M2 Chapter 8: Further Kinematics

Vector Integration

Integrating Vectors

We can similarly integrate the $\boldsymbol{i}$ and $\boldsymbol{j}$ components to get from acceleration to velocity and velocity to displacement.

[Textbook] A particle P is moving in a plane. At time t seconds, its velocity $\boldsymbol{v}$ ms^{-1} is given by

$$\boldsymbol{v} = 3t\boldsymbol{i} + \frac{1}{2}t^2\boldsymbol{j}, \qquad t \geq 0$$

When $t = 0$, the position vector of P with respect to a fixed O is $(2\boldsymbol{i} - 3\boldsymbol{j})$ m. Find the position vector of P at time t seconds.

$$r = \int \begin{pmatrix} 3t \\ \frac{1}{2}t^2 \end{pmatrix} dt = \begin{pmatrix} \frac{3}{2}t^2 \\ \frac{1}{6}t^3 \end{pmatrix} + \boldsymbol{c}$$

The constant of integration is a vector.

When $t = 0$,

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \boldsymbol{c} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \rightarrow \quad \boldsymbol{c} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\therefore \boldsymbol{r} = \begin{pmatrix} \frac{3}{2}t^2 \\ \frac{1}{6}t^3 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}t^2 + 2 \\ \frac{1}{6}t^3 - 3 \end{pmatrix} \text{ m}$$

Further Example

[Textbook] A particle P is moving in a plane so that, at time t seconds, its acceleration is $(4\boldsymbol{i} - 2t\boldsymbol{j})$ ms^{-2}. When $t = 3$, the velocity of P is $6\boldsymbol{i}$ ms^{-1} and the position vector of P is $(20\boldsymbol{i} + 3\boldsymbol{j})$ m with respect to a fixed origin O. Find:

(a) the angle between the direction of motion of P and $\boldsymbol{i}$ when $t = 2$

(b) the distance of P from O when $t = 0$.

a

?

b

?

Further Example

[Textbook] A particle P is moving in a plane so that, at time t seconds, its acceleration is $(4\boldsymbol{i} - 2t\boldsymbol{j})$ ms^{-2}. When $t = 3$, the velocity of P is $6\boldsymbol{i}$ ms^{-1} and the position vector of P is $(20\boldsymbol{i} + 3\boldsymbol{j})$ m with respect to a fixed origin O. Find:

(a) the angle between the direction of motion of P and $\boldsymbol{i}$ when $t = 2$

(b) the distance of P from O when $t = 0$.

a The direction of motion is the velocity:

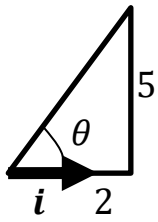
$$\boldsymbol{v} = \int \begin{pmatrix} 4 \\ -2t \end{pmatrix} dt = \begin{pmatrix} 4t \\ -t^2 \end{pmatrix} + c$$

When $t = 3$, $\boldsymbol{v} = \begin{pmatrix} 12 \\ -9 \end{pmatrix} + \boldsymbol{c} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ ← Use $t = 3$ to get constant of integration.

$$\therefore \boldsymbol{c} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 12 \\ -9 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$$

$$\therefore \boldsymbol{v} = \begin{pmatrix} 4t - 6 \\ -t^2 + 9 \end{pmatrix} ms^{-1}$$

When $t = 2$, $\boldsymbol{v} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$



$$\theta = \tan^{-1}\left(\frac{5}{2}\right) = 68.2°$$

b

$$\boldsymbol{r} = \int \begin{pmatrix} 4t - 6 \\ -t^2 + 9 \end{pmatrix} dt$$

$$= \begin{pmatrix} 2t^2 - 6t \\ -\frac{1}{3}t^3 + 9t \end{pmatrix} + \boldsymbol{c}$$

If $t = 3$, $\boldsymbol{r} = \begin{pmatrix} 0 \\ 18 \end{pmatrix} + \boldsymbol{c} = \begin{pmatrix} 20 \\ 3 \end{pmatrix}$

$$\therefore \boldsymbol{c} = \begin{pmatrix} 20 \\ -15 \end{pmatrix}$$

$$\therefore \boldsymbol{r} = \begin{pmatrix} 2t^2 - 6t + 20 \\ -\frac{1}{3}t^3 + 9t - 15 \end{pmatrix}$$

When $t = 0$, $\boldsymbol{r} = \begin{pmatrix} 20 \\ -15 \end{pmatrix}$

$$OP = \sqrt{20^2 + 15^2} = 25 \text{ m}$$

Test Your Understanding

Edexcel M2(Old) Jan 2013 Q4

At time t seconds the velocity of a particle P is $[(4t - 5)\mathbf{i} + 3\mathbf{j}]$ m s^{-1}. When $t = 0$, the position vector of P is $(2\mathbf{i} + 5\mathbf{j})$ m, relative to a fixed origin O.

(a) Find the value of t when the velocity of P is parallel to the vector $\mathbf{j}$.

(1)

(b) Find an expression for the position vector of P at time t seconds.

(4)

A second particle Q moves with constant velocity $(-2\mathbf{i} + c\mathbf{j})$ m s^{-1}. When $t = 0$, the position vector of Q is $(11\mathbf{i} + 2\mathbf{j})$ m. The particles P and Q collide at the point with position vector $(d\mathbf{i} + 14\mathbf{j})$ m.

(c) Find

(i) the value of c,

(ii) the value of d.

(5)

(a) ?

(b) ?

(c) ?

Test Your Understanding

Edexcel M2(Old) Jan 2013 Q4

At time t seconds the velocity of a particle P is $[(4t-5)\mathbf{i}+3\mathbf{j}]$ m s^{-1}. When $t=0$, the position vector of P is $(2\mathbf{i}+5\mathbf{j})$ m, relative to a fixed origin O.

(a) Find the value of t when the velocity of P is parallel to the vector $\mathbf{j}$.

(1)

(b) Find an expression for the position vector of P at time t seconds.

(4)

A second particle Q moves with constant velocity $(-2\mathbf{i}+c\mathbf{j})$ m s^{-1}. When $t=0$, the position vector of Q is $(11\mathbf{i}+2\mathbf{j})$ m. The particles P and Q collide at the point with position vector $(d\mathbf{i}+14\mathbf{j})$ m.

(c) Find

(i) the value of c,

(ii) the value of d.

(5)

(a)	$t=\frac{5}{4}$	M1
(b)	$\mathbf{r}=(2t^2-5t)\mathbf{i}+3t\mathbf{j}(+\mathbf{c})$	A1
		DM1
	$t=0 \quad 2\mathbf{i}+5\mathbf{j}=\mathbf{c}$	A1
	$\mathbf{r}=(2t^2-5t)\mathbf{i}+3t\mathbf{j}+(2\mathbf{i}+5\mathbf{j})$	
	$(2t^2-5t+2)\mathbf{i}+(3t+5)\mathbf{j}$	B1

(c)	$\mathbf{r}_Q=11\mathbf{i}+2\mathbf{j}-2t\mathbf{i}+ct\mathbf{j}$		
	$(11-2t)\mathbf{i}+(2+ct)\mathbf{j}$		
	$\mathbf{r}_P=(2t^2-5t+2)\mathbf{i}+(3t+5)\mathbf{j}$		$2t^2-5t$
	$\mathbf{r}_Q=\mathbf{r}_P=d\mathbf{i}+14\mathbf{j}$		
	$3t+5=14$	$2t^2-3t-9$	M1
		$(2t+3)(t-3)=0$	
		$t=3$	
	$t=3$	A1 ft	A1
	$2+ct=14 \Rightarrow c=4$		A1 ft
	$d=11-2\times3=5$ or		
	$d=2\times3^2-5\times3+2 \Rightarrow d=5$		
	Alt: $2t^2-5t+2=11-2t=d \Rightarrow t=\frac{11-d}{2}$		
	$2\left(\frac{11-d}{2}\right)^2-5\left(\frac{11-d}{2}\right)+2=d,$		
	$d^2-19d+70=0=(d-5)(d-14)$		

Exercise 8.5

Pearson Stats/Mechanics Year 2
Pages 73-74

Homework Exercise

1 A particle P starts from rest at a fixed origin O. The acceleration of P at time t seconds (where $t \geqslant 0$) is $(6t^2\mathbf{i} + (8 - 4t^3)\mathbf{j})\,\text{m s}^{-2}$. Find:

a the velocity of P when $t = 2$ **(3 marks)**

b the position vector of P when $t = 4$. **(3 marks)**

2 A particle P is moving in a plane with velocity $\mathbf{v}\,\text{m s}^{-1}$ at time t seconds where

$$\mathbf{v} = (3t^2 + 2)\mathbf{i} + (6t - 4)\mathbf{j},\ t \geqslant 0$$

When $t = 2$, P has position vector $9\mathbf{j}$ m with respect to a fixed origin O. Find:

a the distance of P from O when $t = 0$ **(4 marks)**

b the acceleration of P at the instant when it is moving parallel to the vector $\mathbf{i}$. **(4 marks)**

3 At time t seconds, where $t \geqslant 0$, the particle P is moving in a plane with velocity $\mathbf{v}\,\text{m s}^{-1}$ and acceleration $\mathbf{a}\,\text{m s}^{-2}$, where $\mathbf{a} = (2t - 4)\mathbf{i} + 6 \sin t\mathbf{j}$.

Given that P is instantaneously at rest when $t = \frac{\pi}{2}$ seconds, find:

a $\mathbf{v}$ in terms of π and t **(5 marks)**

b the exact speed of P when $t = \frac{3\pi}{2}$ **(3 marks)**

Homework Exercise

4 At time t seconds (where $t \geqslant 0$), the particle P is moving in a plane with acceleration $\mathbf{a}\,\text{m s}^{-2}$, where

$$\mathbf{a} = (5t - 3)\mathbf{i} + (8 - t)\mathbf{j}$$

When $t = 0$, the velocity of P is $(2\mathbf{i} - 5\mathbf{j})\,\text{m s}^{-1}$. Find:

a the velocity of P after t seconds **(3 marks)**

b the value of t for which P is moving parallel to $\mathbf{i} - \mathbf{j}$ **(4 marks)**

c the speed of P when it is moving parallel to $\mathbf{i} - \mathbf{j}$. **(3 marks)**

5 At time t seconds (where $t \geqslant 0$), a particle P is moving in a plane with acceleration $(2\mathbf{i} - 2t\mathbf{j})\,\text{m s}^{-2}$. When $t = 0$, the velocity of P is $2\mathbf{j}\,\text{m s}^{-1}$ and the position vector of P is $6\mathbf{i}$ m with respect to a fixed origin P.

a Find the position vector of P at time t seconds. **(5 marks)**

At time t seconds (where $t \geqslant 0$), a second particle Q is moving in the plane with velocity $((3t^2 - 4)\mathbf{i} - 2t\mathbf{j})\,\text{m s}^{-1}$. The particles collide when $t = 3$.

b Find the position vector of Q at time $t = 0$. **(4 marks)**

6 At time $t = 0$ a particle P is at rest at a point with position vector $(4\mathbf{i} - 6\mathbf{j})$ m with respect to a fixed origin O. The acceleration of P at time t seconds (where $t \geqslant 0$) is $((4t - 3)\mathbf{i} - 6t^2\mathbf{j})\,\text{m s}^{-2}$. Find:

a the velocity of P when $t = \frac{1}{2}$ **(5 marks)**

b the position vector of P when $t = 6$. **(5 marks)**

Homework Exercise

7 At time t seconds (where $t \geqslant 0$) the particle P is moving in a plane with acceleration $\mathbf{a}\,\text{m s}^{-2}$, where $\mathbf{a} = (8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}$.

When $t = 2$, the velocity of P is $(16\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$. Find:

a the velocity of P after t seconds **(4 marks)**

b the value of t when P is moving parallel to $\mathbf{i}$. **(3 marks)**

8 At time t seconds the velocity of a particle P is $((4t - 3)\mathbf{i} + 4\mathbf{j})\,\text{m s}^{-1}$. When $t = 0$, the position vector of P is $(\mathbf{i} + 2\mathbf{j})$ m, relative to a fixed origin O.

a Find an expression for the position vector of P at time t seconds. **(4 marks)**

A second particle Q moves with constant velocity $(5\mathbf{i} + k\mathbf{j})\,\text{m s}^{-1}$.
When $t = 0$, the position vector of Q is $(11\mathbf{i} + 5\mathbf{j})$ m.

b Given that the particles P and Q collide, find:

i the value of k

ii the position vector of the point of collision. **(6 marks)**

Challenge

A particle P is moving in a plane. At time t seconds, P is moving with velocity $\mathbf{v}\,\text{m s}^{-1}$, where $\mathbf{v} = 3t\cos t\mathbf{i} + 5t\mathbf{j}$. Given that P is initially at the point with position vector $4\mathbf{i} + \mathbf{j}$ m relative to a fixed origin O, find the position vector of P when $t = \frac{\pi}{2}$.

Homework Answers

1 **a** $16\mathbf{i}\,\text{m s}^{-1}$ **b** $128\mathbf{i} - 192\mathbf{j}\,\text{m}$

2 **a** 13 m **b** $(4\mathbf{i} + 6\mathbf{j})\,\text{m s}^{-2}$

3 **a** $\mathbf{v} = \left(t^2 - 4t + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 6\cos t\mathbf{j}$ **b** $2\pi^2 - 4\pi$

4 **a** $\left(\left(\frac{5t^2}{2} - 3t + 2\right)\mathbf{i} + \left(8t - \frac{t^2}{2} - 5\right)\mathbf{j}\right)\text{m s}^{-1}$

b $t = \frac{1}{2}$ **c** $\frac{9\sqrt{2}}{8}\,\text{m s}^{-1}$

5 **a** $\left((t^2 + 6)\mathbf{i} + \left(2t - \frac{t^3}{3}\right)\mathbf{j}\right)\text{m}$ **b** $6\mathbf{j}\,\text{m}$

6 **a** $\left(-\mathbf{i} - \frac{1}{4}\mathbf{j}\right)\text{m s}^{-1}$ **b** $(94\mathbf{i} - 654\mathbf{j})\,\text{m}$

7 **a** $((2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j})\,\text{m s}^{-1}$

b $t = \frac{7}{4}$

8 **a** $\mathbf{r} = (2t^2 - 3t + 1)\mathbf{i} + (4t + 2)\mathbf{j}\,\text{m}$

b **i** 3.4 **ii** $\mathbf{r} = 36\mathbf{i} + 22\mathbf{j}$

Challenge

$\mathbf{r} = \left(\frac{3\pi}{2} + 1\right)\mathbf{i} + \left(\frac{5\pi^2}{8} + 1\right)\mathbf{j}$