§2.5 Independence

Example 1. Roulette wheel

A Roulette (a wheel gamble) has 18 red, 18 black, and 2 green. If you spin the wheel 2 times, what is the probability of getting 2 red.

R_1 : the first result is red.
R_2 : the second result is red.
$P(R_1 \cap R_2) = P(R_2 \cap R_1) = P(R_1)P(R_2|R_1) = (18/38)^2$.

Definition.

The sets A and B are called **independent** if

$$P(A \cap B) = P(A)P(B).$$

Recall our Theorem $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$.

If If A and B are not empty set, we have the following equivalent definition.

Theorem.

The sets A and B are independent iff

$$P(A|B) = P(A) \text{ iff } P(B|A) = P(B)$$

From the theorem, independent means the the probability of A does not depend on the result of B, vice versa.

Example 2. Draw a card from a standard poker deck.

Event A: the card is a King.

Event B: the card is a Diamond.

not mutually exclusive, but are independent

Are the sets A and B independent?

Check: $P(A \cap B) \overset{?}{=} P(A)P(B)$

$$\frac{1}{52} = \frac{4}{52} \cdot \frac{13}{52} \checkmark$$

They are independent

$P(A \cap B) = P(B|A) \cdot P(A)$

$\Rightarrow$ independent means $P(B|A) = P(B)$

Remark: It is important not to confuse "mutually exclusive" and "independence". In the above example, A and B are not disjoint.

Consider Event C: the card is a Jack. Then A and C are disjoint but not independent.)

Example 3. Let A and B be two independent events on S, and $P(A) = 0.3$ and $P(B) = 0.8$. Find $P(A \cup B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.8 - (0.3 \cdot 0.8)$$
$$= 0.86$$

Example 4. Roll two fair 6-sides dice.

Consider the sets: A={first roll =3}, B={sum=8}, C={sum=7}, D={first roll =1},

(1) Are the sets A and B independent? Are they disjoint?

$$P(A \cap B) \stackrel{?}{=} P(A)P(B)$$
$$\frac{1}{36} \stackrel{?}{=} \frac{6}{36} \cdot \frac{5}{36}$$

not independent

$A \cap B \neq \emptyset \therefore$ not disjoint

(2) Are the sets A and C independent? Are they disjoint?

$A \cap C = \{(3,4)\}$ $\quad P(A \cap C) = P(A)P(C)$

not m.e., independent $\quad \frac{1}{36} = \frac{6}{36} \cdot \frac{6}{36}$ ✓

(3) Are the sets B and D independent? Are they disjoint?

More than Two Sets:

Definition.

The sets A, B, and C are called **independent** if:
(1) $P(A \cap B \cap C) = P(A)P(B)P(C)$, and
(2) $P(A \cap B) = P(A)P(B)$,
$P(A \cap C) = P(A)P(C)$,
$P(B \cap C) = P(B)P(C)$.

if A, B, C

Most of the time, we know independent from the real world questions. (For example, roll a coin or dice n times.)

Then we can use one side of the property: If A_1, A_2, ..., A_n are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

Example 6. Consider a string of 20 Christmas tree lights connected in series. Suppose the probability that a light bulb fails is 2%. What is the probability that the string fails?

P(at lest 1 light fails)

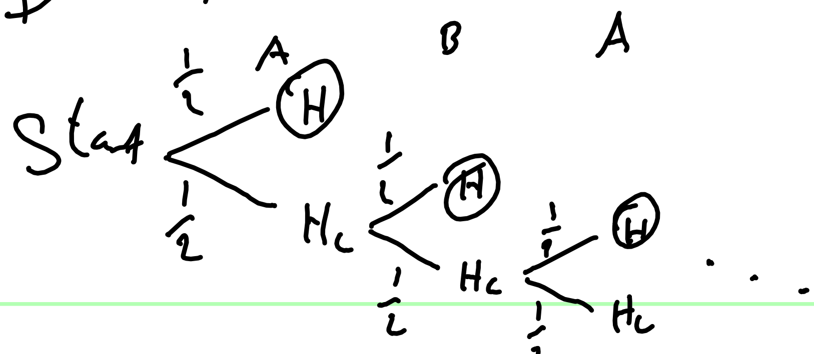
$= 1 - P(\text{no fails})$

$= 1 - (0.98)^{20} \approx 0.33239$

Example 7. Players A and B toss a fair coin in order. The first player to throw a head wins and ends the game. What are their respective chances of winning?

A wins if 1^{st} head occurs on turn 1, 3, 5, ... etc.

B wins if 1^{st} head occurs on turn 2, 4, 6 ... etc.



$P(A) = P(H) + P(TTH) + P(TTTTH) + \dots$

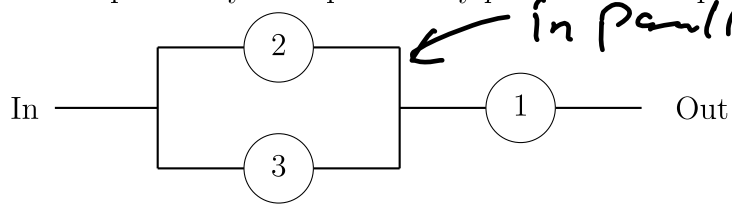
$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} \dots$

$r = \frac{1}{4}$,

$= \dfrac{\frac{1}{2}}{1-\frac{1}{4}} = \dfrac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$

$P(B) = 1 - P(A)$
$= 1/3$

Example 8. (Circuit Problem.) Consider the following 3-switch circuit. Suppose each component fail independently with probability p. What is the probability of the circuit fail?



in parallel

in series

in

$F_i = i^{th}$ component fails, $P(F_i) = p$

$P((F_2 \cap F_3) \cup F_1)$

$= P(F_2 \cap F_3) + P(F_1) - P(F_1 \cap F_2 \cap F_3)$

$\boxed{= p^2 + p - p^3}$

Example 9. Roll a unfair (biased) coin 9 times. (Or, roll 9 coin once.) Suppose the probability of getting Head is $P(H) = p = 1/3$.

Find: (1). P(all Heads)

By independent, P(all heads) $= P(H_1 \cap H_2 \cap H_3 \ldots) = P(H_1)P(H_2)\cdots$

$= \left(\frac{1}{3}\right)^9$

(2). P(no Head)

$P(\text{no head}) = P(\text{All } T) = \left(\frac{2}{3}\right)^9$

(3). P(Exactly one Head)

$= \binom{9}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^8 = 0.1170$

(4). P(Exactly three Heads)

$\binom{9}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^6 = 0.2731$