P1 Chapter 4: Transforming Graphs

Cubic Graphs

Polynomial Graphs

In Chapter 2 we briefly saw that a **polynomial** expression is of the form:

$$a + bx + cx^2 + dx^3 + ex^3 + \cdots$$

where a, b, c, d, e, … are constants (which could be 0).

The **order** of a polynomial is its highest power.

Order	Name
0	Constant (e.g. "4")
1	Linear (e.g. "$2x - 1$")
2	Quadratic (e.g. "$x^2 + 3$")
3	Cubic
4	Quartic
5	Quintic

These are covered in Chapter 5.

Chapter 2 explored the graphs for these.

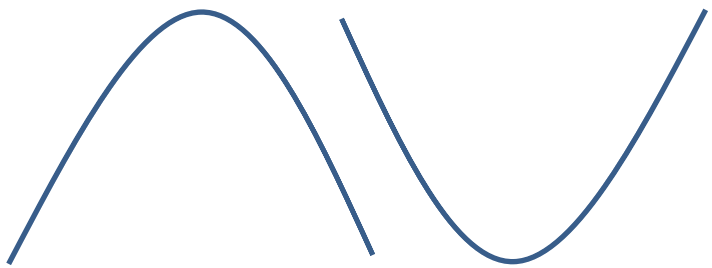
We will cover these now.

While these are technically beyond the A Level syllabus, we will look at how to sketch polynomials in general.

Polynomial Graphs

Order:

2



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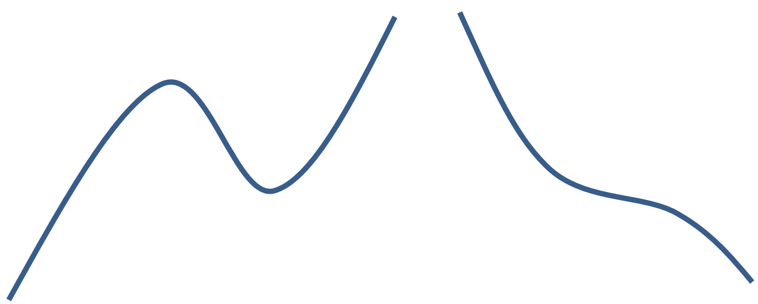

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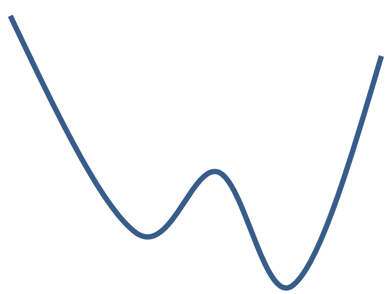


What property connects the order of the polynomial and the shape?

The number of 'turns' is one less than the order, e.g. a cubic has 2 'turns', a quartic 3 'turns'.

Bro Note: …Actually this is not strictly true, e.g. consider $y = x^4$, which has a U shape. But this is because multiple turns are being squashed into a single point.

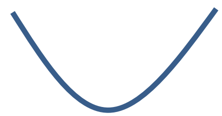
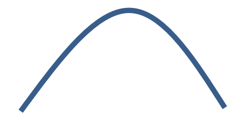
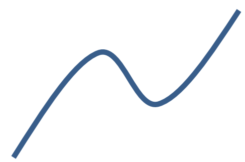
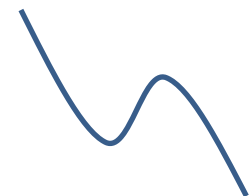
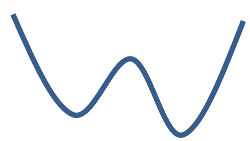
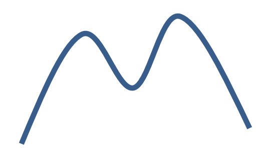
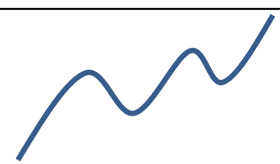
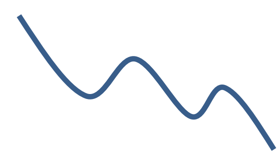
In Chapter 2 how did we tell what way up a quadratic is, and why does this work?

For a quadratic $y = ax^2 + bx + c$, i.f. $a > 0$, we had a 'valley' shape. This is because if x was a large positive value, ax^2 would be large and positive, thus the graph's y value tends towards infinity.

We would write:
"As $x \to \infty$, $y \to \infty$" where "$\to$" means "tends towards".

Polynomial Graphs

e.g. If $y = 2x^2 + 3$, try a large positive value like $x = 1000$. We can see we'd get a large positive y value. Thus as $x \to \infty, y \to \infty$

Equation	If $a > 0$	Resulting Shape	If $a < 0$	Resulting Shape
$y = ax^2 + bx + c$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^3 + bx^2 + cx + d$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	
$y = ax^4 + bx^3 + cx^2 + dx + e$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to \infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to -\infty$	
$y = ax^5 + bx^4 + \cdots$	As $x \to \infty, y \to \infty$ As $x \to -\infty, y \to -\infty$		As $x \to \infty, y \to -\infty$ As $x \to -\infty, y \to \infty$	

If $a > 0$, what therefore can we say about the shape if:

- **The order is odd**: It goes uphill (from left to right)
- **The order is even**: The tails go upwards.

(And we have the opposite if $a < 0$)

Cubics

Sketch the curve with equation
$$y = (x-2)(1-x)(1+x)$$

Sketch the curve with equation
$$y = x^2(x-1)$$

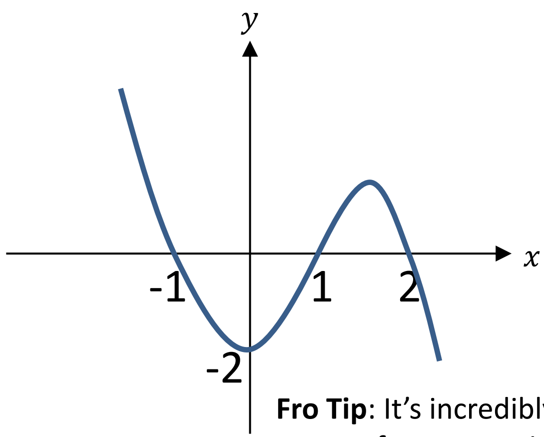
Features you must consider:

Shape? If we expanded, x^3 term would be negative, so 'downhill' shape.

Fro Tip: No need to expand out the whole thing. Just mentally consider the x terms multiplied together.

Roots? If $y = 0$, $x = 2, 1\ or\ -1$

***y*-intercept?** If $x = 0, y = -2$



Fro Tip: It's incredibly easy to forget to write in one of the intercepts. So don't!

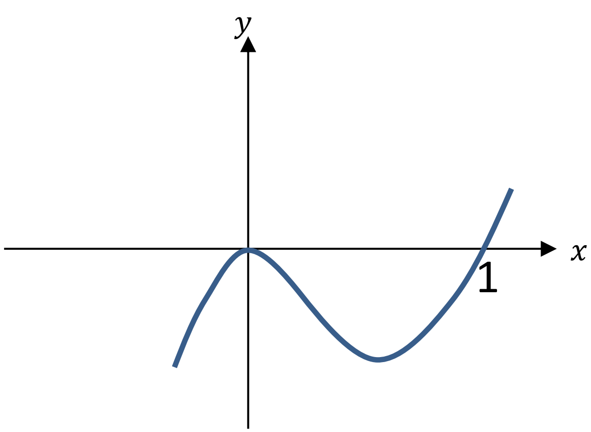
Shape? x^3 term is positive so 'uphill' shape.

Roots? $x = 0\ or\ x = 1$

However as root of 0 is **repeated** (because the factor of x appears twice), the curve **touches** at $x = 0$.

This is sort of because the curve crosses at 0 then immediately crosses at 0 again!

***y*-intercept?** If $x = 0, y = 0$



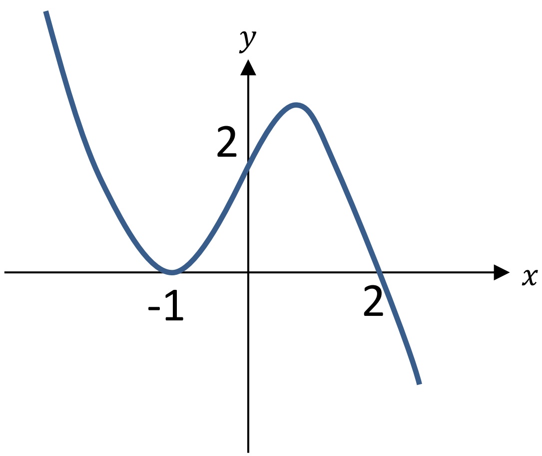
Cubics

Sketch the curve with equation
$$y = (2 - x)(x + 1)^2$$

Shape? Downhill.

Roots? $x = 2, or\ x = -1$
Curve crosses as 2, but touches at -1 (again, because of repeated root)

y-intercept? 2

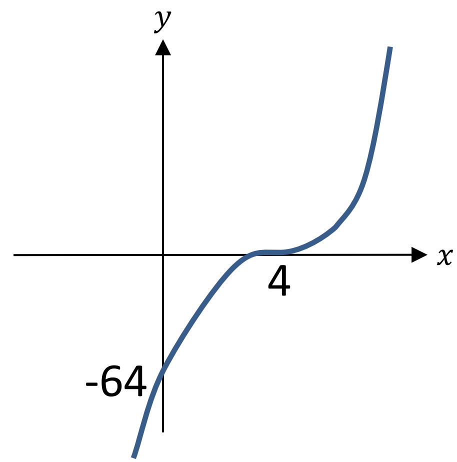


Sketch the curve with equation
$$y = (x - 4)^3$$

Shape? Uphill.

Roots? $x = 4$
But root is triple repeated.
We have a **point of inflection** at $x = 4$.

y-intercept?



Exam Notes: You might be able to see we get this shape at $x = 4$ because as the root is triple repeated, the curve crosses at 4, then crosses again, then crosses again, hence ending up in the same direction and the line becoming momentarily horizontal.

A point of inflection is where the curve goes from 'convex' to 'concave' (or vice versa), i.e. curves in one direction before and curves in another direction after. You might have encountered these terms in Physics.

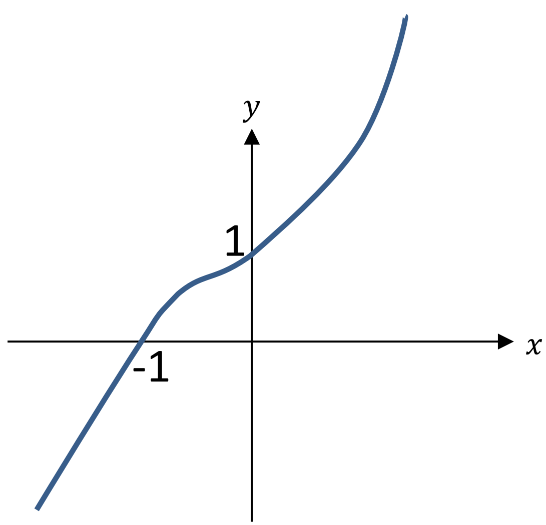
Cubics with Limited Roots

Sketch the curve with equation
$$y = (x+1)(x^2+x+1)$$

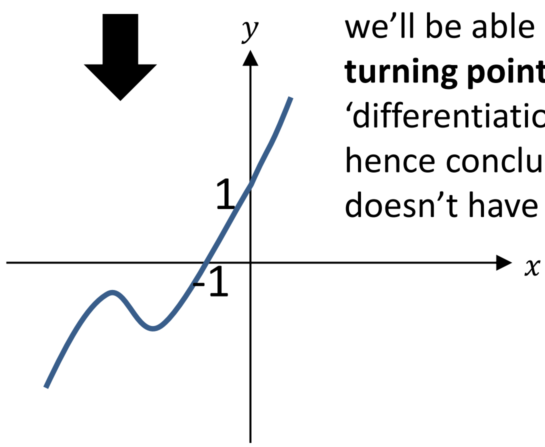
Shape? Uphill.

Roots? Either $x+1=0$ (giving root of -1) or $x^2+x+1=0$.
This does not have any solutions as the discriminant is -3.
Thus -1 is the only root.

***y*-intercept?** 1



We don't have enough information to determine the exact shape. It could for example have been:



However, in Chapter 12, we'll be able to work **turning points** using 'differentiation', and hence conclude that it doesn't have any!

Finding the equation yourself

Edexcel C1 May 2013(R) Q9

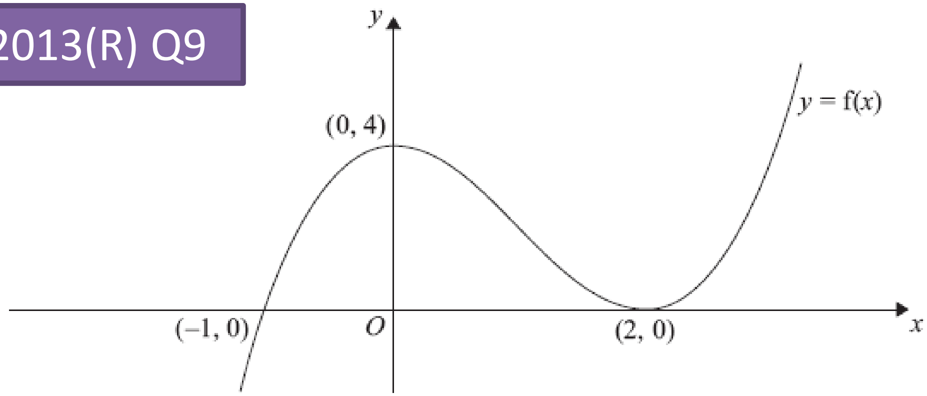


Figure 1 shows a sketch of the curve C with equation $y = f(x)$.
The curve C passes through the point (–1, 0) and touches the x-axis at the point (2, 0).
The curve C has a maximum at the point (0, 4).
The equation of the curve C can be written in the form.

$$y = x^3 + ax^2 + bx + c$$

where a, b and c are integers.
(a) Calculate the values of a, b, c.

?

Finding the equation yourself

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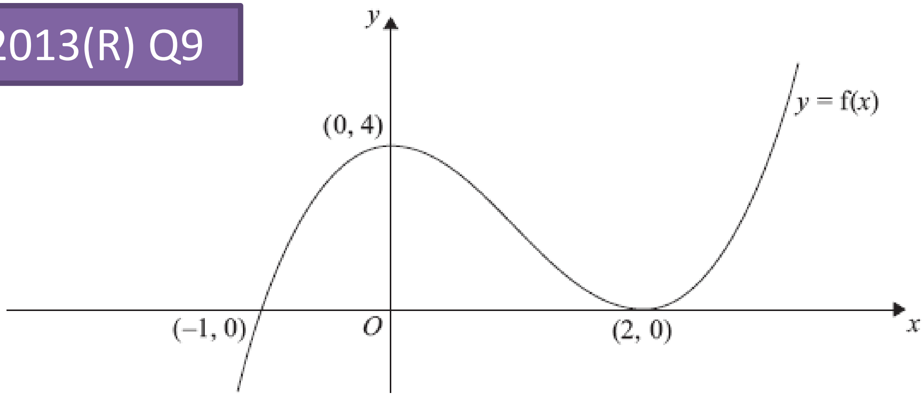

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where a, b and c are integers.
(a) Calculate the values of a, b, c.

If it crosses at $(-1,0)$ we must have $(x+1)$.
If it touches at $(2,0)$ we must have $(x-2)^2$

$$\therefore \ y = (x-2)^2(x+1)$$
$$= (x^2 - 4x + 4)(x+1)$$
$$= x^3 + x^2 - 4x^2 - 4x + 4x + 4$$
$$= x^3 - 3x^2 + 4$$
$$\therefore a = -3, b = 0, c = 4$$

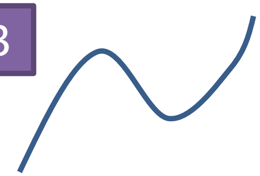
Test Your Understanding

1 Sketch the curve with equation $y = x(x-3)^2$

?

2 Sketch the curve with equation $y = -(x+2)^3$

?

3 

A curve has this shape, touches the x axis at 3 and crosses the x axis at -2. Give a suitable equation for this graph.

?

☠ Sketch the curve with equation $y = 2x^2(x-1)(x+1)^3$

?

Test Your Understanding

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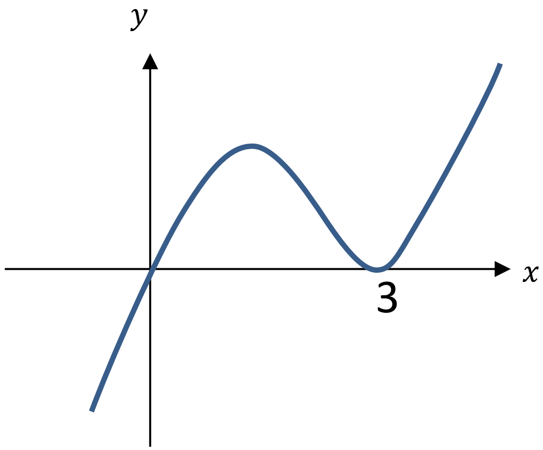

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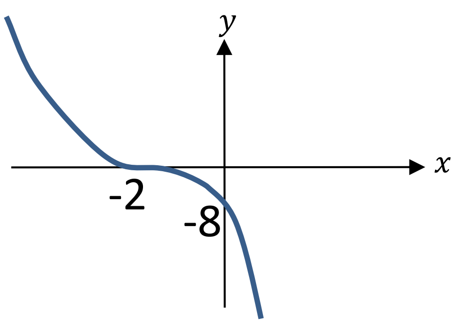

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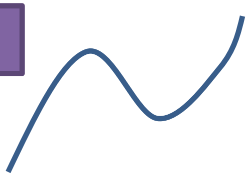

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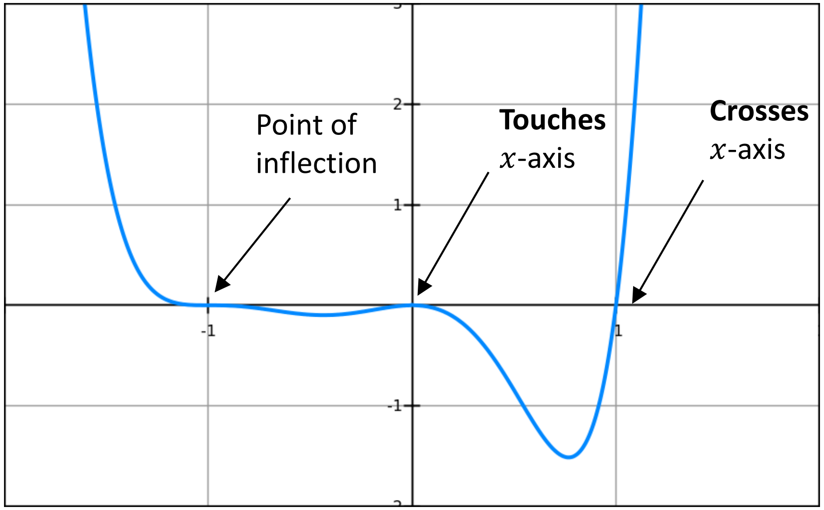

Exercise 4.1

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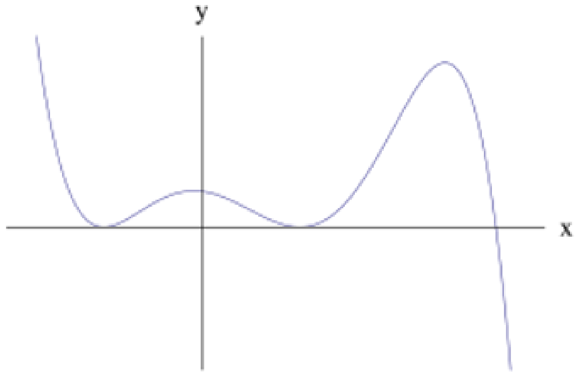
Extension

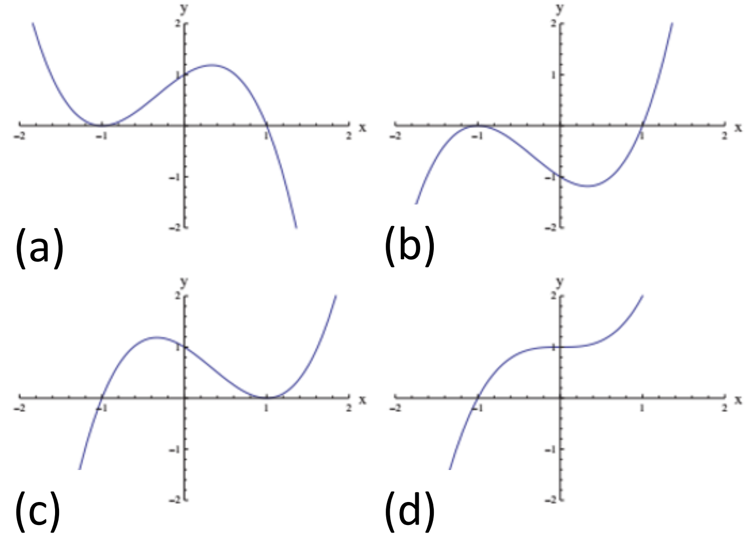
1 [MAT 2012 1E] Which one of the following equations could possibly have the graph given below?



A) $y = (3 - x)^2(3 + x)^2(1 - x)$
B) $y = -x^2(x - 9)(x^2 - 3)$
C) $y = (x - 6)(x - 2)^2(x + 2)^2$
D) $y = (x^2 - 1)^2(3 - x)$

?

2 [MAT 2011 1A] A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axis?



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Exercise 4.1

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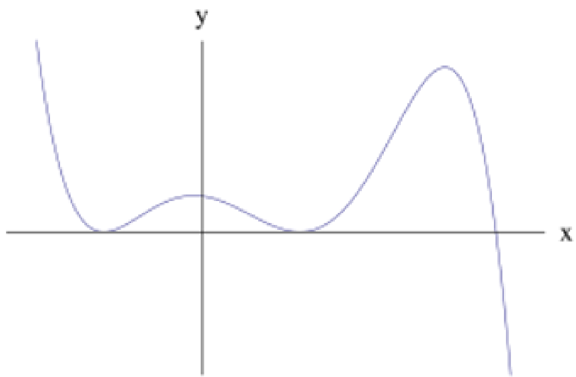

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Solution: D

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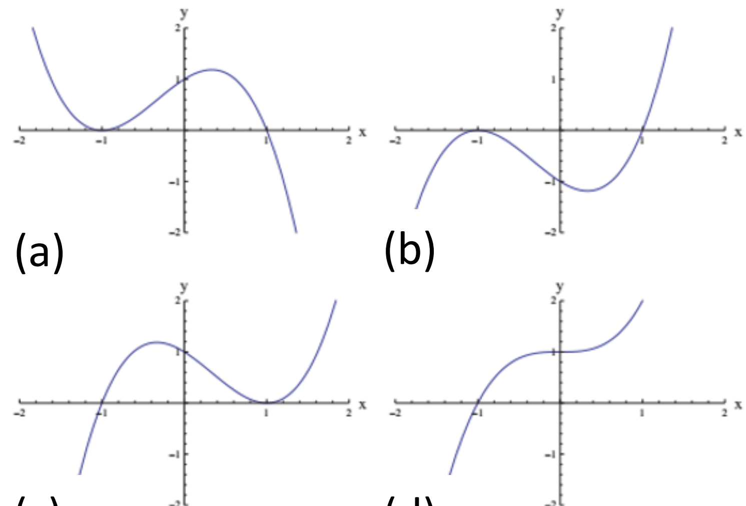


(a) (b) (c) (d)

Cubics can sometimes be factorised by pairing the terms: $\boldsymbol{x^2(x - 1) - 1(x - 1) = (x^2 - 1)(x - 1) = (x + 1)(x - 1)^2}$ $\therefore$ **(c)**

Homework Exercise

1 Sketch the following curves and indicate clearly the points of intersection with the axes:

a $y = (x - 3)(x - 2)(x + 1)$ **b** $y = (x - 1)(x + 2)(x + 3)$

c $y = (x + 1)(x + 2)(x + 3)$ **d** $y = (x + 1)(1 - x)(x + 3)$

e $y = (x - 2)(x - 3)(4 - x)$ **f** $y = x(x - 2)(x + 1)$

g $y = x(x + 1)(x - 1)$ **h** $y = x(x + 1)(1 - x)$

i $y = (x - 2)(2x - 1)(2x + 1)$ **j** $y = x(2x - 1)(x + 3)$

2 Sketch the curves with the following equations:

a $y = (x + 1)^2(x - 1)$ **b** $y = (x + 2)(x - 1)^2$ **c** $y = (2 - x)(x + 1)^2$

d $y = (x - 2)(x + 1)^2$ **e** $y = x^2(x + 2)$ **f** $y = (x - 1)^2x$

g $y = (1 - x)^2(3 + x)$ **h** $y = (x - 1)^2(3 - x)$ **i** $y = x^2(2 - x)$

j $y = x^2(x - 2)$

3 Factorise the following equations and then sketch the curves:

a $y = x^3 + x^2 - 2x$ **b** $y = x^3 + 5x^2 + 4x$ **c** $y = x^3 + 2x^2 + x$

d $y = 3x + 2x^2 - x^3$ **e** $y = x^3 - x^2$ **f** $y = x - x^3$

g $y = 12x^3 - 3x$ **h** $y = x^3 - x^2 - 2x$ **i** $y = x^3 - 9x$

j $y = x^3 - 9x^2$

4 Sketch the following curves and indicate the coordinates of the points where the curves cross the axes:

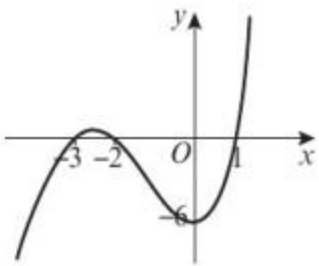
a $y = (x - 2)^3$ **b** $y = (2 - x)^3$ **c** $y = (x - 1)^3$ **d** $y = (x + 2)^3$

e $y = -(x + 2)^3$ **f** $y = (x + 3)^3$ **g** $y = (x - 3)^3$ **h** $y = (1 - x)^3$

i $y = -(x - 2)^3$ **j** $y = -\left(x - \frac{1}{2}\right)^3$

Homework Exercise

5 The graph of $y = x^3 + bx^2 + cx + d$ is shown opposite, where b, c and d are real constants.



a Find the values of b, c and d. **(3 marks)**

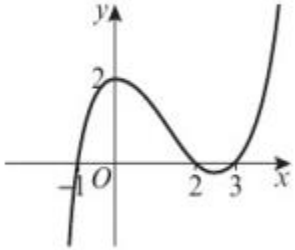
b Write down the coordinates of the point where the curve crosses the y-axis. **(1 mark)**

Problem-solving

Start by writing the equation in the form $y = (x - p)(x - q)(x - r)$.

6 The graph of $y = ax^3 + bx^2 + cx + d$ is shown opposite, where a, b, c and d are real constants.

Find the values of a, b, c and d. **(4 marks)**



7 Given that $f(x) = (x - 10)(x^2 - 2x) + 12x$

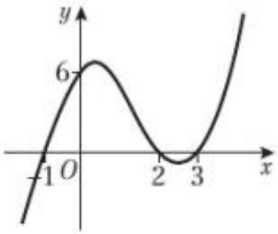
a Express $f(x)$ in the form $x(ax^2 + bx + c)$ where a, b and c are real constants. **(3 marks)**

b Hence factorise $f(x)$ completely. **(2 marks)**

c Sketch the graph of $y = f(x)$ showing clearly the points where the graph intersects the axes. **(3 marks)**

Homework Answers

1 **a**



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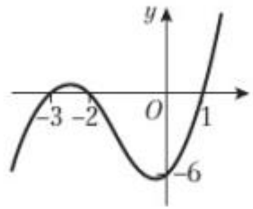

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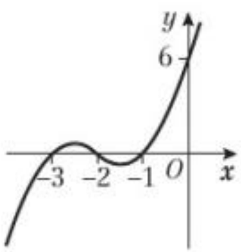

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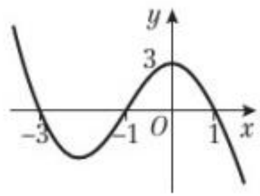

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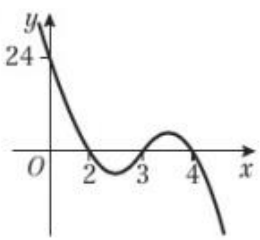

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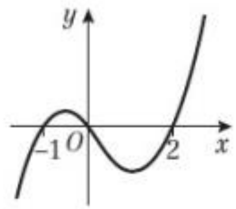

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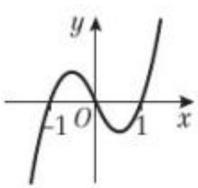

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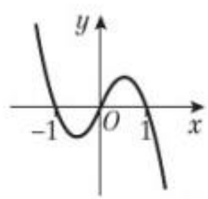

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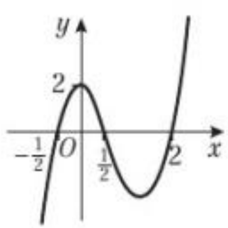

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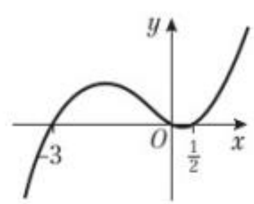
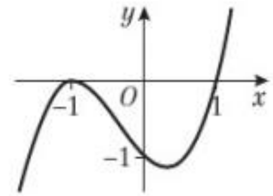


2 **a**



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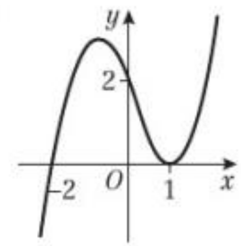
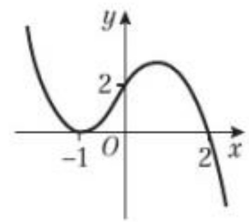


c



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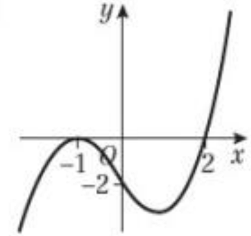

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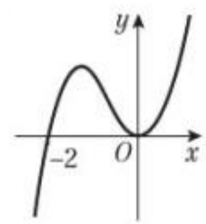

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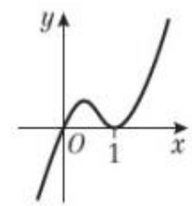

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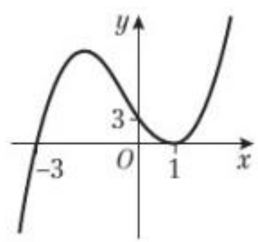

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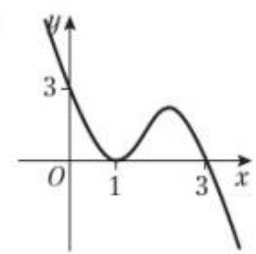
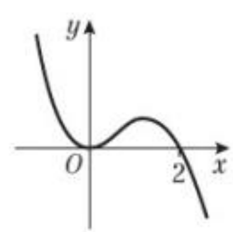


i



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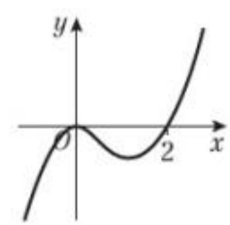
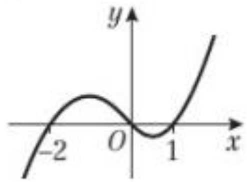
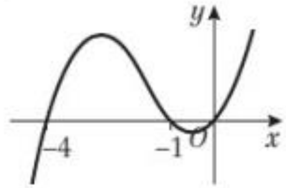


Homework Answers

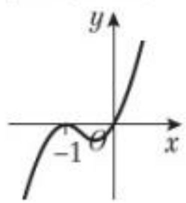
3 a $y = x(x + 2)(x - 1)$



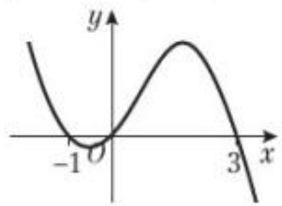
b $y = x(x + 4)(x + 1)$



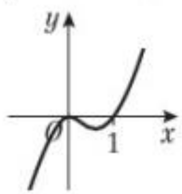
c $y = x(x + 1)^2$



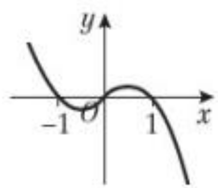
d $y = x(x + 1)(3 - x)$



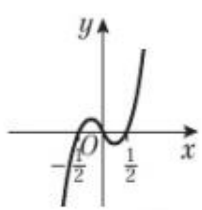
e $y = x^2(x - 1)$



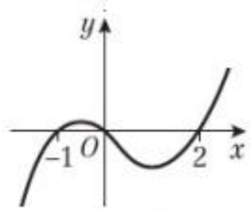
f $y = x(1 - x)(1 + x)$



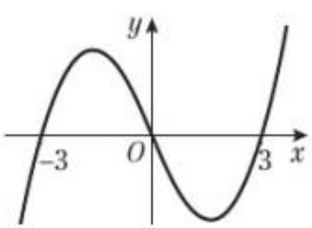
g $y = 3x(2x - 1)(2x + 1)$



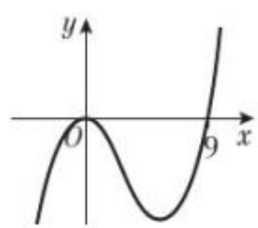
h $y = x(x + 1)(x - 2)$



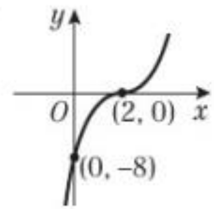
i $y = x(x - 3)(x + 3)$



j $y = x^2(x - 9)$



4 a



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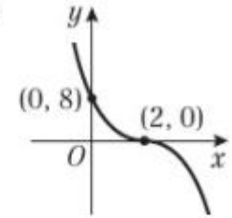

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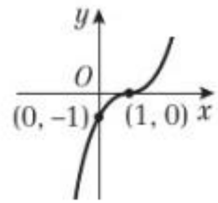

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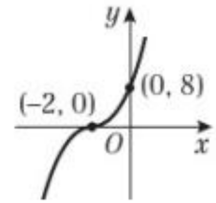

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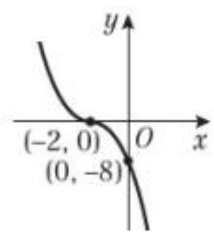

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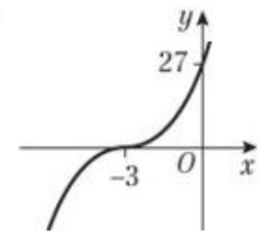

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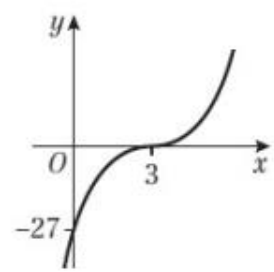

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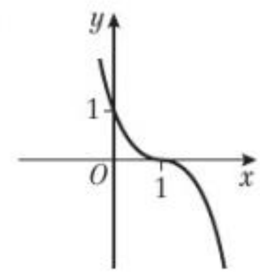
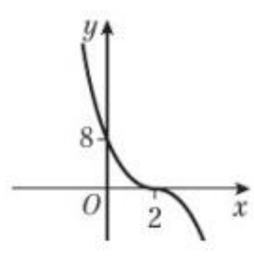


i



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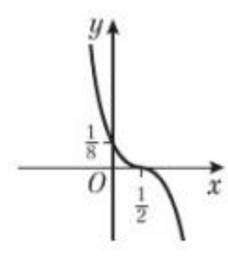

Homework Answers

5 **a** $b = 4, c = 1, d = -6$

b $(0, -6)$

6 $a = \frac{1}{3}, b = -\frac{4}{3}, c = \frac{1}{3}, d = 2$

7 **a** $x(x^2 - 12x + 32)$

b $x(x - 8)(x - 4)$

c