ΘΕΜΑ 4: Bayes

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

α) Το σύνορο θα βρεθεί όταν

$$P(\omega_1)\, P(x|\omega_1) = P(\omega_2) \cdot P(x|\omega_2)$$

$$P(x|\omega_1) = \frac{1}{\sqrt{(2\pi)^2 \cdot |\Sigma_1|}}\, e^{-\frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)}$$

• για ω_1

$$\det(\Sigma_1) = 1{,}2^2 - 0{,}4^2 = 1{,}28$$

$$\Sigma_1^{-1} = \frac{1}{1{,}28}\begin{pmatrix} 1{,}2 & 0{,}4 \\ 0{,}4 & 1{,}2 \end{pmatrix} = \begin{pmatrix} 0{,}93 & 0{,}31 \\ 0{,}31 & 0{,}93 \end{pmatrix}$$

$$(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1) = \begin{pmatrix} x_1 - 3 & x_2 - 3 \end{pmatrix} \begin{pmatrix} 0{,}93 & 0{,}31 \\ 0{,}31 & 0{,}93 \end{pmatrix} \begin{pmatrix} x_1 - 3 \\ x_2 - 3 \end{pmatrix}$$

$$= 0{,}93x_1^2 + 0{,}62x_1x_2 - 7{,}44x_1 + 0{,}93x_2^2 + 22{,}32 - 7{,}44x_2$$

• για ω_2

$$\det(\Sigma_2) = 1{,}2^2 - 0{,}4^2 = 1{,}28$$

$$\Sigma_2^{-1} = \frac{1}{1{,}28}\begin{pmatrix} 1{,}2 & -0{,}4 \\ -0{,}4 & 1{,}2 \end{pmatrix} = \begin{pmatrix} 0{,}93 & -0{,}31 \\ -0{,}31 & 0{,}93 \end{pmatrix}$$

$$(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2) = 0{,}93x_1^2 - 0{,}62x_1x_2 - 7{,}44x_1 + 0{,}93x_2^2 + 44{,}64 - 7{,}44x_2$$

$$P(\omega_1)\cdot \frac{1}{\cancel{2\pi|\Sigma_1|^{\frac{1}{2}}}} e^{-\frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)} = P(\omega_2)\frac{1}{\cancel{2\pi|\Sigma_2|^{\frac{1}{2}}}} e^{-\frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)}$$

$$\Rightarrow \quad P(\omega_1)\, e^{-\frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1)} = P(\omega_2)\, e^{-\frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)}$$

$$\Rightarrow \ln(P(\omega_1)) - \frac{1}{2}(x-\mu_1)^T\Sigma_1^{-1}(x-\mu_1) = \ln(P(\omega_2)) - \frac{1}{2}(x-\mu_2)^T\Sigma_2^{-1}(x-\mu_2)$$

$$\Rightarrow \ln(P(\omega_1)) - \cancel{\frac{0,93}{2}x_1^2} - \frac{0,62}{2}x_1x_2 + \cancel{\frac{7,44}{2}x_1} - \cancel{\frac{0,93}{2}x_2^2} - \frac{22,32}{2} + \cancel{\frac{7,44}{2}x_2}$$

$$= \ln(P(\omega_2)) - \cancel{\frac{0,93}{2}x_1^2} + \frac{0,62}{2}x_1x_2 + \cancel{\frac{7,44}{2}x_1} - \cancel{\frac{0,93}{2}x_2^2} - \frac{44,64}{2} + \cancel{\frac{7,44}{2}x_2}$$

$$\Rightarrow \ln(P(\omega_1)) - 0,31x_1x_2 - 11,16 = \ln(P(\omega_2)) + 0,31x_1x_2 - 22,32$$

$$\Rightarrow \ln(P(\omega_1)) - \ln(P(\omega_2)) + 11,16 = 0,62\cdot x_1x_2$$

$$\Rightarrow \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 11,16 = 0,62\, x_1x_2$$

$$\Rightarrow \boxed{x_1 = \frac{\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 11,16}{0,62\, x_2}}, \quad \underline{\text{με } x_2 \neq 0}$$

ή $\boxed{x_1x_2 = 1,61\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 18}$

e) Άρα $\Sigma_1 = \Sigma_2 = \Sigma = \begin{pmatrix} 1{,}2 & 0{,}4 \\ 0{,}4 & 1{,}2 \end{pmatrix}$

με $\det(\Sigma) = \det(\Sigma_1) = 1{,}28$

και $\Sigma^{-1} = \Sigma_1^{-1} = \begin{pmatrix} 0{,}93 & 0{,}31 \\ 0{,}31 & 0{,}93 \end{pmatrix}$ από a) ερώτημα

- Για ω_1

$$(x-\mu_1)^T \Sigma^{-1} (x-\mu_1) = 0{,}93x_1^2 + 0{,}62x_1x_2 - 7{,}44x_1 + 0{,}93x_2^2 + 22{,}32 - 7{,}44x_2$$

από a) ερώτημα.

- Για ω_2

$$(x-\mu_2)^T \Sigma^{-1} (x-\mu_2) = (x_1-6 \quad x_2-6) \begin{pmatrix} 0{,}93 & 0{,}31 \\ 0{,}31 & 0{,}93 \end{pmatrix} \begin{pmatrix} x_1-6 \\ x_2-6 \end{pmatrix} =$$

$$= 0{,}93x_1^2 + 0{,}62x_1x_2 - 14{,}88x_1 + 0{,}93x_2^2 + 89{,}28 - 14{,}88x_2$$

Άρα

$$P(\omega_1)P(x|\omega_1) = P(\omega_2)P(x|\omega_2)$$

$$\Rightarrow \ln(P(\omega_1)) - \frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1) = \ln(P(\omega_2)) - \frac{1}{2}(x-\mu_2)^T \Sigma^{-1}(x-\mu_2)$$

$$\Rightarrow \ln(P(\omega_1)) - \cancel{\frac{0{,}93}{2}x_1^2} - \cancel{\frac{0{,}62}{2}x_1x_2} + \frac{7{,}44}{2}x_1 - \cancel{\frac{0{,}93}{2}x_2^2} - \frac{22{,}32}{2} + \frac{7{,}44}{2}x_2 =$$

$$\ln(P(\omega_2)) - \cancel{\frac{0{,}93}{2}x_1^2} - \cancel{\frac{0{,}62}{2}x_1x_2} + \frac{14{,}88}{2}x_1 - \cancel{\frac{0{,}93}{2}x_2^2} - \frac{89{,}28}{2} + \frac{14{,}88}{2}x_2$$

$$\Rightarrow \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) + 3{,}72x_1 - 11{,}16 + 3{,}72x_2 = 7{,}44x_1 - 44{,}64 + 7{,}44x_2$$

$$\Rightarrow \boxed{x_1 = \frac{\ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right) - 3{,}72x_2 + 33{,}48}{3{,}72}}$$