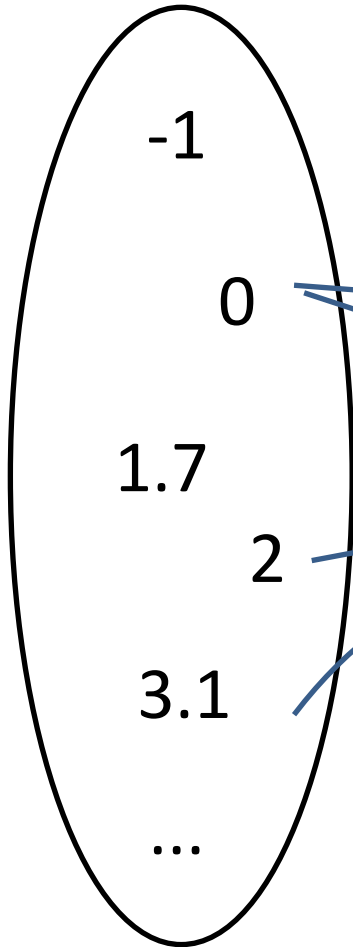

P2 Chapter 2: Graphing Functions

3D Functions as Maps

What is a mapping?

Inputs



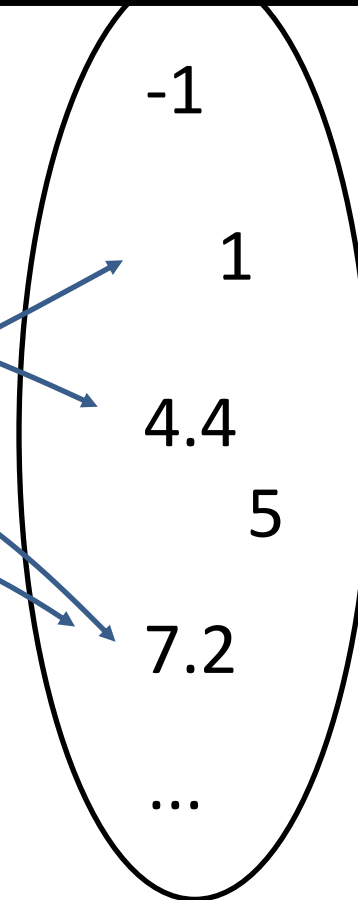
The mapping might be completely arbitrary, or might have some underlying rule, e.g.
 $x \rightarrow 2x$
(meaning each value is mapped to twice its value)

Also notice that **one input might map to multiple outputs**, or multiple inputs to one output.

Notice also that not all values in the set of inputs necessarily have a mapping to a value in the set of outputs.

A **mapping** is something which maps one set of numbers to another.

Outputs



 The **domain** is the set of possible inputs.

 The **range** is the set of possible outputs.

What is a function?

✎ A function is: a mapping such that every element of the domain is mapped to exactly one element of the range.

Notation: $f(x) = 2x + 1$ $f: x \rightarrow 2x + 1$

$f(x)$ refers to the output of the function.

Function?

Tip: Use the 'vertical ray test'. If a vertically fired ray can hit the curve multiple times, it is NOT a function.

Note: We can illustrate a mapping/function graphically, by plotting a point (x, y) if x maps to y . We write $y = f(x)$ to mean "make y the output of the function".

For each input (x value), we only get one output (y value)

No

Yes

$$f(x) = 2^x$$

Domain: $x \in \mathbb{R}$
(i.e. all real values)

No

Yes

$$f(x) = \sqrt{x}$$

No

Yes

Domain: $x \in \mathbb{R}$

We can't square root a negative number, but the input set is \mathbb{R} , so some inputs don't map to a value.

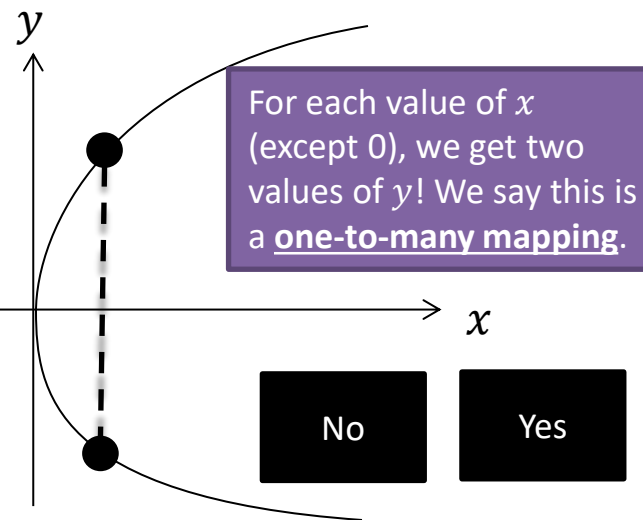
$$f(x) = \pm\sqrt{x}$$

Domain: $x \geq 0$

No

Yes

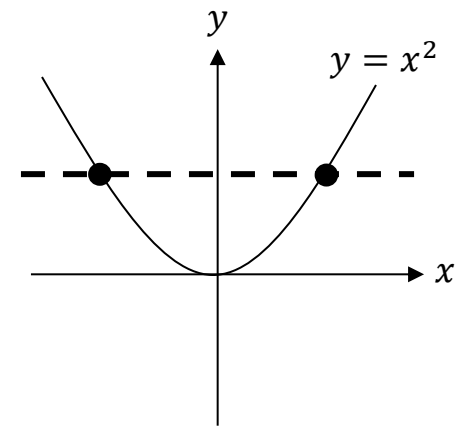
$f(4) = 2$ but $f(4) = -2$ also. This is one-to-many so not a function.



One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

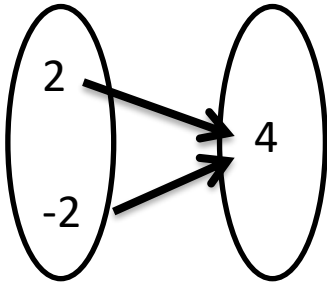
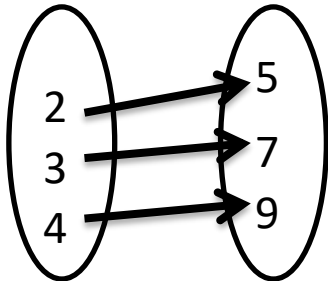
Type	Description	Example
Many-to-one function	?	?
One-to-one function	?	?

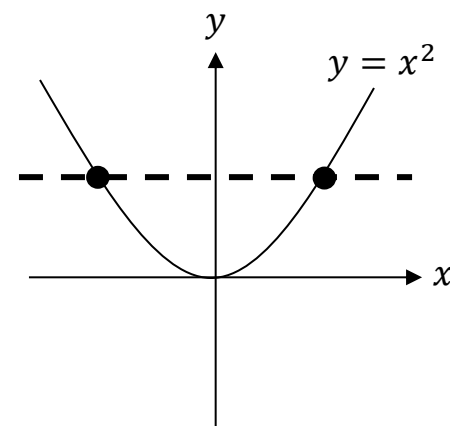


You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

One-to-one vs Many-to-one

While functions permit an input only to be mapped to one output, there's nothing stopping multiple different inputs mapping to the same output.

Type	Description	Example
Many-to-one function	Multiple inputs can map to the same output. 	$f(x) = x^2$ e.g. $f(2) = 4$ $f(-2) = 4$
One-to-one function	Each output has one input and vice versa. 	$f(x) = 2x + 1$



You can use the 'horizontal ray test' to see if a function is one-to-one or many-to-one.

Further Examples

It is often helpful to sketch the function to reason about the range.

[Textbook] Find the range of each of the following functions.

- a) $f(x) = 3x - 2$, domain $\{1, 2, 3, 4\}$
- b) $g(x) = x^2$, domain $\{x \in \mathbb{R}, -5 \leq x \leq 5\}$
- c) $h(x) = \frac{1}{x}$, domain $\{x \in \mathbb{R}, 0 < x \leq 3\}$

State if the functions are one-to-one or many-to-one.

We use x to refer to the input, and $f(x)$ to refer to the output.

Thus your ranges should be in terms of $f(x)$.

a

?

b

?

c

?

Further Examples

It is often helpful to sketch the function to reason about the range.

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State if the functions are one-to-one or many-to-one.

We use x to refer to the input, and $f(x)$ to refer to the output.

Thus your ranges should be in terms of $f(x)$.

a

$$f(1) = 1$$

$$f(2) = 4$$

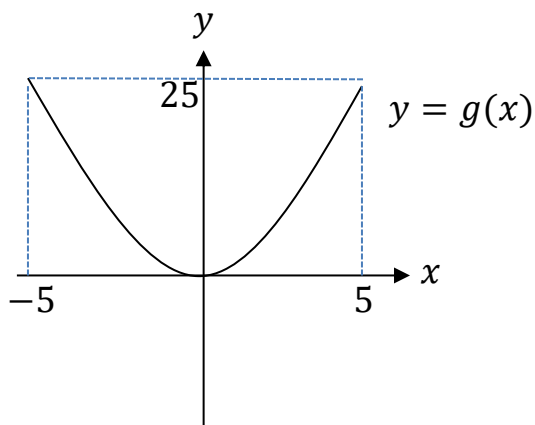
$$f(3) = 7$$

$$f(4) = 10$$

Therefore range is $\{1, 4, 7, 10\}$

$f(x)$ is one-to-one.

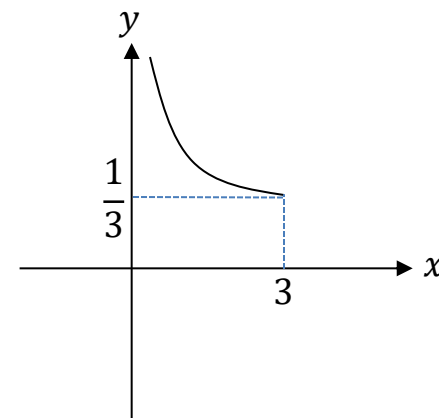
b



Using graph, range is
 $0 \leq g(x) \leq 25$

$g(x)$ is many-to-one.

c



Using graph, range is
 $h(x) \geq \frac{1}{3}$

$h(x)$ is one-to-one.

Piecewise Functions

A 'piecewise function' is one which is defined in parts: we can use different rules for different intervals within the domain.

[Textbook] The function $f(x)$ is defined by

$$f: x \rightarrow \begin{cases} 5 - 2x, & x < 1 \\ x^2 + 3, & x \geq 1 \end{cases}$$

- a) Sketch $y = f(x)$, and state the range of $f(x)$.
- b) Solve $f(x) = 19$

a

?

b

?

Piecewise Functions

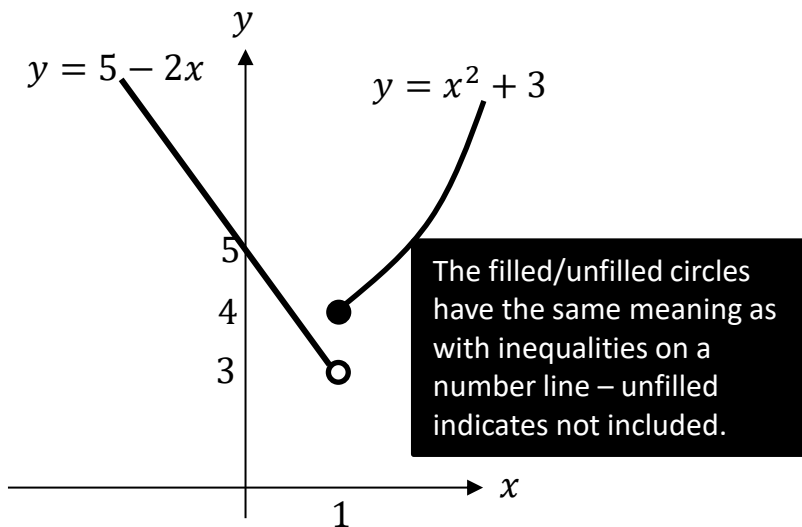
A 'piecewise function' is one which is defined in parts: we can use different rules for different intervals within the domain.

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- a) Sketch $y = f(x)$, and state the range of $f(x)$.
- b) Solve $f(x) = 19$

a



The filled/unfilled circles have the same meaning as with inequalities on a number line – unfilled indicates not included.

$f(x) > 3$
(as the 3 is not included)

b

Using the graph, the range is $f(x) > 3$

When $x \geq 1$:

$$x^2 + 3 = 19$$

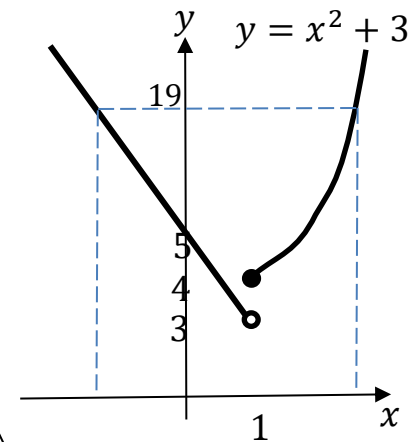
$$x = \pm 4$$

$$x = 4$$

When $x < 1$:

$$5 - 2x = 19$$

$$x = -7$$



$x = -2$ corresponds to a part of the curve which was never used.

Test Your Understanding

Edexcel C4 June 2012 Q6a

The function f is defined by

$$f: x \rightarrow e^x + 2, \quad x \in \mathbb{R}$$

State the range of f .

?

Edexcel C4 June 2010 Q4d

The function g is defined by

$$g: x \rightarrow x^2 - 4x + 1, \quad x \in \mathbb{R}, 0 \leq x \leq 5$$

Find the range of g .

Hint: Identify the minimum point first, as this may or may not affect the range.

Extra Hint: Carefully consider the stated domain.

?

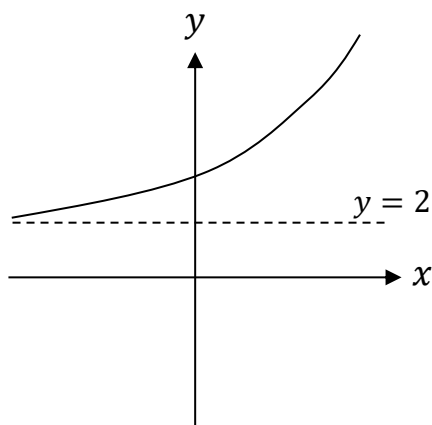
Test Your Understanding

Edexcel C4 June 2012 Q6a

The function f is defined by

$$f: x \rightarrow e^x + 2, \quad x \in \mathbb{R}$$

State the range of f .



$$f(x) > 2$$

Notice the range doesn't include 2, as the line never reaches the asymptote.

Edexcel C4 June 2010 Q4d

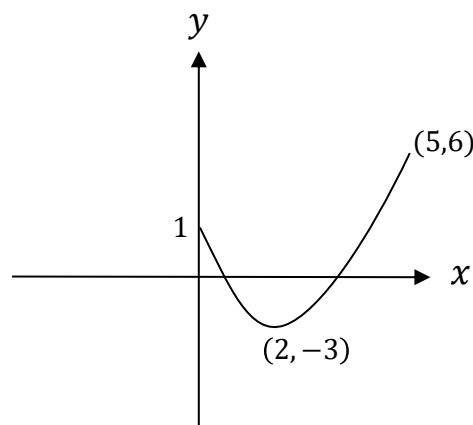
The function g is defined by

$$g: x \rightarrow x^2 - 4x + 1, \quad x \in \mathbb{R}, 0 \leq x \leq 5$$

Find the range of g .

Hint: Identify the minimum point first, as this may or may not affect the range.

Extra Hint: Carefully consider the stated domain.



$$\begin{aligned} x^2 - 4x + 1 \\ &= (x - 2)^2 - 4 + 1 \\ &= (x - 2)^2 - 3 \end{aligned}$$

So minimum point is
 $(2, -3)$

At two end points of curve:

$$f(0) = 1$$

$$f(5) = 6$$

Therefore range:

$$-3 \leq f(x) \leq 6$$

Summary of Domain/Range

It is important that you can identify the range for common graphs, using a suitable sketch:

$$f(x) = x^2, \quad x \in \mathbb{R}$$

$$\text{Range: } f(x) \geq 0$$

$$f(x) = \frac{1}{x}, \quad x \in \mathbb{R}, x \neq 0$$

$$\text{Range: } f(x) \neq 0$$

$$f(x) = \ln x, \quad x \in \mathbb{R}, x > 0$$

$$\text{Range: } f(x) \in \mathbb{R}$$

$$f(x) = e^x, \quad x \in \mathbb{R},$$

$$\text{Range: } f(x) > 0$$

$$f(x) = x^2 + 2x + 9, \quad x \in \mathbb{R}$$

$$\text{Range: } f(x) \geq 8$$

Be careful in noting the domain – it may be ‘restricted’, which similarly restricts the range. Again, use a sketch!

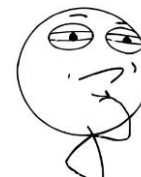
$$f(x) = x^2, \quad x \in \mathbb{R}, -1 \leq x \leq 4$$

$$\text{Range: } 0 \leq f(x) \leq 16$$

Just for your interest...

What is the difference between the notation

$$f(x) = 2x + 1 \text{ and } f: x \rightarrow 2x + 1?$$



$f: x \rightarrow 2x + 1$ means “the value of f is a mapping from x to $2x + 1$ ”.

You're used to variables, e.g. x , representing numerical values. But we've also seen that the value of a variable can be a vector, e.g. $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, sets, e.g. $A = \{1, 2, 3\}$ and so on. So when we use f on its own, its 'value' is a mapping, in this case with the value $x \rightarrow 2x + 1$.

This notation therefore places more emphasis on the value of f , and its 'value' as a mapping.

$f(x) = 2x + 1$ means “the output of f is $2x + 1$ ”.

It's easy to think that the notation “ $f(x)$ ” refers to the function. It doesn't! The f is the function, and the “ (x) ” appendage obtains the output of the function when the input is x . Therefore $f(x)$ refers specifically to the output of the function, which is why we write the range of a function in terms of $f(x)$ (and not in terms of f).

This notation therefore places more emphasis on the output of f .

Consequence 1

To solve an equation means to find the values of the variables, e.g. the “solution” of $2x + 1 = 5$ is $x = 2$.

To solve a **functional equation** means to find the ‘values’ of f .

$$\text{Solve } f(x + y) = f(x)f(y)$$

One solution to this equation is $f: x \rightarrow 2^x$ because $f(x + y) = 2^{x+y}$ and $f(x)f(y) = 2^x 2^y = 2^{x+y}$. To fully solve this functional equation means to find **all** functions which satisfy the equation.

See <http://www.drfrstmaths.com/resources/resource.php?rid=165>

Consequence 2

A bit of Computer Science!

In many programming languages, we can pass functions as the parameters of a method, when a variable is allowed to have a function as its value.

We could code a function `map` which takes a list, say a , and applies a function f to each item of this list.

e.g. `map (x→x+1, [1, 2, 3])` would output `[2, 3, 4]`.

```
function map(f, a) {  
  let b be a new list  
  for(i from 1 to size(a)) {  
    bi = f(ai)  
  }  
  return b  
}
```

Exercise 2.2

Pearson Pure Mathematics Year 2/AS

Page 7

Homework Exercise

1 For each of the following functions:

- i draw the mapping diagram
- ii state if the function is one-to-one or many-to-one
- iii find the range of the function.

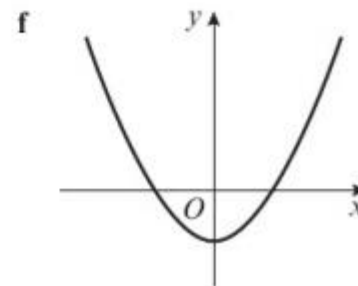
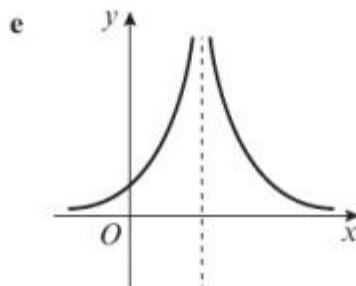
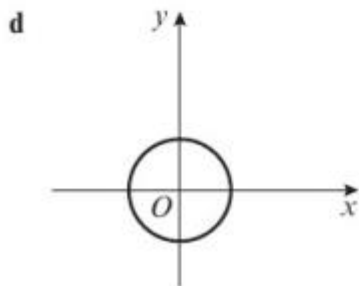
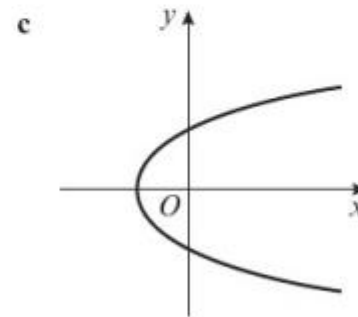
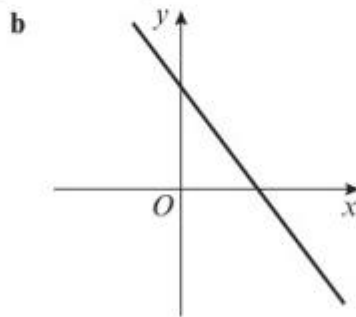
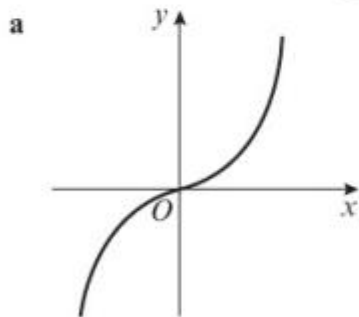
a $f(x) = 5x - 3$, domain $\{x = 3, 4, 5, 6\}$

b $g(x) = x^2 - 3$, domain $\{x = -3, -2, -1, 0, 1, 2, 3\}$

c $h(x) = \frac{7}{4 - 3x}$, domain $\{x = -1, 0, 1\}$

2 For each of the following mappings:

- i State whether the mapping is one-to-one, many-to-one or one-to-many.
- ii State whether the mapping could represent a function.



Homework Exercise

3 Calculate the value(s) of a , b , c and d given that:

a $p(a) = 16$ where $p: x \mapsto 3x - 2, x \in \mathbb{R}$

b $q(b) = 17$ where $q: x \mapsto x^2 - 3, x \in \mathbb{R}$

c $r(c) = 34$ where $r: x \mapsto 2(2^x) + 2, x \in \mathbb{R}$

d $s(d) = 0$ where $s: x \mapsto x^2 + x - 6, x \in \mathbb{R}$

4 For each function:

i represent the function on a mapping diagram, writing down the elements in the range

ii state whether the function is one-to-one or many-to-one.

a $f(x) = 2x + 1$ for the domain $\{x = 1, 2, 3, 4, 5\}$

b $g: x \mapsto \sqrt{x}$ for the domain $\{x = 1, 4, 9, 16, 25, 36\}$

c $h(x) = x^2$ for the domain $\{x = -2, -1, 0, 1, 2\}$

d $j: x \mapsto \frac{2}{x}$ for the domain $\{x = 1, 2, 3, 4, 5\}$

e $k(x) = e^x + 3$ for the domain $\{x = -2, -1, 0, 1, 2\}$

Notation

Remember, \sqrt{x} means the positive square root of x .

5 For each function:

i sketch the graph of $y = f(x)$

ii state the range of $f(x)$

iii state whether $f(x)$ is one-to-one or many-to-one.

a $f: x \mapsto 3x + 2$ for the domain $\{x \geq 0\}$

b $f(x) = x^2 + 5$ for the domain $\{x \geq 2\}$

c $f: x \mapsto 2 \sin x$ for the domain $\{0 \leq x \leq 180\}$

d $f: x \mapsto \sqrt{x+2}$ for the domain $\{x \geq -2\}$

e $f(x) = e^x$ for the domain $\{x \geq 0\}$

f $f(x) = 7 \log x$, for the domain, $\{x \in \mathbb{R}, x > 0\}$

Homework Exercise

- 6 The following mappings f and g are defined on all the real numbers by

$$f(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x \geq 4 \end{cases} \quad g(x) = \begin{cases} 4 - x, & x < 4 \\ x^2 + 9, & x > 4 \end{cases}$$

- a** Explain why $f(x)$ is a function and $g(x)$ is not. **b** Sketch $y = f(x)$.
c Find the values of: **i** $f(3)$ **ii** $f(10)$ **d** Solve $f(a) = 90$.

- 7 The function s is defined by

$$s(x) = \begin{cases} x^2 - 6, & x < 0 \\ 10 - x, & x \geq 0 \end{cases}$$

- a** Sketch $y = s(x)$.
b Find the value(s) of a such that $s(a) = 43$.
c Solve $s(x) = x$.

Problem-solving

The solutions of $s(x) = x$ are the values in the domain that get mapped to themselves in the range.

- 8 The function p is defined by

$$p(x) = \begin{cases} e^{-x}, & -5 \leq x < 0 \\ x^3 + 4, & 0 \leq x \leq 4 \end{cases}$$

- a** Sketch $y = p(x)$.
b Find the values of a , to 2 decimal places, such that $p(a) = 50$.

(3 marks)

(4 marks)

Homework Exercise

- 9 The function h has domain $-10 \leq x \leq 6$, and is linear from $(-10, 14)$ to $(-4, 2)$ and from $(-4, 2)$ to $(6, 27)$.

a Sketch $y = h(x)$. (2 marks)

b Write down the range of $h(x)$. (1 mark)

c Find the values of a , such that $h(a) = 12$. (4 marks)

Problem-solving

The graph of $y = h(x)$ will consist of two line segments which meet at $(-4, 2)$.

- 10 The function g is defined by $g(x) = cx + d$ where c and d are constants to be found. Given $g(3) = 10$ and $g(8) = 12$ find the values of c and d .

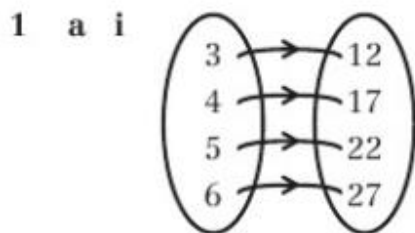
- 11 The function f is defined by $f(x) = ax^3 + bx - 5$ where a and b are constants to be found. Given that $f(1) = -4$ and $f(2) = 9$, find the values of the constants a and b .

- 12 The function h is defined by $h(x) = x^2 - 6x + 20$ and has domain $x \geq a$. Given that $h(x)$ is a one-to-one function find the smallest possible value of the constant a . (6 marks)

Problem-solving

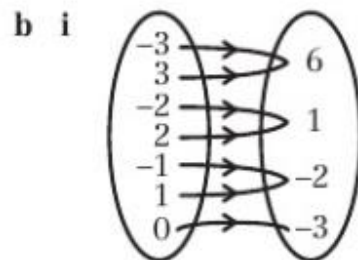
First complete the square for $h(x)$.

Homework Answers



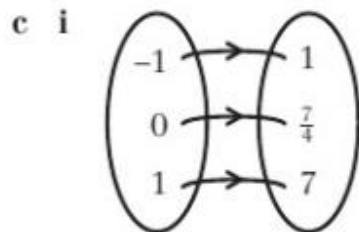
ii one-to-one

iii $\{f(x) = 12, 17, 22, 27\}$



ii many-to-one

iii $\{g(x) = -3, -2, 1, 6\}$

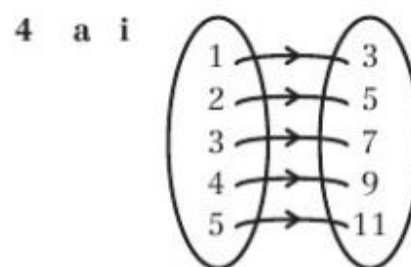


ii one-to-one

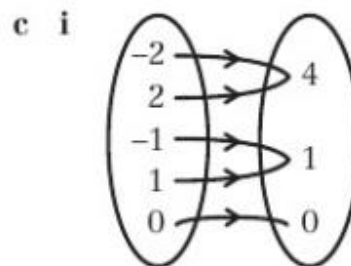
iii $\{h(x) = 1, \frac{7}{4}, 7\}$

- 2 a i one-to-one ii function
 b i one-to-one ii function
 c i one-to-many ii not a function
 d i one to many ii not a function
 e i one to one
 ii not valid at the asymptote, so not a function.
 f i many to one ii function

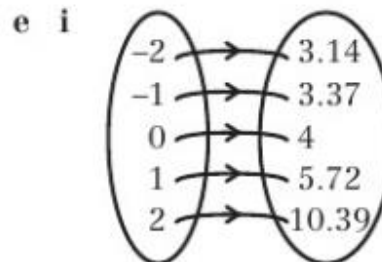
3 a 6 b $\pm 2\sqrt{5}$ c 4 d 2, -3



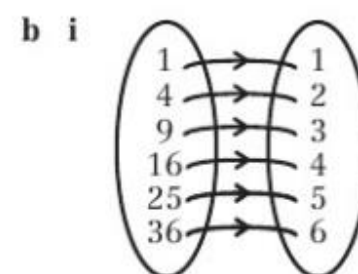
ii one-to-one



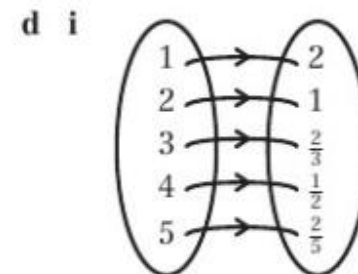
ii many-to-one



ii one-to-one

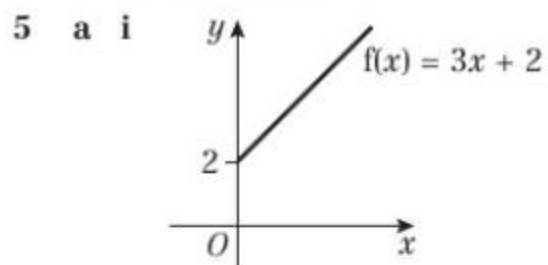


ii one-to-one

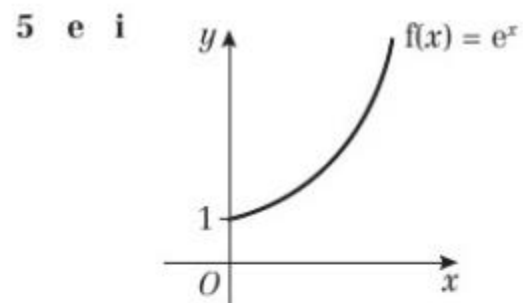


ii one-to-one

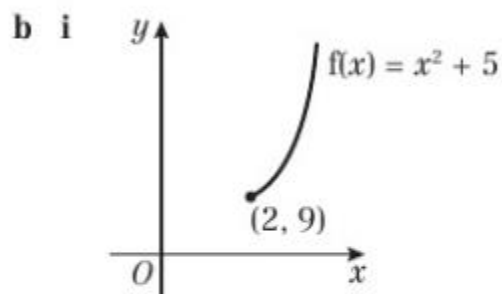
Homework Answers



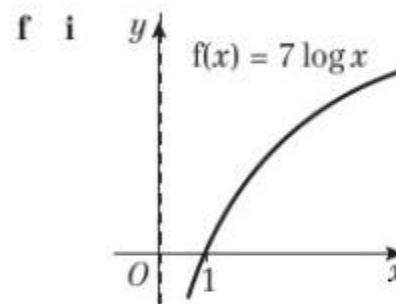
- ii $f(x) \geq 2$
 iii one-to-one



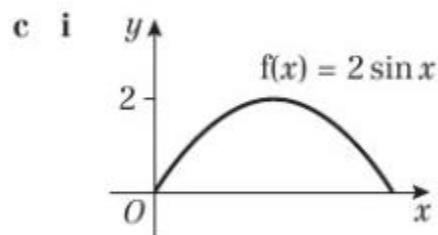
- ii $f(x) \geq 1$
 iii one-to-one



- ii $f(x) \geq 9$
 iii one-to-one

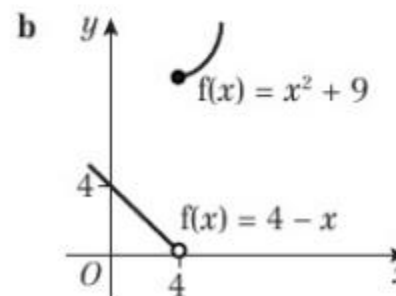


- ii $f(x) \in \mathbb{R}$
 iii one-to-one

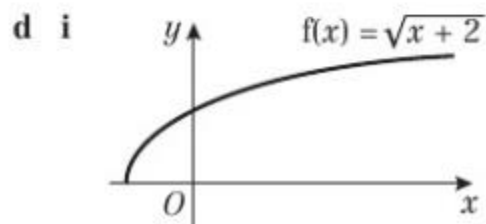


- ii $0 \leq f(x) \leq 2$
 iii many-to-one

6 a $g(x)$ is not a function because it is not defined for $x = 4$



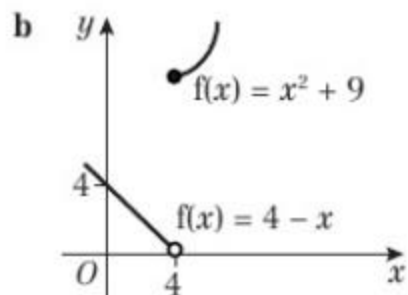
- c i 1 ii 109
 d $a = -86$ or $a = 9$



- ii $f(x) \geq 0$
 iii one-to-one

Homework Answers

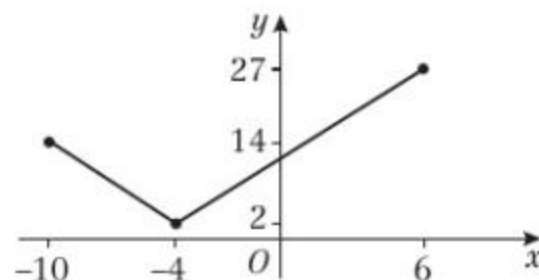
6 a $g(x)$ is not a function because it is not defined for $x = 4$



c i 1 ii 109

d $a = -86$ or $a = 9$

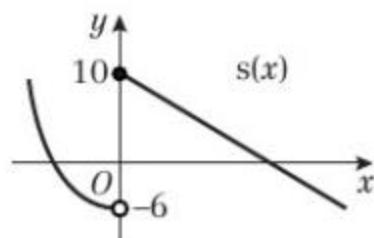
9 a



b Range $\{2 \leq h(x) \leq 27\}$

c $a = -9, a = 0$

7 a



b -7

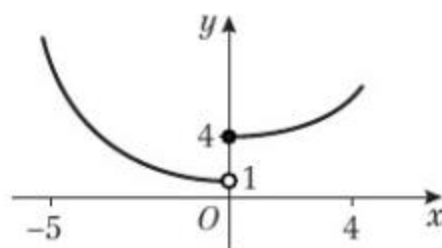
c -2 and 5

10 $c = \frac{2}{5}, d = \frac{44}{5}$

11 $a = 2, b = -1$

12 $a = 3$

8 a



b $a = -3.91$ or $a = 3.58$