

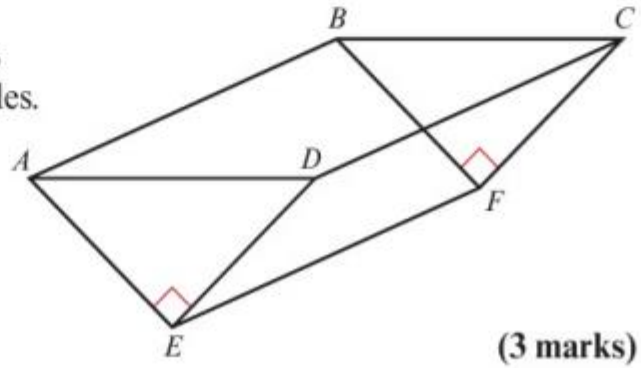
b use differentiation to find the positive value of x for which V is stationary. **(3 marks)**

c Prove that this value of x gives a maximum value of V. **(2 marks)**

d Find this maximum value of V. **(1 mark)**

e Determine the percentage of the sheet metal used in the lid when V is a maximum. **(2 marks)**

29 The diagram shows an open tank for storing water, $ABCDEF$. The sides $ABFE$ and $CDEF$ are rectangles. The triangular ends ADE and BCF are isosceles, and $\angle AED = \angle BFC = 90°$. The ends ADE and BCF are vertical and EF is horizontal.



Given that $AD = x$ metres:

a show that the area of triangle ADE is $\frac{1}{4}x^2$ m^2 **(3 marks)**

Given also that the capacity of the container is 4000 m^3 and that the total area of the two triangular and two rectangular sides of the container is S m^2:

b show that $S = \dfrac{x^2}{2} + \dfrac{16\,000\sqrt{2}}{x}$ **(4 marks)**

Given that x can vary:

c use calculus to find the minimum value of S. **(6 marks)**

d justify that the value of S you have found is a minimum. **(2 marks)**

Challenge

a Find the first four terms in the binomial expansion of $(x + h)^7$, in ascending powers of h.

b Hence prove, from first principles, that the derivative of x^7 is $7x^6$.

1 $\mathrm{f}'(x) = \lim_{h \to 0} \dfrac{10(x+h)^2 - 10x^2}{h} = \lim_{h \to 0} \dfrac{20xh + 10h^2}{h}$

$= \lim_{h \to 0} (20x + 10h) = 20x$

2 **a** y-coordinate of $B = (\delta x)^3 + 3(\delta x)^2 + 6\delta x + 4$

Gradient $= \dfrac{((\delta x)^3 + 3(\delta x)^2 + 6\delta x + 4) - 4}{(1 + \delta x) - 1}$

$= \dfrac{(\delta x)^3 + 3(\delta x)^2 + 6\delta x}{(\delta x)} = (\delta x)^2 + 3\delta x + 6$

b 6

3 $4, 11\frac{3}{4}, 17\frac{25}{27}$

4 $2, 2\frac{2}{3}$

5 $(2, -13)$ and $(-2, 15)$

6 **a** $1 - \dfrac{9}{x^2}$ **b** $x = \pm 3$

7 $\frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$

8 **a** $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$ **b** $(4, 16)$

9 **a** $x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$ **b** $1 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$ **c** $4\frac{1}{16}$

10 $6x^2 + \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-2}$

11 $a = 1, b = -4, c = 5$

12 **a** $3x^2 - 10x + 5$

b **i** $\frac{1}{3}$ **ii** $y = 2x - 7$ **iii** $\frac{7}{2}\sqrt{5}$

13 $y = 9x - 4$ and $9y + x = 128$

14 **a** $\left(\frac{4}{5}, -\frac{2}{5}\right)$ **b** $\frac{1}{5}$

15 P is $(0, -1)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x - 4$

Gradient at $P = -4$, so L is $y = -4x - 1$.

$-4x - 1 = x^3 - 2x^2 - 4x - 1 \Rightarrow x^2(x - 2) = 0$

$x = 2 \Rightarrow y = -9$, so Q is $(2, -9)$

Distance $PQ = \sqrt{(2-0)^2 + (-9-(-1))^2} = \sqrt{68} = 2\sqrt{17}$

16 **a** $x = 4, y = 20$

b $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{3}{4}x^{-\frac{1}{2}} + 96x^{3}$

At $x = 4$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{15}{8} > 0$

$(4, 20)$ is a local minimum.

17 $(1, -11)$ and $\left(\frac{7}{3}, -\frac{329}{27}\right)$

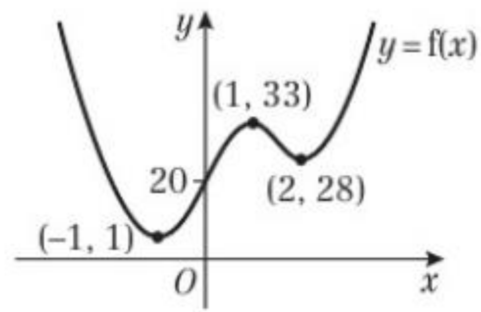
18 **a** $7\frac{31}{32}$

b $\mathrm{f}'(x) = \left(x - \dfrac{1}{x}\right)^2 \geqslant 0$ for all values of x

19 $(1, 4)$

20 **a** $(1, 33)$ maximum, $(2, 28)$ and $(-1, 1)$ minimum

b



21 a $\dfrac{250}{x^2} - 2x$ **b** $(5, 125)$

22 a $P\left(x, 5 - \frac{1}{2}x^2\right)$

$OP^2 = (x-0)^2 + \left(5 - \frac{1}{2}x^2 - 0\right)^2$

$= \frac{1}{4}x^4 - 4x^2 + 25$

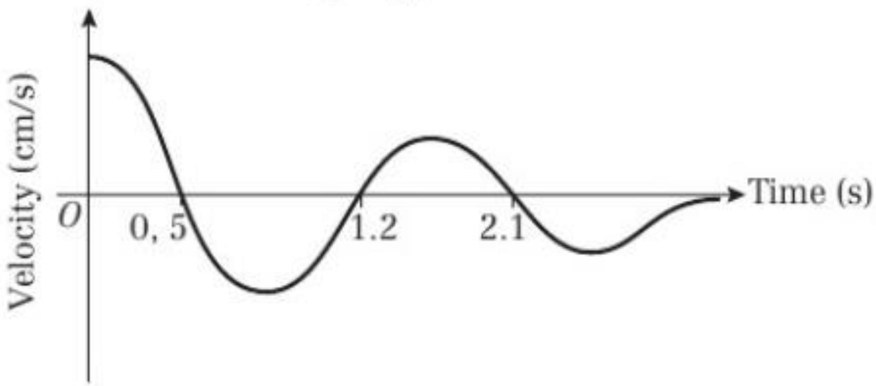
b $x = \pm 2\sqrt{2}$ or $x = 0$

c $OP = 3$; $f''(x) \geqslant 0$ so minimum when $x = \pm 2\sqrt{2}$, maximum when $x = 0$

23 a $3 + 5(3) + 3^2 - 3^3 = 0$ therefore C on curve

b A is $(-1, 0)$; B is $\left(\frac{5}{3}, 9\frac{13}{27}\right)$

24



25 $\dfrac{10}{3}, \dfrac{2300\pi}{27}$

26 $\dfrac{dA}{dx} = 4\pi x - \dfrac{2000}{x^2}$

$\dfrac{dA}{dx} = 0$: $4\pi x = \dfrac{2000}{x^2} \rightarrow x^3 = \dfrac{2000}{4\pi} = \dfrac{500}{\pi}$

27 a $y = 1 - \dfrac{x}{2} - \dfrac{\pi x}{4}$

b $R = xy + \dfrac{\pi}{2}\left(\dfrac{x}{2}\right)^2$

$= x\left(1 - \dfrac{x}{2} - \dfrac{\pi x}{4}\right) + \dfrac{\pi x^2}{8}$

$= x - \dfrac{x^2}{2} - \dfrac{\pi x^2}{4} + \dfrac{\pi x^2}{8}$

$= \dfrac{x}{8}(8 - 4x - \pi x)$

c $\dfrac{2}{4+\pi}\,\text{m}^2$ $(0.280\,\text{m}^2)$

28 a $\pi x^2 + 2\pi x + \pi x^2 + 2\pi x h = 80\pi$

$h = \dfrac{40 - x - x^2}{x}$

$V = \pi x^2 h = \pi x^2\left(\dfrac{40 - x - x^2}{x}\right)$

$= \pi(40x - x^2 - x^3)$

b $\frac{10}{3}$ **c** $\dfrac{d^2V}{dx^2} < 0 \therefore$ maximum

d $\dfrac{2300\pi}{27}$ **e** $22\frac{2}{9}\%$

29 **a** Length of short sides $= \dfrac{x}{\sqrt{2}}$

Area $= \dfrac{1}{2} \times$ base $\times$ height

$$= \frac{1}{2}\left(\frac{x^2}{2}\right) = \frac{1}{4}x^2\,\text{m}^2$$

b Let l be length of EF.

$$\frac{1}{4}x^2 l = 4000 \Rightarrow l = \frac{16\,000}{x^2}$$

$$S = 2\left(\frac{1}{4}x^2\right) + \frac{2xl}{\sqrt{2}}$$

$$= \frac{1}{2}x^2 + \frac{32\,000x}{\sqrt{2}\,x^2} = \frac{x^2}{2} + \frac{16\,000\sqrt{2}}{x}$$

c $x = 20\sqrt{2}$, $S = 1200\,\text{m}^2$

d $\dfrac{\text{d}^2S}{\text{d}x^2} > 0$

Challenge

a $x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3$

b $\dfrac{\text{d}}{\text{d}x}(x^7) = \lim\limits_{h\to 0}\dfrac{(x+h)^7 - x^7}{h} = \lim\limits_{h\to 0}\dfrac{7x^6h + 21x^5h^2 + 35x^4h^3}{h}$

$$= \lim_{h\to 0}(7x^6 + 21x^5h + 35x^4h^2) = 7x^6$$