P1 Chapter 2: Quadratics

Discriminants

Starter

How many **distinct** real solutions do each of the following have?

$x^2 - 12x + 36 = 0$

?

$x^2 + x + 3 = 0$

?

$x^2 - 2x - 1 = 0$

?

Starter

How many **distinct** real solutions do each of the following have?

$x^2 - 12x + 36 = 0$ $\quad$ $\boldsymbol{x = 6}$ **(1 distinct solution)**

$x^2 + x + 3 = 0$ $\quad$ $x = \frac{-1 \pm \sqrt{-11}}{2}$

We can't square root -11,
Therefore no real solutions.

$x^2 - 2x - 1 = 0$ $\quad$ $\boldsymbol{x = 1 \pm \sqrt{2}}$ **(2 distinct solutions)**

The Discriminant

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Fro Note: Roots of a **function** f are the values of x such that $f(x) = 0$.
Similarly the roots of an **equation** are solutions to an equation in the form $f(x) = 0$

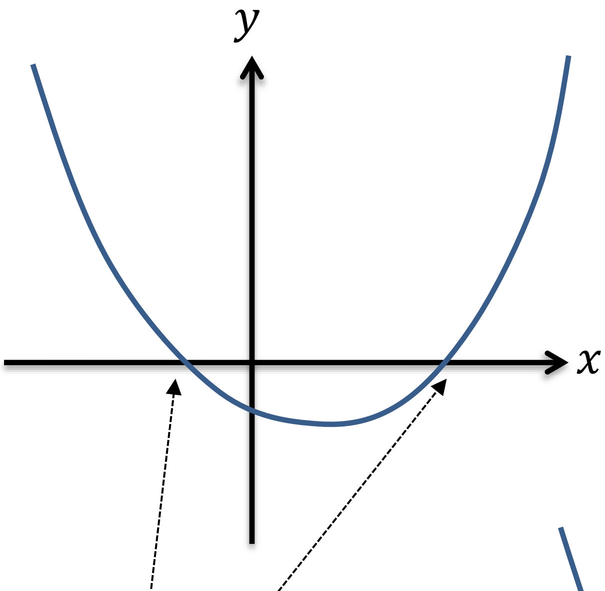
Looking at this formula, when in general do you think we have:

- No real roots? $\boldsymbol{b^2 - 4ac < 0}$
- Equal roots? $\boldsymbol{b^2 - 4ac = 0}$
- Two distinct roots? $\boldsymbol{b^2 - 4ac > 0}$

Because adding 0 or subtracting 0 in the quadratic formula gives the same value.

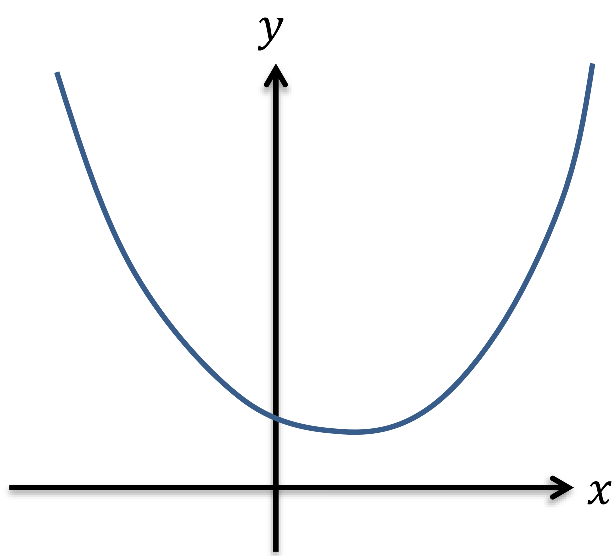
$b^2 - 4ac$ is known as the discriminant.

Discriminant Types



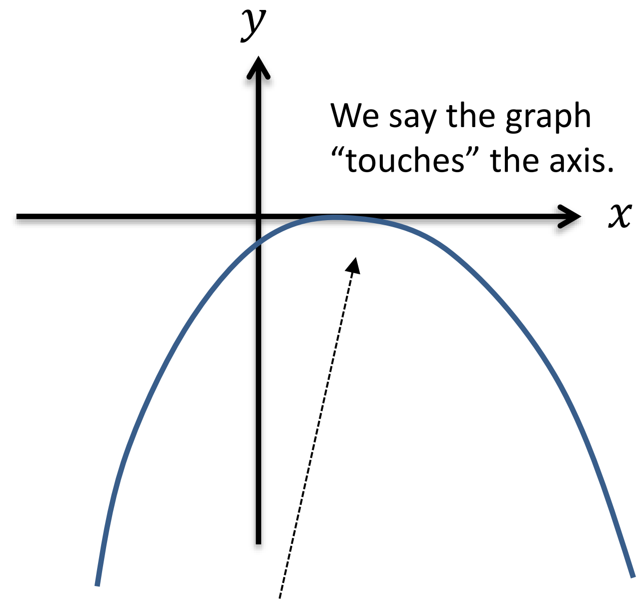
Distinct real roots

$$b^2 - 4ac > 0$$



No real roots

$$b^2 - 4ac < 0$$

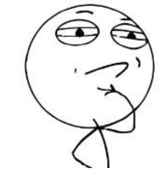


Equal roots

$$b^2 - 4ac = 0$$

Just for your interest...

Why do we say "Equal Roots" not "One root"?



$$x^2 - 12x + 36 = 0$$

Using the quadratic formula gives us the same value in both $+$ and $-$ cases: $x = 6$.

You might wonder why we say "*it has one repeated root*" or "*it has equal roots*", i.e. indicating we have **2 roots** (but with the same value). Why not say it has 1 root?

It is due to the **Fundamental Theorem of Algebra**:

"Every polynomial of order n has exactly n roots."

A **polynomial** is an expression with non-negative integer powers of x, i.e. $a + bx + cx^2 + dx^3 + \cdots$ All linear, quadratic and cubic expressions are examples of polynomials.

The **order** of a polynomial is its highest power of x. So the order of a quadratic is 2, and a cubic 3.

These roots might be repeated or might not be 'real' roots. $\sqrt{-1}$ is known as a **complex number**, which you will encounter if you do FM. But **it is still a value**!

The theorem means that a quadratic (order 2) will **always** have 2 roots. This is why you should say "no **real** roots" when $b^2 - 4ac < 0$ rather than "no roots", because there are still roots – it's just they're not 'real'! Similarly we must say "*equal roots*" because there are still 2 roots.

Despite the theorem being a simple statement, it was only until 1806 that it was first proven by **Argand**. Clearly by using the quadratic formula we can show a quadratic equation has 2 roots. We can use similar formulas to show that a cubic has 3 roots and a quartic 4 roots. But there is **provably** no such formulae for order 5 (quintics) and beyond. So we have to prove for example that 5 roots exist for a quintic, despite us having no way to find these exact roots!

One side result of the Fundamental Theorem of Algebra is that every polynomial can be written as a product of linear and/or quadratic expressions.

Leibniz claimed in 1702 that a polynomial of the form $x^4 + a^4$ cannot be written in this way. He then got completely burned by Euler in 1742 who managed to do so:

$$x^4 + a^4 = \left(x^2 + a\sqrt{2}\,x + a^2\right)\left(x^2 - a\sqrt{2}\,x + a^2\right)$$

There are various other 'Fundamental Laws'. The '**FL of Arithmetic**' you encountered at KS3, which states that "*every positive integer > 1 can be written as a product of primes in one way only*". You will encounter the '**FL of Calculus**' in Chapter 13.

Quickfire Questions

Equation	Discriminant	Number of Distinct Real Roots
$x^2 + 3x + 4 = 0$	?	?
$x^2 - 4x + 1 = 0$	?	?
$x^2 - 4x + 4 = 0$	?	?
$2x^2 - 6x - 3 = 0$	?	?
$x - 4 - 3x^2 = 0$	?	?
$1 - x^2 = 0$	?	?

Quickfire Questions

Equation	Discriminant	Number of Distinct Real Roots
$x^2 + 3x + 4 = 0$	-7	0
$x^2 - 4x + 1 = 0$	12	2
$x^2 - 4x + 4 = 0$	0	1
$2x^2 - 6x - 3 = 0$	60	2
$x - 4 - 3x^2 = 0$	-47	0
$1 - x^2 = 0$	4	2

Problems involving the discriminant

8. The equation $x^2 + 2px + (3p + 4) = 0$, where p is a positive constant, has equal roots.

(*a*) Find the value of p. **(4)**

(*b*) For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$. **(2)**

a) $a =$? $b =$? $c =$?

?

b) ?

Tip: Always start by writing out a, b and c explicitly.

Problems involving the discriminant

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(a) Find the value of p.

(4)

(b) For this value of p, solve the equation $x^2 + 2px + (3p + 4) = 0$.

(2)

a) $a = 1, \quad b = 2p, \quad c = 3p + 4$

$$(2p)^2 - 4(1)(3p+4) = 0$$
$$4p^2 - 12p - 16 = 0$$
$$p^2 - 3p - 4 = 0$$
$$(p+1)(p-4) = 0$$
$$p = 4$$

Tip: Always start by writing out a, b and c explicitly.

b) When $p = 4$: $\quad x^2 + 8x + 16 = 0$

$$(x+4)^2 = 0, \qquad x = -4$$

Test Your Understanding

$$x^2 + 5kx + (10k + 5) = 0$$

where k is a constant.

Given that this equation has equal roots, determine the value of k.

?

Find the range of values of k for which $x^2 + 6x + k = 0$ has two distinct real solutions.

?

Test Your Understanding

$$x^2 + 5kx + (10k + 5) = 0$$

where k is a constant.

Given that this equation has equal roots, determine the value of k.

$$a = 1, \qquad b = 5k, \qquad c = 10k + 5$$
$$(5k)^2 - 4(1)(10k + 5) = 0$$
$$25k^2 - 40k - 20 = 0$$
$$5k^2 - 8k - 4 = 0$$
$$(5k + 2)(k - 2) = 0$$
$$k = 2$$

Find the range of values of k for which $x^2 + 6x + k = 0$ has two distinct real solutions.

$$a = 1, b = 6, c = k$$
$$36 - 4(1)(k) = 36 - 4k > 0$$
$$36 > 4k$$
$$k < 9$$

Exercise 2.5

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Extension Questions:

1 [MAT 2009 1C] Given a real constant c, the equation

$$x^4 = (x - c)^2$$

Has four real solutions (including possible repeated roots) for:

A) $c \leq \frac{1}{4}$

B) $-\frac{1}{4} \leq c \leq \frac{1}{4}$

C) $c \leq -\frac{1}{4}$

D) all values of c

?

2 [MAT 2006 1B] The equation $(2 + x - x^2)^2 = 16$ has how many real root(s)?

?

3 [MAT 2011 1B] A rectangle has perimeter P and area A. The values P and A must satisfy:

A) $P^3 > A$

B) $A^2 > 2P + 1$

C) $P^2 \geq 16A$

D) $PA > A + P$

?

Exercise 2.5

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Extension Questions:

1 [MAT 2009 1C] Given a real constant c, the equation

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C) $c \leq -\frac{1}{4}$
D) all values of c

Square rooting: $x^2 = \pm(x-c)$
Case 1:

$$x^2 - x + c = 0$$

Discriminant: $1 - 4c \geq 0 \therefore c \leq \frac{1}{4}$
Case 2:

$$x^2 + x - c = 0$$

Discriminant: $1 + 4c \geq 0 \therefore c \geq -\frac{1}{4}$
Answer is B.

2 [MAT 2006 1B] The equation $(2 + x - x^2)^2 = 16$ has how many real root(s)?

$$2 + x - x^2 = \pm 4$$

First case:

$$2 + x - x^2 = 4$$
$$x^2 - x + 2 = 0$$

As $b^2 - 4ac < 0$ no real roots.

Second case:

$$2 + x - x^2 = -4$$
$$x^2 - x - 6 = 0$$
$$(x+2)(x-3) = 0$$
$$x = -2 \text{ or } x = 3$$

So 2 distinct real roots.

3 [MAT 2011 1B] A rectangle has perimeter P and area A. The values P and A must satisfy:

A) $P^3 > A$
B) $A^2 > 2P + 1$
C) $P^2 \geq 16A$
D) $PA > A + P$

Let x and y be the width and height. Then $A = xy, P = 2x + 2y$. Substituting:

$$P = 2x + 2\left(\frac{A}{x}\right)$$
$$Px = 2x^2 + 2A$$
$$2x^2 - Px + 2A = 0$$

Discriminant: $a = 2, b = -P, c = 2A$

$$(-P)^2 - 4(2)(2A) \geq 0$$
$$P^2 - 16A \geq 0$$
$$P^2 \geq 16A$$

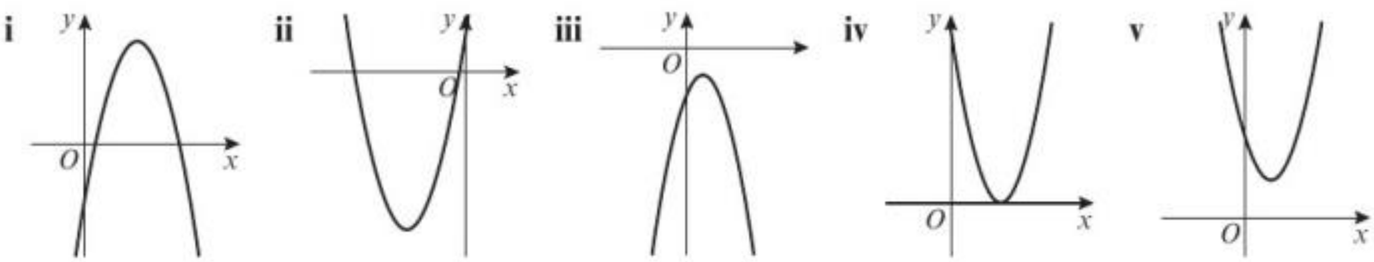
Homework Exercise

1 **a** Calculate the value of the discriminant for each of these five functions:

i $f(x) = x^2 + 8x + 3$ **ii** $g(x) = 2x^2 - 3x + 4$ **iii** $h(x) = -x^2 + 7x - 3$

iv $j(x) = x^2 - 8x + 16$ **v** $k(x) = 2x - 3x^2 - 4$

b Using your answers to part **a**, match the same five functions to these sketch graphs.



2 Find the values of k for which $x^2 + 6x + k = 0$ has two real solutions. **(2 marks)**

3 Find the value of t for which $2x^2 - 3x + t = 0$ has exactly one solution. **(2 marks)**

4 Given that the function $f(x) = sx^2 + 8x + s$ has equal roots, find the value of the positive constant s. **(2 marks)**

5 Find the range of values of k for which $3x^2 - 4x + k = 0$ has no real solutions. **(2 marks)**

6 The function $g(x) = x^2 + 3px + (14p - 3)$, where p is an integer, has two equal roots.

a Find the value of p. **(2 marks)**

b For this value of p, solve the equation $x^2 + 3px + (14p - 3) = 0$. **(2 marks)**

Homework Exercise

7 $h(x) = 2x^2 + (k + 4)x + k$, where k is a real constant.

  a Find the discriminant of $h(x)$ in terms of k. **(3 marks)**

  b Hence or otherwise, prove that $h(x)$ has two distinct real roots for all values of k. **(3 marks)**

Problem-solving

If a question part says 'hence or otherwise' it is usually easier to use your answer to the previous question part.

Challenge

a Prove that, if the values of a and c are given and non-zero, it is always possible to choose a value of b so that $f(x) = ax^2 + bx + c$ has distinct real roots.

b Is it always possible to choose a value of b so that $f(x)$ has equal roots? Explain your answer.

Homework Answers

1	**a**	**i** 52		**ii** −23		**iii** 37	
		iv 0		**v** −44			
	b	**i** h(x)		**ii** f(x)		**iii** k(x)	
		iv j(x)		**v** g(x)			

2 $k < 9$

3 $t = \frac{9}{8}$

4 $s = 4$

5 $k > \frac{4}{3}$

6 **a** $p = 6$ **b** $x = -9$

7 **a** $k^2 + 16$

b k^2 is always positive so $k^2 + 16 > 0$

Challenge

a Need $b^2 > 4ac$. If $a, c > 0$ or $a, c < 0$, choose b such that $b > \sqrt{4ac}$. If $a > 0$ and $c < 0$ (or vice versa), then $4ac < 0$, so $4ac < b^2$ for all b.

b Not if one of a or c are negative as this would require b to be the square root of a negative number. Possible if both negative or both positive.