ΘΕΜΑ 1: Λογιστική Παλινδρόμηση: Αναλυτική εύρεση κλίσης (Gradient).

- $h_\theta(x) = f(\theta^T x) = \dfrac{1}{1+e^{-\theta^T x}}$, $\theta = [\theta_1, \theta_2, \dots, \theta_n]^T$
 $x = [x_1, x_2, \dots, x_n]^T$

Θα υπολογίσουμε την παράγωγο $\dfrac{\partial h_\theta(x)}{\partial \theta_j}$ η οποία θα μας χρειαστεί αργότερα.

$$\frac{\partial}{\partial \theta_j}\left(h_\theta(x)\right)$$

$$= \frac{\partial}{\partial \theta_j}\left(\frac{1}{1+e^{-\theta^T x}}\right)$$

$$= \frac{-1}{(1+e^{-\theta^T x})^2} \cdot \frac{\partial}{\partial \theta_j}\left(1+e^{-\theta^T x}\right)$$

$$= -\frac{1}{(1+e^{-\theta^T x})^2} \cdot \frac{\partial}{\partial \theta_j}\left(e^{-\theta^T x}\right)$$

$$= -\frac{e^{-\theta^T x}}{(1+e^{-\theta^T x})^2} \frac{\partial}{\partial \theta_j}\left(-\theta^T x\right)$$

$$= \frac{e^{-\theta^T x}}{(1+e^{-\theta^T x})^2} \frac{\partial}{\partial \theta_j}\left(\sum_{k=1}^{n} \theta_k \cdot x_k\right)$$

$$= \frac{x_j e^{-\theta^T x}}{\left(1+e^{-\theta^T x}\right)^2}$$

Άρα $$\frac{\partial}{\partial \theta_j}\left[h_\theta(x)\right] = \frac{x_j e^{-\theta^T x}}{\left(1+e^{-\theta^T x}\right)^2} \quad (1)$$

Θα υπολογίσουμε και την ποσότητα $1-h_\theta(x)$

$$1-h_\theta(x) = 1-\frac{1}{1+e^{-\theta^T x}} = \frac{e^{-\theta^T x}}{1+e^{-\theta^T x}} \quad (2)$$

<u>Ζητούμενο:</u>

$$J(\theta) = \frac{1}{m}\sum_{i=1}^{m}\left(-y^{(i)}\ln\left(h_\theta(x^{(i)})\right) - \left(1-y^{(i)}\right)\ln\left(1-h_\theta(x^{(i)})\right)\right)$$

$$\frac{\partial}{\partial \theta_j}\left(J(\theta)\right) = \frac{1}{m}\sum_{i=1}^{m} -y^{(i)}\frac{1}{h_\theta(x^{(i)})}\frac{\partial}{\partial \theta_j}\left(h_\theta(x^{(i)})\right) + \left(1-y^{(i)}\right)\frac{1}{1-h_\theta(x^{(i)})}\frac{\partial}{\partial \theta_j}\left(h_\theta(x^{(i)})\right)$$

(1) και (2) $\Rightarrow$

$$= \frac{1}{m}\sum_{i=1}^{m} -y^{(i)}\left(1+e^{-\theta^T x^{(i)}}\right)\left(\frac{x_j^{(i)} e^{-\theta^T x^{(i)}}}{\left(1+e^{-\theta^T x}\right)^2}\right) + \left(1-y^{(i)}\right)\frac{\cancel{1+e^{-\theta^T x^{(i)}}}}{\cancel{e^{-\theta^T x}}}\left(\frac{x_j^{(i)} \cancel{e^{-\theta^T x^{(i)}}}}{\left(1+e^{-\theta^T x}\right)^2}\right)$$

$$= \frac{1}{m}\sum_{i=1}^{m} -y^{(i)} x_j^{(i)}\frac{e^{-\theta^T x^{(i)}}}{\left(1+e^{-\theta^T x^{(i)}}\right)} + \frac{\left(1-y^{(i)}\right)x_j^{(i)}}{\left(1+e^{-\theta^T x^{(i)}}\right)}$$

$$= \frac{1}{m} \sum_{i=1}^{m} -y^{(i)} x_j^{(i)} \left(1 - h_\theta(x^{(i)})\right) + (1 - y^{(i)}) \cdot x_j^{(i)} h_\theta(x^{(i)})$$

$$= \frac{1}{m} \sum_{i=1}^{m} -y^{(i)} x_j^{(i)} + y^{(i)} \cancel{x_j^{(i)} h_\theta(x^{(i)})} + x_j h_\theta(x^{(i)}) - \cancel{y^{(i)} x_j^{(i)} h_\theta(x^{(i)})}$$

$$= \boxed{\frac{1}{m} \sum_{i=1}^{n} x_j \left(h_\theta(x^{(i)}) - y^{(i)}\right)}$$