§6.2 z-test for μ
§6.3 z-test for p
§7-4 t-test for μ

Example 1: It is known that the SAT math test score is a random variable $X \sim N(528, 117^2)$. A team in education company claims that they have discovered a new teaching method that increases test scores. The team use the new teaching method in a class of 40 students and found that the scores averaged 560. What should the team conclude about the sample results? Can they claim that the new teaching method significantly increases the score.

■ These sections present individual components of a hypothesis test. We should know and understand the following:

- How to identify the **null hypothesis** and **alternative hypothesis** from a given claim, and how to express both in symbolic form.
- How to calculate the value of the **test statistic**, given a claim and sample data.
- How to identify the **critical value**(s), given a significance level.
- How to identify the **P-value**, given a value of the test statistic.
- How to state the **conclusion** about a claim in simple and nontechnical terms.

■ Hypotheses

H_0: Null hypothesis A statement that the value of a population parameter (such as proportion, mean, or standard deviation) is equal to some hypothesized value.

H_a: Alternative hypothesis The statement that the parameter has a value that somehow differs from the null hypothesis. The alternative hypothesis is denoted by **H_1** or **H_a** or **H_A**. The symbolic form of the alternative hypothesis must use one of these symbols: ≠, <, >.

Null hypothesis Alt. hypothesis

In example 1, H_0: $\mu = 528$ H_a: $\mu > 528$

Example 2: The California Highway patrol claims that the average speed of cars traveling on I-15 between San Bernardino and San Diego on a typical Saturday is 70 miles per hour.

- I believe that the average speed is higher.

H_0: $\mu = 70$ H_a: $\mu > 70$ Right tailed test

- I believe that the average speed is slower.

H_0: $\mu = 70$ H_a: $\mu < 70$ Left tailed test

- I believe that the average speed is different from 70.

H_0: $\mu = 70$ H_a: $\mu \neq 70$ 2-tailed test

Same H_0

Note: The null hypothesis is always stated using the *equal* sign! When researchers conduct a study, they are usually looking for evidence to support their claim. Thus, the claim should be stated as the alternative hypothesis. The alternative hypothesis is also called the research hypothesis.

■ Steps in Conducting a Significance Test

1. State $\mathbf{H_0}$ and $\mathbf{H_a}$ symbolically.

Test on Population Mean μ

a) Left-tailed test (or lower tailed test): H_0: $\mu = \mu_0$ vs H_a: $\mu < \mu_0$
b) Right-tailed test (or upper tailed test): H_0: $\mu = \mu_0$ vs H_a: $\mu > \mu_0$
c) Two-tailed test (or two-sided test): H_0: $\mu = \mu_0$ vs H_a: $\mu \neq \mu_0$

Test on Population Proportion p

a) Left-tailed test (or lower tailed test): H_0: $p = p_0$ vs H_a: $p < p_0$
b) Right-tailed test (or upper tailed test): H_0: $p = p_0$ vs H_a: $p > p_0$
c) Two-tailed test (or two-sided test): H_0: $p = p_0$ vs H_a: $p \neq p_0$

2. Determine $\boldsymbol{\alpha}$ (level of significance)

The **significance level,** symbolized by α, is the probability that the test statistic will fall in the critical region when the null hypothesis is actually true. α is the maximum acceptable probability of making an error, called a Type I error.

Type I error: Reject H_0 incorrectly (H_0 is true)

If α is not given, we assume $\alpha = 0.05$ (5%).

3. Determine the test statistic.

The **test statistic** (or **test value**) is the numerical value obtained from a sample. It is used in making a decision about whether the null hypothesis should be rejected and is found by converting the sample statistic to a z-score or t-score.

$$\textbf{Test statistic} = \frac{\text{estimate} - \text{hypothesized value}}{\text{standard deviation of the estimate}}$$

a) Test statistic for mean: $z^* = \dfrac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$ or $t^* = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$

sample proportion

b) Test statistic for proportion: $z^* = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}}$

4. Determine the rejection region by looking at H_a

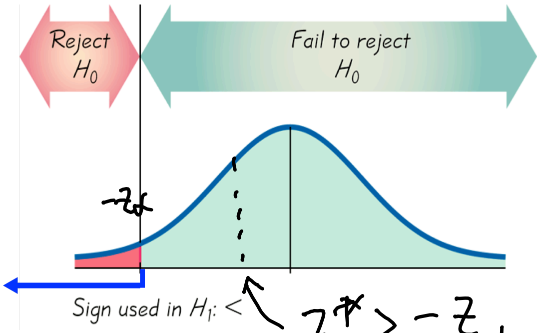
■ **Decision Rule - Critical Value Method**

The **rejection region** (or **critical region**) is the set of all values of the test statistic that cause us to reject the null hypothesis.

a) Left-tailed test: H_0: $\mu=\mu_0$ vs H_a: $\mu<\mu_0$ or H_0: $p=p_0$ vs H_a: $p<p_0$

The critical region is in the extreme left regions (tail) under the curve.

$$\text{Reject } H_0 \text{ if } z^* < -z_\alpha$$
$$\text{Reject } H_0 \text{ if } t^* < -t_{\alpha,df}$$

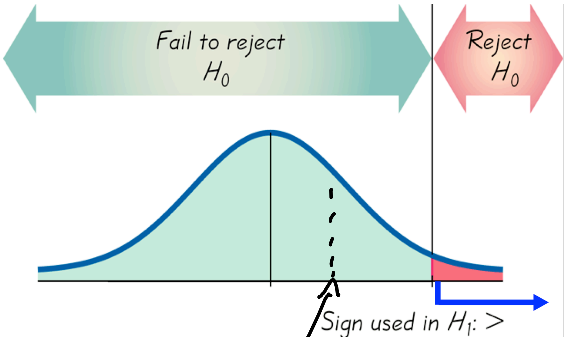


$z^* > -z_\alpha$ ∴ fail to reject

b) Right-tailed test: H_0: $\mu=\mu_0$ vs H_a: $\mu>\mu_0$ or H_0: $p=p_0$ vs H_a: $p>p_0$

The critical region is in the extreme right region (tail) under the curve.

$$\text{Reject } H_0 \text{ if } z^* > z_\alpha$$
$$\text{Reject } H_0 \text{ if } t^* > t_{\alpha,df}$$

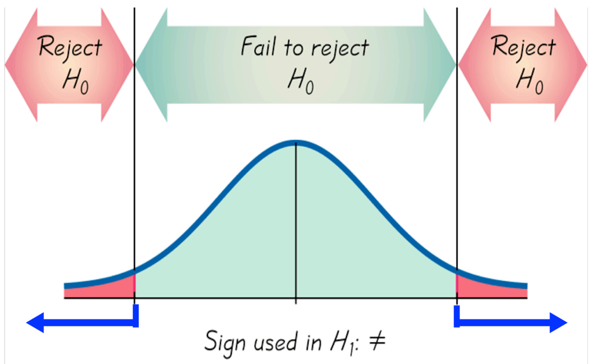


$z^* < z_\alpha$ ∴ fail to reject

c) Two-sided test (or two-tailed test): H_0: $\mu = \mu_0$ vs H_a: $\mu \neq \mu_0$ or H_0: $p = p_0$ vs H_a: $p \neq p_0$

The critical region is in the two extreme regions (tails) under the curve.

$$\text{Reject } H_0 \text{ if } z^* < -z_{\alpha/2} \text{ or } z^* > z_{\alpha/2}$$
$$\text{Reject } H_0 \text{ if } t^* < -t_{\alpha/2,df} \text{ or } t^* > t_{\alpha/2,df}$$



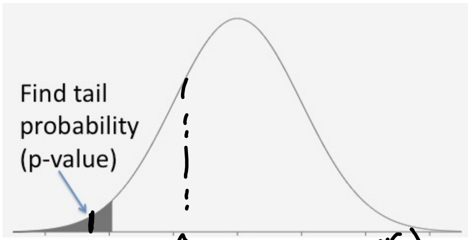
■ Decision Rule - *P*-value Method

The ***P*-value** (or p-value or probability value) is the probability of getting a value of the test statistic that is at least as extreme as the one representing the sample data, assuming that the null hypothesis is true.

The null hypothesis H_0 is **rejected** if the *P*-value is very small, such as 0.05 or less.

a) **Left**-tailed test:

***P*-value** = area to the **left** of the test statistic = $P(z < z^*)$
or $= P(t < t^*)$

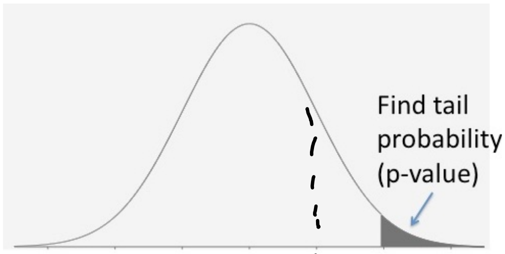


$P(z<z^*) > \alpha$, ∴ fail to reject

$P(z<z^*) < \alpha$, reject H_0

b) **Right**-tailed test:

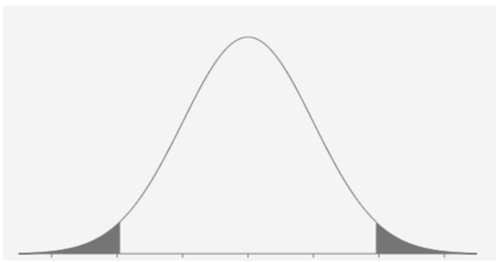
***P*-value** = area to the **right** of the test statistic = $P(z > z^*)$
or $= P(t > t^*)$



$P(z>z^*) > \alpha$ ∴ fail to reject

c) **Two**-tailed test:

***P*-value** = **twice** the area in the tail beyond the test statistic = $2\,P(z > |z^*|)$
or $= 2\,P(t > |t^*|)$



<u>*P*-value method:</u>

If *P*-value $\le \alpha$, reject H_0.
If *P*-value $> \alpha$, fail to reject H_0.

5. Conclusion: Either **fail to reject H_0** or **reject H_0**

Note: In this course, we use the terms "accept H_0" and "fail to reject H_0" interchangeably.

Example 3: Evaluate the appropriate test statistic and state your decision. Calculate both the rejection region and the P-values to conduct the hypothesis tests above.

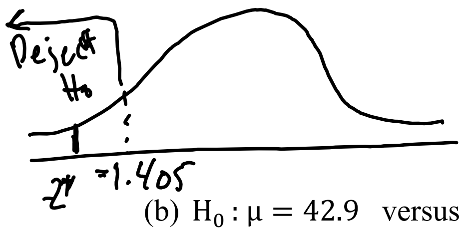
(a) $H_0 : \mu = 120$ versus $H_a : \mu < 120$; $\bar{y}=114.2$, $n=25$, $\sigma=18$, $\alpha = 0.08$

$\alpha = 0.08$

$z_\alpha = -1.405$

$\bar{y} = 114.2,\ n = 25,\ \sigma = 18$

$z^* = \dfrac{114.2-120}{\left(\frac{18}{\sqrt{25}}\right)} = -1.611$



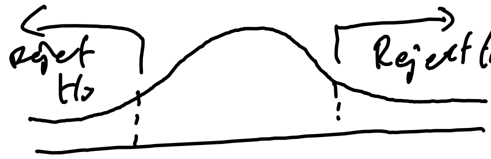
$z^* < z_\alpha \Rightarrow$ reject H_0.

$P(Z < -1.611) = \text{normcdf}(-10^{99}, -1.611, 0, 1)$

$= 0.0535 < 0.08$

$\Rightarrow$ reject H_0

(b) $H_0 : \mu = 42.9$ versus $H_a : \mu \neq 42.9$; $\bar{y}=45.1$, $n=16$, $\sigma=3.2$, $\alpha = 0.01$



$z^* = \dfrac{45.1-42.9}{\left(\frac{3.2}{\sqrt{16}}\right)}$

$= 2.75$

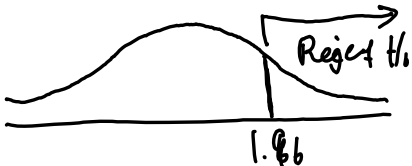
$\alpha = 0.01$

$z_{\frac{\alpha}{2}} = \pm 2.576$

$z^* > z_{\frac{\alpha}{2}} \Rightarrow$ reject H_0

$P(Z > 2.75) = 0.002 < 0.005 \Rightarrow$ reject H_0

(c) $H_0 : \mu = 14.2$ versus $H_a : \mu > 14.2$; $\bar{y}=15.8$, $n=9$, $\sigma=4.1$, $\alpha = 0.05$



$z^* = \dfrac{15.8-14.2}{\left(\frac{4.1}{\sqrt{9}}\right)} = 1.17$

$\alpha = 0.05$

$z_\alpha = 1.96$

$z^* < z_\alpha \Rightarrow$ fail to reject H_0

$P(Z > 1.17) = 0.1208 > 0.05 \Rightarrow$ fail to reject H_0

■ **Calculator:**

***z*-test for μ:** STAT >> TESTS >> **#1: ZTest**

***t*-test for μ:** STAT >> TESTS >> **#2: TTest**

***z*-test for p:** STAT >> TESTS >> **#5: 1-PropZTest**

$P(a < z < b) =$ **normalcdf**$(a, b, 0, 1)$ 2nd >> Vars >> **#2: normalcdf**

$P(a < t < b) =$ **tcdf**(a, b, df) 2nd >> Vars >> **#6: tcdf**

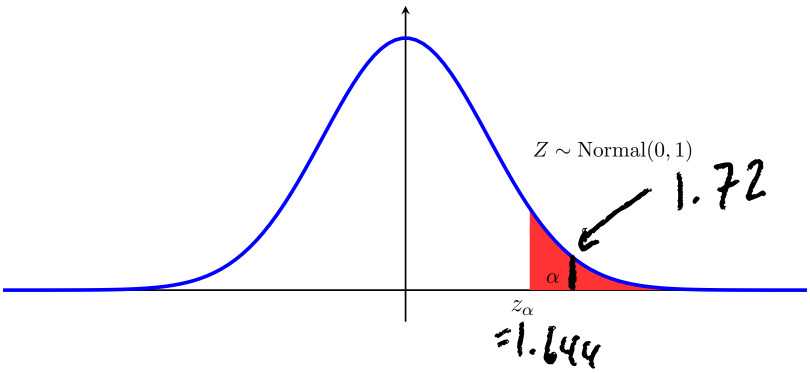
Example 1: (cont'd) It is known that the SAT math test scores is a random variable $X \sim N(528, 117^2)$. A team in education company claims that they have discovered a new teaching method that increases test scores. The team use the new teaching method in a class of 40 students and found that the scores averaged 560. What should the team conclude about the sample results? Can they claim that the new teaching method significantly increases the score?

Sol.

- H_0: $\mu = 528$ vs. H_a: $\mu > 528$
- α is not given, so we assume $\alpha = 0.05$. $\rightarrow 1.644$
- Test value:

$$z^* = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{560 - 528}{\left(\frac{117}{\sqrt{40}}\right)} = 1.72$$

- Critical value:

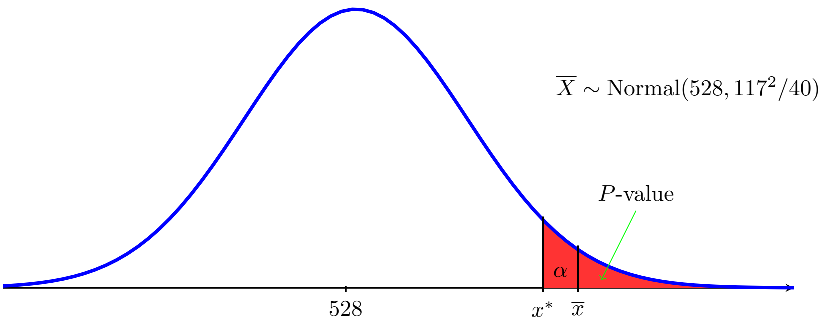


reject H_0

$P(X > z^*) = 0.047$

< 0.05

$\Rightarrow$ reject H_0.



- Decision rule:

  Critical value method:

  See previous page

  P-value method:

  P-value = probability of getting a sample mean ≥ 560 for a sample of size 40 given that the null hypothesis is true

- Conclusion:

Example 4: A random sample of 30-quart cartons of ice cream was taken from a large production run. If their mean fat content was 12.6 percent with a standard deviation of 1.25 percent, is this reason to believe the average fat content in this type of ice cream is more than 12 percent? Use the 0.01 level of significance.

$\mu = 12$ $\quad H_0: \mu = 12$ vs. $H_a: \mu > 12$

$\bar{x} = 12.6$

$s = 1.25$ $\quad t^*_{29} = \dfrac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = 2.629$

$n = 30$

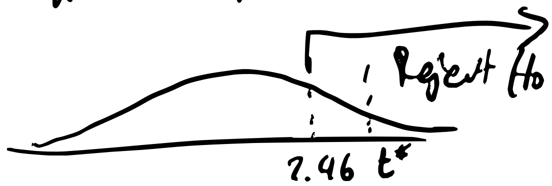
$df = 29$

P-Value Method:

$P(T > 2.62) = 0.006 < 0.01 \Rightarrow$ reject H_0

CV method:

$t_\alpha = t_{0.01, 29} = 2.46$



Conclusion:
There is "strong evidence to conclude that this ice cream has a higher mean fat % than 12% with 99% confidence

Example 5: An oceanographer wants to test, on the basis of a random sample of size 35, whether the average depth of the ocean in a certain area is 72.4 fathoms. At the .05 level of significance, what will the oceanographer decide if she gets a sample mean of 73.2? Assume the population standard deviation is 2.1.

$\sigma = 2.1$ $\quad H_0: \mu = 72.4$ vs. $H_a: \mu \neq 72.4$

Z Test...

$n = 35$

$z^* = 2.25 > +1.96 \Rightarrow$ reject H_0

$\bar{x} = 73.2$

$\mu = 72.4$ $\quad P = 0.024 < 0.05$

$z_\alpha = \pm 1.96$ $\quad \Rightarrow$ Reject H_0

Manually find CV & P-Value

$z_{\frac{\alpha}{2}} = \text{invNorm}(.975, 0, 1) = \pm 1.96$, $\quad P = 2P(Z \geq |z^*|) = 0.024$

use invNorm to calculate Z-score

Example 6: Suppose that, for the U.S. data in Example 2 in the notes 7.2-7.3, we want to test whether the U.S. average is different from the reported U.K. average at the 0.05 significance level. Specifically, we want to test

$$H_0: \ \mu = 3.7 \qquad \text{vs} \qquad H_0: \ \mu \neq 3.7$$

← 2 tailed test

t test

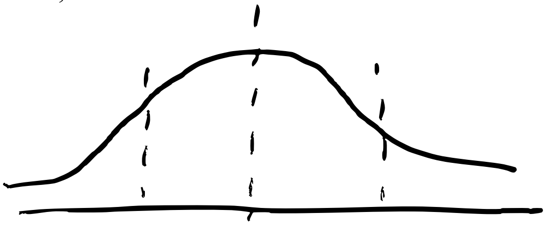
Could also use TTest

Recall that n = 8, $\bar{x}$ = 6.75, and s = 3. 88.

$n = 8$

$\bar{x} = 6.75$

$s = 3.88$



$df = 7$

$\mu = 3.7$

$$t_7^* = \frac{6.75 - 3.7}{\left(\frac{3.88}{\sqrt{8}}\right)} = 2.22 < t_{\frac{\alpha}{2}}$$

$\Rightarrow$ fail to reject H_0

sub from 1 in other

$t_{\frac{\alpha}{2}} = \text{invT}(0.025, 7) = \pm 2.36$

$2P(T \geq t_7^*) = 0.0619 > 0.05$

$\Rightarrow$ fail to reject H_0

Example 7: A random sample of 12 students in a high school typing class averaged 73.2 words per minute with a standard deviation of 7.9 on a typing test. What can we conclude, at the .05 level of significance, regarding the claim that students at the school average less than 75 words per minute on the typing test?

$H_0: \mu = 75$ vs. $H_a: \mu < 75$ ← left tailed test

$\bar{x} = 73.2$

$s = 7.9$

$\alpha = 0.05$

$n = 12$

$df = 11$

$$t_{11}^* = -0.789 = \left[\frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}\right] > -1.795$$

$P = 0.2234 > 0.05$

$\Rightarrow$ fail to reject

CV: $t_{0.05,11} = -1.795 = \text{invT}(0.05, 11)$

Manual P: $P = P(t < -0.789)$

$= 0.2234$

Example 8: Five mature dogs of a certain breed weigh 66, 63, 64, 62, and 65 pounds. A kennel club claims that the average weight for this breed is 60 pounds. Using .05 level of significance, is there reason to believe that the average weight is above the established claim?

$H_0: \mu = 60$, vs. $H_a: \mu > 60$

Data: 66, 63, 64, 62, 65

T Test (data)

$S = 1.5811$

$\bar{X} = 64$

$\alpha = 0.05$

$\mu = 60$

$t^* = 5.657$

$p = 0.0024 < \alpha$

$\Rightarrow$ reject H_0

CV: $t_{0.05,4} = 2.132$

p-value: $p = tcdf(5.657, 10^{99}, 4) = 0.0024$

Example 9: A local newspaper claims that 25% of the its readers ~~regularly clip coupons from~~ the newspaper. To see whether this number is reasonable, 185 readers were surveyed and 44 of them indicated that they regularly clipped coupons. Conduct a test of interest allowing at the 0.1 level of significance.

$H_0: p = 0.25$ vs. $H_a: p \neq 0.25$, $q = 0.75$

$\hat{p} = \frac{44}{185} = 0.2378$

$\hat{q} = 1 - \hat{p} = 0.7622$

$\alpha = 0.1$

$z_\alpha =$

1-PropZTest

$z^* = -0.382$

p-value $= 0.7024 > 0.1$

$\Rightarrow$ accept H_0.

$\hat{p} \sim N\left(p, \left(\sqrt{\frac{pq}{n}}\right)^2\right)$

Example 10: Tony believes that callers to his home fail to leave a message on his answering machine 30% of the time. In order to test this, he refuses to answer his phone on 50 random calls and observe whether or not the caller leaves a message. 10 callers do not leave a message. Conduct the test whether his claim is too high at .01 level of significance.

$H_0: p = 0.3$ vs. $H_a: p < 0.3$

$\hat{p} = 0.2$

$\hat{q} = 0.8$

$n = 50$

$x = 10$

1-PropZTest

$z^* = -1.543$

$p = 0.0614 > 0.01$

$\Rightarrow$ fail to reject H_0

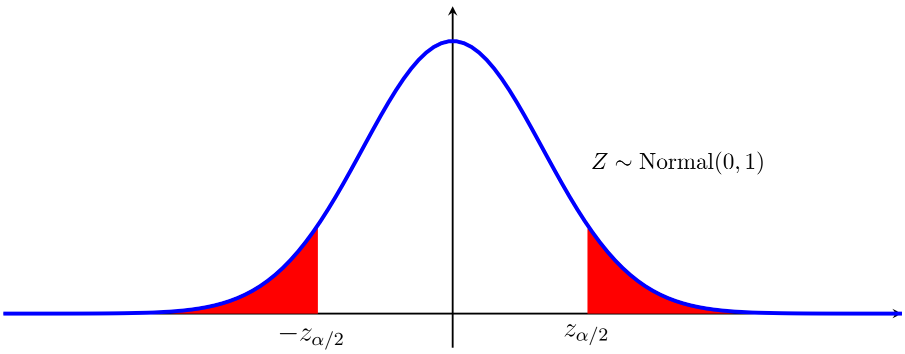
$P = P(Z < -1.543)$

$= \text{normalcdf}(-10^{99}, -1.543, 0, 1)$

Example 11: A city council member claims that 45% of her constituency is "very concerned" about drug trafficking. To see whether this claim is too low, a sample of 265 citizens was taken and 135 of them indicated that they were "very concerned" about drug trafficking. Test at the .05 level of significance.

Note: TWO-SIDED SIGNIFICANCE TESTS AND CONFIDENCE INTERVALS

A level α two-sided significance test rejects a hypothesis H_0: $\mu = \mu_0$ exactly when the value μ_0 falls outside a level $1 - \alpha$ confidence interval for μ.



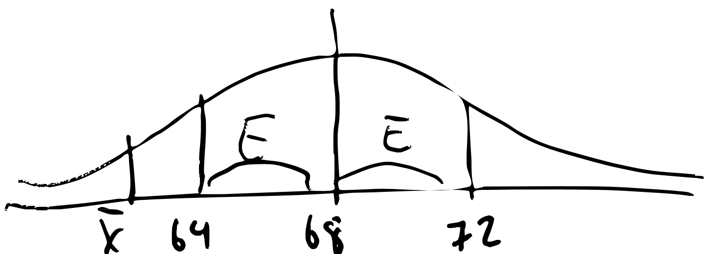
Example 12: A 95% confidence interval for a population mean is (57, 65).
(a) Can you reject the null hypothesis that $\mu = 68$ at the 5% significance level? Explain.

$$\bar{X} = \frac{65+57}{2} = 61$$

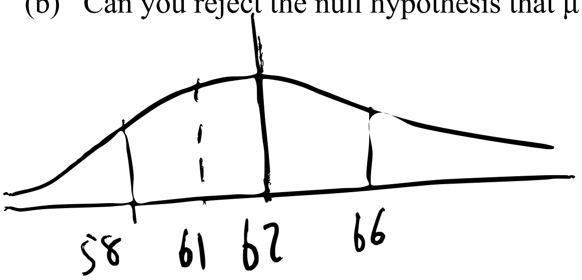
$$E = 4$$

$\bar{X} < \mu - E \Rightarrow \bar{X}$ belongs to the rejection region.

$\therefore$ reject H_0



(b) Can you reject the null hypothesis that $\mu = 62$ at the 5% significance level? Explain.



We fail to reject H_0