P2 Chapter 4: Binomial Terms

Chapter Practice

Key Points

1 This form of the binomial expansion can be applied to negative or fractional values of n to obtain an infinite series:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots + \frac{n(n-1)\ldots(n-r+1)x^r}{r!} + \ldots$$

The expansion is valid when $|x| < 1$, $n \in \mathbb{R}$.

2 The expansion of $(1 + bx)^n$, where n is negative or a fraction, is valid for $|bx| < 1$, or $|x| < \frac{1}{|b|}$.

3 The expansion of $(a + bx)^n$, where n is negative or a fraction, is valid for $\left|\frac{b}{a}x\right| < 1$ or $|x| < \left|\frac{a}{b}\right|$.

Chapter Exercises

1 For each of the following,

i find the binomial expansion up to and including the x^3 term

ii state the range of values of x for which the expansion is valid.

a $(1-4x)^3$ **b** $\sqrt{16+x}$ **c** $\dfrac{1}{1-2x}$ **d** $\dfrac{4}{2+3x}$

e $\dfrac{4}{\sqrt{4-x}}$ **f** $\dfrac{1+x}{1+3x}$ **g** $\left(\dfrac{1+x}{1-x}\right)^2$ **h** $\dfrac{x-3}{(1-x)(1-2x)}$

2 Use the binomial expansion to expand $\left(1-\dfrac{1}{2}x\right)^{\frac{1}{2}}$, $|x|<2$ in ascending powers of x, up to and including the term in x^3, simplifying each term. **(5 marks)**

3 **a** Give the binomial expansion of $(1+x)^{\frac{1}{2}}$ up to and including the term in x^3.

b By substituting $x=\frac{1}{4}$, find an approximation to $\sqrt{5}$ as a fraction.

4 The binomial expansion of $(1+9x)^{\frac{2}{3}}$ in ascending powers of x up to and including the term in x^3 is $1+6x+cx^2+dx^3$, $|x|<\frac{1}{9}$

a Find the value of c and the value of d. **(4 marks)**

b Use this expansion with your values of c and d together with an appropriate value of x to obtain an estimate of $(1.45)^{\frac{2}{3}}$. **(2 marks)**

c Obtain $(1.45)^{\frac{2}{3}}$ from your calculator and hence make a comment on the accuracy of the estimate you obtained in part **b**. **(1 mark)**

5 In the expansion of $(1 + ax)^{\frac{1}{2}}$ the coefficient of x^2 is -2.

a Find the possible values of a.

b Find the corresponding coefficients of the x^3 term.

6 $f(x) = (1 + 3x)^{-1}$, $|x| < \frac{1}{3}$

a Expand $f(x)$ in ascending powers of x up to and including the term in x^3. **(5 marks)**

b Hence show that, for small x:

$$\frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3.$$ **(4 marks)**

c Taking a suitable value for x, which should be stated, use the series expansion in part **b** to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places. **(3 marks)**

7 When $(1 + ax)^n$ is expanded as a series in ascending powers of x, the coefficients of x and x^2 are -6 and 27 respectively.

a Find the values of a and n. **(4 marks)**

b Find the coefficient of x^3. **(3 marks)**

c State the values of x for which the expansion is valid. **(1 mark)**

8 Show that if x is sufficiently small then $\frac{3}{\sqrt{4 + x}}$ can be approximated by $\frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$.

Chapter Exercises

9 **a** Expand $\frac{1}{\sqrt{4-x}}$, where $|x| < 4$, in ascending powers of x up to and including the term in x^2. Simplify each term. **(5 marks)**

b Hence, or otherwise, find the first 3 terms in the expansion of $\frac{1+2x}{\sqrt{4-x}}$ as a series in ascending powers of x. **(4 marks)**

10 **a** Find the first four terms of the expansion, in ascending powers of x, of $(2+3x)^{-1}$, $|x| < \frac{2}{3}$ **(4 marks)**

b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of:

$\frac{1+x}{2+3x}$, $|x| < \frac{2}{3}$ **(3 marks)**

11 **a** Use the binomial theorem to expand $(4+x)^{-\frac{1}{2}}$, $|x| < 4$, in ascending powers of x, up to and including the x^3 term, giving each answer as a simplified fraction. **(5 marks)**

b Use your expansion, together with a suitable value of x, to obtain an approximation to $\frac{\sqrt{2}}{2}$. Give your answer to 4 decimal places. **(3 marks)**

Chapter Exercises

12 $q(x) = (3 + 4x)^{-3}$, $|x| < \frac{3}{4}$

Find the binomial expansion of $q(x)$ in ascending powers of x, up to and including the term in the x^2. Give each coefficient as a simplified fraction. **(5 marks)**

13 $g(x) = \dfrac{39x + 12}{(x + 1)(x + 4)(x - 8)}$, $|x| < 1$

$g(x)$ can be expressed in the form $g(x) = \dfrac{A}{x + 1} + \dfrac{B}{x + 4} + \dfrac{C}{x - 8}$

a Find the values of A, B and C. **(4 marks)**

b Hence, or otherwise, find the series expansion of $g(x)$, in ascending powers of x, up to and including the x^2 term. Simplify each term. **(7 marks)**

14 $f(x) = \dfrac{12x + 5}{(1 + 4x)^2}$, $|x| < \dfrac{1}{4}$

For $x \neq -\dfrac{1}{4}$, $\dfrac{12x + 5}{(1 + 4x)^2} = \dfrac{A}{1 + 4x} + \dfrac{B}{(1 + 4x)^2}$, where A and B are constants.

a Find the values of A and B. **(3 marks)**

b Hence, or otherwise, find the series expansion of $f(x)$, in ascending powers of x, up to and including the term x^2, simplifying each term. **(6 marks)**

15 $q(x) = \dfrac{9x^2 + 26x + 20}{(1 + x)(2 + x)}$, $|x| < 1$

a Show that the expansion of $q(x)$ in ascending powers of x can be approximated to $10 - 2x + Bx^2 + Cx^3$ where B and C are constants to be found. **(7 marks)**

b Find the percentage error made in using the series expansion in part **a** to estimate the value of $q(0.1)$. Give your answer to 2 significant figures. **(4 marks)**

Challenge

Obtain the first four non-zero terms in the expansion, in ascending powers of x, of the function $f(x)$ where $f(x) = \dfrac{1}{\sqrt{1 + 3x^2}}$, $3x^2 < 1$.

Chapter Answers

1 **a** **i** $1 - 12x + 48x^2 - 64x^3$ **ii** all x

b **i** $4 + \frac{x}{8} - \frac{x^2}{512} + \frac{x^3}{16384}$ **ii** $|x| < 16$

c **i** $1 + 2x + 4x^2 + 8x^3$ **ii** $|x| < \frac{1}{2}$

d **i** $2 - 3x + \frac{9x^2}{2} - \frac{27x^3}{4}$ **ii** $|x| < \frac{2}{3}$

e **i** $2 + \frac{x}{4} + \frac{3x^2}{64} + \frac{5x^3}{512}$ **ii** $|x| < 4$

f **i** $1 - 2x + 6x^2 - 18x^3$ **ii** $|x| < \frac{1}{3}$

g **i** $1 + 4x + 8x^2 + 12x^3$ **ii** $|x| < 1$

h **i** $-3 - 8x - 18x^2 - 38x^3$ **ii** $|x| < \frac{1}{2}$

2 $1 - \frac{x}{4} - \frac{x^2}{32} - \frac{x^3}{128}$

3 **a** $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$ **b** $\frac{1145}{512}$

4 **a** $c = -9, d = 36$ **b** 1.282

c calculator = 1.28108713, approximation is correct to 2 decimal places.

5 **a** $a = 4$ or $a = -4$

b coefficient of $x^3 = 4$, coefficient of $x^3 = -4$.

6 **a** $1 - 3x + 9x^2 - 27x^3$

b $(1 + x)(1 - 3x + 9x^2 - 27x^3)$
$= 1 - 3x + 9x^2 - 27x^3 + x - 3x^2 + 9x^3$
$= 1 - 2x + 6x^2 - 18x^3$

c $x = 0.01$, $0.980\,58$

7 **a** $n = -2, a = 3$ **b** -108

c $|x| < \frac{1}{3}$

8 For small values of x ignore powers of x^3 and higher.
$\frac{1}{\sqrt{4+x}} = \frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256}$, $\frac{3}{\sqrt{4+x}} = \frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$

9 **a** $\frac{1}{2} + \frac{x}{16} + \frac{3}{256}x^2$ **b** $\frac{1}{2} + \frac{17}{16}x + \frac{35}{256}x^2$

10 **a** $\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3$ **b** $\frac{1}{2} - \frac{x}{4} + \frac{3}{8}x^2 - \frac{9}{16}x^3$

11 **a** $\frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256} - \frac{5x^3}{2048}$ **b** 0.6914

12 $\frac{1}{27} - \frac{4}{27}x + \frac{32}{81}x^2$

13 **a** $A = 1, B = -4, C = 3$ **b** $-\frac{3}{8} - \frac{51}{64}x + \frac{477}{512}x^2$

14 **a** $A = 3$ and $B = 2$ **b** $5 - 28x + 144x^2$

15 **a** $10 - 2x + \frac{5}{2}x^2 - \frac{11}{4}x^3$, so $B = \frac{5}{2}$ and $C = -\frac{11}{4}$

b Percent error = 0.0027%

Challenge

$1 - \frac{3x^2}{2} + \frac{27x^4}{8} - \frac{135x^6}{16}$