P1 Chapter 14: Logarithms

Natural Logs

Natural Logarithms

We have previously seen that $y = \log_a x$ is the inverse function of $y = a^x$.
We also saw that e^x is "**the**" exponential function.
The inverse of e^x is $\log_e x$, but because of its special importance, it has its own function name!

✎ The inverse of $y = e^x$ is $y = \ln x$

$$\ln e^x = \boldsymbol{x}$$
$$e^{\ln x} = \boldsymbol{x}$$

Since "e to the power of" and "ln of" are inverse functions, they cancel each other out!

Solve $e^x = 5$

$$x = \ln 5$$

"ln both sides". On the LHS it cancels out the "e to the power of"

Solve $2\ln x + 1 = 5$

$$\ln x = 2$$
$$x = e^2$$

Do "e to the power of" each side. On the LHS it cancels out the ln.

Quadratics in e^x

In previous chapters we've already dealt with quadratics in disguise, e.g. "quadratic in sin". We therefore just apply our usual approach: either make a suitable substitution so the equation **is** then quadratic, or (strongly recommended!) **go straight for the factorisation**.

Solve $e^{2x} + 2e^x - 15 = 0$

?

Solve $e^x - 2e^{-x} = 1$

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Quadratics in e^x

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Solve $e^{2x} + 2e^x - 15 = 0$

Note that $e^{2x} = (e^x)^2$ therefore

$$(e^x + 5)(e^x - 3) = 0$$
$$e^x = -5 \text{ or } e^x = 3$$

Exponential functions are always positive therefore:

$$x = \ln 3$$

Solve $e^x - 2e^{-x} = 1$

First write negative powers as fractions:

$$e^x - \frac{2}{e^x} = 1$$
$$(e^x)^2 - 2 = e^x$$
$$(e^x)^2 - e^x - 2 = 0$$
$$(e^x + 1)(e^x - 2) = 0$$
$$e^x = -1 \text{ or } e^x = 2$$
$$x = \ln 2$$

Test Your Understanding

Solve $\ln(3x+1) = 2$

?

Solve $e^{2x} + 5e^{x} = 6$

?

Solve $2^{x}e^{x+1} = 3$ giving your answer as an exact value.

?

Test Your Understanding

Solve $\ln(3x+1) = 2$

$$3x + 1 = e^2$$
$$x = \frac{e^2 - 1}{3}$$

Solve $e^{2x} + 5e^x = 6$

$$e^{2x} + 5e^x - 6 = 0$$
$$(e^x + 6)(e^x - 1) = 0$$
$$e^x = -6 \; or \; e^x = 1$$
$$x = \ln 1 = 0$$

Solve $2^x e^{x+1} = 3$ giving your answer as an exact value.

$$\ln 2^x e^{x+1} = \ln 3$$
$$\ln 2^x + \ln e^{x+1} = \ln 3$$
$$x \ln 2 + x + 1 = \ln 3$$
$$x(\ln 2 + 1) = \ln 3 - 1$$
$$x = \frac{\ln 3 - 1}{\ln 2 + 1}$$

Exercise 14.7

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Homework Exercise

1 Solve these equations, giving your answers in exact form.

a $e^x = 6$ **b** $e^{2x} = 11$ **c** $e^{-x+3} = 20$

d $3e^{4x} = 1$ **e** $e^{2x+6} = 3$ **f** $e^{5-x} = 19$

2 Solve these equations, giving your answers in exact form.

a $\ln x = 2$ **b** $\ln(4x) = 1$ **c** $\ln(2x + 3) = 4$

d $2\ln(6x - 2) = 5$ **e** $\ln(18 - x) = \frac{1}{2}$ **f** $\ln(x^2 - 7x + 11) = 0$

3 Solve these equations, giving your answers in exact form.

a $e^{2x} - 8e^x + 12 = 0$ **b** $e^{4x} - 3e^{2x} = -2$

c $(\ln x)^2 + 2\ln x - 15 = 0$ **d** $e^x - 5 + 4e^{-x} = 0$

e $3e^{2x} + 5 = 16e^x$ **f** $(\ln x)^2 = 4(\ln x + 3)$

Hint All of the equations in question 3 are quadratic equations in a function of x.

Hint First in part **d** multiply each term by e^x.

4 Find the exact solutions to the equation $e^x + 12e^{-x} = 7$. **(4 marks)**

5 Solve these equations, giving your answers in exact form.

a $\ln(8x - 3) = 2$ **b** $e^{5(x-8)} = 3$ **c** $e^{10x} - 8e^{5x} + 7 = 0$

d $(\ln x - 1)^2 = 4$

6 Solve $3^x e^{4x-1} = 5$, giving your answer in the form $\frac{a + \ln b}{c + \ln d}$

(5 marks)

Hint Take natural logarithms of both sides and then apply the laws of logarithms.

Homework Exercise

7 Officials are testing athletes for doping at a sporting event. They model the concentration of a particular drug in an athlete's bloodstream using the equation $D = 6e^{\frac{-t}{10}}$ where D is the concentration of the drug in mg/l and t is the time in hours since the athlete took the drug.

a Interpret the meaning of the constant 6 in this model.

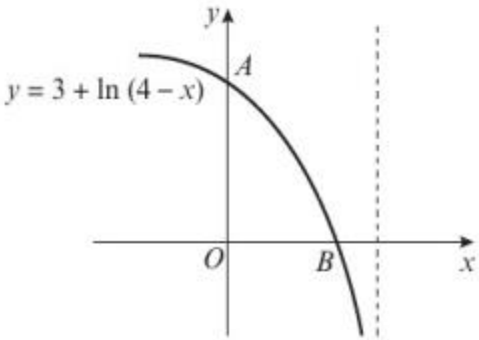
b Find the concentration of the drug in the bloodstream after 2 hours.

c It is impossible to detect this drug in the bloodstream if the concentration is lower than 3 mg/l. Show that this happens after $t = -10\ln\left(\frac{1}{2}\right)$ and convert this result into hours and minutes.

8 The graph of $y = 3 + \ln(4 - x)$ is shown to the right.

a State the exact coordinates of point A. **(1 mark)**

b Calculate the exact coordinates of point B. **(3 marks)**



Challenge

The graph of the function $g(x) = Ae^{Bx} + C$ passes through $(0, 5)$ and $(6, 10)$. Given that the line $y = 2$ is an asymptote to the graph, show that $B = \frac{1}{6}\ln\left(\frac{8}{3}\right)$.

Homework Answers

1 **a** $\ln 6$ **b** $\frac{1}{2}\ln 11$ **c** $3 - \ln 20$
d $\frac{1}{4}\ln\left(\frac{1}{3}\right)$ **e** $\frac{1}{2}\ln 3 - 3$ **f** $5 - \ln 19$

2 **a** e^2 **b** $\frac{e}{4}$ **c** $\frac{1}{2}e^4 - \frac{3}{2}$
d $\frac{1}{6}(e^{\frac{1}{2}} + 2)$ **e** $18 - e^{\frac{1}{2}}$ **f** $2, 5$

3 **a** $\ln 2, \ln 6$ **b** $\frac{1}{2}\ln 2, 0$ **c** e^3, e^{-5}
d $\ln 4, 0$ **e** $\ln 5, \ln\left(\frac{1}{3}\right)$ **f** e^6, e^{-2}

4 $\ln 3, 2\ln 2$

5 **a** $\frac{1}{8}(e^2 + 3)$ **b** $\frac{1}{5}(\ln 3 + 40)$ **c** $\frac{1}{5}\ln 7, 0$
d e^3, e^{-1}

6 $\dfrac{1 + \ln 5}{4 + \ln 3}$

7 **a** The initial concentration of the drug in mg/l
b 4.91 mg/l
c $3 = 6e^{-\frac{t}{10}}$
$\frac{1}{2} = e^{-\frac{t}{10}}$
$\ln\left(\frac{1}{2}\right) = -\dfrac{t}{10}$
$t = -10\ln\left(\frac{1}{2}\right) = 6.931\ldots = 6$ hours 56 minutes

8 **a** $(0, 3 + \ln 4)$ **b** $(4 - e^{-3})$

Challenge

As $y = 2$ is an asymptote, $C = 2$.
Substituting $(0, 5)$ gives $5 = Ae^0 + 2$, so A is 3.
Substituting $(6, 10)$ gives $10 = 3e^{6B} + 2$.
Rearranging this gives $B = \frac{1}{6}\ln\left(\frac{8}{3}\right)$.