Dominic Becker
Math 4446 Project 1
3/21/2024

1a.) We have the following piecewise polynomials and conditions given for the quadratic spline.

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 & [x_0, x_1] \\ S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 & [x_i, x_{i+1}] \\ S_{i+1}(x) = a_{i+1} + b_{i+1}(x - x_{i+1}) + c_{i+1}(x - x_{i+1})^2 & [x_{i+1}, x_{i+2}] \end{cases}$$

$$S'(x) = \begin{cases} S_0'(x) = b_0 + 2c_0(x - x_0) & [x_0, x_1] \\ S_i'(x) = b_i + 2c_i(x - x_i) & [x_i, x_{i+1}] \\ S_{i+1}'(x) = b_{i+1} + 2c_{i+1}(x - x_{i+1}) & [x_{i+1}, x_{i+2}] \end{cases}$$

Conditions:

1. $S_i(x_i) = f(x_i)$ 2. $S_i(x_{i+1}) = f(x_{i+1})$ 3. $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$ 4. $S_0'(x_0) = f'(x_0)$

First, we must find a_0, b_0, and c_0. This can be done with the following equations:

$$a_0 = S_0(x_0) = f(x_0)$$

$$b_0 = S_0'(x_0) = f'(x_0)$$

$$c_0 = \frac{S_0(x_1) - a_0 - b_0(x_1 - x_0)}{(x_1 - x_0)^2} = \frac{f(x_1) - a_0 - b_0(x_1 - x_0)}{(x_1 - x_0)^2}$$

Now we must find a_{i+1}, b_{i+1}, and c_{i+1} for $\mathrm{i} = 0,1,\ldots,\mathrm{n}-2$. We get the following:

$$a_{i+1} = a_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 = f(x_{i+1})$$

$$b_{i+1} = b_i + 2c_i(x_{i+1} - x_i)$$

$$c_{i+1} = \frac{S_{i+1}(x_{i+2}) - a_{i+1} - b_{i+1}(x_{i+2} - x_{i+1})}{(x_{i+2} - x_{i+1})^2}$$

Using these equations, we can get a_i, b_i, and c_i for $\mathrm{i} = 0,1,\ldots,\mathrm{n}-1$.

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1b.


```
function [pp] = quadspline(x,y,k)
%QUADSPLINE - computes the quadratic spline with given slope at the left
% endpoint.
% -Input:
% x - the vector of x_i values
% y - the vector of f(x_i)
% k - derivative f'(x) at left endpoint
% -Output :
% pp - A piecewise polynomial structure for the quadratic spline
%% initialize coefficient matrix
n = length(x)-1;
A = zeros(n,3);
%% Calculate a_0, b_0, and c_0
A(1,3) = y(1);
A(1,2) = k;
A(1,1) = (y(2)-A(1,3)-A(1,2)*(x(2)-x(1)))/(x(2)-x(1))^2;
%% Complete the coefficient matrix for a_i, b_i, and c_i
for i = 1:n-1
    A(i+1,3) = A(i,3)+A(i,2)*(x(i+1)-x(i))+A(i,1)*(x(i+1)-x(i))^2;
    A(i+1,2) = A(i,2)+2*A(i,1)*(x(i+1)-x(i));
    A(i+1,1) = (y(i+2)-A(i+1,3)-A(i+1,2)*(x(i+2)-x(i+1)))/(x(i+2)-x(i+1))^2;
end
%% Store spline in pp format
pp = mkpp(x,A);
```

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2a.) The order of accuracy for this quadratic spline should be three. This is because the spline uses quadratic polynomials where the error is contained in the cubic term with h^3. α is consistent with this expectation.


```
function [S] = P2a(a)
%P2a — Creates a table of numerical error and order of accuracy for k
% piecewise polynomials, each with n_k polynomials, at h_k stepsize for
% f = xe^(-x) using quadratic spline.
% —Input:
% a - number of piecewise polynomials to compute
% —Output :
% S — A table of values for k, n_k, h_k, numerical error, and order of
% accuracy of each piecewise polynomial
%% set the function to interpolate
f = @(x) x*exp(-x);
%% initialize each column of the output table
n = zeros(a,1);
k = zeros(a,1);
h = zeros(a,1);
enum = zeros(a,1);
alpha = zeros(a,1);
%% loop for each entry in every vector previously initialized
for i = 1:a
%% calculate ith entry for k, n_k, and h_k
    k(i) = i;
    n(i) = 2^(i+1);
    h(i) = 2^(1-i);
%% calculate x and f(x) vectors to use in quadspline
    X = 0:h(i):n(i);
    Y = zeros(n(i)+1,1);
    for l = 0:h(i):n(i)
        Y(l/h(i)+1) = f(l);
    end
    pp = quadspline(X,Y,1);
%% calculate numerical error and order of accuracy for each quadspline
    for m = 0:0.001:4
        if abs(f(m)-ppval(pp,m)) > enum(i)
            enum(i) = abs(f(m)-ppval(pp,m));
        end
    end
    if i ~= 1
        alpha(i) = log(enum(i)/enum(i-1))/log(h(i)/h(i-1));
    end
end
alpha(1) = "NAN";
%% create a results table for k, n_k, h_k, numerical error, and order or accuracy
R = table(k,n,h,enum,alpha,'VariableNames',{'k','n','h','error','accuracy'});
S = table(R,'VariableNames',{'Results'});
```

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```
a = 5;
P2a(a)
```

```
ans =

  5×1 table

                       Results
    k    n       h          error         accuracy
    ________________________________________________

    1     4        1       0.057408          NaN
    2     8      0.5      0.0084533       2.7637
    3    16     0.25      0.0011163       2.9208
    4    32    0.125     0.00014239       2.9708
    5    64   0.0625      1.795e-05       2.9878
```

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2b.) The splines would be ranked as follows:

1. Clamped cubic spline – 4^{th} order of accuracy
    - Advantages: Smallest error
    - Disadvantages: Needs $f'(x)$ at both endpoints

2. Not-a-knot spline – 4^{th} order of accuracy
    - Advantages: 4^{th} order of accuracy, does not need endpoint conditions
    - Disadvantages: Third derivatives must be continuous

3. Quadratic spline – 3^{rd} order of accuracy
    - Advantages: Does not need to compute a d_i column
    - Disadvantages: Needs $f'(x)$ at one endpoint

4. Natural cubic spline – 2^{nd} order of accuracy
    - Advantages: Simple endpoint conditions
    - Disadvantages: Low order of accuracy

Natural cubic spline does not work well for this problem since the boundary conditions are set so that $f''(0) = f''(4) = 0$. Since $f''(x) = xe^{-x} - 2e^{-x}$, $f''(0) = -2 \neq 0$, and $f''(4) = 2e^{-4} \neq 0$, the boundary conditions of the natural cubic spline inhibit its performance interpolating $f(x)$.

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```
function [T1,T2,T3,T4] = P2b(a)
%P2b — Creates a table of numerical error and order of accuracy for k
% piecewise polynomials, each with n_k polynomials, at h_k stepsize for
% f = xe^(-x) using quadratic spline and the three cubic splines.
% —Input:
% a - number of piecewise polynomials to compute
% —Output :
% T1 — A table of values for k, n_k, h_k, numerical error, and order of
% accuracy of each piecewise polynomial using a quadratic spline
% T2 — A table of values for k, n_k, h_k, numerical error, and order of
% accuracy of each piecewise polynomial using a natural cubic spline
% T3 — A table of values for k, n_k, h_k, numerical error, and order of
% accuracy of each piecewise polynomial using a clamped cubic spline
% T4 — A table of values for k, n_k, h_k, numerical error, and order of
% accuracy of each piecewise polynomial using a not-a-knot cubic spline
%% set the function to interpolate
f = @(x) x.*exp(-x);
%% setup x and f(x) vectors for quadratic, natural, and not-a-knot splines
x = [0,1,2,3,4];
y = [0,exp(-1),2*exp(-2),3*exp(-3),4*exp(-4)];
%% setup f(x) vector with f'(0) and f'(4) for clamped spline
y2 = [1,0,exp(-1),2*exp(-2),3*exp(-3),4*exp(-4),exp(-4)-4*exp(-4)];
%% calculate the four splines and store in pp format
pp1 = quadspline(x,y,1);
pp2 = myspline(x,y,1);
pp3 = myspline(x,y2,2);
pp4 = myspline(x,y,3);
%% graph the four splines along with the original function f, and the 5 nodes
hold on
xq = 0:0.001:4;
fplot(f,[0,4],'black')
plot(xq,ppval(pp1,xq),'red');
plot(xq,ppval(pp2,xq),'green');
plot(xq,ppval(pp3,xq),'blue');
plot(xq,ppval(pp4,xq),'cyan');
scatter(x,y,'filled','black');
legend('f','quad','natural','clamped','not-a-knot','nodes');
%% create an error and accuracy table for each of the four splines
T1 = P2a(a);
T2 = mysplinetable(a,1);
T3 = mysplinetable(a,2);
T4 = mysplinetable(a,3);
```

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```
function [S] = mysplinetable(a,b)
%P2a — Creates a table of numerical error and order of accuracy for k
% piecewise polynomials, each with n_k polynomials, at h_k stepsize for
% f = xe^(-x) using either natural, clamped, or not-a-knot cubic spline.
% —Input:
% a - number of piecewise polynomials to compute
% —Output :
% S — A table of values for k, n_k, h_k, numerical error, and order of
% accuracy of each piecewise polynomial
%% set the function to interpolate
f = @(x) x*exp(-x);
%% initialize each column of the output table
n = zeros(a,1);
k = zeros(a,1);
h = zeros(a,1);
enum = zeros(a,1);
alpha = zeros(a,1);
%% loop for each entry in every vector previously initialized
for i = 1:a
%% calculate ith entry for k, n_k, and h_k
    k(i) = i;
    n(i) = 2^(i+1);
    h(i) = 2^(1-i);
%% calculate x and f(x) vectors to use in myspline
    X = 0:h(i):n(i);
    Y = zeros(n(i)+1,1);
    for l = 0:h(i):n(i)
        Y(l/h(i)+1) = f(l);
    end
%% calculate f'(0) and f'(4) for clamped spline
    if b == 2
        Y = zeros(n(i)+3,1);
        Y(1) = 1;
        Y(n(i)/h(i)+3) = exp(-4)-4*exp(-4);
        for l = 0:h(i):n(i)
            Y(l/h(i)+2) = f(l);
        end
    end
%% use myspline to create either a natural, clamped, or not-a-knot spline
    pp = myspline(X,Y,b);
%% calculate numerical error and order of accuracy for each quadspline
    for m = 0:0.001:4
        if abs(f(m)-ppval(pp,m)) > enum(i)
            enum(i) = abs(f(m)-ppval(pp,m));
        end
    end
    if i ~= 1
        alpha(i) = log(enum(i)/enum(i-1))/log(h(i)/h(i-1));
    end
```



```
end
alpha(1) = "NAN";
%% create a results table for k, n_k, h_k, numerical error, and order or accuracy
R = table(k,n,h,enum,alpha,'VariableNames',{'k','n','h','error','accuracy'});
S = table(R,'VariableNames',{'Results'});
```

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```
a = 5;
[quadratic,natural,clamped,notaknot] = P2b(a)
```

```
quadratic =

  5×1 table

                    Results
    k    n       h          error        accuracy
    ______________________________________________

    1     4        1       0.057408          NaN
    2     8      0.5      0.0084533       2.7637
    3    16     0.25      0.0011163       2.9208
    4    32    0.125     0.00014239       2.9708
    5    64   0.0625      1.795e-05       2.9878


natural =

  5×1 table

                    Results
    k    n       h          error        accuracy
    ______________________________________________

    1     4        1       0.080493          NaN
    2     8      0.5       0.023188       1.7955
    3    16     0.25      0.0060438       1.9399
    4    32    0.125       0.001528       1.9838
    5    64   0.0625     0.00038312       1.9957


clamped =

  5×1 table

                    Results
    k    n       h          error        accuracy
    ______________________________________________

    1     4        1       0.007331          NaN
    2     8      0.5     0.00056371        3.701
    3    16     0.25     3.8241e-05       3.8818
    4    32    0.125     2.4723e-06       3.9512
    5    64   0.0625     1.5678e-07        3.979
```

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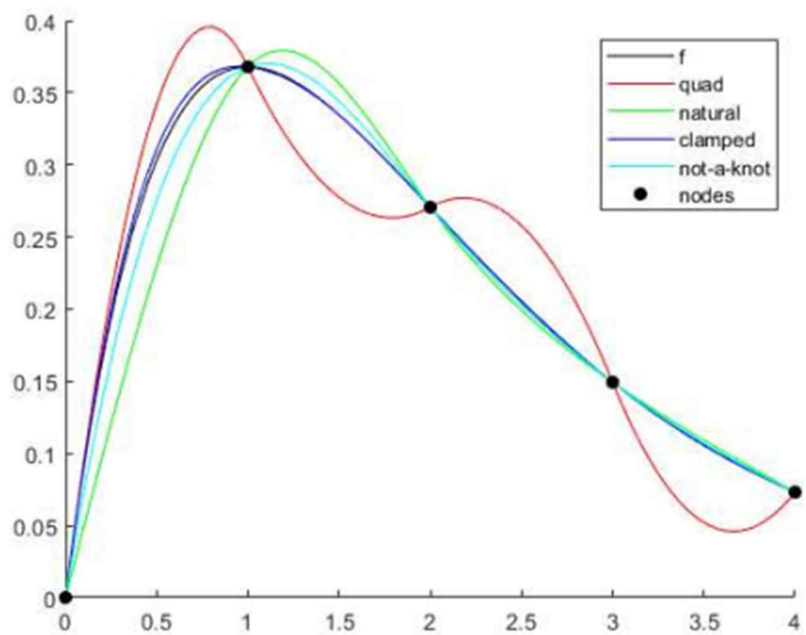
```
notaknot =

  5×1 table

                  Results
 k     n       h          error        accuracy
 ________________________________________________

 1     4         1       0.031593          NaN
```

```
 2     8       0.5       0.003714       3.0886
 3    16      0.25     0.00032003       3.5367
 4    32     0.125     2.3494e-05       3.7678
 5    64    0.0625     1.5913e-06        3.884
```



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3a.) quadspline2 will maintain the order of accuracy of quadspline since $f'(x_0)$ was derived using the five-point forward difference formula. Since the order of accuracy for the five-point forward difference formula is greater than that of quadspline, quadspline will maintain its order of accuracy.

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3b.)

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```
function [pp] = quadspline2(x,y)
%QUADSPLINE2 — computes the quadratic spline with slope at the left
% endpoint approximated by finite difference.
% —Input:
% x — the vector of x_i values
% y — the vector of f(x_i)
% —Output :
% pp — A piecewise polynomial structure for the quadratic spline
%% initialize coefficient matrix
n = length(x)-1;
A = zeros(n,3);
%% Calculate a_0, b_0, and c_0 using finite difference approximation
A(1,3) = y(1);
A(1,2) = finitder(0.001);
A(1,1) = (y(2)-A(1,3)-A(1,2)*(x(2)-x(1)))/(x(2)-x(1))^2;
%% Complete the coefficient matrix for a_i, b_i, and c_i
for i = 1:n-1
    A(i+1,3) = A(i,3)+A(i,2)*(x(i+1)-x(i))+A(i,1)*(x(i+1)-x(i))^2;
    A(i+1,2) = A(i,2)+2*A(i,1)*(x(i+1)-x(i));
    A(i+1,1) = (y(i+2)-A(i+1,3)-A(i+1,2)*(x(i+2)-x(i+1)))/(x(i+2)-x(i+1))^2;
end
%% Store spline in pp format
pp = mkpp(x,A);
```

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```
function [f0] = finitder(h)
%FINITDER — computes the slope at the left endpoint using five-point
% forward difference
% —Input:
% h — distance between x_i and x_{i+1}
% —Output :
% f0 — f'(0) when f = xe^(-x)
%% First five interpolation points of the function given in 2a
f = @(x) x.*exp(-x);
X0 = [0,h,2*h,3*h,4*h];
Y0 = f(X0);
%% approximate f'(0) with five-point forward difference formula
f0 = (-25*Y0(1)+48*Y0(2)-36*Y0(3)+16*Y0(4)-3*Y0(5))/(12*h);
```

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3c.) Yes, the convergence order is the same.


```
function [S] = P3c(a)
%P2a — Creates a table of numerical error and order of accuracy for k
% piecewise polynomials, each with n_k polynomials, at h_k stepsize for
% f = xe^(-x) using quadratic spline with finite difference approximation.
% —Input:
% a - number of piecewise polynomials to compute
% —Output :
% S — A table of values for k, n_k, h_k, numerical error, and order of
% accuracy of each piecewise polynomial
%% set the function to interpolate
f = @(x) x*exp(-x);
%% initialize each column of the output table
n = zeros(a,1);
k = zeros(a,1);
h = zeros(a,1);
enum = zeros(a,1);
alpha = zeros(a,1);
%% loop for each entry in every vector previously initialized
for i = 1:a
%% calculate ith entry for k, n_k, and h_k
    k(i) = i;
    n(i) = 2^(i+1);
    h(i) = 2^(1-i);
%% calculate x and f(x) vectors to use in quadspline2
    X = 0:h(i):n(i);
    Y = zeros(n(i)+1,1);
    for l = 0:h(i):n(i)
        Y(l/h(i)+1) = f(l);
    end
    pp = quadspline2(X,Y);
%% calculate numerical error and order of accuracy for each quadspline
    for m = 0:0.001:4
        if abs(f(m)-ppval(pp,m)) > enum(i)
            enum(i) = abs(f(m)-ppval(pp,m));
        end
    end
    if i ~= 1
        alpha(i) = log(enum(i)/enum(i-1))/log(h(i)/h(i-1));
    end
end
alpha(1) = "NAN";
%% create a results table for k, n_k, h_k, numerical error, and order or accuracy
R = table(k,n,h,enum,alpha,'VariableNames',{'k','n','h','error','accuracy'});
S = table(R,'VariableNames',{'Results'});
```

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```
a = 5;
P3c(a)
```

```
ans =

  5×1 table
```

Results

k	n	h	error	accuracy
1	4	1	0.057408	NaN
2	8	0.5	0.0084533	2.7637
3	16	0.25	0.0011163	2.9208
4	32	0.125	0.00014239	2.9708
5	64	0.0625	1.795e-05	2.9878

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3d.) Aside from different finite difference approximations of $f'(x_0)$, I cannot think of any other ways to compute the quadratic spline without $f'(x_0)$. Since we need b_0, and b_0 comes from the derivative of x_0, we need the derivative of x_0 for the quadratic spline.