P2 Chapter 4: Binomial Terms

General Binomials

Dealing with $(a + bx)^n$

Find first four terms in the binomial expansion of $\sqrt{4 + x}$
State the values of x for which the expansion is valid.

$$(4 + x)^{\frac{1}{2}} = \left[4\left(1 + \frac{1}{4}x\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(1 + \frac{1}{4}x\right)^{\frac{1}{2}}$$

$$= 2(1 + \left(\frac{1}{2}\right)\left(\frac{1}{4}x\right) + \frac{\frac{1}{2} \times -\frac{1}{2}}{2!}\left(\frac{1}{4}x\right)^2 + \frac{\left(\frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2}\right)}{3!}\left(\frac{1}{4}x\right)^3$$

$$= 2\left(1 + \frac{1}{8}x - \frac{1}{128}x^2 + \frac{1}{1024}x^3 - \cdots\right)$$

$$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 - \cdots$$

We need it in the form $(1 + x)^n$. So factorise the 4 out.

Valid if $\left|\frac{1}{4}x\right| < 1 \quad \Rightarrow \quad |x| < 4$

Remember for the expansion of $(1 + x)^n$ to be valid then $|x| < 1$. In this case the 'x' is in fact $\frac{1}{4}x$.

Quickfire First Step

What would be the first step in finding the Binomial expansion of each of these?

Binomial expansion valid if:

$(2+x)^{-3} \Rightarrow$? | ?

$(9+2x)^{\frac{1}{2}} \Rightarrow$? | ?

$(8-x)^{\frac{1}{3}} \Rightarrow$? | ?

$(5-2x)^{-3} \Rightarrow$? | ?

$(16+3x)^{-\frac{1}{2}}$? | ?

Quickfire First Step

What would be the first step in finding the Binomial expansion of each of these?

Binomial expansion valid if:

$$(2+x)^{-3} \Rightarrow \mathbf{\frac{1}{8}\left(1+\frac{1}{2}x\right)^{-3}} \qquad \left|\frac{1}{2}x\right| < 1 \;\rightarrow\; |x| < 2$$

$$(9+2x)^{\frac{1}{2}} \Rightarrow \mathbf{3\left(1+\frac{2}{9}x\right)^{\frac{1}{2}}} \qquad \left|\frac{2}{9}x\right| < 1 \;\rightarrow\; |x| < \frac{9}{2}$$

$$(8-x)^{\frac{1}{3}} \Rightarrow \mathbf{2\left(1-\frac{1}{8}x\right)^{\frac{1}{3}}} \qquad \left|\frac{1}{8}x\right| < 1 \;\rightarrow\; |x| < 8$$

$$(5-2x)^{-3} \Rightarrow \mathbf{\frac{1}{25}\left(1-\frac{2}{5}x\right)^{-3}} \qquad \left|\frac{2}{5}x\right| < 1 \;\rightarrow\; |x| < \frac{5}{2}$$

$$(16+3x)^{-\frac{1}{2}} \Rightarrow \mathbf{\frac{1}{4}\left(1+\frac{3}{16}x\right)^{-\frac{1}{2}}} \qquad \left|\frac{3}{16}x\right| < 1 \;\rightarrow\; |x| < \frac{16}{3}$$

Test Your Understanding

Edexcel C4 June 2013 (Withdrawn) Q1

(a) Find the binomial expansion of

$$\sqrt{(9 + 8x)}, \quad |x| < \frac{9}{8}$$

in ascending powers of x, up to and including the term in x^2.
Give each coefficient as a simplified fraction.

(5)

(b) Use your expansion to estimate the value of $\sqrt{(11)}$, giving your answer as a single fraction.

(3)

(a)

?

?

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(a) $\sqrt{(9+8x)} = (9+8x)^{\frac{1}{2}} = \underline{(9)^{\frac{1}{2}}}\left(1+\frac{8x}{9}\right)^{\frac{1}{2}} = \underline{3}\left(1+\frac{8x}{9}\right)^{\frac{1}{2}}$	$\underline{(9)^{\frac{1}{2}}}$ or $\underline{3}$ outside brackets	<u>B1</u>
	Expands $(1+**x)^{\frac{1}{2}}$ to give a simplified or an un-simplified $1+(\frac{1}{2})(**x)$;	M1;
$= 3\left[\underline{1+(\tfrac{1}{2})(**x);+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(**x)^2+\ldots}\right]$ with $** \neq 1$	A correct simplified or an un-simplified [..........] expansion with candidate's followed through $(**x)$	A1 $\sqrt{\ }$
$= 3\left[\underline{1+\left(\frac{1}{2}\right)\left(\frac{8x}{9}\right);+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\frac{8x}{9}\right)^2+\ldots}\right]$	Award SC M1 if you see $\frac{1}{2}(**x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(**x)^2$ or $1+\ldots+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(**x)^2$	
$= 3\left[1+\frac{4}{9}x;-\frac{8}{81}x^2+\ldots\right]$	$3\left[1+\frac{4}{9}x;\ldots\right]$ or SC $K\left[1+\frac{4}{9}x-\frac{8}{81}x^2+\ldots\right]$	A1 oe
$= 3+\frac{4}{3}x;-\frac{8}{27}x^2+\ldots$	$-\frac{8}{27}x^2$	A1

(b) $\sqrt{11} = \sqrt{(9+8x)} \Rightarrow \underline{x = \frac{1}{4}}$	$\underline{x=\frac{1}{4}}$	B1 oe
$\sqrt{11} \approx 3+\frac{4}{3}\left(\frac{1}{4}\right)-\frac{8}{27}\left(\frac{1}{4}\right)^2 \quad \left\{=3+\frac{1}{3}-\frac{1}{54}\right\}$	Substitutes their x into their binomial expansion	M1
$= 3\frac{17}{54} = \frac{179}{54}$	$3\frac{17}{54}$ or $\frac{179}{54}$.	A1

Exercise 4.2

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Extension

[AEA 2006 Q1]

(a) For $|y| < 1$, write down the binomial series expansion of $(1 - y)^{-2}$ in ascending powers of y up to and including the term in y^3.

(b) Hence, or otherwise, show that

$$1 + \frac{2x}{1+x} + \frac{3x^2}{(1+x)^2} + \cdots + \frac{rx^{r-1}}{(1+x)^{r-1}} + \cdots$$

can be written in the form $(a + x)^n$. Write down the values of the integers a and n.

(c) Find the set of values of x for which the series in part (b) is convergent.

(a) ? a

(b) ? b

(c) ? c

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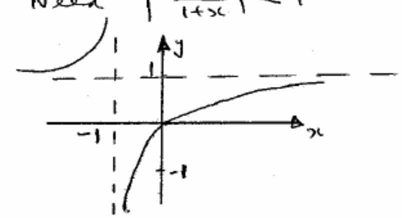
can be written in the form $(a+x)^n$. Write down the values of the integers a and n.

(c) Find the set of values of x for which the series in part (b) is convergent.

(a)	$(1-y)^{-2} = 1 + 2y + 3y^2 + 4y^3 + \dots$	B1 (1)
(b)	$S = 1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \dots$ Identify $y = \frac{x}{x+1}$	M1
	$\Rightarrow S = 1 + 2y + 3y^2 + \dots$	A1
	$= \left(1 - \frac{x}{1+x}\right)^{-2}$	
	$= \underline{(1+x)^{+2}}$, so $\underline{a = 1, n = 2}$	M1, A1 (4)
(c)	Need $\left\lvert \frac{x}{1+x} \right\rvert < 1$ correct condition	B1
	 Critical value is $\frac{x}{x+1} = -1$	M1
	$\Rightarrow x = -\frac{1}{2}$	
	$\therefore \underline{x > -\frac{1}{2}}$	A1 (3)

Homework Exercise

1 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $\sqrt{4+2x}$ **b** $\frac{1}{2+x}$ **c** $\frac{1}{(4-x)^2}$ **d** $\sqrt{9+x}$

e $\frac{1}{\sqrt{2+x}}$ **f** $\frac{5}{3+2x}$ **g** $\frac{1+x}{2+x}$ **h** $\sqrt{\frac{2+x}{1-x}}$

Hint Write part **g** as $1-\frac{1}{x+2}$

2 $f(x)=(5+4x)^{-2}$, $|x|<\frac{5}{4}$

Find the binomial expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction. **(5 marks)**

3 $m(x)=\sqrt{4-x}$, $|x|<4$

a Find the series expansion of $m(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(4 marks)**

b Show that, when $x=\frac{1}{9}$, the exact value of $m(x)$ is $\frac{\sqrt{35}}{3}$ **(2 marks)**

c Use your answer to part **a** to find an approximate value for $\sqrt{35}$, and calculate the percentage error in your approximation. **(4 marks)**

Homework Exercise

4 The first three terms in the binomial expansion of $\frac{1}{\sqrt{a+bx}}$ are $3 + \frac{1}{3}x + \frac{1}{18}x^2 + \ldots$

a Find the values of the constants a and b.

b Find the coefficient of the x^3 term in the expansion.

5 $f(x) = \frac{3 + 2x - x^2}{4 - x}$

Prove that if x is sufficiently small, $f(x)$ may be approximated by $\frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2$.

6 **a** Expand $\frac{1}{\sqrt{5+2x}}$, where $|x| < \frac{5}{2}$, in ascending powers of x up to and including the term in x^2, giving each coefficient in simplified surd form. **(5 marks)**

b Hence or otherwise, find the first 3 terms in the expansion of $\frac{2x-1}{\sqrt{5+2x}}$ as a series in ascending powers of x. **(4 marks**

Homework Exercise

7 **a** Use the binomial theorem to expand $(16 - 3x)^{\frac{1}{4}}$, $|x| < \frac{16}{3}$ in ascending powers of x, up to and including the term in x^2, giving each term as a simplified fraction. **(4 marks)**

b Use your expansion, with a suitable value of x, to obtain an approximation to $\sqrt[4]{15.7}$. Give your answer to 3 decimal places. **(2 marks)**

8 $g(x) = \frac{3}{4 - 2x} - \frac{2}{3 + 5x}$, $|x| < \frac{1}{2}$

a Show that the first three terms in the series expansion of $g(x)$ can be written as $\frac{1}{12} + \frac{107}{72}x - \frac{719}{432}x^2$. **(5 marks)**

b Find the exact value of $g(0.01)$. Round your answer to 7 decimal places. **(2 marks)**

c Find the percentage error made in using the series expansion in part **a** to estimate the value of $g(0.01)$. Give your answer to 2 significant figures. **(3 marks)**

Homework Answers

1 **a** **i** $2+\frac{x}{2}-\frac{x^2}{16}+\frac{x^3}{64}$ **ii** $|x|<2$

b **i** $\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}$ **ii** $|x|<2$

c **i** $\frac{1}{16}+\frac{x}{32}+\frac{3x^2}{256}+\frac{x^3}{256}$ **ii** $|x|<4$

d **i** $3+\frac{x}{6}-\frac{x^2}{216}+\frac{x^3}{3888}$ **ii** $|x|<9$

e **i** $\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{8}x+\frac{3\sqrt{2}}{64}x^2-\frac{5\sqrt{2}}{256}x^3$ **ii** $|x|<2$

f $\frac{5}{3}-\frac{10}{9}x+\frac{20}{27}x^2-\frac{40}{81}x^3$ **ii** $|x|<\frac{3}{2}$

g **i** $\frac{1}{2}+\frac{1}{4}x-\frac{1}{8}x^2+\frac{1}{16}x^3$ **ii** $|x|<2$

h **i** $\sqrt{2}+\frac{3\sqrt{2}}{4}x+\frac{15\sqrt{2}}{32}x^2+\frac{51\sqrt{2}}{128}x^3$ **ii** $|x|<1$

2 $\frac{1}{25}-\frac{8}{125}x+\frac{48}{625}x^2-\frac{256}{3125}x^3$

3 **a** $2-\frac{x}{4}-\frac{x^2}{64}$

b $\mathrm{m}(x)=\sqrt{\frac{35}{9}}=\frac{\sqrt{35}}{\sqrt{9}}=\frac{\sqrt{35}}{3}$

c 5.91609 (correct to 5 decimal places), % error = 1.38×10^{-4}%

4 **a** $a=\frac{1}{9}, b=-\frac{2}{81}$ **b** $\frac{5}{486}$

5 For small values of x ignore powers of x^3 and higher.

$(4-x)^{-1}=\frac{1}{4}+\frac{x}{16}+\frac{x^2}{64}+\dots$

Multiply by $(3+2x-x^2)=\frac{3}{4}+\frac{x}{2}-\frac{x^2}{4}+\frac{3x}{16}+\frac{x^2}{8}+\frac{3x^2}{64}$

$=\frac{3}{4}+\frac{11}{16}x-\frac{5}{64}x^2$

6 **a** $\frac{1}{\sqrt{5}}-\frac{x}{5\sqrt{5}}+\frac{3x^2}{50\sqrt{5}}$ **b** $-\frac{1}{\sqrt{5}}+\frac{11x}{5\sqrt{5}}-\frac{23x^2}{50\sqrt{5}}$

7 **a** $2-\frac{3}{32}x-\frac{27}{4096}x^2$ **b** 1.991

8 **a** $\frac{3}{4-2x}=\frac{3}{4}+\frac{3x}{8}+\frac{3x^2}{16}, \frac{2}{3+5x}=\frac{2}{3}-\frac{10x}{9}+\frac{50x^2}{27}$

$\frac{3}{4-2x}-\frac{2}{3+5x}=\frac{1}{12}+\frac{107}{72}x-\frac{719}{432}x^2$

b 0.0980311

c 0.0032%