Lookback Prophet Inequalities

Ziyad Benomar, Dorian Baudry, Vianney Perchet

Abstract

Prophet inequalities are fundamental optimal stopping problems, where a decision-maker observes sequentially items with values sampled independently from known distributions, and must decide at each new observation to either stop and gain the current value or reject it irrevocably and move to the next step. This model is often too pessimistic and does not adequately represent real-world online selection processes. Potentially, rejected items can be revisited and a fraction of their value can be recovered. To analyze this problem, we consider general decay functions $D_1, D_2, \ldots$, quantifying the value to be recovered from a rejected item, depending on how far it has been observed in the past. We analyze how lookback improves, or not, the competitive ratio in prophet inequalities in different order models. We show that, under mild monotonicity assumptions on the decay functions, the problem can be reduced to the case where all the decay functions are equal to the same function $x \mapsto \gamma x$, where $\gamma = \inf_{x>0} \inf_{j \geq 1} D_j(x)/x$. Consequently, we focus on this setting and refine the analyses of the competitive ratios, with upper and lower bounds expressed as increasing functions of γ.

1 Introduction

Optimal stopping problems constitute a classical paradigm of decision-making under uncertainty [Dyn63], with the primary challenge lying in determining the opportune moment to take action. Typically, in online algorithms, these problems are formalized as variations of the secretary problem [Gar70, Fer89, Lin61] or the prophet inequality [KS77, KS78, SC84]. In the context of the prophet inequality, the decision-maker observes a finite sequence of items, each having a value drawn independently from a known probability distribution. Upon encountering a new item, the decision-maker faces the choice of either accepting it and concluding the selection process or irreversibly rejecting it, with the objective of maximizing the value of the selected item. However, while the prophet inequality problem is already used in scenarios such as posted-price mechanism design [CHMS10, HKS07], it might present a pessimistic model of real-world online selection problems. Indeed, it is in general possible in practice to revisit previously rejected items and potentially recover them or at least recover a fraction of their value.

For instance, imagine an individual navigating a city in search of a restaurant. When encountering one, they have the choice to stop and dine at this place, continue their search, or revisit a previously passed option, incurring a utility cost that is proportional to the distance of backtracking. In another example drawn from the real estate market, homeowners receive offers from potential buyers. The decision to accept or reject an offer can be revisited later, although buyer interest may have changed, resulting in a potentially lower offer or even a lack of interest. Lastly, in the financial domain, an agent may choose to sell an asset at its current price or opt for a lookback put option, allowing them to sell at the asset's highest price over a specified future period. To make a meaningful comparison between the two, one must account for factors such as discounting (time value of money) and the cost of the option.

To encompass diverse scenarios, we propose a general way to quantify the cost incurred by the decision-maker for retrieving a previously rejected value.

Definition 1.1 (Decay functions). *Let $\mathcal{D} = (D_1, D_2, \dots)$ be a sequence of non-negative functions defined on $[0, \infty)$. It is a sequence of decay functions if*

1. *$D_1(x) \le x$ for all $x \ge 0$,*
2. *the sequence $(D_j(x))_{j\ge1}$ is non-increasing for all $x \ge 0$,*
3. *the function D_j is non-decreasing for all $j \ge 1$.*

In the context of decay functions $\mathcal{D}$, if an item with value x is rejected at a particular step, the algorithm has the potential to recover $D_j(x)$ after j subsequent steps. The three conditions defining decay functions serve as fundamental prerequisites for the $\mathcal{D}$-prophet problem. The first and second conditions ensure that the recoverable value of a rejected item can only diminish over time, while the final condition implies that an increase in the observed value x corresponds to an increase in the potential recovered value after j steps. Although the non-negativity assumption on decay functions is non-essential, we retain it for convenience, as we can easily revert to this assumption by considering that the algorithm has a reward of zero by not selecting any item.

The motivating examples that we introduced can be modeled respectively with decay functions of the form $D_j(x) = x - c_j$ where $(c_j)_{j\ge1}$ is a non-decreasing positive sequence, $D_j(x) = \xi_j x$ with $\xi_j \sim \mathcal{B}(p_j)$ and $(p_j)_{j\ge1}$ a non-increasing sequence of probabilities, and $D_j(x) = \lambda^j x$ with $\lambda \in [0, 1]$. In one of these examples (housing market), the natural model is to use a *random decay function*: the buyer makes the same offer if they are still interested, and offers 0 otherwise. Definition 1.1 can be easily extended to consider this case. However, to enhance the clarity of the presentation, we only discuss the deterministic case in the rest of the paper. In Appendix A, we explain how all the proofs and theorems proposed in the next sections can be generalized to that case.

The $\mathcal{D}$-prophet inequality For any sequence $\mathcal{D}$ of decay functions, we define the $\mathcal{D}$-prophet inequality problem, where the decision maker, knowing $\mathcal{D}$, observes sequentially the values $X_1, \ldots, X_n$, with X_i drawn from a known distribution F_i for all $i \in [n]$. If they decide to stop at some step τ, then instead of gaining X_τ as in the classical prophet inequality, they can choose to select the current item X_τ and have its full value, or select any item X_i with $i < \tau$ among the rejected ones but only recover a fraction $D_{\tau-i}(X_i) \leq X_i$ of its value. Therefore, if an algorithm ALG stops at step τ its reward is

$$\begin{aligned}\mathsf{ALG}^{\mathcal{D}}(X_1, \ldots, X_n) &= \max\{X_\tau, D_1(X_{\tau-1}), D_2(X_{\tau-2}), \ldots, D_{\tau-1}(X_1)\} \\ &= \max_{0\leq i\leq \tau-1}\{D_i(X_{\tau-i})\} ,\end{aligned}$$

with the convention $D_0(x) = x$. If the algorithm does not stop at any step before n, then its reward is $\mathsf{ALG}^{\mathcal{D}}(X_1, \ldots, X_n) = \max_{1\leq i\leq n}\{D_i(X_{\pi(n-i+1)})\}$, which corresponds to $\tau = n + 1$.

Remark 1.1. *As in the standard prophet inequality, an algorithm is defined by its stopping time, i.e., the rule set to decide whether to stop or not. Hence, if $\mathcal{D}$ and $\mathcal{D}'$ are two different sequences of decay functions, any algorithm for the $\mathcal{D}$-prophet inequality, although its stopping time might depend on the particular sequence of functions $\mathcal{D}$, is also an algorithm for the $\mathcal{D}'$-prophet inequality. Consider for example an algorithm ALG with stopping time $\tau(\mathcal{D})$ that depends on $\mathcal{D}$. Its reward in the $\mathcal{D}'$-prophet inequality is $\mathsf{ALG}^{\mathcal{D}'}(X_1, \ldots, X_n) = \max_{0\leq i\leq \tau-1}\{D'_i(X_{\tau(\mathcal{D})-i})\}$.*

Observation order Several variants of the prophet inequality problem have been studied, depending on the order of observations. The standard model is the adversarial (or fixed) order: given an instance, the ordering of $F_1, \ldots, F_n$ is fixed by an adversary [KS77, KS78]. In other variants, the order in which the items are observed is not adversarial. A first case is the *order selection* model [CHMS10, BGL$^+$21, PT22], where the decision-maker can choose in which order they observe the samples of $F_1, \ldots, F_n$. The second setting is the *random order* model [EHLM17, CSZ21], where the observation order is chosen uniformly at random. Another model in which the order is no longer important is the IID model [HK82, Ker86, CFH$^+$21], where all the values are sampled independently from the same distribution F. The $\mathcal{D}$-prophet inequality is well-defined in each of these different order models: if the items are observed in the order $X_{\pi(1)}, \ldots, X_{\pi(n)}$ with π a permutation of $[n]$, then the reward of the algorithm becomes $\mathsf{ALG}^{\mathcal{D}}(X_1, \ldots, X_n) = \max_{0\leq i\leq \tau-1}\{D_i(X_{\pi(\tau-i)})\}$. In this paper, we delve into the study of the $\mathcal{D}$-prophet inequality in the adversarial order, random order and IID models, providing both lower and upper bounds for each of them. Naturally, the lower bounds for the random order model and the upper bounds for the IID model can be extended to the order selection model.

Competitive ratio In the $\mathcal{D}$-prophet inequality, an input instance I is a finite sequence of probability distributions $(F_1, \ldots, F_n)$. Thus, for any instance I, we denote by $\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I)]$ the expected reward of ALG given I as input, and we denote by $\mathbb{E}[\mathsf{OPT}(I)]$ the expected maximum of independent random variables $(X_i)_{i\in[n]}$, where $X_i \sim F_i$. With these notations, we define the competitive ratio, which will be used to measure the quality of the algorithms.

Definition 1.2 (Competitive ratio)**.** *Let $\mathcal{D}$ be a sequence of decay functions and ALG an algorithm. We define the competitive ratio of ALG by*

$$\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) = \inf_I \frac{\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} ,$$

where the infimum is taken over all the tuples of all sizes of non-negative distributions with finite expectation.

An algorithm is said to be α-competitive if its competitive ratio is at least α, which means that for any possible instance I, the algorithm guarantees a reward of at least $\alpha\mathbb{E}[\mathsf{OPT}(I)]$. The notion of competitive ratio is used more broadly in competitive analysis as a metric to evaluate online algorithms [BEY05, Kar92, MPT94, MMS88, Sei00], as it lower bounds their performance even in worst-case scenarios, when the input instance is chosen adversarially.

1.1 Main results

It is trivial that non-zero decay functions $\mathcal{D}$ guarantee a better reward compared to the classical prophet inequality. However, in general this is not sufficient to conclude that the standard upper bounds or the competitive ratio of a given algorithm can be improved. Hence, a first key question is: what condition on $\mathcal{D}$ is necessary to surpass the conventional upper bounds of the classical prophet inequality? Surprisingly, the answer hinges solely on the constant $\gamma_{\mathcal{D}}$, defined as follows,

$$\gamma_{\mathcal{D}} = \inf_{x>0} \inf_{j\geq 1} \left\{ \frac{D_j(x)}{x} \right\} .$$

In the adversarial order model, we demonstrate that the optimal competitive ratio achievable in the $\mathcal{D}$-prophet inequality is determined by the parameter $\gamma_{\mathcal{D}}$ alone. Additionally, in both the random order and IID models, we demonstrate the essential requirement of $\gamma_{\mathcal{D}} > 0$ for enhancing the upper bounds of the classical prophet inequality. In particular, this implies that no improvement can be made with decay functions of the form $D_j(x) = x - c_j$ with $c_j > 0$, or $D_j(x) = \lambda^j x$ with $\lambda \in [0, 1)$. Subsequently, we develop algorithms and provide upper bounds in the $\mathcal{D}$-prophet inequality, uniquely dependent on the parameter $\gamma_{\mathcal{D}}$.

In the adversarial order, random order, and IID models, we establish these results through the following procedural steps:

Step 1 Using the monotonicity of the sequence $(D_j)_j$, we obtain that it converges to a limit decay function $D_\infty : x \mapsto \inf_{j\geq 1} D_j(x)$, and we prove the following theorem.

Theorem 2.1 (informal)**.** *If β is an upper bound on the competitive ratios of all algorithms in the D_∞-prophet inequality, i.e. where all the decay functions are equal to D_∞, then it is also an upper bound in the $\mathcal{D}$-prophet inequality.*

Reciprocally, given that $D_j \geq D_\infty$, any upper bound in the $\mathcal{D}$-prophet inequality applies in the D_∞-prophet inequality. This shows that establishing upper bounds can be entirely reduced to establishing them in D_∞-prophet inequality.

Step 2 Using the previous reduction, we introduce the constant $\gamma_{\mathcal{D}}$, which can be written as $\inf_{x>0} D_\infty(x)/x$, and we first prove that the condition $\gamma_{\mathcal{D}} > 0$ is necessary for surpassing the upper bounds of the prophet inequality.

Proposition 3.2. *if $\gamma_{\mathcal{D}} = 0$, then any upper bound in the classical prophet inequality is also an upper bound in the D_∞(respectively the $\mathcal{D}$)-prophet inequality.*

Then, assuming that $\gamma_{\mathcal{D}} > 0$, we give a weaker proposition, allowing us to establish upper bounds in the $\mathcal{D}$-prophet inequality, not necessarily tight ones, using upper bounds from the $\gamma_{\mathcal{D}}$-prophet inequality, where all the decay functions are equal to $x \mapsto \gamma_{\mathcal{D}} x$.

Proposition 3.3. *Let $a, b > 0$ and I an instance of random variables $(X_1, \dots, X_n)$ satisfying $X_i \in \{0, a, b\}$ a.s. for all $i \in [n]$. If the reward of any algorithm in the $\gamma_{\mathcal{D}}$-prophet inequality, given the input instance I, is at most $\beta \mathbb{E}[\mathsf{OPT}(I)]$, then β is an upper bound on the competitive ratios of all algorithms in the D_∞(respectively the $\mathcal{D}$)-prophet inequality.*

Finally, observing that $D_j(x) \geq \gamma_{\mathcal{D}} x$ for all $x \geq 0$ and $j \geq 1$, any guarantees on algorithms in the $\gamma_{\mathcal{D}}$-prophet inequality remain in the $\mathcal{D}$-prophet inequality. This means that, by studying the $\gamma_{\mathcal{D}}$-prophet inequality, it is possible to prove upper and lower bounds in the $\mathcal{D}$-prophet inequality.

Step 3 We fix $\gamma \in [0, 1]$ and study the γ-prophet inequality. Observe that $\gamma = 0$ corresponds to the classical prophet inequality, while $\gamma = 1$ corresponds to the offline setting, where the decision-maker can observe all the items before selecting one. We first show that, for any order model, given lower and upper bounds α and β in the classical prophet inequality, $\max(\alpha, \gamma)$ and $\big((1-\gamma)\beta + \gamma\big)$ are respectively lower and upper bounds in the γ-prophet inequality (see Proposition 4.2). Then, for each of the adversarial order, random order, and IID models, we give guarantees on single-threshold algorithms, accepting the first item with a value more than the threshold, and we prove upper bounds using instances satisfying the condition in Proposition 3.3.

Theorem 4.3 (Adversarial order)**.** *The competitive ratio of any algorithm is at most $\frac{1}{2-\gamma}$, and there exists a single-threshold algorithm with competitive ratio $\frac{1}{2-\gamma}$.*

In particular, using the previous results, this implies that the optimal competitive ratio in the $\mathcal{D}$-prophet inequality is $\frac{1}{2-\gamma_{\mathcal{D}}}$, which only depends on $\gamma_{\mathcal{D}}$.

Theorem 4.4 (Random order)**.** *The competitive ratio of any algorithm ALG in the γ-prophet inequality with random order satisfies*

$$\mathsf{CR}(\mathsf{ALG}) \leq (1-\gamma)^{3/2}(\sqrt{3-\gamma} - \sqrt{1-\gamma}) + \gamma \ .$$

Furthermore, denoting by p_γ the unique solution to the equation $1 - (1-\gamma)p = \frac{1-p}{-\log p}$, the single-threshold algorithm ALG_θ with $\Pr_{X_1 \sim F_1, \dots, X_n \sim F_n}(\max_{i \in [n]} X_i \leq \theta) = p_\gamma$ satisfies

$$\mathsf{CR}^\gamma(\mathsf{ALG}_\theta) \geq 1 - (1-\gamma)p_\gamma \ .$$

Note that for $\gamma = 0$, the upper bound becomes $\sqrt{3} - 1$, which is the upper bound proven in [CSZ21]. The equation satisfied by p_0 can be easily solved and we obtain $p_0 = 1/e$, which provides the best competitive ratio achievable by a single-threshold algorithm in the random order prophet inequality [EHLM17, CSZ21].

Theorem 4.5 (IID model)**.** *The competitive ratio of any algorithm in the IID γ-prophet inequality is at most*

$$U(\gamma) = 1 - (1-\gamma)\frac{e^2 \log(3-\gamma) - (2-\gamma)}{2(2e^2-1) - (3e^2-1)\gamma} \ .$$

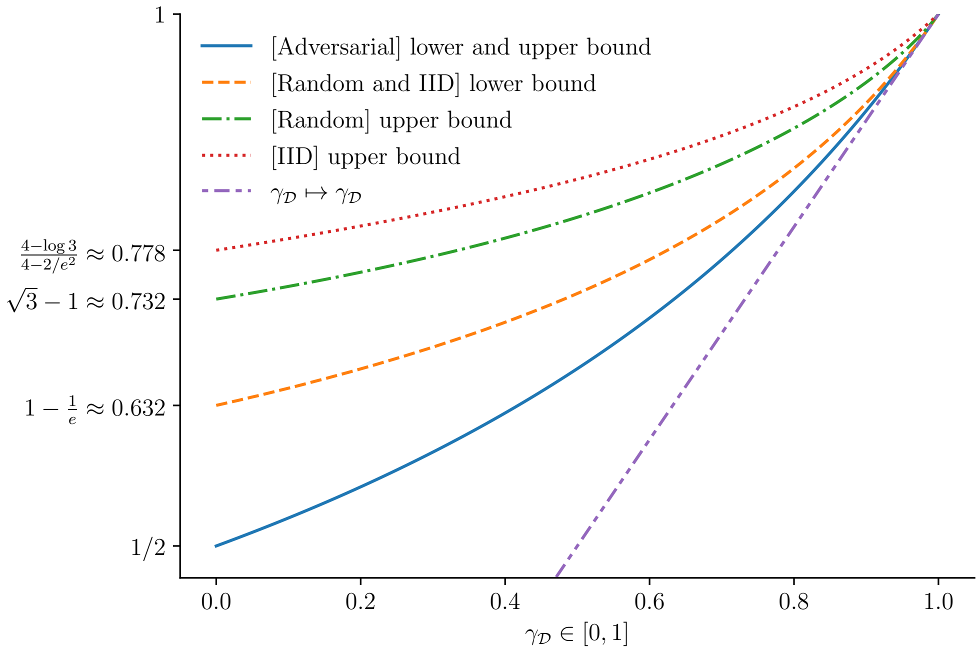

Figure 1: Lower and upper bounds on the competitive ratio in the $\mathcal{D}$-prophet inequality depending on $\gamma_{\mathcal{D}}$, in the adversarial order (Thm 4.3), random order (Thm 4.4) and IID (Thm 4.5) models

In particular, U is increasing, $U(0) = \frac{4-\log 3}{2(2-\frac{1}{e}^2)} \approx 0.778$ and $U(1) = 1$. Furthermore, there exists a single-threshold algorithm ALG_θ satisfying

$$\mathsf{CR}^\gamma(\mathsf{ALG}_\theta) \geq 1 - (1-\gamma)p_\gamma \ ,$$

where p_γ is defined in Theorem 4.4.

The upper bound for the IID case does not match the tight upper bound (approximately 0.745) when $\gamma = 0$ [HK82], but it is proven using arbitrary large instances satisfying the property described in Proposition 3.2, and thus can be extended to the $\mathcal{D}$-prophet inequality with $\gamma = \gamma_{\mathcal{D}}$ by Theorem 2.1. While the lower bound is exactly the same as in the random order model, we demonstrate it with a simpler proof, leveraging that the values are IID. Moreover, we give better guarantees depending on the size n of the instance (Lemma 4.6), that asymptotically converge to the lower bound. For $\gamma = 0$, it becomes $(1 - \frac{1}{e})$, which is proven to be the best competitive ratio of a single-threshold algorithm also in the IID prophet inequality [LWW22].

Using results from Steps 1 and 2, all these lower and upper bounds hold for the $\mathcal{D}$-prophet inequality when instantiated with $\gamma_{\mathcal{D}}$. We furthermore illustrate these bounds in Figure 1, comparing them with the identity function $\gamma \mapsto \gamma$, which is a trivial lower bound.

1.2 Main proof techniques

The first step of the proof involves reducing the problem to the D_∞-prophet inequality, which is characterized by a single function rather than a sequence. To prove Theorem 2.1, the main idea is to create an alternative instance J from any given instance I, in which an arbitrarily large number of zero values are inserted between every consecutive observations drawn from distributions in I. When the decision-maker observes a non-zero value in instance J, any preceding non-zero value X

is positioned so far in the past that only $D_\infty(X)$ can be recovered. While this is straightforward in the adversarial order model, it necessitates more elaborate arguments in the random order model, due to the lack of control over the positions of zeros, and in the IID model, because the number of zero and non-zero values in the instance J constructed is itself a random variable.

In the second step, the core idea is that rescaling an instance, i.e. considering $(rX_i)_{i\in[n]}$ instead of $(X_i)_{i\in[n]}$, has no impact in the classical prophet inequality. However, in the D_∞-prophet inequality, rescaling can be exploited to adjust the ratio $\frac{D_\infty(rx)}{rx}$. By considering instances with random variables taking values in $\{0, a, b\}$ almost surely, where $a < b$, any reasonable algorithm facing such an instance would never reject the value b. Consequently, the value it recovers from rejected items is either $D_\infty(0) = 0$ or $D_\infty(a)$. Rescaling this instance by a factor $r = s/a$ and taking the ratio to the expected maximum, the ratio $\frac{D_\infty(s)}{s}$ appears, with s a free parameter that can be chosen to satisfy $\frac{D_\infty(s)}{s} \to \inf_{x>0} \frac{D_\infty(x)}{x} = \gamma_{\mathcal{D}}$.

Finally, in the γ-prophet inequality, we establish upper bounds similarly as in the classical prophet inequality, by constructing hard instances where no algorithm's reward can exceed a certain fraction of the expected maximum. Then, we derive lower bounds for some tailored single-threshold algorithms ALG_θ. In the proof, we use that the reward for these algorithms is expressed as $\mathsf{ALG}^\gamma_\theta(X_1, \ldots, X_n) = \mathsf{ALG}^0_\theta(X_1, \ldots, X_n) + \gamma \max_{i\in[n]} X_i \mathbb{1}_{\max_{i\in[n]} X_i < \theta}$. The first term, corresponding to the reward in the classical prophet inequality, can be lower bounded using standard methods. The challenge is then to control the additional term to demonstrate improvement when $\gamma > 0$. We propose different techniques for each of the adversarial order, random order, and IID models.

1.3 Related work

Prophet inequalities Prophet inequalities seek to measure how the value selected by an online algorithm compares to the maximum value overall selected by the optimal offline algorithm. The first prophet inequality was proven by Krengel and Sucheston [KS77, KS78] in the setting where the items are observed in a fixed order, stating that the decision-maker can guarantee an expected value of at least half the expected maximum of the item's values, and that this bound is the best possible. It was later shown that this same competitive ratio can be obtained with a simple threshold algorithm [SC84], accepting the first value above a certain carefully chosen threshold. Combining ideas from [SC84] and [KW12], many thresholds can guarantee a competitive ratio of 1/2, such as the median or the mean of the maximum value of all the items. Surprisingly, [RWW19] proved that, even without knowing the distributions of the items' values, the same competitive ratio can be obtained when only given one sample of each distribution beforehand. Many variants and related problems have been studied, including, for example, the matroid prophet inequality [KW12, FSZ16], matchings [PPSW21, FGL14a, EFGT20, DFKL20], combinatorial auctions [FGL14b, DKL20], prophet inequality with advice [DKT+21], and variants with fairness considerations [CCDNF21, AK22]. For a more comprehensive and historical overview, we refer the interested reader to surveys on the problem such as [CFH+19, Luc17].

Different order models The *prophet secretary* problem, a fundamental variant of the prophet inequality, combines aspects of both the prophet inequality and the secretary problem, with items observed in a uniformly random order, deviating from the adversarial order model. First introduced in [EHLM17] with a $(1 - \frac{1}{e})$-competitive algorithm, subsequent advancements include a slightly improved ratio of $(1 - \frac{1}{e} + \frac{1}{400}) \approx 0.6346$ in [ACK18], followed by a competitive ratio of 0.669 in [CSZ21], which currently stands as the best-known lower bound. The same paper establishes an upper bound of $\sqrt{3} - 1 \approx 0.732$. Very recently, [BC23] demonstrated a tighter upper bound of

0.7254. Therefore, a disparity persists between the currently established lower and upper bounds, and addressing this gap remains an engaging and actively pursued open question.

Another setting is the *order selection* model, where the decision-maker can choose the order in which the items are observed, knowing their distributions. A first $(1 - \frac{1}{e}) \approx 0.632$ competitive algorithm was proposed in [CHMS10], and more recently this ratio was improved to 0.654 in [BGL$^+$21] then 0.725 in [PT22]. Nonetheless, the separation between the random order and order selection models remained uncertain until a very recent development [BC23], giving a 0.7258-competitive algorithm for the order selection model and a 0.7254 upper bound for the random order model. The best-known upper bound for the order selection model is ≈ 0.745, and it is proven using IID instances [HK82, CFH$^+$21] where the observation order no longer matters.

The IID model is itself a very interesting case, and its study dates back to [HK82], showing that the reward of the optimal (dynamic programming) algorithm, given an instance of n IID random variables, is at least a fraction $\frac{1}{a_n}$ of the expected maximum, with $(a_n)_n$ a recursively constructed sequence satisfying $1.1 < a_n < 1.6$. It was proved later in [Ker86] that $a_n \leq \frac{e}{e-1}$ and $a_n \to 1/\beta^* \approx \frac{1}{0.745}$. Finally, the IID prophet inequality was completely solved in [CFH$^+$21], where the authors show that $1/\beta*$ is exactly the competitive ratio achieved by the optimal dynamic programming algorithm, thus it constitutes an upper bound on the competitive ratio of any algorithm, and they give a simpler adaptive threshold algorithm matching this upper bound.

Related settings The ability of the decision-maker to revisit items seen in the past has already been investigated in other online selection problems. For instance, in the multiple-choice prophet inequality problem [ASC00], the algorithm can select up to k items and its reward is the maximum selected value. This means that the algorithm can choose up to k items to be revisited by the end of the process, for taking a final acceptance or rejection decision. A variant where the reward is the sum of selected items has also been studied in [Ken87, HKS07, Ala14]. Another similar setting is called Pandora's box problem, first introduced in [Wei78], where each item X_i is placed in a box requiring a cost c_i to be opened. The distribution of X_i and the cost c_i are known beforehand, but X_i is only revealed when the box i is opened. The decision-maker can freely choose the order in which they open the boxes and they can stop at any time. Their gain is the maximum observed value minus the sum of the costs of all the open boxes. An optimal algorithm was designed for the problem in [Wei78], and a simpler elegant proof of its optimality was given in [KWW16].

1.4 Open questions and future work

While we provided a full characterization of the optimal competitive ratio in the $\mathcal{D}$-prophet inequality with adversarial observation order, and demonstrated the necessity of $\gamma_{\mathcal{D}} > 0$ to surpass classical prophet inequality upper bounds in the random and IID models, the bounds established with $\gamma_{\mathcal{D}} > 0$ are not tight and do not match with state-of-the-art results as $\gamma_{\mathcal{D}} \to 0$ [CSZ21, CFH$^+$21, BC23]. This limitation stems from the restricted set of instances utilized to establish upper bounds (see Proposition 3.2), a remaining challenge consists therefore in devising a method to establish upper bounds in the D_∞-prophet inequality without confining the analysis to such instances. On the other hand, the limitation on lower bounds is due to the exclusive use of single-threshold algorithms. Our analysis relies on the reward formula for single-threshold algorithms and does not easily extend to more elaborate stopping times, such as multiple-threshold algorithms.

The exploration of the IID model in the γ-prophet inequality is of particular interest, because a tight upper bound exists in the classical prophet inequality, along with an adaptive threshold algorithm matching it [CFH$^+$21], while this is still an open question in the prophet secretary problem. Furthermore, the order selection model remains unexplored. Although the results of the second step of the proof (Section 3) hold for the order selection model, it remains uncertain whether the reduction from $\mathcal{D}$ to D_∞ is applicable in this context. The only bounds that can be deduced in this paper are the generic lower and upper bounds presented in Proposition 4.2, the lower bounds on the random order model, and the upper bounds on the IID model.

2 From $\mathcal{D}$-prophet to the D_∞-prophet inequality

Let us consider a sequence $\mathcal{D}$ of decay functions. By Definition 1.1, for any $x \in \mathbb{R}^+$ the sequence $(D_j(x))_{j\geq 1}$ converges, since it is non-increasing and non-negative. Hence, there exists a mapping D_∞ such that for any $x \geq 0$, $\lim_{j\to\infty} D_j(x) = D_\infty(x)$. Furthermore, we can easily verify that D_∞ is non-decreasing and satisfies $D_\infty(x) \in [0, x]$ for all $x \geq 0$.

Thanks to these properties, we obtain that $(D_\infty)_{j\geq 1}$ also satisfies Definition 1.1, and is hence a valid sequence of decay functions. We thus refer to the corresponding problem as the D_∞-prophet inequality. Since $D_j \geq D_\infty$ for any $j \geq 1$, it is straightforward that the stopping problem with the decay functions D_∞ would be less favorable to the decision-maker. More precisely, for any random variables $X_1, \ldots, X_n$, any observation order π, and any algorithm ALG with stopping time τ, it holds that

$$\mathsf{ALG}^{\mathcal{D}}(X_1, \ldots, X_n) \coloneqq \max\{X_{\pi(\tau)}, \max_{i<\tau} D_{\tau-i}(X_{\pi(i)})\} \geq \max\{X_{\pi(\tau)}, \max_{i<\tau} D_\infty(X_{\pi(i)})\} ,$$

which corresponds to the output of ALG (with the same decision rule) when all the decay functions are equal to D_∞. Therefore, any guarantees established for algorithms in the D_∞-prophet inequality naturally extend to the $\mathcal{D}$-prophet inequality. However, it remains uncertain whether the $\mathcal{D}$-prophet inequality can yield improved competitive ratios compared to the D_∞-prophet inequality. In the following, we prove that this is not the case, for all the observation orders presented in Section 1.

Theorem 2.1. *Let D_∞ be the pointwise limit of the sequence of decay functions $\mathcal{D} = (D_j)_{j\geq 1}$. Then for any instance $I = (F_1, \ldots, F_n)$ of non-negative distributions, in the adversarial order and the random order models it holds that*

$$\forall \mathsf{ALG} : \mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \leq \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} , \tag{1}$$

where the supremum is taken over all the online algorithms A. In the IID model, the same inequality holds with an additional $O(n^{-1/3})$ term, which depends only on the size n of the instance.

In the rest of this section, we provide a complete proof of the theorem, with specific arguments for each observation order. Before that, let us first detail its main implications.

Corollary 2.1.1. *In the adversarial order and the random order models, if $\bar{\mathsf{A}}_\infty$ be an optimal algorithm for the D_∞-prophet inequality, i.e. with maximal competitive ratio, then $\bar{\mathsf{A}}_\infty$ is also optimal for the $\mathcal{D}$-prophet inequality. Moreover, it holds that*

$$\mathsf{CR}^{\mathcal{D}}(\bar{\mathsf{A}}_\infty) = \mathsf{CR}^{D_\infty}(\bar{\mathsf{A}}_\infty) .$$

Proof. Let us denote $\mathsf{A}_{*,\infty}$ the algorithm taking optimal decisions for any instance in the D_∞-prophet inequality (obtained via dynamic programming). Then, by Theorem 2.1 we obtain for adversarial and random order models that

$$\sup_{\mathsf{ALG}} \mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \leq \inf_I \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} = \inf_I \frac{\mathbb{E}[\mathsf{A}_{*,\infty}^{D_\infty}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} = \mathsf{CR}^{D_\infty}(\mathsf{A}_{*,\infty}) = \sup_{\mathsf{A}} \mathsf{CR}^{D_\infty}(\mathsf{A}) . \tag{2}$$

Since $\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \geq \mathsf{CR}^{D_\infty}(\mathsf{ALG})$ for any algorithm, we deduce that (2) is an equality. If we consider now any algorithm $\bar{\mathsf{A}}_\infty$ that is optimal for the D_∞-prophet inequality, not necessarily $\mathsf{A}_{*,\infty}$, then Equation (2) provides

$$\mathsf{CR}^{\mathcal{D}}(\bar{\mathsf{A}}_\infty) \geq \mathsf{CR}^{D_\infty}(\bar{\mathsf{A}}_\infty) = \sup_{\mathsf{ALG}} \mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \ .$$

The previous inequality is again an equality, and it implies that $\bar{\mathsf{A}}_\infty$ is also optimal, in the sense of the competitive ratio, for the $\mathcal{D}$-prophet inequality, and

$$\mathsf{CR}^{\mathcal{D}}(\bar{\mathsf{A}}_\infty) = \mathsf{CR}^{D_\infty}(\bar{\mathsf{A}}_\infty) \ .$$

□

A direct consequence of this result is that, in the adversarial and the random order models, for any sequence of decay functions $\mathcal{D}$, the asymptotic decay D_∞ entirely determines the competitive ratio that is achievable and the upper bounds for the $\mathcal{D}$-prophet inequality. Therefore, we can restrict our analysis to algorithms designed for the problem with identical decay function.

In the IID model, the same conclusion holds if the worst-case instances are arbitrarily large, making the additional $O(n^{-1/3})$ term vanish. This is the case in particular in the classical prophet inequality [HK82].

2.1 Proof of Theorem 2.1

Before presenting the formal proof of the theorem, let us detail some intuitions. Indeed, while we use different techniques for each observation order considered, all the proofs share the same underlying idea. Given any instance I, it consists in building an alternative instance J such that any algorithm designed for the $\mathcal{D}$-prophet inequality and seeing observations from J cannot perform better than if it was receiving rewards from the instance I and facing the constant decay function D_∞. To do this, we essentially introduce an arbitrarily large number of zero values between two successive observations drawn from distributions belonging to I. Hence, under J the algorithm cannot recover much more than a fraction $D_\infty(X)$ for any past observation X collected from $F \in I$.

Alternative instance in each model In the adversarial case, implementing this idea is straightforward, since nature can build J by directly inserting a number m of zeros between each pair of consecutive values, and the result is obtained by making m arbitrarily large. For the random order model we use the same instance J, but extra steps are needed to use that the number of steps between two non-zero values is very large with high probability.

Moving to the IID model, an instance I is defined by a pair (F, n), where F is a non-negative distribution, and n is the size of the instance. In this scenario, we consider an instance consisting of $m > n$ IID random variables $(Y_i)_{i\in[m]}$, each sampled from F with probability n/m, and equal to zero with the remaining probability. We again achieve the desired result by letting m be arbitrarily large compared to n. However, in contrast to the adversarial and random order models, the number of variables sampled from F is not fixed; it follows a Binomial distribution with parameters $(m, n/m)$. We control this variability by using concentration inequalities, which is the cause of the additional term $O(n^{-1/3})$, independent of the distribution F, introduced in Theorem 2.1.

Auxiliary result The efficiency of the proof scheme introduced in the previous paragraph relies on the following key argument: if $(D_j)_{j\geq 1}$ converges pointwise to D_∞, then for any algorithm A and any instance I, the output of A when all the decay functions are equal to D_m converges to its output when all the decay functions are equal to D_∞. If $X_1, \ldots, X_n$ are the realizations of I

observed by A and if σ is the order in which they are observed, then denoting τ the stopping time of A we can write that

$$\begin{aligned}
&\mathbb{E}[\mathsf{A}^{D_m}(I)] - \mathbb{E}[\mathsf{A}^{D_\infty}(I)] \\
&\qquad = \mathbb{E}[\max\{X_{\sigma(\tau)}, D_m(\max_{i<\tau} X_{\sigma(i)})\}] - \mathbb{E}[\max\{X_{\sigma(\tau)}, D_\infty(\max_{i<\tau} X_{\sigma(i)})\}] \\
&\qquad \leq \max_{\substack{\pi\in\mathcal{S}_n \\ q\in[n]}} \left\{ \mathbb{E}[\max\{X_{\pi(q)}, D_m(\max_{i<q} X_{\pi(i)})\}] - \mathbb{E}[\max\{X_{\pi(q)}, D_\infty(\max_{i<q} X_{\pi(i)})\}] \right\} ,
\end{aligned}$$

where $\mathcal{S}_n$ is the set of all permutations of $[n]$. The latter upper bound is independent of σ and A. We show in the following lemma that it converges to 0 when $m \to \infty$.

Lemma 2.2. *Let $\mathcal{S}_n$ be the set of all permutations of $[n]$. For any fixed instance $I = (F_1, \dots, F_n)$, considering $X_i \sim F_i$ for all $i \in [n]$, define for all $m \geq 1$*

$$\epsilon_m(I) = \max_{\substack{\pi\in\mathcal{S}_n \\ q\in[n]}} \left\{ \mathbb{E}[\max\{X_{\pi(q)}, D_m(\max_{i<q} X_{\pi(i)})\}] - \mathbb{E}[\max\{X_{\pi(q)}, D_\infty(\max_{i<q} X_{\pi(i)})\}] \right\} .$$

If $\mathbb{E}[\max_{i\in[n]} X_i] < \infty$, then $\lim_{m\to\infty} \epsilon_m(I) = 0$.

Proof. Let us denote $f_1, \dots, f_n$ the respective probability density functions of $X_1, \dots, X_q$. For any $q \in [n]$ and $\pi \in \mathcal{S}_n$, let us define for all $m \geq 0$ the function $\varphi_m^{\pi,q} : [0,\infty)^q \to [0,\infty)$ by $\varphi_m^{\pi,q}(x_1, \dots, x_q) = \max\{x_{\pi(q)}, D_m(\max_{i<q} x_{\pi(i)})\} - \max\{x_{\pi(q)}, D_\infty(\max_{i<q} x_{\pi(i)})\}$. $\varphi_m^{\pi,q}$ is positive because $D_m \geq D_\infty$. The sequence $(\varphi_m^{\pi,q})_m$ is non-increasing, converges to 0 pointwise, and is dominated by $(x_1, \dots, x_q) \mapsto \max_{i\in[q]} x_i$, which is integrable with respect to the probability measure $(x_1, \dots, x_q) \mapsto \prod_{i=1}^q f(x_i)$. Therefore, using the dominated convergence theorem, we deduce that $\lim_{m\to\infty} \mathbb{E}[\varphi_m^{\pi,q}(X_1, \dots, X_q)] = 0$. It follows that

$$\lim_{m\to\infty} \epsilon_m(I) = \lim_{m\to\infty} \Big(\max_{\substack{\pi\in\mathcal{S}_n \\ q\in[n]}} \mathbb{E}[\varphi_m^{\pi,q}(X_1, \dots, X_q)] \Big) = 0 .$$

□

Let us now detail the full proof of Theorem 2.1.

Proof of Theorem 2.1. We provide a separate proof for each of the adversarial order, random order and IID models.

Adversarial order Let $I = (F_1, \dots, F_n)$ be any instance and $X_i \sim F_i$ for all $i \in [n]$. Consider the instance $I_m = (Y_1, \dots, Y_{mn})$, where $Y_{km} \sim F_k$ for any $k \in [n]$ and $Y_i = 0$ a.s. for all $i \notin \{m, 2m, \dots, mn\}$. It is clear that no reasonable algorithm would stop at a zero value: if the current observation is 0 it is preferable to wait for a non-null value, or it would have been preferable to stop at the previous non-null value. Hence, τ is a multiple of m: $\tau = \rho m$ for some $\rho \in \mathbb{N}^\star$. Given that $D_j(0) = 0$ for all j and the sequence $(D_j)_j$ is non-increasing, we have that

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m)] &= \mathbb{E}[\max_{i \leq \tau} D_{\tau - i}(Y_i)] \\
&= \mathbb{E}[\max_{k \leq \rho} D_{\tau - km}(Y_{km})] \\
&= \mathbb{E}[\max_{k \leq \rho} D_{\rho m - km}(X_k)] \\
&= \mathbb{E}[\max\{X_\rho, \max_{k < \rho} D_{(\rho - k)m}(X_k)\}] \\
&\leq \mathbb{E}[\max\{X_\rho, \max_{k < \rho} D_m(X_k)\}] \\
&\leq \mathbb{E}[\max\{X_\rho, \max_{k < \rho} D_\infty(X_k)\}] + \epsilon_m(I) \ ,
\end{aligned}$$

where $\epsilon_m(I)$ is defined in Lemma 2.2. We can then use that the first right-hand term is the output of some other algorithm that would choose a stopping time ρ when facing I in the context of the D_∞-prophet inequality. More precisely, consider the algorithm A_m which, given any instance $I = (F_1, \ldots, F_n)$, simulates the behavior of ALG facing the sequence I_m, where at each step $i \in [mn]$

- if $i \notin \{m, \ldots, nm\}$: ALG observes $Y_i = 0$,
- otherwise, if $i = km$ for some $k \in [n]$: A_m observes X_k and ALG observes $Y_{km} = X_k$
- if ALG stops on $Y_{\rho m}$, then A_m also stops, and its reward is X_ρ.

$\mathsf{A}_m(X_1, \ldots, X_n)$ stops at the same value as $\mathsf{ALG}(Y_1, \ldots, Y_m)$, their reward in the D_∞-prophet inequality is the same, and since $\max_{i \in [n]} X_i = \max_{i \in [mn]} Y_i$ this yields to

$$\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \leq \frac{\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m)]}{\mathbb{E}[\mathsf{OPT}(I_m)]} \leq \frac{\mathbb{E}[\mathsf{A}_m^{D_\infty}(I)] + \epsilon_m(I)}{\mathbb{E}[\mathsf{OPT}(I)]} \leq \sup_{\mathsf{A}\text{: algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} + \frac{\epsilon_m(I)}{\mathbb{E}[\mathsf{OPT}(I)]} \ ,$$

and taking the limit when $m \to \infty$ gives the result, by making the second term vanish.

Random order Let $I = (F_1, \ldots, F_n)$ be an instance of distributions and $X_i \sim F_i$ for $i \in [n]$. Using the notation δ_0 for the Dirac distribution in 0, we consider $I_m = (F_1, \ldots, F_n, \delta_0, \ldots, \delta_0)$ containing m copies of δ_0 so that the observations from this instance always contain at least m null values. Let $Y_1, \ldots, Y_m$ be a realization of this instance. For simplicity, say that $Y_i = X_i$ for $i \in [n]$, and $Y_i = 0$ for $i > n$.

We first show that when $m \to \infty$, since the observation order is drawn uniformly at random, the algorithm observes a large number of zeros between every two random variables drawn from $(F_1, \ldots, F_n)$. Let us denote by π the uniformly random order in which the observations are received, i.e. the algorithms observes $Y_{\pi(1)}, Y_{\pi(2)}, \ldots$, and let $\ell \geq 1$ be some positive integer, and $t_1, \ldots, t_n$ be the increasing indices in which the variables $Y_1, \ldots, Y_n$ are observed, i.e. $t_1 < \ldots < t_n$ and $\{t_1, \ldots t_n\} = \{\pi^{-1}(1), \ldots, \pi^{-1}(n)\}$. Therefore, any observation outside $\{Y_{\pi(t_1)}, \ldots, Y_{\pi(t_1)}\}$ is zero.

Using the notation $L = \min_{i\in[n-1]} |t_{i+1} - t_i|$, we obtain that

$$\begin{aligned}
\Pr(L \le \ell) &= \Pr(\cup_{i=1}^{n-1} \{t_{i+1} - t_i \le \ell\}) \\
&= \Pr\left(\cup_{k=1}^{n} \cup_{j=1}^{k-1} \left\{|\pi^{-1}(k) - \pi^{-1}(j)| \le \ell\right\}\right) \\
&\le \frac{n(n-1)}{2} \Pr(|\pi^{-1}(1) - \pi^{-1}(2)| \le \ell) \\
&= \frac{n(n-1)}{2} \Pr(\big(\cup_{k=1}^{n+m} (\pi^{-1}(1) = k, \pi^{-1}(2) \in \{k-\ell, \dots, k+\ell\} \setminus \{k\}) \\
&\le \frac{n(n-1)}{2} \times (n+m) \times \frac{1}{n+m} \times \frac{2\ell}{n+m-1} \\
&\le \frac{n^2 \ell}{m} \,.
\end{aligned}$$

Taking $\ell = \sqrt{m}$, we find that $\Pr(L \le \ell) \le n^2/\sqrt{m}$. Therefore, for any algorithm ALG, observing that the reward of ALG^D is at most $\max_{i\in[n]} X_i$ a.s., and by independence of $\max_{i\in[n]} X_i$ and L, we deduce that

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m)] &= \mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m)\mathbb{1}_{L>\ell}] + \mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m)\mathbb{1}_{L\le\ell}] \\
&\le \mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m) \mid L > \ell] + \mathbb{E}[(\max_{i\in[n]} X_i)\mathbb{1}_{L\le\ell}] \\
&\le \mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m) \mid L > \ell] + \mathbb{E}[\max_{i\in[n]} X_i] \frac{n^2}{\sqrt{m}} \,.
\end{aligned} \tag{3}$$

Let us denote τ the stopping time of ALG and $t_\rho = \max_{j\in[n]}\{t_j : t_j \le \tau\}$ the last time when a variable $(X_j)_{j\in[n]}$ was observed by ALG. The sequence of functions $(D_j)_{j\ge1}$ is non-increasing, hence

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m) \mid L > \ell] &= \mathbb{E}[\max_{i\in[\tau]} D_{\tau-i}(Y_{\pi(i)}) \mid L > \ell] \\
&= \mathbb{E}[\max_{j\le i} D_{\tau-t_j}(Y_{\pi(t_j)}) \mid L > \ell] && (4) \\
&\le \mathbb{E}[\max_{j\le i} D_{t_\rho-t_j}(Y_{\pi(t_j)}) \mid L > \ell] && (5) \\
&= \mathbb{E}[\max\left\{Y_{\pi(t_\rho)}, \max_{j<\rho} D_{t_\rho-t_j}(Y_{\pi(t_j)})\right\} \mid L > \ell] \\
&\le \mathbb{E}[\max\left\{Y_{\pi(t_\rho)}, \max_{j<\rho} D_{\ell}(Y_{\pi(t_j)})\right\}] \,, && (6)
\end{aligned}$$

Equation (4) holds because the only non-zero values up to step τ are $(Y_{\pi(t_j)})_{j\in[\rho]}$. Inequality (5) uses that the sequence $(D_j)_{j\ge1}$ is non-increasing, and (6) uses, in addition to that, the independence of L and $(Y_{\pi(t_j)})_{j\in[n]}$. We now argue that the term $\mathbb{E}[\max\left\{Y_{\pi(t_\rho)}, \max_{j<\rho} D_\ell(Y_{\pi(t_j)})\right\}]$ is the expected reward of an algorithm in the D_ℓ-inequality. Given that π is a uniform random permutation of $[n+m]$ and by definition of $t_1, \dots, t_n$, the application $\sigma : k \in [n] \mapsto \pi(t_k)$ is a random permutation of $[n]$. Therefore we consider the algorithm A_m that receives as input the instance $I = (F_1, \dots, F_n)$, then considers the array $u = (1, \dots, 1, 0, \dots, 0)$ composed of n values equal to 1 and m zero values, and a uniformly random permutation π of $[n+m]$, then simulates $\mathsf{ALG}^D(I_m)$ as follows: at each step $j \in [n+m]$

- if $u_{\pi(j)} = 0$, then ALG observes the value $Y_{\pi(j)} = 0$,
- if $u_{\pi(j)} = 1$, then A_m observes the next value $X_{\sigma(k)}$, and ALG observes $Y_{\pi(j)} = X_{\sigma(k)}$,

- when ALG decides to stop, A_m also stops, and its reward is the current value $X_{\sigma(k)}$.

With this construction, $(Y_j)_{j\in[n+m]}$ is indeed a realization of the instance I_m, and A_m stops on the last value sampled from $F_1,\dots,F_n$ observed by ALG. Therefore, denoting ρ the stopping time of A_m, and $\epsilon_\ell(I)$ as defined in Lemma 2.2, we have

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m) \mid L > \ell] &\le \mathbb{E}[\max\left\{Y_{\pi(t_\rho)}, \max_{j<\rho} D_\ell(Y_{\pi(t_j)})\right\}] \\
&= \mathbb{E}[\max\left\{X_{\sigma(\rho)}, \max_{j<\rho} D_\ell(X_{\sigma(t_j)})\right\}] \\
&= \mathbb{E}[\mathsf{A}_m^{D_\ell}(I)] \\
&\le \mathbb{E}[\mathsf{A}_m^{D_\infty}(I)] + \epsilon_\ell(I) \\
&\le \sup_{\mathsf{A}:\text{algo}} \mathbb{E}[\mathsf{A}^{D_\infty}(I)] + \epsilon_\ell(I)\ .
\end{aligned}$$

Taking $\ell = \sqrt{m}$ and substituting into Equation (3), then observing that $\mathbb{E}[\mathsf{OPT}(I)] = \mathbb{E}[\mathsf{OPT}(I_m)]$, gives that

$$\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \le \frac{\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(I_m)]}{\mathbb{E}[\mathsf{OPT}(I_m)]} \le \sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} + \frac{\epsilon_{\sqrt{m}}(I)}{\mathbb{E}[\mathsf{OPT}(I)]} + \frac{n^2}{\sqrt{m}}\ .$$

Finally, taking $m \to \infty$ and using Lemma 2.2, we deduce that

$$\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \le \sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]}\ ,$$

which completes the proof for the random order.

IID random variables For any probability distribution F on $[0,\infty)$ and for any $n \ge 1$ we denote $\mathbb{E}[\mathsf{OPT}(F,n)]$ the expected maximum of n independent random variables drawn from F, and for any algorithm ALG we denote $\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(F,n)]$ its expected output when given n IID variable sampled from F as input. The proof of Theorem 2.1 for this last model is much more technical than for previous models, so we first prove several auxiliary results that we will later use to provide a concise proof of the last part of the theorem.

Lemma 2.3. *For any probability distribution F and $n \ge 1, \Delta \ge 0$, we have*

$$\mathbb{E}[\mathit{OPT}(F, n+\Delta)] \le \left(1 + \frac{\Delta}{n}\right) \mathbb{E}[\mathit{OPT}(F,n)]\ .$$

Proof. We first write

$$\begin{aligned}
\Pr(\mathsf{OPT}(F, n+\Delta) > x) &= 1 - F(x)^{n+\Delta} \\
&= \left(1 + F(x)^n \frac{1 - F(x)^\Delta}{1 - F(x)^n}\right)(1 - F(x)^n) \\
&= \left(1 + F(x)^n \frac{1 - F(x)^\Delta}{1 - F(x)^n}\right) \Pr(\mathsf{OPT}(F,n) > x)\ ,
\end{aligned}$$

and then use that

$$F(x)^\Delta = e^{\Delta \log(F(x))} \ge 1 + \Delta \log(F(x)) = 1 - \frac{\Delta}{n} \log(1/F(x)^n) \ge 1 - \frac{\Delta}{n}\left(\frac{1 - F(x)^n}{F(x)^n}\right)\ ,$$

so we directly obtain

$$F(x)^n \frac{1-F(x)^\Delta}{1-F(x)^n} \le \frac{\Delta}{n} \ ,$$

which gives that

$$\Pr(\mathsf{OPT}(F, n+\Delta) > x) \le \left(1 + \frac{\Delta}{n}\right) \Pr(\mathsf{OPT}(F,n) > x) \ .$$

As we consider non-negative random variables, it follows directly that

$$\mathbb{E}(\mathsf{OPT}(F, n+\Delta)) \le \left(1 + \frac{\Delta}{n}\right) \mathbb{E}(\mathsf{OPT}(F,n)) \ .$$

$\square$

Lemma 2.4. *Let $N \sim \mathcal{B}(m, \varepsilon)$ and let $n := \mathbb{E}[N] = \varepsilon m$, then we have*

$$\mathbb{E}[\mathit{OPT}(F,N)\mathbb{1}_{N \ge n+n^{2/3}}] \le \frac{6}{n^{1/3}} \mathbb{E}[\mathit{OPT}(F, n+n^{2/3})] \ ,$$
$$\mathbb{E}[\mathit{OPT}(F,N)] \le \left(1 + \frac{3}{n^{1/3}}\right) \mathbb{E}[\mathit{OPT}(F, n+n^{2/3})] \ .$$

Proof. Let $\Delta, s > 0$ such that $\Delta \le s$. For any $k \ge 1$ let $W_k = [s+(k-1)\Delta, s+k\Delta)$. $(W_k)_{k\ge1}$ is a partition of $[s, \infty)$, thus we have

$$\begin{aligned}
\mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N\ge s}] &= \sum_{k=1}^{\infty} \mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N \in W_k}] \\
&\le \sum_{k=1}^{\infty} \mathbb{E}[\mathsf{OPT}(F, s+k\Delta)\mathbb{1}_{N \in W_k}] \\
&= \sum_{k=1}^{\infty} \mathbb{E}[\mathsf{OPT}(F, s+k\Delta)] \Pr(N \in W_k) \\
&\le \sum_{k=1}^{\infty} \left(1 + \frac{k\Delta}{s}\right) \mathbb{E}[\mathsf{OPT}(F,s)] \Pr(N \in W_k) \\
&= \left(\Pr(N \ge s) + \frac{\Delta}{s} \sum_{k=1}^{\infty} k \Pr(N \in W_k)\right) \mathbb{E}[\mathsf{OPT}(F,s)] \ ,
\end{aligned}$$

and observing that

$$\sum_{k=1}^{\infty} k \Pr(N \in W_k) = \sum_{k=1}^{\infty}\sum_{\ell=0}^{k-1} \Pr(N \in W_k) = \sum_{\ell=0}^{\infty}\sum_{k=\ell+1}^{\infty} \Pr(N \in W_k) = \sum_{\ell=0}^{\infty} \Pr(N \ge s + \ell\Delta) \ ,$$

we obtain, given $\Delta \le s$, that

$$\begin{aligned}
\mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N\ge s}] &\le \left(\Pr(N \ge s) + \frac{\Delta}{s} \sum_{k=0}^{\infty} \Pr(N \ge s + k\Delta)\right) \mathbb{E}[\mathsf{OPT}(F,s)] &(7)\\
&\le \left(2 \sum_{k=0}^{\infty} \Pr(N \ge s+k\Delta)\right) \ . &(8)
\end{aligned}$$

N is a Bernoulli random variable with expectation n. Therefore, Chernoff's inequality gives for any $\delta \geq 0$ that

$$\Pr(N \geq (1+\delta)n) \leq \exp\left(-\frac{\delta^2 n}{2+\delta}\right) \leq \exp\left(-\frac{\min(\delta,\delta^2)n}{3}\right) ,$$

where the second inequality can be derived by treating separately $\delta < 1$ and $\delta \geq 1$. In particular, for any $k \geq 1$, taking $\delta = \frac{k\Delta}{n}$ such that $\Delta \leq n$ yields

$$\Pr(N \geq n + k\Delta) \leq \exp\left(-\frac{\min(k\Delta, k^2\Delta^2/n)}{3}\right) \leq \exp\left(-\frac{k\min(\Delta,\Delta^2/n)}{3}\right) = \exp\left(-\frac{k\Delta^2}{3n}\right) .$$

Substituting this Inequality into (8) with $s = n + \Delta$, we obtain

$$\begin{aligned}
\mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N\geq n+\Delta}] &\leq \left(2\sum_{k=1}^{\infty}\Pr(N \geq n+k\Delta)\right)\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] \\
&\leq \left(2\sum_{k=1}^{\infty}\exp\left(-\frac{k\Delta^2}{3n}\right)\right)\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] \\
&= \frac{2}{\exp\left(\frac{\Delta^2}{3n}\right)-1}\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] \\
&\leq \frac{6n}{\Delta^2}\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] ,
\end{aligned}$$

and taking $\Delta = n^{2/3}$ proves the first inequality of the lemma.

Let us move now to the second inequality. We have

$$\mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N<s}] \leq \mathbb{E}[\mathsf{OPT}(F,s)\mathbb{1}_{N<s}] = \mathbb{E}[\mathsf{OPT}(F,s)]\Pr(N < s) ,$$

and thus, using Inequality (7), again with $s = n + \Delta$ and $\Delta = n^{2/3}$, it follows that

$$\begin{aligned}
\mathbb{E}[\mathsf{OPT}(F,N)] &= \mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N<s}] + \mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N\geq s}] \\
&\leq \left(1 + \frac{\Delta}{s}\sum_{k=0}^{\infty}\Pr(N \geq s+k\Delta)\right)\mathbb{E}[\mathsf{OPT}(F,s)] \\
&\leq \left(1 + \sum_{k=1}^{\infty}\Pr(N \geq n+k\Delta)\right)\mathbb{E}[\mathsf{OPT}(F,s)] \\
&\leq \left(1 + \frac{3}{n^{1/3}}\right)\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] .
\end{aligned}$$

□

Lemma 2.5. *Let $\delta_1,\ldots,\delta_m \overset{iid}{\sim} \mathcal{B}(\varepsilon)$, $N = \sum_{i=1}^m \delta_i$, $n = \mathbb{E}[N] = \varepsilon m$ and $L = \min_{i\neq j}\{|i-j| : \delta_i = 1, \delta_j = 1\}$, then for any $\ell \geq 0$ we have*

$$\mathbb{E}[\mathit{OPT}(F,N)\mathbb{1}_{L\leq\ell}] \leq \frac{4m\ell\varepsilon^2}{(1-\varepsilon)^2}\mathbb{E}[\mathit{OPT}(F,n+n^{2/3})] .$$

Proof. The random variables N and L are not independent, thus we need to adequately compute the distribution of L conditional to N. For any $\ell \geq 0$ and $k \geq 2$ we have

$$\begin{aligned}
\Pr(L \leq \ell, N = s) &= \Pr\left(L \leq \ell, \sum_{i=1}^{m} \delta_k = s\right) \\
&= \Pr\left(\cup_{i=1}^{m} \cup_{j=\max(1,i-\ell)}^{i-1} \big(\delta_i = \delta_j = 1, \sum_{i=1}^{m} \delta_k = s\big)\right) \\
&\leq m\ell \Pr\left(\delta_1 = \delta_2 = 1, \sum_{i=3}^{m} \delta_k = s\right) \\
&= m\ell\varepsilon^2 \binom{m-2}{s} \varepsilon^s (1-\varepsilon)^{m-s-2},
\end{aligned}$$

therefore

$$\begin{aligned}
\Pr(L \leq \ell \mid N = s) &= \frac{\Pr(L \leq \ell, N = s)}{\Pr(N = s)} \\
&\leq \frac{m\ell\varepsilon^2 \binom{m-2}{s} \varepsilon^s (1-\varepsilon)^{m-s-2}}{\binom{m}{s} \varepsilon^s (1-\varepsilon)^{m-s}} \\
&\leq \frac{m\ell\varepsilon^2}{(1-\varepsilon)^2} \ .
\end{aligned}$$

Using this inequality and Lemma 2.4, we deduce that

$$\begin{aligned}
\mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{L \leq \ell}] &= \mathbb{E}[\mathsf{OPT}(F,N) \Pr(L \leq \ell \mid N)] \\
&\leq \frac{m\ell\varepsilon^2}{(1-\varepsilon)^2} \mathbb{E}[\mathsf{OPT}(F,N)] \\
&\leq \frac{m\ell\varepsilon^2}{(1-\varepsilon)^2} \left(1 + \frac{3}{n^{1/3}}\right) \mathbb{E}[\mathsf{OPT}(F, n + n^{2/3})] \\
&\leq \frac{4m\ell\varepsilon^2}{(1-\varepsilon)^2} \mathbb{E}[\mathsf{OPT}(F, n + n^{2/3})] \ .
\end{aligned}$$

□

Using the previous lemmas, we can now prove the theorem. Let $m > n \geq 1$, $\Delta = n^{2/3}$, $\varepsilon = n/m$ and let Q be the probability distribution of a random variable that is equal to 0 with probability $1 - \varepsilon$, and drawn from F with probability ε.

Let us consider m i.i.d. variables $Y_1, \ldots, Y_m \sim Q$, and for each $i \in [m]$ we denote by δ_i the indicator that Y_i is drawn from F. Define $N = \sum_{i=1}^{m} \delta_i \sim \mathcal{B}(m, \varepsilon)$ the number of random variables Y_i drawn from the distribution F. In the following, we upper bound the competitive ratio of any algorithm by analyzing its ratio on this particular instance. For this, we first provide a lower bound on $\mathbb{E}[\mathsf{OPT}(Q,m)]$ using Lemma 2.3, and obtain

$$\mathbb{E}[\mathsf{OPT}(F, n-\Delta)] \geq \frac{1}{1 + \frac{2\Delta}{n}} \mathbb{E}[\mathsf{OPT}(F, n+\Delta)] \geq \left(1 - \frac{2\Delta}{n}\right) \mathbb{E}[\mathsf{OPT}(F, n+\Delta)] \ ,$$

thus we have

$$\begin{aligned}
\mathbb{E}[\mathsf{OPT}(Q,m)] &= \mathbb{E}[\mathsf{OPT}(F,N)] \\
&\geq \mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N\geq n-\Delta}] \\
&\geq \mathbb{E}[\mathsf{OPT}(F,n-\Delta)\mathbb{1}_{N\geq n-\Delta}] \\
&= \mathbb{E}[\mathsf{OPT}(F,n-\Delta)]\Pr(N\geq n-\Delta) \\
&\geq \left(1-\frac{2\Delta}{n}\right)\Pr(N\geq n-\Delta)\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] \\
&\geq \left(1-\frac{2\Delta}{n}-\Pr(N<n-\Delta)\right)\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] \\
&\geq \left(1-2n^{-1/3}-\exp(-n^{1/3}/2)\right)\mathbb{E}[\mathsf{OPT}(F,n+\Delta)] \\
&\geq \left(1-4n^{-1/3}\right)\mathbb{E}[\mathsf{OPT}(F,n+\Delta)]\ , \qquad (9)
\end{aligned}$$

where, for the last three inequalities, we used respectively Bernoulli's inequality, Chernoff bound, then $e^{-y}\leq 1/y$.

Then, we upper bound the reward of any algorithm given the instance (Q,m) as input. Let $L=\min_{i\neq j}\{|i-j| : \delta_i=1,\delta_j=1\}$ the smallest gap between two successive variables Y_i drawn from F, and let $t_1<\ldots<t_N$ the indices for which $\delta_i=1$. We have for any algorithm ALG and positive integer ℓ that

$$\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)] = \mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)\mathbb{1}_{N\geq n+\Delta \text{ or } L\leq\ell}] + \mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)\mathbb{1}_{N<n+\Delta, L>\ell}]\ . \qquad (10)$$

Using Lemma 2.4 and Lemma 2.5, the first term can be bounded as follows

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)\mathbb{1}_{N\geq n+\Delta \text{ or } L\leq\ell}] &\leq \mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N\geq n+\Delta \text{ or } L\leq\ell}] \\
&\leq \mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{N\geq n+\Delta}] + \mathbb{E}[\mathsf{OPT}(F,N)\mathbb{1}_{L\leq\ell}] \\
&\leq \left(\frac{6}{n^{1/3}}+\frac{4m\ell\varepsilon^2}{(1-\varepsilon)^2}\right)\mathbb{E}[\mathsf{OPT}(F,n+n^{2/3})]\ .
\end{aligned}$$

Considering that $m\geq 2n$ (i.e. $\varepsilon=n/m\leq 1/2$) and taking $\ell=\sqrt{m}$, we obtain

$$\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)\mathbb{1}_{N\geq n+\Delta \text{ or } L\leq\ell}] \leq \left(\frac{6}{n^{1/3}}+\frac{16n^2}{m}\right)\mathbb{E}[\mathsf{OPT}(F,n+n^{2/3})]\ . \qquad (11)$$

Regarding the second term in Equation (10), let τ be the stopping time of ALG and $t_\rho=\max\{j\leq\tau : \delta_j=1\}$ the last value sampled from F and observed by ALG before it stops. We have

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)\mathbb{1}_{N<n+\Delta,L>\ell}] &\leq \mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)\mid N<n+\Delta, L>\ell] \\
&= \mathbb{E}[\max_{i\in[\tau]} D_{\tau-i}(Y_i)\mid N<n+\Delta, L>\ell] \\
&= \mathbb{E}[\max_{j\in[\rho]} D_{\tau-t_j}(Y_i)\mid N<n+\Delta, L>\ell] \\
&\leq \mathbb{E}[\max_{j\in[\rho]} D_{t_\rho-t_j}(Y_i)\mid N<n+\Delta, L>\ell] \\
&= \mathbb{E}[\max\left\{Y_{t_\rho}, \max_{j<\rho} D_{t_\rho-t_j}(Y_{t_j})\right\}\mid N<n+\Delta, L>\ell] \\
&\leq \mathbb{E}[\max\left\{Y_{t_\rho}, \max_{j<\rho} D_\ell(Y_{t_j})\right\}\mid N<n+\Delta]\ .
\end{aligned}$$

We then prove that the last term is the reward of an algorithm A_m in the D_ℓ-prophet inequality. Let us A_m be the algorithm that takes as input an instance $X_1, \dots, X_{n+\Delta-1}$ of $n+\Delta$ IID random variables, then simulates $\mathsf{ALG}^{\mathcal{D}}(Q,m) \mid N < n+\Delta$ as follows: let $\delta_1, \dots, \delta_m \overset{\text{iid}}{\sim} \mathcal{B}(n/m)$ set $N_{\mathsf{A}} = 0$ and for each $i \in [m]$

- if $\delta_i = 0$: ALG observes the value $Y_i = 0$,
- if $\delta_i = 1$: increment N, then A_m observes the next value X_k, and ALG observes $Y_i = X_k$,
- if $N_{\mathsf{A}} = n + \Delta - 1$ or ALG stops, then A_m also stops.

When ALG decides to stop, the current value observed by A_m is X_ρ: the last value Y_{t_ρ} observed by ALG such that $\delta_{t_\rho} = 0$. Observe that stopping when $N_{\mathsf{A}} = n + \Delta + 1$, is equivalent to letting ALG observe zero values until the end, and stopping when ALG stops. Hence, the variables $Y_1, \dots, Y_m$ have the same distribution as m IID samples from Q conditional to $N < n + \Delta$. Denoting ρ the stopping time of A_m and $\epsilon_\ell(F, n+\Delta)$ as defined in Lemma 2.2, we deduce that

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m) \mid N < n+\Delta, L > \ell] &\leq \mathbb{E}[\max\left\{Y_{t_\rho}, \max_{j<\rho} D_\ell(Y_{t_j})\right\} \mid N < n+\Delta] \\
&= \mathbb{E}[\max\left\{X_\rho, \max_{j<\rho} D_\ell(X_j)\right\}] \\
&= \mathbb{E}[\mathsf{A}_m^{D_\ell}(F, n+\Delta)] \\
&\leq \mathbb{E}[\mathsf{A}_m^{D_\infty}(F, n+\Delta)] + \epsilon_\ell(F, n+\Delta)\ . \qquad (12)
\end{aligned}$$

Substituting (11) and (12) in (10), with $\ell = \sqrt{m}$, yields

$$\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)] \leq \left(\frac{6}{n^{1/3}} + \frac{16n^2}{m}\right) \mathbb{E}[\mathsf{OPT}(F, n+\Delta)] + \mathbb{E}[\mathsf{A}_m^{D_\infty}(F, n+\Delta)] + \epsilon_{\sqrt{m}}(F, n+\Delta)\ ,$$

and using Inequality 9, it follows that

$$\begin{aligned}
\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) &\leq \frac{\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(Q,m)]}{\mathbb{E}[\mathsf{OPT}(Q,m)]} \\
&\leq \frac{\frac{6}{n^{1/3}} + \frac{16n^2}{m}}{1 - \frac{4}{n^{1/3}}} + \frac{\mathbb{E}[\mathsf{A}_m^{D_\infty}(F, n+\Delta)] + \epsilon_{\sqrt{m}}(F, n+\Delta)}{(1 - \frac{4}{n^{1/3}})\mathbb{E}[\mathsf{OPT}(F, n+\Delta)]} \\
&\leq \frac{\frac{6}{n^{1/3}} + \frac{16n^2}{m}}{1 - \frac{4}{n^{1/3}}} + \frac{1}{1 - \frac{4}{n^{1/3}}} \left(\frac{\epsilon_{\sqrt{m}}(F, n+\Delta)}{\mathbb{E}[\mathsf{OPT}(F, n+\Delta)]} + \sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(F, n+\Delta)]}{\mathbb{E}[\mathsf{OPT}(F, n+\Delta)]}\right)\ ,
\end{aligned}$$

taking the limit $m \to \infty$ and using Lemma 2.2 gives

$$\begin{aligned}
\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) &\leq \frac{\frac{6}{n^{1/3}}}{1 - \frac{4}{n^{1/3}}} + \frac{1}{1 - \frac{4}{n^{1/3}}} \left(\sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(F, n+\Delta)]}{\mathbb{E}[\mathsf{OPT}(F, n+\Delta)]}\right) \\
&= \frac{6}{n^{1/3} - 4} + \left(1 + \frac{4}{n^{1/3} - 4}\right) \left(\sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(F, n+\Delta)]}{\mathbb{E}[\mathsf{OPT}(F, n+\Delta)]}\right) \\
&\leq \frac{10}{n^{1/3} - 4} + \sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(F, n+n^{2/3})]}{\mathbb{E}[\mathsf{OPT}(F, n+n^{2/3})]}\ .
\end{aligned}$$

where the last inequality holds because $\mathbb{E}[\mathsf{A}^{D_\infty}(F, n+\Delta)] \leq \mathbb{E}[\mathsf{OPT}(F, n+\Delta)]$ for any algorithm A. From here, the statement of the theorem can be deduced by observing that, for $k = n + n^{2/3}$, we have $n \geq (n + n^{2/3})/2 = k/2$, thus $n^{1/3} \geq k^{1/3}/2$, and we obtain

$$\begin{aligned}\mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) &\leq \frac{20}{k^{1/3} - 8} + \sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(F, k)]}{\mathbb{E}[\mathsf{OPT}(F, k)]} \\ &= \sup_{\mathsf{A}:\text{algo}} \frac{\mathbb{E}[\mathsf{A}^{D_\infty}(F, k)]}{\mathbb{E}[\mathsf{OPT}(F, k)]} + O\left(\frac{1}{k^{1/3}}\right) .\end{aligned}$$

□

3 From D_∞-prophet to the $\gamma_{\mathcal{D}}$-prophet inequality

As discussed in Section 2, Theorem 2.1 implies that, for either establishing upper bounds or guarantees on the competitive ratios of algorithms, it is sufficient to study the D_∞-prophet inequality, where all the decay functions are equal to D_∞. The remaining question is then to determine which functions D_∞ allow to improve upon the upper bounds of the classical prophet inequality. Before tackling this question, let us make some observations regarding algorithms in the D-prophet inequality.

Definition 3.1. *An algorithm ALG is rational in the D_∞-prophet inequality if, when observing a sequence $X_1, \ldots, X_n$, it rejects all the items X_i such that $X_i \leq D_\infty(\max_{j<i} X_j)$.*

Lemma 3.1. *For any rational algorithm ALG in the D_∞-prophet inequality, if we denote τ its stopping time, then for any instance $I = (F_1, \ldots, F_n)$ and $X_i \sim F_i$ for all $i \in [n]$ we have*

$$\mathsf{ALG}^{D_\infty}(X_1, \ldots, X_n) = \mathsf{ALG}^0(X_1, \ldots, X_n) + D_\infty(\max_{i\in[n]} X_i)\mathbb{1}_{\tau=n+1} .$$

Moreover, the optimal dynamic programming algorithm in the D_∞-prophet inequality is rational.

Proof. Let ALG be any rational algorithm in the D_∞-prophet inequality. If ALG stops at some step $\tau \in [n]$, then by definition we have that $X_\tau > D_\infty(\max_{j<\tau} X_j)$, and thus $\mathsf{ALG}^{D_\infty}(X_1, \ldots, X_n) = \mathsf{ALG}^0(X_1, \ldots, X_n)$. Otherwise, if it stops at $\tau = n+1$, then its reward is $\max_{i\in[n]} D_\infty(X_i) = D_\infty(\max_{i\in[n]} X_i)$, because D_∞-is non increasing.

On the other hand, let A_* be the optimal dynamic programming algorithm for the D_∞-prophet inequality. At any step i, if $X_i < D_\infty(\max_{j<i} X_j)$, then stopping at i gives a reward of $D_\infty(\max_{j<i} X_j)$, while by rejecting X_i, the final reward is guaranteed to be at least $D_\infty(\max_{j<i} X_j)$. Thus rejecting X_i can only increase the reward, it is therefore the optimal decision. □

The best competitive ratio in the D_∞-prophet inequality is achieved, among others, by the optimal dynamic programming algorithm A_*. Therefore, to prove upper bounds on the competitive ratio of any algorithm, it suffices to prove it on the competitive ratio of A_*. Hence it suffices to prove it on rational algorithms. We use this observation to prove the next propositions.

Proposition 3.2. *In the D_∞-prophet inequality, if $\inf_{x>0} \frac{D_\infty(x)}{x} = 0$, then it holds, in any order model, that*

$$\forall \mathsf{ALG} : \mathsf{CR}^{D_\infty}(\mathsf{ALG}) \leq \sup_{\mathsf{A}} \mathsf{CR}^0(\mathsf{A}) , \quad (13)$$

where the supremum is taken over all the online algorithms A.

Proof. Let us place ourselves in any order model, or in the IID model. Assume that $\inf_{x>0} \frac{D_\infty(x)}{x} = 0$, then there exist a sequence $(s_k)_{k\geq 1}$ such that $\lim_{k\to\infty} \frac{D_\infty(s_k)}{s_k} = 0$.

Let $I = (F_1, \dots, F_n)$ an instance of non-negative random variables with finite expectation, and $X_i \sim F_i$ for all $i \in [n]$. Let ALG be a rational algorithm for the D_∞-prophet inequality and let us denote τ its stopping time. Denoting $X^* := \max_{i\in[n]} X_i$ and using Lemma 3.1, we have for any constant $C > 0$ that that

$$
\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{D_\infty}(I)] &= \mathbb{E}[\mathsf{ALG}^0(I)\mathbb{1}_{\tau\leq n}] + \mathbb{E}[D_\infty(X^*)\mathbb{1}_{\tau=n+1}] \\
&\leq \mathbb{E}[\mathsf{ALG}^0(I)] + \mathbb{E}[D_\infty(C)] + \mathbb{E}[D_\infty(X^*)\mathbb{1}_{X^*>C}] \\
&\leq \sup_{\mathsf{A}} \mathbb{E}[\mathsf{A}^0(I)] + \mathbb{E}[D_\infty(C)] + \mathbb{E}[X^*\mathbb{1}_{X^*>C}] \ . \qquad (14)
\end{aligned}
$$

Let $k \geq 1$ a positive integer, M a positive constant, and consider the instance I^k of random variables $(X_1^k, \dots, X_n^k)$ with $X_i^k \sim \frac{s_k}{M} X_i$ for all $i \in [n]$. We have that $\mathsf{OPT}(I^k) = \frac{s_k}{M}\mathsf{OPT}(I)$ and $\mathbb{E}[\mathsf{A}^0(I^k)] = \frac{s_k}{M}\mathbb{E}[\mathsf{A}^0(I)]$ for any algorithm A. Therefore, using Inequality (14) with I^k instead of I and $C = \frac{s_k}{M}$, then dividing by $\mathsf{OPT}(I^k)$ gives that

$$
\begin{aligned}
\mathsf{CR}^{D_\infty}(\mathsf{ALG}) \leq \frac{\mathbb{E}[\mathsf{ALG}^{D_\infty}(I^k)]}{\mathbb{E}[\mathsf{OPT}(I^k)]} &\leq \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} + \frac{D_\infty(s_k)}{\frac{s_k}{M}\mathbb{E}[\mathsf{OPT}(I)]} + \frac{\mathbb{E}[\frac{s_k}{M} X^*\mathbb{1}_{X^*>M}]}{\frac{s_k}{M}\mathbb{E}[\mathsf{OPT}(I)]} \\
&= \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} + \left(\frac{M}{\mathbb{E}[\mathsf{OPT}(I)]}\right)\frac{D_\infty(s_k)}{s_k} + \frac{\mathbb{E}[X^*\mathbb{1}_{X^*>M}]}{\mathbb{E}[\mathsf{OPT}(I)]} \ ,
\end{aligned}
$$

and taking the limit $k \to \infty$, we obtain

$$
\mathsf{CR}^{D_\infty}(\mathsf{ALG}) \leq \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} + \frac{\mathbb{E}[X^*\mathbb{1}_{X^*>M}]}{\mathbb{E}[\mathsf{OPT}(I)]} \ ,
$$

and since X^* has finite expectation, taking the limit $M \to \infty$ gives

$$
\mathsf{CR}^{D_\infty}(\mathsf{ALG}) \leq \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} \ .
$$

Finally, taking the infimum over all instances, we obtain that

$$
\mathsf{CR}^{D_\infty}(\mathsf{ALG}) \leq \sup_{\mathsf{A}} \mathsf{CR}^0(\mathsf{A}) \ .
$$

As in the proof of Corollary 2.1.1, permuting $\inf_I$ and $\sup_{\mathsf{A}}$ is possible because there is an algorithm (dynamic programming) achieving the supremum for any instance. By Lemma 3.1, the inequality above holds for in particular for the optimal dynamic programming algorithm, which has a maximal competitive ratio. Therefore, the inequality remains true for any other algorithm A, not necessarily rational. $\square$

Proposition 3.2 implies that, if $\inf_{x>0} \frac{D_\infty(x)}{x} = 0$, then, in any order model, any upper bound on the competitive ratios of all algorithms in the classical prophet inequality is also an upper bound on the competitive ratios of all algorithm in the D_∞-prophet inequality. Consequently, for surpassing the upper bounds of the classical prophet inequality, it is necessary to have, for some $\gamma > 0$, that $D_\infty(x) \geq \gamma x$ for all $x \geq 0$. Furthermore, the next Proposition allows giving upper bounds in the D_∞-prophet inequality that depend only on $\inf_{x>0} \frac{D_\infty(x)}{x}$.

Proposition 3.3. *Let us denote $\gamma = \inf_{x>0} \frac{D_\infty(x)}{x}$, and let $a < b$ be two positive real numbers. Consider an instance I with distributions of $(X_1, \ldots, X_n)$ such that $X_i \in \{0, a, b\}$ a.s. for all $i \in [n]$, then in any order model and for any algorithm ALG we have that*

$$\mathit{CR}^{D_\infty}(\mathit{ALG}) \leq \sup_{\mathit{A}} \frac{\mathbb{E}[\mathit{A}^\gamma(I)]}{\mathbb{E}[\mathit{OPT}(I)]} \ ,$$

where $\mathbb{E}[\mathit{A}^\gamma(I)]$ is to the reward of A if all the decay functions were equal to $x \mapsto \gamma x$.

Proof. Let us place ourselves in any order model or in the IID model. Since $\gamma = \inf_{x>0} \frac{D_\infty(x)}{x}$, there exists a sequence $(s_k)_{k\geq 1}$ of positive numbers such that $\lim_{k\to\infty} \frac{D_\infty(s_k)}{s_k} = \gamma$.

For the random variables $X_1, \ldots, X_n$, in any order model, it is clear that the optimal decision when observing a zero value is to reject it, and when observing the value b is to accept it. Let ALG be an algorithm satisfying this property and let τ be its stopping time. If $\tau = n+1$ then necessarily $\max_{i\in[n]} X_i \neq b$, and the reward of ALG in that case is $D_\infty(a)$ if $\max_{i\in[n]} X_i$ and 0 otherwise. In particular, ALG is rational in the D_∞-prophet inequality and we have by Lemma 3.1 that

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{D_\infty}(I)] &= \mathbb{E}[\mathsf{ALG}^0(I)] + \mathbb{E}[D_\infty(\max_{i\in[n]} X_i)\mathbb{1}_{\tau=n+1}] \\
&= \mathbb{E}[\mathsf{ALG}^0(I)] + D_\infty(a)\Pr(\tau = n+1, \max_{i\in[n]} X_i = a) \ . \qquad (15)
\end{aligned}$$

Consider now the instance I^k of random variables $(X_1^k, \ldots, X_n^k)$ where $X_i^k = \frac{s_k}{a} X_i$ for all $i \in [n]$. I^k satisfies the same assumptions as I with $a^k = s_k$ and $b^k = \frac{s_k b}{a}$, and we have $\mathbb{E}[\mathsf{OPT}(I^k)] = \frac{s_k}{a}\mathbb{E}[\mathsf{OPT}(I)]$, $\mathbb{E}[\mathsf{ALG}^0(I^k)] = \frac{s_k}{a}\mathbb{E}[\mathsf{ALG}^0(I)]$ and $(\max_{i\in[n]} X_i^k = a^k) \iff (\max_{i\in[n]} X_i = a)$. It follows that

$$\begin{aligned}
\frac{\mathbb{E}[\mathsf{ALG}^{D_\infty}(I^k)]}{\mathbb{E}[\mathsf{OPT}(I^k)]} &= \frac{\mathbb{E}[\mathsf{ALG}^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} + \frac{D_\infty(s_k)}{\frac{s_k}{a}\mathbb{E}[\mathsf{OPT}(I)]}\Pr(\tau = n+1, \max_{i\in[n]} X_i = a) \\
&= \frac{\mathbb{E}[\mathsf{ALG}^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} + \left(\frac{D_\infty(s_k)}{s_k}\right)\frac{a}{\mathbb{E}[\mathsf{OPT}(I)]}\Pr(\tau = n+1, \max_{i\in[n]} X_i = a) \ .
\end{aligned}$$

Taking the limit $k \to \infty$ gives

$$\begin{aligned}
\mathsf{CR}^{D_\infty}(\mathsf{ALG}) &\leq \frac{\mathbb{E}[\mathsf{ALG}^0(I)] + \gamma a \Pr(\tau = n+1, \max_{i\in[n]} X_i = a)}{\mathbb{E}[\mathsf{OPT}(I)]} \\
&= \frac{\mathbb{E}[\mathsf{ALG}^0(I)] + \mathbb{E}[\gamma(\max_{i\in[n]} X_i)\mathbb{1}_{\tau=n+1}]}{\mathbb{E}[\mathsf{OPT}(I)]} \ .
\end{aligned}$$

ALG is also reasonable in the γ-prophet inequality. Therefore, using Lemma 3.1, we deduce that

$$\mathsf{CR}^{D_\infty}(\mathsf{ALG}) \leq \frac{\mathbb{E}[\mathsf{ALG}^\gamma(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} \leq \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^\gamma(I)]}{\mathbb{E}[\mathsf{OPT}(I)]}.$$

This upper bound is true for the optimal dynamic programming algorithm, since it rejects all zeros and accepts b, therefore the upper bound also holds for any other algorithm. $\square$

As a consequence of Proposition 3.3, if $\inf_{x>0} \frac{D_\infty(x)}{x} = \gamma$, then any upper bound obtained in the γ-prophet inequality (when the decay functions are all equal to $x \mapsto \gamma x$) using instances of random variables $(X_1, \ldots, X_n)$, such that $X_i \in \{0, a, b\}$ a.s. for all i, is also an upper bound in the D_∞-prophet inequality.

Implication Let us consider any sequence $\mathcal{D}$ of decay functions, and define

$$\gamma_{\mathcal{D}} := \inf_{x>0} \left\{ \frac{D_\infty(x)}{x} \right\} = \inf_{x>0} \inf_{j\geq 1} \left\{ \frac{D_j(x)}{x} \right\} .$$

For any $x > 0$ and $j \geq 1$ it holds that $D_j(x) \geq \gamma_{\mathcal{D}} x$, therefore, any guarantees on the competitive ratio of an algorithm in the $\gamma_{\mathcal{D}}$-prophet inequality are valid in the $\mathcal{D}$-prophet inequality, under any order model. Furthermore, combining Theorem 2.1 and Proposition 3.3, we obtain that for any instance I of random variables taking values in a set $\{0, a, b\}$ it holds that

$$\forall \mathsf{ALG} : \mathsf{CR}^{\mathcal{D}}(\mathsf{ALG}) \leq \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^{\gamma_{\mathcal{D}}}(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} ,$$

with an additional term of order $O(n^{-1/3})$ in the IID model. In the particular case where $\gamma_{\mathcal{D}} = 0$, Proposition 3.3 with Theorem 2.1 give a stronger result, showing that no algorithm can surpass the upper bounds of the classical prophet inequality. This is true also for the IID model since the instances used to prove the tight upper bound of ≈ 0.745 are of arbitrarily large size [HK82].

Therefore, by studying the γ-prophet inequality for $\gamma \in [0, 1]$, we can prove upper bounds and lower bounds on the $\mathcal{D}$-prophet inequality for any sequence $\mathcal{D}$ of decay functions.

4 The γ-prophet inequality

We study in this section the γ-prophet inequality, where all the decay functions are equal to $x \mapsto \gamma x$, for some $\gamma \in [0, 1]$. For any algorithm ALG with stopping time τ and random variables $X_1, \ldots, X_n$, if the observation order is π, we use the notation

$$\mathsf{ALG}^\gamma(X_1, \ldots, X_n) = \max\{X_{\pi(\tau)}, \gamma X_{\pi(\tau-1)}, \ldots, \gamma X_{\pi(1)}\} .$$

and we denote by $\mathsf{CR}^\gamma(\mathsf{ALG})$ the competitive ratio of any algorithm ALG in this setting. In the following, we provide theoretical guarantees for the γ-prophet inequality. For each observation order, we first derive upper bounds on the competitive ratio of any algorithm, depending on γ, and using only hard instances satisfying the condition in Proposition 3.2, and for the IID model these instances are of arbitrarily large size, which guarantees that they can be extended to the $\mathcal{D}$-prophet inequality with $\gamma = \gamma_{\mathcal{D}}$. For $\gamma = 0$, these bounds match standard results obtained in the literature, while for $\gamma = 1$ all ratios are equal to 1, since the decision-maker can see all of the values and then pick the maximum. Then, we lower bound the competitive ratio of carefully designed single-threshold algorithm. Furthermore, for $\gamma \in (0, 1)$ these results always strictly improve over the trivial lower bound γ (see Figure 1 for a summary).

Remark 4.1. *In the proofs of lower bounds, we will only consider instances with continuous distributions, and the thresholds θ considered are such that $\Pr(\max_{i\in[n]} X_i \geq \theta) = g(\gamma, n)$, where g is a constant depending on γ, the order model and the size of the instance. Such a threshold is always guaranteed to exist when the distributions are continuous. However, as in the prophet inequality, the algorithms can be easily adapted to non-continuous distributions by allowing stochastic tie-breaking. A detailed strategy for doing this can be found for example in [CSZ21].*

Generic bounds Before delving into the study of the different models, we provide generic lower and upper bounds, which depend solely on the bounds of the classical prophet inequality and γ.

Proposition 4.2. *In any order model, if α is a lower bound in the classical prophet inequality, and β an upper bound, then, in the γ-prophet inequality*

1. *there exists a trivial algorithm with a competitive ratio of at least* $\max\{\gamma, \alpha\}$*,*

2. *the competitive ratio of any algorithm is at most* $(1-\gamma)\beta + \gamma$*.*

Proof. For the lower bound, it suffices to consider the following trivial algorithm: if $\alpha > \gamma$ then run A_α, and if $\gamma > \alpha$ then observe all the items then select the one with maximum value.

For the upper bound, let $I = (F_1, \dots, F_n)$ be an instance of the problem and $X_i \sim F_i$ for all i, and let $\beta_I := \sup_{\mathsf{A}} \frac{\mathbb{E}[\mathsf{A}^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]}$. Let A be any algorithm, τ its stopping time, and $Y_\tau = \max_{i<\tau} X_{\pi(i)}$, where π is the observation order. With the previous notations, we can write that $\mathbb{E}[\mathsf{A}^\gamma(I)] = \mathbb{E}[\max(X_{\pi(\tau)}, \gamma Y_\tau)]$. For any $x, y \geq 0$, the application $\gamma \mapsto \max(x, \gamma y)$ is convex on $[0,1]$, hence it can be upper bounded by $(1-\gamma)x + \gamma \max(x,y)$. Therefore

$$\begin{aligned}
\mathbb{E}[\mathsf{A}^\gamma(I)] &\leq (1-\gamma)\mathbb{E}[X_{\pi(\tau)}] + \gamma\mathbb{E}[\max(X_{\pi(\tau)}, Y_\tau)] \\
&\leq (1-\gamma)\mathbb{E}[\mathsf{A}^0(I)] + \gamma\mathbb{E}[\mathsf{OPT}(I)] \\
&\leq \big((1-\gamma)\beta_I + \gamma\big)\mathbb{E}[\mathsf{OPT}(I)] \; .
\end{aligned}$$

Therefore, $\mathsf{CR}^\gamma(\mathsf{ALG}) \leq ((1-\gamma)\beta_I + \gamma)$. Taking the infimum over all the instances I gives the result. Indeed, if we denote A_* the optimal dynamic programming algorithm for the standard prophet inequality, then

$$\inf_I \beta_I = \inf_I \frac{\mathbb{E}[\mathsf{A}_*^0(I)]}{\mathbb{E}[\mathsf{OPT}(I)]} = \mathsf{CR}^0(\mathsf{A}_*) \leq \beta \; .$$

□

4.1 Adversarial order

We first consider the adversarial order model, and prove that the best achievable competitive ratio is $\frac{1}{2-\gamma}$. This result interpolates the competitive ratios of $1/2$ in the classical prophet inequality, and a competitive ratio of 1 when $\gamma = 1$. Then, we provide a single-threshold algorithm that matches this result, therefore fully solving the γ-prophet inequality in this setting.

Theorem 4.3. *In the adversarial order model, the competitive ratio of any algorithm for the γ-prophet inequality is at most $\frac{1}{2-\gamma}$. Furthermore, there exists a single threshold algorithm with a competitive ratio of at least $\frac{1}{2-\gamma}$: given any instance $(F_1, \dots, F_n)$, this result is achieved with the threshold θ satisfying $\Pr_{X_1 \sim F_1, \dots, X_n \sim F_n}(\max_{i \in [n]} X_i \leq \theta) = \frac{1}{2-\gamma}$.*

Proof. We first prove the upper bound, and then analyze the single threshold algorithm proposed in the theorem.

Upper bound Let $\varepsilon \in (0, 1-\gamma)$, and let $a = \frac{1}{1-(1-\varepsilon)\gamma}$ (such that $1 + (1-\varepsilon)\gamma a = a$). Let X_1, X_2 the two random variables defined by $X_1 = a$ almost surely and

$$X_2 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1-\varepsilon \end{cases} \; .$$

Stopping at the first step gives a reward of a, while stopping at the second step gives

$$\mathbb{E}[\max(\gamma a, X_2)] = \varepsilon \times \frac{1}{\varepsilon} + (1-\varepsilon) \times \gamma a = 1 + (1-\varepsilon)\gamma a = a \; ,$$

hence the expected output of any algorithm is exactly equal to a. On the other hand

$$\mathbb{E}[\max(X_1, X_2)] = 1 + (1-\varepsilon)a \; ,$$

and we deduce that, for any algorithm ALG for the γ-prophet inequality, we have

$$\mathsf{CR}(\mathsf{ALG}) \leq \frac{\mathbb{E}[\mathsf{ALG}(X_1, X_2)]}{\mathbb{E}[\max(X_1, X_2)]} = \frac{a}{1 + (1-\varepsilon)a} ,$$

and this is true for any $\varepsilon \in (0, 1-\gamma)$, thus taking $\varepsilon \to 0$ gives

$$\mathsf{CR}(\mathsf{ALG}) \leq \frac{\frac{1}{1-\gamma}}{1 + \frac{1}{1-\gamma}} = \frac{1}{2-\gamma} .$$

Lower bound Consider an algorithm with an acceptance threshold θ, i.e. that accepts the first value larger than θ. Let $I = (F_1, \dots, F_n)$ be any instance, such that $X_i \sim F_i$ for all i, and let us define $X^* = \max_{i \in [n]} X_i$ and $p = \Pr(X^* \leq \theta)$. In the classical prophet inequality, if no value is larger than θ then the reward of the algorithm is zero, and we have the classical lower bound

$$\mathbb{E}[\mathsf{ALG}^0(I)] \geq (1-p)\theta + p\mathbb{E}[(X^* - \theta)_+],$$

For the γ-prophet, if no value is larger than θ (i.e $X^* \leq \theta$), then the algorithm gains γX^* instead of 0. Therefore, it holds that

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^\gamma(I)] &= \mathbb{E}[\mathsf{ALG}^0(I)] + \mathbb{E}[X^* \mathbb{1}_{X^* \leq \theta}] \\
&\geq (1-p)\theta + p\mathbb{E}[(X^* - \theta)_+] + \gamma \mathbb{E}[X^* \mathbb{1}_{X^* \leq \theta}] \\
&= (1-p)\theta + p\mathbb{E}[(X^* - \theta)\mathbb{1}_{X^* > \theta}] + \gamma \mathbb{E}[X^* \mathbb{1}_{X^* \leq \theta}] \\
&= (1-p)\theta + p(\mathbb{E}[X^* \mathbb{1}_{X^* > \theta}] - (1-p)\theta) + \gamma \mathbb{E}[X^* \mathbb{1}_{X^* \leq \theta}] \\
&= (1-p)^2\theta + p\mathbb{E}[X^* \mathbb{1}_{X^* > \theta}] + \gamma \mathbb{E}[X^* \mathbb{1}_{X^* \leq \theta}] ,
\end{aligned}$$

and observing that

$$\theta = \frac{\mathbb{E}[\theta \mathbb{1}_{X^* \leq \theta}]}{p} \geq \frac{\mathbb{E}[X^* \mathbb{1}_{X^* \leq \theta}]}{p} ,$$

we deduce the lower bound

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^\gamma(I)] &\geq p\mathbb{E}[X^* \mathbb{1}_{X^* > \theta}] + \left(\gamma + \frac{(1-p)^2}{p}\right) \mathbb{E}[X^* \mathbb{1}_{X^* \leq \theta}] \\
&\geq \min\left\{p, \gamma + \frac{(1-p)^2}{p}\right\} \mathbb{E}[X^*] .
\end{aligned}$$

The right term is maximized for p satisfying $p = \gamma + \frac{(1-p)^2}{p}$, that leads to

$$\begin{aligned}
p = \gamma + \frac{(1-p)^2}{p} &\iff p^2 = \gamma p + 1 - 2p + p^2 \\
&\iff p = \frac{1}{2-\gamma} .
\end{aligned}$$

Hence, by choosing a threshold θ satisfying $\Pr(X^* \leq \theta) = \frac{1}{2-\gamma}$ we obtain a competitive ratio of at least $\frac{1}{2-\gamma}$. □

4.2 Random order

Consider now that the items are observed in a uniformly random order $X_{\pi(1)}, \ldots, X_{\pi(n)}$, and $X^* = \max_{i\in[n]} X_i$. As for the adversarial model, we first prove an upper bound on the competitive ratio as a function of γ, and then prove a lower bound for a single-threshold algorithm. However, for this model we observe a gap between the two bounds, as illustrated in Figure 1.

We first prove an upper bound that depends on γ, matching the upper bound $\sqrt{3}-1$ [CSZ21] when $\gamma = 0$ and equal to 1 when $\gamma = 1$. While for the lower bound, we give a single-threshold algorithm with a competitive ratio of at least $(1-\frac{1}{e})$ when $\gamma = 0$, which is the competitive ratio of the best single threshold algorithm in the classical prophet inequality [EHLM17, CSZ21], and equal to 1 for $\gamma = 1$.

Theorem 4.4. *The competitive ratio of any algorithm ALG in the γ-prophet inequality with random order satisfies*

$$\mathit{CR}(\mathit{ALG}) \leq (1-\gamma)^{3/2}(\sqrt{3-\gamma} - \sqrt{1-\gamma}) + \gamma \ .$$

Furthermore, denoting by p_γ is the unique solution to the equation $1-(1-\gamma)p = \frac{1-p}{-\log p}$, the single-threshold algorithm ALG_θ with $\Pr_{X_1\sim F_1,\ldots,X_n\sim F_n}(\max_{i\in[n]} X_i \leq \theta) = p_\gamma$ satisfies

$$\mathit{CR}^\gamma(\mathit{ALG}) \geq 1-(1-\gamma)p_\gamma \ .$$

Proof. We first prove the upper bound, and then derive the analysis for single threshold algorithms.

Upper bound Let $a > 0$, and let $X_1, \ldots, X_{n+1}$ be independent random variables such that $X_{n+1} = a$ a.s. and for $1 \leq i \leq n$

$$X_i \sim \begin{cases} n & \text{w.p. } \frac{1}{n^2} \\ 0 & \text{w.p. } 1-\frac{1}{n^2} \end{cases} .$$

Any reasonable algorithm skips zero values and stops when observing the value n. The only strategic decision to make is thus to stop or not when observing $X_{n+1} = a$. By analyzing the dynamic programming solution $\mathsf{ALG}_\star$ we obtain that the optimal decision rule is to skip a if it is observed before a certain step j, and select it otherwise. The step j corresponds to the time when the expectation of the future reward is less than a. Let π be the random order in which the variables are observed. Then, if $\pi^{-1}(n+1) < j$, i.e. if the value a is observed before time j, $X_{n=1}$ is not selected. The output of this algorithm is hence n if at least one random variable equals n, and γa otherwise. This leads to

$$\begin{aligned}\mathbb{E}[\mathsf{ALG}_\star^\gamma(X) \mid \pi^{-1}(n+1) < j] &= n\left(1-\left(1-\frac{1}{n^2}\right)^n\right) + \gamma a\left(1-\frac{1}{n^2}\right)^n \\ &\leq 1+\gamma a \ ,\end{aligned}$$

where we used the inequality $\left(1-\frac{1}{n^2}\right)^n \geq 1-\frac{1}{n}$. On the other hand, if $\pi^{-1}(n) \geq j$, then a is selected if the value n has not been observed before it, hence for any $i \geq j$ we have

$$\begin{aligned}\mathbb{E}[\mathsf{ALG}_\star^\gamma(X) \mid \pi^{-1}(n+1) = i] &= n\left(1-\left(1-\frac{1}{n^2}\right)^{i-1}\right) + a\left(1-\frac{1}{n^2}\right)^{i-1} \\ &\leq \frac{i-1}{n} + a \ ,\end{aligned}$$

we deduce that

$$
\begin{aligned}
\mathbb{E}[\mathsf{ALG}^\gamma_\star(X)] &\leq (1+\gamma a)\Pr(\pi^{-1}(n) \leq j-1) + \sum_{i=j}^{n+1}\left(\frac{i-1}{n} + a\right)\Pr(\pi^{-1}(n) = i)\\
&= \frac{j-1}{n+1}(1+\gamma a) + \frac{1}{n+1}\sum_{i=j}^{n+1}\left(\frac{i-1}{n} + a\right)\\
&= (1-(1-\gamma)a)\frac{j}{n} - \frac{1}{2}\left(\frac{j}{n}\right)^2 + \frac{1}{2} + a + o(1)\\
&\leq 1 + 2\gamma a + (1-\gamma)^2 a^2 + o(1)\ ,
\end{aligned}
$$

where the last inequality is obtained by maximizing over j/n. Finally, we directly obtain that

$$
\begin{aligned}
\mathbb{E}[\max_i X_i] &= n\left(1 - \left(1 - \frac{1}{n^2}\right)^n\right) + a\left(1 - \frac{1}{n^2}\right)^n\\
&= 1 + a + o(1)\ ,
\end{aligned}
$$

so for any algorithm ALG we obtain that

$$
\mathsf{CR}^\gamma(\mathsf{ALG}) \leq \mathsf{CR}^\gamma(\mathsf{ALG}_\star) \leq \frac{1 + 2\gamma a + (1-\gamma)^2 a^2}{1+a}\ .
$$

The function above is minimized for $a = \sqrt{\frac{3-\gamma}{1-\gamma}} - 1$, which translates to

$$
\mathsf{CR}^\gamma(\mathsf{ALG}) \leq (1-\gamma)^{3/2}(\sqrt{3-\gamma} - \sqrt{1-\gamma}) + \gamma\ .
$$

Lower bound We still denote by $I = (F_1, \dots, F_n)$ the input instance and $X_i \sim F_i$ for all $i \in [n]$. Let ALG be the algorithm with single threshold θ, then it is direct that

$$
\mathsf{ALG}^\gamma(I) = \mathsf{ALG}^0(I) + \gamma X^* \mathbb{1}_{X^* < \theta}\ . \tag{16}
$$

We start by giving a lower bound on $\mathbb{E}[\mathsf{ALG}^0]$. We use from [CSZ21] (Theorem 2.1) that for any $x < \theta$ it holds that

$$
\Pr(\mathsf{ALG}^0(I) \geq x) = \Pr(\mathsf{ALG}^0(I) \geq \theta) = \Pr(X^* \geq \theta) = 1 - p\ ,
$$

and for $x \geq \theta$ it holds that

$$
\Pr(\mathsf{ALG}^0(I) \geq x) \geq \frac{1-p}{-\log p}\Pr(X^* \geq x)\ ,
$$

from which we deduce that

$$
\begin{aligned}
\mathbb{E}[\mathsf{ALG}^0(I)] &= \int_0^\infty \Pr(\mathsf{ALG}^0(I) \geq x)dx\\
&\geq (1-p)\theta + \frac{1-p}{-\log p}\int_\theta^\infty \Pr(X^* \geq x)dx\ .
\end{aligned} \tag{17}
$$

On the other hand, we obtain that

$$
\begin{aligned}
\mathbb{E}[X^*\mathbb{1}_{X^*<\theta}] &= \int_0^\infty \Pr(X^*\mathbb{1}_{X^*<\theta} \geq x)dx = \int_0^\infty \Pr(x \leq X^* < \theta)dx \\
&= \int_0^\theta (\Pr(X^* > x) - \Pr(X^* \geq \theta))dx \\
&= \int_0^\theta \Pr(X^* > x)dx - (1-p)\theta\ . \qquad (18)
\end{aligned}
$$

Using (16), (17) and (18) we deduce that

$$
\begin{aligned}
\mathbb{E}[\mathsf{ALG}^\gamma(I)] &\geq (1-p)\theta + \frac{1-p}{-\log p}\int_\theta^\infty \Pr(X^* \geq x)dx + \gamma\int_0^\theta \Pr(X^* \geq x)dx - \gamma(1-p)\theta \\
&= (1-\gamma)(1-p)\theta + \gamma\int_0^\theta \Pr(X^* \geq x)dx + \frac{1-p}{-\log p}\int_\theta^\infty \Pr(X^* \geq x)dx \\
&\geq ((1-\gamma)(1-p)+\gamma)\int_0^\theta \Pr(X^* \geq x)dx + \frac{1-p}{-\log p}\int_\theta^\infty \Pr(X^* \geq x)dx \\
&\geq \min\left\{(1-\gamma)(1-p)+\gamma\,,\ \frac{1-p}{-\log p}\right\}\left(\int_0^\theta \Pr(X^* \geq x)dx + \int_\theta^\infty \Pr(X^* \geq x)dx\right) \\
&= \min\left\{1-(1-\gamma)p\,,\ \frac{1-p}{-\log p}\right\}\mathbb{E}[X^*]\ .
\end{aligned}
$$

Finally, choosing $p = p_\gamma$ gives the result. □

Before moving to the IID case, we propose in the following a more explicit lower bound based on Theorem 4.4. While the equation defining p_γ cannot be solved analytically, the solution can easily be computed numerically for any $\gamma \in [0,1]$. The following corollary is obtained by considering a threshold close to the optimal one for any γ.

Corollary 4.4.1. *In the random order model, a single threshold algorithm with a threshold θ satisfying $\Pr(\max_{i\in[n]} \geq \theta) = \frac{1/e}{1-(1-1/e)\gamma}$ has a competitive ratio of at least $1 - \frac{(1-\gamma)/e}{1-(1-1/e)\gamma}$.*

Proof. For $p = \frac{1/e}{1-(1-1/e)\gamma}$, we have immediately that

$$
1-(1-\gamma)p = 1 - \frac{(1-\gamma)/e}{1-(1-1/e)\gamma}\ ,
$$

and $p \in [1/e, 1]$ for any $\gamma \in [0,1]$. Since the function $x \mapsto (1-x)/\log(1/x)$ is concave, we can lower bound it on $[1/e, 1]$ by $x \mapsto 1 - 1/e + \frac{x-1/e}{e-1}$, which is the line intersecting it in $1/e$ and 1. Therefore we have

$$
\frac{1-p}{-\log p} \geq 1 - 1/e + \frac{p-1/e}{e-1} = 1 - \frac{(1-\gamma)/e}{1-(1-1/e)\gamma}\ .
$$

Finally, using the previous theorem, this choice of p guarantees a competitive ratio of at least

$$
\min\left\{1-(1-\gamma)p\,,\ \frac{1-p}{-\log p}\right\} = 1 - \frac{(1-\gamma)/e}{1-(1-1/e)\gamma}\ .
$$

□

4.3 IID Random Variables

In this section, we analyze the case where the decision-maker faces a sequence of IID random variables. In the classical IID prophet inequality, [HK82] showed that the competitive ratio of any algorithm is at most ≈ 0.745. The proof of this upper bound is hard to generalize for the IID γ-prophet inequality. As an alternative, we prove a slightly weaker upper bound, which equals ≈ 0.778 for $\gamma = 0$ and 1 for $\gamma = 1$, and to prove it we use instances of arbitrarily large size satisfying the condition of Proposition 3.2. Then we present a single-threshold algorithm with the same competitive ratio as the random order algorithm. However, the proof is different, leveraging the fact that the variables are identically distributed. More precisely, we introduce a single-threshold algorithm with guarantees that depends on the size n of the instance, then we show that its competitive ratio is at least that of the algorithm presented in Theorem 4.4, with equality when n approaches infinity. While the IID model is considered to be conceptually simpler compared to the random order model, it is not surprising to see that employing single-threshold algorithms does not lead to enhanced performance in the IID case. Indeed, [LWW22] established that no single-threshold algorithm can achieve a competitive ratio better than $1 - 1/e$ in the standard prophet inequality with IID random variables, which is also achieved with a single-threshold algorithm for the random order model.

In the following, we denote by $\mathbb{E}[\mathsf{ALG}^\gamma(F, n)]$ the expected output of any algorithm ALG when given instance of n IID random variables with distribution F, and we denote by $\mathbb{E}[\mathsf{OPT}(F, n)]$ their expected maximum.

Theorem 4.5. *The competitive ratio of any algorithm in the IID γ-prophet inequality is at most*

$$U(\gamma) = 1 - (1-\gamma)\frac{e^2 \log(3-\gamma) - (2-\gamma)}{2(2e^2-1) - (3e^2-1)\gamma} .$$

In particular, U is increasing, $U(0) = \frac{4-\log 3}{2(2-\frac{1}{e}^2)} \approx 0.778$ and $U(1) = 1$. Furthermore, there exists a single-threshold algorithm ALG_θ satisfying

$$\mathsf{CR}^\gamma(\mathsf{ALG}_\theta) \geq 1 - (1-\gamma)p_\gamma ,$$

where p_γ is defined in Theorem 4.4.

Proof. We first prove the upper bound, and then we give the single-threshold algorithm satisfying the lower bound.

Upper bound We consider an instance similar to the one used in the proof of Theorem 4.4. Let $a, x > 0$, and let $X_1, \ldots, X_n$ be IID random variables with he following the distribution F defined by

$$X_1 \sim \begin{cases} n & \text{w.p. } \frac{1}{n^2} \\ a & \text{w.p. } \frac{x}{n} \\ 0 & \text{w.p. } 1 - \frac{x}{n} - \frac{1}{n^2} \end{cases} .$$

A reasonable algorithm would always reject the value 0 and accept the value n. However, if the algorithm faces an item with value a, it must decide to either accept it, or reject it with a guarantee of recovering γa at the end. By analyzing the dynamic programming algorithm $\mathsf{ALG}_\star$, we find that the optimal decision is to reject a if observed before a certain step j, and accept it otherwise. Let us denote τ the stopping time of $\mathsf{ALG}_\star$. By convention, we write $\tau = n + 1$ to say that no value was selected by the algorithm, in which case the reward is $\gamma \max_{i \in [n]} X_i$.

If $\tau \le j-1$ then necessarily $X_\tau = n$, because $\mathsf{ALG}_\star$ rejects the value a if it is met before step j, and if $\tau = n+1$ then $\max_{i\in[n]} X_i \in \{0, a\}$, because otherwise the algorithm would have selected the value n and stopped earlier. It follows that the expected output of $\mathsf{ALG}_\star$ on this instance is

$$\begin{aligned}\mathbb{E}[\mathsf{ALG}_\star^\gamma(F,n)] =& n\Pr(\tau < j) + \sum_{i=j}^{n} \mathbb{E}[X_i \mid \tau = i] \Pr(\tau = i)\\ &+ \gamma a \Pr(\tau = n+1, \max_{i\in[n]} X_i = a) \ . \end{aligned} \tag{19}$$

Let us now compute the terms above one by one.

$$\Pr(\tau < j) = \Pr(\exists i \in [j-1] : X_i = n) = 1 - \left(1 - \frac{1}{n^2}\right)^{j-1} \le \frac{j}{n^2} \ ,$$

where we used Bernoulli's inequality $(1 - 1/n^2)^{j-1} \ge 1 - \frac{j-1}{n^2} > 1 - \frac{j}{n^2}$. For $i \in \{j, \dots, n\}$, $\mathsf{ALG}_\star$ stops at i if $X_i \in \{a, n\}$ and if it has not stopped before, i.e. $X_k \in \{0, a\}$ for all $k < j$ and $X_k = 0$ for all $k \in \{j, \dots, i-1\}$, hence

$$\begin{aligned}\Pr(\tau = i) &= \Pr(\forall k < j : X_k \ne n \text{ and } \forall j \le k \le i-1 : X_k = 0 \text{ and } X_i \ne 0)\\ &= \left(1 - \frac{1}{n^2}\right)^{j-1} \left(1 - \frac{x}{n} - \frac{1}{n^2}\right)^{i-j} \Pr(X_i \ne 0)\\ &\le \left(1 - \frac{x}{n}\right)^{i-j} \Pr(X_i \ne 0) \ ,\end{aligned}$$

the second equality is true by independence, and the last inequality holds because $1 - \frac{1}{n^2} \le 1$ and $1 - \frac{x}{n} - \frac{1}{n^2} \le 1 - \frac{x}{n}$. By independence of the variables $(X_k)_k$, we also have that

$$\mathbb{E}[X_i \mid \tau = i] = \mathbb{E}[X_i \mid X_i \ne 0] = \frac{\mathbb{E}[X_i]}{\Pr(X_i \ne 0)} = \frac{1 + ax}{n \Pr(X_i \ne 0)} \ .$$

Finally, the event $(\tau = n+1, \max_{i\in[n]} X_i = a)$ is equivalent $(\max_{i\in[j-1]} X_i = a, \forall k \ge j : X_k = 0)$. In fact, the algorithm does not stop before $n+1$ if and only if $X_k \ne n$ for all $k < j$ and $X_k = 0$ for all $j \le k \le n$, and under these conditions, it holds that $\max_{i\in[n]} X_i = \max_{i\in[j-1]} X_i$. Therefore

$$\begin{aligned}\Pr(\tau = n+1, \max_{i\in[n]} X_i = a) &= \Pr(\max_{i\in[j-1]} X_i = a, \forall k \ge j : X_k = 0)\\ &\le \Pr(\max_{i\in[j-1]} X_i \ne 0) \Pr(\forall k \ge j : X_k = 0)\\ &= \left(1 - \left(1 - \frac{x}{n} - \frac{1}{n^2}\right)^{j-1}\right) \left(1 - \frac{x}{n} - \frac{1}{n^2}\right)^{n-j}\\ &= \left(1 - e^{-\frac{xj}{n}} + o(1)\right)\left(e^{-x + \frac{xj}{n}} + o(1)\right)\\ &= \left(e^{\frac{xj}{n}} - 1\right)e^{-x} + o(1) \ .\end{aligned}$$

All in all, we obtain by substituting into 19 that

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\gamma}_{\star}(F,n)] &\leq \frac{j}{n} + \left(\frac{1+ax}{n}\right)\sum_{i=j}^{n}\left(1-\frac{x}{n}\right)^{i-j} + \gamma a e^{-x}\left(e^{\frac{xj}{n}}-1\right) + o(1)\\
&= \frac{j}{n} + \left(\frac{1+ax}{n}\right)\frac{1-(1-x/n)^{n-j+1}}{x/n} + \gamma a e^{-x}\left(e^{\frac{xj}{n}}-1\right) + o(1)\\
&= \frac{j}{n} + \left(\frac{1}{x}+a\right)\left(1-e^{-x+\frac{xj}{n}}+o(1)\right) + \gamma a e^{-x}\left(e^{\frac{xj}{n}}-1\right) + o(1)\\
&= \frac{j}{n} - \Big[\big(\tfrac{1}{x}+(1-\gamma)a\big)\Big]e^{\frac{xj}{n}} + \frac{1}{x} + a - \gamma a e^{-x} + o(1)\\
&\leq \max_{s>0}\left\{s - \Big[\big(\tfrac{1}{x}+(1-\gamma)a\big)\Big]e^{xs}\right\} + \frac{1}{x} + (1-\gamma e^{-x})a + o(1)\\
&= -\frac{1}{x}\big(\log(1+(1-\gamma)ax)+1-x\big) + \frac{1}{x} + (1-\gamma e^{-x})a + o(1)\\
&= -\frac{1}{x}\log(1+(1-\gamma)ax) + 1 + (1-\gamma e^{-x})a + o(1)\ .
\end{aligned}$$

On the other hand, we have that

$$\Pr(\max_{i\in[n]} X_i = n) = 1-\left(1-\frac{1}{n^2}\right)^n = \frac{1}{n}+o(1/n)\ ,$$

$$\Pr(\max_{i\in[n]} X_i = 0) = \left(1-\frac{x}{n}-\frac{1}{n^2}\right) = e^{-x}+o(1)\ ,$$

$$\Pr(\max_{i\in[n]} X_i = a) = 1-\Pr(\max_{i\in[n]} X_i = 0)-\Pr(\max_{i\in[n]} X_i = n) = 1-e^{-x}+o(1)\ ,$$

therefore, the expected maximum value is

$$\begin{aligned}
\mathbb{E}[\max_{i\in[n]} X_i] &= n\Pr(\max_{i\in[n]} X_i = n) + a\Pr(\max_{i\in[n]} X_i = a)\\
&= 1+\big(1-e^{-x}\big)a + o(1)\ .
\end{aligned}$$

We deduce that

$$\begin{aligned}
\frac{\mathbb{E}[\mathsf{ALG}^{\gamma}_{\star}(F,n)]}{\mathbb{E}[\max_{i\in[n]} X_i]} &\leq \frac{-\frac{1}{x}\log(1+(1-\gamma)ax)+1+(1-\gamma e^{-x})a}{1+\big(1-e^{-x}\big)a} + o(1)\\
&= 1-\frac{\frac{1}{x}\log(1+(1-\gamma)ax)-(1-\gamma)ae^{-x}}{1+\big(1-e^{-x}\big)a} + o(1)\ .
\end{aligned}$$

Consequently, for any $a, x > 0$ and for any algorithm ALG we have

$$\begin{aligned}
\mathsf{CR}(\mathsf{ALG}) &\leq \mathsf{CR}(\mathsf{ALG}_{\star})\\
&\leq \lim_{n\to\infty}\frac{\mathbb{E}[\mathsf{ALG}^{\gamma}_{\star}(F,n)]}{\mathbb{E}[\max_{i\in[n]} X_i]}\\
&\leq 1-\frac{\log(1+(1-\gamma)ax)-(1-\gamma)axe^{-x}}{x+\big(1-e^{-x}\big)ax}\ .
\end{aligned}$$

In particular, for $x = 2$ and $a = \frac{1-\gamma/2}{1-\gamma}$ we find that

$$\begin{aligned}
\mathsf{CR}(\mathsf{ALG}) &\leq 1 - \frac{\log(3-\gamma) - (2-\gamma)e^{-2}}{2 + \frac{2-\gamma}{1-\gamma}(1-e^{-2})} \\
&= 1 - (1-\gamma)\frac{e^2\log(3-\gamma) - (2-\gamma)}{2(2e^2-1) - \gamma(3e^2-1)} \\
&= U(\gamma) \ .
\end{aligned}$$

This proves the upper bound stated in the theorem, and we can verify that it is increasing, and satisfies $U(0) = \frac{4-\log 3}{4-2/e^2}$ $U(1) = 1$

Lower bound depending on n We describe the algorithm and give a first lower bound on its reward depending on the size of the instance in the following lemma.

Lemma 4.6. *Let $a_{n,\gamma}$ be the unique solution of the equation $\left(\frac{1}{(1-a/n)^n} - 1\right)\left(\frac{1}{a} - 1\right) = \gamma$, then for any IID instance $X_1, \ldots, X_n$, the reward of the algorithm with single threshold θ satisfying $\Pr(X_1 > \theta) = \frac{a_{n,\gamma}}{n}$ is at least $\frac{1}{a_{n,\gamma}}\big(1 - (1 - \frac{a_{n,\gamma}}{n})^n\big)\mathbb{E}[\max_{i\in[n]} X_i]$.*

Proof of Lemma 4.6. Let F be the cumulative distribution function of X_1, $a > 0$ and ALG the algorithm with single threshold θ such that $1 - F(\theta) = \frac{a}{n}$. We denote $X^* = \max_{i\in[n]} X_i$. As in the previous proofs, we will begin by lower bounding $\mathsf{ALG}^0(F, n)$. For any $i \in [n]$, ALG stops at step i if and only if $X_i > \theta$ and all the previous items were rejected, i.e. $X_j \leq \theta$ for all $j < i$. Thus we can write

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^0(F, n)] &= \mathbb{E}\Big[\sum_{i=1}^n (X_i \mathbb{1}_{X_i > \theta})\mathbb{1}_{\forall j<i: X_j \leq \theta}\Big] \\
&= \sum_{i=1}^n F(\theta)^{i-1}\mathbb{E}[X_i \mathbb{1}_{X_i>\theta}] \\
&= \frac{1 - F(\theta)^n}{1 - F(\theta)}\mathbb{E}[X_1 \mathbb{1}_{X_1>\theta}] \\
&= \frac{1 - F(\theta)^n}{a} \times n\mathbb{E}[X_1 \mathbb{1}_{X_1>\theta}] \ ,
\end{aligned} \tag{20}$$

where the second equality is true by the independence of the random variables $(X_i)_i$. On the other hand, we can upper bound $\mathbb{E}[X^* \mathbb{1}_{X^*>\theta}]$ as follows

$$\begin{aligned}
\mathbb{E}[X^* \mathbb{1}_{X^*>\theta}] &\leq \Pr(X^* > \theta)\theta + \mathbb{E}[(X^* - \theta)_+] \\
&\leq \Pr(X^* > \theta)\theta + \mathbb{E}\Big[\sum_{i=1}^n (X_i - \theta)_+\Big] \\
&= \Pr(X^* > \theta)\theta + n\big(\mathbb{E}[X_1 \mathbb{1}_{X_1>\theta}] - \Pr(X_1 > \theta)\theta\big) \\
&= (1 - F(\theta)^n - a)\theta + n\mathbb{E}[X_1 \mathbb{1}_{X_1>\theta}] \ .
\end{aligned}$$

Using the definition of θ, the independence of $(X_i)_i$ then Bernoulli's inequality we have that

$$\Pr(X^* > \theta) = 1 - F(\theta)^n = 1 - \Big(1 - \frac{a}{n}\Big)^n \leq 1 - (1 - n \times \frac{a}{n}) = a \ ,$$

and observing that $\theta = \frac{\mathbb{E}[\theta \mathbb{1}_{X^* \le \theta}]}{F(\theta)^n} \ge \frac{\mathbb{E}[X^* \mathbb{1}_{X^* \le \theta}]}{F(\theta)^n}$, we deduce that

$$\mathbb{E}[X^* \mathbb{1}_{X^* > \theta}] \le -\left(1 - \frac{1-a}{F(\theta)^n}\right) \mathbb{E}[X^* \mathbb{1}_{X^* \le \theta}] + n\mathbb{E}[X_1 \mathbb{1}_{X_1 > \theta}] \ .$$

by substituting into (20), we obtain

$$\mathbb{E}[\mathsf{ALG}^0(F,n)] \ge \frac{1-F(\theta)^n}{a} \left(\mathbb{E}[X^* \mathbb{1}_{X^* > \theta}] + \left(1 - \frac{1-a}{F(\theta)^n}\right) \mathbb{E}[X^* \mathbb{1}_{X^* \le \theta}] \right) \ .$$

Finally, the reward in the γ-prophet inequality is

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^\gamma(F,n)] &= \mathbb{E}[\mathsf{ALG}^0(F,n)] + \gamma \mathbb{E}[X^* \mathbb{1}_{X^* < \theta}] \\
&\ge \frac{1-F(\theta)^n}{a} \mathbb{E}[X^* \mathbb{1}_{X^* > \theta}] + \left(\frac{1-F(\theta)^n}{a} \left(1 - \frac{1-a}{F(\theta)^n}\right) + \gamma \right) \mathbb{E}[X^* \mathbb{1}_{X^* < \theta}] \\
&\ge \min\left\{ \frac{1-F(\theta)^n}{a}, \frac{1-F(\theta)^n}{a} \left(1 - \frac{1-a}{F(\theta)^n}\right) + \gamma \right\} \mathbb{E}[X^*] \ .
\end{aligned}$$

The equation $\frac{1-F(\theta)^n}{a} = \frac{1-F(\theta)^n}{a}\left(1 - \frac{1-a}{F(\theta)^n}\right) + \gamma$, is equivalent to

$$\left(\frac{1}{(1-a/n)^n} - 1 \right) \left(\frac{1}{a} - 1 \right) = \gamma \ , \tag{21}$$

and for any $n \ge 2$ the function $a \mapsto \left(\frac{1}{(1-a/n)^n} - 1\right)\left(\frac{1}{a} - 1\right)$ is decreasing on $(0, 1]$ and goes from 1 to 0, thus Equation (21) admits a unique solution $a_{n,\gamma}$, and taking $a = a_{n,\gamma}$ guarantees a reward of $\frac{1-F(\theta)^n}{a_{n,\gamma}} \mathbb{E}[X^*] = \frac{1-(1-\frac{a_{n,\gamma}}{n})^n}{a_{n,\gamma}} \mathbb{E}[X^*]$. □

Lower bound on the competitive ratio We will prove that the previous algorithm has a competitive ratio of at least $(1-(1-\gamma)p_\gamma)$, where p_γ, first introduced in Theorem 4.4, is the unique solution of the equation $(1-(1-\gamma)p) = \frac{1-p}{-\log p}$, which is equivalent to $\left(\frac{1}{p} - 1\right)\left(\frac{1}{\log(1/p)} - 1\right) = \gamma$.

Let $a_\gamma = -\log(p_\gamma)$. It follows from the definition of p_γ that a_γ is the unique solution of the equation $(e^a - 1)(\frac{1}{a} - 1) = \gamma$. For any $n \ge 2$ and $x \ge 0$ we have that $(1-x/n)^n \le e^{-x}$, hence, by definition of $a_{n,\gamma}$ and a_γ

$$\begin{aligned}
\left(\frac{1}{e^{-a_{n,\gamma}}} - 1 \right) \left(\frac{1}{a_{n,\gamma}} - 1 \right) &\le \left(\frac{1}{(1-\frac{a_{n,\gamma}}{n})^n} - 1 \right) \left(\frac{1}{a_{n,\gamma}} - 1 \right) \\
&= \gamma \\
&= \left(\frac{1}{e^{-a_\gamma}} - 1 \right) \left(\frac{1}{a_\gamma} - 1 \right) \ .
\end{aligned} \tag{22}$$

Moreover, the function $x \mapsto (e^x - 1)(1/x - 1)$ is decreasing on $(0, 1)$. In fact its derivative at any point $x \in (0, 1)$ is

$$\begin{aligned}
\frac{d}{dx}\left[\left(\frac{1}{e^{-x}} - 1 \right) \left(\frac{1}{x} - 1 \right) \right] &= \left(\frac{1}{x} - 1 \right) e^x - \frac{e^x - 1}{x^2} \\
&= \frac{1}{x^2}\left(1 - x^2 - (1-x)e^x\right) \\
&= \frac{1-x}{x^2}(1 + x - e^x) < 0 \ .
\end{aligned}$$

It follows from (22) that $a_\gamma \le a_{n,\gamma}$. Finally, given that $x \mapsto \frac{1-e^{-x}}{x}$ is non-increasing on $(0,1]$, we deduce that

$$\frac{1-(1-\frac{a_{n,\gamma}}{n})^n}{a_{n,\gamma}} \ge \frac{1-e^{-a_{n,\gamma}}}{a_{n,\gamma}} \ge \frac{1-e^{-a_\gamma}}{a_\gamma} .$$

We deduce that the competitive ratio of the algorithm described in Theorem 4.5 is at least $\frac{1-e^{-a_\gamma}}{a_\gamma} = \frac{1-p_\gamma}{\log(1/p_\gamma)} = 1-(1-\gamma)p_\gamma$.

□

References

[ACK18] Yossi Azar, Ashish Chiplunkar, and Haim Kaplan. Prophet secretary: Surpassing the 1-1/e barrier. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 303–318, 2018.

[AK22] Makis Arsenis and Robert Kleinberg. Individual fairness in prophet inequalities. In *Proceedings of the 23rd ACM Conference on Economics and Computation*, pages 245–245, 2022.

[Ala14] Saeed Alaei. Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers. *SIAM Journal on Computing*, 43(2):930–972, 2014.

[ASC00] David Assaf and Ester Samuel-Cahn. Simple ratio prophet inequalities for a mortal with multiple choices. *Journal of applied probability*, 37(4):1084–1091, 2000.

[BC23] Archit Bubna and Ashish Chiplunkar. Prophet inequality: Order selection beats random order. In *Proceedings of the 24th ACM Conference on Economics and Computation*, pages 302–336, 2023.

[BEY05] Allan Borodin and Ran El-Yaniv. *Online computation and competitive analysis*. cambridge university press, 2005.

[BGL+21] Hedyeh Beyhaghi, Negin Golrezaei, Renato Paes Leme, Martin Pál, and Balasubramanian Sivan. Improved revenue bounds for posted-price and second-price mechanisms. *Operations Research*, 69(6):1805–1822, 2021.

[CCDNF21] Jose Correa, Andres Cristi, Paul Duetting, and Ashkan Norouzi-Fard. Fairness and bias in online selection. In *International conference on machine learning*, pages 2112–2121. PMLR, 2021.

[CFH+19] Jose Correa, Patricio Foncea, Ruben Hoeksma, Tim Oosterwijk, and Tjark Vredeveld. Recent developments in prophet inequalities. *ACM SIGecom Exchanges*, 17(1):61–70, 2019.

[CFH+21] José Correa, Patricio Foncea, Ruben Hoeksma, Tim Oosterwijk, and Tjark Vredeveld. Posted price mechanisms and optimal threshold strategies for random arrivals. *Mathematics of operations research*, 46(4):1452–1478, 2021.

[CHMS10] Shuchi Chawla, Jason D Hartline, David L Malec, and Balasubramanian Sivan. Multiparameter mechanism design and sequential posted pricing. In *Proceedings of the forty-second ACM symposium on Theory of computing*, pages 311–320, 2010.

[CSZ21] Jose Correa, Raimundo Saona, and Bruno Ziliotto. Prophet secretary through blind strategies. *Mathematical Programming*, 190(1-2):483–521, 2021.

[DFKL20] Paul Dutting, Michal Feldman, Thomas Kesselheim, and Brendan Lucier. Prophet inequalities made easy: Stochastic optimization by pricing nonstochastic inputs. *SIAM Journal on Computing*, 49(3):540–582, 2020.

[DKL20] Paul Dütting, Thomas Kesselheim, and Brendan Lucier. An o (log log m) prophet inequality for subadditive combinatorial auctions. *ACM SIGecom Exchanges*, 18(2):32–37, 2020.

[DKT+21] Ilias Diakonikolas, Vasilis Kontonis, Christos Tzamos, Ali Vakilian, and Nikos Zarifis. Learning online algorithms with distributional advice. In *International Conference on Machine Learning*, pages 2687–2696. PMLR, 2021.

[Dyn63] Evgenii Borisovich Dynkin. The optimum choice of the instant for stopping a markov process. *Soviet Mathematics*, 4:627–629, 1963.

[EFGT20] Tomer Ezra, Michal Feldman, Nick Gravin, and Zhihao Gavin Tang. Online stochastic max-weight matching: prophet inequality for vertex and edge arrival models. In *Proceedings of the 21st ACM Conference on Economics and Computation*, pages 769–787, 2020.

[EHLM17] Hossein Esfandiari, MohammadTaghi Hajiaghayi, Vahid Liaghat, and Morteza Monemizadeh. Prophet secretary. *SIAM Journal on Discrete Mathematics*, 31(3):1685–1701, 2017.

[Fer89] Thomas S Ferguson. Who solved the secretary problem? *Statistical science*, 4(3):282–289, 1989.

[FGL14a] Michal Feldman, Nick Gravin, and Brendan Lucier. Combinatorial auctions via posted prices. In *Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms*, pages 123–135. SIAM, 2014.

[FGL14b] Michal Feldman, Nick Gravin, and Brendan Lucier. Combinatorial auctions via posted prices. In *Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms*, pages 123–135. SIAM, 2014.

[FSZ16] Moran Feldman, Ola Svensson, and Rico Zenklusen. Online contention resolution schemes. In *Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms*, pages 1014–1033. SIAM, 2016.

[Gar70] Martin Gardner. Mathematical games. *Scientific american*, 222(6):132–140, 1970.

[HK82] Theodore P Hill and Robert P Kertz. Comparisons of stop rule and supremum expectations of iid random variables. *The Annals of Probability*, pages 336–345, 1982.

[HKS07] Mohammad Taghi Hajiaghayi, Robert Kleinberg, and Tuomas Sandholm. Automated online mechanism design and prophet inequalities. In *AAAI*, volume 7, pages 58–65, 2007.

[Kar92] Richard M Karp. On-line algorithms versus off-line algorithms: How much. In *Algorithms, Software, Architecture: Information Processing 92: Proceedings of the IFIP 12th World Computer Congress*, volume 1, page 416, 1992.

[Ken87] Douglas P Kennedy. Prophet-type inequalities for multi-choice optimal stopping. *Stochastic Processes and their applications*, 24(1):77–88, 1987.

[Ker86] Robert P Kertz. Stop rule and supremum expectations of iid random variables: a complete comparison by conjugate duality. *Journal of multivariate analysis*, 19(1):88–112, 1986.

[KS77] Ulrich Krengel and Louis Sucheston. Semiamarts and finite values. 1977.

[KS78] Ulrich Krengel and Louis Sucheston. On semiamarts, amarts, and processes with finite value. *Probability on Banach spaces*, 4:197–266, 1978.

[KW12] Robert Kleinberg and Seth Matthew Weinberg. Matroid prophet inequalities. In *Proceedings of the forty-fourth annual ACM symposium on Theory of computing*, pages 123–136, 2012.

[KWW16] Robert Kleinberg, Bo Waggoner, and E Glen Weyl. Descending price optimally coordinates search. *arXiv preprint arXiv:1603.07682*, 2016.

[Lin61] Denis V Lindley. Dynamic programming and decision theory. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 10(1):39–51, 1961.

[Luc17] Brendan Lucier. An economic view of prophet inequalities. *ACM SIGecom Exchanges*, 16(1):24–47, 2017.

[LWW22] Bo Li, Xiaowei Wu, and Yutong Wu. Query efficient prophet inequality with unknown iid distributions. *arXiv preprint arXiv:2205.05519*, 2022.

[MMS88] Mark Manasse, Lyle McGeoch, and Daniel Sleator. Competitive algorithms for online problems. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, pages 322–333, 1988.

[MPT94] Rajeev Motwani, Steven Phillips, and Eric Torng. Nonclairvoyant scheduling. *Theoretical computer science*, 130(1):17–47, 1994.

[PPSW21] Christos Papadimitriou, Tristan Pollner, Amin Saberi, and David Wajc. Online stochastic max-weight bipartite matching: Beyond prophet inequalities. In *Proceedings of the 22nd ACM Conference on Economics and Computation*, pages 763–764, 2021.

[PT22] Bo Peng and Zhihao Gavin Tang. Order selection prophet inequality: From threshold optimization to arrival time design. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 171–178. IEEE, 2022.

[RWW19] Aviad Rubinstein, Jack Z Wang, and S Matthew Weinberg. Optimal single-choice prophet inequalities from samples. *arXiv preprint arXiv:1911.07945*, 2019.

[SC84] Ester Samuel-Cahn. Comparison of threshold stop rules and maximum for independent nonnegative random variables. *the Annals of Probability*, pages 1213–1216, 1984.

[Sei00] Steven S Seiden. A guessing game and randomized online algorithms. In *Proceedings of the thirty-second annual ACM symposium on Theory of computing*, pages 592–601, 2000.

[Wei78] Martin Weitzman. *Optimal search for the best alternative*, volume 78. Department of Energy, 1978.

A Random decay functions

While we only studied deterministic decay functions in the paper, it is also possible to have scenarios with random decay functions. Consider for example that rejected items remain available after j steps with a probability p_j, this is modeled by $D_j(x) = \xi_j x$ with ξ_j a Bernoulli random variable with parameter p_j. We explain in this section how the definitions and our results extend to this case.

Definition A.1 (Random process)**.** *Let $\mathcal{X}$ is a non-empty set. A random process o $\mathcal{X}$ is a collection of random variables $\{Z_x\}_{x\in\mathcal{X}}$. Two random processes $\mathcal{Z} = \{Z_x\}_{x\in\mathcal{X}}$ and $\mathcal{Z}' = \{Z'_x\}_{x\in\mathcal{X}'}$ are independent if any finite sub-process of $\mathcal{Z}$ is independent of any sub-process of $\mathcal{Z}'$. For simplicity, let us say that the random processes $\{Z^1_x\}_{x\in\mathcal{X}^1}, \ldots, \{Z^m_x\}_{x\in\mathcal{X}^m}$ are mutually independent if, for any $x_1 \in \mathcal{X}_1, \ldots, x_m \in \mathcal{X}_m$, the random variables $Z^1_{x_1}, \ldots, Z^m_{x_m}$ are mutually independent.*

Definition A.2 (Random decay functions)**.** *Let $\mathcal{D} = (D_1, D_2, \ldots)$ be a sequence of mutually independent random processes. We say that $\mathcal{D}$ is a sequence of random decay functions if*

1. $\Pr(D_j(x) \notin [0, x]) = 0$ *for any* $x \geq 0$ *and* $j \geq 1$,
2. $j \in \mathbb{N}_{\geq 1} \mapsto \Pr(D_j(x) \geq a)$ *is non-increasing for any* $x, a \geq 0$,
3. $x \geq 0 \mapsto \Pr(D_j(x) \geq a)$ *is non-decreasing for any* $j \in \mathbb{N}_{\geq 1}$ *and* $a \geq 0$.

The second condition asserts that the random variable $D_{j-1}(x)$ has first-order stochastic dominance over $D_j(x)$. Along with the first condition, reflect that the distributions of the rejected values become progressively smaller. The last condition indicates that for any integer $j \geq 1$ and non-negative real numbers $x < y$, $D_j(y)$ has a first-order stochastic dominance over $D_j(x)$, which means that, as the value of x increases, so does the potential recovered value after j steps.

The decision-maker In the $\mathcal{D}$-prophet inequality with deterministic decay functions, we assumed that the decision-maker has full knowledge of the functions $D_1, D_2 \ldots$. In the randomized setting, we assume instead that the decision-maker knows the distributions of the decay functions, i.e. knows the distribution of the random variables $D_j(x)$ for all $x \geq 0$ and $j \geq 1$. However, they do not observe their values until they decide to stop. The online selection process is therefore as follows: the algorithm knows beforehand the distributions of the decay functions, then at each step, it observes a new item with value X_i, and decides to stop or continue. Once they decide to stop at some time τ, they observe the values $D_1(X_{\tau-1}), \ldots, D_\tau(X_1)$ and then they choose the maximal one. As a consequence, the stopping time τ is independent of the randomness induced by the decay functions. As in the deterministic case, the expected reward of any algorithm ALG can be written as

$$\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(X_1, \ldots, X_n)] = \mathbb{E}[\max_{0\leq i\leq \tau-1}\{D_i(X_{\tau-i})\}] .$$

The limit decay A key result in our paper is the reduction of the problem to the case where all the decay functions are identical, and we prove this reduction by considering the pointwise limit of the decay functions. In the case of random decay functions, instead of the pointwise convergence, it holds for all $x \geq 0$ that the random variables $(D_j(x))_j$ converge in distribution to some random variable $D_\infty(x)$. In fact, for any $x \geq 0$ and $a \geq 0$, the sequence $(\Pr(D_j(x) \geq a))_{j\geq 1}$ is non-increasing and non-negative, thus it converges to some constant $G(x, a)$. Given that $x \mapsto \Pr(D_j(x) \geq a)$ is non-decreasing for any j, we obtain by taking the limit $j \to \infty$ that $x \mapsto G(x, a)$ is non-increasing, and with similar argument we obtain, for any $x \geq 0$, that $G(x, a) = 1$ for all $a \leq 0$ and $G(x, a) = 0$

for all $a > x$. Therefore, $a \mapsto 1 - G(x, a)$ defined the cumulative distribution of a random variable D_∞ such that

- $x \geq 0 \mapsto \Pr(D_\infty(x) \geq a)$ is non-decreasing for all $a \geq 0$,
- $\Pr(D_\infty(x) \notin [0, x]) = 0$ for al $x \geq 0$.

Therefore, for all $x \geq 0$, $D_\infty(x)$ is the limit in distribution of $(D_j(x))_j$, hence a sequence $\mathcal{D}' = (D'_1, D'_2, \ldots)$ of mutually independent random processes such that $D'_j(x) \sim D_\infty(x)$ for any $j \geq 1$ and $x \geq 0$ defines a sequence of decay functions. We say in this case that all the decay functions are identically distributed as D_∞. Moreover, it holds for all $x \geq 0$ that $\mathbb{E}[D_\infty(x)] = \lim_{j\to\infty} \mathbb{E}[D_j(x)] = \inf_{j\geq 1} \mathbb{E}[D_j(x)]$

From there, all the proofs of Section 2 can be easily generalized to the case of random decay functions, and it follows that we can restrict ourselves to studying identically distributed decay functions. Moreover, Proposition 3.2 can be generalized to the case of random decay functions, and the necessary condition for surpassing 1/2 becomes $\inf_{x>0} \frac{\mathbb{E}[D_\infty(x)]}{x} > 0$. Similarly, using that the stopping time τ of the algorithm is independent of randomness induced by D_∞, Proposition 3.3 remains true with $\gamma = \inf_{x>0} \frac{\mathbb{E}[D_\infty(x)]}{x}$.

Lower bounds For establishing lower bounds, observe that, for any random decay functions $\mathcal{D}$, if we denote $H_j(x) = \mathbb{E}[D_j(x)]$ for all x, then $\mathcal{H} = (H_1, H_2, \ldots)$ defines a sequence of deterministic decay functions. Furthermore, for any instance $X_1, \ldots, X_n$ and any algorithm ALG, it holds that

$$\begin{aligned}
\mathbb{E}[\mathsf{ALG}^{\mathcal{D}}(X_1, \ldots, X_n)] &= \mathbb{E}[\max_{0\leq i\leq \tau-1} \{D_i(X_{\tau-i})\}] \\
&= \mathbb{E}\Big[\mathbb{E}[\max_{0\leq i\leq \tau-1} \{D_i(X_{\tau-i})\} \mid \tau, X_1, \ldots, X_n]\Big] \\
&\geq \mathbb{E}\Big[\max_{0\leq i\leq \tau-1} \big\{\mathbb{E}[D_i(X_{\tau-i}) \mid \tau, X_1, \ldots, X_n]\big\}\Big] \\
&= \mathbb{E}\Big[\max_{0\leq i\leq \tau-1} \{H_i(X_{\tau-i})\}\Big] \\
&= \mathbb{E}[\mathsf{ALG}^{\mathcal{H}}(X_1, \ldots, X_n)].
\end{aligned}$$

It follows that lower bounds established for deterministic decay functions can be extended to random decay functions by considering their expectations.

Implications With the previous observations, both the lower and upper bounds, depending on $\gamma_{\mathcal{D}}$ that we proved in the deterministic $\mathcal{D}$-prophet inequality can be generalized to the random $\mathcal{D}$-prophet inequality, by taking

$$\gamma_{\mathcal{D}} = \inf_{x>0} \inf_{j\geq 1} \frac{\mathbb{E}[D_j(x)]}{x} \ .$$