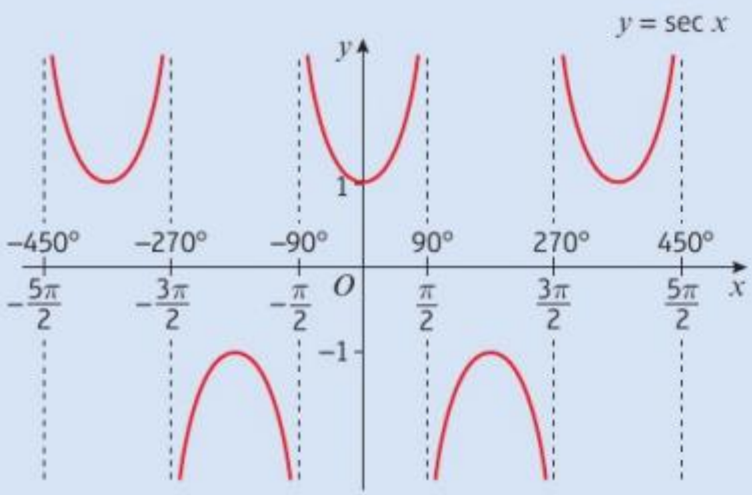
P2 Chapter 1: CoTrigonometry

Chapter Practice

Key Points

1
- $\sec x = \dfrac{1}{\cos x}$ (undefined for values of x for which $\cos x = 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (undefined for values of x for which $\sin x = 0$)
- $\cot x = \dfrac{1}{\tan x}$ (undefined for values of x for which $\tan x = 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$

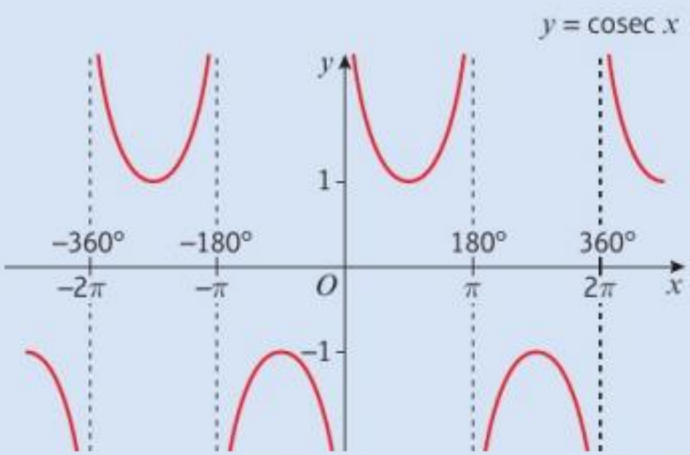
2 The graph of $\boldsymbol{y = \sec x}$, $x \in \mathbb{R}$, has symmetry in the y-axis and has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\cos x = 0$.



- The domain of $y = \sec x$ is $x \in \mathbb{R}$, $x \neq 90°, 270°, 450°, \ldots$ or any odd multiple of 90°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ or any odd multiple of $\frac{\pi}{2}$
- The range of $y = \sec x$ is $y \leqslant -1$ or $y \geqslant 1$.

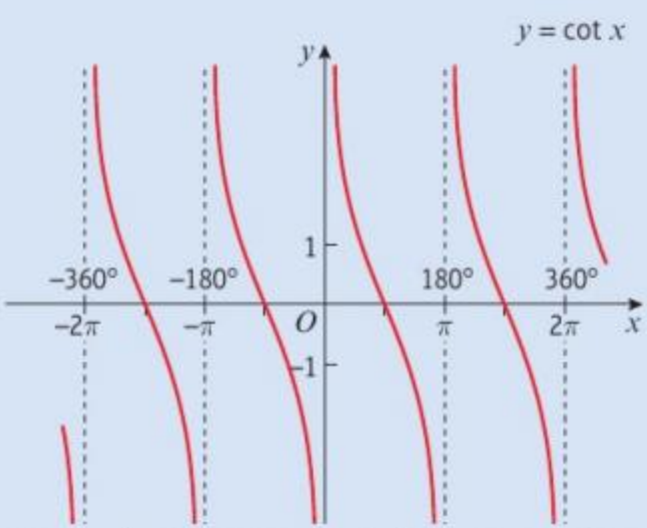
Key Points

3 The graph of $y = \mathbf{cosec}\, x$, $x \in \mathbb{R}$, has period 360° or 2π radians. It has vertical asymptotes at all the values of x for which $\sin x = 0$.



- The domain of $y = \text{cosec}\, x$ is $x \in \mathbb{R}$, $x \neq 0°, 180°, 360°, \ldots$ or any multiple of 180°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi, \ldots$ or any multiple of π
- The range of $y = \text{cosec}\, x$ is $y \leqslant -1$ or $y \geqslant 1$.

4 The graph of $y = \mathbf{cot}\, x$, $x \in \mathbb{R}$, has period 180° or π radians. It has vertical asymptotes at all the values of x for which $\tan x = 0$.



- The domain of $y = \cot x$ is $x \in \mathbb{R}$, $x \neq 0°$, $180°, 360°, \ldots$ or any multiple of 180°.
- In radians the domain is $x \in \mathbb{R}$, $x \neq 0, \pi, 2\pi$, $\ldots$ or any multiple of π.
- The range of $y = \cot x$ is $y \in \mathbb{R}$.

5 $\sec x = k$ and $\text{cosec}\, x = k$ have no solutions for $-1 < k < 1$.

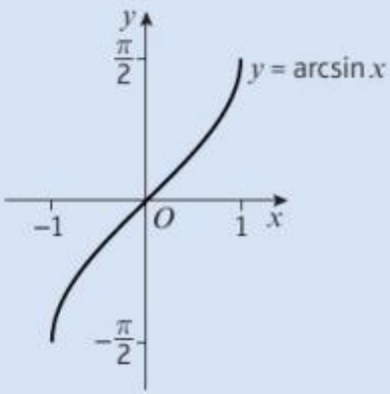
Key Points

6 You can use the identity $\sin^2 x + \cos^2 x \equiv 1$ to prove the following identities:

- $1 + \tan^2 x \equiv \sec^2 x$
- $1 + \cot^2 x \equiv \text{cosec}^2 x$

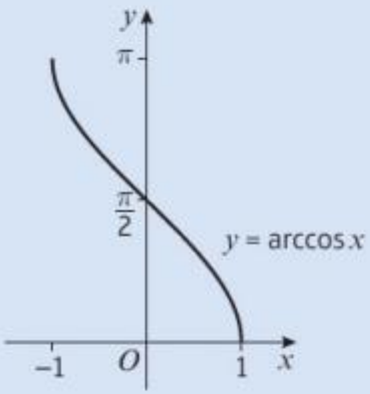
7 The inverse function of $\sin x$ is called **arcsin x**.

- The domain of $y = \arcsin x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}$ or $-90° \leqslant \arcsin x \leqslant 90°$



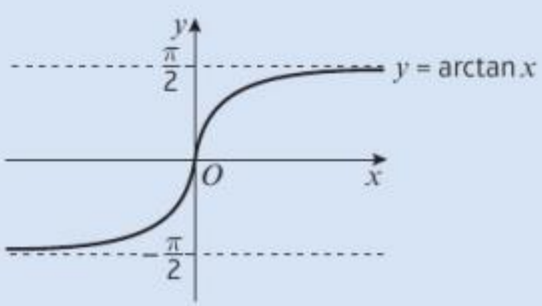
8 The inverse function of $\cos x$ is called **arccos x**.

- The domain of $y = \arccos x$ is $-1 \leqslant x \leqslant 1$
- The range of $y = \arccos x$ is $0 \leqslant \arccos x \leqslant \pi$ or $0° \leqslant \arccos x \leqslant 180°$



9 The inverse function of $\tan x$ is called **arctan x**.

- The domain of $y = \arctan x$ is $x \in \mathbb{R}$
- The range of $y = \arctan x$ is $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$ or $-90° < \arctan x < 90°$



Chapter Exercises

Give any non-exact answers to equations to 1 decimal place.

1 Solve $\tan x = 2\cot x$, in the interval $-180° \leqslant x \leqslant 90°$. **(4 marks)**

2 Given that $p = 2\sec\theta$ and $q = 4\cos\theta$, express p in terms of q. **(4 marks)**

3 Given that $p = \sin\theta$ and $q = 4\cot\theta$, show that $p^2q^2 = 16(1 - p^2)$. **(4 marks)**

4 **a** Solve, in the interval $0 < \theta < 180°$,

i $\operatorname{cosec}\theta = 2\cot\theta$ **ii** $2\cot^2\theta = 7\operatorname{cosec}\theta - 8$

b Solve, in the interval $0 \leqslant \theta \leqslant 360°$,

i $\sec(2\theta - 15°) = \operatorname{cosec} 135°$ **ii** $\sec^2\theta + \tan\theta = 3$

c Solve, in the interval $0 \leqslant x \leqslant 2\pi$,

i $\operatorname{cosec}\left(x + \frac{\pi}{15}\right) = -\sqrt{2}$ **ii** $\sec^2 x = \frac{4}{3}$

5 Given that $5\sin x\cos y + 4\cos x\sin y = 0$, and that $\cot x = 2$, find the value of $\cot y$. **(5 marks)**

6 Prove that:

a $(\tan\theta + \cot\theta)(\sin\theta + \cos\theta) \equiv \sec\theta + \operatorname{cosec}\theta$

b $\dfrac{\operatorname{cosec} x}{\operatorname{cosec} x - \sin x} \equiv \sec^2 x$

c $(1 - \sin x)(1 + \operatorname{cosec} x) \equiv \cos x\cot x$

d $\dfrac{\cot x}{\operatorname{cosec} x - 1} - \dfrac{\cos x}{1 + \sin x} \equiv 2\tan x$

e $\dfrac{1}{\operatorname{cosec}\theta - 1} + \dfrac{1}{\operatorname{cosec}\theta + 1} \equiv 2\sec\theta\tan\theta$

f $\dfrac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{1 + \tan^2\theta} \equiv \cos^2\theta$

Chapter Exercises

7 **a** Prove that $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} \equiv 2\operatorname{cosec} x$. **(4 marks)**

b Hence solve, in the interval $-2\pi \leqslant x \leqslant 2\pi$, $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = -\dfrac{4}{\sqrt{3}}$ **(4 marks)**

8 Prove that $\dfrac{1+\cos\theta}{1-\cos\theta} \equiv (\operatorname{cosec}\theta + \cot\theta)^2$ **(4 marks)**

9 Given that $\sec A = -3$, where $\dfrac{\pi}{2} < A < \pi$,

a calculate the exact value of $\tan A$ **(3 marks)**

b show that $\operatorname{cosec} A = \dfrac{3\sqrt{2}}{4}$ **(3 marks)**

10 Given that $\sec\theta = k$, $|k| \geqslant 1$, and that θ is obtuse, express in terms of k:

a $\cos\theta$ **b** $\tan^2\theta$ **c** $\cot\theta$ **d** $\operatorname{cosec}\theta$

11 Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec\left(x + \dfrac{\pi}{4}\right) = 2$, giving your answers in terms of π. **(5 marks)**

12 Find, in terms of π, the value of $\arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$. **(4 marks)**

13 Solve, in the interval $0 \leqslant x \leqslant 2\pi$, the equation $\sec^2 x - \frac{2\sqrt{3}}{3}\tan x - 2 = 0$, giving your answers in terms of π. **(5 marks)**

14 **a** Factorise $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2$. **(2 marks)**

b Hence solve $\sec x \operatorname{cosec} x - 2\sec x - \operatorname{cosec} x + 2 = 0$, in the interval $0 \leqslant x \leqslant 360°$. **(4 marks)**

15 Given that $\arctan(x - 2) = -\frac{\pi}{3}$, find the value of x. **(3 marks)**

16 On the same set of axes sketch the graphs of $y = \cos x$, $0 \leqslant x \leqslant \pi$, and $y = \arccos x$, $-1 \leqslant x \leqslant 1$, showing the coordinates of points at which the curves meet the axes. **(4 marks)**

17 **a** Given that $\sec x + \tan x = -3$, use the identity $1 + \tan^2 x \equiv \sec^2 x$ to find the value of $\sec x - \tan x$. **(3 marks)**

b Deduce the values of:

i $\sec x$ **ii** $\tan x$ **(3 marks)**

c Hence solve, in the interval $-180° \leqslant x \leqslant 180°$, $\sec x + \tan x = -3$. **(3 marks)**

18 Given that $p = \sec\theta - \tan\theta$ and $q = \sec\theta + \tan\theta$, show that $p = \frac{1}{q}$ **(4 marks)**

19 **a** Prove that $\sec^4\theta - \tan^4\theta = \sec^2\theta + \tan^2\theta$. **(3 marks)**

b Hence solve, in the interval $-180° \leqslant \theta \leqslant 180°$, $\sec^4\theta = \tan^4\theta + 3\tan\theta$. **(4 marks)**

Chapter Exercises

20 **a** Sketch the graph of $y = \sin x$ and shade in the area representing $\int_0^{\frac{\pi}{2}} \sin x \, dx$.

b Sketch the graph of $y = \arcsin x$ and shade in the area representing $\int_0^1 \arcsin x \, dx$.

c By considering the shaded areas explain why $\int_0^{\frac{\pi}{2}} \sin x \, dx + \int_0^1 \arcsin x \, dx = \frac{\pi}{2}$

21 Show that $\cot 60° \sec 60° = \frac{2\sqrt{3}}{3}$

22 **a** Sketch, in the interval $-2\pi \leqslant x \leqslant 2\pi$, the graph of $y = 2 - 3\sec x$. **(3 marks)**

b Hence deduce the range of values of k for which the equation $2 - 3\sec x = k$ has no solutions. **(2 marks)**

23 **a** Sketch the graph of $y = 3\arcsin x - \frac{\pi}{2}$, showing clearly the exact coordinates of the end-points of the curve. **(4 marks)**

b Find the exact coordinates of the point where the curve crosses the x-axis. **(3 marks)**

24 **a** Prove that for $0 < x \leqslant 1$, $\arccos x = \arctan \frac{\sqrt{1-x^2}}{x}$

b Prove that for $-1 \leqslant x < 0$, $\arccos x = k + \arctan \frac{\sqrt{1-x^2}}{x}$, where k is a constant to be found.

Chapter Answers

1 $-125.3°, \pm 54.7°$

2 $p = \frac{8}{q}$

3 $p^2q^2 = \sin^2\theta \times 4^2\cot^2\theta = 16\sin^2\theta \times \frac{\cos^2\theta}{\sin^2\theta}$

$= 16\cos^2\theta = 16(1 - \sin^2\theta) = 16(1 - p^2)$

4 **a** **i** $60°$

ii $30°, 41.8°, 138.2°, 150°$

b **i** $30°, 165°, 210°, 345°$

ii $45°, 116.6°, 225°, 296.6°$

c **i** $\frac{71\pi}{60}, \frac{101\pi}{60}$

ii $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

5 $-\frac{8}{5}$

6 **a** L.H.S. $\equiv \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)(\sin\theta + \cos\theta)$

$\equiv \frac{(\sin^2\theta + \cos^2\theta)}{\cos\theta\sin\theta}(\sin\theta + \cos\theta)$

$\equiv \frac{\sin\theta}{\sin\theta\cos\theta} + \frac{\cos\theta}{\cos\theta\sin\theta}$

$\equiv \sec\theta + \operatorname{cosec}\theta \equiv$ R.H.S.

b L.H.S. $\equiv \dfrac{\frac{1}{\sin x}}{\frac{1}{\sin x} - \sin x}$

$\equiv \dfrac{\frac{1}{\sin x}}{\frac{1-\sin^2 x}{\sin x}} \equiv \frac{1}{\sin x} \times \frac{\sin x}{\cos^2 x} \equiv \frac{1}{\cos^2 x} \equiv \sec^2 x \equiv$ R.H.S.

c L.H.S. $\equiv 1 - \sin x + \operatorname{cosec} x - 1 \equiv \frac{1}{\sin x} - \sin x$

$\equiv \frac{1-\sin^2 x}{\sin x} \equiv \frac{\cos^2 x}{\sin x} \equiv \cos x \frac{\cos x}{\sin x} \equiv \cos x \cot x$

$\equiv$ R.H.S.

d L.H.S. $\equiv \frac{\cot x(1 + \sin x) - \cos x(\operatorname{cosec} x - 1)}{(\operatorname{cosec} x - 1)(1 + \sin x)}$

$\equiv \frac{\cot x + \cos x - \cot x + \cos x}{\operatorname{cosec} x - 1 + 1 - \sin x} \equiv \frac{2\cos x}{\operatorname{cosec} x - \sin x}$

$\equiv \dfrac{2\cos x}{\frac{1}{\sin x} - \sin x} \equiv \dfrac{2\cos x}{\left(\frac{1-\sin^2 x}{\sin x}\right)} \equiv \frac{2\cos x\sin x}{\cos^2 x}$

$\equiv 2\tan x \equiv$ R.H.S.

e L.H.S. $\equiv \frac{\operatorname{cosec}\theta + 1 + \operatorname{cosec}\theta - 1}{(\operatorname{cosec}^2\theta - 1)} \equiv \frac{2\operatorname{cosec}\theta}{\cot^2\theta}$

$\equiv \frac{2}{\sin\theta} \cdot \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{2\sin\theta}{\cos^2\theta} \equiv \frac{2}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta}$

$\equiv 2\sec\theta\tan\theta \equiv$ R.H.S.

f L.H.S. $\equiv \frac{\sec^2\theta - \tan^2\theta}{\sec^2\theta} \equiv \frac{1}{\sec^2\theta} \equiv \cos^2\theta \equiv$ R.H.S.

7 a L.H.S. $\equiv \dfrac{\sin^2 x + (1+\cos x)^2}{(1+\cos x)\sin x}$

$\equiv \dfrac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1+\cos x)\sin x} \equiv \dfrac{2+2\cos x}{(1+\cos x)\sin x}$

$\equiv \dfrac{2(1+\cos x)}{(1+\cos x)\sin x} \equiv \dfrac{2}{\sin x} \equiv 2\operatorname{cosec} x$

b $-\frac{\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

8 R.H.S. $\equiv \left(\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\right)^2 \equiv \dfrac{(1+\cos\theta)^2}{\sin^2\theta} \equiv \dfrac{(1+\cos\theta)^2}{1-\cos^2\theta}$

$\equiv \dfrac{(1+\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} \equiv \dfrac{1+\cos\theta}{1-\cos\theta} \equiv$ L.H.S.

9 a $-2\sqrt{2}$

b $\operatorname{cosec}^2 A = 1 + \cot^2 A = 1 + \frac{1}{8} = \frac{9}{8}$

$\Rightarrow \operatorname{cosec} A = \pm\dfrac{3}{2\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{4}$

As A is obtuse, $\operatorname{cosec} A$ is +ve, $\Rightarrow \operatorname{cosec} A = \dfrac{3\sqrt{2}}{4}$

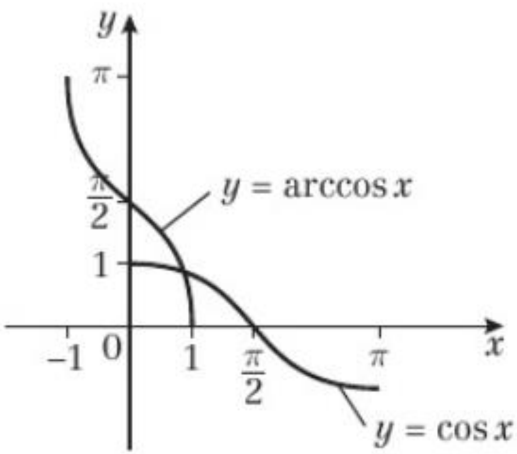
10 a $\dfrac{1}{k}$ **b** $k^2 - 1$ **c** $-\dfrac{1}{\sqrt{k^2-1}}$ **d** $-\dfrac{k}{\sqrt{k^2-1}}$

11 $\frac{\pi}{12}, \frac{17\pi}{12}$ **12** $\frac{\pi}{3}$ **13** $\frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

14 a $(\sec x - 1)(\operatorname{cosec} x - 2)$ **b** 30°, 150°

15 $2 - \sqrt{3}$

16



17 a $-\frac{1}{3}$ **b i** $-\frac{5}{3}$, **ii** $-\frac{4}{3}$ **c** 126.9°

18 $pq = (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = \sec^2\theta - \tan^2\theta$

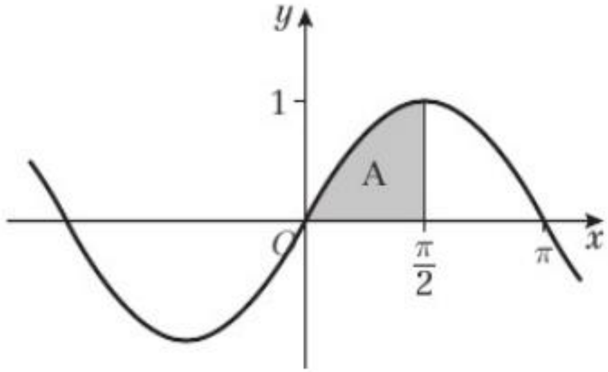
$= 1 \Rightarrow p = \dfrac{1}{q}$

19 a L.H.S. $= (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

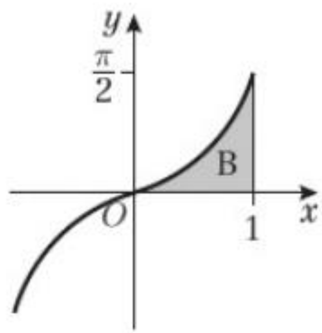
$= 1 \times (\sec^2\theta + \tan^2\theta) = \sec^2\theta + \tan^2\theta =$ R.H.S.

b −153.4°, −135°, 26.6°, 45°

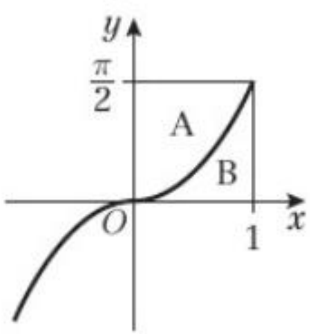
20 a



b



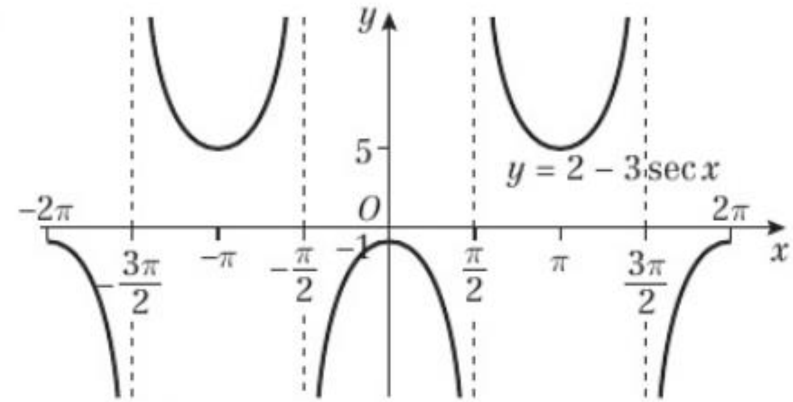
c The regions A and B fit together to make a rectangle.



Area $= 1 \times \frac{\pi}{2} = \frac{\pi}{2}$

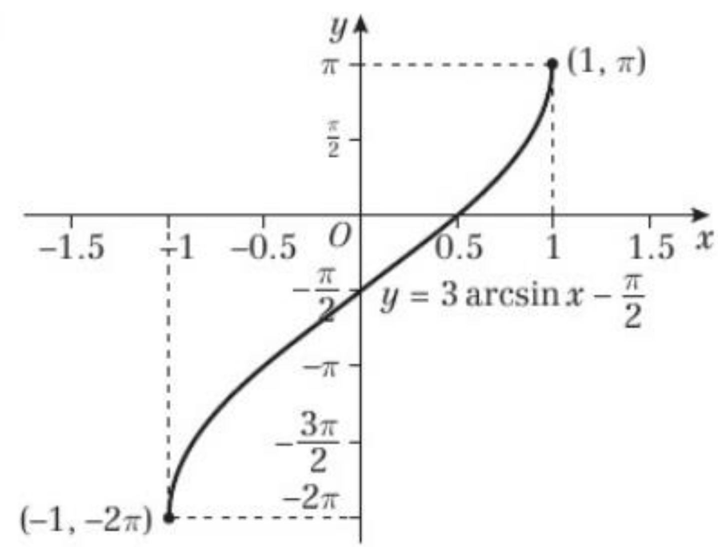
21 $\cot 60° \sec 60° = \frac{1}{\tan 60°} \times \frac{1}{\cos 60°} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

22 a



b $-1 < k < 5$

23 a



b $\left(\frac{1}{2}, 0\right)$

24 **a** Let $y = \arccos x$. So $\cos y = x$, $\sin y = \sqrt{1 - x^2}$.

Thus $\tan y = \dfrac{\sqrt{1 - x^2}}{x}$, which is valid for $x \in (0, 1]$.

Therefore $\arccos x = \arctan \dfrac{\sqrt{1 - x^2}}{x}$ for $0 < x \leq 1$.

b Letting $y = \arccos x$, $x \in [-1, 0) \Rightarrow y \in \left(\dfrac{\pi}{2}, \pi\right]$

$\tan y = \dfrac{\sin y}{\cos y} = \dfrac{\sqrt{1 - x^2}}{x}$

$\arctan \dfrac{\sqrt{1 - x^2}}{x}$ gives values in the range $\left(-\dfrac{\pi}{2}, 0\right]$,

so for $y \in \left(\frac{\pi}{2}, \pi\right]$ you need to add π:

$y = \pi + \arctan \dfrac{\sqrt{1 - x^2}}{x}$

Therefore $\arccos x = \pi + \arctan \dfrac{\sqrt{1 - x^2}}{x}$