Constructing efficient universal qudit gate sets for quantum computation

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May 2, 2025

Abstract

Qudit-based quantum computing extends conventional qubit systems by using a higher-dimensional Hilbert space, potentially improving computational efficiency. We introduce the concept of a qudit such that readers with backgrounds in quantum physics, mathematics, or computer science can rigorously grasp their structure and utility. The universality of qudit gate sets is then established upon defining several qudit gates and establishing what an efficient qudit-based quantum computer would require for computation. We then explore how the structure of higher-dimensional space impacts overall computational efficiency, circuit depth, and fault tolerance.

1 Introduction

Quantum information is generally encoded into fundamental units called qubits, where a classical bit $x \in \{0, 1\}$ is represented quantum mechanically by the orthonormal states $|0\rangle$ and $|1\rangle$. This binary encoding forms the foundation of nearly all quantum computing curricula, software, and hardware design. However, many physical systems that implement qubits such as trapped ions, superconducting circuits, and photonic modes naturally possess more than two accessible energy levels. Leveraging these additional levels gives rise to a more general computational model, the qudit.

A qudit encodes classical information in a d-dimensional Hilbert space $\mathcal{H}_d$, where $x \in \{0, 1, \ldots, d-1\}$ is represented by the quantum states $|0\rangle, |1\rangle, \ldots, |d-1\rangle$ [1, 2]. When $d = 2$, this model reduces to the standard qubit, but for $d > 2$, qudits offer new opportunities: increased information density, native access to higher-order logic, and the potential to reduce overall circuit complexity [3, 4]. Additionally, by encoding more information per physical unit, qudits can improve fault tolerance and reduce the overhead associated with quantum error correction [5]. In certain quantum algorithms, such as the Quantum Approximate Optimization Algorithm (QAOA) or quantum simulations of spin-d systems, qudits even offer native advantages in algorithmic expressiveness [6, 7].

Despite these advantages, most quantum computing systems and curricula continue to focus primarily on qubits. The mathematical framework for qubit-based computation grounded in Pauli matrices, Bloch spheres, and simple two-level unitaries is widely taught in undergraduate quantum mechanics [8, 9]. Qudits, on the other hand, require higher level, generalized mathematical tools such as SU(d) Lie groups, generalized d-dimensional unitaries, and higher-dimensional tensor networks. These tools introduce technical complexity that has limited the adoption of qudit models in both theoretical and applied contexts.

Nevertheless, the gap between physical capability and theoretical frameworks is narrowing. Recent experimental results demonstrate high-fidelity control over qutrits and ququarts in superconducting and trapped-ion platforms [5], motivating a renewed interest in understanding how to construct universal gate sets, perform circuit compilation, and reason about computational efficiency in the qudit model.

In this work, we investigate the mathematical foundations and practical implications of qudit-based quantum computing. We begin by formally defining qudits and exploring their encoding in quantum information theory. We then review the necessary conditions for constructing a universal qudit gate set. Finally, we analyze the benefits and trade offs of qudits in terms of circuit depth, algorithmic power, and architectural compatibility.

2 Qudits

While qubits are the standard unit of information in quantum computing, many physical systems such as trapped ions, superconducting transmons, and photonic systems naturally support more than two orthogonal quantum states. In such systems, it is useful to generalize the notion of a qubit to a qudit. Qudits allow a richer computational framework and, in some cases, better alignment with physical device characteristics.

2.1 The qudit

Definition 1. Let $d \in \mathbb{N}$, $d \geq 2$. A **qudit** is a quantum system whose state resides in the Hilbert space $\mathcal{H}_d \in \mathbb{C}^d$, with computational basis

$$B^{(d)} = \{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}. \tag{1}$$

2.2 Encodings

This generalizes the familiar qubit space $\mathbb{C}^2$, recovered when $d = 2$. There exists a natural bijection between the classical d-ary digit set $\{0, 1, \ldots, d-1\}$ and the elements of $B^{(d)}$, enabling the classical digit j to be encoded quantum mechanically as $j \mapsto |j\rangle$ for each $j \in \{0, 1, \ldots, d-1\}$.

Definition 2. Given an n-digit d-ary classical string $x = x_0 x_1 \cdots x_{n-1} \in \{0, \ldots, d-1\}^n$, its **basis encoding** is the quantum state

$$|x\rangle = |x_0\rangle \otimes |x_1\rangle \otimes \cdots \otimes |x_{n-1}\rangle \in \mathcal{H}_d^{\otimes n}. \tag{2}$$

This encoding allows classical data over a d-ary alphabet to be faithfully embedded into an n-qudit

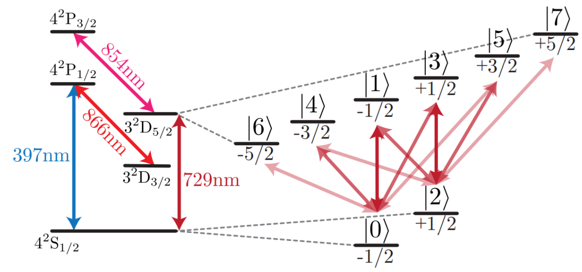

Figure 1: Energy level scheme of $^{40}Ca^+$ for a trapped ion quantum processor where $d = 8$ [12].

quantum register. The full Hilbert space of such a register is the tensor, product [10]

$$\mathcal{H}_d^{\otimes n} = \underbrace{\mathcal{H}_d \otimes \mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d}_{n \text{ times}} \cong \mathbb{C}^{d^n}, \tag{3}$$

with the corresponding computational basis

$$\big\{|i_1\rangle \otimes \cdots \otimes |i_n\rangle \mid i_k \in \{0, \ldots, d-1\}\big\}.$$

For a fixed number of physical units n, the Hilbert space dimension $\dim \mathcal{H}_d^{\otimes n} = d^n$ grows exponentially in d.

Beyond state encoding, qudits are governed by a broader set of operations. The set of all valid single-qudit unitaries forms the Lie group $\mathrm{SU}(d)$, the group of $d \times d$ unitary matrices with unit determinant. This structure generalizes the familiar single qubit gate space $\mathrm{SU}(2)$ [11].

When $d = 2$, we recover the traditional qubit model. But for $d > 2$, the qudit model grants access to increased algorithmic expressiveness [1], improved fault tolerance [5], and potential reductions in circuit complexity [3]. Moreover, for quantum simulation of systems with higher local dimensions (such as spin-d models and trapped ions), qudits are often a more natural choice [7]. See Figure 1 for an example of how the energy levels can be visualized for qudit-based computation.

Although qubit-based computation remains dominant in contemporary platforms and software, qudit-based approaches are increasingly relevant due to the inherent multilevel structure of many physical implementations. As such, understanding qudit encodings is essential for the development of generalized quantum computing frameworks.

3 Universal gate sets

In the qubit model, we use single- and two-qubit gates such as H, T, and CNOT to build quantum circuits. In the qudit model, we generalize these gates to act on d-dimensional systems.

Definition 3. A **single-qudit gate** is a unitary operator $U \in \mathrm{U}(d)$ acting on the single-qudit state $|\psi\rangle \in \mathcal{H}_d$.

3.1 Criteria for universality

Universality in quantum computing refers to the ability of a gate set to approximate any unitary transformation on a given quantum system. For qudits the conditions for universality generalize the familiar qubit setting.

Definition 4. The Lie group $\mathrm{SU}(d)$ is a set such that $\mathrm{SU}(d) \subseteq \mathrm{U}(d)$ and for all $U \in \mathrm{SU}(d)$, $\det U = 1$.

Definition 5. A set of gates $\mathcal{G} = \{g_1, \ldots, g_k\} \subset \mathrm{SU}(d)$ is **universal** if $\langle \mathcal{G} \rangle$ is dense in $\mathrm{SU}(d)$. That is, for every $U \in \mathrm{SU}(d)$ and $\epsilon > 0$, there exists a gate $V \in \langle \mathcal{G} \rangle$ such that $\|U - V\| < \epsilon$ [2, 13, 14].

3.1.1 Single-qudit gate criteria

The universality of a set $\mathcal{G} = \{g_1, \ldots, g_k\} \subset \mathrm{SU}(d)$ can be determined using the adjoint representation [11].

Theorem 1 (Adjoint representation criteria [11])**.** *Let Ad_g be the adjoint representation of $g \in \mathrm{SU}(d)$, defined by $\mathrm{Ad}_g(X) = gXg^{-1}$ where $X \in \mathrm{SU}(d)$. Then $\mathcal{G}$ is universal if*

(A) the only matrices that commute with all Ad_g are scalar multiples of the identity, and

(B) there exists an element $g \in \langle \mathcal{G} \rangle$ such that $0 < \mathrm{dist}(g, Z(\mathrm{SU}(d))) \leq 1/\sqrt{2}$.

Here $Z(\mathrm{SU}(d))$ denotes the center of $\mathrm{SU}(d)$, and dist is the distance between the vectors. These conditions ensure the generated group acts irreducibly on $\mathrm{SU}(d)$ and is not confined to a proper subgroup.

3.1.2 Two-qudit gate criteria

Definition 6. A two-qudit gate $V \in \mathrm{SU}(d^2)$ is **primitive** if it maps any separable state $|\psi\rangle \otimes |\phi\rangle$ to another separable state. That is,

$$V|\psi\rangle \otimes |\phi\rangle = |\psi'\rangle \otimes |\phi'\rangle, \tag{4}$$

for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}_d$. Otherwise, V is **imprimitive** [2].

Theorem 2 (Universality with imprimitive gates [2])**.** *Let $\mathcal{A}$ be the set of all single-qudit gates and V a two-qudit gate. Then the following statements are equivalent.*

(A) $\mathcal{A} \cup \{V\}$ is approximately universal for n-qudit systems.

(B) $\mathcal{A} \cup \{V\}$ is exactly universal.

(C) V is imprimitive.

Imprimitive gates are thus essential for generating entanglement and hence for universal quantum computation. Even though a finite set of such gates cannot be exactly universal (as $\mathrm{SU}(d^n)$ is uncountable), it can be approximately universal. Hence, adding any entangling (imprimitive) two-qudit gate to the set of all single-qudit gates suffices for universality [15].

Vlasov [16] further showed that the combination of any two noncommuting single-qudit gates together with an imprimitive gate is enough to simulate any $U \in \mathrm{SU}(d^n)$.

3.2 Single-qudit gates

Claim 1. There exists a unitary transformation in d-dimensional space that maps any given qudit state to $|d-1\rangle$ [15]. Namely,

$$U_d(\boldsymbol{\alpha})\colon \sum_{i=0}^{d-1} \alpha_i |i\rangle \mapsto |d-1\rangle, \tag{5}$$

where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_{d-1})$.

In this scheme, U_d can be deterministically decomposed into $d-1$ unitary transformations such that

$$U_d = X_d^{(d-1)}(a_{d-1}, b_{d-1}) \cdots X_d^{(2)}(a_2, b_2) X_d^{(1)}(a_1, b_1), \tag{6}$$

$$a_i = \alpha_i, \qquad b_i = \sqrt{\sum_{j=0}^{i-1} \alpha_j^2} \tag{7}$$

with

$$\begin{aligned} &X_d^{(j)}(x, y) \\ &\quad = I_{j-1} \oplus \frac{1}{\sqrt{|x|^2 + |y|^2}} \begin{pmatrix} x & -y \\ y^* & x^* \end{pmatrix} \oplus I_{d-1-j} \end{aligned} \tag{8}$$

and $\oplus$ denotes the direct sum.

Example 1. Let $d = 3$ (a qutrit), and consider the state

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle$$

where $\boldsymbol{\alpha} = (1, 1, 1)/\sqrt{3}$. We wish to construct the unitary $U_3(\boldsymbol{\alpha})$ that maps this state to $|2\rangle$. We compute

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{3}}, & b_1 &= \frac{1}{\sqrt{3}}, \\ a_2 &= \frac{1}{\sqrt{3}}, & b_2 &= \sqrt{\frac{2}{3}}. \end{aligned}$$

Then $U_3(\boldsymbol{\alpha})$ is

$$U_3(\boldsymbol{\alpha}) = X_3^{(2)}(a_2, b_2) X_3^{(1)}(a_1, b_1).$$

Each $X_3^{(j)}(x, y)$ acts as a two-dimensional rotation embedded in $\mathbb{C}^3$:

$$\begin{aligned} X_3^{(1)}(x, y) &= \frac{1}{\sqrt{|x|^2 + |y|^2}} \begin{pmatrix} x & -y \\ y^* & x^* \end{pmatrix} \oplus 1, \\ X_3^{(2)}(x, y) &= 1 \oplus \frac{1}{\sqrt{|x|^2 + |y|^2}} \begin{pmatrix} x & -y \\ y^* & x^* \end{pmatrix}. \end{aligned}$$

Specifically,

$$\begin{aligned} X_3^{(1)}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) &= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ X_3^{(2)}\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}\right) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{3} & -\sqrt{2/3} \\ 0 & \sqrt{2/3} & 1/\sqrt{3} \end{pmatrix}. \end{aligned}$$

Finally multiplying the matrices gives

$$U_3(\boldsymbol{\alpha}) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -\sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}.$$

Then we can show that

$$\begin{aligned} &U_3(\boldsymbol{\alpha})|\psi\rangle \\ &\quad = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -\sqrt{2/3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

as desired. Thus, the composed operation $U_3(\boldsymbol{\alpha})$ maps $|\psi\rangle$ to $|2\rangle$, verifying the decomposition.

Definition 7. The d-dimensional **phase gate** is [15]

$$Z_d(\theta) = \sum_{l=0}^{d-1} \mathrm{e}^{\mathrm{i}(1-\mathrm{sgn}(d-1-l))\theta} |l\rangle\langle l|, \tag{9}$$

where sgn is the signum function. This gate applies a phase shift of θ to the state $|d-1\rangle$ and acts trivially on all other basis states.

Example 2. Let $d = 3$ (a qutrit), and consider again the state given by

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \alpha_2|2\rangle$$

with $\boldsymbol{\alpha} = (1, 1, 1)/\sqrt{3}$. Then $Z_3(\theta)$ is,

$$\begin{aligned} Z_3(\theta) &= \sum_{l=0}^{2} \mathrm{e}^{\mathrm{i}(1-\mathrm{sgn}(2-l))\theta} |l\rangle\langle l| \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| + \mathrm{e}^{\mathrm{i}\theta}|2\rangle\langle 2| \end{aligned}$$

after evaluation of the signum. Applying this to $|\psi\rangle$, we have

$$Z_3(\theta)|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}\mathrm{e}^{\mathrm{i}\theta}|2\rangle.$$

Hence, only the amplitude of $|2\rangle$ acquires a phase shift of $\mathrm{e}^{\mathrm{i}\theta}$, and the other components are unchanged.

Definition 8. The **generalized Hadamard gate** H_d for qudits is [15]

$$H_d|j\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\omega^{ij}|i\rangle, \quad j \in \{0, 1, \ldots, d-1\}, \tag{10}$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/d}$ is the dth root of unity [17]. This is analogous to the **quantum Fourier transform (QFT)** for qubits.

3.3 Multiple-qudit gates

Each primitive gate, $X_d^{(l)}$ or Z_d has two free complex parameters to be controlled. Let R_d represent any primitive gate with two free complex parameters.

Definition 9. The **generalized controlled-qudit gate** is [15]

$$\begin{aligned}\Lambda_2[R_d] &= |x\rangle\langle x| \otimes I_d + |d-1\rangle\langle d-1| \otimes R_d \\ &= \begin{pmatrix} I_{d^2-d} & 0 \\ 0 & R_d \end{pmatrix} = I_{d^2-d} \oplus R_d,\end{aligned}$$

which is a $d^2 \times d^2$ matrix that acts on two qudits, creating the usual "target" and "control" relationship [15].

Definition 10. The **SWAP gate** is a two-qudit operation that exchanges the quantum states of two qudits such that [15]

$$\mathrm{SWAP}|\phi\rangle|\psi\rangle = |\psi\rangle|\phi\rangle. \tag{11}$$

To construct a qudit SWAP gate, one commonly uses the controlled-shift gate $\Lambda_2^m[X_d]$, defined by

$$\Lambda_2[X_d]|x\rangle|y\rangle = |x\rangle|(x+y) \bmod d\rangle, \tag{12}$$

where the addition is performed modulo d. Its inverse operation is given by

$$\Lambda_2[X_d^\dagger]|x\rangle|y\rangle = |x\rangle|(y-x) \bmod d\rangle. \tag{13}$$

In some implementations, a complementary operation K_d may be required, defined by

$$K_d|x\rangle = |d-x\rangle. \tag{14}$$

Since the gate $\Lambda_2 X_d$ is generally not hermitian, it differs from the qubit SWAP gate which is hermitian. One hermitian alternative for the qudit-controlled operation is the **GXOR gate**, defined as

$$\mathrm{GXOR}|x\rangle|y\rangle = |x\rangle|(x-y) \bmod d\rangle. \tag{15}$$

To construct a fully hermitian SWAP operation using GXOR, it may be necessary to combine it with the complement operation K_d [15].

3.4 A universal gate set

Many constructions of universal gate sets for qudits rely on the assumption that the qudit dimension d is prime. This is because finite fields $\mathbb{F}_d$, essential for defining well-behaved modular arithmetic and invertible operations, exist only when d is a prime or a power of a prime [17]. Recent work further confirms that field arithmetic is necessary for qudit universality and circuit design in general qudit systems [18].

Claim 2. The qudit gate set

$$\Gamma_d = \{X_d^{(l)}, Z_d, \Lambda_2[R_d]\} \tag{16}$$

is universal [15, 19].

Proof. Let $U \in \mathrm{SU}(d^n)$ be an arbitrary unitary acting on n qudits. The universality of Γ_d can be proven in three main steps [15].

(1) Any unitary U can be expressed via spectral decomposition as

$$\begin{aligned}U &= \sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\phi_j}|E_j\rangle\langle E_j| = \sum_{j=1}^{N}\omega^{\lambda_j}|E_j\rangle\langle E_j| \\ &= \sum_{j=1}^{N}\omega^{\lambda_j}\Upsilon_j,\end{aligned}$$

where Υ_j are eigenoperators (projectors) and $N = d^n$. Each Υ_j can be synthesized using controlled diagonalizations

$$\Upsilon_j = U_{j,N}^\dagger Z_{j,N} U_{j,N},$$

where $U_{j,N}$ and $Z_{j,N}$ are generalizations of the Fourier and phase gates to d^n-dimensional systems.

(2) These gates $U_{j,N}$ and $Z_{j,N}$ can be decomposed using multi-controlled gates of the form $\Lambda_m[R_d]$. For practical syntheses, these can be built using compositions of $\Lambda_2[R_d]$, which are controlled single-qudit gates.

(3) The primitive gate R_d can be taken to be $X_d^{(l)}$ or Z_d, both of which are realizable directly in prime-dimensional systems due to the existence of a finite field structure [18].

Putting these pieces together, any unitary in $\mathrm{SU}(d^n)$ can be approximated using the gate set Γ_d. □

3.5 Approximation efficiency

The theoretical justification for the efficiency of the proposed gate set Γ_d can be established using complexity bounds from the Solovay–Kitaev constructions for $\mathrm{SU}(d)$ [14, 20, 21].

Theorem 3 (Solovay–Kitaev [14, 20, 21])**.** *Let $\mathcal{G} \subset \mathrm{SU}(d)$ be a finite gate set that generates a dense subgroup of $\mathrm{SU}(d)$, and assume $\mathcal{G}$ is closed under inversion. Then there exists a constant $c > 0$ such that for any unitary $U \in \mathrm{SU}(d)$ and any error tolerance $\epsilon > 0$, there exists a finite sequence of gates $V = g_1 g_2 \cdots g_n$, where each $g_i \in \mathcal{G}$, such that*

$$\|U - V\| \leq \epsilon,$$

and the length of the sequence satisfies

$$\Delta = O\big(\log^c(1/\epsilon)\big).$$

The distinction between the cases $d = 2$ and $d > 2$ is the constant c.

Definition 11. Let $\epsilon > 0$. If a gate set $\mathcal{A} \subset \mathrm{SU}(d^n)$ generates a dense subgroup, then there exists a finite sequence of gates from $\mathcal{A}$ of length Δ such that

$$\Delta \geq \frac{d^2 - 1}{\log_2|\mathcal{A}|} \log \frac{1}{\epsilon} - \frac{\log k_2}{\log_2|\mathcal{A}|} \tag{17}$$

that approximates any unitary in $\mathrm{SU}(d^n)$ within error ϵ, where k_2 is a constant depending on the covering net, which has volume on the order of ϵ^{d^2-1} [14].

This result confirms that if Γ_d generates a dense subgroup of $\mathrm{SU}(d^n)$, then it can approximate any unitary with circuit depth $O\big(\log(1/\epsilon)\big)$, which is asymptotically optimal up to constants [19]. In contrast, if a gate set $\mathcal{A} \subset \mathrm{SU}(d^n)$ is not dense, then no finite-length circuit over $\mathcal{A}$ can approximate all unitaries to arbitrary precision.

Even if we define a perturbed gate set $B(\mathcal{A}, \delta)$, where each gate in $\mathcal{A}$ is perturbed by at most δ, and although $B(\mathcal{A}, \delta)$ has nonzero measure in $\mathrm{SU}(d^n)$, almost all of its elements will be computationally universal. However, sequences of gates from $B(\mathcal{A}, \delta)$ of length Δ remain confined within a distance $\Delta\delta$ of the nondense subgroup generated by $\mathcal{A}$. Thus, for any fixed constant multiple of $\log(1/\epsilon)$, there exist values of ϵ for which such nonuniversal gate sets will necessarily fail to achieve a universal approximation.

This underscores a critical point that universality cannot be salvaged by longer gate sequences or small perturbations of nondense gate sets. The set Γ_d avoids this issue by construction: it contains both shift and phase operators and a controlled entangling operation, ensuring that the group it generates is nontrivial and dense.

Furthermore, universality is robust under small perturbations, as long as the generating set remains of nonzero measure, the efficiency of approximation is preserved. This robustness is important for experimental realizations where gates may suffer imperfections. Consequently, the gate set Γ_d provides an efficient and physically realizable means for approximating arbitrary unitaries over $\mathrm{SU}(d^n)$, with performance scaling logarithmically in precision [19].

4 Computational efficiency

Conventionally, the computational efficiency of a qubit quantum computer is compared to a classical one. Here, we compare the efficiency of qudit-based computation to qubit-based computation [22, 4, 15].

One primary difference in efficiency between qubit and qudit based computing is circuit complexity. A qudit system may be able to reduce the number of gates required. Gate counts for implementing arbitrary unitaries scales more favorably for qudits as

$O(n^2 d^n)$ compared to $O(n^2 2^n)$ for qubits [15]. Additionally, certain logic may be implemented with fewer gates.

4.1 Toffoli gates

An example of this are the Toffoli (CCNOT) gates, particularly on three qubits. The simplest known decomposition of the Toffoli gate using qubits requires six two-qubit (CNOT) gates, shown in Figure 2.

If instead the target qubit is a qutrit, having an additional computational state, the circuit can be reduced to only three CNOT gates. A new Gell-Mann qutrit gate X_a can do the following: $X_a|0\rangle = |2\rangle$ and $X_a|1\rangle = |1\rangle$, with $X_a|2\rangle = |0\rangle$. The new circuit for the Toffoli gate decomposition using X_a is shown in Figure 3.

Qudits additionally enable the implementation of multilevel control gates allowing for more complex operations with smaller circuit depth. An example is the multivalue controlled gate (MVCG), which provides a unique operation to the target qudit for each unique state of the control qudit. For a d-dimensional qudit system, the MVCG for a two-qudit system is given by a $d^2 \times d^2$ matrix

$$\mathrm{MVCG} = \begin{pmatrix} U_0 & 0 & 0 & \cdots & 0 \\ 0 & U_1 & 0 & \cdots & 0 \\ 0 & 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & U_{d-1} \end{pmatrix}. \tag{18}$$

Here each U_i represents a unique unitary qudit operation.

4.2 QAOA

The Quantum Approximate Optimization Algorithm (QAOA) is a hybrid quantum–classical variational algorithm for solving combinatorial optimization problems. While originally formulated for qubit-based systems, recent work has extended QAOA to qudit systems to take advantage of their higher-dimensional Hilbert spaces and greater computational expressiveness [6]. In this generalized setting, the cost and mixer Hamiltonians are defined over d-dimensional spaces, and the variational ansatz employs generalized Pauli operators σ_d^x, σ_d^z, along with controlled multilevel operations drawn from a universal gate set such as Γ_d.

This extension enables QAOA to natively encode problems like graph coloring, scheduling, and clustering without requiring binary reductions, thereby avoiding the overhead of auxiliary qubits or unary encodings. Practical implementations have also been demonstrated to show that QAOA executed on a Rydberg-atom array with native qutrit interactions achieves comparable or improved fidelity using fewer layers and gates than qubit-based equivalents [23]. Furthermore, the ability to directly embed integer-weighted constraints in the cost Hamiltonian simplifies circuit design and improves the optimization landscape. Together, these advantages suggest that qudit-based QAOA not only broadens the scope of problems addressable on near-term quantum hardware, but also offers meaningful improvements in efficiency, depth, and physical feasibility [6, 7, 23].

4.3 Fault tolerance

A method for qudit-based fault-tolerant quantum computation embeds a logical qubit directly into a single d-dimensional system, using its higher-dimensional Hilbert space to host both logical and error-correcting subspaces.

Definition 12. Let $\mathcal{H}_d$ be a d-level Hilbert space, with codewords $|\ell, k\rangle$, where $\ell \in \{0, 1\}$ encodes the logical qubit and $k \in \{0, \ldots, \lfloor d/2 \rfloor - 1\}$ enumerates error syndromes (that is, the type of error, bit-flip or phase-flip).

These codewords satisfy the Knill–Laflamme error correction conditions

$$\langle 0_\mathrm{L}|E_k^\dagger E_j|0_\mathrm{L}\rangle = \langle 1_\mathrm{L}|E_k^\dagger E_j|1_\mathrm{L}\rangle, \tag{19a}$$

$$\langle 0_\mathrm{L}|E_k^\dagger E_j|1_\mathrm{L}\rangle = 0, \tag{19b}$$

ensuring that physical errors E_k are detectable and correctable without disturbing the logical information.

Logical gates $\mathcal{G}_\mathrm{L}$ are implemented as blockwise operations $\mathcal{G} \otimes I$, acting identically across all syndrome

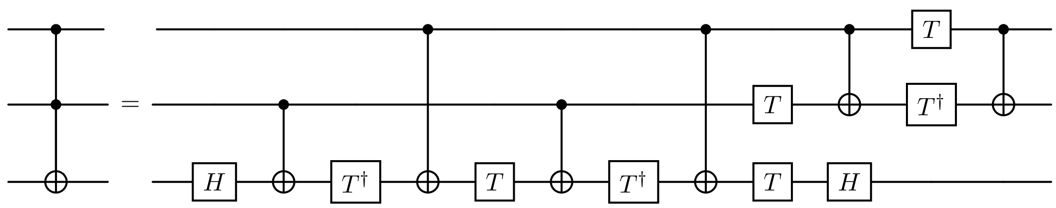

Figure 2: Standard qubit-decomposition of the Toffoli gate with CNOT a gate depth of six.

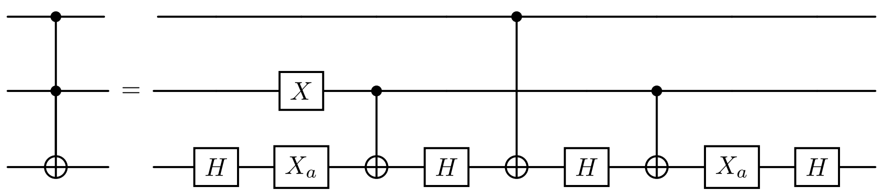

Figure 3: Decomposition of Toffoli gate permitted by introducing a qutrit as the target. The H and X gates are standard qubit gates that are idempotent on $|2\rangle$. Adapted from [15].

subspaces. Syndromes are extracted via a controlled unitary CU acting on an ancilla qudit:

$$CU\colon |\ell, k\rangle|0\rangle \mapsto |\ell, k\rangle|k\rangle, \tag{20}$$

after which recovery operations R_k return $|\ell, k\rangle \to |\ell, 0\rangle$. Readout is achieved by measuring observables Z_{L}, where the operator is defined by its action on the logical code states as $Z_{\mathrm{L}}|0_{\mathrm{L}}\rangle = |0_{\mathrm{L}}\rangle$ and $Z_{\mathrm{L}}|1_{\mathrm{L}}\rangle = -|1_{\mathrm{L}}\rangle$, dependent only on the logical index ℓ.

This embedded approach reduces physical overhead and enables scalable error correction. Numerical simulations show that increasing d linearly can yield exponential suppression of logical error rates, outperforming many multiqubit codes with significantly fewer physical resources [24].

5 Conclusion

Qudit-based quantum computing represents a compelling generalization of the traditional qubit paradigm, offering increased computational expressiveness of the encodings, more compact circuit constructions, and the potential for improved fault tolerance. By leveraging higher-dimensional Hilbert spaces, qudits allow for denser encodings and richer gate operations that align more naturally with many physical quantum systems such as transmons, Rydberg atoms, and trapped ions. In this work, we established the foundational mathematics of qudit systems, defined universal gate sets in the $\mathrm{SU}(d)$ framework, and explored how generalized unitaries can be efficiently compiled using gate sets such as Γ_d. We demonstrated how Solovay–Kitaev bounds can extend to qudits, and how circuit depth scales logarithmically with approximation precision. Further, we examined qudit-based implementations of QAOA and fault-tolerant encoding, highlighting both theoretical and experimental advantages in real contexts.

As experimental control over multi-level systems continues to advance, the theoretical frameworks presented here provide essential guidance for the development of scalable and efficient qudit-based quantum architectures. Future work may investigate gate compilation techniques, robust error correction tailored to prime-dimensional systems, and broader algorithmic applications uniquely suited to the qudit paradigm. Ultimately, mastering qudit computation is a critical step toward realizing the full computational potential of quantum technologies.

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