P1 Chapter 7: Algebraic Methods

The Factor Theorem

The Factor Theorem

$$x^3 + x^2 - 4x - 4 = (x - 2)(x^2 + 3x + 2)$$

We can see that $(x - 2)$ is a factor of $x^3 + x^2 - 4x - 4$.
What would happen if x is 2?

$2 - 2 = 0$ so the RHS, and hence LHS would be 0.
The converse is also true: if we could find a value a such that the LHS is 0 when we substitute in a for x, then $(x - a)$ would be a factor.

✏ The Factor Theorem states that if $f(x)$ is a polynomial then:

- If $f(p) = 0$, then $(x - p)$ is a factor of $f(x)$.
- Conversely, if $(x - p)$ is a factor of $f(x)$, then $f(p) = 0$.

Examples

Show that $(x-2)$ is a factor of x^3+x^2-4x-4.

?

Fully factorise $2x^3+x^2-18x-9$.

?

Examples

Show that $(x-2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Let $f(x) = x^3 + x^2 - 4x - 4$

$$f(2) = 2^3 + 2^2 - 4(2) - 4 = 0$$

$\therefore$ $(x-2)$ is a factor of $x^3 + x^2 - 4x - 4$.

Writing $f(x) = \cdots$ gives us appropriate notation, i.e. $f(2)$, to show we're substituting 2 into the polynomial on the next line.

Fully factorise $2x^3 + x^2 - 18x - 9$.

Let $f(x) = 2x^3 + x^2 - 18x - 9$

$f(1) = 2(1)^3 + 1^2 - 18(1) - 9 = -24$

$f(-1) = 2(-1)^3 + (-1)^2 + 18(-1) - 9 = 24$

...

$f(3) = 0 \quad \therefore \quad (x-3)$ is a factor.

Using algebraic division we find that:

$$2x^3 + x^2 - 18x - 9 = (x-3)(2x^2 + 7x + 3)$$
$$= (x-3)(2x+1)(x+3)$$

Keep on trying values until you find one where $f(p) = 0$. Recommended order: $p = 1, -1, 2, -2, \ldots$
You can use the 'Table' mode on your calculator to try lots of values in a range – see my interactive Classwiz powerpoint.

Tip: If you consider that:

$$2x^3 + x^2 - 18x - 9 = (x-3)(ax^2 + bx + c)$$

it's clear, by considering the expansion of the RHS, that $a = 2$ and $c = 3$. We can determine b by considering for example the x terms in the expansion. This is much faster than algebraic division.

Using Factor Theorem to find unknown coefficients

Given that $2x + 1$ is a factor of $6x^3 + ax^2 + 1$, determine the value of a.

?

Using Factor Theorem to find unknown coefficients

Given that $2x + 1$ is a factor of $6x^3 + ax^2 + 1$, determine the value of a.

$$f(x) = 6x^3 + ax^2 + 1$$

$$f\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + 1$$

$$= -\frac{3}{4} + \frac{1}{4}a + 1 = 0$$

$$\frac{1}{4}a = -\frac{1}{4}$$

$$a = -1$$

Side Notes:
Choose the value of x to substitute that **<u>makes the divisor/factor 0</u>**.
So if $2x + 1 = 0$, then $x = -\frac{1}{2}$, thus find $f\left(-\frac{1}{2}\right)$.
This even works when you're dividing by non-linear factors, e.g. $\frac{1}{x} + 1$, but at A Level they will always be of the form $ax + b$.

Test Your Understanding

Edexcel C2 May 2016 Q2

$$f(x) = 6x^3 + 13x^2 - 4$$

(a) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2)**

(b) Factorise $f(x)$ completely. **(4)**

?

Given that $3x - 1$ is a factor of $3x^3 + 11x^2 + ax + 1$, determine the value of a.

?

Test Your Understanding

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$$f(x) = 6x^3 + 13x^2 - 4$$

(a) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2)**

(b) Factorise $f(x)$ completely. **(4)**

(a)	$f(-2) = 6(-2)^3 + 13(-2)^2 - 4$ $= 0$, **and** so $(x+2)$ is a factor.
(b)	$f(x) = \{(x+2)\}(6x^2 + x - 2)$ $= (x+2)(2x-1)(3x+2)$

Given that $3x - 1$ is a factor of $3x^3 + 11x^2 + ax + 1$, determine the value of a.

$$f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 + 11\left(\frac{1}{3}\right)^2 + \frac{1}{3}a + 1 = 0$$

$$\frac{7}{3} + \frac{1}{3}a = 0 \quad \rightarrow \quad a = -7$$

Exercise 7.1

Pearson Pure Mathematics Year 1/AS
Pages 56

Extension

1 *[MAT 2006 1E]* The cubic

$$x^3 + ax + b$$

Has both $(x-1)$ and $(x-2)$ as factors. Determine the values of a and b.

?

2 [MAT 2009 1I] The polynomial

$$n^2x^{2n+3} - 25nx^{n+1} + 150x^7$$

Has $x^2 - 1$ as a factor

A) for no values of n;
B) for $n = 10$ only;
C) for $n = 15$ only;
D) for $n = 10$ and $n = 15$ only.

?

The **remainder theorem** states that if $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. This similarly works whenever a makes the divisor 0.

3 [MAT 2013 1G] Let $n \geq 2$ be an integer and $p_n(x)$ be the polynomial

$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n)$$

What is the remainder, in terms of n, when $p_n(x)$ is divided by $p_{n-1}(x)$?

?

Exercise 7.1

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Extension

1 *[MAT 2006 1E]* The cubic

$$x^3 + ax + b$$

Has both $(x-1)$ and $(x-2)$ as factors. Determine the values of a and b.

$$\boldsymbol{a = -7, b = 6}$$

2 [MAT 2009 1I] The polynomial

$$n^2 x^{2n+3} - 25nx^{n+1} + 150x^7$$

Has $x^2 - 1$ as a factor

A) for no values of n;
B) for $n = 10$ only;
C) for $n = 15$ only;
D) for $n = 10$ and $n = 15$ only.

$\boldsymbol{x^2 - 1 = (x+1)(x-1)}$ so both factors.

$$\boldsymbol{f(1) = n^2 - 25n + 150 = 0}$$

If $\boldsymbol{n}$ is odd: $\boldsymbol{f(-1) = -n^2 - 25n - 150 = 0}$
If $\boldsymbol{n}$ is even: $\boldsymbol{f(-1) = -n^2 + 25n - 150 = 0}$
Putting this all together, we get option B.

The **remainder theorem** states that if $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. This similarly works whenever a makes the divisor 0.

3 [MAT 2013 1G] Let $n \geq 2$ be an integer and $p_n(x)$ be the polynomial

$$p_n(x) = (x-1) + (x-2) + \cdots + (x-n)$$

What is the remainder, in terms of n, when $p_n(x)$ is divided by $p_{n-1}(x)$?

$$\boldsymbol{p_n(x) = nx - \frac{1}{2}n(n+1)}$$

$$\boldsymbol{p_{n-1}(x) = (n-1)x + \frac{1}{2}n(n-1)}$$

We need to make the divisor 0:

$$\boldsymbol{(n-1)\left(x + \frac{1}{2}n\right) = 0 \quad \rightarrow \quad x = -\frac{n}{2}}$$

Remainder:

$$\boldsymbol{p_n\left(-\frac{n}{2}\right) = n\left(-\frac{n}{2}\right) - \frac{1}{2}n(n+1) = -\frac{1}{2}n}$$

Homework Exercise

1 Use the factor theorem to show that:

a $(x - 1)$ is a factor of $4x^3 - 3x^2 - 1$

b $(x + 3)$ is a factor of $5x^4 - 45x^2 - 6x - 18$

c $(x - 4)$ is a factor of $-3x^3 + 13x^2 - 6x + 8$.

2 Show that $(x - 1)$ is a factor of $x^3 + 6x^2 + 5x - 12$ and hence factorise the expression completely.

3 Show that $(x + 1)$ is a factor of $x^3 + 3x^2 - 33x - 35$ and hence factorise the expression completely.

4 Show that $(x - 5)$ is a factor of $x^3 - 7x^2 + 2x + 40$ and hence factorise the expression completely.

5 Show that $(x - 2)$ is a factor of $2x^3 + 3x^2 - 18x + 8$ and hence factorise the expression completely.

6 Each of these expressions has a factor $(x \pm p)$. Find a value of p and hence factorise the expression completely.

a $x^3 - 10x^2 + 19x + 30$

b $x^3 + x^2 - 4x - 4$

c $x^3 - 4x^2 - 11x + 30$

7 **i** Fully factorise the right-hand side of each equation.

ii Sketch the graph of each equation.

a $y = 2x^3 + 5x^2 - 4x - 3$

b $y = 2x^3 - 17x^2 + 38x - 15$

c $y = 3x^3 + 8x^2 + 3x - 2$

d $y = 6x^3 + 11x^2 - 3x - 2$

e $y = 4x^3 - 12x^2 - 7x + 30$

8 Given that $(x - 1)$ is a factor of $5x^3 - 9x^2 + 2x + a$, find the value of a.

9 Given that $(x + 3)$ is a factor of $6x^3 - bx^2 + 18$, find the value of b.

Homework Exercise

10 Given that $(x - 1)$ and $(x + 1)$ are factors of $px^3 + qx^2 - 3x - 7$, find the values of p and q.

11 Given that $(x + 1)$ and $(x - 2)$ are factors of $cx^3 + dx^2 - 9x - 10$, find the values of c and d.

Problem-solving

Use the factor theorem to form simultaneous equations.

12 Given that $(x + 2)$ and $(x - 3)$ are factors of $gx^3 + hx^2 - 14x + 24$, find the values of g and h.

13 $f(x) = 3x^3 - 12x^2 + 6x - 24$

a Use the factor theorem to show that $(x - 4)$ is a factor of $f(x)$. **(2 marks)**

b Hence, show that 4 is the only real root of the equation $f(x) = 0$. **(4 marks)**

14 $f(x) = 4x^3 + 4x^2 - 11x - 6$

a Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$. **(2 marks)**

b Factorise $f(x)$ completely. **(4 marks)**

c Write down all the solutions of the equation $4x^3 + 4x^2 - 11x - 6 = 0$. **(1 mark)**

15 **a** Show that $(x - 2)$ is a factor of $9x^4 - 18x^3 - x^2 + 2x$. **(2 marks)**

b Hence, find four real solutions to the equation $9x^4 - 18x^3 - x^2 + 2x = 0$. **(5 marks)**

Challenge

$f(x) = 2x^4 - 5x^3 - 42x^2 - 9x + 54$

a Show that $f(1) = 0$ and $f(-3) = 0$.

b Hence, solve $f(x) = 0$.

Homework Answers

1 **a** $f(1) = 0$ **b** $f(-3) = 0$ **c** $f(4) = 0$

2 $(x - 1)(x + 3)(x + 4)$

3 $(x + 1)(x + 7)(x - 5)$

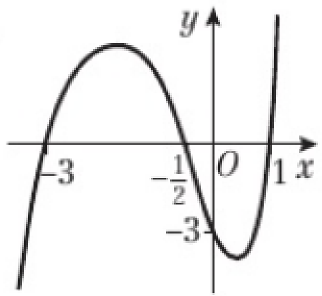
4 $(x - 5)(x - 4)(x + 2)$

5 $(x - 2)(2x - 1)(x + 4)$

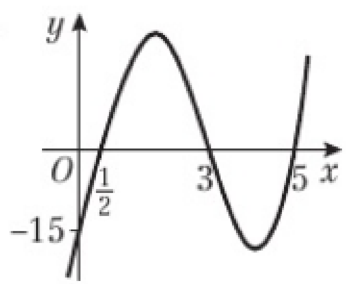
6 **a** $(x + 1)(x - 5)(x - 6)$ **b** $(x - 2)(x + 1)(x + 2)$

c $(x - 5)(x + 3)(x - 2)$

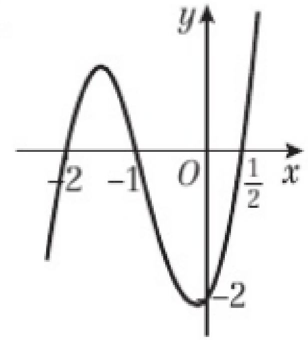
7 **a** **i** $(x - 1)(x + 3)(2x + 1)$ **ii**



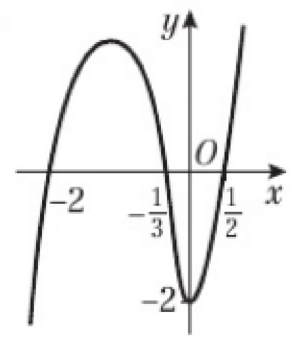
b **i** $(x - 3)(x - 5)(2x - 1)$ **ii**



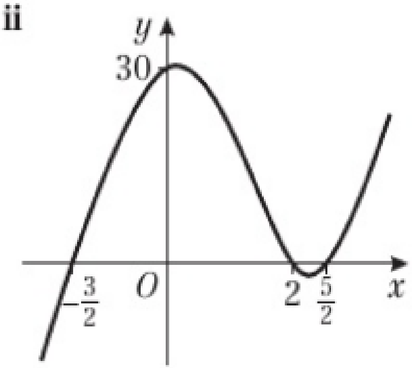
7 **c** **i** $(x + 1)(x + 2)(3x - 1)$ **ii**



d **i** $(x + 2)(2x - 1)(3x + 1)$ **ii**



e **i** $(x - 2)(2x - 5)(2x + 3)$ **ii**



Homework Answers

8 2

9 −16

10 $p = 3, q = 7$

11 $c = 2, d = 3$

12 $g = 3, h = -7$

13 **a** $f(4) = 0$

b $f(x) = (x - 4)(3x^2 + 6)$

For $3x^2 + 6 = 0$, $b^2 - 4ac = -72$ so there are no real roots. Therefore, 4 is the only real root of $f(x) = 0$.

14 **a** $f(-2) = 0$ **b** $(x + 2)(2x + 1)(2x - 3)$

c $x = -2$, $x = -\frac{1}{2}$ and $x = 1\frac{1}{2}$

15 **a** $f(2) = 0$ **b** $x = 0$, $x = 2$, $x = -\frac{1}{3}$ and $x = \frac{1}{3}$

Challenge

a $f(1) = 2 - 5 - 42 - 9 + 54 = 0$

$f(-3) = 162 + 135 - 378 + 27 + 54 = 0$

b $2x^4 - 5x^3 - 42x^2 - 9x + 54$

$= (x - 1)(x + 3)(x - 6)(2x + 3)$

$x = 1$ $x = -3$, $x = 6$, $x = -1.5$