P2 Chapter 6: CoAngle Trigonometry

Secant Identities

New Identities

In Pure-1 we had the Pythagorean Trig Identity:

$$\sin^2 x + \cos^2 x = 1$$

There are two tangent identities to know:

Dividing by $\cos^2 x$:

$$1 + \tan^2 x = \sec^2 x$$

Dividing by $\sin^2 x$:

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

Tip: remember this one by starting with the above then putting 'co-' on front of each trig function.

"Prove that $1 + \tan^2 x \equiv \sec^2 x$."

$$\mathbf{\sin^2 x + \cos^2 x \equiv 1}$$

$$\mathbf{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} \equiv \frac{1}{\cos^2 x}}$$

$$\mathbf{\tan^2 x + 1 \equiv \sec^2 x}$$

Tip: This has been asked in an exam! <u>You must explicitly show each term being divided by $\cos^2 x$.</u>

Examples

[Textbook] Prove that $cosec^4\theta - \cot^4\theta = \frac{1+\cos^2\theta}{1-\cos^2 x}$

?

Solve the equation $4\,cosec^2\theta - 9 = \cot\theta$ in the interval $0 \leq \theta \leq 360°$

?

This is just like in Year-1; if you had a mixture of $\sin\theta$, $\sin^2\theta$, $\cos^2\theta$: you'd change the $\cos^2\theta$ to $1 - \sin^2\theta$ in order to get a quadratic in terms of sin.

Examples

[Textbook] Prove that $cosec^4\theta - \cot^4\theta = \frac{1+\cos^2\theta}{1-\cos^2 x}$

$$LHS = \left(\mathrm{cosec}^2\theta + cot^2\theta\right)\left(\mathrm{cosec}^2\theta - cot^2\theta\right)$$
$$= \mathrm{cosec}^2\theta + cot^2\theta$$
$$= \frac{1}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$$
$$= \frac{1 + cos^2\theta}{1 - cos^2 x} = RHS$$

Solve the equation $4\,cosec^2\theta - 9 = \cot\theta$ in the interval $0 \le \theta \le 360°$

$$4\left(1 + \cot^2\theta\right) - 9 = \cot\theta$$
$$4\cot^2\theta - \cot\theta - 5 = 0$$
$$(4\cot\theta - 5)(\cot\theta + 1) = 0$$
$$\therefore \cot\theta = \frac{5}{4}\ or \cot\theta = -1$$
$$\tan\theta = \frac{4}{5}\ or \tan\theta = -1$$
$$\theta = 135°, 315°$$

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Test Your Understanding

Edexcel C3 June 2013 (R)

6. (ii) Solve, for $0 \leq \theta < 2\pi$, the equation

$$3\sec^2\theta + 3\sec\theta = 2\tan^2\theta$$

You must show all your working. Give your answers in terms of π.

(6)

?

Q

Solve, for $0 \leq x < 2\pi$, the equation

$$2cosec^2 x + \cot x = 5$$

giving your solutions to 3sf.

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$$3\sec^2\theta + 3\sec\theta = 2(\sec^2\theta - 1)$$
$$\sec^2\theta + 3\sec\theta + 2 = 0 \quad \rightarrow (\sec\theta + 2)(\sec\theta + 1) = 0$$
$$\frac{1}{\cos\theta} = -2 \quad \rightarrow \quad \cos\theta = -\frac{1}{2} \quad \rightarrow \quad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$
$$\frac{1}{\cos\theta} = -1 \quad \rightarrow \quad \cos\theta = -1 \quad \rightarrow \quad \theta = \pi$$

Q

Solve, for $0 \leq x < 2\pi$, the equation

$$2cosec^2 x + \cot x = 5$$

giving your solutions to 3sf.

$$2(1 + \cot^2 x) + \cot x - 5 = 0$$
$$2 + 2\cot^2 x + \cot x - 5 = 0$$
$$2\cot^2 x + \cot x - 3 = 0$$
$$(2\cot x + 3)(\cot x - 1) = 0$$
$$\cot x = -\frac{3}{2} \quad or \quad \cot x = 1$$
$$\tan x = -\frac{2}{3} \quad or \quad \tan x = 1$$
$$\boldsymbol{x = 2.55, \quad 5.70, \quad 0.785, \quad 3.93}$$

Exercise 6.4

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Homework Exercise

Give answers to 3 significant figures where necessary.

1 Simplify each of the following expressions.

a $1 + \tan^2 \frac{1}{2}\theta$ **b** $(\sec\theta - 1)(\sec\theta + 1)$ **c** $\tan^2\theta(\operatorname{cosec}^2\theta - 1)$

d $(\sec^2\theta - 1)\cot\theta$ **e** $(\operatorname{cosec}^2\theta - \cot^2\theta)^2$ **f** $2 - \tan^2\theta + \sec^2\theta$

g $\dfrac{\tan\theta\sec\theta}{1 + \tan^2\theta}$ **h** $(1 - \sin^2\theta)(1 + \tan^2\theta)$ **i** $\dfrac{\operatorname{cosec}\theta\cot\theta}{1 + \cot^2\theta}$

j $(\sec^4\theta - 2\sec^2\theta\tan^2\theta + \tan^4\theta)$ **k** $4\operatorname{cosec}^2 2\theta + 4\operatorname{cosec}^2 2\theta\cot^2 2\theta$

2 Given that $\operatorname{cosec} x = \dfrac{k}{\operatorname{cosec} x}$, where $k > 1$, find, in terms of k, possible values of $\cot x$.

3 Given that $\cot\theta = -\sqrt{3}$, and that $90° < \theta < 180°$, find the exact values of:

a $\sin\theta$ **b** $\cos\theta$

4 Given that $\tan\theta = \frac{3}{4}$, and that $180° < \theta < 270°$, find the exact values of:

a $\sec\theta$ **b** $\cos\theta$ **c** $\sin\theta$

5 Given that $\cos\theta = \frac{24}{25}$, and that θ is a reflex angle, find the exact values of:

a $\tan\theta$ **b** $\operatorname{cosec}\theta$

Homework Exercise

6 Prove the following identities.

a $\sec^4\theta - \tan^4\theta \equiv \sec^2\theta + \tan^2\theta$

b $\operatorname{cosec}^2 x - \sin^2 x \equiv \cot^2 x + \cos^2 x$

c $\sec^2 A(\cot^2 A - \cos^2 A) \equiv \cot^2 A$

d $1 - \cos^2\theta \equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$

e $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} \equiv 1 - 2\sin^2 A$

f $\sec^2\theta + \operatorname{cosec}^2\theta \equiv \sec^2\theta\operatorname{cosec}^2\theta$

g $\operatorname{cosec} A\sec^2 A \equiv \operatorname{cosec} A + \tan A\sec A$

h $(\sec\theta - \sin\theta)(\sec\theta + \sin\theta) \equiv \tan^2\theta + \cos^2\theta$

7 Given that $3\tan^2\theta + 4\sec^2\theta = 5$, and that θ is obtuse, find the exact value of $\sin\theta$.

8 Solve the following equations in the given intervals.

a $\sec^2\theta = 3\tan\theta,\ 0 \leqslant \theta \leqslant 360°$

b $\tan^2\theta - 2\sec\theta + 1 = 0,\ -\pi \leqslant \theta \leqslant \pi$

c $\operatorname{cosec}^2\theta + 1 = 3\cot\theta,\ -180° \leqslant \theta \leqslant 180°$

d $\cot\theta = 1 - \operatorname{cosec}^2\theta,\ 0 \leqslant \theta \leqslant 2\pi$

e $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta,\ 0 \leqslant \theta \leqslant 360°$

f $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta,\ 0 \leqslant \theta \leqslant \pi$

g $\tan^2 2\theta = \sec 2\theta - 1,\ 0 \leqslant \theta \leqslant 180°$

h $\sec^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 1,\ 0 \leqslant \theta \leqslant 2\pi$

9 Given that $\tan^2 k = 2\sec k$,

a find the value of $\sec k$ **(4 marks)**

b deduce that $\cos k = \sqrt{2} - 1$. **(2 marks)**

c Hence solve, in the interval $0 \leqslant k \leqslant 360°$, $\tan^2 k = 2\sec k$, giving your answers to 1 decimal place. **(3 marks)**

Homework Exercise

10 Given that $a = 4\sec x$, $b = \cos x$ and $c = \cot x$,

a express b in terms of a **(2 marks)**

b show that $c^2 = \dfrac{16}{a^2 - 16}$ **(3 marks)**

11 Given that $x = \sec\theta + \tan\theta$,

a show that $\dfrac{1}{x} = \sec\theta - \tan\theta$. **(3 marks)**

b Hence express $x^2 + \dfrac{1}{x^2} + 2$ in terms of θ, in its simplest form. **(5 marks)**

12 Given that $2\sec^2\theta - \tan^2\theta = p$ show that $\operatorname{cosec}^2\theta = \dfrac{p-1}{p-2}$, $p \neq 2$. **(5 marks)**

Homework Answers

1 **a** $\sec^2\left(\frac{1}{2}\theta\right)$ **b** $\tan^2\theta$ **c** 1
d $\tan\theta$ **e** 1 **f** 3
g $\sin\theta$ **h** 1 **i** $\cos\theta$
j 1 **k** $4\operatorname{cosec}^4 2\theta$

2 $\pm\sqrt{k-1}$

3 **a** $\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$

4 **a** $-\frac{5}{4}$ **b** $-\frac{4}{5}$ **c** $-\frac{3}{5}$

5 **a** $-\frac{7}{24}$ **b** $-\frac{25}{7}$

6 **a** L.H.S. $\equiv (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$
$\equiv 1(\sec^2\theta + \tan^2\theta) = $ R.H.S.

b L.H.S. $\equiv (1 + \cot^2 x) - (1 - \cos^2 x)$
$\equiv \cot^2 x + \cos^2 x = $ R.H.S.

c L.H.S. $\equiv \dfrac{1}{\cos^2 A}\left(\dfrac{\cos^2 A}{\sin^2 A} - \cos^2 A\right) \equiv \dfrac{1}{\sin^2 A} - 1$
$\equiv \operatorname{cosec}^2 A - 1 = \cot^2 A = $ R.H.S.

d R.H.S. $\equiv \tan^2\theta \times \cos^2\theta \equiv \dfrac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \equiv \sin^2\theta$
$\equiv 1 - \cos^2\theta = $ L.H.S.

e L.H.S. $= \dfrac{1 - \tan^2 A}{\sec^2 A} \equiv \cos^2 A\left(1 - \dfrac{\sin^2 A}{\cos^2 A}\right)$
$\equiv \cos^2 A - \sin^2 A \equiv (1 - \sin^2 A) - \sin^2 A$
$\equiv 1 - 2\sin^2 A = $ R.H.S.

6 **e** L.H.S. $= \dfrac{1 - \tan^2 A}{\sec^2 A} \equiv \cos^2 A\left(1 - \dfrac{\sin^2 A}{\cos^2 A}\right)$
$\equiv \cos^2 A - \sin^2 A \equiv (1 - \sin^2 A) - \sin^2 A$
$\equiv 1 - 2\sin^2 A = $ R.H.S.

f L.H.S. $= \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta} \equiv \dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}$
$\equiv \dfrac{1}{\cos^2\theta\sin^2\theta} \equiv \sec^2\theta\operatorname{cosec}^2\theta = $ R.H.S.

g L.H.S. $= \operatorname{cosec} A(1 + \tan^2 A) \equiv \operatorname{cosec} A\left(1 + \dfrac{\sin^2 A}{\cos^2 A}\right)$
$\equiv \operatorname{cosec} A + \dfrac{1}{\sin A} \cdot \dfrac{\sin^2 A}{\cos^2 A} \equiv \operatorname{cosec} A + \dfrac{\sin A}{\cos A} \cdot \dfrac{1}{\cos A}$
$\equiv \operatorname{cosec} A + \tan A \sec A = $ R.H.S.

h L.H.S. $= \sec^2\theta - \sin^2\theta \equiv (1 + \tan^2\theta) - (1 - \cos^2\theta)$
$\equiv \tan^2\theta + \cos^2\theta \equiv $ R.H.S.

7 $\dfrac{\sqrt{2}}{4}$

8 **a** $20.9°, 69.1°, 201°, 249°$ **b** $\pm\dfrac{\pi}{3}$
c $-153°, -135°, 26.6°, 45°$ **d** $\dfrac{\pi}{2}, \dfrac{3\pi}{4}, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}$
e $120°$ **f** $0, \dfrac{\pi}{4}, \pi$
g $0°, 180°$ **h** $\dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{5\pi}{4}, \dfrac{4\pi}{3}$

Homework Answers

9 **a** $1 + \sqrt{2}$

b $\cos k = \dfrac{1}{1+\sqrt{2}} = \dfrac{\sqrt{2}-1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2} - 1$

c $65.5°, 294.5°$

10 **a** $b = \dfrac{4}{a}$

b $c^2 = \cot^2 x = \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{b^2}{1-b^2} = \dfrac{\left(\frac{4}{a}\right)^2}{1-\left(\frac{4}{a}\right)^2}$

$= \dfrac{16}{a^2} \times \dfrac{a^2}{(a^2-16)} = \dfrac{16}{a^2-16}$

11 **a** $\dfrac{1}{x} = \dfrac{1}{\sec\theta + \tan\theta} = \dfrac{\sec\theta - \tan\theta}{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}$

$= \dfrac{\sec\theta - \tan\theta}{(\sec^2\theta - \tan^2\theta)} = \dfrac{\sec\theta - \tan\theta}{1}$

b $x^2 + \dfrac{1}{x^2} + 2 = \left(x + \dfrac{1}{x}\right)^2 = (2\sec\theta)^2 = 4\sec^2\theta$

12 $p = 2(1 + \tan^2\theta) - \tan^2\theta = 2 + \tan^2\theta$

$\Rightarrow \tan^2\theta = p - 2 \Rightarrow \cot^2\theta = \dfrac{1}{p-2}$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta = 1 + \dfrac{1}{p-2} = \dfrac{(p-2)+1}{p-2} = \dfrac{p-1}{p-2}$