P1 Chapter 10: Trigonometry Equations

Equations and Identities

Quadratics in sin/cos/tan

We saw that an equation can be 'quadratic in' something, e.g. $x - 2\sqrt{x} + 1 = 0$ is 'quadratic in $\sqrt{x}$', meaning that $\sqrt{x}$ could be replaced with another variable, say y, to produce a quadratic equation $y^2 - 2y + 1 = 0$.

Solve $5\sin^2 x + 3\sin x - 2 = 0$ in the interval $0 \leq x \leq 360°$.

Method 1: Use a substitution.

Let $y = \sin x$

Then $5y^2 + 3y - 2 = 0$

$(5y - 2)(y + 1) = 0$

$y = \frac{2}{5}$ or $y = -1$

$\therefore \sin x = \frac{2}{5}$ or $\sin x = -1$

$\boldsymbol{x = 23.6°, 156.4°,}$

$\boldsymbol{or\ x = 270°}$

Method 2: Factorise without substitution.

This is the same, but we 'imagine' $\sin x$ as a single variable and hence factorise immediately.

$(5\sin x - 2)(\sin x + 1) = 0$

$\sin x = \frac{2}{5}$ or $\sin x = -1$

$\boldsymbol{x = 23.6°, 156.4°,}$

$\boldsymbol{or\ x = 270°}$

Note: Method 2 is best provided you feel confident with it. Method 1 is good practice until you build up confidence.

More Examples

Solve $\tan^2\theta = 4$ in the interval $0 \leq x \leq 360°$.

?

Solve $2\cos^2 x + 9\sin x = 3\sin^2 x$ in the interval $-180° \leq x \leq 180°$.

?

Tip: We have an identity involving sin^2 and cos^2, so it makes sense to change the squared one that would match all the others.

More Examples

Solve $\tan^2\theta = 4$ in the interval $0 \leq x \leq 360°$.

$\tan\theta = 2 \;\; or \;\; \tan\theta = -2$

$\theta = 63.4°, 243.4°$

$or\ \theta = -63.4°, 116.6°, 296.6°$

Missing the negative case would result in the loss of multiple marks. Beware!

$-63.4°$ was outside the range so we had to add $180°$ twice.

Solve $2\cos^2 x + 9\sin x = 3\sin^2 x$ in the interval $-180° \leq x \leq 180°$.

$2(1-\sin^2 x) + \sin x = 3\sin^2 x$

$2 - 2\sin^2 x + \sin x = 3\sin^2 x$

$5\sin^2 x - \sin x - 2 = 0$

$(5\sin x + 1)(\sin x - 2) = 0$

$\sin x = -\frac{1}{5}\ or\ \cancel{\sin x = 2}$

$x = -168.5°, -11.5°$

Tip: We have an identity involving sin^2 and cos^2, so it makes sense to change the squared one that would match all the others.

Test Your Understanding

Edexcel C2 Jan 2010 Q2

(*a*) Show that the equation

$$5 \sin x = 1 + 2 \cos^2 x$$

can be written in the form

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(2)

(*b*) Solve, for $0 \leq x < 360°$,

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(4)

?

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(4)

(a)	$5\sin x = 1 + 2\left(1 - \sin^2 x\right)$	M1
	$2\sin^2 x + 5\sin x - 3 = 0$ **(*)**	A1cso (2)
(b)	$(2s-1)(s+3) = 0$ **giving** s **=**	M1
	$[\sin x = -3$ has no solution$]$ **so** $\sin x = \frac{1}{2}$	A1
	$\therefore \quad x = 30,\ 150$	B1, B1ft (4) **[6]**

Exercise 10.6

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Extension

1 *[MAT 2010 1C]* In the range $0 \leq x < 360°$, the equation
$$\sin^2 x + 3\sin x\cos x + 2\cos^2 x = 0$$
Has how many solutions?

?

2 *[MAT 2014 1E]* As x varies over the real numbers, the largest value taken by the function $(4\sin^2 x + 4\cos x + 1)^2$ equals what?

?

Exercise 10.6

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Extension

1 *[MAT 2010 1C]* In the range $0 \leq x < 360°$, the equation

$$\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$$

Has how many solutions?

There are multiple ways to do this, including factorising LHS to $(\sin x + \cos x)(\sin x + 2\cos x)$, but dividing by $\cos^2 x$ gives:

$$\tan^2 x + 3\tan x + 2 = 0$$
$$(\tan x + 1)(\tan x + 2) = 0$$
$$\tan x = -1 \text{ or } \tan x = -2$$

tan always gives a pair of solutions per $360°$, so there are 4 solutions.

2 *[MAT 2014 1E]* As x varies over the real numbers, the largest value taken by the function $(4\sin^2 x + 4\cos x + 1)^2$ equals what?

$$\left(4 - 4\cos^2 x + 4\cos x + 1\right)^2$$
$$= \left(-4\cos^2 x + 4\cos x + 5\right)^2$$
$$= \left(6 - (1 - 2\cos x)^2\right)^2$$

We can make $\cos x = \frac{1}{2}$, thus giving a maximum value of $6^2 = 36$.

Homework Exercise

1 Find the values of θ, in the interval $0 \leqslant \theta \leqslant 360°$, for which:

a $\sin 4\theta = 0$

b $\cos 3\theta = -1$

c $\tan 2\theta = 1$

d $\cos 2\theta = \frac{1}{2}$

e $\tan \frac{1}{2}\theta = -\frac{1}{\sqrt{3}}$

f $\sin(-\theta) = \frac{1}{\sqrt{2}}$

2 Solve the following equations in the interval given:

a $\tan(45° - \theta) = -1,\ 0 \leqslant \theta \leqslant 360°$

b $2\sin(\theta - 20°) = 1,\ 0 \leqslant \theta \leqslant 360°$

c $\tan(\theta + 75°) = \sqrt{3},\ 0 \leqslant \theta \leqslant 360°$

d $\sin(\theta - 10°) = -\frac{\sqrt{3}}{2},\ 0 \leqslant \theta \leqslant 360°$

e $\cos(70° - x) = 0.6,\ 0 \leqslant \theta \leqslant 180°$

3 Solve the following equations in the interval given:

a $3\sin 3\theta = 2\cos 3\theta,\ 0 \leqslant \theta \leqslant 180°$

b $4\sin(\theta + 45°) = 5\cos(\theta + 45°),\ 0 \leqslant \theta \leqslant 450°$

c $2\sin 2x - 7\cos 2x = 0,\ 0 \leqslant x \leqslant 180°$

d $\sqrt{3}\sin(x - 60°) + \cos(x - 60°) = 0,\ -180° \leqslant x \leqslant 180°$

4 Solve for $0 \leqslant x \leqslant 180°$ the equations:

a $\sin(x + 20°) = \frac{1}{2}$ **(4 marks)**

b $\cos 2x = -0.8$, giving your answers to 1 decimal place. **(4 marks)**

5 Find all the solutions, in the interval $0 \leqslant x \leqslant 360°$, to the equation $8\sin^2 x + 6\cos x - 9 = 0$ giving each solution to one decimal place.

(6 marks)

Homework Exercise

6 Find, for $0 \leqslant x \leqslant 360°$, all the solutions of $\sin^2 x + 1 = \frac{7}{2}\cos^2 x$ giving each solution to one decimal place. **(6 marks)**

7 Show that the equation $2\cos^2 x + \cos x - 6 = 0$ has no solutions. **(3 marks)**

8 **a** Show that the equation $\cos^2 x = 2 - \sin x$ can be written as $\sin^2 x - \sin x + 1 = 0$. **(2 marks)**

b Hence show that the equation $\cos^2 x = 2 - \sin x$ has no solutions. **(3 marks)**

Problem-solving

If you have to answer a question involving the number of solutions to a quadratic equation, see if you can make use of the discriminant.

9 $\tan^2 x - 2\tan x - 4 = 0$

a Show that $\tan x = p \pm \sqrt{q}$ where p and q are numbers to be found. **(3 marks)**

b Hence solve the equation $\tan^2 x - 2\tan x - 4 = 0$ in the interval $0 \leqslant x \leqslant 540°$. **(5 marks)**

Challenge

1 Solve the equation $\cos^2 3\theta - \cos 3\theta = 2$ in the interval $-180° \leqslant \theta \leqslant 180°$.

2 Solve the equation $\tan^2(\theta - 45°) = 1$ in the interval $0 \leqslant \theta \leqslant 360°$.

Homework Answers

1 **a** 60°, 120°, 240°, 300°
b 45°, 135°, 225°, 315°
c 0°, 180°, 199°, 341°, 360°
d 77.0°, 113°, 257°, 293°
e 60°, 300°
f 204°, 336°
g 30°, 60°, 120°, 150°, 210°, 240°, 300°, 330°

2 **a** ±45°, ±135° **b** –180°, –117°, 0°, 63.4°, 180°
c ±114° **d** 0°, ±75.5°, ±180°

3 **a** 72.0°, 144° **b** 0°, 60°
c No solutions in range

4 **a** ±41.8°, ±138° **b** 38.2°, 142°

5 60°, 75.5°, 284.5°, 300°

6 48.2°, 131.8°, 228.2°, 311.8°

7 $2\cos^2 x + \cos x - 6 = (2\cos x - 3)(\cos x + 2)$
There are no solutions to $\cos x = -2$ or to $\cos x = \frac{3}{2}$

8 **a** $1 - \sin^2 x = 2 - \sin x$
Rearrange to get $\sin^2 x - \sin x + 1 = 0$
b The equation has no real roots as $b^2 - 4ac < 0$

9 **a** $p = 1$, $q = 5$
b 72.8°, 129.0°, 252.8°, 309.0°, 432.8°, 489.0°

Challenge

1 –180°, –60°, 60°, 180°

2 0°, 90°, 180°, 270°, 360°