

Functional Programming

SS 2025

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Functors, Applicatives, and Monads

We will now discuss the `Functor`, `Applicative`, and `Monad` family of type classes. Each of these type classes (or: algebras) is more powerful than the last.

- `Monad`, in particular, will finally provide an answer to what lies behind the ominous `IO a` type—as in `main :: IO ()`—which somehow allows to cleanly integrate side effects ☠ (e.g., I/O, state, non-determinism, ...) into the pure Haskell language.

Type Class **Functor**

Type class **Functor** embodies the **application of a function to the elements (or: inside) of a structure**, while leaving the structure (or: outside) alone.

- Examples:
 - `map :: (a -> b) -> [a] -> [b]`
 - `mapTree :: (a -> b) -> Tree a -> Tree b`
- In general:

```
class Functor f where  
  fmap :: (a -> b) -> f a -> f b
```

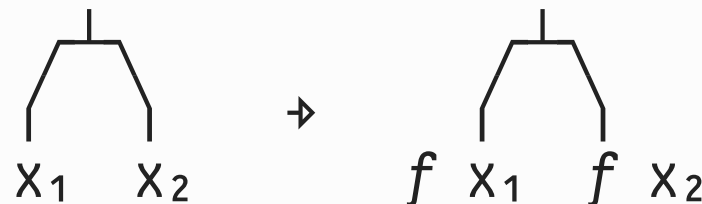
- **NB:** `f` is a type constructor that receives exactly one type argument. (**Functor** is also called a *constructor class*.)

Functors

- **Example:** Instances of constructor class `Functor`:

```
-- map f [x1,...,xn] = [f x1,...,f xn]. Behaves like a functor. ✓
instance Functor [] where
  fmap = map
```

```
instance Functor Tree where
  fmap = mapTree
```



- Again **NB:** Both, `[]` and `Tree`, are **type constructors**. Applying them to a type t yields regular types `[] t` (\equiv `[t]`) and `Tree t`.

Interlude: Kinds

- We know that Haskell distinguishes **types** and **type constructors**:
 - **Types** have values (e.g., `Bool` has values `True`, `False`).
 - **Type constructors** do not (e.g., no value has “type” `Maybe`).
- Type constructors build new types from existing types:

❶ <code>Maybe Bool</code>	✓ is a type
❷ <code>Maybe (Maybe Bool)</code>	✓ is a type
❸ <code>Maybe Maybe</code>	✗ is nonsense

- We find a similar situation on the level of values:

❶ <code>True</code>	✓ is a value (<code>True :: Bool</code>)
❷ <code>not True</code>	✓ is a value (<code>not :: Bool -> Bool</code>)
❸ <code>not not</code>	✗ is nonsense

Kinds: “Types for Types”

Spot the correspondence:

- On the **value level**, we have **types** that describe which function applications make sense.
- On the **type level**, we have **kinds** that describe which type constructor applications make sense.
- Types and type constructors have **kinds** (read *** as “any type”):

Kind	describes...	Examples
<i>*</i>	types	<code>Float</code> , <code>Bool</code> , <code>[Char]</code> , <code>[(Int,Int)]</code>
<i>* -> *</i>	unary type constructors	<code>Maybe</code> , <code>[]</code>
<i>* -> * -> *</i>	binary type constructors	<code>Either</code> , <code>(,)</code> , <code>(->)</code>

$$\begin{array}{ccc}
 & * & -> & * \\
 & \uparrow & & \uparrow \\
 \text{type argument} & & & \text{result type}
 \end{array}$$

Kinds of Type Classes

- Type classes are also kinded to avoid nonsense constructions:

❶	<code>Eq Bool</code>	✓ is a constraint
❷	<code>Eq (Maybe Bool)</code>	✓ is a constraint
❸	<code>Eq Maybe</code>	✗ is nonsense

- Kinds for type classes (again: read `*` as “*any type*”). In GHCi:

```
> :kind Eq
Eq :: * -> Constraint
> :kind Show
Show :: * -> Constraint
> :kind Functor
Functor :: (* -> *) -> Constraint  -- [✗]
```

[✗] Only unary type constructors can be instances of `Functor`!

- $$\begin{aligned} a \rightarrow b &\equiv (-\rightarrow) \ a \ b \\ (a, \ b) &\equiv (,) \ a \ b \end{aligned}$$

- [illegible]

More Functor Instances

- If the kind of `Either e` is `* -> *`, we should be able to define an instance of `Functor` for it:

```
data Either e a = Left e | Right a
```

↑
↑

⇒ `fmap` operates on the second (last) argument `a` of the type constructor

- Works indeed:

```
instance Functor (Either e) where
  fmap _ (Left err) = Left err
  fmap f (Right x)  = Right (f x)
```

- Functors `f` thus need *not* be containers (like `[]`, `Tree`), but can also describe **values** (of type `a`) **in a context** `f`.

More Functor Instances

- Make type constructors `Flagged`, `Indexed` instances of `Functor`:

```
instances Functor Flagged where           -- Flagged a ≡ (Bool, a)
  fmap f (b, x) = (b, f x)
```

```
instance Functor Indexed where          -- Indexed a ≡ Int -> a
  fmap f g = f . g
```

- Check: Do these still fit with our intuition that `fmap :: (a -> b) -> f a -> f b` applies a function to a value in context `f`? ✓

Stacked Functors

Since functors have kind $* \rightarrow *$ we can build “stacks” of functors f_i applied to some initial type t (of kind $*$):

$$f_n (\dots (f_2 (f_1 t)) \dots)$$

- **Example:** stack of depth $n = 4$ with functors (outer to inner)
 $f_4 = [\circ]$, $f_3 = (\text{String}, \circ)$, $f_2 = \text{Maybe } \circ$, $f_1 = [\circ]$ (of Char):

```
-- (Excerpt of) characters in "The Force Awakens"
tfa :: [(String, Maybe String)]
tfa = [ ("Rey" , Nothing),
        ("Han" , Just "Solo"),
        ("Finn", Nothing),
        ("Kylo", Just "Ren") ]
```

\Rightarrow Can use n -fold composition of `fmap` to “reach into” nested structure and apply a function to contained values at depth n .

Functor Laws

- Any Functor is expected to adhere to the two **functor laws**:
 1. `fmap id ≡ id`
 2. `fmap f . fmap g ≡ fmap (f . g)`
- The two laws capture the essence of the functor idea: `fmap` applies a function to values *inside* the structure (container, context), leaving the structure alone.
- ⚠ Haskell does *not* enforce these laws. Our implementations of `fmap` are *expected* to behave as shown above.

Deriving `Functor` Instances

- Note that the `Functor` instance for `(Pred i)` is generic and could have been **derived automatically**. General “recipe”:
 - To implement `fmap :: (a -> b) -> f a -> f b` for functor `f`:
 1. Apply function to all contained values of type `a` in the structure.
 2. Recurse into substructures of type `f a`.
 3. Leave everything else untouched.
- 💡 Haskell can derive `Functor` instances for algebraic datatypes with language extension `DeriveFunctor` (use GHC compiler pragma `{-# LANGUAGE DeriveFunctor #-}`).

Type Class `Applicative`

Essence of a functor `f a`: we can use `fmap` to apply a function to the insides of a structure/context. Now, type class `Applicative`:

- **Both**, the function to apply and its argument(s), reside in a structure/context. Two steps to apply (via operator `<*>`, read: *"tie fighter"*):
 1. extract function and argument(s) from their structures,
 2. apply function, place result in structure again:
- Compare (**NB**: `fmap f e` may also be written as `f <$> e`):

<code>(<code>\$</code>)</code>	<code>::</code>	<code>(a -> b) -> a -> b</code>
<code>(<\$>)</code>	<code>:: Functor f =></code>	<code>(a -> b) -> f a -> f b</code>
<code>(<*>)</code>	<code>:: Applicative f =></code>	<code>f (a -> b) -> f a -> f b</code>

Type Class `Applicative`

Haskell's type class `Applicative`:

```
class Functor f => Applicative f where  
  pure  :: a -> f a  
  (<*>) :: f (a -> b) -> f a -> f b
```

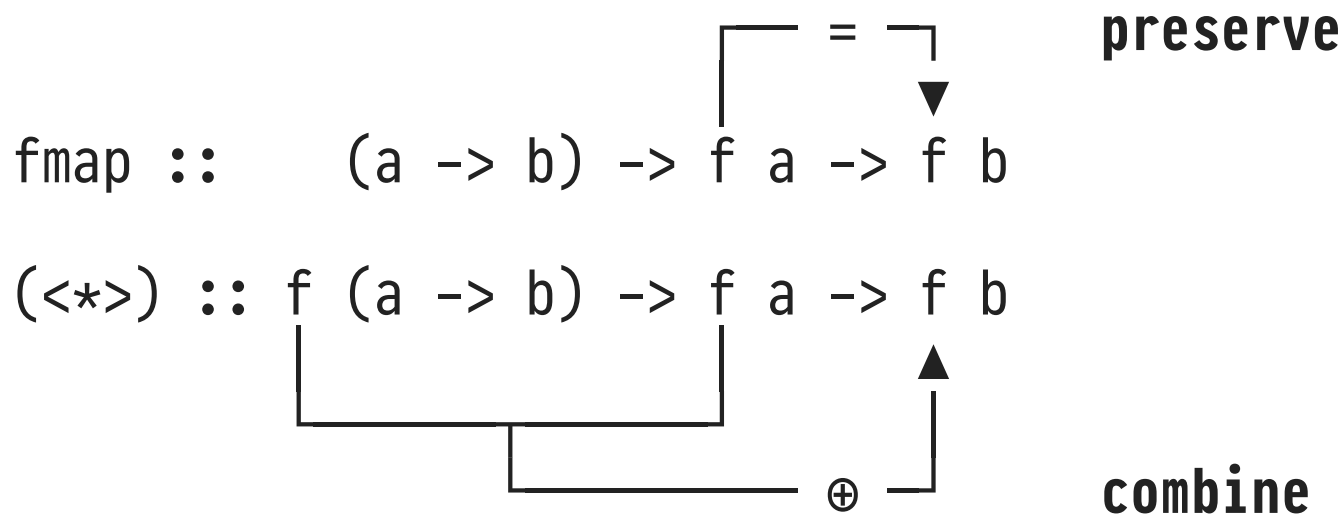
- **Notes:**

- Every `Applicative` also is a `Functor` (`f` is also referred to as *applicative functor*), i.e., we need `fmap`.
- `pure f` places regular function (any value, actually) `f` into a trivial structure/context. Thus:

```
fmap f e = pure f <*> e
```

Applicative: Function *and* Argument in Structure/Context


- With **Functor** and **fmap** we had **one bit of structure** that we needed to preserve. With **Applicative** and **(<*>)** we have **two bits of structure** that we need to combine:
 - **Applicative** embodies
 1. **function application** on the level of values, *and*
 2. **combination** on the level of structures:



Combining Structure/Context

Examples (of structure/context combination \oplus):

```
Prelude> Just (+ 2) <*> Just 40
Just 42
Prelude> Just (+ 2) <*> Nothing
Nothing
Prelude> Nothing <*> Just 40
Nothing
Prelude> Nothing <*> Nothing
Nothing
Prelude> [(+ 1),(* 10)] <*> [1,2,3]
[2,3,4,10,20,30]
Prelude> ([True], (+ 2)) <*> ([False], 40)
([True,False],42)
Prelude> (True, (+ 2)) <*> (False, 40)      ⚡
```



how to combine two Booleans?

Interlude: Type Class `Monoid`

Type class `Monoid a` represents *combinable* values of type `a` with a neutral element:

```
class Monoid a where
  mempty  :: a           -- neutral element for mappend
  mappend :: a -> a -> a  -- associative combine/ $\oplus$ , also: (<>)
  mconcat :: [a] -> a    --  $x_1 <> x_2 <> \dots <> x_n$ 
```

- **Examples** of monoids (`mempty`, `(<>)`):

- `(0, (+))`, `(1, (*))`
- `(True, (&&))`, `(False, (||))`
- `(empty, union)`
- `([], (++)`

[see module `Data.Set`: `(\emptyset , \cup)`]

Sample **Applicative** Instances

```
instance Applicative Maybe where  
  pure x = Just x
```

```
  Just f <*> Just x = Just (f x)  
  _ <*> _ = Nothing
```

```
instance Applicative ((,) c) where  
  pure x =
```

```
  (c1, f) <*> (c2, x) =
```

```
instance Applicative [] where  
  pure x =
```

```
  fs <*> xs =
```

An Application of **Applicative**: Input Validation

- The upcoming example uses **Applicative** operator variant ***>**:

$$\begin{array}{c}
 \text{preserve value} \\
 (*>) :: f\ a \rightarrow f\ b \rightarrow f\ b \\
 \begin{array}{c}
 \overbrace{f\ a \rightarrow f\ b}^{\text{preserve value}} \\
 \underbrace{\quad \oplus \quad}_{\text{combine structures}}
 \end{array}
 \end{array}$$

- Default definition of ***>** in class **Applicative**:

$$\begin{array}{c}
 \text{const id } u = \text{id} \quad \text{id } v = v \quad \text{value of } v \text{ preserved } \checkmark \\
 \begin{array}{c}
 \overbrace{\text{const id } u = \text{id} \quad \text{id } v = v}^{\text{value of } v \text{ preserved } \checkmark} \\
 \underbrace{\quad \oplus \quad}_{\text{structure of } u, v \text{ combined } \checkmark}
 \end{array}
 \end{array}$$

combined, but simply yields structure of u since **pure** builds a neutral structure: $\text{pure id } <*> x \equiv x$