P2 Chapter 1: Algebra Techniques

Algebraic Division

Dealing with Improper Fractions

In Pure Year 1, we saw that the '**degree**' of a polynomial is the highest power, e.g. a quadratic has degree 2.

An algebraic fraction is **improper** if the degree of the numerator is **at least** the degree of the denominator.

$$\frac{x^2-3}{x+2} \qquad \frac{x+1}{x-1} \qquad \frac{x^3-x^2+3}{x^2-x}$$

A partial fraction is still improper if the degree is the same top and bottom.

Questions might take one of two forms:

- Do the division to express as a quotient and a remainder, e.g. $\frac{x+1}{x-1} \rightarrow 1+\frac{2}{x-1}$
- Express as partial fractions, e.g. $\frac{x^2+x}{(x+1)(x-2)} = A+\frac{B}{x+1}+\frac{C}{x-2}$

Reducing to Quotient and Remainder

You know for example that as $7 \div 3 = 2\ rem\ 1$, we could write:

$$\frac{7}{3} = 2 + \frac{1}{3}$$

Similarly in general:

$$\frac{F(x)}{divisor} = Q(x) + \frac{remainder}{divisor}$$

Quotient

If $\frac{x^2+5x-9}{x+2} = Ax + B + \frac{C}{x+2}$, determine the values of A, B and C.

Note first that after dividing $(x^2 + 5x - 9)$ by $(x + 2)$, $Ax + B$ is the quotient and C the remainder:

$$\begin{array}{r|l} & x + 3 \\ \hline x + 2 & x^2 + 5x - 9 \\ & \underline{x^2 + 2x} \\ & 3x - 9 \\ & \underline{3x + 6} \\ & -15 \end{array}$$

$$\frac{x^2 + 5x - 9}{x + 2} = x + 3 - \frac{15}{x + 2}$$

i.e. $A = 1, B = 3, C = -15$

Fro Tip: The degree of the quotient is the degree of the original expression minus the degree of the divisor. In this case, the quotient $Ax + B$ has degree $2 - 1 = 1$.

Test Your Understanding

Edexcel C4 June 2013 Q1

Given that

$$\frac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 - 4}, \quad x \neq \pm 2$$

find the values of the constants a, b, c, d and e.

(4)

Fro Tip: There's a missing x term in the numerator and missing x term in the denominator. Use $+0x$ to avoid gaps.

?

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$$x^2(+0x) - 4 \overline{)\, 3x^4 - 2x^3 - 5x^2 + (0x) - 4}$$

$$\begin{array}{r} 3x^2 - 2x + 7 \\ \hline 3x^4 + 0x^3 - 12x^2 \\ -2x^3 + 7x^2 + 0x \\ \underline{-2x^3 + 0x^2 + 8x} \\ 7x^2 - 8x - 4 \\ \underline{7x^2 + 0x - 28} \\ -8x + 24 \end{array}$$

$a = 3$ **B1**

$$x^2(+0x) - 4 \overline{)\, 3x^4 - 2x^3 - 5x^2 + (0x) - 4}$$

$$\begin{array}{r} 3x^2 - 2x \ldots\ldots \\ \hline 3x^4 + 0x^3 - 12x^2 \\ -2x^3 + \ldots\ldots\ldots\ldots \\ \underline{-2x^3 + \ldots\ldots\ldots\ldots} \end{array}$$

Long division as far as **M1**

Two of $b = -2$ $c = 7$ $d = -8$ $e = 24$ **A1**

All four of $b = -2$ $c = 7$ $d = -8$ $e = 24$ **A1**

An alternative (but probably harder) approach is to write:

$3x^4 - 2x^3 - 5x^2 - 4$

$\equiv (ax^2 + bx + c)(x^2 - 4) + dx + e$

and to then compare coefficients.

Dealing with Improper Fractions

Q Split $\frac{3x^2-3x-2}{(x-1)(x-2)}$ into partial fractions.

Method 1: Algebraic Division

$$\frac{3x^2-3x-2}{(x-1)(x-2)} \equiv \frac{3x^2-3x-2}{x^2-3x+2}$$

Dividing algebraically gives:

$$3+\frac{6x-8}{x^2-3x+2}$$

Turn numerator back:

$$=3+\frac{6x-8}{(x-1)(x-2)}$$

Let $\frac{6x-8}{(x-1)(x-2)} \equiv \frac{A}{x-1}+\frac{B}{x-2}$

$A=2$

$B=4$

So $\frac{3x^2-3x-2}{(x-1)(x-2)} \equiv \mathbf{3}+\frac{\mathbf{2}}{x-1}+\frac{\mathbf{4}}{x-2}$

Method 2: Using One Identity

Let:

$$\frac{3x^2-3x-2}{(x-1)(x-2)} \equiv A+\frac{B}{x-1}+\frac{C}{x-2}$$

$$3x^2-3x-2$$
$$\equiv A(x-1)(x-2)+B(x-2)+C(x-1)$$

If $x=2$: $4=C$

If $x=1$: $-2=-B \quad \rightarrow \quad B=2$

Comparing coefficients of x^2:

$3=A$

Note: Mark-schemes give "using-one-identity" as "Method 1" – implying more standard!

Test Your Understanding

C4 Jan 2013 Q3

Express $\dfrac{9x^2 + 20x - 10}{(x+2)(3x-1)}$ in partial fractions.

(4)

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Test Your Understanding

C4 Jan 2013 Q3

Express $\dfrac{9x^2+20x-10}{(x+2)(3x-1)}$ in partial fractions. **(4)**

Method 1: Using one identity

$$\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv A + \frac{B}{(x+2)} + \frac{C}{(3x-1)}$$

$A = 3$	their constant term = 3	B1
$9x^2+20x-10 \equiv A(x+2)(3x-1)+B(3x-1)+C(x+2)$	Forming a correct identity.	B1
Either x^2: $9 = 3A$, x: $20 = 5A+3B+C$ constant: $-10 = -2A-B+2C$ or	Attempts to find the value of either one of their B or their C from their identity.	M1
$x=-2 \Rightarrow 36-40-10=-7B \Rightarrow -14=-7B \Rightarrow B=2$	Correct values for their B and their C, which are found using a correct identity.	A1
$x=\frac{1}{3} \Rightarrow 1+\frac{20}{3}-10=\frac{7}{3}C \Rightarrow -\frac{7}{3}=\frac{7}{3}C \Rightarrow C=-1$		

Exercise 1.5

Pearson Pure Mathematics Year 2/AS

Page 4

Homework Exercise

1 $\dfrac{x^3 + 2x^2 + 3x - 4}{x + 1} \equiv Ax^2 + Bx + C + \dfrac{D}{x + 1}$

Find the values of the constants A, B, C and D. **(4 marks)**

2 Given that $\dfrac{2x^3 + 3x^2 - 4x + 5}{x + 3} \equiv ax^2 + bx + c + \dfrac{d}{x + 3}$ find the values of a, b, c and d. **(4 marks)**

3 $\mathrm{f}(x) = \dfrac{x^3 - 8}{x - 2}$

Show that $\mathrm{f}(x)$ can be written in the form $px^2 + qx + r$ and find the values of p, q and r. **(4 marks)**

4 Given that $\dfrac{2x^2 + 4x + 5}{x^2 - 1} \equiv m + \dfrac{nx + p}{x^2 - 1}$ find the values of m, n and p. **(4 marks)**

5 Find the values of the constants A, B, C and D in the following identity:

$8x^3 + 2x^2 + 5 \equiv (Ax + B)(2x^2 + 2) + Cx + D$ **(4 marks)**

6 $\dfrac{4x^3 - 5x^2 + 3x - 14}{x^2 + 2x - 1} \equiv Ax + B + \dfrac{Cx + D}{x^2 + 2x - 1}$

Find the values of the constants A, B, C and D. **(4 marks)**

Homework Exercise

7 $g(x) = \dfrac{x^4 + 3x^2 - 4}{x^2 + 1}$. Show that $g(x)$ can be written in the form $px^2 + qx + r + \dfrac{sx + t}{x^2 + 1}$ and find the values of p, q, r, s and t. **(4 marks)**

8 Given that $\dfrac{2x^4 + 3x^3 - 2x^2 + 4x - 6}{x^2 + x - 2} \equiv ax^2 + bx + c + \dfrac{dx + e}{x^2 + x - 2}$ find the values of a, b, c, d and e. **(5 marks)**

9 Find the values of the constants A, B, C, D and E in the following identity:

$3x^4 - 4x^3 - 8x^2 + 16x - 2 \equiv (Ax^2 + Bx + C)(x^2 - 3) + Dx + E$ **(5 marks)**

10 **a** Fully factorise the expression $x^4 - 1$. **(2 marks)**

b Hence, or otherwise, write the algebraic fraction $\dfrac{x^4 - 1}{x + 1}$ in the form $(ax + b)(cx^2 + dx + e)$ and find the values of a, b, c, d and e. **(4 marks)**

Homework Exercise

1 $g(x) = \dfrac{x^2 + 3x - 2}{(x-1)(x-2)}$. Show that $g(x)$ can we written in the form $A + \dfrac{B}{x-1} + \dfrac{C}{x-2}$ and find the values of the constants A, B and C. **(4 marks)**

2 Given that $\dfrac{x^2 - 10}{(x-2)(x+1)} \equiv A + \dfrac{B}{x-2} + \dfrac{C}{x+1}$, find the values of the constants A, B and C. **(4 marks)**

3 Find the values of the constants A, B, C and D in the following identity:

$$\frac{x^3 - x^2 - x - 3}{x(x-1)} \equiv Ax + B + \frac{C}{x} + \frac{D}{x-1}$$

(5 marks)

4 Show that $\dfrac{-3x^3 - 4x^2 + 19x + 8}{x^2 + 2x - 3}$ can be expressed in the form $A + Bx + \dfrac{C}{(x-1)} + \dfrac{D}{(x+3)}$, where A, B, C and D are constants to be found. **(5 marks)**

5 $p(x) \equiv \dfrac{4x^2 + 25}{4x^2 - 25}$

Show that $p(x)$ can be written in the form $A + \dfrac{B}{2x-5} + \dfrac{C}{2x+5}$, where A, B and C are constants to be found. **(4 marks)**

6 Given that $\dfrac{2x^2 - 1}{x^2 + 2x + 1} \equiv A + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$, find the values of the constants A, B and C. **(4 marks)**

Homework Exercise

7 By factorising the denominator, express the following as partial fractions:

a $\dfrac{4x^2 + 17x - 11}{x^2 + 3x - 4}$ **b** $\dfrac{x^4 - 4x^3 + 9x^2 - 17x + 12}{x^3 - 4x^2 + 4x}$

8 Given that $\dfrac{6x^3 - 7x^2 + 3}{3x^2 + x - 10} \equiv Ax + B + \dfrac{C}{3x - 5} + \dfrac{D}{x + 2}$, find the values of the constants A, B, C and D. **(6 marks)**

9 $q(x) = \dfrac{8x^3 + 1}{4x^2 - 4x + 1}$

Show that $q(x)$ can be written in the form $Ax + B + \dfrac{C}{2x - 1} + \dfrac{D}{(2x - 1)^2}$ and find the values of the constants A, B, C and D. **(6 marks)**

10 $h(x) = \dfrac{x^4 + 2x^2 - 3x + 8}{x^2 + x - 2}$

Show that $h(x)$ can be written as $Ax^2 + Bx + C + \dfrac{D}{x + 2} + \dfrac{E}{x - 1}$ and find the values of A, B, C, D and E. **(5 marks)**

Homework Answers

1 $A = 1, B = 1, C = 2, D = -6$

2 $a = 2, b = -3, c = 5, d = -10$

3 $p = 1, q = 2, r = 4$

4 $m = 2, n = 4, p = 7$

5 $A = 4, B = 1, C = -8$ and $D = 3$.

6 $A = 4, B = -13, C = 33$ and $D = -27$

7 $p = 1, q = 0, r = 2, s = 0$ and $t = -6$

8 $a = 2, b = 1, c = 1, d = 5$ and $e = -4$

9 $A = 3, B = -4, C = 1, D = 4, E = 1$

10 **a** $(x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$

b $(x - 1)(x^2 + 1), a = 1, b = -1, c = 1, d = 0$ and $e = 1$.