P2 Chapter 3: Sequences and Series

Arithmetic Series

Series

A **series** is a <u>**sum**</u> of terms in a sequence.
You will encounter 'series' in many places in A Level:

Arithmetic Series (this chapter!)

Sum of terms in an arithmetic sequence.

$$2+5+8+11$$

Binomial Series (Later in Year 2)

You did Binomial expansions in Year 1. But when the power is negative or fractional, we end up with an infinite series.

$$\sqrt{1+x}=(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{6}x^3-\cdots$$

Taylor Series (Further Maths)

Expressing a function as an infinite series, consisting of polynomial terms.

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

Extra Notes: A 'series' usually refers to an <u>infinite</u> sum of terms in a sequence. If we were just summing some finite number of them, we call this a <u>partial sum of the series</u>.

e.g. The '*Harmonic Series*' is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, which is infinitely many terms. But a we could get a partial sum, e.g. $H_3=1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6}$

However, in this syllabus, the term 'series' is used to mean either a finite or infinite addition of terms.

Terminology: A '*power series*' is an infinite polynomial with increasing powers of x. There is also a chapter on power series in the Further Stats module.

Arithmetic Series

n^{th} term

$$u_n = a + (n-1)d$$

Sum of first n terms

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Let's prove it!

Example:

Take an arithmetic sequence 2, 5, 8, 11, 14, 17, …

$$S_5 = 2 \quad + 5 \quad + 8 + 11 + 14$$

Reversing:

$$S_5 = 14 + 11 + 8 + 5 \quad + 2$$

The idea is that each pair of terms, at symmetrically opposite ends, adds to the same number.

Adding these:

$$2S_5 = 16 + 16 + 16 + 16 + 16 = 16 \times 5 = 80$$

$$\therefore S_5 = 40$$

Proving more generally:

$$S_n = a \quad + (a+d) + (a+2d) + \cdots + (a + (n-1)d)$$

$$S_n = (a + (n-1)d) + \cdots + (a+2d) + (a+d) + a$$

Adding:

$$2S_n = (2a + (n-1)d) + \cdots + (2a + (n-1)d) = n(2a + (n-1)d)$$

$$\therefore S_n = \frac{n}{2}(2a + (n-1)d)$$

Fro Exam Note: The proof has been an exam question before. It's also a university interview favourite!

Alternative Formula

$$a + (a + d) + \cdots + L$$

Suppose last term was L.
We saw earlier that each opposite pair of terms (first and last, second and second last, etc.) added to the same total, in this case $a + L$.

There are $\frac{n}{2}$ pairs, therefore:

$$S_n = \frac{n}{2}(a + L)$$

Examples

Find the sum of the first 30 terms of the following arithmetic sequences...

1. $2 + 5 + 8 + 11 + 14 \ldots$ $S_{30} =$?
2. $100 + 98 + 96 + \cdots$ $S_{30} =$?
3. $p + 2p + 3p + \cdots$ $S_{30} =$?

Fro Tips: Again, explicitly write out "$a = \cdots, d = \cdots, n = \cdots$". You're less likely to make incorrect substitutions into the formula.

Make sure you write $S_n = \cdots$ so you make clear to yourself (and the examiner) that you're finding the sum of the first n terms, not the nth term.

Find the minimum number of terms for the sum of $4 + 9 + 14 + \cdots$ to exceed 2000.

?

Examples

Find the sum of the first 30 terms of the following arithmetic sequences...

1. $2 + 5 + 8 + 11 + 14 \ldots$ $S_{30} = 1365$
2. $100 + 98 + 96 + \cdots$ $S_{30} = 2130$
3. $p + 2p + 3p + \cdots$ $S_{30} = 465p$

Fro Tips: Again, explicitly write out "$a = \cdots, d = \cdots, n = \cdots$". You're less likely to make incorrect substitutions into the formula.

Make sure you write $S_n = \cdots$ so you make clear to yourself (and the examiner) that you're finding the sum of the first n terms, not the nth term.

Find the minimum number of terms for the sum of $4 + 9 + 14 + \cdots$ to exceed 2000.

$$\boldsymbol{S_n > 2000, \qquad a = 4, \qquad d = 5}$$
$$\boldsymbol{\frac{n}{2}[2a + (n-1)d] > 2000}$$
$$\boldsymbol{\frac{n}{2}[8 + (n-1)5] > 2000}$$
$$\boldsymbol{5n^2 + 3n - 4000 > 0}$$
$$\boldsymbol{n < -28.5 \text{ or } n > 27.9}$$

So 28 terms needed.

Test Your Understanding

Edexcel C1 Jan 2012 Q9

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P.
Salary increases by £($2T$) each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £(P + 1800).
Salary increases by £T each year, forming an arithmetic sequence.

(*a*) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$£(10P + 90T).$$

(2)

For the 10-year period, the total earned is the same for both salary schemes.

(*b*) Find the value of T.

?

(4)

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850.

(*c*) Find the value of P.

?

(3)

Test Your Understanding

Edexcel C1 Jan 2012 Q9

9. A company offers two salary schemes for a 10-year period, Year 1 to Year 10 inclusive.

Scheme 1: Salary in Year 1 is £P.
Salary increases by £($2T$) each year, forming an arithmetic sequence.

Scheme 2: Salary in Year 1 is £(P + 1800).
Salary increases by £T each year, forming an arithmetic sequence.

(*a*) Show that the total earned under Salary Scheme 1 for the 10-year period is

$$£(10P + 90T).$$

(2)

For the 10-year period, the total earned is the same for both salary schemes.

(*b*) Find the value of T.

$T = 400$

(4)

For this value of T, the salary in Year 10 under Salary Scheme 2 is £29 850.

(*c*) Find the value of P.

$P = £24450$

(3)

Exercise 3.2

Pearson Pure Mathematics Year 2/AS
Page 18

Extension

1 [MAT 2007 1J]
The inequality

$$(n+1)+(n^4+2)+(n^9+3)+\cdots$$
$$+(n^{10000}+100)>k$$

Is true for all $n \geq 1$. It follows that

A) $k < 1300$
B) $k^2 < 101$
C) $k \geq 101^{10000}$
D) $k < 5150$

?

2 [AEA 2010 Q2]
The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$.
Giving simplified answers in terms of p and q, find

a) The common different of the terms in this series,
b) The first term of the series,
c) The sum of the first $(p+q)$ terms of the series.

Solution on next slide.

3 [MAT 2008 1I]
The function $S(n)$ is defined for positive integers n by
$S(n) =$ sum of digits of n
For example, $S(723) = 7 + 2 + 3 = 12$.
The sum
$S(1) + S(2) + S(3) + \cdots + S(99)$
equals what?

?

Exercise 3.2

Pearson Pure Mathematics Year 2/AS
Page 18

Extension

1 [MAT 2007 1J]
The inequality

$$(n+1)+(n^4+2)+(n^9+3)+\cdots$$
$$+(n^{10000}+100)>k$$

Is true for all $n \geq 1$. It follows that

A) $k < 1300$
B) $k^2 < 101$
C) $k \geq 101^{10000}$
D) $k < 5150$

k will be largest when n is its smallest, so let $n = 1$.
Each of the n terms are therefore 1, giving the LHS:

$$\mathbf{100+(1+2+3+\cdots+100)}$$
$$\mathbf{100+\frac{100}{2}(2+99\times 1)=5150}$$

The answer is D.

2 [AEA 2010 Q2]
The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$.
Giving simplified answers in terms of p and q, find

a) The common different of the terms in this series,
b) The first term of the series,
c) The sum of the first $(p+q)$ terms of the series.

Solution on next slide.

3 [MAT 2008 1I]
The function $S(n)$ is defined for positive integers n by
$S(n) =$ sum of digits of n
For example, $S(723) = 7 + 2 + 3 = 12$.
The sum
$S(1) + S(2) + S(3) + \cdots + S(99)$
equals what?

When we sum all digits, we can separately consider sum of all units digits, and the sum of all tens digits.
Each of 1 to 9 occurs ten times as units digit, so sum is $\mathbf{10\times(1+2+\cdots+9)=450}$
Similarly each of 1 to 9 occurs ten times as tens digit, thus total is $\mathbf{450+450=900}$.

Solution to Extension Q2

[AEA 2010 Q2]
The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p \neq q$.
Giving simplified answers in terms of p and q, find

a) The common different of the terms in this series,
b) The first term of the series,
c) The sum of the first $(p+q)$ terms of the series.

(a)	$q = \frac{p}{2}(2a+(p-1)d)$ and $p = \frac{q}{2}(2a+(q-1)d)$	M1 A1	Attempt one sum formula Both correct expressions
	$2\left(\frac{q}{p} - \frac{p}{q}\right) = d(p-1-q+1)$	dM1 A1	Eliminate a. Dep on 1st M1 Must use 2 indep. eqns Correct elimination of a
	$d = \frac{2(q^2-p^2)}{pq(p-q)}$; $\underline{d = \frac{-2(p+q)}{pq}}$	A1 (5)	Correct simplified $d =$
(b)	$2a = \frac{2q}{p} + \frac{(p-1)2(q+p)}{pq}$; $a = \frac{q^2(q-1)-p^2(p-1)}{pq(q-p)}$	M1 dM1	Substitute for d in a correct sum formula i.e. eqn in a only Rearrange to $a =$. Dep M1
	$\underline{\frac{q^2+qp+p^2-p-q}{pq}}$ or $\underline{\frac{q^2+(p-1)(q+p)}{pq}}$ or $\underline{\frac{p^2+(q-1)(q+p)}{pq}}$	A1 (3)	Correct single fraction with denom $= pq$
(c)	$S_{p+q} = \frac{p+q}{2}\left(\frac{2q}{p} + \frac{(p-1)2(q+p)}{pq} + \frac{-2(p+q)}{pq}(p+q-1)\right)$	M1	Attempt sum formula with $n = (p+q)$ and ft their a and d
	$= \frac{p+q}{2}\left[\frac{2(q^2+qp+p^2-p-q)}{pq} - \frac{2(p+q-1)(p+q)}{pq}\right]$	M1	Attempt to simplify- denominator $= pq$ or $2pq$
	$\frac{p+q}{pq}[-pq] = -[p+q]$	A1 (3) **[11]**	A1for $-(p+q)$ (S+ for concise simplification/factorising)

Homework Exercise

1 Find the sums of the following series.

a $3 + 7 + 11 + 14 + \ldots$ (20 terms)

b $2 + 6 + 10 + 14 + \ldots$ (15 terms)

c $30 + 27 + 24 + 21 + \ldots$ (40 terms)

d $5 + 1 + -3 + -7 + \ldots$ (14 terms)

e $5 + 7 + 9 + \ldots + 75$

f $4 + 7 + 10 + \ldots + 91$

g $34 + 29 + 24 + 19 + \ldots + -111$

h $(x + 1) + (2x + 1) + (3x + 1) + \ldots + (21x + 1)$

Hint For parts **e** to **h**, start by using the last term to work out the number of terms in the series.

2 Find how many terms of the following series are needed to make the given sums.

a $5 + 8 + 11 + 14 + \ldots = 670$

b $3 + 8 + 13 + 18 + \ldots = 1575$

c $64 + 62 + 60 + \ldots = 0$

d $34 + 30 + 26 + 22 + \ldots = 112$

Hint Set the expression for S_n equal to the total and solve the resulting equation to find n.

3 Find the sum of the first 50 even numbers.

4 Find the least number of terms for the sum of $7 + 12 + 17 + 22 + 27 + \ldots$ to exceed 1000.

Homework Exercise

5 The first term of an arithmetic series is 4. The sum to 20 terms is -15. Find, in any order, the common difference and the 20th term.

6 The sum of the first three terms of an arithmetic series is 12. If the 20th term is -32, find the first term and the common difference.

7 Prove that the sum of the first 50 natural numbers is 1275.

Problem-solving

Use the same method as Example 4.

8 Show that the sum of the first $2n$ natural numbers is $n(2n + 1)$.

9 Prove that the sum of the first n odd numbers is n^2.

10 The fifth term of an arithmetic series is 33. The tenth term is 68. The sum of the first n terms is 2225.

a Show that $7n^2 + 3n - 4450 = 0$. **(4 marks)**

b Hence find the value of n. **(1 mark)**

Homework Exercise

11 An arithmetic series is given by $(k + 1) + (2k + 3) + (3k + 5) + \ldots + 303$

a Find the number of terms in the series in terms of k. **(1 mark)**

b Show that the sum of the series is given by $\dfrac{152k + 46208}{k + 2}$ **(3 marks)**

c Given that $S_n = 2568$, find the value of k. **(1 mark)**

12 **a** Calculate the sum of all the multiples of 3 from 3 to 99 inclusive,

$$3 + 6 + 9 + \ldots + 99$$ **(3 marks)**

b In the arithmetic series

$$4p + 8p + 12p + \ldots + 400$$

where p is a positive integer and a factor of 100,

i find, in terms of p, an expression for the number of terms in this series.

ii Show that the sum of this series is $200 + \dfrac{20\,000}{p}$ **(4 marks)**

c Find, in terms of p, the 80th term of the arithmetic sequence

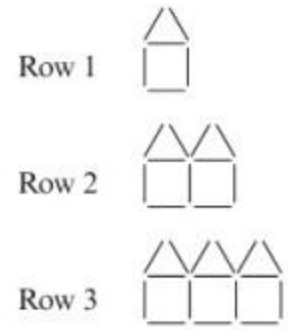
$$(3p + 2), (5p + 3), (7p + 4), \ldots,$$

giving your answer in its simplest form. **(2 marks)**

Homework Exercise

13 Joanna has some sticks that are all of the same length.
She arranges them in shapes as shown opposite and has made the following 3 rows of patterns.

She notices that 6 sticks are required to make the single pentagon in the first row, 11 sticks in the second row and for the third row she needs 16 sticks.



a Find an expression, in terms of n, for the number of sticks required to make a similar arrangement of n pentagons in the nth row. **(3 marks)**

Joanna continues to make pentagons following the same pattern. She continues until she has completed 10 rows.

b Find the total number of sticks Joanna uses in making these 10 rows. **(3 marks)**

Joanna started with 1029 sticks. Given that Joanna continues the pattern to complete k rows but does not have enough sticks to complete the $(k + 1)$th row:

c show that k satisfies $(5k - 98)(k + 21) \leqslant 0$ **(4 marks)**

d find the value of k. **(2 marks)**

Challenge

An arithmetic sequence has nth term $u_n = \ln 9 + (n - 1)\ln 3$. Show that the sum of the first n terms $= a \ln 3^{n^2+3n}$ where a is a rational number to be found.

Homework Answers

1 **a** 820 **b** 450 **c** −1140
d −294 **e** 1440 **f** 1425
g −1155 **h** $231x + 21$

2 **a** 20 **b** 25 **c** 65 **d** 4 or 14

3 2550 **4** 20

5 $d = -\frac{1}{2}$, 20th term = −5.5 **6** $a = 6, d = -2$

7 $S_{50} = 1 + 2 + 3 + \dots + 50$
$S_{50} = 50 + 49 + 48 + \dots + 1$
$2 \times S_{50} = 50(51) \Rightarrow S_{50} = 1275$

8 $S_{2n} = 1 + 2 + 3 + \dots + 2n$
$S_{2n} = 2n + (2n - 1) + (2n - 2) + \dots + 1$
$2 \times S_n = 2n(2n + 1) \Rightarrow S_n = n(2n + 1)$

9 $S_n = 1 + 3 + 5 + \dots + (2n - 3) + (2n - 1)$
$S_n = (2n - 1) + (2n - 3) + \dots + 5 + 3 + 1$
$2 \times S_n = n(2n) \Rightarrow S_n = n^2$

10 **a** $a + 4d = 33,\ a + 9d = 68$

$d = 7,\ a = 5$ so $S_n = \frac{n}{2}[2(5) + (n - 1)7]$

$\Rightarrow 2225 = \frac{n}{2}(7n + 3) \Rightarrow 7n^2 + 3n - 4450 = 0$

b 25

11 **a** $\frac{304}{k + 2}$

b $S_n = \frac{152}{k + 2}(k + 1 + 303) = \frac{152k + 46\,208}{k + 2}$

c 17

12 **a** 1683

b **i** $\frac{100}{p}$

ii $S_{\frac{100}{p}} = \frac{50}{p}\left[8p + \left(\frac{100 - p}{p}\right)4p\right]$

$S_{\frac{100}{p}} = \frac{50}{p}[4p + 400] = 200\left[1 + \frac{100}{p}\right]$

c $161p + 81$

13 **a** $5n + 1$ **b** 285

c $S_k = \frac{k}{2}[2(6) + (k - 1)5] = \frac{k}{2}(5k + 7)$

$\frac{k}{2}(5k + 7) \leqslant 1029$

$5k^2 + 7k - 2058 \leqslant 0$

$(5k - 98)(k + 21) \leqslant 0$

d $k = 19$

Challenge

$S_n = \frac{n}{2}(2\ln 9 + (n - 1)\ln 3) = \frac{n}{2}(\ln 81 - \ln 3 + n\ln 3)$

$= \frac{n}{2}(\ln 27 + n\ln 3) = \frac{n}{2}(\ln 3^3 + \ln 3^n)$

$= \frac{n}{2}(\ln 3^{n+3}) = \frac{1}{2}(\ln 3^{n^2+3n}) \Rightarrow a = \frac{1}{2}$