P1 Chapter 8: Binomial Expansion

Coefficient Formula

Getting a single term in the expansion

In the expansion of $(a+b)^n$ the general term is given by $\binom{n}{r} a^{n-r} b^r$

Expression	Power of x in term wanted.	Term in expansion
$(a+x)^{10}$	3	?
$(2x-1)^{75}$	50	?
$(3-x)^{12}$	7	?
$(3x+4)^{16}$	3	?

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Expression	Power of x in term wanted.	Term in expansion
$(a+x)^{10}$	3	$\binom{10}{3}a^7x^3$ **Note**: The two powers add up to 10.
$(2x-1)^{75}$	50	$\binom{75}{50}(-1)^{25}(2x)^{50}$
$(3-x)^{12}$	7	$\binom{12}{7}(3)^5(-x)^7$
$(3x+4)^{16}$	3	$\binom{16}{3}(4)^{13}(3x)^3$

Getting a single term in the expansion

The coefficient of x^4 in the expansion of $(1+qx)^{10}$ is 3360.
Find the possible value(s) of the constant q.

Term is:

?

Therefore:

?

Getting a single term in the expansion

The coefficient of x^4 in the expansion of $(1+qx)^{10}$ is 3360. Find the possible value(s) of the constant q.

Term is:

$$\binom{10}{4}(1^6)(qx)^4 = 210q^4x^4$$

Therefore:

$$210q^4 = 3360$$
$$q^4 = 16$$
$$q = \pm 2$$

Test Your Understanding

In the expansion of $(1 + ax)^{10}$, where a is a non-zero constant the coefficient of x^3 is double the coefficient of x^2. Find the value of a.

?

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In the expansion of $(1 + ax)^{10}$, where a is a non-zero constant the coefficient of x^3 is double the coefficient of x^2. Find the value of a.

x^2 term: $\binom{10}{2}(1^8)(ax)^2 = 45a^2x^2$

x^3 term: $\binom{10}{3}(1^7)(ax)^3 = 120a^3x^3$

$$\therefore 120a^3 = 2(45a^2)$$
$$120a^3 = 90a^2$$
$$4a^3 - 3a^2 = 0$$
$$a^2(4a - 3) = 0$$
$$a = 0 \; or \; a = \frac{3}{4}$$

But a is non-zero, so $a = \frac{3}{4}$

Exercise 8.4

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Extension

1 *[MAT 2014 1G]* Let n be a positive integer. The coefficient of x^3y^5 in the expansion of $(1+xy+y^2)^n$ equals:

A) n

B) 2^n

C) $\binom{n}{3}\binom{n}{5}$

D) $4\binom{n}{4}$

E) $\binom{n}{8}$

?

2 [STEP I 2013 Q6] By considering the coefficient of x^r in the series for $(1+x)(1+x)^n$, or otherwise, obtain the following relation between binomial coefficients:

$$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$

?

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1 *[MAT 2014 1G]* Let n be a positive integer. The coefficient of x^3y^5 in the expansion of $(1+xy+y^2)^n$ equals:

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D) $4\binom{n}{4}$
E) $\binom{n}{8}$

Try to imagine n brackets written out. To get x^3y^5, we must have chosen xy from 3 brackets, y^2 from one and 1 from the remaining brackets. That's $\binom{n}{3}$ choices for the xy term, and $n-3$ choices for the y^2 term.
Using the definition of the choose function, you can show that $4\binom{n}{4}=(n-3)\binom{n}{3}$

2 [STEP I 2013 Q6] By considering the coefficient of x^r in the series for $(1+x)(1+x)^n$, or otherwise, obtain the following relation between binomial coefficients:

$$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$

Noting that $(1+x)(1+x)^n=(1+x)^{n+1}$, the x^r term is $\binom{n+1}{r}x^r$.
But the x^r term could be obtained either by 1 in the first bracket multiplied by x^r term in the second, giving $1\times\binom{n}{r}x^r$, or the x in the first bracket multiplied by the x^{r-1} term in the second, $x\times\binom{n}{r-1}x^{r-1}=\binom{n}{r-1}x^r$. Thus comparing coefficients:

$$\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$

Homework Exercise

1 Find the coefficient of x^3 in the binomial expansion of:

a $(3 + x)^5$ **b** $(1 + 2x)^5$ **c** $(1 - x)^6$ **d** $(3x + 2)^5$

e $(1 + x)^{10}$ **f** $(3 - 2x)^6$ **g** $(1 + x)^{20}$ **h** $(4 - 3x)^7$

i $\left(1 - \frac{1}{2}x\right)^6$ **j** $\left(3 + \frac{1}{2}x\right)^7$ **k** $\left(2 - \frac{1}{2}x\right)^8$ **l** $\left(5 + \frac{1}{4}x\right)^5$

2 The coefficient of x^2 in the expansion of $(2 + ax)^6$ is 60. Find two possible values of the constant a.

Problem-solving

$a = 2$, $b = ax$, $n = 6$. Use brackets when you substitute ax.

3 The coefficient of x^3 in the expansion of $(3 + bx)^5$ is -720. Find the value of the constant b.

4 The coefficient of x^3 in the expansion of $(2 + x)(3 - ax)^4$ is 30. Find two possible values of the constant a.

5 When $(1 - 2x)^p$ is expanded, the coefficient of x^2 is 40. Given that $p > 0$, use this information to find:

a the value of the constant p **(6 marks)**

b the coefficient of x **(1 mark)**

c the coefficient of x^3 **(2 marks)**

Problem-solving

You will need to use the definition of $\binom{n}{r}$ to find an expression for $\binom{p}{2}$.

6 **a** Find the first three terms, in ascending powers of x, of the binomial expansion of $(5 + px)^{30}$, where p is a non-zero constant. **(2 marks)**

b Given that in this expansion the coefficient of x^2 is 29 times the coefficient of x work out the value of p. **(4 marks)**

Homework Exercise

7 a Find the first four terms, in ascending powers of x, of the binomial expansion of $(1 + qx)^{10}$, where q is a non-zero constant. **(2 marks)**

b Given that in the expansion of $(1 + qx)^{10}$ the coefficient of x^3 is 108 times the coefficient of x, work out the value of q. **(4 marks)**

8 a Find the first three terms, in ascending powers of x of the binomial expansion of $(1 + px)^{11}$, where p is a constant. **(2 marks)**

b The first 3 terms in the same expansion are 1, $77x$ and qx^2, where q is a constant. Find the value of p and the value of q. **(4 marks)**

9 a Write down the first three terms, in ascending powers of x, of the binomial expansion of $(1 + px)^{15}$, where p is a non-zero constant. **(2 marks)**

b Given that, in the expansion of $(1 + px)^{15}$, the coefficient of x is $(-q)$ and the coefficient of x^2 is $5q$, find the value of p and the value of q. **(4 marks)**

10 In the binomial expansion of $(1 + x)^{30}$, the coefficients of x^9 and x^{10} are p and q respectively.

Find the value of $\frac{q}{p}$. **(4 marks)**

Challenge

Find the coefficient of x^4 in the binomial expansion of: **a** $(3 - 2x^2)^9$ **b** $\left(\frac{5}{x} + x^2\right)^8$

Homework Exercise

1 **a** 90 **b** 80 **c** –20
d 1080 **e** 120 **f** –4320
g 1140 **h** –241 920 **i** –2.5
j 354.375 **k** –224 **l** 3.90625

2 $a = \pm\frac{1}{2}$

3 $b = -2$

4 $1, \dfrac{5 \pm \sqrt{105}}{8}$

5 **a** $p = 5$ **b** –10 **c** –80

6 **a** $5^{30} + 5^{29} \times 30px + 5^{28} \times 435p^2x^2$
b $p = 10$

7 **a** $1 + 10qx + 45q^2x^2 + 120q^3x^3$
b $q = \pm 3$

8 **a** $1 + 11px + 55p^2x^2$
b $p = 7, q = 2695$

9 **a** $1 + 15px + 105p^2x^2$
b $p = -\frac{5}{7}, q = 10\frac{5}{7}$

10 $\dfrac{q}{p} = 2.1$

Challenge

a 314 928 **b** 43 750