Continued fractions

1 Definitions

S:def

A continued fraction (CF) is defined by two sequences $\{a_i : i \in \mathbb{N}\}$ and $\{b_i : i \in \mathbb{N}\}$, which can be integers or real numbers or even things like $7x^{15}$. Symbolically the CF can be represented as

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \dots}}}$$

or (overleaf style)

$$\frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ddots}}}$$

or in a tidier form

$$\frac{a_1}{b_1+}\;\frac{a_2}{b_2+}\;\frac{a_3}{b_3+}\dots,$$

or

$$a_1\big/\big(b_1 + a_2\big/\big(b_2 + a_3\big/\big(\dots$$

or

$$\frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \dots$$

or

$$a_1/b_1 + a_2/b_2 + a_3/\dots$$

$$/$$

$$b_1$$

where the ... can stand for infinitely many more terms. I don't think it needs to assumed that the sequence

fn.def <1>
$$f_n := \frac{a_1}{b_1+} \quad \frac{a_2}{b_2+} \dots \quad \frac{a_n}{b_n} = a_1\Big/\Big(b_1{+}a_2\Big/\Big(b_2{+}a_3\Big/\Big(\dots/(b_{n-1}{+}a_n/b_n)\dots\Big)\Big)$$

converges in any usual sense to the expression with the ....

Many mathematicians (for example, https://mathworld.wolfram.com/Erf.html) work with the function

$$\text{ERFC}(x) := G(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 2\bar{\Phi}(x\sqrt{2}),$$

which is called the complementary error function.

Baricz (2008, page 1367) stated the Laplace CF for the Mills ratio to be

Baricz <2>
$$\bar{\Phi}(x)/\phi(x) = \frac{1}{x+} \quad \frac{1}{x+} \quad \frac{2}{x+} \quad \frac{3}{x+}$$

David (2005, page) gave the CF in the form originally used by Laplace (Cel. Mech. page 255)

David <3>
$$2xe^{x^2} \int_x^\infty e^{-t^2} dt = \frac{1}{1+} \quad \frac{q}{1+} \quad \frac{2q}{1+} \quad \frac{3q}{1+} \quad \dots$$

where $q = 1/(2x^2)$. Acton (1990, page 313) gave the same formula, although written in slightly different notation.

Shenton (1954) gave a 'new' CF

Shenton <4>
$$\phi(x) \int_0^x \phi(t)\, dt = \frac{x}{1-} \quad \frac{x^2}{3+} \quad \frac{2x^2}{5-} \dots$$

It is not obvious to me whether the three CFs are actually equivalent, in the sense that they lead to the same approximations f_n for a fixed n.

2 From CF to ratio

S:rat

Acton (1990, pages 16-19, 279) The approximation f_n can be written as a ratio P_n/Q_n where there are simple recurrence formulas for P_n and Q_n. I find it best to work out a few special cases to see the pattern.

Clearly $f_1 = P_1/Q_1$ where

$$P_1 = a_1 \quad \text{AND} \quad Q_1 = b_1.$$

Replace b_1 by $B_1 = b_1 + a_2/b_2$ to get

$$f_2 = \frac{a_1}{b_1 + a_2/b_2} = \frac{b_2 a_1}{b_2 b_1 + a_2} = \frac{P_2}{Q_2}$$

where

$$P_2 = b_2 P_1 \quad \text{AND} \quad Q_2 = b_2 Q_1 + a_2.$$

Then replace b_2 by $B_2 = b_2 + a_3/b_3$ to get

$$f_3 = \frac{B_2 P_1}{B_2 Q_2 + a_2} = \frac{b_2 P_1 + P_1(a_3/b_3)}{b_2 Q_1 + a_2 + Q_1(a_3/b_3)} = \frac{P_2 + P_1(a_3/b_3)}{Q_2 + Q_1(a_3/b_3)} = \frac{P_3}{Q_3}$$

where

$$P_3 = b_3 P_2 + a_3 P_1 \quad \text{AND} \quad Q_3 = b_3 Q_2 + a_3 Q_1.$$

Then replace b_3 by $B_3 = b_3 + a_4/b_4$ to get

$$f_4 = \frac{B_3 P_2 + a_3 P_1}{B_3 Q_2 + a_3 Q_1} = \frac{P_3 + P_2(a_4/b_4)}{Q_3 + (a_4/b_4)Q_2} = \frac{P_4}{Q_4}$$

where

$$P_4 = b_4 P_3 + a_4 P_2 \quad \text{AND} \quad Q_4 = b_4 Q_3 + a_4 P_2.$$

If we define $P_0 = 0$ and $Q_0 = 1$ then f_2 fits the same pattern,

$$P_2 = b_2 P_1 + a_2 P_0 \quad \text{AND} \quad Q_2 = b_2 Q_1 + a_2 Q_0.$$

The whole recursion can then be summarized by the initialization

$$\begin{aligned} P_0 &= 0 \quad \text{AND} \quad P_1 = a_1 \\ Q_0 &= 1 \quad \text{AND} \quad Q_1 = b_1 \end{aligned}$$

and the update

$$\left.\begin{aligned} P_i &= b_i P_{i-1} + a_i P_{i-2} \\ Q_i &= b_i Q_{i-1} + a_i Q_{i-2} \end{aligned}\right\} \quad \text{for } i \geq 2.$$

For example

$$P_2 =$$

3 Comparisons

S:compare

References

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