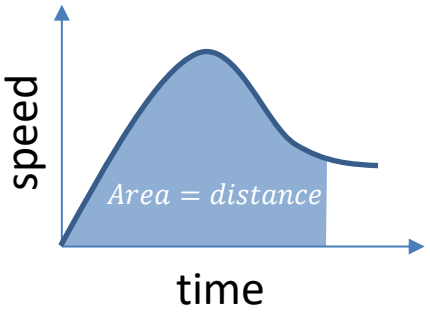
P1 Chapter 13: Integration

Definite Integrals

Definite Integration

So far we've seen integration as '**the opposite of differentiation**', allowing us to find $y = f(x)$ when we know the gradient function $y = f'(x)$.

In practical settings however the most useful use of integration is that **it finds the area under a graph**. Remember at GCSE for example when you estimated the area under a speed-time graph, using trapeziums, to get the distance?
If you knew the equation of the curve, you could get the exact area!

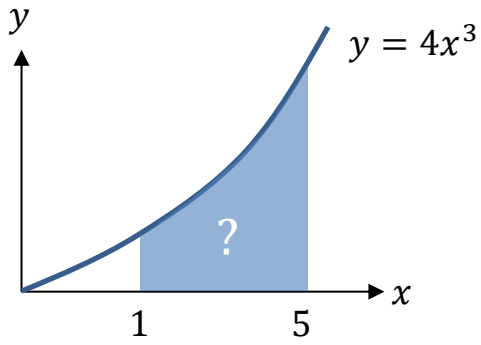


Before we do this, we need to understand how to find a **definite integral**:

These are known as **limits**, which give the values of x we're finding the area between.

We integrate as normal, but put expression in **square brackets**, meaning **we still need to evaluate the integrated expression using the limits**.

$$\int_1^5 4x^3\,dx = [x^4]_1^5$$
$$= (5^4) - (1^4)$$
$$= 624$$



Write $(\ldots) - (\ldots)$ and evaluate the expression for each of the limits, top one first.

Another Example

$$\int_{-3}^{3} x^2 + 1 \, dx = \left[\frac{1}{3}x^3 + x\right]_{-3}^{3}$$

We **DON'T have a constant of integration** when doing definite integration. I'll explain why later.

$$= \left(\frac{1}{3}(3)^3 + 3\right) - \left(\frac{1}{3}(-3)^3 - 3\right)$$
$$= 12 - -12$$
$$= 24$$

Write out you working EXACTLY as seen here. The $(\ldots) - (\ldots)$ brackets are particularly crucial as you'll otherwise likely make a sign error.

Use of Technology:
You can use the $[\![\int_b^a \square]\!]$ button on your calculator to evaluate definite integrals.
<u>But only use it to check your answer.</u>

Problem Solving

Given that P is a constant and $\int_1^5 (2Px + 7)\, dx = 4P^2$, show that there are two possible values for P and find these values.

?

Remember: P is a constant, so just treat it as a number.

Problem Solving

Given that P is a constant and $\int_1^5 (2Px + 7)\, dx = 4P^2$, show that there are two possible values for P and find these values.

$$\int_1^5 (2Px + 7)\, dx = [Px^2 + 7x]_1^5$$
$$= (25P + 35) - (P + 7)$$
$$= 24P + 28$$

$$\therefore 24P + 28 = 4P^2$$
$$P^2 - 6P - 7 = 0$$
$$(P + 1)(P - 7) = 0$$
$$P = -1 \text{ or } 7$$

Remember: P is a constant, so just treat it as a number.

Exercise 13.4

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(Classes in a rush may want to skip this exercise and go to the next section, which continues definite integration, but in the context of areas under graphs).

Extension

1 *[MAT 2009 1A]* The smallest value of

$$I(a) = \int_0^1 (x^2 - a)^2 \, dx$$

as a varies, is what?

?

2 *[MAT 2015 1D]* Let $f(x) = \int_0^1 (xt)^2 \, dt$ and $g(x) = \int_0^x t^2 \, dt$

Let $A > 0$. Which of the following statements are true?

A) $g\big(f(A)\big)$ is always bigger than $f\big(g(A)\big)$

B) $f\big(g(A)\big)$ is always bigger than $g\big(f(A)\big)$

C) They are always equal.

D) $f\big(g(A)\big)$ is bigger if $A < 1$, and $g\big(f(A)\big)$ is bigger if $A > 1$.

E) $g\big(f(A)\big)$ is bigger if $A < 1$, and $f\big(g(A)\big)$ is bigger if $A > 1$.

?

Exercise 13.4

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Extension

1 *[MAT 2009 1A]* The smallest value of

$$I(a) = \int_0^1 (x^2 - a)^2 \, dx$$

as a varies, is what?

$$\boldsymbol{I(a) = \int_0^1 x^4 - 2ax^2 + a^2 \; dx}$$

$$\boldsymbol{= \left[\frac{1}{5}x^5 - \frac{2}{3}ax^3 + a^2x\right]_0^1}$$

$$\boldsymbol{= \frac{1}{5} - \frac{2}{3}a + a^2}$$

We then use differentiation or completing the square to find that the minimum value of $\frac{1}{5} - \frac{2}{3}a + a^2$ is $\frac{4}{45}$.

2 *[MAT 2015 1D]* Let $f(x) = \int_0^1 (xt)^2 \, dt$ and $g(x) = \int_0^x t^2 \, dt$

Let $A > 0$. Which of the following statements are true?

A) $g\big(f(A)\big)$ is always bigger than $f\big(g(A)\big)$

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E) $g\big(f(A)\big)$ is bigger if $A < 1$, and $f\big(g(A)\big)$ is bigger if $A > 1$.

(Official Sln) Evaluating: $f(x) = \frac{x^2}{3}$ and $g(x) = \frac{x^3}{3}$. Hence $g\big(f(A)\big) = \frac{A^6}{3^4}$ and $f\big(g(A)\big) = \frac{A^6}{3^3}$. Hence answer is (B).

Homework Exercise

1 Evaluate the following definite integrals:

a $\int_2^5 x^3\,dx$ **b** $\int_1^3 x^4\,dx$

c $\int_0^4 \sqrt{x}\,dx$ **d** $\int_1^3 \frac{3}{x^2}\,dx$

Watch out You must not use a calculator to work out definite integrals in your exam. You need to use calculus and show clear algebraic working.

2 Evaluate the following definite integrals:

a $\int_1^2 \left(\frac{2}{x^3} + 3x\right)dx$ **b** $\int_0^2 (2x^3 - 4x + 5)dx$ **c** $\int_4^9 \left(\sqrt{x} - \frac{6}{x^2}\right)dx$ **d** $\int_1^8 (x^{-\frac{1}{3}} + 2x - 1)dx$

3 Evaluate the following definite integrals:

a $\int_1^3 \frac{x^3 + 2x^2}{x}\,dx$ **b** $\int_3^6 \left(x - \frac{3}{x}\right)^2 dx$ **c** $\int_0^1 x^2\left(\sqrt{x} + \frac{1}{x}\right)dx$ **d** $\int_1^4 \frac{2 + \sqrt{x}}{x^2}\,dx$

4 Given that A is a constant and $\int_1^4 (6\sqrt{x} - A)dx = A^2$, show that there are two possible values for A and find these values. **(5 marks)**

5 Use calculus to find the value of $\int_1^9 (2x - 3\sqrt{x})dx$. **(5 marks)**

Homework Exercise

6 Evaluate $\int_4^{12} \frac{2}{\sqrt{x}} \mathrm{d}x$, giving your answer in the form $a + b\sqrt{3}$, where a and b are integers. **(4 marks)**

7 Given that $\int_1^k \frac{1}{\sqrt{x}} \mathrm{d}x = 3$, calculate the value of k. **(4 marks)**

8 The speed, v ms^{-1}, of a train at time t seconds is given by $v = 20 + 5t$, $0 \leqslant t \leqslant 10$.

The distance, s metres, travelled by the train in 10 seconds is given by $s = \int_0^{10} (20 + 5t) \mathrm{d}t$. Find the value of s.

Problem-solving

You might encounter a definite integral with an unknown in the limits. Here, you can find an expression for the definite integral in terms of k then set that expression equal to 3.

Challenge

Given that $\int_k^{3k} \frac{3x + 2}{8} \mathrm{d}x = 7$ and $k > 0$, calculate the value of k.

Homework Answers

1 **a** $152\frac{1}{4}$ **b** $48\frac{2}{5}$ **c** $5\frac{1}{3}$ **d** 2

2 **a** $5\frac{1}{4}$ **b** 10 **c** $11\frac{5}{6}$ **d** $60\frac{1}{2}$

3 **a** $16\frac{2}{3}$ **b** $46\frac{1}{2}$ **c** $\frac{11}{14}$ **d** $2\frac{1}{2}$

4 $A = -7$ or 4

5 28

6 $-8 + 8\sqrt{3}$

7 $k = \frac{25}{4}$

8 450 m

Challenge

$k = 2$