DSAA 2043 | Design and Analysis of Algorithms

香港科技大学（广州）
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY (GUANGZHOU)

Advanced Data Structures

- Binary Search Trees
- AVL Trees
- Red-Black Trees
- Heaps

Yanlin Zhang & Wei Wang | DSAA 2043 Spring 2025

Binary Search Tree

Binary Search Tree

Sorted Array:

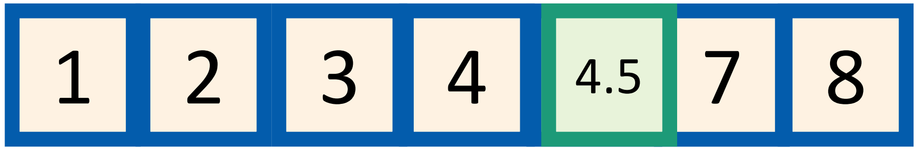
1	2	3	4	5	6	7	8

Linked list (not necessarily sorted):

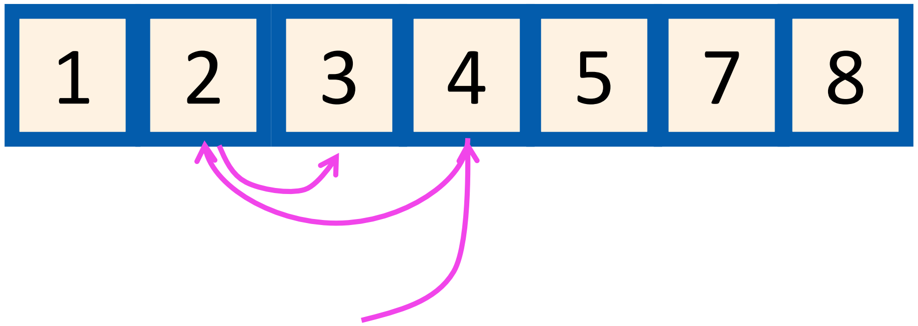
5 → 4 → 1 → 3 → 2 → 7 → 6 → 8

Array

- $O(n)$ INSERT/DELETE:
  - First, find the relevant element (we'll see how below), and then move a bunch elements in the array:

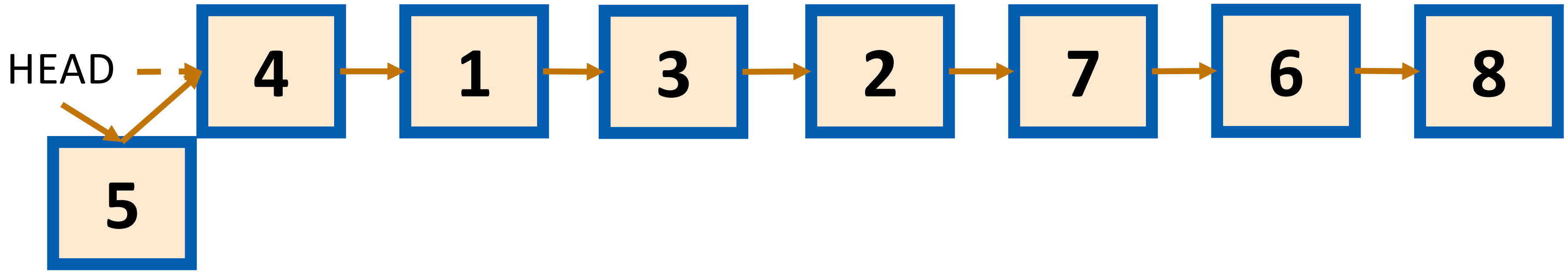


- $O(\log(n))$ SEARCH (if sorted):

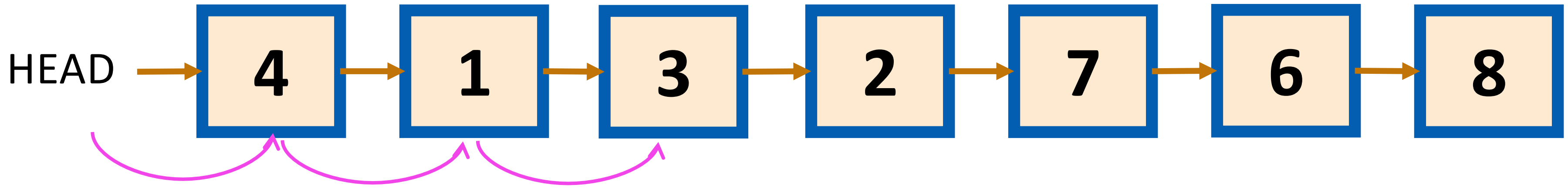


Linked Lists

- $O(1)$ INSERT (manipulating pointers)



- $O(n)$ SEARCH/DELETE:



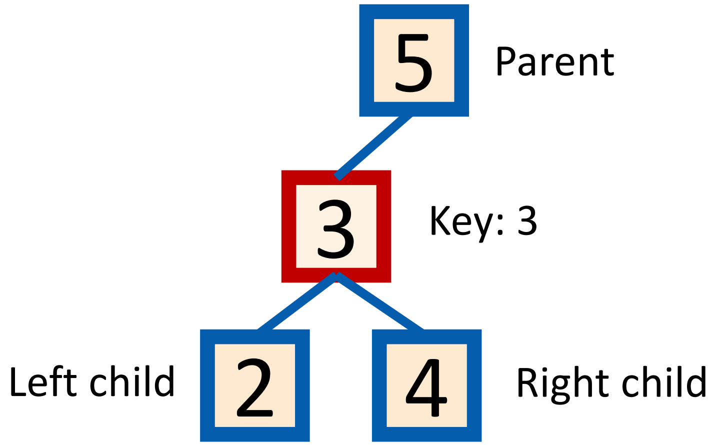
eg, search for 3 (and then you could delete it by manipulating pointers).

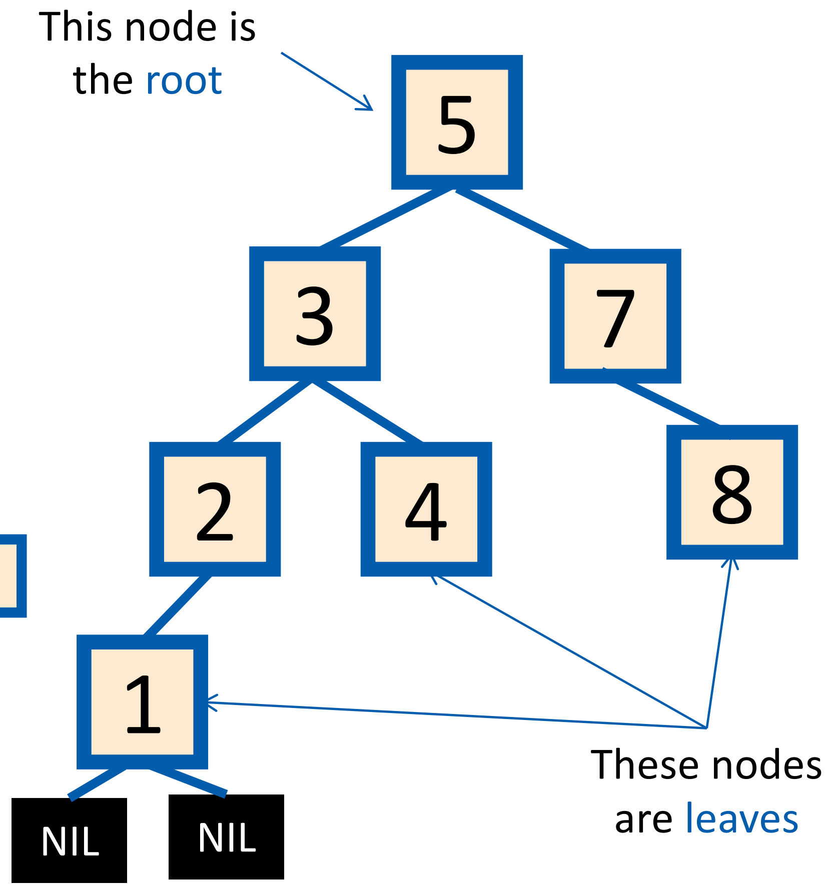
Binary Search Tree

	Arrays	Linked Lists	(Balanced) Binary Search Trees
Search	$O(n)$ ($O(\log n)$ if sorted)	$O(n)$	$O(\log n)$
Delete	$O(n)$	$O(n)$	$O(\log n)$
Insert	$O(n)$	$O(1)$	$O(\log n)$

Binary Tree Terminology

Each node has two children





$\boxed{2}$ is a descendant of $\boxed{5}$ $\boxed{5}$ is an ancestor of $\boxed{2}$

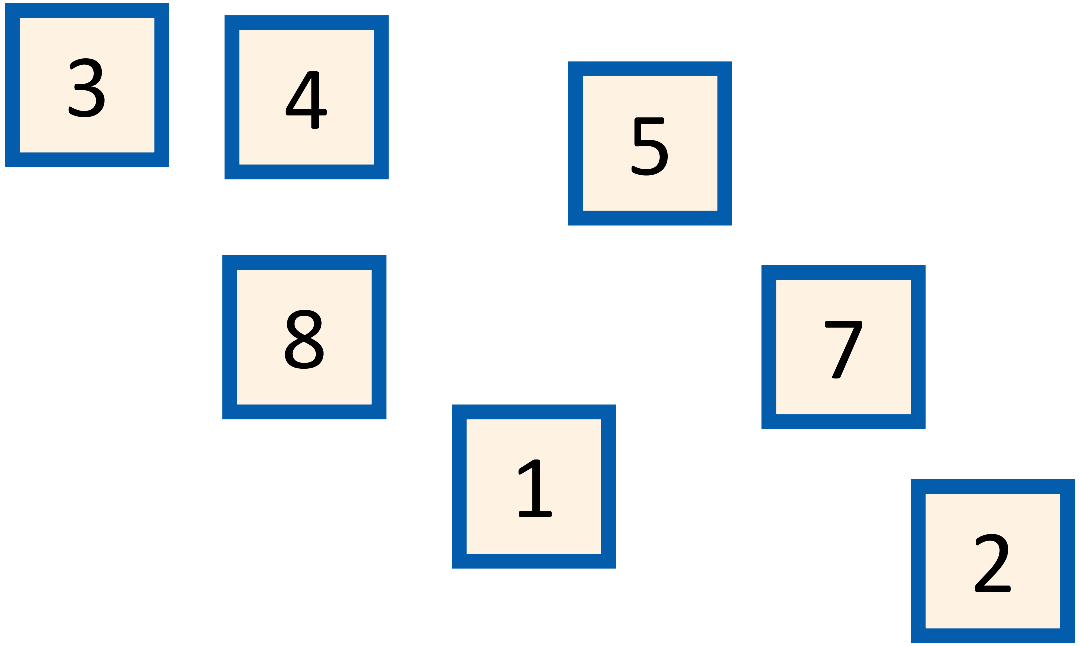
Both children of $\boxed{1}$ are NIL (usually not drawn)

The height of this tree is 3

\# of edges in the longest path

Binary Search Tree

- A BST is a binary tree such that:
  - Every LEFT descendant of a node has key less than that node.
  - Every RIGHT descendant of a node has key larger than that node.



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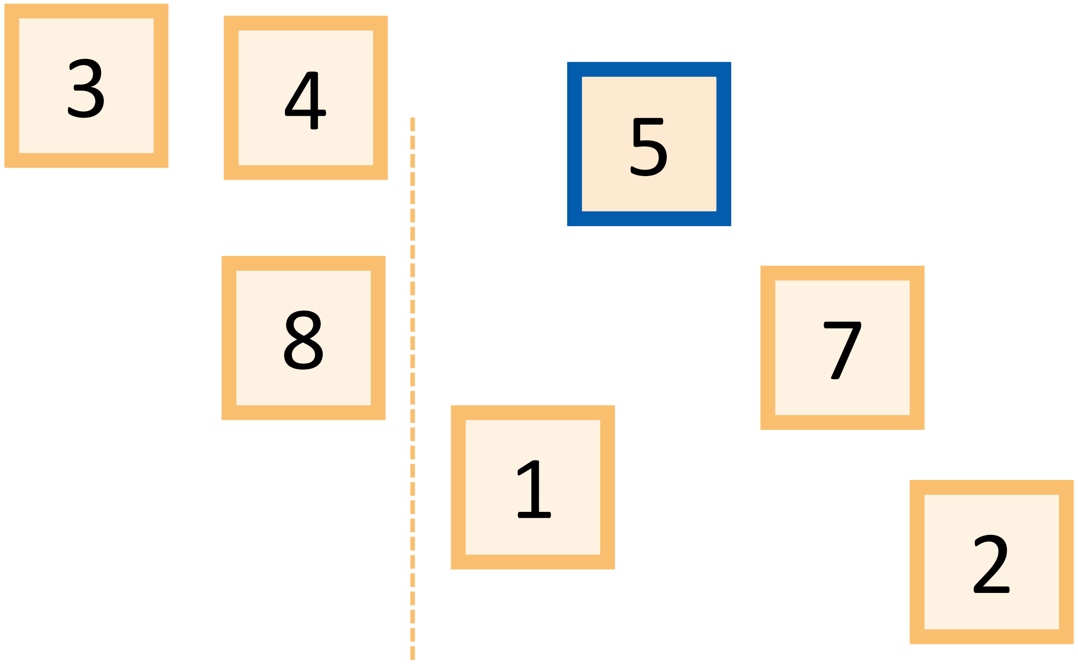

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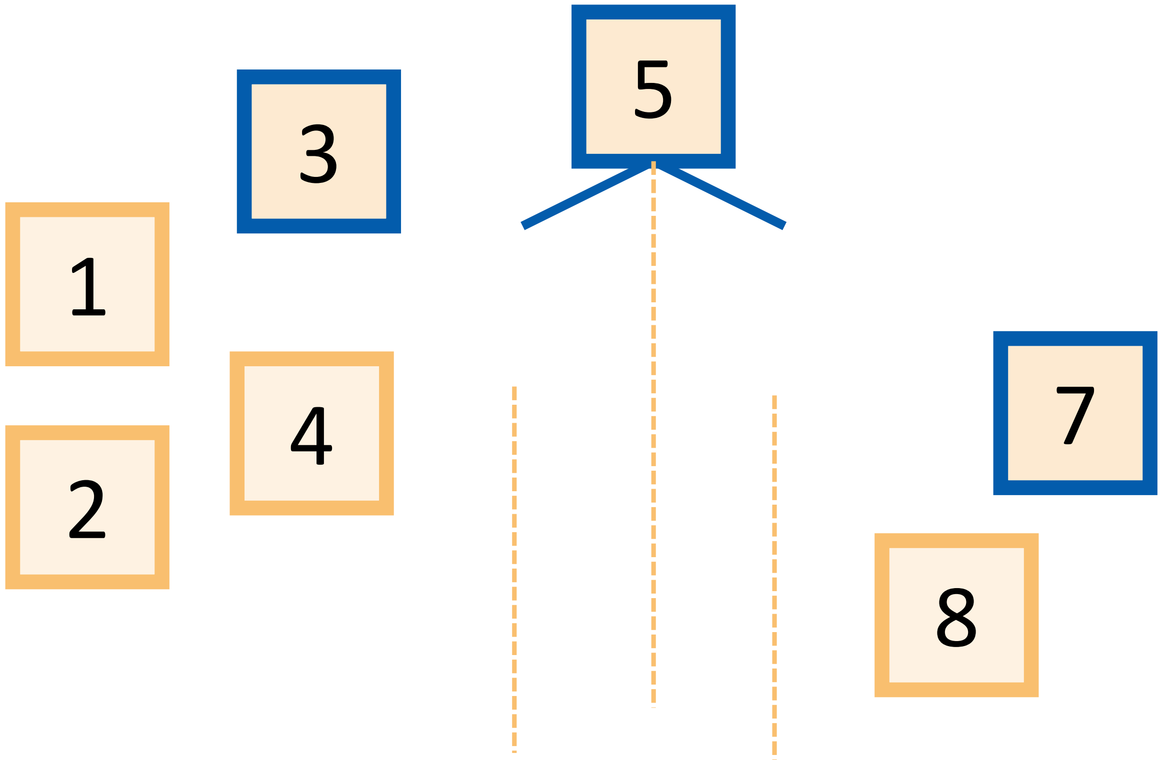

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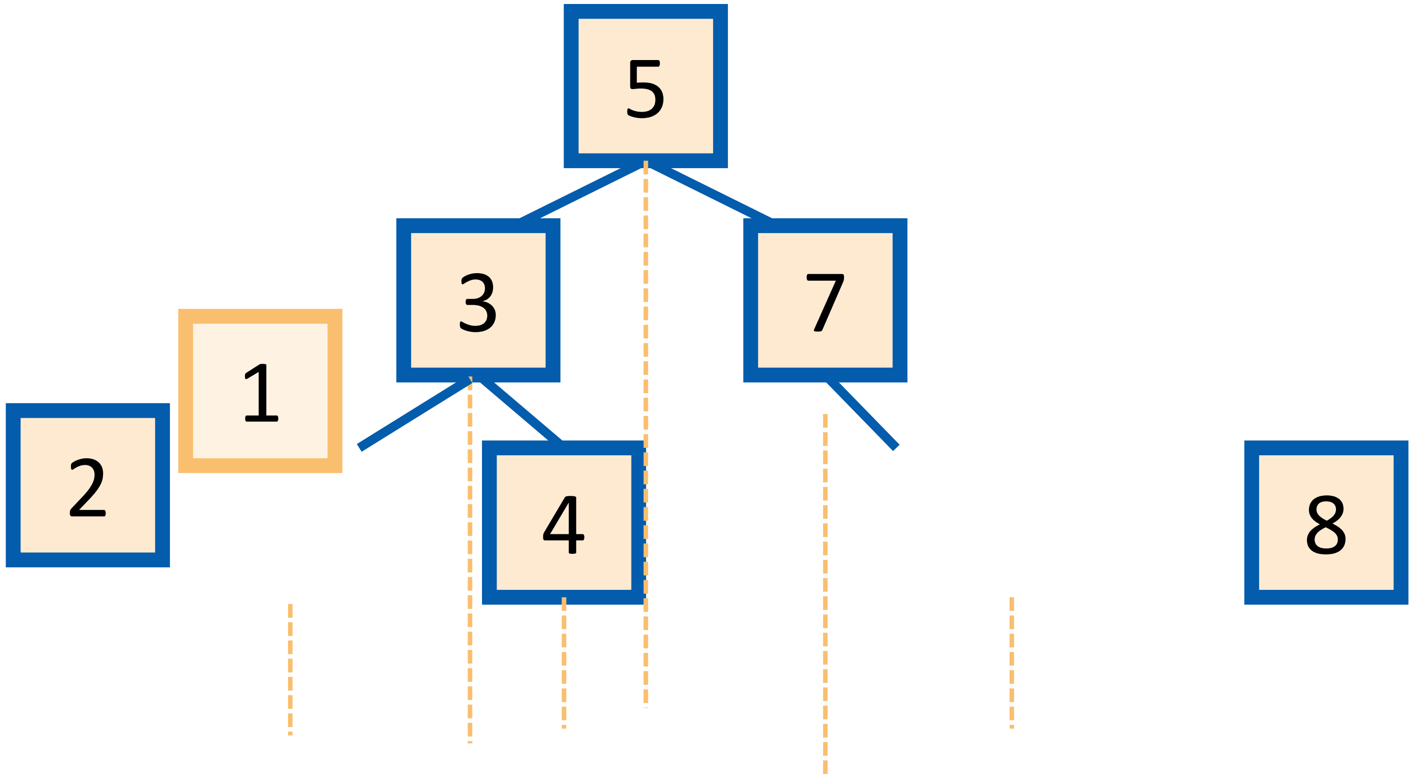

Binary Search Tree

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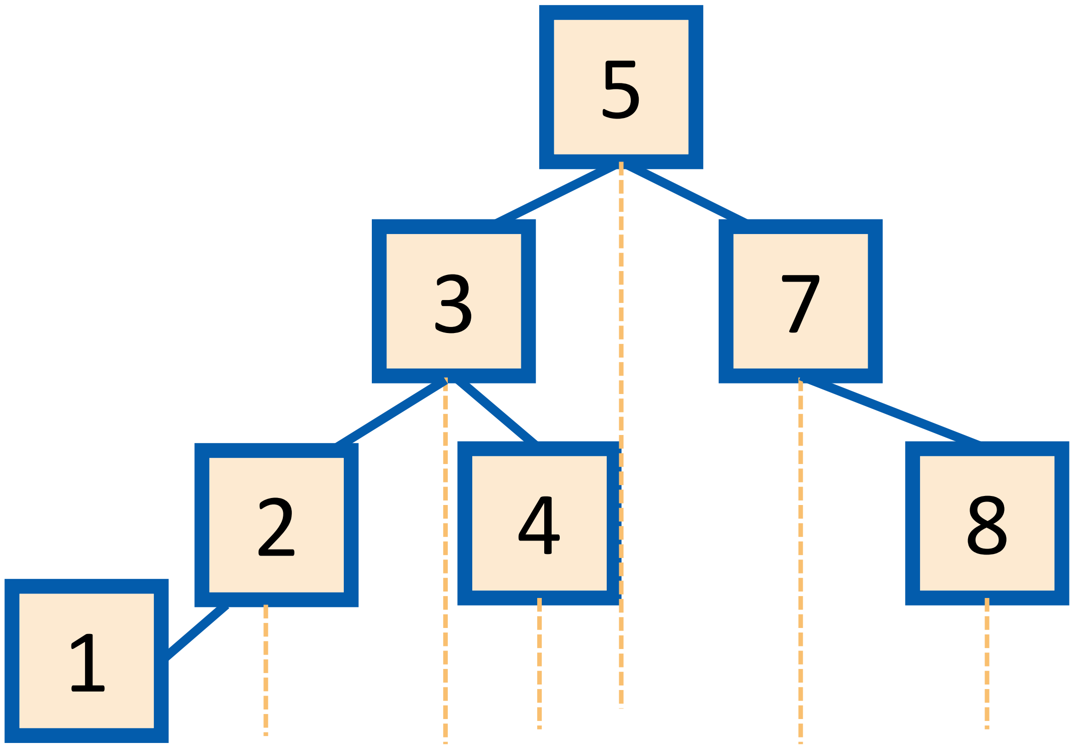
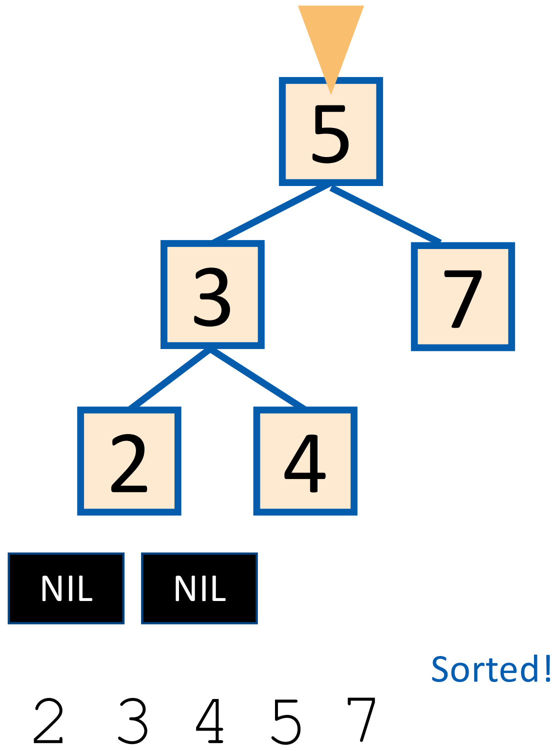


Q: Is this the only binary search tree I could possibly build with these values?

Traversal

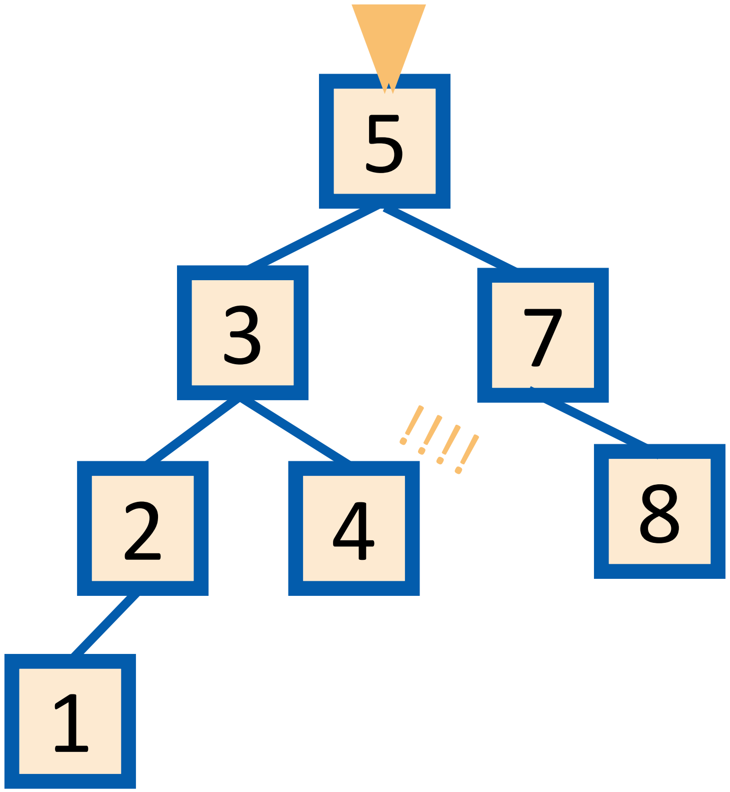
- Output all the elements in sorted order!

- inOrderTraversal(x):
  - if x!= NIL:
    - inOrderTraversal(x.left)
    - print(x.key)
    - inOrderTraversal(x.right)



Pre-order / post-order traversal?

Search



EXAMPLE: Search for 4.

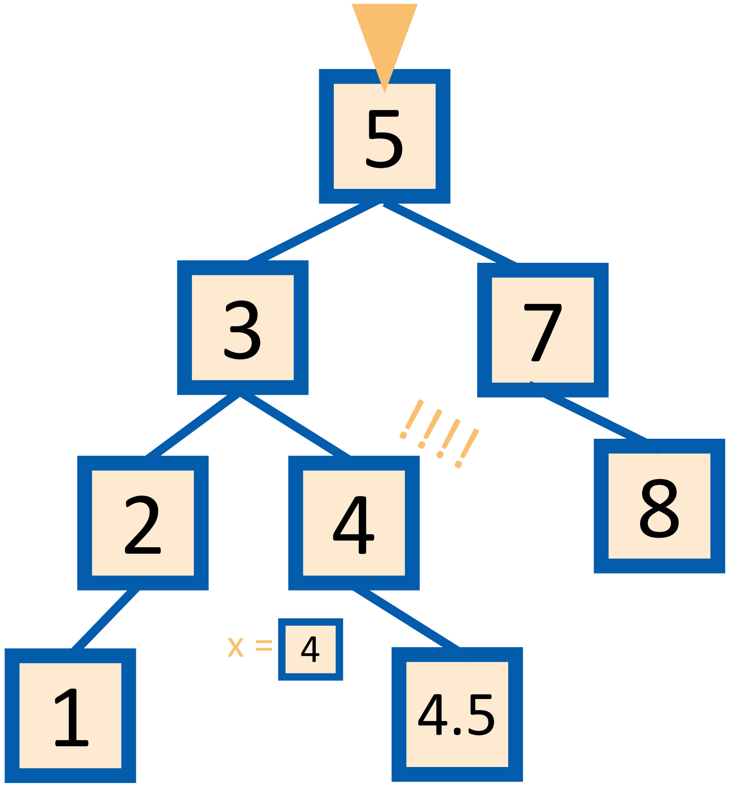
EXAMPLE: Search for 4.5

- Sometimes, it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

Semantics:

- find the largest element in the collection that is no larger than the search key
- Largest predecessor query

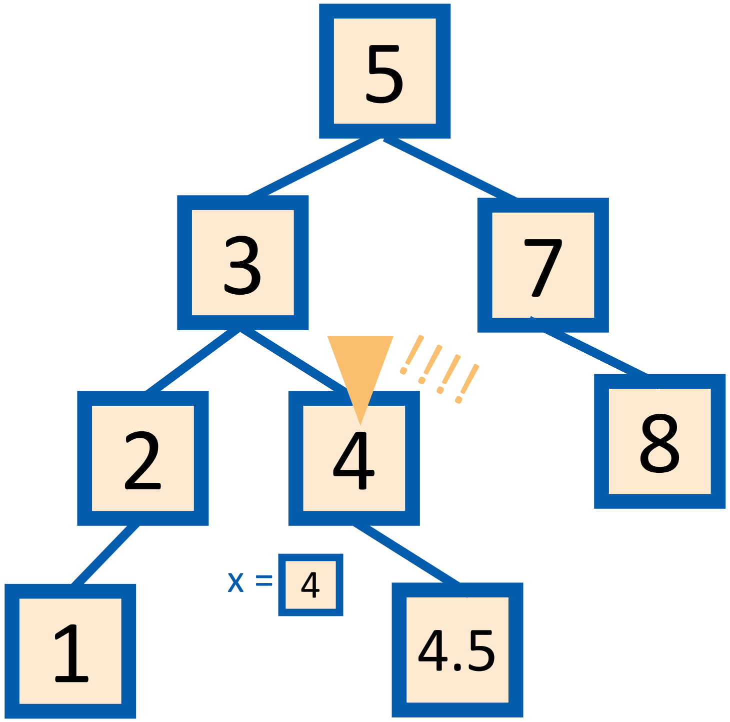
Insert



EXAMPLE: Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **Insert** a new node with desired key at x…

Insert

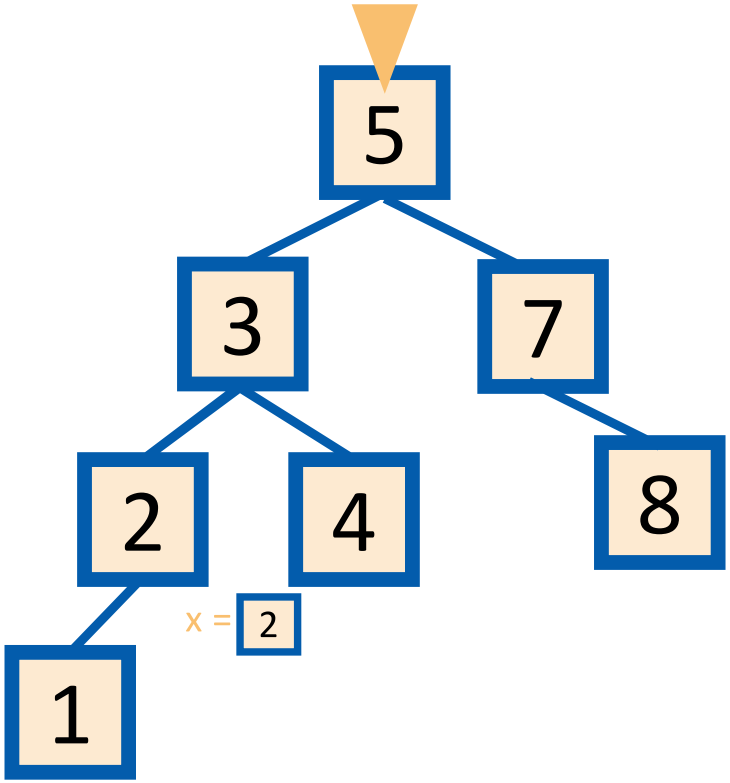


EXAMPLE: Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - if key > x.key:
    - Make a new node with the correct key, and put it as the right child of x
  - if key < x.key:
    - Make a new node with the correct key, and put it as the left child of x
  - if x.key == key:
    - **return**

Semantics?

Delete

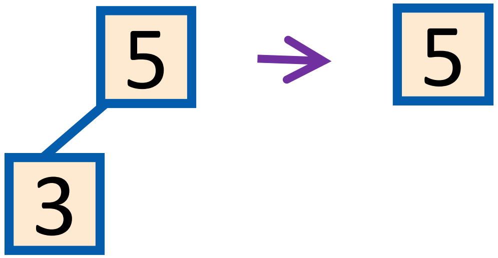


EXAMPLE: Delete 2

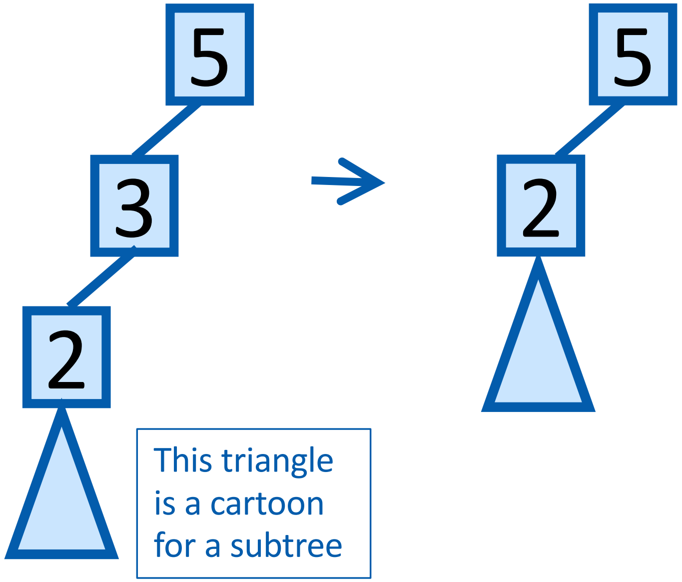
- DELETE(key):
  - x = SEARCH(key)
  - **if** x.key == key:
    - ….delete x….

This is a bit more complicated…

Delete



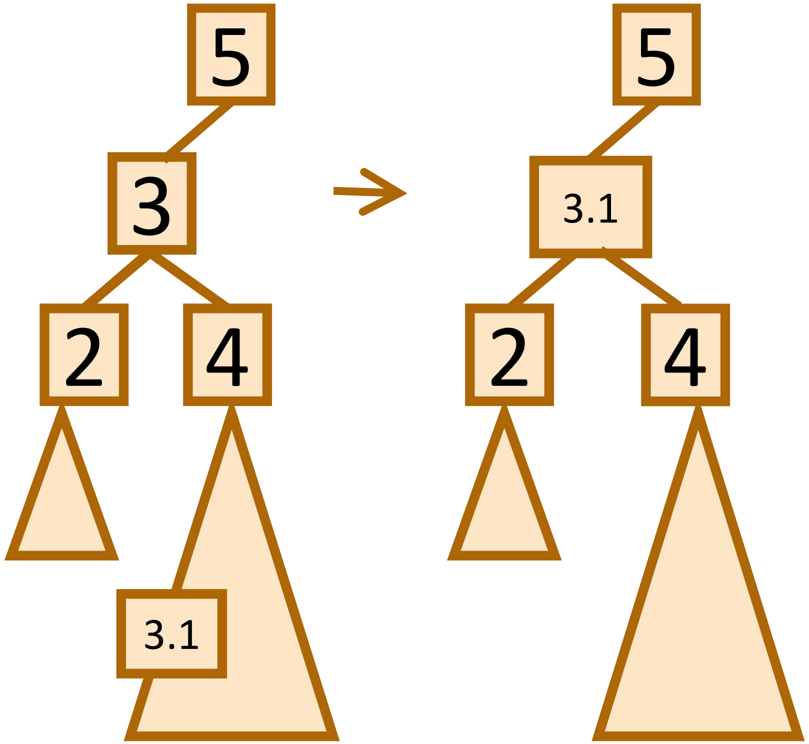
Case 1: if 3 is a leaf,
just delete it.



Case 2: if 3 has just one child,
move that up.

Delete

Case 3: if 3 has two children,
replace 3 with it's immediate successor.
(aka, next biggest element after 3)



- Does this maintain the BST property?
  - Yes

Why?

- How do we find the immediate successor?
  - SEARCH for 3 in the subtree under 3.right
- How do we remove it when we find it?
  - If [3.1] has 0 or 1 children, do one of the previous cases
- What if [3.1] has two children?
  - It doesn't

Why?

More Operations

Best case (when?)

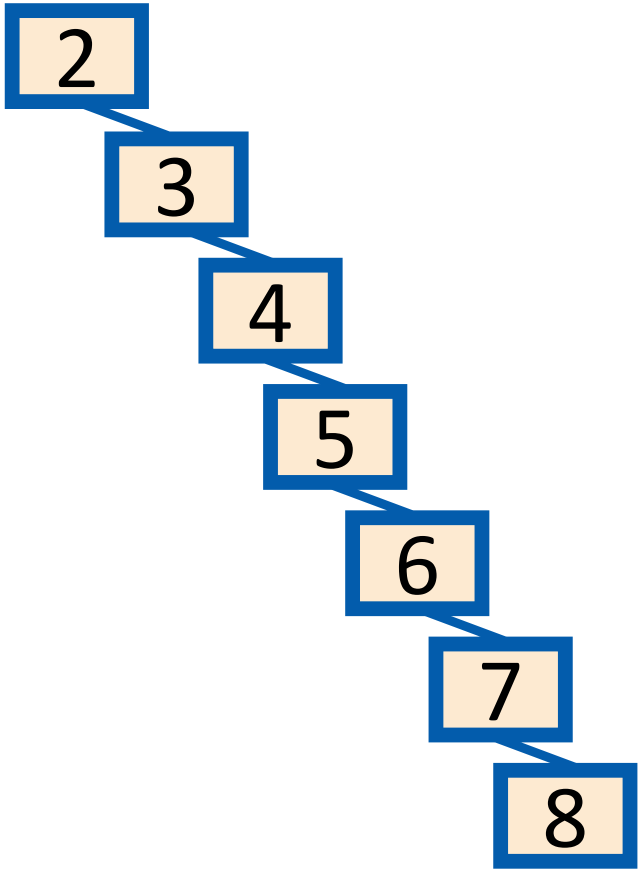
Worst case (when?)

- findmin(x): finds the minimum of the tree rooted at x
- findmax(x): finds the max of the tree rooted at x
- deletemin(): finds the minimum of the tree and delete it

Time complexities of them?

The Importance of Being Balanced

- This is a valid binary search tree
- The version with n nodes has depth n, **not** $\Theta(\log(n))$



Balanced BST Strategy

- Augment every node with some property
- Define a local invariant on property
- Show (prove) that invariant guarantees $\Theta(\log n)$ height
- Design algorithms to maintain property and the invariant

AVL Trees

AVL Trees

An AVL (Adelson-Velskii and Landis) tree is a binary search tree that also meets the following rule

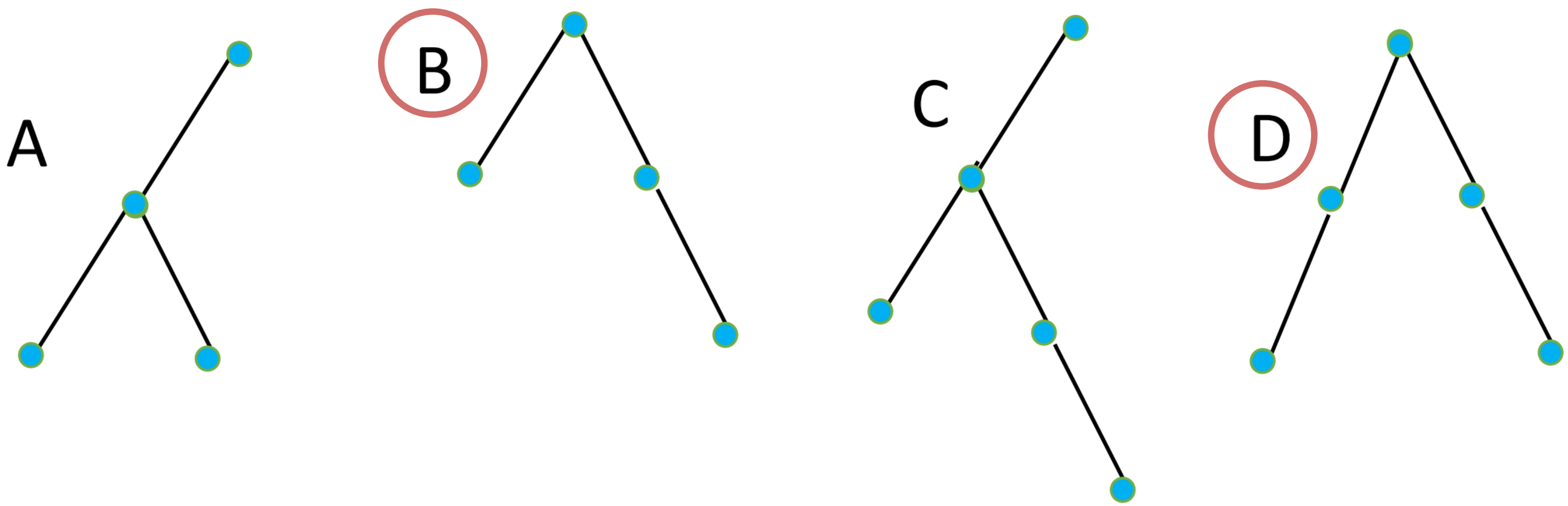
AVL condition: For every node, the height of its left subtree and right subtree differ by at most 1.

Height of a tree:
Maximum number of edges on a path from the root to a leaf.

A tree with one node has height 0.
A null tree (no nodes) has height -1.

AVL Trees

Which one(s) is balanced according to AVL's definition?



AVL Trees

An AVL tree is a binary search tree that also meets the following rule

AVL condition: For every node, the height of its left subtree and right subtree differ by at most 1.

This will avoid the $\Theta(n)$ behavior! We have to check:

1. We must be able to maintain this property when inserting/deleting.
2. Such a tree must have height $\Theta(\log n)$.

Bounding the Height

- Let $n(h)$ be the minimum number of nodes in an AVL tree of height h.
- If we can say $n(h)$ is big, we'll be able to say that a tree with n nodes has a small height.

- So…what's $n(h)$?
- $n(h) = \begin{cases} 1, & \text{if } h = 0 \\ 2, & \text{if } h = 1 \\ n(h-1) + n(h-2) + 1, & \text{otherwise} \end{cases}$

Bounding the Height

- Hey! That's a recurrence!
- Recurrences can describe any kind of function, not just running time of code!
- $n(h) = \begin{cases} 1, & \text{if } h = 0 \\ 2, & \text{if } h = 1 \\ n(h-1) + n(h-2) + 1, & \text{otherwise} \end{cases}$
- We could use tree method, but it's a little…weird.
- It'll be easier if we change things just a bit:
- $n(h) \geq \begin{cases} 1, & \text{if } h = 0 \\ 2, & \text{if } h = 1 \\ n(h-2) + n(h-2) + 1, & \text{otherwise} \end{cases}$

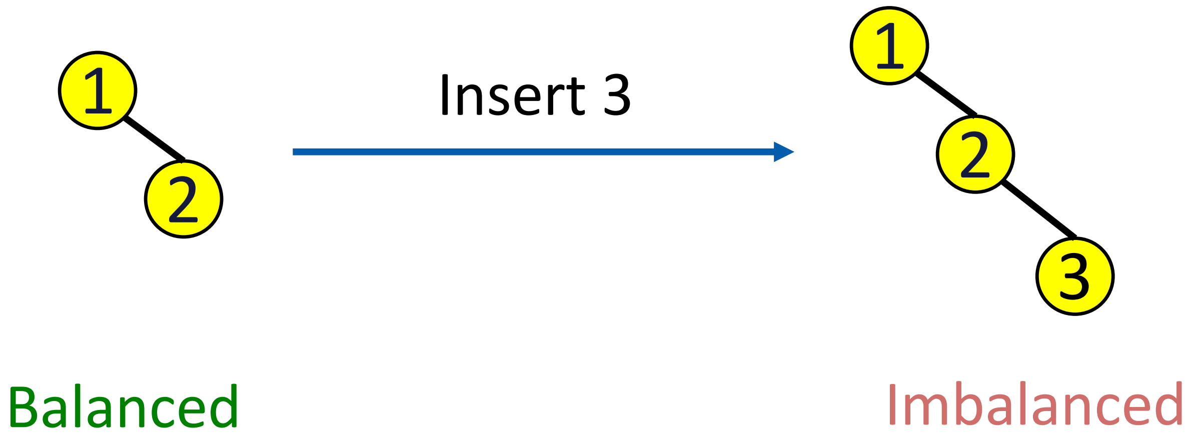
Bounding the Height

$$
\begin{aligned}
n(h) &= n(h-1) + n(h-2) + 1 \\
&> 2n(h-2) \\
&> 2 \times 2n(h-4) \\
&> 2^{\frac{h}{2}} \\
h &< 2\log n(h)
\end{aligned}
$$

Hence, $h = \Theta(\log n)$.

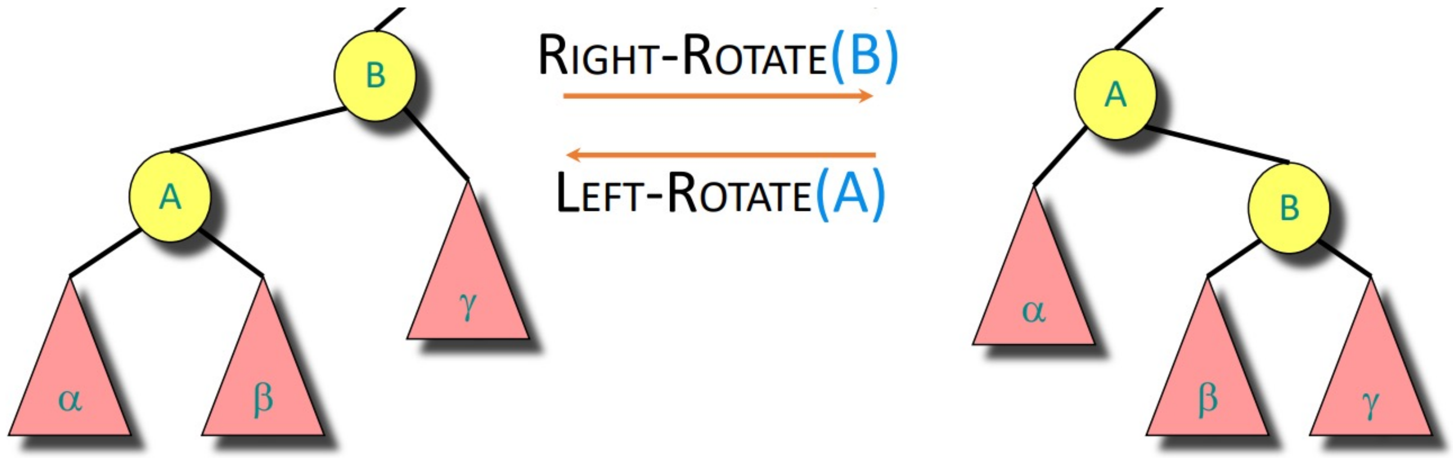
Insertion

What happens if when the AVL condition is violated after insertion?

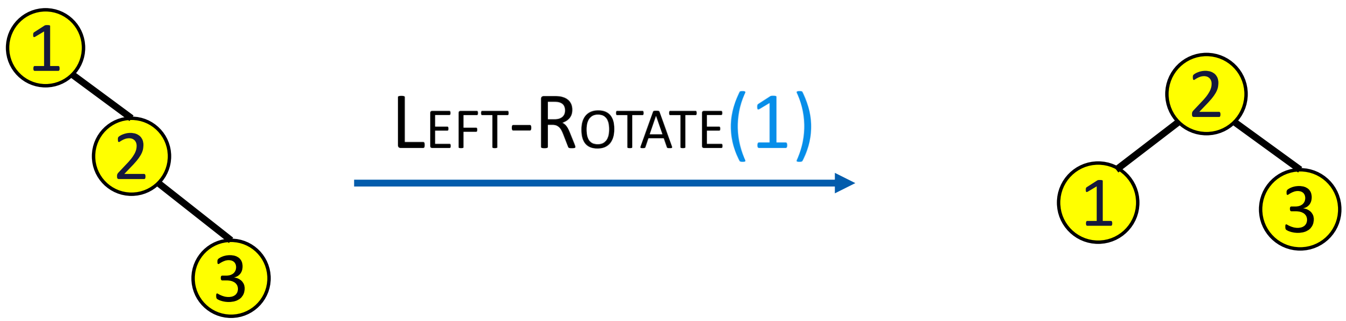


Insertion

Rotations!

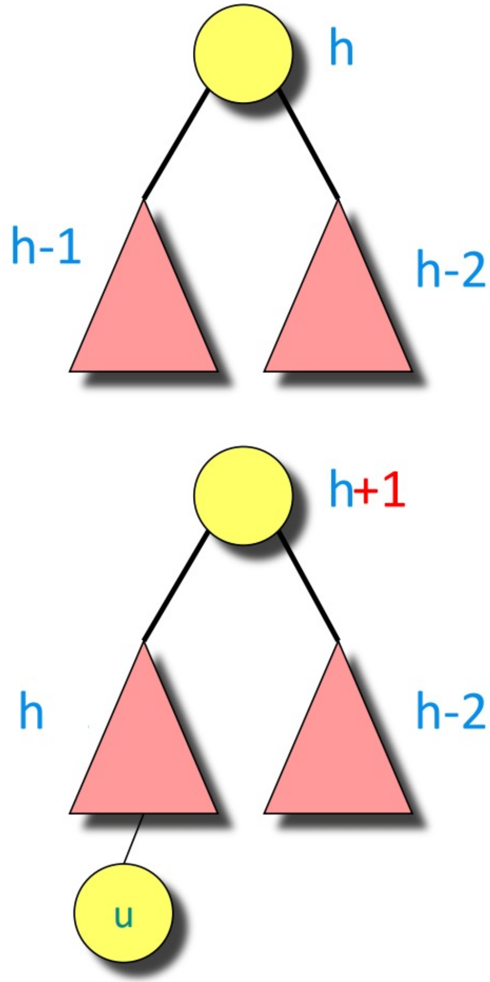


Rotations can reduce the height!



Insertion / Deletion

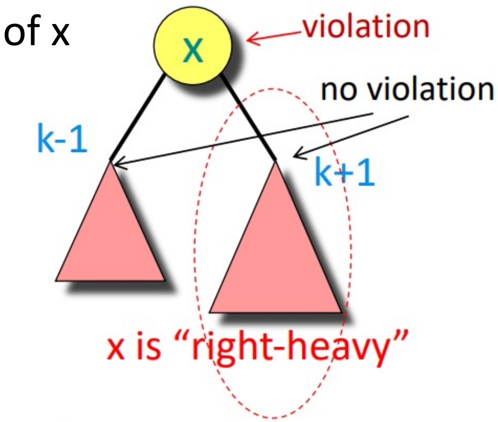
- Insert new node u as in the simple BST
  - Can create imbalance
- Work your way up the tree, restoring the balance
- Similar issue/solution when deleting a node

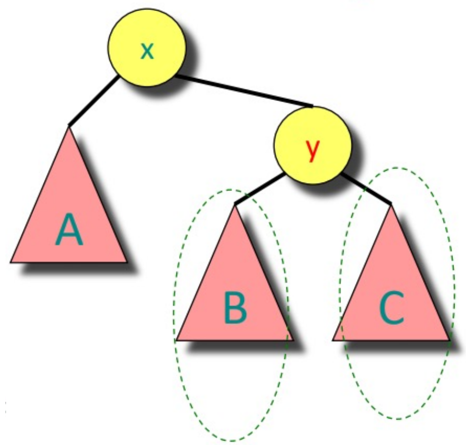


Balancing

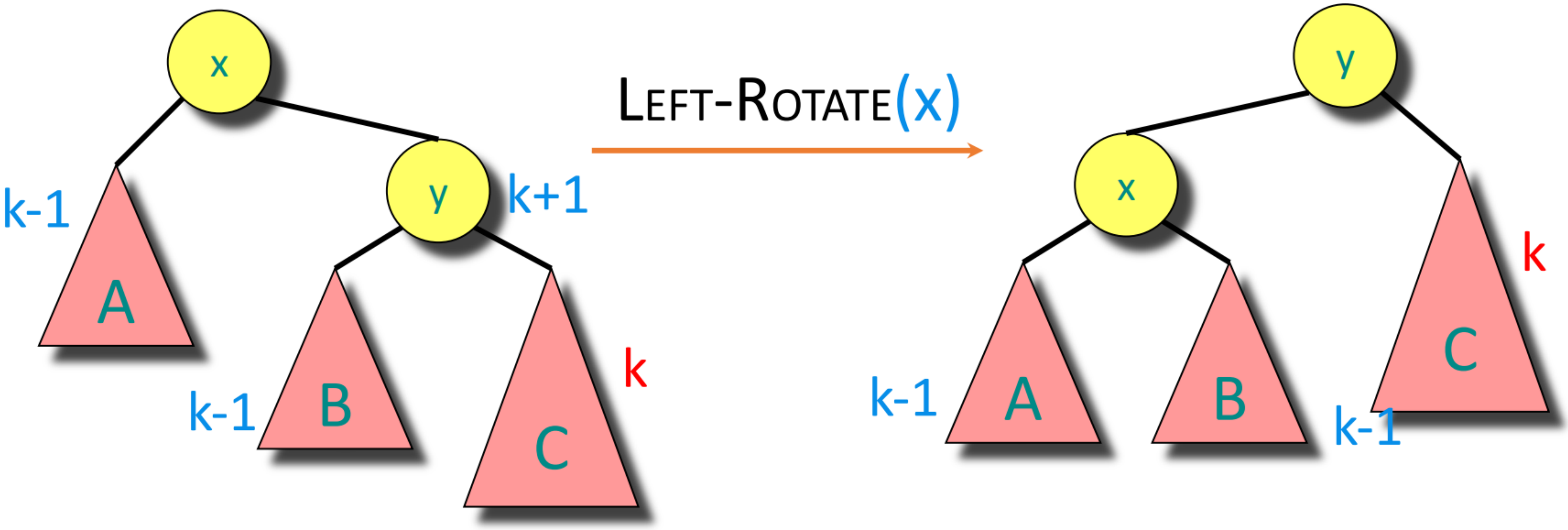
- Let x be the lowest "violating" node
  - we will try to correct that and move up the tree
- Assume that x is "**right-heavy**"
  - we analyze more the right subtree of x
  - y is the right child of x
- Scenarios
  - Case 1: y is right-heavy / **balanced**
  - Case 2: y is **left-heavy**

The right child of x has +2 height than the left child of x



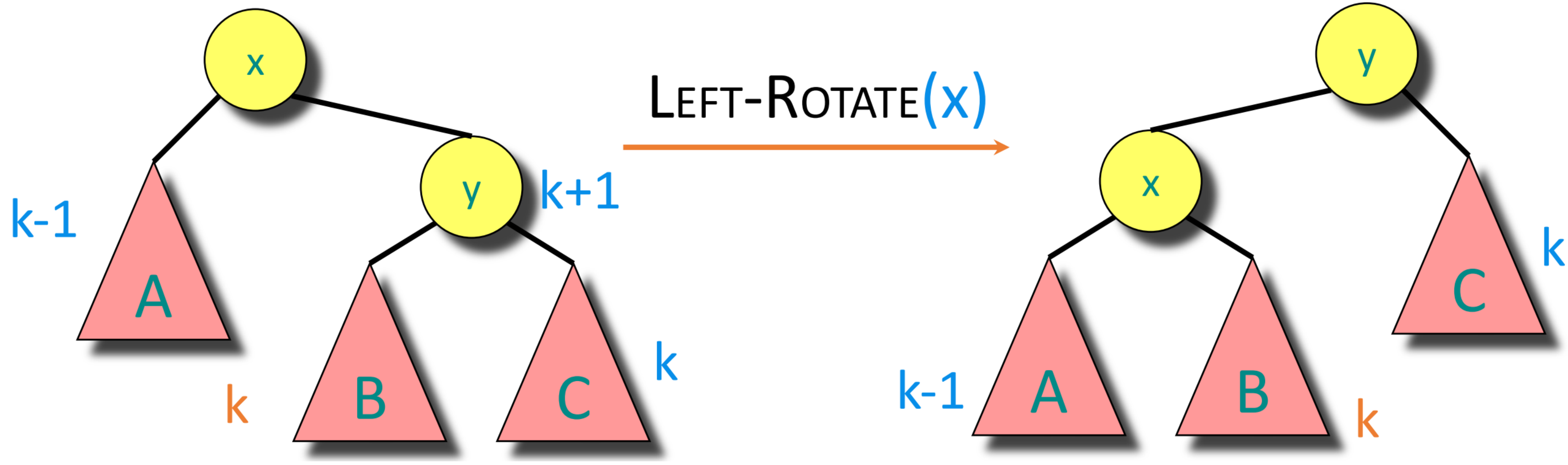


Case 1.1: y is right-heavy



Balancing

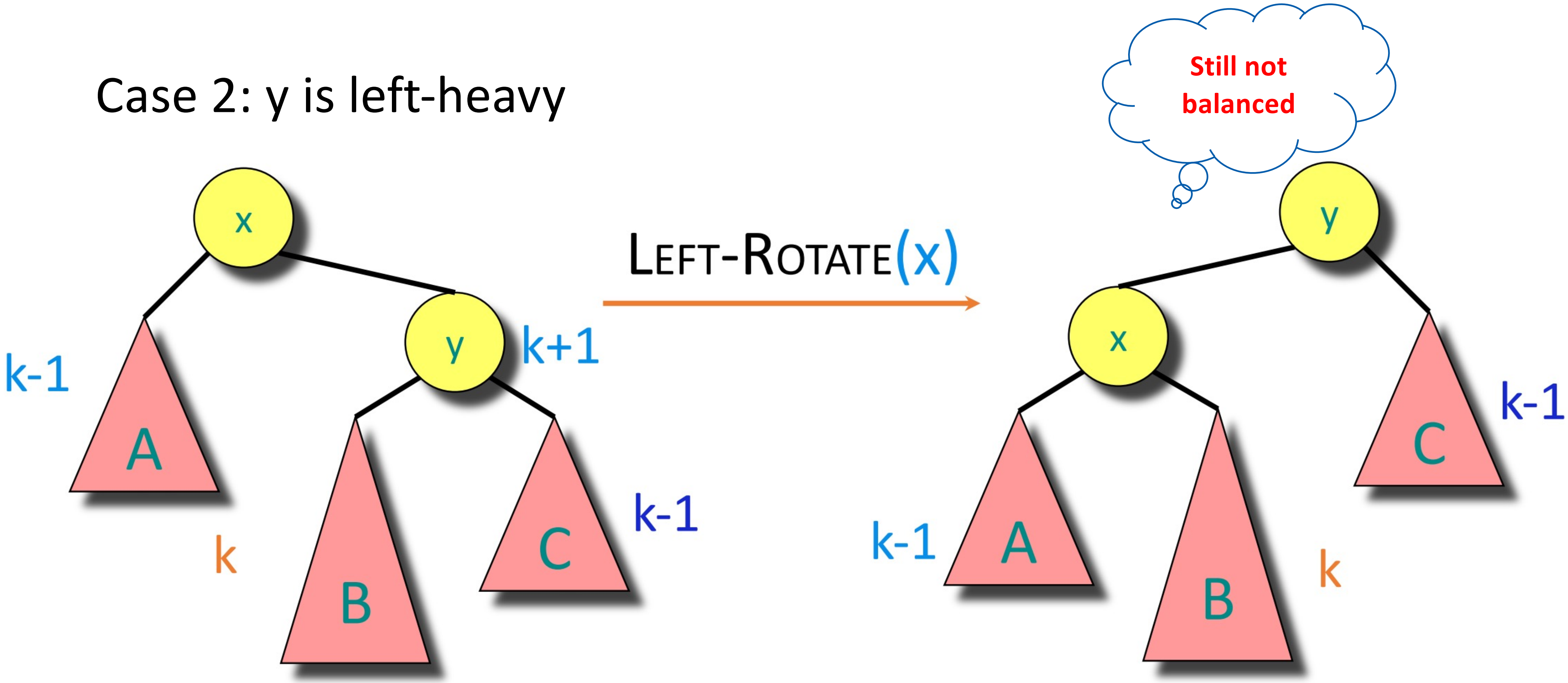
Case 1.2: y is balanced



Same as Case 1.1

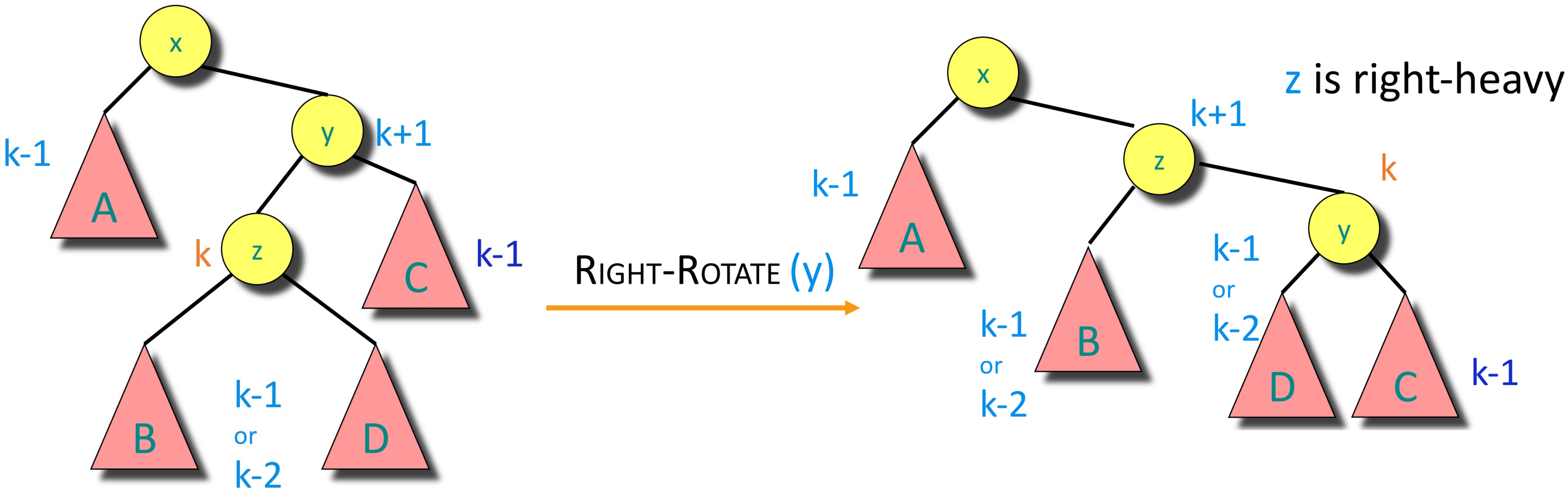
Balancing

Case 2: y is left-heavy



Balancing

Case 2: y is left-heavy



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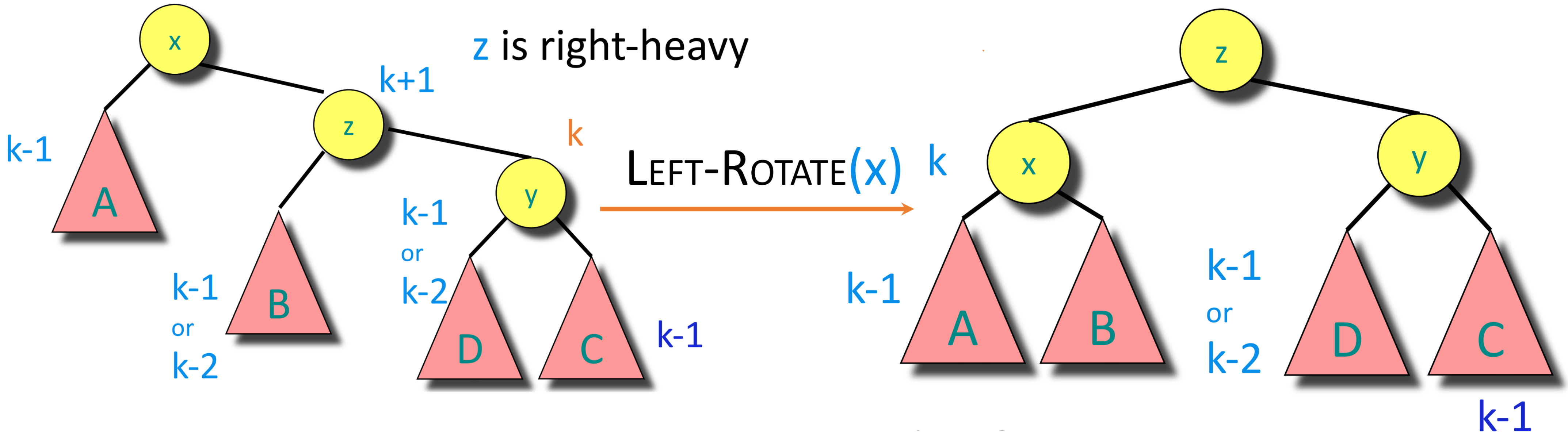
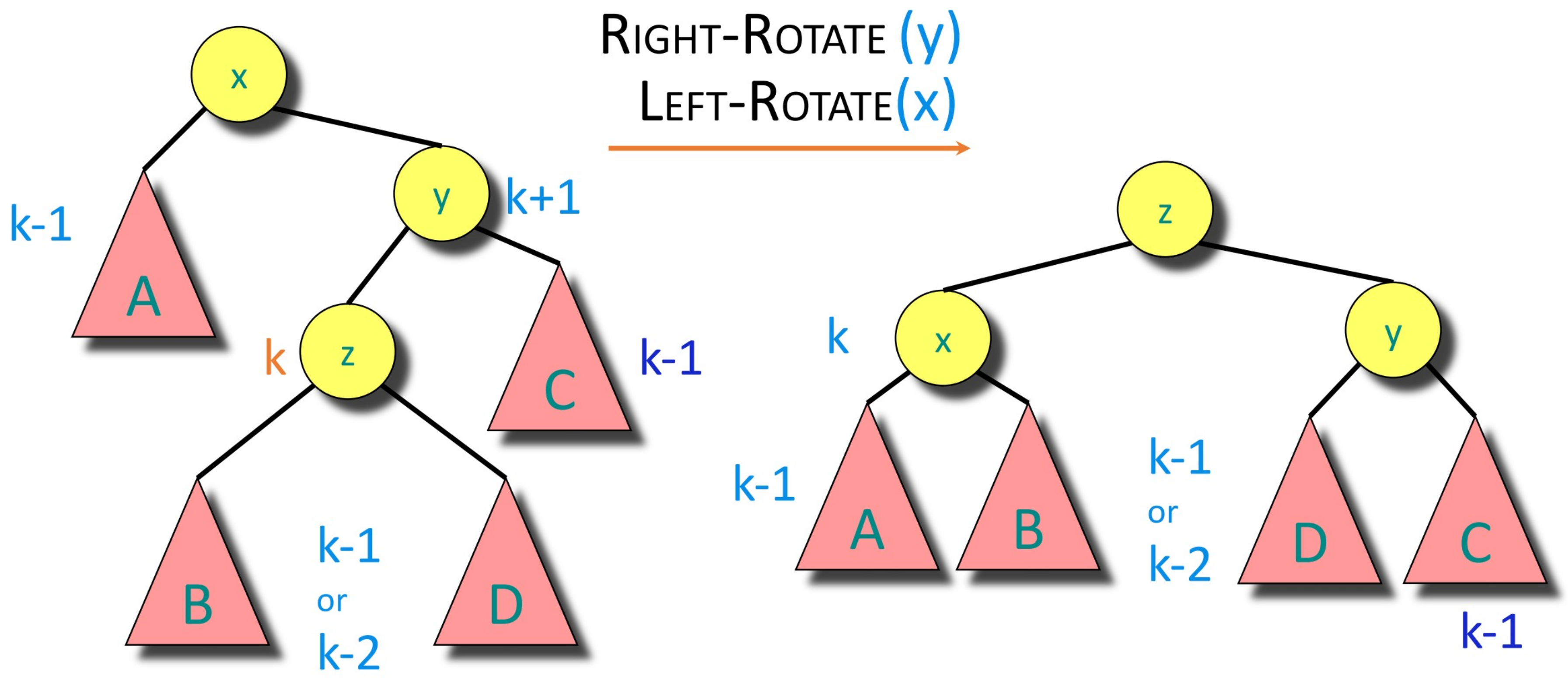


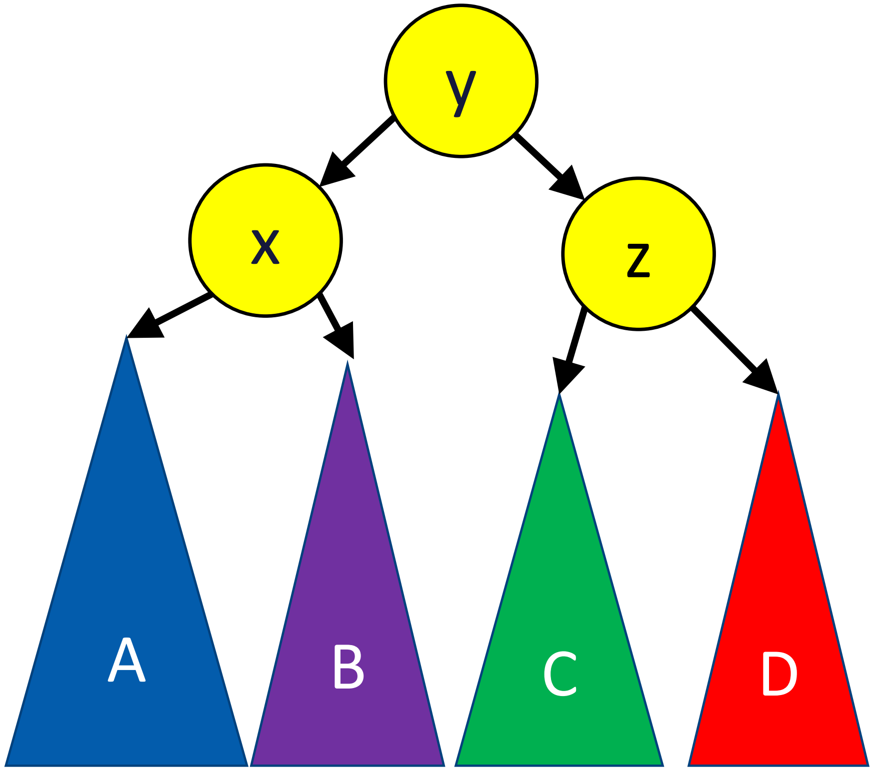
Balancing

Case 2: y is left-heavy (final solution)



Four Types of Rotations

To summarize



Insert location	Solution
Left subtree of left child **(A)**	Single right rotation
Right subtree of left child **(B)**	Double (left-right) rotation
Left subtree of right child **(C)**	Double (right-left) rotation
Right subtree of right child **(D)**	Single left rotation

Other Self-Balancing Trees

- “Red-black trees” work on a similar principle to AVL trees.
- “Splay trees”: Get $O(\log n)$ amortized bounds for all operations.
- “Scapegoat trees”: worst case $O(\log n)$ search complexity. Others are same as splay trees.
- “Treaps” – a BST and heap in one (!)

Similar tradeoffs to AVL trees.

Red-Black Trees

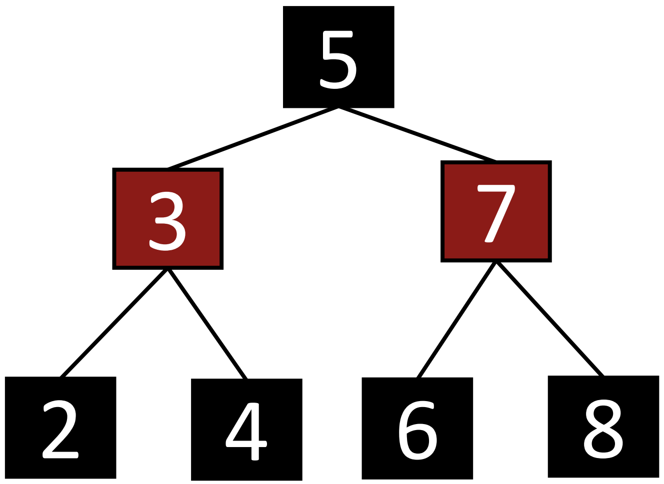
Red-Black Trees

- AVL trees requires more rotations during insertion/deletion due to relatively strict balancing.

- What if we relax the constraint a bit and use some proxy of balancing?

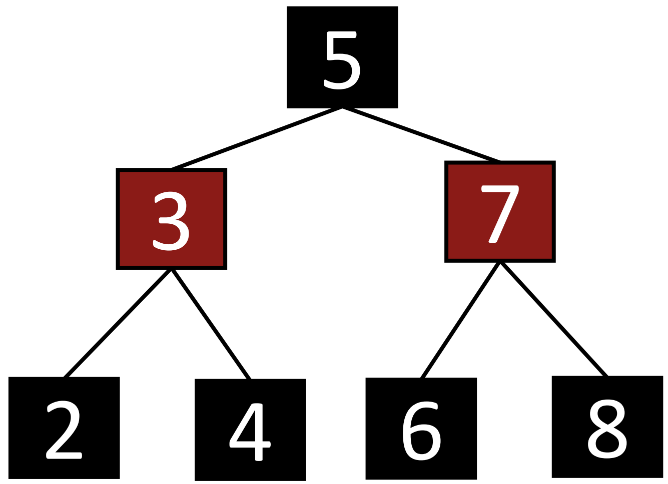
Red-Black tree!

Maintain balance by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.



Red-Black Trees

- Every node is colored **red** or **black**.
- The root node is a **black** node.
- NIL children count as **black** nodes.
- Children of a **red** node are **black** nodes.
- For all nodes x:
  - all paths from x to NIL's have the same number of **black** nodes on them.



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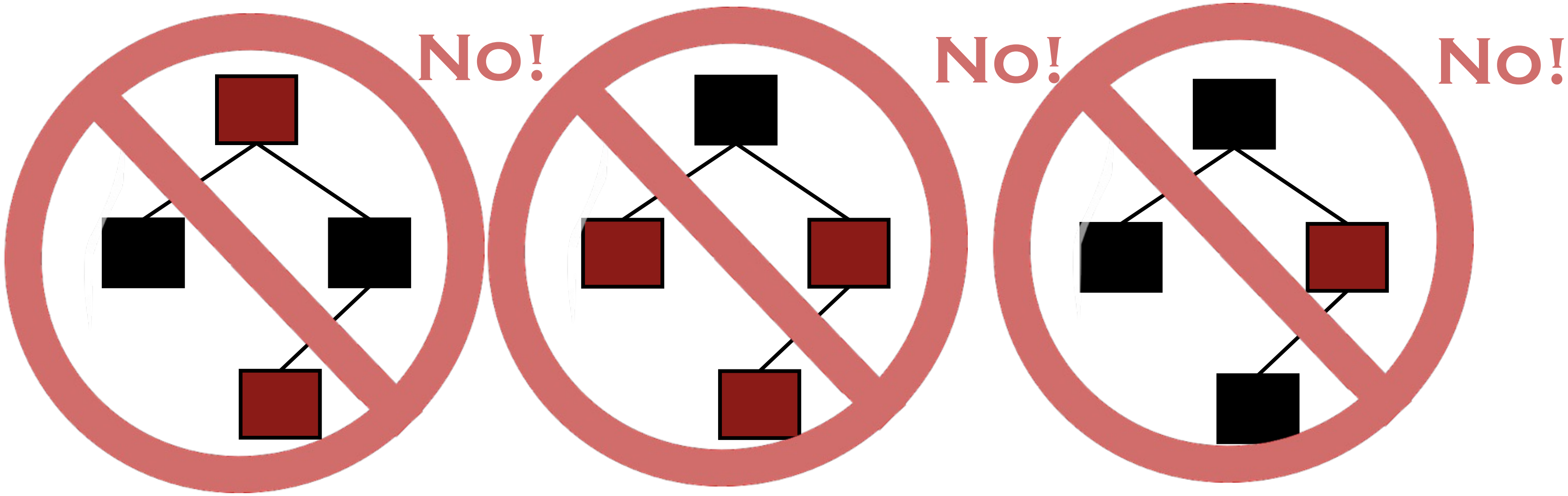
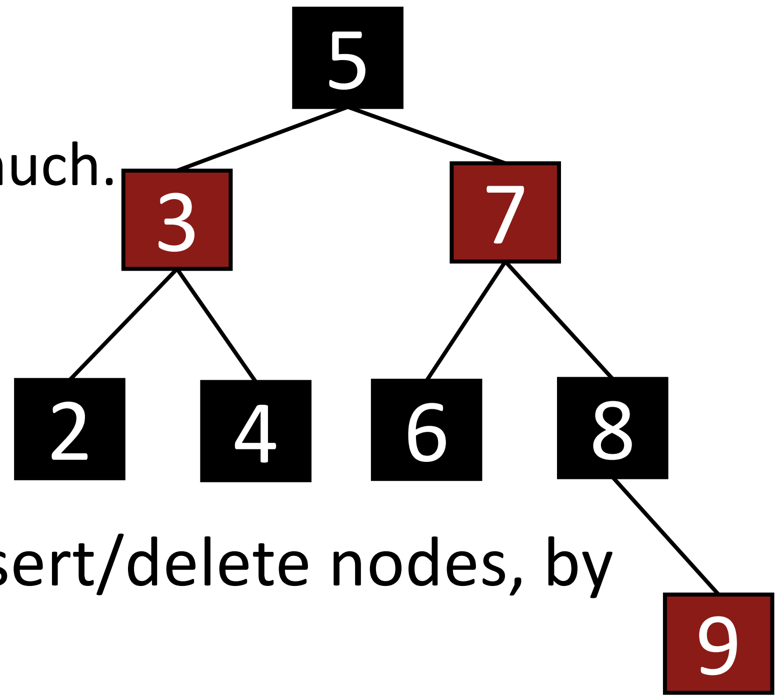
Which of these
are red-black trees?
(NIL nodes not drawn)

1 minute think
1 minute share



Why These Rules?

- This is pretty balanced.
  - The **black nodes** are balanced
  - The **red nodes** don’t mess things up too much.

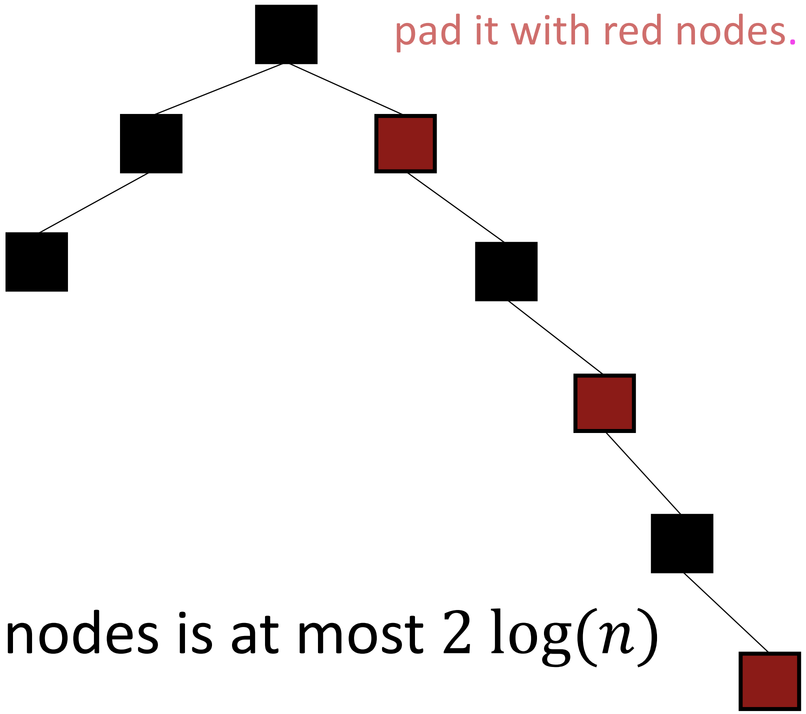


- We can maintain this property as we insert/delete nodes, by using rotations or color flipping.

Why These Rules?

- This is “pretty balanced”.

One path can be at most twice as long another if we pad it with red nodes.



- Conjecture:
  - the height of a **red**-**black** tree with n nodes is at most $2\log(n)$

Why These Rules?

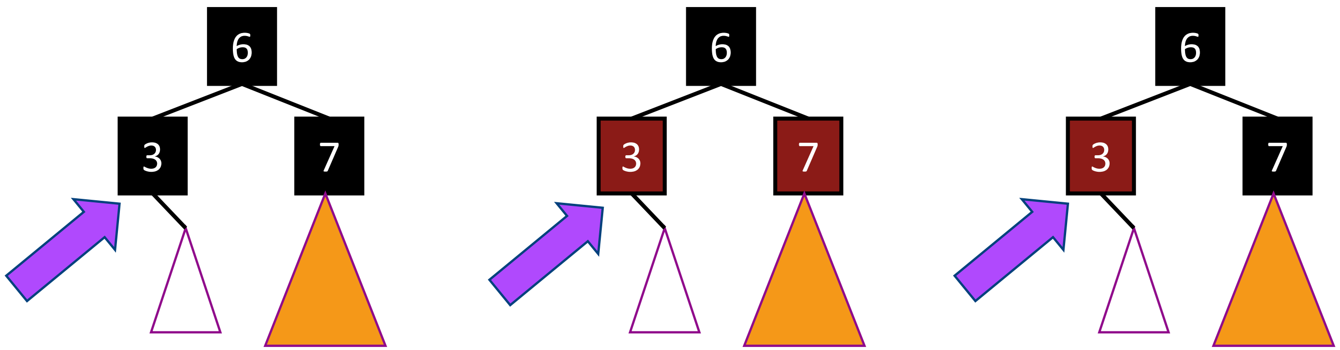
The height of a RB-tree with n non-NIL nodes is at most $2\log_2(n+1)$.

- Prove it?

Insert / Delete

- Since the insertion and deletion in RB Trees are complicated, you don’t need to master the details of them.
  - You should know what the “proxy for balance” property is and why it ensures approximate balance.
  - You should know **that** this property can be efficiently maintained, but you do not need to know the details of how.

Many cases

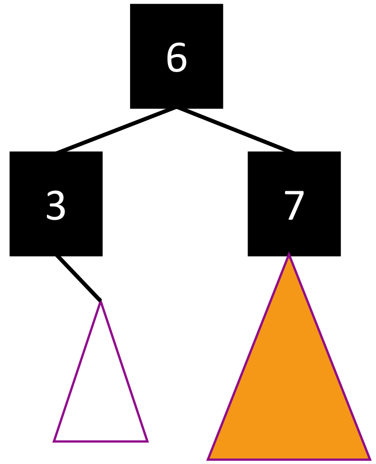


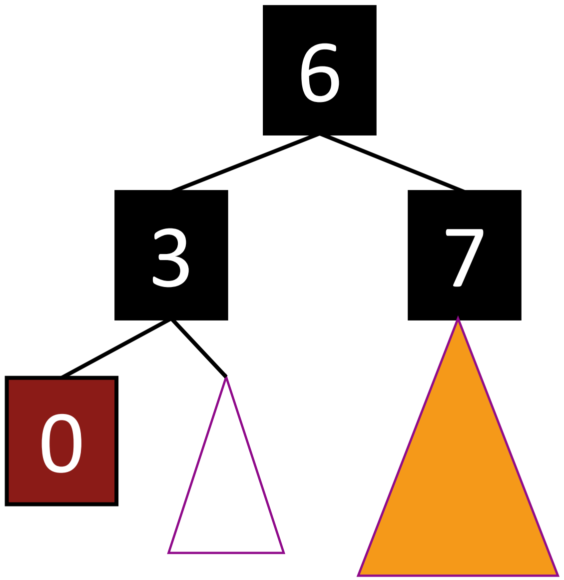
- Suppose we want to insert 0

- 3 "important" cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

Insert: Case 1

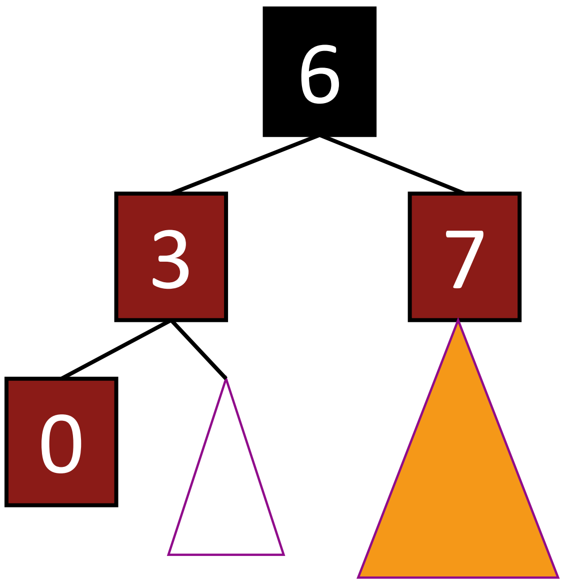
- Make a new **red node**.
- Insert it as you would normally.

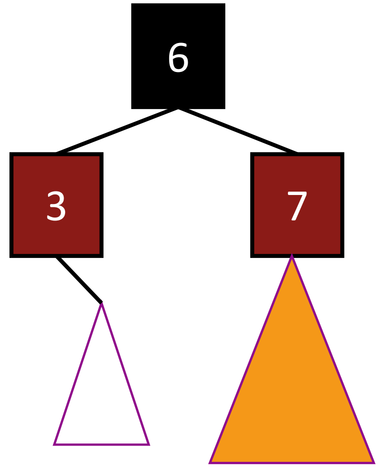




Insert: Case 2

- Make a new **red node**.
- Insert it as you would normally?
- Fix things up if needed.





What if it looks like this?

Insert: Case 2

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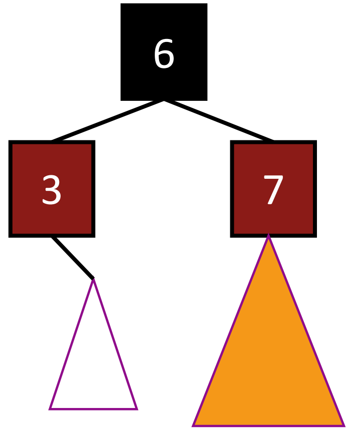
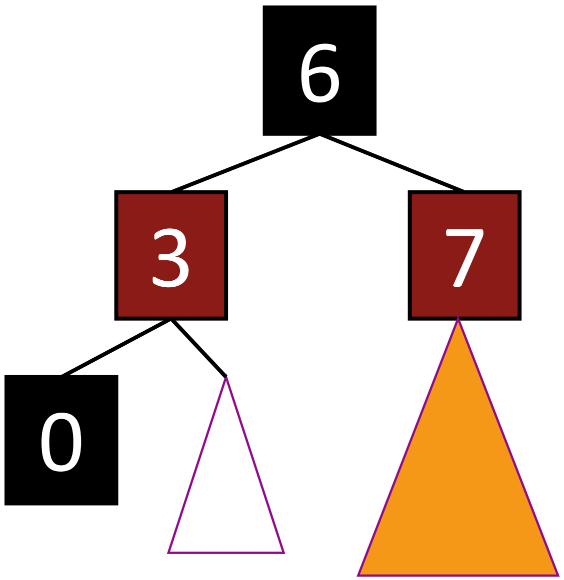
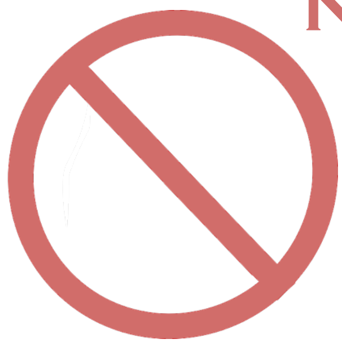


What if it looks like this?



Can't we just insert
0 as a **black node?**

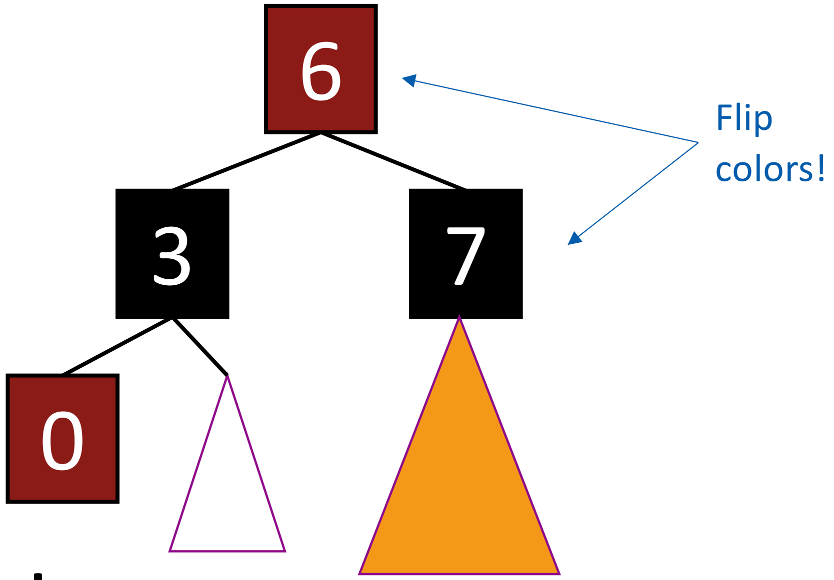
No!



One more black node in this path!

Insert: Case 2

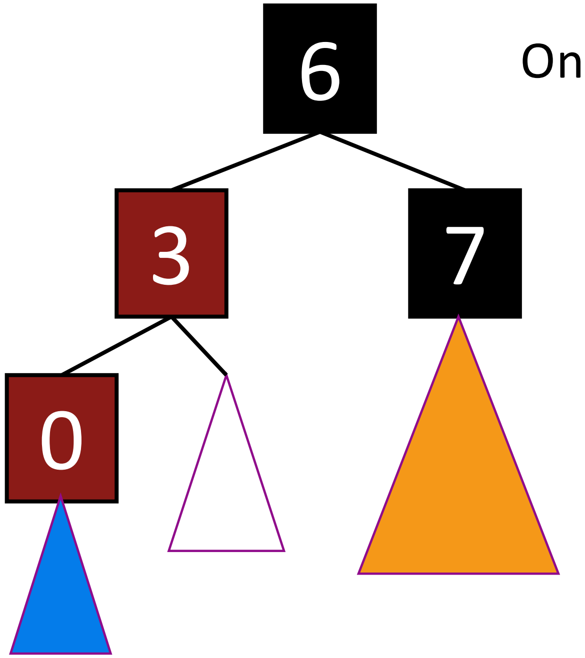
- An important observation: The root can be switched from red to black without violating any rule.



- Add 0 as a **red** node.
- Flip the colors of its parent and uncle.
- Pass the **red** to the top.
- Flip the color of the root from **red** to **black**.

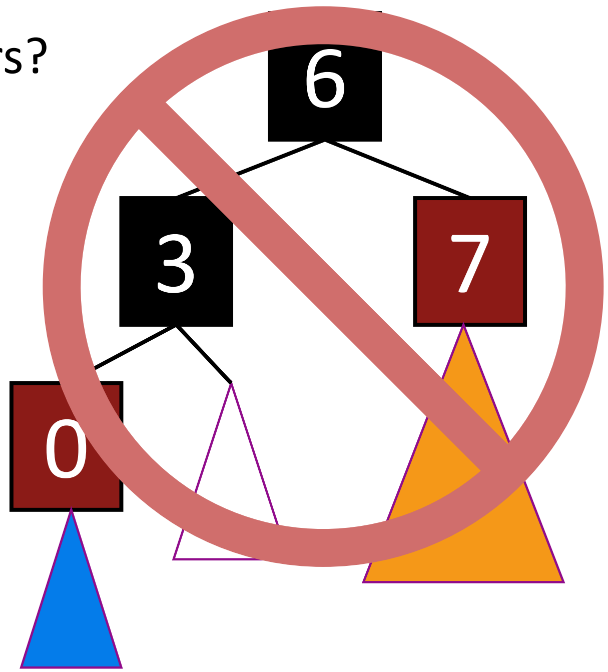
Insert: Case 3

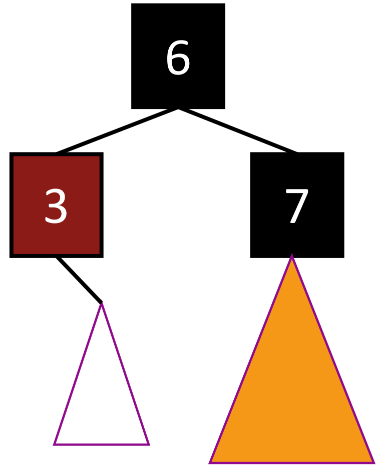
- Make a new **red node**.
- Insert it as you would normally?
- Fix things up if needed.



Only flip colors?

No!

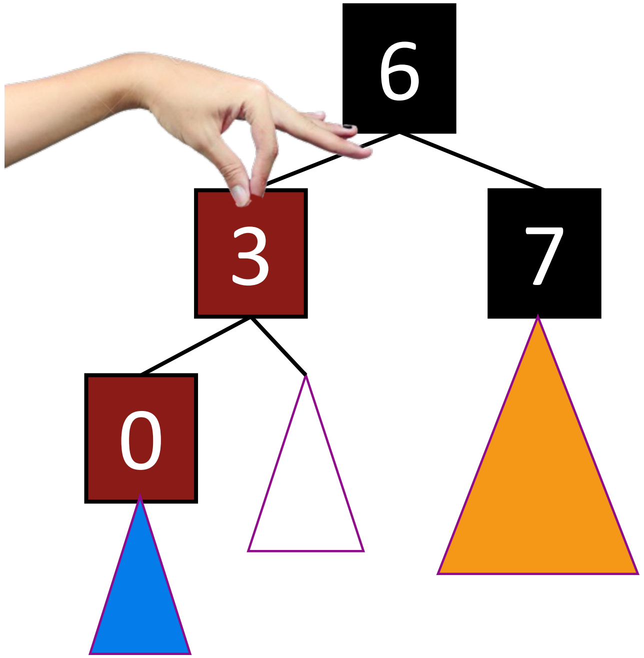




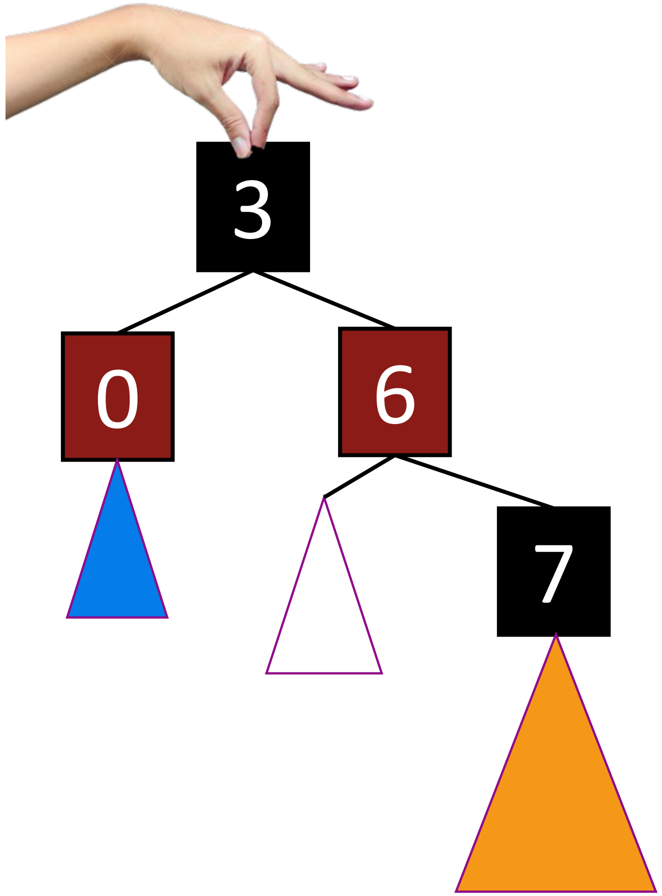
What if it looks like this?

Insert: Case 3

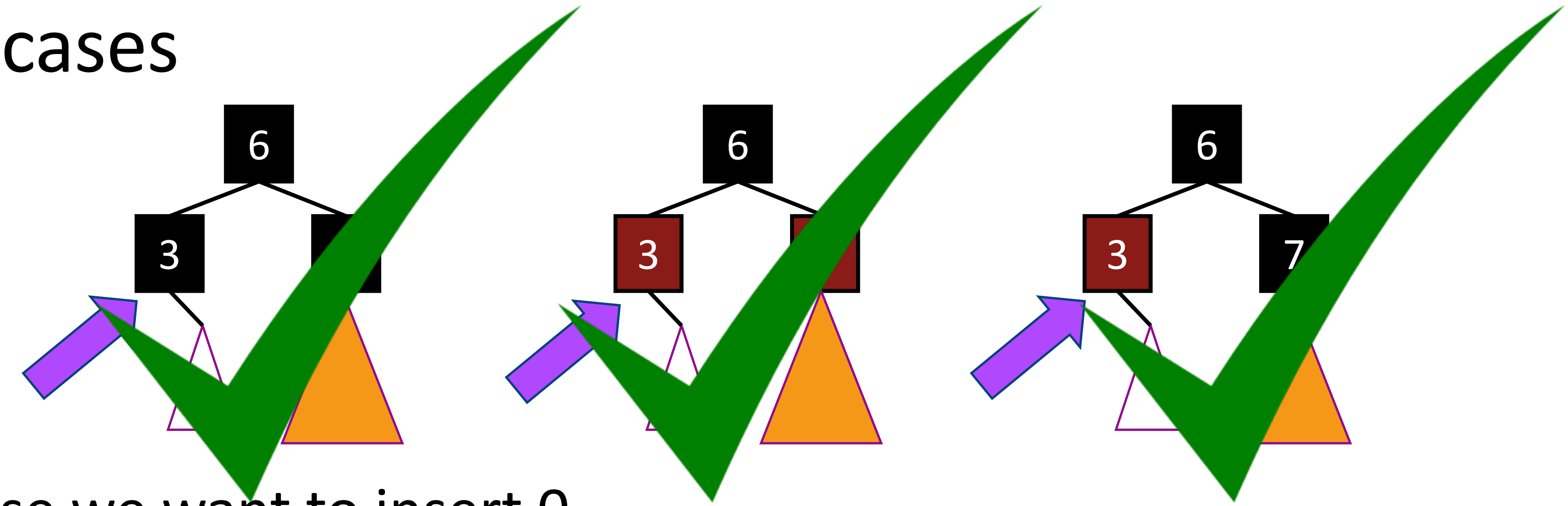
- Recall Rotations:



Rotate
+
Flip color



Many cases



- Suppose we want to insert 0

- 3 “important” cases for different colorings of the existing tree, and there are 9 more cases for all of the various symmetries of these 3 cases.

(Binary) Heaps

Revisiting FindMin

- Application: Find the smallest (or highest priority) item quickly
  – Operating system needs to schedule jobs according to priority instead of FIFO
  – Event simulation (bank customers arriving and departing, ordered according to when the event happened)
  – Find student with highest grade, employee with highest salary etc.

Priority Queue ADT

- Priority Queue can efficiently do:
  - FindMin (and DeleteMin)
  - Insert

- What if we use…
  - Lists: If sorted, what is the run time for Insert and FindMin? Unsorted?
  - Binary Search Trees: What is the run time for Insert and FindMin?
  - Hash Tables (Maybe next lecture): What is the run time for Insert and FindMin?

Less Flexibility → More Speed

- Lists
  - If sorted: FindMin is O(1) but Insert is O(N)
  - If not sorted: Insert is O(1) but FindMin is O(N)
- Balanced Binary Search Trees (BSTs)
  - Insert is O(log N) and FindMin is O(log N)
- BSTs look good but…
  - BSTs are efficient for all Finds, not just FindMin
  - We only need FindMin

Better than a speeding BST

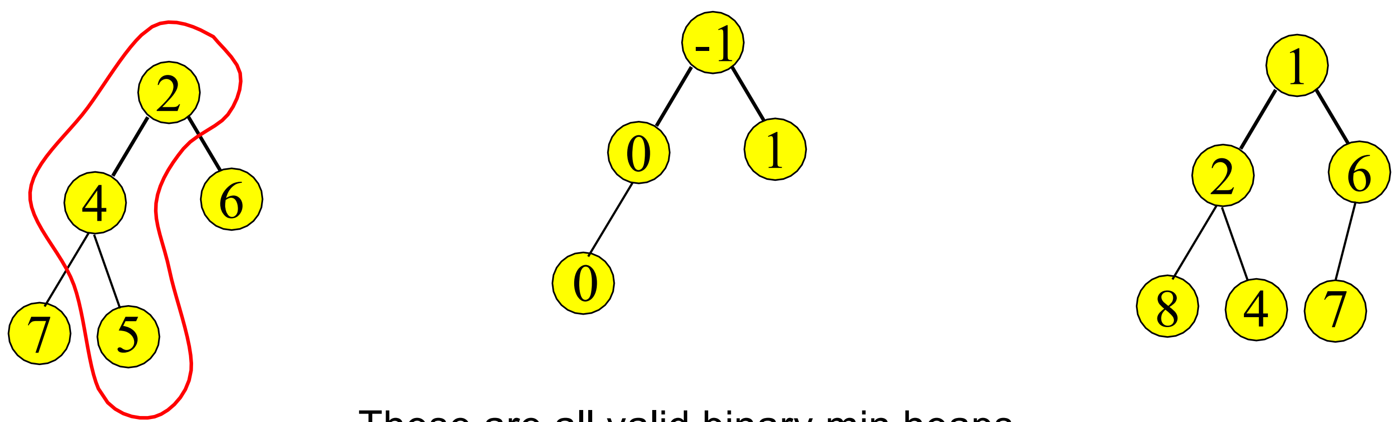
- Can we do better than Balanced Binary Search Trees?
  - Very limited requirements: Insert, FindMin, DeleteMin
  - The goals are:
    - FindMin is $O(1)$
    - Insert is $O(\log N)$
    - DeleteMin is $O(\log N)$

Binary Heaps

- A binary heap is a binary tree (NOT a BST) that is:
  - Complete: the tree is completely filled except possibly the bottom level, which is filled from left to right
  - Satisfies the **heap order property**
    - every node is less than or equal to its children (MinHeap, the default)
    - or every node is greater than or equal to its children (for MaxHeap)
- The root node is always the smallest node
  - or the largest, depending on the heap order (for MaxHeap)

Heap order property

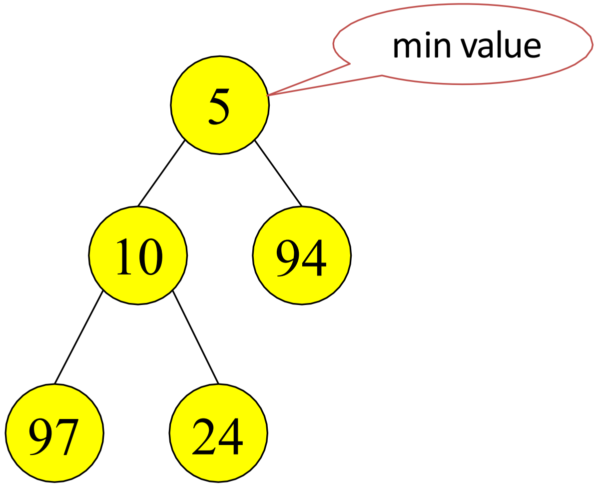
- A heap provides limited ordering information
- Each *path* is sorted, but the subtrees are not sorted relative to each other
  - A binary heap is NOT a binary search tree



These are all valid binary min heaps

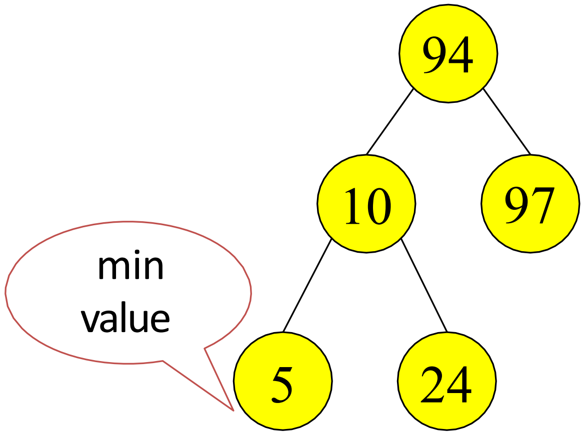
Binary Heap vs Binary Search Tree

Binary Heap



Parent is less than both left and right children

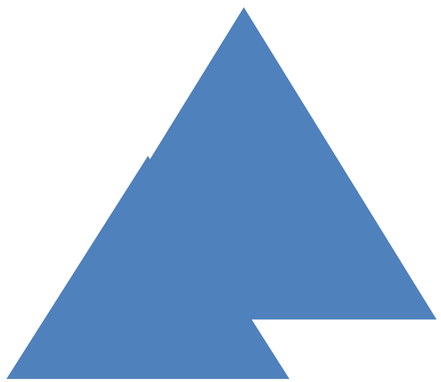
Binary Search Tree



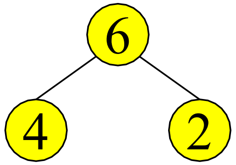
Parent is greater than left child, less than right child

Structure Property

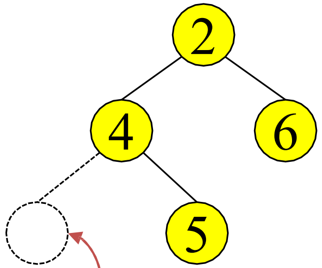
- A binary heap is a complete tree
  - All nodes are in use except for possibly the right end of the bottom row



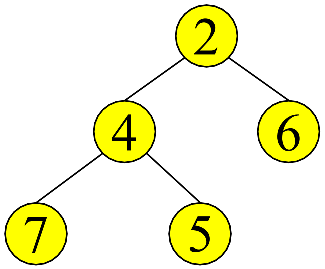
Examples



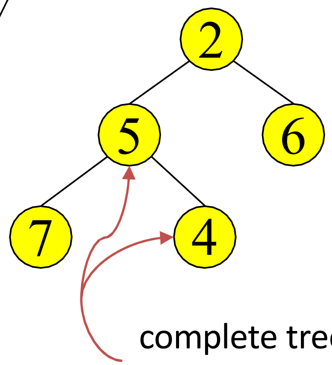
complete tree,
heap order is "max"



not
complete



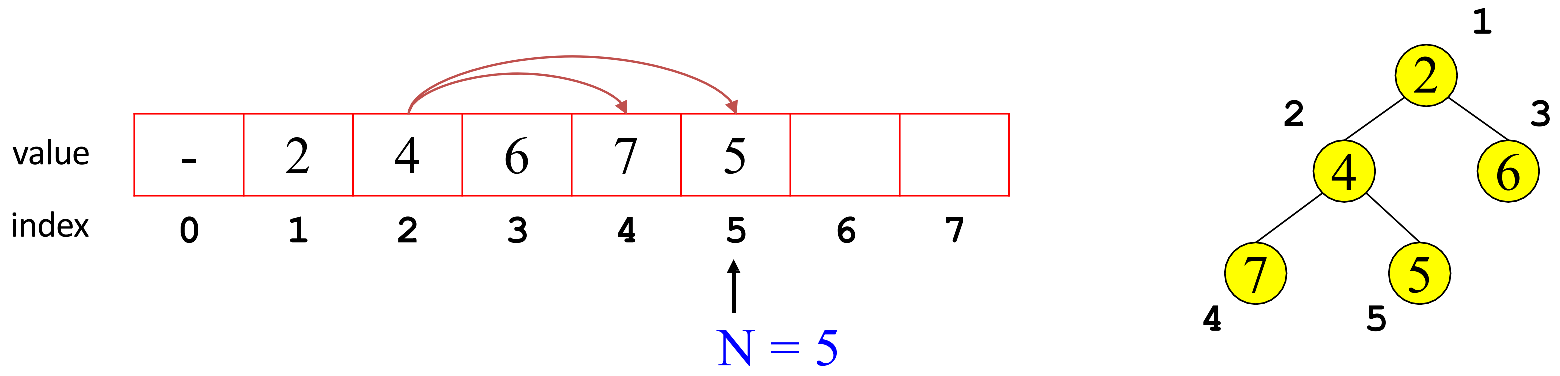
complete tree,
heap order is
"min"



complete tree, but
min heap order is
broken

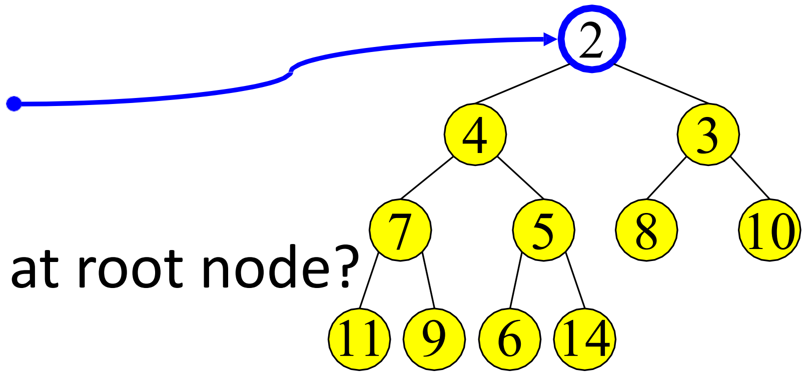
Array Implementation (Implicit Pointers)

- Root node = A[1]
- Children of A[i] = A[2i], A[2i + 1]
- Parent of A[j] = A[j // 2]
- Keep track of current size N (number of nodes)



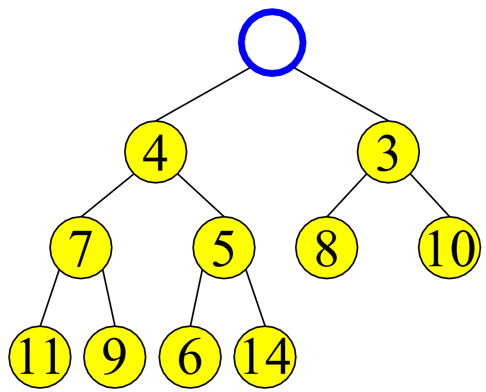
FindMin and DeleteMin

- FindMin: Easy!
  - Return root value A[1]
  - Run time = ?
- DeleteMin:
  - Delete (and return) value at root node?



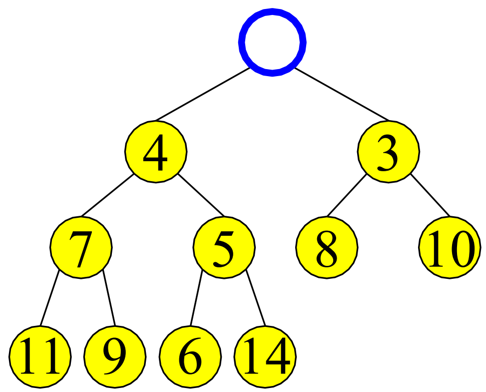
Maintain the Structure Property

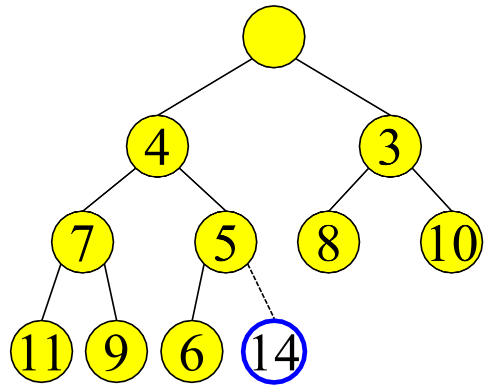
- Delete (and return) value at root node



Maintain the Structure Property

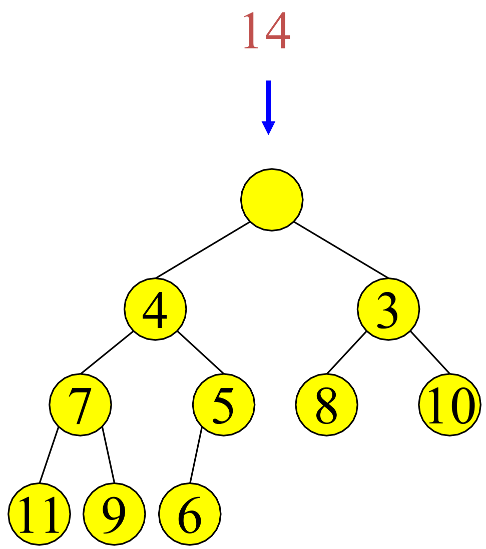
- We now have a “Hole” at the root
  - Need to fill the hole with another value
- When we get done, the tree will have one less node and **must still be complete**



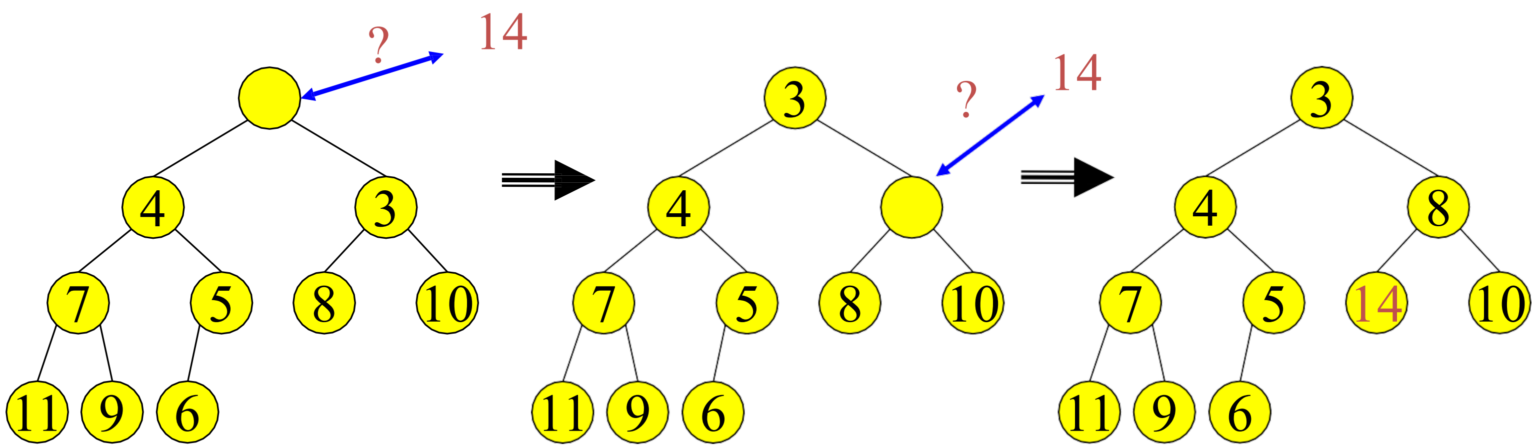


Maintain the Heap Property

- The last value has lost its node
  - we need to find a new place for it



DeleteMin: Percolate Down



- Keep comparing with children A[2i] and A[2i + 1]
- Copy smaller child up and go down one level
- Done if both children are ≥ item or reached a leaf node
- What is the run time?

Percolate Down

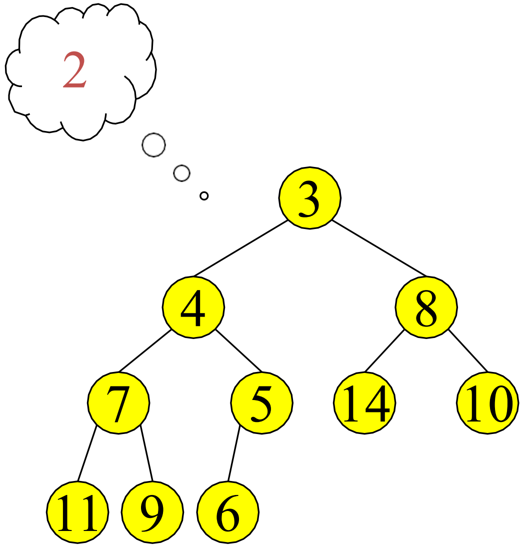
```
PercDown(i: integer, x: integer): {
// N is the number elements, i is the hole, x is the value to insert
      Case {
No child               2i > N: A[i] := x;       // At bottom
One child at the end   2i = N: if A[2i] < x then A[i]:= A[2i]; A[2i] := x
                               else A[i] := x
Two Children           2i < N: if A[2i] < A[2i+1] then j := 2i
                               else j := 2i+1
                               if A[j] < x then
                                   A[i]:= A[j]; PercDown(j, x);
                               else A[i] := x
      }
}
```

DeleteMin: Run Time Analysis

- Run time is $O(depth\ of\ heap)$
- A heap is a complete binary tree
- Depth of a complete binary tree of N nodes?
  - depth = $\log(N)$
- Run time of DeleteMin is $O(\log N)$

Insert

- Add a value to the tree
- Structure and heap order properties must still be correct when we are done



Maintain the Structure Property

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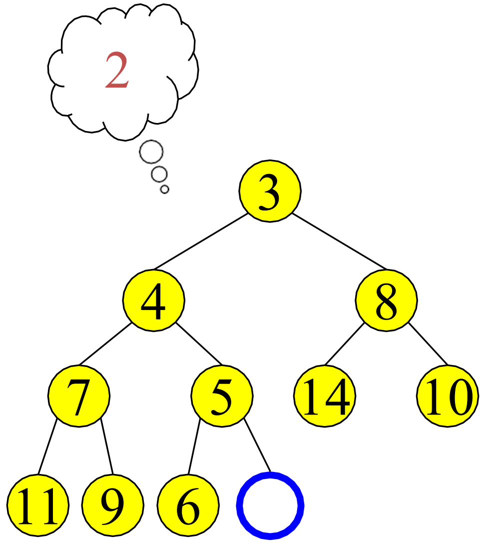
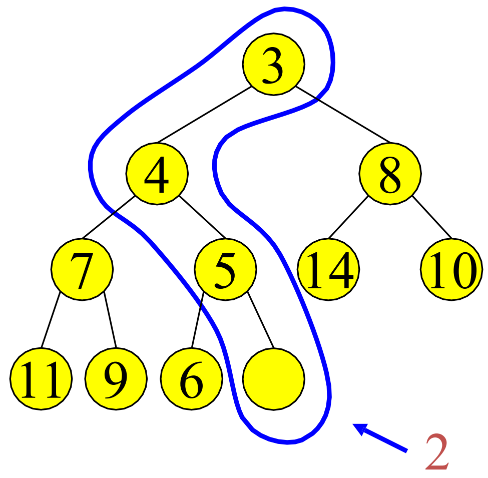
- The only valid place for a new node in a complete tree is at the end of the array
- We need to decide on the correct value for the new node, and adjust the heap accordingly

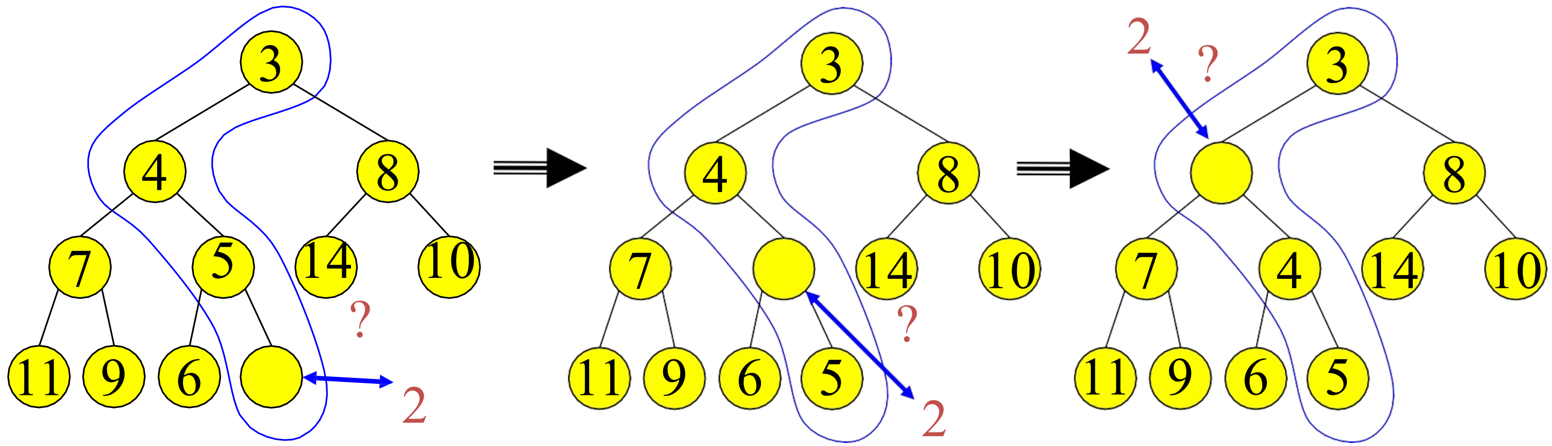


Maintain the Heap Property

- The new value goes where?

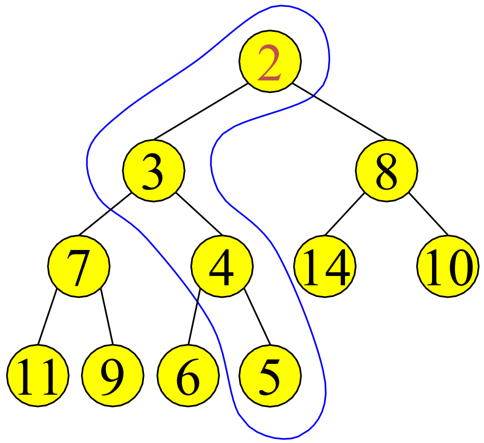


Insert: Percolate Up



- Start at last node and keep comparing with parent A[i/2]
- If parent larger, copy parent down and go up one level
- Done if parent ≤ item or reached top node A[1]

Insert: Done



- Run time?

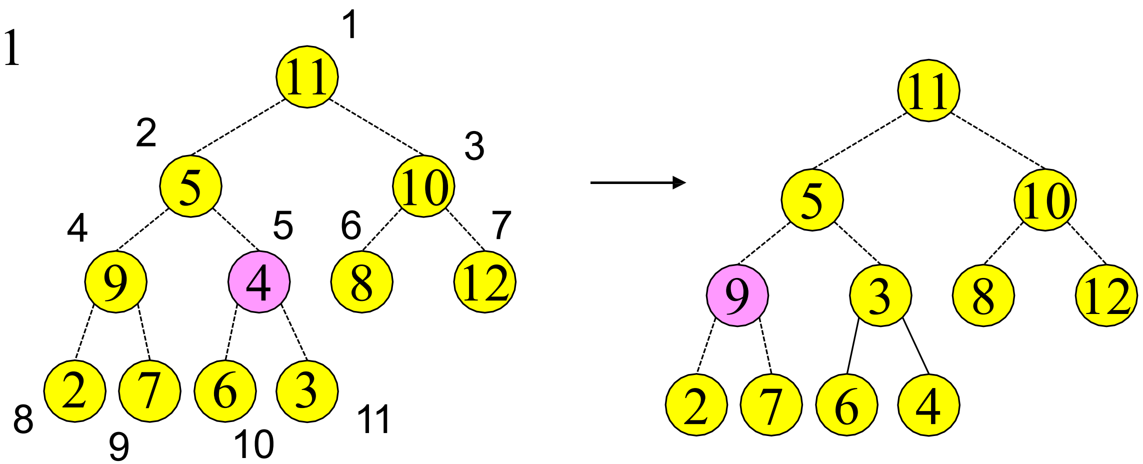
Binary Heap Analysis

- Space needed for heap of N nodes: O(MaxN)
  - An array of size MaxN, plus a variable to store the size N
- Time
  - FindMin: O(1)
  - DeleteMin and Insert: O(log N)
  - BuildHeap from N inputs ???

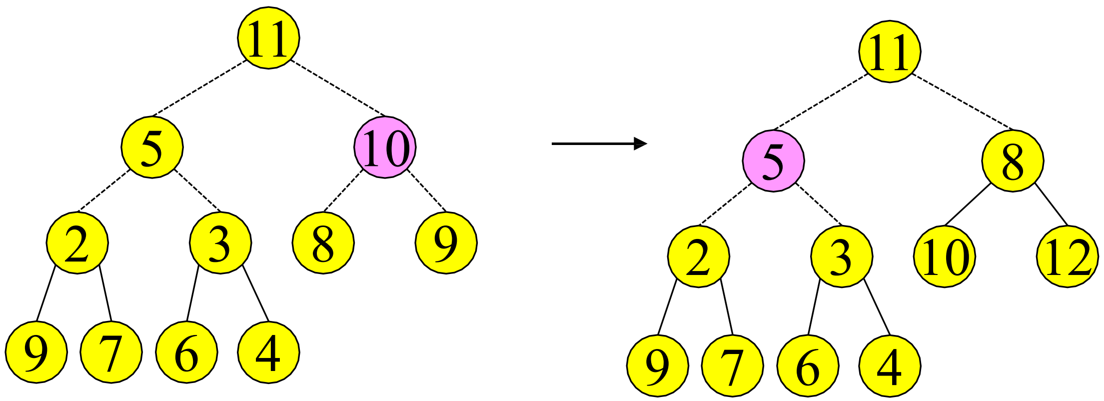
Build Heap


```
BuildHeap {
  for i = N/2 to 1
     PercDown(i, A[i])
}
```

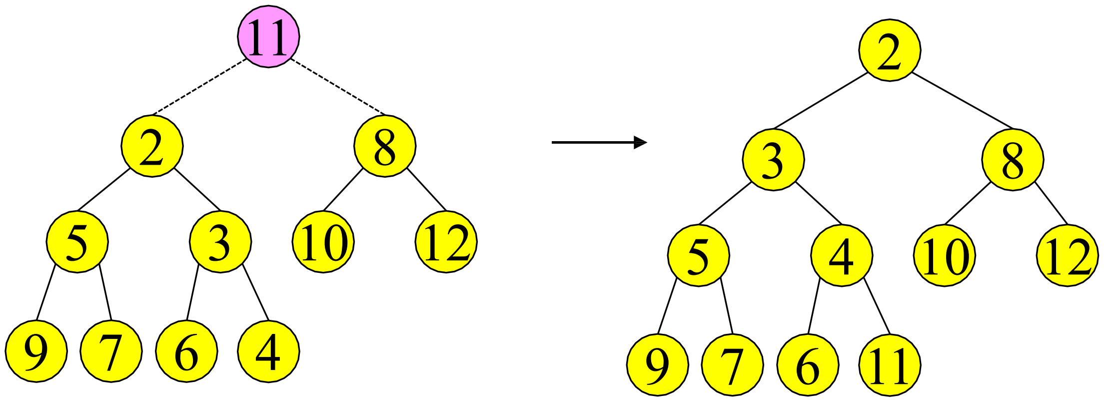
N=11



Build Heap



Build Heap



Time Complexity

- Naïve considerations:
  - $n/2$ calls to `PercDown`, each takes $c \cdot \log(n)$
  - Total: $c \cdot n \cdot \log(n)$
- More careful considerations:
  - Only $O(n)$

Analysis of Build Heap

Assume $n = 2^{h+1} - 1$ where h is height of the tree

- Thus, level h has 2^h nodes but there is nothing to `PercDown`
- At level $h-1$ there are 2^{h-1} nodes, each might percolate down 1 level
- At level $h-j$, there are 2^{h-j} nodes, each might percolate down j levels

$$T(n) = \sum_{j=0}^{h} j2^{h-j} = \sum_{j=0}^{h} j\frac{2^h}{2^j}$$

Total Time = $O(n)$

Other Heap Operations

- Find(X, H): Find the element X in heap H of N elements
  - What is the running time? $O(N)$
- FindMax(H): Find the maximum element in H
- Where FindMin is $O(1)$
  - What is the running time? $O(N)$
- We sacrificed performance of these operations in order to get $O(1)$ performance for FindMin

Other Heap Operations

- DecreaseKey(P,Δ,H): Decrease the key value of node at position P by a positive amount Δ, e.g., to increase priority
  - First, subtract Δ from current value at P
  - Heap order property may be violated
  - so percolate up to fix
  - Running Time: $O(\log N)$

Other Heap Operations

- Delete(P,H): E.g. Delete a job waiting in queue that has been preemptively terminated by user
  - Use DecreaseKey(P, Δ,H) followed by DeleteMin
  - Running Time: $O(\log N)$

- Merge(H1,H2): Merge two heaps H1 and H2 of size $O(N)$. H1 and H2 are stored in two arrays.
  - Can do $O(N)$ Insert operations: $O(N \log N)$ time
  - Better: Copy H2 at the end of H1 and use `BuildHeap`. Running Time: $O(N)$

Other Heap Operations

- Merge(H1,H2): Merge two heaps H1 and H2 of size $O(N)$. H1 and H2 are stored in two arrays.
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Heap Sort

- Idea: `buildHeap` then call `deleteMin` *n* times

```
input = buildHeap(...);
output = new E[n];
for (int i = 0; i < n; i++) {
        output[i] = deleteMin(input);
}
```

- Runtime?
  - Best-case ___________
  - Worst-case ________
  - Average-case _______
- Stable? ________
- In-place? ______

Heap Sort

- Idea: `buildHeap` then call `deleteMin` n times

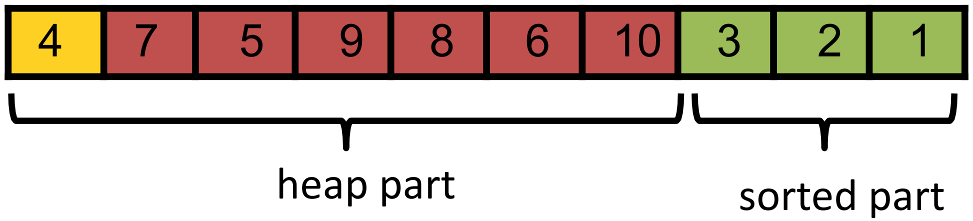
```
input = buildHeap(...);
output = new E[n];
for (int i = 0; i < n; i++) {
        output[i] = deleteMin(input);
}
```

- Runtime?
  - Best-case, Worst-case, and Average-case: $O(n\log(n))$
- Stable? No.
- In-place? No. But it could be, with a slight trick...

In-place Heap Sort

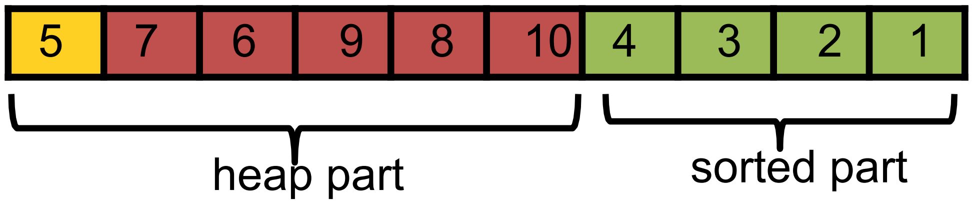
But this reverse sorts – how would you fix that?

- Treat the initial array as a heap (via `buildHeap`)
- When you delete the `i`th element, put it at `arr[n-i]`
  - That array location isn't needed for the heap anymore!



put the min at the end of the heap data

`arr[n-i]= deleteMin()`



AVL Sort?

Sure, we can also use an AVL tree to:

- `Insert` each element: total time $O(n\log n)$
- Repeatedly `deleteMin`: total time $O(n \log n)$
  - Better: in-order traversal $O(n)$, but still $O(n \log n)$ overall
- But this cannot be done in-place and has worse constant factors than heap sort