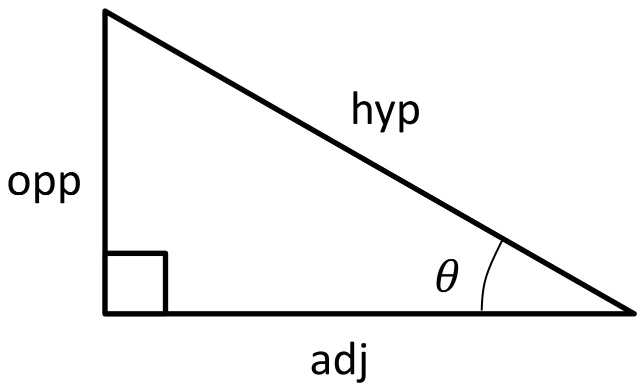
P1 Chapter 9: Trigonometric Ratios

Cosine Rule

RECAP :: Right-Angled Trigonometry

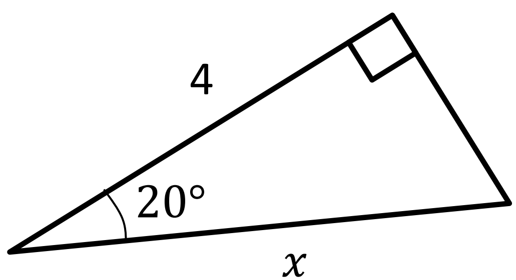


You are probably familiar with the formula: $\sin(\theta) = \frac{opp}{hyp}$

But what is the *conceptual* definition of sin ?

sin is a <u>function</u> which <u>inputs an angle</u> and gives the <u>ratio</u> between the opposite and hypotenuse.

Remember that a ratio just means the 'relative size' between quantities (in this case lengths). For this reason, sin/cos/tan are known as "trigonometric ratios".

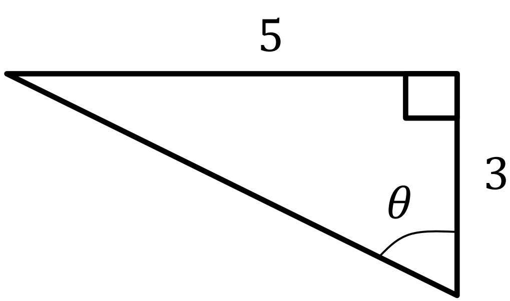


Find x.

$$\cos(20) = \frac{4}{x}$$

$$x = \frac{4}{\cos(20)} = 11.7\ (3sf)$$

Tip: You can swap the thing you're dividing by and the result. e.g. $\frac{8}{2} = 4\ \rightarrow \frac{8}{4} = 2$. I call this the '*swapsie trick*'.



Find θ.

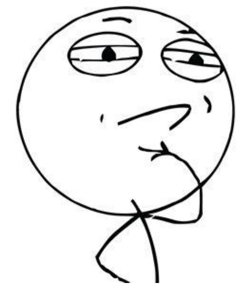
$$\tan(\theta) = \frac{5}{3}$$

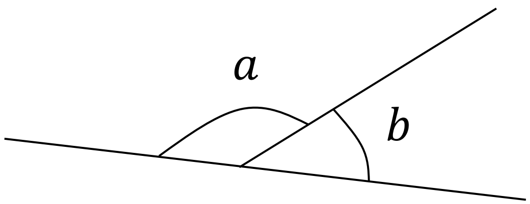
$$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.0°$$

Note: You may have been taught "use tan^{-1} whenever you're finding an angle", and therefore write the second line directly. This is fine, but I prefer to always write the first line, then see the problem as a 'changing the subject' one. We need to remove the tan on front of the θ, so apply tan^{-1} to each side of the equation to 'cancel out' the tan on the LHS.

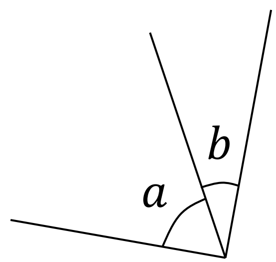
Just for your interest…

Have you ever wondered why “cosine” contains the word “sine”?

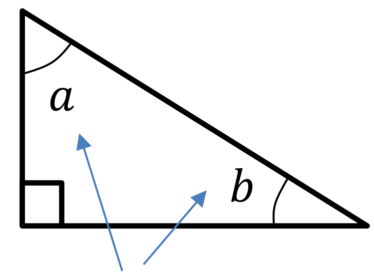




Supplementary Angles
add to 180°

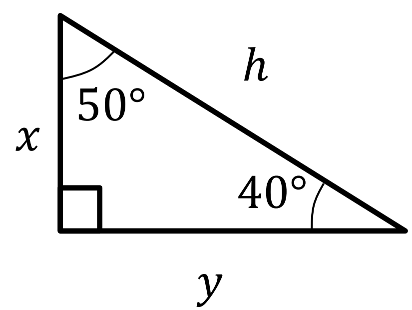


Complementary Angles
add to 90°



Therefore these angles are complementary.

i.e. The **cosine** of an angle is the **sine** of the **complementary** angle.
Hence **cosine = COMPLEMENTARY SINE**



$$\cos(50) = \frac{\boldsymbol{x}}{\boldsymbol{h}}$$

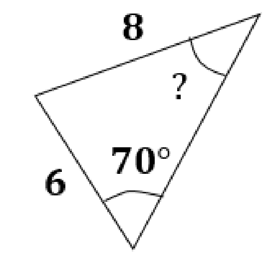
$$\sin(40) = \frac{\boldsymbol{x}}{\boldsymbol{h}}$$

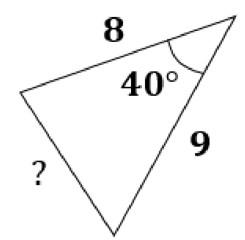
$$\therefore \mathbf{cos(50) = sin(40)}$$

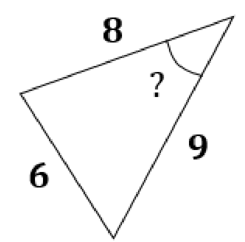
OVERVIEW: Finding missing sides and angles

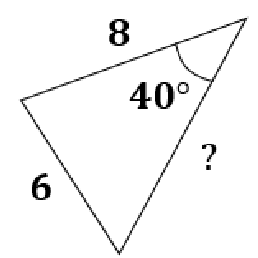
When triangles are not right-angled, we can no longer use simple trigonometric ratios, and must use the cosine and sine rules.

You have	You want	Use
#1: Two angle-side opposite pairs	Missing angle or side in one pair	Sine rule
#2 Two sides known and a missing side opposite a known angle	Remaining side	Cosine rule
#3 All three sides	An angle	Cosine rule
#4 Two sides known and a missing side <u>not</u> opposite known angle	Remaining side	Sine rule twice



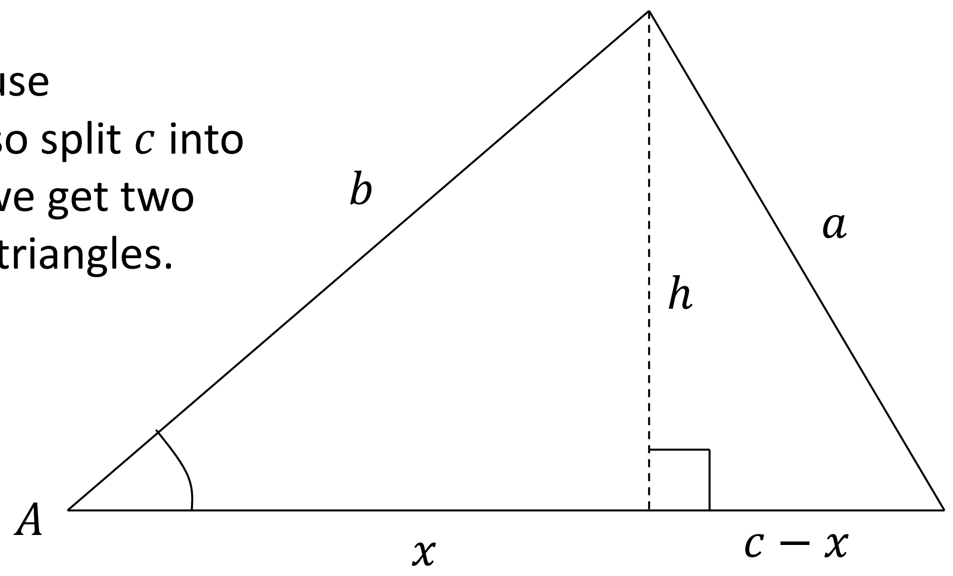






Proof of Cosine Rule

We want to use Pythagoras, so split c into two so that we get two right-angled triangles.



By Pythagoras: $h^2 + x^2 = b^2$ and $h^2 + (c - x)^2 = a^2$
Subtracting to eliminate h:

?

Proof of Cosine Rule

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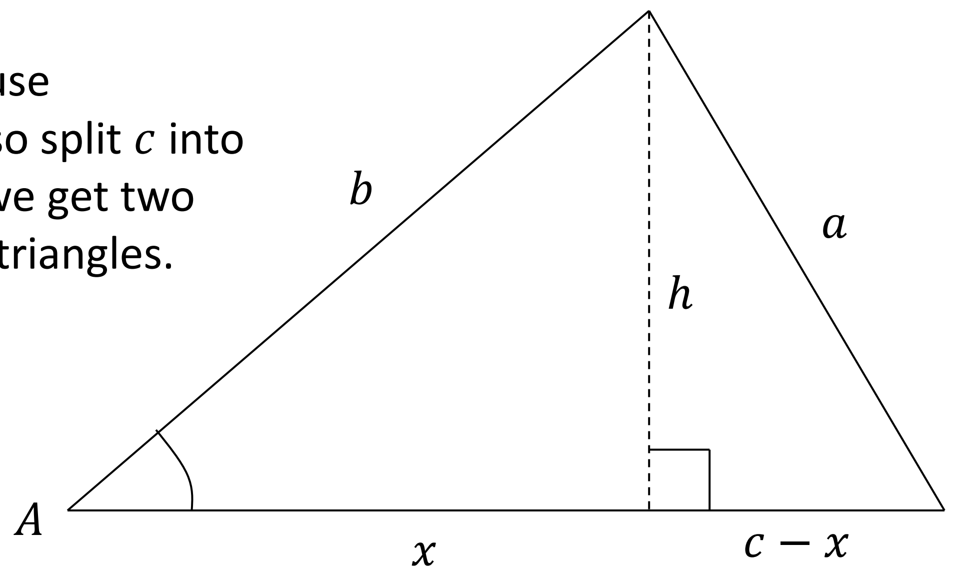


By Pythagoras: $h^2 + x^2 = b^2$ and $h^2 + (c - x)^2 = a^2$
Subtracting to eliminate h:

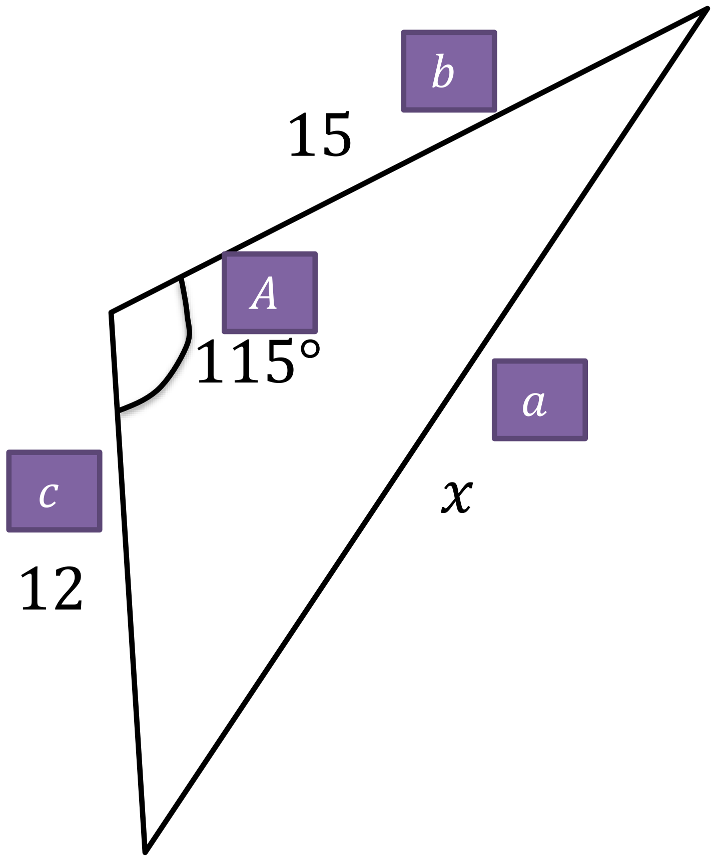
$$x^2 - (c - x)^2 = b^2 - a^2$$
$$a^2 = b^2 + c^2 - 2cx$$

But $x = b \cos A$

hence

Cosine Rule

We use the **cosine rule** whenever we have **three sides** (and an angle) involved.



Proof at end of PowerPoint.

Cosine Rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

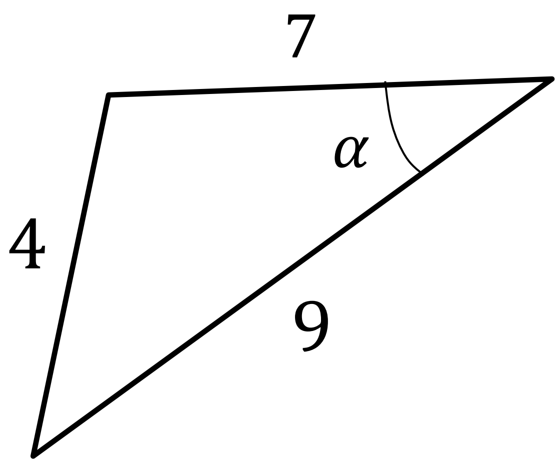
The only angle in formula is A, so label angle in diagram A, label opposite side a, and so on (b and c can go either way).

$$x^2 = 15^2 + 12^2 - (2 \times 15 \times 12 \times \cos 115)$$
$$x^2 = 521.14257\ldots$$
$$x = 22.83$$

Dealing with Missing Angles

You have	You want	Use
All three sides	An angle	Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Tip: The brackets are not needed, but students who forget about BIDMAS see $4^2 = (7^2 + 9^2 - 2 \times 7 \times 9)\cos\alpha$ and hence incorrectly simplify to $16 = 4\cos\alpha$

?

?

?

?

Label sides then substitute into formula.

Simplify each bit of formula.

Rearrange (I use 'subtraction swapsie trick' to swap thing you're subtracting and result)

Textbook Note: The textbook presents the rearrangement of the cosine rule: $\cos A = \frac{b^2+c^2-a^2}{2bc}$ to find missing angles. I'd personally advise against using this as: (a) It's another formula to remember. (b) Anything that gives you less practice of manipulating/rearranging equations is probably a bad thing. (c) You won't get to use the swapsie trick. ☹

Dealing with Missing Angles

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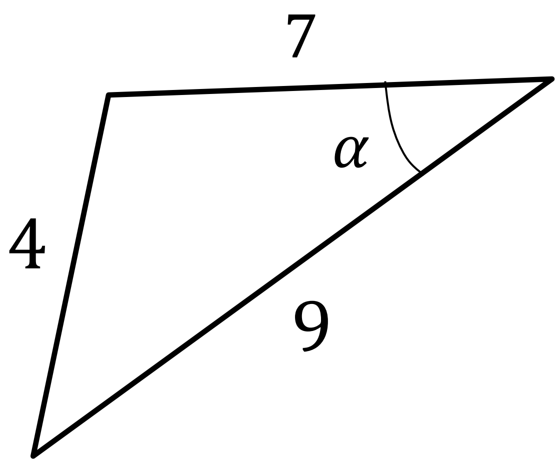

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$$4^2 = 7^2 + 9^2 - (2 \times 7 \times 9 \times \cos\alpha)$$
$$16 = 130 - 126\cos\alpha$$
$$126\cos\alpha = 130 - 16$$
$$\cos\alpha = \frac{114}{126}$$
$$\alpha = \cos^{-1}\left(\frac{114}{126}\right) = 25.2°$$

Label sides then substitute into formula.

Simplify each bit of formula.

Rearrange (I use 'subtraction swapsie trick' to swap thing you're subtracting and result)

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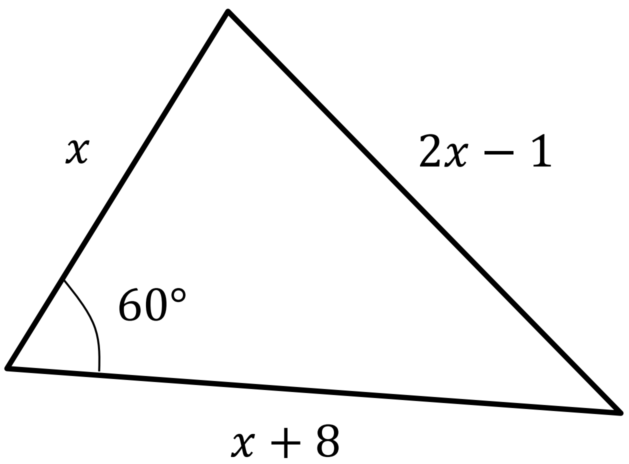
Harder Ones



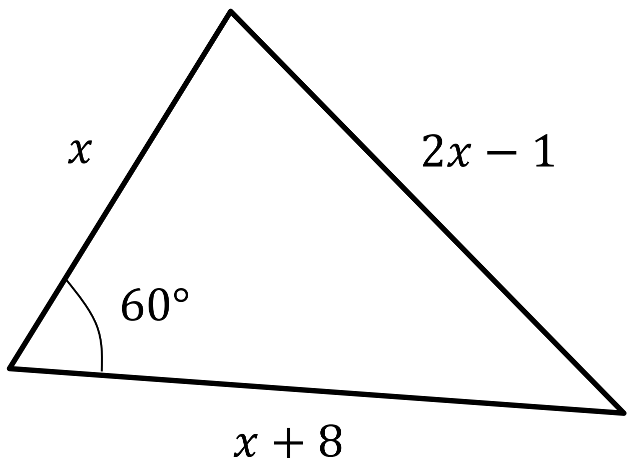
Determine the value of x.

?

[From textbook] Coastguard station B is 8 km, on a bearing of $060°$, from coastguard station A. A ship C is 4.8 km on a bearing of $018°$, away from A. Calculate how far C is from B.

?

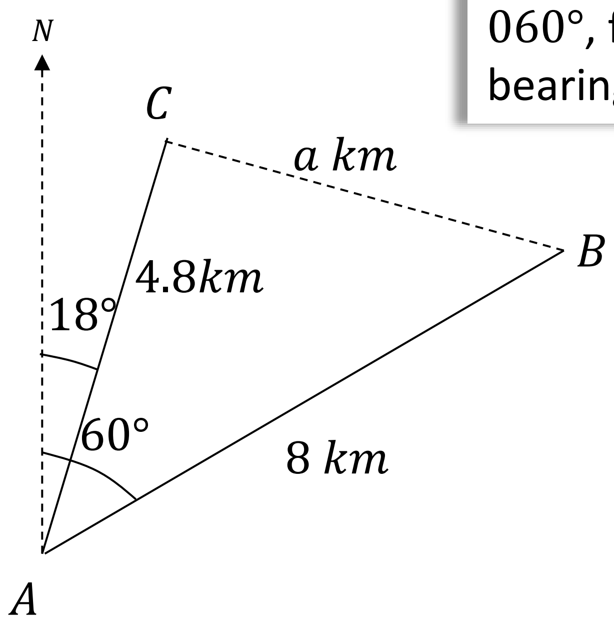
Harder Ones



Determine the value of x.

$$(2x-1)^2 = x^2 + (x+8)^2 - 2x(x+8)\cos 60°$$
$$4x^2 - 4x + 1 = x^2 + x^2 + 16x + 64 - x^2 - 8x$$
$$3x^2 - 12x - 63 = 0$$
$$x^2 - 4x - 21 = 0$$
$$(x+3)(x-7) = 0$$
$$\therefore x = 7$$

[From textbook] Coastguard station B is 8 km, on a bearing of $060°$, from coastguard station A. A ship C is 4.8 km on a bearing of $018°$, away from A. Calculate how far C is from B.

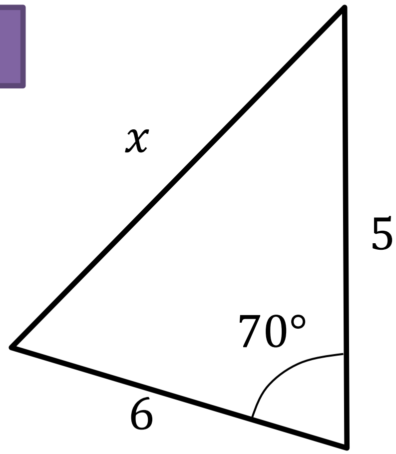


$$\boldsymbol{a^2 = 4.8^2 + 8^2 - (2 \times 4.8 \times 8 \times \cos(42°))}$$
$$\boldsymbol{a = 5.47\quad (3sf)}$$

C is 5.47 km from coastguard station B.

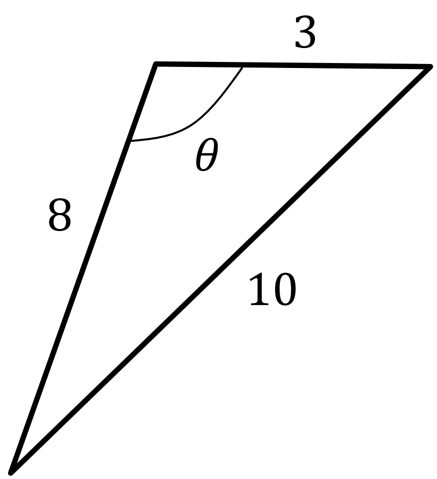
Test Your Understanding

1



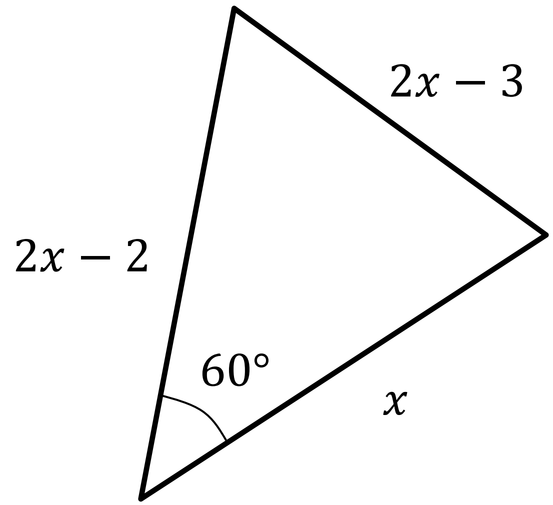
$x =$?

2



$\theta =$?

3



$x =$?

Fro Note: You will get an obtuse angle whenever you inverse cos a negative value.

Exercise 9A

Pearson Pure Mathematics Year 1/AS
Pages 177-179

Extension

1 *[STEP I 2009 Q4i]*
The sides of a triangle have lengths $p-q$, p and $p+q$, where $p>q>0$. The largest and smallest angles of the triangle are α and β respectively. Show by means of the cosine rule that

$$4(1-\cos\alpha)(1-\cos\beta)=\cos\alpha+\cos\beta$$

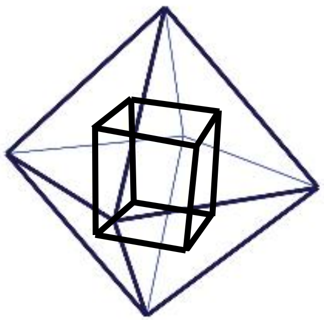
?

2 [STEP I 2007 Q5] Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.

(i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$

(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

Solutions for Q2 on next slide.



Exercise 9A

Pearson Pure Mathematics Year 1/AS
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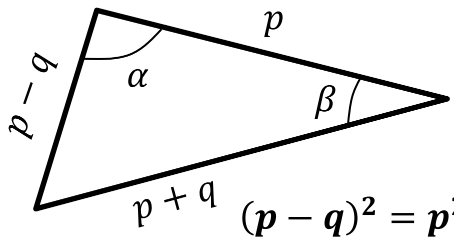
Note that the longest side of a triangle is opposite the largest angle, and the shortest opposite the smallest angle. Thus:



$$(\boldsymbol{p}-\boldsymbol{q})^2=\boldsymbol{p}^2+(\boldsymbol{p}+\boldsymbol{q})^2-2\boldsymbol{p}(\boldsymbol{p}+\boldsymbol{q})\cos\boldsymbol{\beta}$$
$$(\boldsymbol{p}+\boldsymbol{q})^2=\boldsymbol{p}^2+(\boldsymbol{p}-\boldsymbol{q})^2-2\boldsymbol{p}(\boldsymbol{p}-\boldsymbol{q})\cos\boldsymbol{\alpha}$$

We can manipulate and combine these two equations to get the desired equation above.

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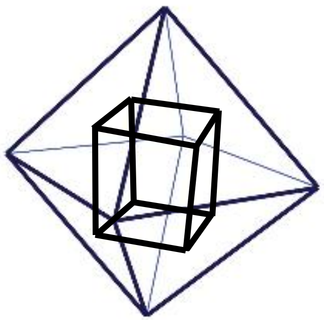

Solutions to Extension Question 2

[STEP I 2007 Q5] Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.

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(Official solutions) Big, clear diagram essential!

(i)

?

(ii)

?

Solutions to Extension Question 2

[STEP I 2007 Q5] Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.

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(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

(Official solutions) Big, clear diagram essential!

(i) Let the side length of the octahedron be $2k$. Then sloping "height" of a triangular face is $k\sqrt{3}$.
Also, the vertical height of the whole octahedron is $2k\sqrt{2}$.
Therefore, by the cosine rule, $8k^2 = 3k^2 + 3k^2 - 2(k\sqrt{3})(k\sqrt{3})\cos A$
Hence $A = arcos\left(-\frac{2k^2}{6k^2}\right) = \arccos\left(-\frac{1}{3}\right)$

(ii) The centre of each face is on any median of the equilateral triangle that is the face, and the centre is two-thirds of the way along the median from any vertex.
This is a quotable fact, but can be worked out from a diagram, using the fact that the centre of an equilateral triangle is equidistant from the three vertices: the centre divides the median in the ratio $1{:}\cos 60°$.
The feet of the two medians from the apex of the octahedron in two adjacent triangles are $k\sqrt{2}$ apart.
Therefore, by similarity, adjacent centres of the triangular faces are $\frac{2}{3} \times k\sqrt{2}$ apart. Therefore, the volume of the cube (whose vertices are the centres of the faces) is

$$\left(\frac{2}{3} \times k\sqrt{2}\right)^3 = \frac{16k^3\sqrt{2}}{27}$$

and the volume of the octahedron is $2 \times \frac{4k^2 \times k\sqrt{2}}{3} = \frac{8k^3\sqrt{2}}{3}$.
Hence the ratio of the volume of the octahedron to the volume of the cube is $9{:}\,2$.

Exercise 9.1

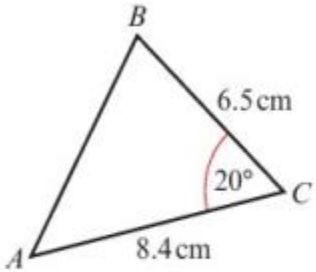
Pearson Pure Mathematics Year 1/AS
Page 68

Homework Exercise

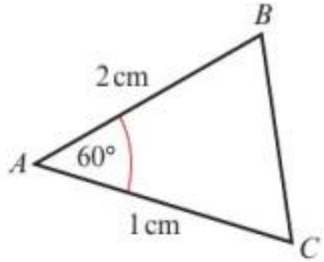
Give answers to 3 significant figures, where appropriate.

1 In each of the following triangles calculate the length of the missing side.

a



b



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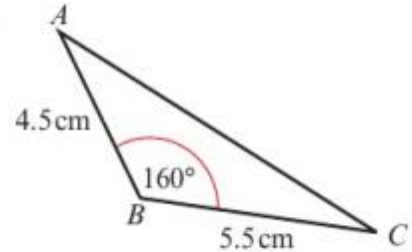
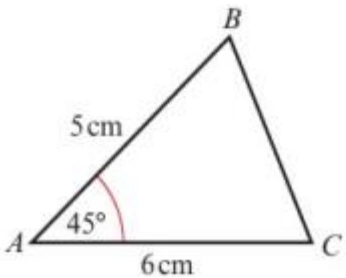
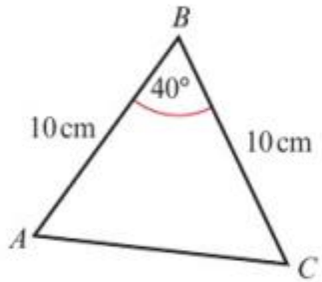


d



e



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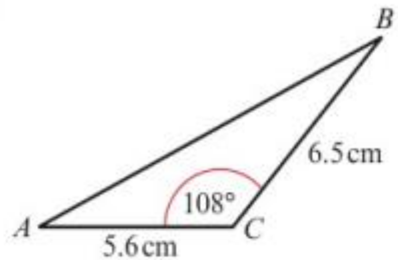
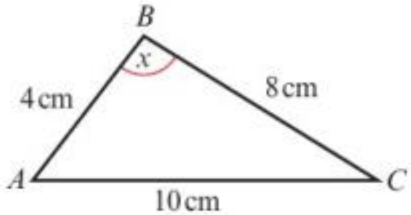


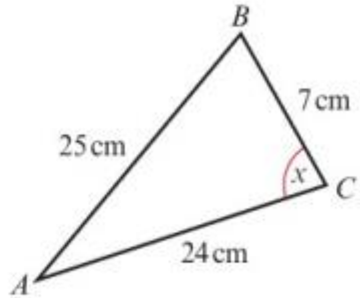
Homework Exercise

2 In the following triangles calculate the size of the angle marked x:

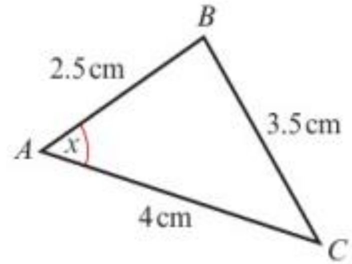
a



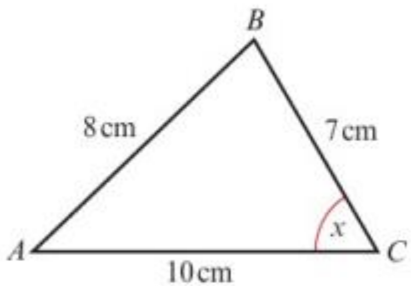
b



c



d



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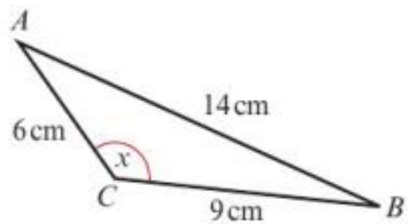

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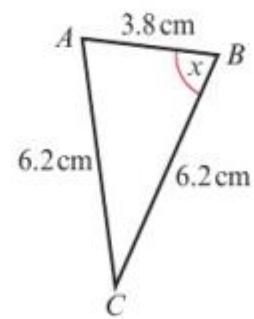
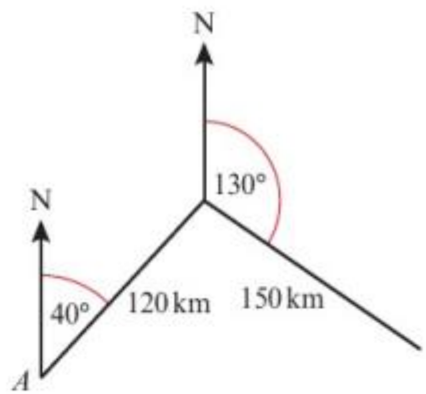


3 A plane flies from airport A on a bearing of 040° for 120 km and then on a bearing of 130° for 150 km. Calculate the distance of the plane from the airport.



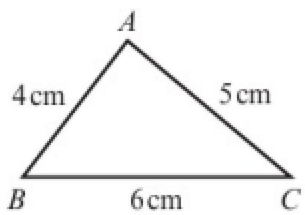
Homework Exercise

4 From a point A a boat sails due north for 7 km to B. The boat leaves B and moves on a bearing of 100° for 10 km until it reaches C. Calculate the distance of C from A.

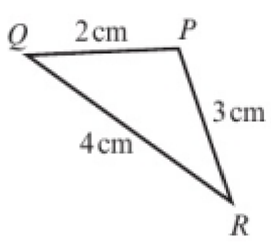
5 A helicopter flies on a bearing of 080° from A to B, where $AB = 50$ km.
It then flies for 60 km to a point C.
Given that C is 80 km from A, calculate the bearing of C from A.

6 The distance from the tee, T, to the flag, F, on a particular hole on a golf course is 494 yards.
A golfer's tee shot travels 220 yards and lands at the point S, where $\angle STF = 22°$.
Calculate how far the ball is from the flag.

7 Show that $\cos A = \frac{1}{8}$



8 Show that $\cos P = -\frac{1}{4}$



Homework Exercise

9 In $\triangle ABC$, $AB = 5\,\text{cm}$, $BC = 6\,\text{cm}$ and $AC = 10\,\text{cm}$.
Calculate the size of the smallest angle.

10 In $\triangle ABC$, $AB = 9.3\,\text{cm}$, $BC = 6.2\,\text{cm}$ and $AC = 12.7\,\text{cm}$.
Calculate the size of the largest angle.

11 The lengths of the sides of a triangle are in the ratio $2:3:4$.
Calculate the size of the largest angle.

12 In $\triangle ABC$, $AB = (x - 3)\,\text{cm}$, $BC = (x + 3)\,\text{cm}$, $AC = 8\,\text{cm}$ and $\angle BAC = 60°$.
Use the cosine rule to find the value of x.

13 In $\triangle ABC$, $AB = x\,\text{cm}$, $BC = (x - 4)\,\text{cm}$, $AC = 10\,\text{cm}$ and $\angle BAC = 60°$.
Calculate the value of x.

14 In $\triangle ABC$, $AB = (5 - x)\,\text{cm}$, $BC = (4 + x)\,\text{cm}$, $\angle ABC = 120°$ and $AC = y\,\text{cm}$.

a Show that $y^2 = x^2 - x + 61$.

b Use the method of completing the square to find the minimum value of y^2, and give the value of x for which this occurs.

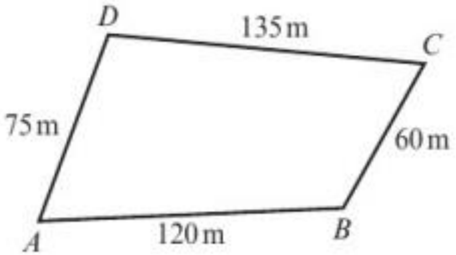
Homework Exercise

15 In $\triangle ABC$, $AB = x$ cm, $BC = 5$ cm, $AC = (10 - x)$ cm.

a Show that $\cos \angle ABC = \dfrac{4x - 15}{2x}$

b Given that $\cos \angle ABC = -\frac{1}{7}$, work out the value of x.

16 A farmer has a field in the shape of a quadrilateral as shown.



Problem-solving

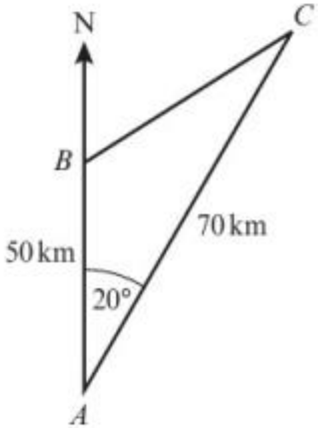
You will have to use the cosine rule twice. Copy the diagram and write any angles or lengths you work out on your copy.

The angle between fences AB and AD is 74°. Find the angle between fences BC and CD.

17 The diagram shows three cargo ships, A, B and C, which are in the same horizontal plane. Ship B is 50 km due north of ship A and ship C is 70 km from ship A. The bearing of C from A is 020°.

a Calculate the distance between ships B and C, in kilometres to 3 s.f. **(3 marks)**

b Calculate the bearing of ship C from ship B. **(4 marks)**



Homework Answers

1 **a** 3.19 cm **b** 1.73 cm ($\sqrt{3}$ cm) **c** 9.85 cm
d 4.31 cm **e** 6.84 cm **f** 9.80 cm

2 **a** 108(.2)° **b** 90° **c** 60°
d 52.6° **e** 137° **f** 72.2°

3 192 km

4 11.2 km

5 128.5° or 031.5° (Angle BAC = 48.5°)

6 302 yards (301.5…)

7 Using the cosine rule $\frac{5^2 + 4^2 - 6^2}{2 \times 5 \times 4} = \frac{1}{8}$

8 Using the cosine rule $\frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3} = -\frac{1}{4}$

9 $ACB = 22.3°$

10 $ABC = 108(.4)°$

11 104° (104.48)°

12 4.4 cm

13 42 cm

14 **a** $y^2 = (5 - x)^2 + (4 + x)^2 - 2(5 - x)(4 + x)\cos 120°$
$= 25 - 10x + x^2 + 16 + 8x + x^2 - 2(20 + x - x^2)\left(-\frac{1}{2}\right)$
$= x^2 - x + 61$

b Minimum $AC^2 = 60.75$; it occurs for $x = \frac{1}{2}$

15 **a** $\cos\angle ABC = \frac{x^2 + 5^2 - (10 - x)^2}{2x \times 5}$

$$= \frac{20x - 75}{10x} = \frac{4x - 15}{2x}$$

b 3.5

16 65.3°

17 **a** 28.7 km **b** 056.6°