M2 Chapter 8: Further Kinematics

Acceleration Functions

Variable Acceleration in One Dimension

$$\begin{tikzcd} s \arrow[d, bend left, "\frac{d}{dt}"] \\ v \arrow[u, bend left, "\int v\,dt"] \arrow[d, bend left, "\frac{d}{dt}"] \\ a \arrow[u, bend left, "\int a\,dt"] \end{tikzcd}$$

In Mechanics Yr1 we saw that velocity was the rate of change of displacement, and thus $v = \frac{ds}{dt}$. Similarly, acceleration is the rate of change of velocity, and thus $a = \frac{dv}{dt}$

Let's stick to one-dimension for the moment, but you may need to **differentiate more complex functions of t that use Pure Year 2 techniques**.

[Textbook] A particle is moving in a straight line with acceleration at time t seconds given by

$$a = \cos 2\pi t \text{ ms}^{-2}, \qquad t \geq 0$$

The velocity of the particle at time $t = 0$ is $\frac{1}{2\pi}$ ms^{-1}. Find:

(a) an expression for the velocity at time t seconds
(b) the maximum speed
(c) the distance travelled in the first 3 seconds.

a

$$v = \int \cos 2\pi t \, dt$$

$$= \frac{1}{2\pi} \sin 2\pi t + c$$

Remember with 'reverse chain rule', we divide by constant in front of variable.

When $t = 0$, $v = 0 + c = \frac{1}{2\pi}$

$$\therefore v = \frac{1}{2\pi} \sin 2\pi t + \frac{1}{2\pi}$$

b

Maximum value of sin is 1, so

$$v_{max} = \frac{1}{2\pi} \times 1 + \frac{1}{2\pi} = \frac{1}{\pi} \text{ ms}^{-1}$$

c

$$s = \frac{1}{2\pi} \int_0^3 (\sin 2\pi t + 1) dt$$

Finding area under velocity-time graph.

Can tidy up integral by factorising out common factor.

$$= \frac{1}{2\pi} \left[-\frac{1}{2\pi} \cos 2\pi t + t \right]_0^3 = \cdots$$

$$= 0.477 \, m \, (3sf)$$

Test Your Understanding

[Textbook] A particle of mass 6kg is moving on the positive x-axis. At time t seconds the displacement, s, of the particle from the origin is given by

$$s = 2t^{\frac{3}{2}} + \frac{e^{-2t}}{3} \text{ m}, \qquad t \geq 0$$

(a) Find the velocity of the particle when $t = 1.5$.
Given that the particle is acted on by a single force of variable magnitude F N which acts in the direction of the positive x-axis,
(b) Find the value of F when $t = 2$

a

?

b

?

Recap: Due to the chain rule,

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

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$$v = \frac{ds}{dt} = 3t^{\frac{1}{2}} - \frac{2}{3}e^{-2t}\ ms^{-1}$$

When $t = 1.5$ seconds:

$$v = 3 \times 1.5^{0.5} - \frac{2}{3}e^{-3} = 3.64\ ms^{-1}$$

Recap: Due to the chain rule,

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

b

$$a = \frac{dv}{dt} = 1.5t^{-0.5} + \frac{4}{3}e^{-4} = 1.0850\ldots ms^{-2}$$

$$F = ma = 6 \times 1.0850\ldots = 6.51 \text{ N}$$

Exercise 8.3

Pearson Stats/Mechanics Year 2

Pages 71-72

Homework Exercise

1 A particle P moves in a straight line. The acceleration, a, of P at time t seconds is given by $a = 1 - \sin \pi t \text{ m s}^{-2}$, where $t \geqslant 0$.

When $t = 0$, the velocity of P is 0 m s^{-1} and its displacement is 0 m. Find expressions for:

a the velocity at time t seconds

b the displacement at time t seconds.

2 A particle moving in a straight line has acceleration a, given by

$$a = \sin 3\pi t \text{ m s}^{-2},\ t \geqslant 0$$

At time t seconds the particle has velocity $v \text{ m s}^{-1}$ and displacement s m. Given that when $t = 0$, $v = \dfrac{1}{3\pi}$ and $s = 1$, find:

a an expression for v in terms of t

b the maximum speed of the particle

c an expression for s in terms of t.

3 An object moves in a straight line from a point O. At time t seconds the object has acceleration, a, where

$$a = -\cos 4\pi t \text{ m s}^{-2},\ 0 \leqslant t \leqslant 4$$

When $t = 0$, the velocity of the object is 0 m s^{-1} and its displacement is 0 m. Find:

a an expression for the velocity at time t seconds

b the maximum speed of the object

c an expression for the displacement of the object at time t seconds

d the maximum distance of the object from O

e the number of times the object changes direction during its motion.

Problem-solving

In part **e**, consider the number of times the velocity changes sign.

Homework Exercise

5 A particle P moves in a straight line so that, at time t seconds, its displacement, s m, from a fixed point O on the line is given by

$$s = \begin{cases} \frac{1}{2}t, & 0 \leqslant t \leqslant 6 \\ \sqrt{t+3}, & t > 6 \end{cases}$$

Find:

a the velocity of P when $t = 4$

b the velocity of P when $t = 22$.

6 A particle P moves in a straight line so that, at time t seconds, its displacement from a fixed point O on the line is given by

$$s = \begin{cases} 3^t + 3t, & 0 \leqslant t \leqslant 3 \\ 24t - 36, & 3 < t \leqslant 6 \\ -252 + 96t - 6t^2, & t > 6 \end{cases}$$

Find:

a the velocity of P when $t = 2$

b the velocity of P when $t = 10$

c the greatest positive displacement of P from O

d the values of s when the speed of P is $18\,\text{m}\,\text{s}^{-1}$.

7 A particle moves in a straight line. At time t seconds after it begins its motion, the acceleration of the particle is $3\sqrt{t}\,\text{m}\,\text{s}^{-2}$ where $t > 0$.

Given that after 1 second the particle is moving with velocity $2\,\text{m}\,\text{s}^{-1}$, find the time taken for the particle to travel 16 m.

Homework Exercise

8 A runner takes part in a race in which competitors have to sprint 200 m in a straight line.
At time t seconds after starting, her displacement, s, from the starting position is modelled as:

$$s = k\sqrt{t},\ 0 \leqslant t \leqslant T$$

Given that the runner completes the race in 25 seconds,

a find the value of k and the value of T **(2 marks)**

b find the speed of the runner when she crosses the finish line **(3 marks)**

c criticise this model for small values of t. **(2 marks)**

9 A particle is moving in a straight line. At time t seconds, where $t \geqslant 0$, the acceleration of P is $a\,\mathrm{m\,s^{-2}}$ and the velocity $v\,\mathrm{m\,s^{-1}}$ of P is given by

$$v = 2 + 8\sin kt$$

where k is a constant.

The initial acceleration of P is $4\,\mathrm{m\,s^{-2}}$.

a Find the value of k. **(3 marks)**

Using the value of k found in part **a**,

b find, in terms of π, the values of t in the interval $0 \leqslant t \leqslant 4\pi$ for which $a = 0$ **(2 marks)**

c show that $4a^2 = 64 - (v - 2)^2$ **(5 marks)**

d find the maximum velocity and the maximum acceleration. **(2 marks)**

Homework Exercise

10 A particle P moves on the x-axis. At time t seconds the velocity of P is $v\,\mathrm{m\,s^{-1}}$ in the direction of x increasing, where v is given by

$$v = \begin{cases} 10t - 2t^{\frac{3}{2}}, & 0 \leqslant t \leqslant 4 \\ 24 - \left(\dfrac{t-4}{2}\right)^4, & t > 4 \end{cases}$$

When $t = 0$, P is at the origin O.

Find:

a the greatest speed of P in the interval $0 \leqslant t \leqslant 4$ **(4 marks)**

b the distance of P from O when $t = 4$ **(3 marks)**

c the time at which P is instantaneously at rest for $t > 4$ **(1 mark)**

d the total distance travelled by P in the first 10 seconds of its motion. **(7 marks)**

Homework Answers

1 **a** $v = t + \frac{\cos \pi t}{\pi} - \frac{1}{\pi}$ **b** $s = \frac{t^2}{2} + \frac{\sin \pi t}{\pi^2} - \frac{t}{\pi}$

2 **a** $v = -\frac{\cos 3\pi t}{3\pi} + \frac{2}{3\pi}$ **b** $\frac{1}{\pi}$

c $s = -\frac{\sin 3\pi t}{9\pi^2} + \frac{2t}{3\pi} + 1$

3 **a** $v = -\frac{\sin 4\pi t}{4\pi}$ **b** $\frac{1}{4\pi}$

c $s = \frac{\cos 4\pi t}{16\pi^2} - \frac{1}{16\pi^2}$ **d** $\frac{1}{8\pi^2}$ **e** 16

4 **a** $1.18\,\mathrm{m\,s^{-1}}$ **b** $-0.152\,\mathrm{m\,s^{-2}}$

c $-0.759\,\mathrm{N}$

5 **a** $0.5\,\mathrm{m\,s^{-1}}$ **b** $0.1\,\mathrm{m\,s^{-1}}$

6 **a** $12.9\,\mathrm{m\,s^{-1}}$ in the direction of s increasing

b $24\,\mathrm{m\,s^{-1}}$ in the direction of s decreasing

c 132 m

d 20.8 m and 118.5 m

7 3.31 s

8 **a** $k = 40$, $T = 25$ **b** $4\,\mathrm{m\,s^{-1}}$

c $v = \frac{20}{\sqrt{t}}$, so for small t, the value of v is large

e.g. $t = 0.01$, $v = 200\,\mathrm{m\,s^{-1}}$, so not realistic for small t.

9 **a** $k = \frac{1}{2}$ **b** $t = \pi, 3\pi$

c $a = 4\cos\left(\frac{t}{2}\right)$, $4a^2 = 64\cos^2\left(\frac{t}{2}\right)$

$v = 2 + 8\sin\left(\frac{t}{2}\right)$, $(v - 2)^2 = 64\sin^2\left(\frac{t}{2}\right)$

$4a^2 = 64 - (v - 2)^2 \Rightarrow 64\cos^2\left(\frac{t}{2}\right) = 64 - 64\sin^2\left(\frac{t}{2}\right)$

$\Rightarrow \cos^2\left(\frac{t}{2}\right) = 1 - \sin^2\left(\frac{t}{2}\right)$

$\Rightarrow \cos^2 T + \sin^2 T = 1$

d $10\,\mathrm{m\,s^{-1}}$, $4\,\mathrm{m\,s^{-2}}$

10 **a** $24\,\mathrm{m\,s^{-1}}$ **b** 54.4 m **c** 8.43 s **d** 101.2 m