P2 Chapter 5: Radians

Chapter Practice

Key Points

1 • 2π radians $= 360°$ • π radians $= 180°$ • 1 radian $= \dfrac{180°}{\pi}$

2 • $30° = \dfrac{\pi}{6}$ radians • $45° = \dfrac{\pi}{4}$ radians • $60° = \dfrac{\pi}{3}$ radians

• $90° = \dfrac{\pi}{2}$ radians • $180° = \pi$ radians • $360° = 2\pi$ radians

3 You need to learn the exact values of the trigonometric ratios of these angles measured in radians.

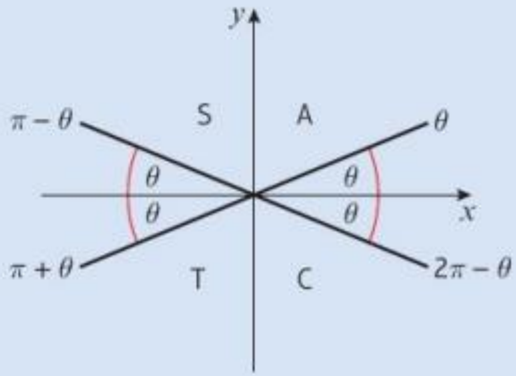
• $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$ • $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ • $\tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

• $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ • $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$ • $\tan\dfrac{\pi}{3} = \sqrt{3}$

• $\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ • $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ • $\tan\dfrac{\pi}{4} = 1$

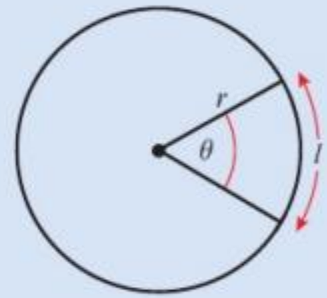
4 You can use these rules to find sin, cos or tan of any positive or negative angle measured in radians using the corresponding acute angle made with the x-axis, θ.

- $\sin(\pi - \theta) = \sin\theta$
- $\sin(\pi + \theta) = -\sin\theta$
- $\sin(2\pi - \theta) = -\sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(\pi + \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$

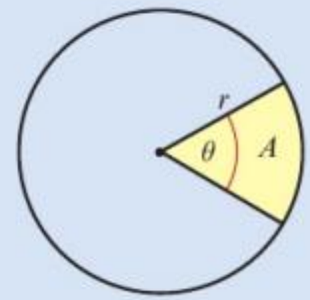


Key Points

5 To find the arc length l of a sector of a circle use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



6 To find the area A of a sector of a circle use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



7 The area of a segment in a circle of radius r is

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

8 When θ is small and measured in radians:

- $\sin\theta \approx \theta$
- $\tan\theta \approx \theta$
- $\cos\theta \approx 1 - \frac{\theta^2}{2}$

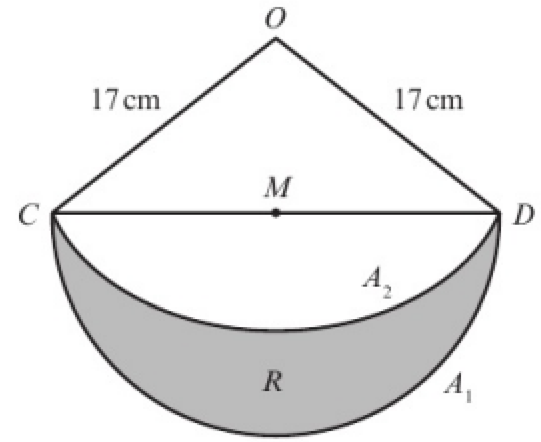
Chapter Exercises

1 Triangle ABC is such that $AB = 5$ cm, $AC = 10$ cm and $\angle ABC = 90°$.
An arc of a circle, centre A and radius 5 cm, cuts AC at D.

a State, in radians, the value of $\angle BAC$.

b Calculate the area of the region enclosed by BC, DC and the arc BD.

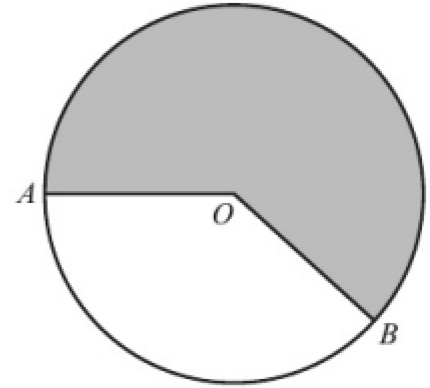
2 The diagram shows the triangle OCD with $OC = OD = 17$ cm and $CD = 30$ cm. The midpoint of CD is M. A semicircular arc A_1, with centre M is drawn, with CD as diameter.
A circular arc A_2 with centre O and radius 17 cm, is drawn from C to D. The shaded region R is bounded by the arcs A_1 and A_2. Calculate, giving answers to 2 decimal places:



a the area of the triangle OCD **(4 marks)**

b the area of the shaded region R. **(5 marks)**

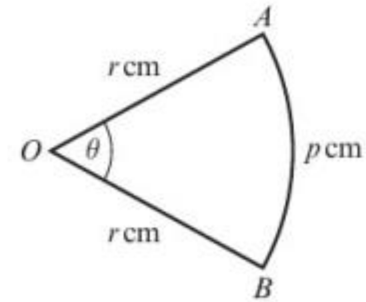
3 The diagram shows a circle, centre O, of radius 6 cm.
The points A and B are on the circumference of the circle.
The area of the shaded major sector is 80 cm^2.
Given that $\angle AOB = \theta$ radians, where $0 < \theta < \pi$, calculate:



a the value, to 3 decimal places, of θ **(3 marks)**

b the length in cm, to 2 decimal places, of the minor arc AB. **(2 marks)**

4 The diagram shows a sector OAB of a circle, centre O and radius r cm. The length of the arc AB is p cm and $\angle AOB$ is θ radians.



a Find θ in terms of p and r. **(2 marks)**

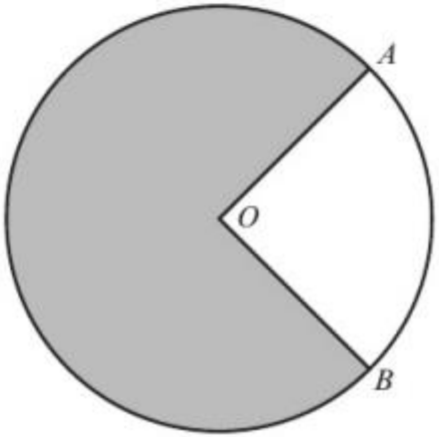
b Deduce that the area of the sector is $\frac{1}{2}pr$ cm^2. **(2 marks)**

Given that $r = 4.7$ and $p = 5.3$, where each has been measured to 1 decimal place, find, giving your answer to 3 decimal places:

c the least possible value of the area of the sector **(2 marks)**

d the range of possible values of θ. **(3 marks)**

5 The diagram shows a circle centre O and radius 5 cm. The length of the minor arc AB is 6.4 cm.



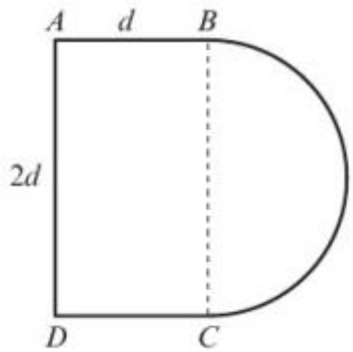
a Calculate, in radians, the size of the acute angle AOB. **(2 marks)**

The area of the minor sector AOB is R_1 cm^2 and the area of the shaded major sector is R_2 cm^2.

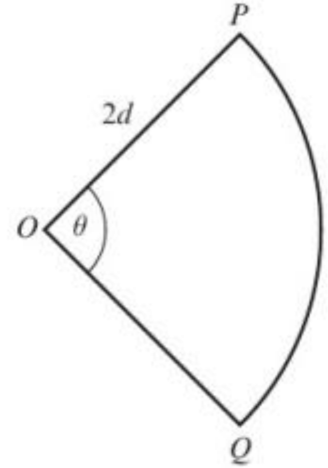
b Calculate the value of R_1. **(2 marks)**

c Calculate $R_1 : R_2$ in the form $1 : p$, giving the value of p to 3 significant figures. **(3 marks)**

6 The diagrams show the cross-sections of two drawer handles. Shape X is a rectangle $ABCD$ joined to a semicircle with BC as diameter. The length $AB = d$ cm and $BC = 2d$ cm. Shape Y is a sector OPQ of a circle with centre O and radius $2d$ cm. Angle POQ is θ radians.



Shape X



Shape Y

Given that the areas of shapes X and Y are equal,

a prove that $\theta = 1 + \frac{\pi}{4}$ **(5 marks)**

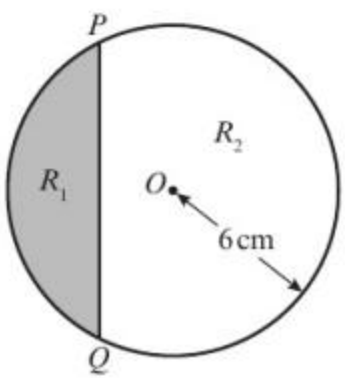
Using this value of θ, and given that $d = 3$, find in terms of π:

b the perimeter of shape X **(3 marks)**

c the perimeter of shape Y. **(3 marks)**

d Hence find the difference, in mm, between the perimeters of shapes X and Y. **(1 mark)**

7 The diagram shows a circle centre O and radius 6 cm. The chord PQ divides the circle into a minor segment R_1 of area A_1 cm^2 and a major segment R_2 of area A_2 cm^2. The chord PQ subtends an angle θ radians at O.



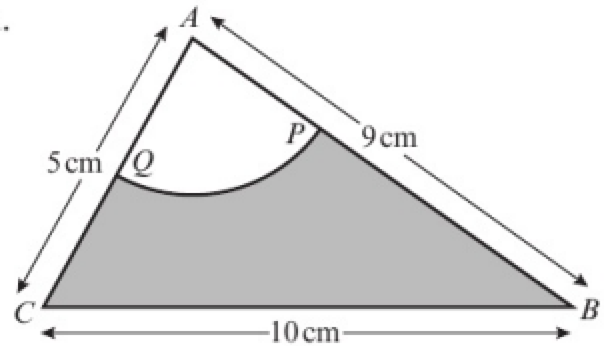
a Show that $A_1 = 18(\theta - \sin\theta)$. **(2 marks)**

Given that $A_2 = 3A_1$,

b show that $\sin\theta = \theta - \frac{\pi}{2}$ **(4 marks)**

Chapter Exercises

8 Triangle ABC has $AB = 9$ cm, $BC = 10$ cm and $CA = 5$ cm. A circle, centre A and radius 3 cm, intersects AB and AC at P and Q respectively, as shown in the diagram.



a Show that, to 3 decimal places, $\angle BAC = 1.504$ radians. **(2 marks)**

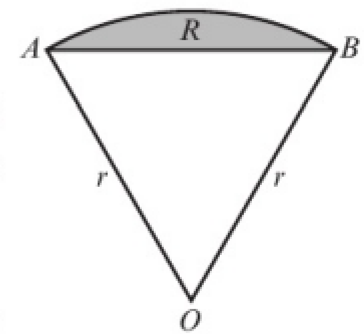
b Calculate:

i the area, in cm², of the sector APQ

ii the area, in cm², of the shaded region $BPQC$

iii the perimeter, in cm, of the shaded region $BPQC$. **(8 marks)**

9 The diagram shows the sector OAB of a circle of radius r cm. The area of the sector is 15 cm² and $\angle AOB = 1.5$ radians.



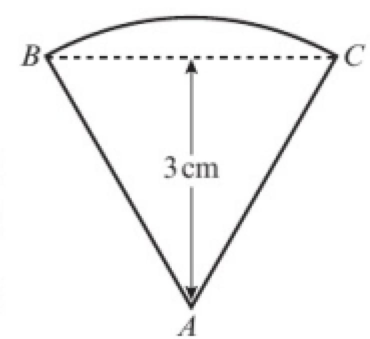
a Prove that $r = 2\sqrt{5}$. **(2 marks)**

b Find, in cm, the perimeter of the sector OAB. **(3 marks)**

The segment R, shaded in the diagram, is enclosed by the arc AB and the straight line AB.

c Calculate, to 3 decimal places, the area of R. **(2 marks)**

10 The shape of a badge is a sector ABC of a circle with centre A and radius AB, as shown in the diagram. The triangle ABC is equilateral and has perpendicular height 3 cm.

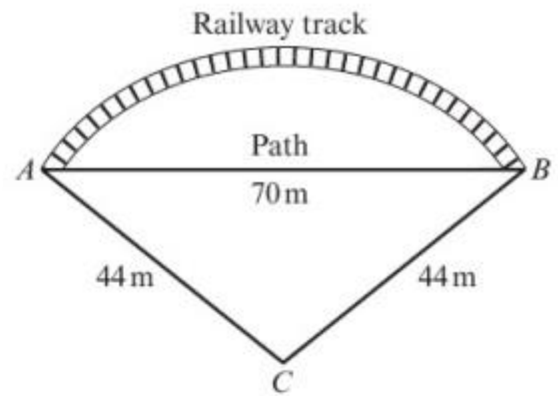


a Find, in surd form, the length of AB. **(2 marks)**

b Find, in terms of π, the area of the badge. **(2 marks)**

c Prove that the perimeter of the badge is $\frac{2\sqrt{3}}{3}(\pi + 6)$ cm. **(4 marks)**

11 There is a straight path of length 70 m from the point A to the point B. The points are joined also by a railway track in the form of an arc of the circle whose centre is C and whose radius is 44 m, as shown in the diagram.



a Show that the size, to 2 decimal places, of $\angle ACB$ is 1.84 radians. **(2 marks)**

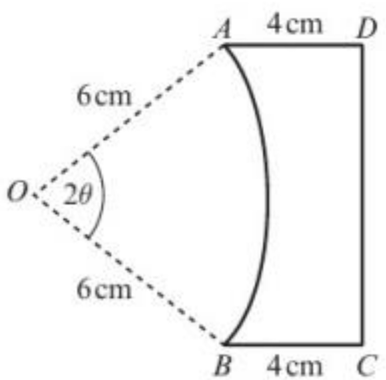
b Calculate:

i the length of the railway track

ii the shortest distance from C to the path

iii the area of the region bounded by the railway track and the path. **(6 marks)**

12 The diagram shows the cross-section $ABCD$ of a glass prism. $AD = BC = 4$ cm and both are at right angles to DC. AB is the arc of a circle, centre O and radius 6 cm. Given that $\angle AOB = 2\theta$ radians, and that the perimeter of the cross-section is $2(7 + \pi)$ cm,



a show that $(2\theta + 2\sin\theta - 1) = \frac{\pi}{3}$

b verify that $\theta = \frac{\pi}{6}$

c find the area of the cross-section.

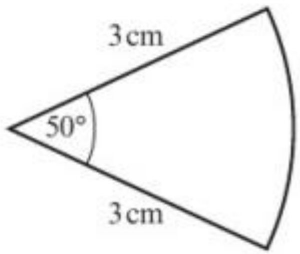
13 Two circles C_1 and C_2, both of radius 12 cm have centres O_1 and O_2 respectively. O_1 lies on the circumference of C_2; O_2 lies on the circumference of C_1. The circles intersect at A and B, and enclose the region R.

a Show that $\angle AO_1B = \frac{2\pi}{3}$

b Hence write down, in terms of π, the perimeter of R.

c Find the area of R, giving your answer to 3 significant figures.

14 A teacher asks a student to find the area of the following sector.
The attempt is shown below.



$$\begin{aligned} \text{Area} &= \tfrac{1}{2}r^2\theta \\ &= \tfrac{1}{2} \times 3^2 \times 50 \\ &= 225 \text{ cm}^2 \end{aligned}$$

a Identify the mistake made by the student. **(1 mark)**

b Calculate the correct area of the sector. **(2 marks)**

15 When θ is small, find the approximate values of:

a $\dfrac{\cos\theta - 1}{\theta\tan 2\theta}$

b $\dfrac{2(1 - \cos\theta) - 1}{\tan\theta - 1}$

Chapter Exercises

16 **a** When θ is small, show that the expression $\dfrac{7 + 2\cos 2\theta}{\tan 2\theta + 3}$ can be written as $3 - 2\theta$. **(3 marks)**

b Hence write down the value of $\dfrac{7 + 2\cos 2\theta}{\tan 2\theta + 3}$ when θ is small. **(1 mark)**

17 **a** When θ is small, show that the equation

$$32\cos 5\theta + 203\tan 10\theta = 182$$

can be written as

$$40\theta^2 - 2030\theta + 15 = 0$$

(4 marks)

b Hence, find the solutions of the equation

$$32\cos 5\theta + 203\tan 10\theta = 182$$

(3 marks)

c Comment on the validity of your solutions. **(1 mark)**

18 When θ is small, find the approximate value of $\cos^4\theta - \sin^4\theta$.

19 Solve the following equations for θ, giving your answers to 3 significant figures where appropriate, in the intervals indicated.

a $3\sin\theta = 2$, $0 \leqslant \theta \leqslant \pi$

b $\sin\theta = -\cos\theta$, $-\pi \leqslant \theta \leqslant \pi$

c $\tan\theta + \dfrac{1}{\tan\theta} = 2$, $0 \leqslant \theta \leqslant 2\pi$

d $2\sin^2\theta - \sin\theta - 1 = \sin^2\theta$, $-\pi \leqslant \theta \leqslant \pi$

20 **a** Sketch the graphs of $y = 5\sin x$ and $y = 3\cos x$ on the same axes ($0 \leqslant x \leqslant 2\pi$), marking on all the points where the graphs cross the axes.

b Write down how many solutions there are in the given range for the equation $5\sin x = 3\cos x$.

c Solve the equation $5\sin x = 3\cos x$ algebraically, giving your answers to 3 significant figures.

21 **a** Express $4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right)$ as a single trigonometric function. **(1 mark)**

b Hence solve $4\sin\theta - \cos\left(\frac{\pi}{2} - \theta\right) = 1$ in the interval $0 \leqslant \theta \leqslant 2\pi$. Give your answers to 3 significant figures. **(3 marks)**

22 Find the values of x in the interval $0 < x < \frac{3\pi}{2}$ which satisfy the equation

$$\frac{\sin 2x + 0.5}{1 - \sin 2x} = 2$$ **(6 marks)**

23 A teacher asks two students to solve the equation $2\cos^2 x = 1$ for $-\pi \leqslant x \leqslant \pi$.
The attempts are shown below.

Student A:

$\cos x = \pm\frac{1}{\sqrt{2}}$

Reject $-\frac{1}{\sqrt{2}}$ as cosine cannot be negative

$x = \frac{\pi}{4}$ or $x = -\frac{\pi}{4}$

Student B:

$2\cos^2 x = 1$

$\cos x = \pm\frac{1}{2}$

$x = \frac{\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}$

a Identify the mistake made by Student A. **(1 mark)**

b Identify the mistake made by Student B. **(1 mark)**

c Calculate the correct solutions to the equation. **(4 marks)**

Chapter Exercises

24 A teacher asks a student to solve the equation $2\tan 2x = 5$ for $0 \leqslant x \leqslant 2\pi$.
The attempt is shown below.

$2\tan 2x = 5$
$\tan 2x = 2.5$
$2x = 1.19, 4.33$
$x = 0.595$ rad or 2.17 rad (3 s.f.)

Problem-solving

Solve the equation yourself then compare your working with the student's answer.

a Identify the mistake made by the student. **(1 mark)**

b Calculate the correct solutions to the equation. **(4 marks)**

25 **a** Show that the equation

$$5\sin x = 1 + 2\cos^2 x$$

can be written in the form

$$2\sin^2 x + 5\sin x - 3 = 0$$ **(2 marks)**

b Solve, for $0 \leqslant x < 2\pi$,

$$2\sin^2 x + 5\sin x - 3 = 0$$ **(4 marks)**

26 **a** Show that the equation

$$4\sin^2 x + 9\cos x - 6 = 0$$

can be written as

$$4\cos^2 x - 9\cos x + 2 = 0$$ **(2 marks)**

b Hence solve, for $0 \leqslant x < 4\pi$,

$$4\sin^2 x + 9\cos x - 6 = 0$$

giving your answers to 1 decimal place. **(6 marks)**

Chapter Exercises

27 **a** Show that the equation

$$\tan 2x = 5\sin\ 2x$$

can be written in the form

$$(1 - 5\cos 2x)\sin 2x = 0$$ **(2 marks)**

b Hence solve, for $0 \leqslant x \leqslant \pi$,

$$\tan 2x = 5\sin 2x$$

giving your answers to 1 decimal place where appropriate. You must show clearly how you obtained your answers. **(5 marks)**

28 **a** Sketch, for $0 \leqslant x \leqslant 2\pi$, the graph of $y = \cos\left(x + \frac{\pi}{6}\right)$. **(2 marks)**

b Write down the exact coordinates of the points where the graph meets the coordinate axes. **(3 marks)**

c Solve, for $0 \leqslant x \leqslant 2\pi$, the equation

$$y = \cos\left(x + \frac{\pi}{6}\right) = 0.65,$$

giving your answers in radians to 2 decimal places. **(5 marks)**

29 Solve, for $0 \leqslant x \leqslant \pi$, the equation

$$\sin\left(3x + \frac{\pi}{3}\right) = 0.45$$

giving your answers in radians to two decimal places. **(5 marks)**

Chapter Exercises

Challenge

Use the small angle approximations to determine whether the following equations have any solutions close to $\theta = 0$. In each case, state whether each root of the resulting quadratic equation is likely to correspond to a solution of the original equation.

a $9\sin\theta\tan\theta + 25\tan\theta = 6$

b $2\tan\theta + 3 = 5\cos 4\theta$

c $\sin 4\theta = 37 - 2\cos 2\theta$

Chapter Answers

1 **a** $\frac{\pi}{3}$ **b** 8.56 cm^2

2 **a** 120 cm^2 **b** 161.07 cm^2

3 **a** 1.839 **b** 11.03 cm

4 **a** $\frac{p}{r}$

b Area $= \frac{1}{2}r^2\theta = \frac{1}{2}r^2\frac{p}{r} = \frac{1}{2}pr\text{ cm}^2$

c $12.206\,25\text{ cm}^2$

d $1.105 \leq \theta \leq 1.151$ (3 d.p.)

5 **a** 1.28 **b** 16 **c** $1:3.91$

6 **a** Area of shape $X = 2d^2 + \frac{1}{2}d^2\pi$

Area of shape $Y = \frac{1}{2}(2d)^2\theta$

$2d^2 + \frac{1}{2}d^2\pi = \frac{1}{2}(2d)^2\theta$

$2d^2 + \frac{1}{2}d^2\pi = 2d^2\theta \Rightarrow 1 + \frac{1}{4}\pi = \theta$

b $(3\pi + 12)\text{ cm}$ **c** $\left(18 + \frac{3\pi}{2}\right)\text{ cm}$ **d** 12.9 mm

7 **a** $A_1 = \frac{1}{2} \times 6^2 \times \theta - \frac{1}{2} \times 6^2 \times \sin\theta = 18(\theta - \sin\theta)$

b $A_2 = \pi \times 6^2 - 18(\theta - \sin\theta) = 36\pi - 18(\theta - \sin\theta)$

Since $A_2 = 3A_1$

$36\pi - 18(\theta - \sin\theta) = 3 \times 18(\theta - \sin\theta)$

$36\pi - 18(\theta - \sin\theta) = 54(\theta - \sin\theta)$

$36\pi = 72(\theta - \sin\theta)$

$\frac{1}{2}\pi = \theta - \sin\theta$

$\sin\theta = \theta - \frac{\pi}{2}$

8 **a** $10^2 = 5^2 + 9^2 - 2 \times 5 \times 9\cos A$

$100 = 25 + 81 - 90\cos A$

$-6 = -90\cos A$

$\frac{1}{15} = \cos A$

$A = \cos^{-1}\left(\frac{1}{15}\right) = 1.504$

b **i** 6.77 cm^2 **ii** 15.7 cm^2 **iii** 22.5 cm

9 **a** $\frac{1}{2}r^2 \times 1.5 = 15 \Rightarrow r^2 = 20$

$r = \sqrt{20} = 2\sqrt{5}$

b 15.7 cm **c** 5.025 cm^2

10 **a** $2\sqrt{3}\text{ cm}$ **b** $2\pi\text{ cm}^2$

c Perimeter $= 2\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} \times \frac{\pi}{3} = \frac{2\sqrt{3}}{3}(\pi + 6)$

11 **a** $70^2 = 44^2 + 44^2 - 2 \times 44 \times 44\cos C$

$\cos C = -\frac{257}{968}$

$C = \cos^{-1}\left(-\frac{257}{968}\right) = 1.84$

b **i** 80.9 m **ii** 26.7 m **iii** 847 m^2

12 a Arc $AB = 6 \times 2\theta = 12\theta$
Length DC = Chord AB
$a = 2 \times 6 \sin\theta = 12 \sin\theta$
Perimeter $ABCD = 12\theta + 4 + 12 \sin\theta + 4 = 2(7 + \pi)$
$12\theta + 12 \sin\theta + 8 = 2(7 + \pi)$
$6\theta + 6 \sin\theta - 3 = \pi$
$2\theta + 2 \sin\theta - 1 = \frac{\pi}{3}$

b $2 \times \frac{\pi}{6} + 2 \sin\left(\frac{\pi}{6}\right) - 1 = \frac{\pi}{3} + 2 \times \frac{1}{2} - 1 = \frac{\pi}{3}$

c $20.7\,\text{cm}^2$

13 a $O_1A = O_2A = 12$, as they are radii of their respective circles.
$O_1O_2 = 12$, as O_2 is on the circumference of C_1 and hence is a radius (and vice versa).
Therefore,
O_1AO_2 is an equilateral triangle $\Rightarrow \angle AO_1O_2 = \frac{\pi}{3}$.
By symmetry, $\angle BO_1O_2$ is $\frac{\pi}{3} \Rightarrow \angle AO_1B = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

b 16π cm **c** $177\,\text{cm}^2$

14 a Student has used an angle measured in degrees – it needs to be measured in radians to use that formula.

b $\frac{5\pi}{4}\,\text{cm}^2$

15 a $-\frac{1}{4}$ **b** $\theta + 1$

16 a $\frac{7 + 2\cos 2\theta}{\tan 2\theta + 3} \approx \frac{7 + 2\left(1 - \frac{(2\theta)^2}{2}\right)}{2\theta + 3}$
$= \frac{7 + 2\left(1 - \frac{4\theta^2}{2}\right)}{2\theta + 3} = \frac{9 - 4\theta^2}{2\theta + 3}$
$= \frac{(3 + 2\theta)(3 - 2\theta)}{2\theta + 3} = 3 - 2\theta$

b 3

17 a $32 \cos 5\theta + 203 \tan 10\theta = 182$
$32\left(1 - \frac{(5\theta)^2}{2}\right) + 203\,(10\theta) = 182$
$32 - 16(25\theta^2) + 2030\theta = 182$
$0 = 400\theta^2 - 2030\theta + 150$
$0 = 40\theta^2 - 203\theta + 15$

b $5, \frac{3}{40}$

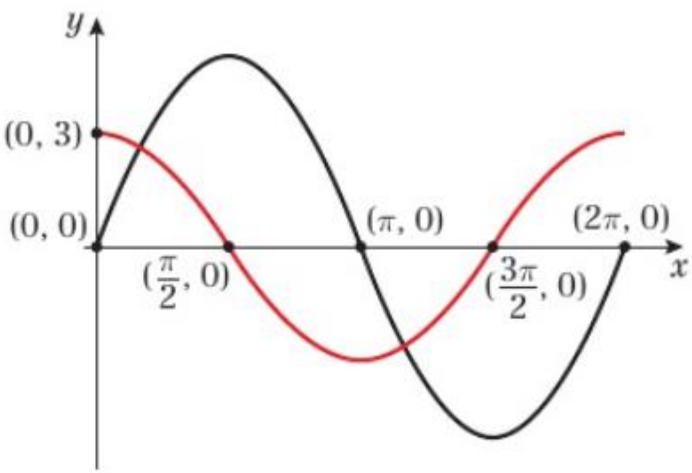
c 5 is not valid as it is not "small". $\frac{3}{40}$ is "small" so is valid.

18 $1 - 2\theta^2$

19 a 0.730, 2.41 **b** $-\frac{\pi}{4}, \frac{3\pi}{4}$

c $\frac{\pi}{4}, \frac{5\pi}{4}$ **d** $-2.48, -0.666$

20 a



b 2 solutions

c 0.540, 3.68

21 a $3 \sin\theta$

b 0.340, 2.80

22 $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

23 a Cosine can be negative so do not reject $-\frac{1}{\sqrt{2}}$. Cosine squared cannot be negative but the student has already square rooted it so no need to reject $-\frac{1}{\sqrt{2}}$.

b Rearranged incorrectly – square rooted incorrectly

c $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

24 a Not found all the solutions

b 0.595, 2.17, 3.74, 5.31

25 a $5 \sin x = 1 + 2\cos^2 x \Rightarrow 5 \sin x = 1 + 2(1 - \sin^2 x) \Rightarrow$
$2 \sin^2 x + 5 \sin x - 3 = 0$

b $\frac{\pi}{6}, \frac{5\pi}{6}$

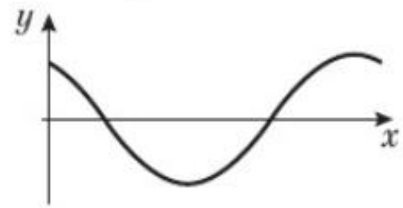
26 a $4 \sin^2 x + 9 \cos x - 6 = 0 \Rightarrow$
$4(1 - \cos^2 x) + 9 \cos x - 6 = 0 \Rightarrow$
$4 \cos^2 x - 9 \cos x + 2 = 0$

b 1.3, 5.0, 7.6, 11.2

27 a $\tan 2x = 5 \sin 2x \Rightarrow \frac{\sin 2x}{\cos 2x} = 5 \sin 2x \Rightarrow$
$(1 - 5 \cos 2x) \sin 2x = 0$

b $0, 0.7, \frac{\pi}{2}, 2.5, \pi$

28 a



b $\left(0, \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{3}, 0\right), \left(\frac{4\pi}{3}, 0\right)$

c 0.34, 4.90

29 $x = 0.54, 1.90$ or 2.64 (2 d.p)

Challenge

a $\theta = \frac{2}{9}$ or $\theta = -3$
$\theta = \frac{2}{9}$ is small, so this value is valid. $\theta = -3$ is not small so this value is not valid. Small in this context is "close to 0".

b $\theta = -\frac{1}{4}$ or $\theta = \frac{1}{5}$
Both θ could be considered "small" in this case so both are valid.

c No solutions