P1 Chapter 10: Trigonometry Equations

Trigonometric Identities

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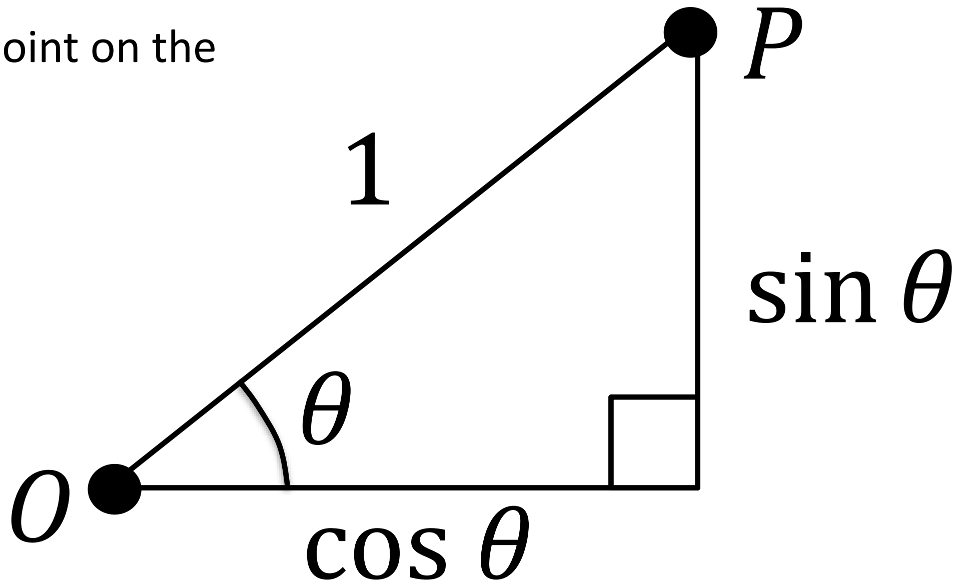
Returning to our point on the unit circle...



1 Then $\boldsymbol{tan\ \theta = \dfrac{sin\ \theta}{cos\ \theta}}$

2 Pythagoras gives you...

$$\boldsymbol{sin^2\theta + cos^2\theta = 1}$$

$\sin^2\theta$ is a shorthand for $(\sin\theta)^2$. It does NOT mean the sin is being squared – this does not make sense as sin is a function, and not a quantity that we can square!

Application of identities #1: Proofs

Prove that $1 - \tan\theta \sin\theta \cos\theta \equiv \cos^2\theta$

$LHS =$?

$=$?

$=$?

$=$?

$$\tan x = \frac{\sin(x)}{\cos(x)}$$
$$\sin^2 x + \cos^2 x = 1$$

Recall that $\equiv$ means 'equivalent to', and just means the LHS is **always** equal to the RHS for all values of θ.
From Chapter 7 ('Proofs') we saw that usually the best method is to manipulate one side (e.g. LHS) until we get to the other (RHS).

Application of identities #1: Proofs

Prove that $1 - \tan\theta \sin\theta \cos\theta \equiv \cos^2\theta$

$$LHS = 1 - \frac{\sin\theta}{\cos\theta}\sin\theta\cos\theta$$

Tip #1: Turn any tan's into sin's and cos's.

$$\tan x = \frac{\sin(x)}{\cos(x)}$$
$$\sin^2 x + \cos^2 x = 1$$

$$= 1 - \frac{\sin^2\theta\, cos\theta}{\cos\theta}$$
$$= 1 - \sin^2\theta$$
$$= \cos^2\theta = RHS$$

Recall that $\equiv$ means 'equivalent to', and just means the LHS is **always** equal to the RHS for all values of θ.
From Chapter 7 ('Proofs') we saw that usually the best method is to manipulate one side (e.g. LHS) until we get to the other (RHS).

More Examples

Edexcel C2 June 2012 Paper 1 Q16

Prove that $\tan\theta + \frac{1}{\tan\theta} \equiv \frac{1}{\sin\theta\cos\theta}$

$LHS \equiv$?

$\equiv$?

$\equiv$?

$\equiv$?

Fro Tip #2: In any addition/subtraction involving at least one fraction (with trig functions), always combine algebraically into one.

Simplify $5 - 5\sin^2\theta$

?

Fro Tip #3: Look out for $1 - \sin^2\theta$ and $1 - \cos^2\theta$. Students often don't spot that these can be simplified.

More Examples

Edexcel C2 June 2012 Paper 1 Q16

Prove that $\tan\theta + \dfrac{1}{\tan\theta} \equiv \dfrac{1}{\sin\theta\cos\theta}$

$$LHS \equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\equiv \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$\equiv \frac{1}{\sin\theta\cos\theta} \equiv RHS$$

Tip #2: In any addition/subtraction involving at least one fraction (with trig functions), always combine algebraically into one.

Simplify $5 - 5\sin^2\theta$

$$\equiv 5(1 - \sin^2\theta)$$
$$\equiv 5\cos^2\theta$$

Tip #3: Look out for $1 - \sin^2\theta$ and $1 - \cos^2\theta$. Students often don't spot that these can be simplified.

Test Your Understanding

Prove that $\frac{\tan x \cos x}{\sqrt{1-\cos^2 x}} \equiv 1$

?

Prove that $\frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$

?

AQA IGCSE Further Maths Worksheet

Prove that $\tan^2\theta \equiv \frac{1}{\cos^2\theta} - 1$

?

Test Your Understanding

Prove that $\frac{\tan x \cos x}{\sqrt{1-\cos^2 x}} \equiv 1$

$$LHS \equiv \frac{\frac{\sin x}{\cos x}\cos x}{\sqrt{\sin^2 x}} \equiv \frac{\sin x}{\sin x} \equiv 1$$

Prove that $\frac{\cos^4\theta - \sin^4\theta}{\cos^2\theta} \equiv 1 - \tan^2\theta$

$$LHS \equiv \frac{(\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)}{\cos^2\theta}$$

$$\equiv \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta} \equiv \frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$\equiv 1 - \tan^2\theta \equiv RHS$$

AQA IGCSE Further Maths Worksheet

Prove that $\tan^2\theta \equiv \frac{1}{\cos^2\theta} - 1$

$$LHS \equiv \frac{\sin^2\theta}{\cos^2\theta}$$

$$\equiv \frac{1-\cos^2\theta}{\cos^2\theta}$$

$$\equiv \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta} - 1 \equiv RHS$$

Exercise 10.3

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Extension:

[MAT 2008 1C] The simultaneous equations in x, y,

$$(\cos\theta)x - (\sin\theta)y = 2$$
$$(\sin\theta)x + (\cos\theta)y = 1$$

are solvable:

A) for all values of θ in range $0 \leq \theta < 2\pi$
B) except for one value of θ in range $0 \leq \theta < 2\pi$
C) except for two values of θ in range $0 \leq \theta < 2\pi$
D) except for three values of θ in range $0 \leq \theta < 2\pi$

?

Exercise 10.3

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For convenience let $s = \sin\theta$ and $c = \cos\theta$. As we'd usually do for simultaneous equations, we could make coefficients of x terms the same:

$$\boldsymbol{scx - s^2y = 2s}$$
$$\boldsymbol{scx + c^2y = c}$$

Then subtracting:

$$\boldsymbol{(s^2 + c^2)y = c - 2s}$$
$$\boldsymbol{\therefore y = c - 2s}$$

Similarly making y terms the same, we yield $x = 2c + s$

x, y are defined for every value of θ, so the answer is (A). Why might it have not been (A)? Suppose $x = \frac{2\cos\theta + \sin\theta}{\sin\theta}$. This would not be defined whenever $\sin\theta = 0$.

Homework Exercise

1 Simplify each of the following expressions:

a $1 - \cos^2 \frac{1}{2}\theta$

b $5\sin^2 3\theta + 5\cos^2 3\theta$

c $\sin^2 A - 1$

d $\dfrac{\sin\theta}{\tan\theta}$

e $\dfrac{\sqrt{1-\cos^2 x}}{\cos x}$

f $\dfrac{\sqrt{1-\cos^2 3A}}{\sqrt{1-\sin^2 3A}}$

g $(1+\sin x)^2 + (1-\sin x)^2 + 2\cos^2 x$

h $\sin^4\theta + \sin^2\theta\cos^2\theta$

i $\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta$

2 Given that $2\sin\theta = 3\cos\theta$, find the value of $\tan\theta$.

3 Given that $\sin x \cos y = 3\cos x \sin y$, express $\tan x$ in terms of $\tan y$.

4 Express in terms of $\sin\theta$ only:

a $\cos^2\theta$

b $\tan^2\theta$

c $\cos\theta\tan\theta$

d $\dfrac{\cos\theta}{\tan\theta}$

e $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

5 Using the identities $\sin^2 A + \cos^2 A \equiv 1$ and/or $\tan A = \dfrac{\sin A}{\cos A}$ $(\cos A \neq 0)$, prove that:

a $(\sin\theta + \cos\theta)^2 \equiv 1 + 2\sin\theta\cos\theta$

b $\dfrac{1}{\cos\theta} - \cos\theta \equiv \sin\theta\tan\theta$

c $\tan x + \dfrac{1}{\tan x} \equiv \dfrac{1}{\sin x\cos x}$

d $\cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 = 1 - 2\sin^2 A$

e $(2\sin\theta - \cos\theta)^2 + (\sin\theta + 2\cos\theta)^2 \equiv 5$

f $2 - (\sin\theta - \cos\theta)^2 \equiv (\sin\theta + \cos\theta)^2$

g $\sin^2 x\cos^2 y - \cos^2 x\sin^2 y \equiv \sin^2 x - \sin^2 y$

Homework Exercise

6 Find, without using your calculator, the values of:

a $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{5}{12}$ and θ is acute.

b $\sin\theta$ and $\cos\theta$, given that $\cos\theta = -\frac{3}{5}$ and θ is obtuse.

c $\cos\theta$ and $\tan\theta$, given that $\sin\theta = -\frac{7}{25}$ and $270° < \theta < 360°$.

7 Given that $\sin\theta = \frac{2}{3}$ and that θ is obtuse, find the exact value of: **a** $\cos\theta$ **b** $\tan\theta$

8 Given that $\tan\theta = -\sqrt{3}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$ **b** $\cos\theta$

9 Given that $\cos\theta = \frac{3}{4}$ and that θ is reflex, find the exact value of: **a** $\sin\theta$ **b** $\tan\theta$

10 In each of the following, eliminate θ to give an equation relating x and y:

a $x = \sin\theta,\ y = \cos\theta$ **b** $x = \sin\theta,\ y = 2\cos\theta$

c $x = \sin\theta,\ y = \cos^2\theta$ **d** $x = \sin\theta,\ y = \tan\theta$

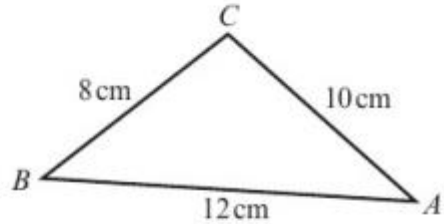
e $x = \sin\theta + \cos\theta,\ y = \cos\theta - \sin\theta$

Problem-solving

In part **e** find expressions for $x + y$ and $x - y$.

Homework Exercise

11 The diagram shows the triangle ABC with $AB = 12$ cm, $BC = 8$ cm and $AC = 10$ cm.

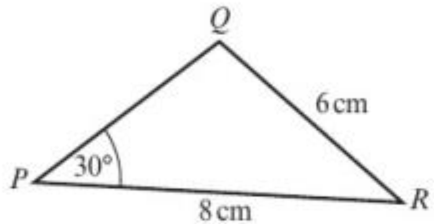


a Show that $\cos B = \frac{9}{16}$ **(3 marks)**

b Hence find the exact value of $\sin B$. **(2 marks)**

Hint Use the cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ ← **Section 9.1**

12 The diagram shows triangle PQR with $PR = 8$ cm, $QR = 6$ cm and angle $QPR = 30°$.



a Show that $\sin Q = \frac{2}{3}$ **(3 marks)**

b Given that Q is obtuse, find the exact value of $\cos Q$ **(2 marks)**

Homework Answers

1 **a** $\sin^2\frac{\theta}{2}$ **b** 5 **c** $-\cos^2 A$

d $\cos\theta$ **e** $\tan x$ **f** $\tan 3A$

g 4 **h** $\sin^2\theta$ **i** 1

2 $1\frac{1}{2}$

3 $\tan x - 3\tan y$

4 **a** $1-\sin^2\theta$ **b** $\dfrac{\sin^2\theta}{1-\sin^2\theta}$ **c** $\sin\theta$

d $\dfrac{1-\sin^2\theta}{\sin\theta}$ **e** $1-2\sin^2\theta$

5 (One outline example of a proof is given)

a $\text{LHS} = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$
$= 1 + 2\sin\theta\cos\theta$
$= \text{RHS}$

b $\text{LHS} = \dfrac{1-\cos^2\theta}{\cos\theta} = \dfrac{\sin^2\theta}{\cos\theta} = \sin\theta \times \dfrac{\sin\theta}{\cos\theta}$
$= \sin\theta\tan\theta = \text{RHS}$

c $\text{LHS} = \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x + \cos x}$
$= \dfrac{1}{\sin x + \cos x} = \text{RHS}$

d $\text{LHS} = \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$
$= 2(1-\sin^2 A) - 1 = 1 - 2\sin^2 A = \text{RHS}$

e $\text{LHS} = (4\sin^2\theta - 4\sin\theta\cos\theta + \cos^2\theta)$
$+ (\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta)$
$= 5(\sin^2\theta + \cos^2\theta) = 5 = \text{RHS}$

f $\text{LHS} = 2 - (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)$
$= 2(\sin^2\theta + \cos^2\theta) - (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)$
$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$
$= (\sin\theta + \cos\theta)^2 = \text{RHS}$

g $\text{LHS} = \sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y$
$= \sin^2 x - \sin^2 y = \text{RHS}$

6 **a** $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$

b $\sin\theta = \frac{4}{5}$, $\tan\theta = -\frac{4}{3}$

c $\cos\theta = \frac{24}{25}$, $\tan\theta = -\frac{7}{24}$

7 **a** $-\dfrac{\sqrt{5}}{3}$ **b** $-\dfrac{2\sqrt{5}}{5}$

8 **a** $-\dfrac{\sqrt{3}}{2}$ **b** $\dfrac{1}{2}$

9 **a** $-\dfrac{\sqrt{7}}{4}$ **b** $-\dfrac{\sqrt{7}}{3}$

10 **a** $x^2 + y^2 = 1$

b $4x^2 + y^2 = 4$ $\left(\text{or } x^2 + \dfrac{y^2}{4} = 1\right)$

c $x^2 + y = 1$

d $x^2 = y^2(1-x^2)$ $\left(\text{or } x^2 + \dfrac{x^2}{y^2} = 1\right)$

e $x^2 + y^2 = 2$ $\left(\text{or } \dfrac{(x+y)^2}{4} + \dfrac{(x-y)^2}{4} = 1\right)$

11 **a** Using cosine rule: $\cos B = \dfrac{8^2 + 12^2 - 10^2}{2 \times 8 \times 12} = \dfrac{9}{16}$

b $\dfrac{\sqrt{175}}{16}$

12 **a** Using sine rule: $\sin Q = \dfrac{\sin 30}{6} \times 8 = \dfrac{2}{3}$

b $-\dfrac{\sqrt{5}}{3}$