§3.5 Expected Values

Expected value is a generalization of the concept "average".

Definition: Expected Value of **discrete** random variable

If X is a discrete random variable with probability function $p_X(k)$, then the **expected value** (or Mean) of X is

$$E(X) = \sum_{\text{all } k} k \cdot p_X(k).$$

Example 1. (In \$ 3.3) Suppose you have \$10 and you go to gamble. Each time, you will either win or lose \$1. Each time, the probability that you will win is $\frac{1}{3}$.

We set k as the number of your winning times. The random variable Y as the amount of your money and we already calculated that $Y = 4 + 2k$. We already have the **pdf** of Y as

k	0	1	2	3	4	5	6
$P(X = k)$	0.0878	0.2634	0.3294	0.2195	0.0823	0.0164	0.0014
$P(Y = 4 + 2k)$	0.0878	0.2634	0.3294	0.2195	0.0823	0.0164	0.0014
$4 + 2k$	4	6	8	10	12	14	16

Q: What is the expect value of Y?

$$E(Y) = \sum_{\text{all } k} (4+2k) \cdot P(Y = 4+2k)$$
$$= 4 \cdot P(Y=4) + 6 \cdot P(Y=6) \ldots = \$8$$

Q: What is the expect value of k?

$$E(X) = \sum_{k \in X} kP(k) = 0(0.0878) + 1(0.2634) \ldots 6(0.0014)$$
$$= 2$$

Proposition.

$$E(aX + b) = aE(X) + b$$

Example 2. (Practice) Toss **2** fair 6-sided dice.

Let X be the **difference** of the two numbers (large−small).

Generally:

X, Y RV, a, b constant

$$E[aX + bY]$$

$$= aE[X] + bE[Y]$$

The sample space S has 36 sample points given by

$$S = \left\{ \begin{array}{llllll} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

So $X(S) = \{0, 1, 2, 3, 4, 5\}$.

What is the **pdf** of X?

k	0	1	2	3	4	5
$p_X(k)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

What is the expect value of X?

$0 \cdot \frac{6}{36} + 1 \cdot \frac{10}{36} + 2 \cdot \frac{8}{36} + 3 \cdot \frac{6}{36} + 4 \cdot \frac{4}{36} \cdot 5 \cdot \frac{2}{36}$

$= 35/18$, so just less than 2

Theorem. Expected value of binomial distribution

Suppose X is a binomial random variable with parameters n and p. That is $p_X(k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$. Then

$$E(X) = np.$$

In a multiple-choice test, there are 100 questions and each with five possible answers. Let X be the number of correct answers just by guessing. Then X binomial random variable with parameters $n = 100$ and $p = \frac{1}{5}$. So, by Theorem, $E(X) = np = 20$.

We would "expect" to get 20 correct answers by "intuition".

Proof.

pdf for binary RV

$$
\begin{aligned}
E(X) &= \overbrace{\sum_{k=0}^{n} k\binom{n}{k} \cdot p^k(1-p)^{n-k}}^{\text{pdf for binary RV}} \qquad \text{def. of } \binom{n}{k} \\
&= \sum_{k=0}^{n} k\frac{n!}{(n-k)!k!} \cdot p^k(1-p)^{n-k} \qquad \frac{k}{k!} = \frac{1}{(k-1)!} \\
&= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k-1)!} \cdot p^k(1-p)^{n-k} \qquad n! \Rightarrow (n-1)! \\
&= np\sum_{k=1}^{n} \binom{n-1}{k-1} \cdot p_x^{k-1}(1-p)_y^{n-k} \\
&= np(\underbrace{p+1-p}_{\text{cancels}})^{n-1} \\
&= np
\end{aligned}
$$

factor out an np together

binomial thm.

$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$

$x = p$ $\quad n = n-1$

$y = 1-p$

Definition: (Expected Value of continuous random variable)

If X is a continuous random variable with probability function $p_X(k)$, then the **expected value** (or Mean) of X is

$$E(X) = \int_{-\infty}^{\infty} x \cdot p_X(x)\, dx.$$

Example 3. The **pdf** for a continuous random variable Y is

$$p_Y(y) = \frac{3}{8}(y^2+1) \text{ for } -1 \le y \le 1.$$

this is how we know it's a continuous distribution

Find the expected value (mean) $E(Y)$.

The expected value of Y is

$$E(Y) = \int_{-1}^{1} y \cdot \frac{3}{8}(y^2+1)\, dy$$

$$= \frac{3}{8}\int_{-1}^{1} y^3 + y\, dy = \frac{3}{8}\left[\frac{y^4}{4} + \frac{y^2}{2}\right]_{-1}^{1} = 0$$

Example 4. Let X be an exponential random variable,

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x \ge 0.$$

Find the expected value of X.

The expected value of X is

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_{0}^{\infty} x\lambda e^{-\lambda x} dx \\ &= \left[-xe^{-\lambda x}\right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx \\ &= 0 + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty} \\ &= \lim_{t\to\infty}\left(\frac{1}{-\lambda e^{\lambda t}} + \frac{1}{\lambda}\right) \\ &= \frac{1}{\lambda} \end{aligned}$$

$u = x$, $dv = \lambda e^{-\lambda x} dx$
$du = 1 dx$ $v = -e^{-\lambda x}$
$uv - \int v\, du$
$\left[-xe^{-\lambda x} - \int -e^{-\lambda x} dx\right]_0^{\infty}$

make a quotient, apply L'Hôpital's

For the third equality, we need integral by parts for $u = x$ and $v' = \lambda e^{-\lambda x}$. Then $u' = 1$ and $v = -e^{-\lambda x}$. So, $\int uv'dx = uv - \int u'vdx$.

Sometimes, the mean is not enough to describe the variable. Especially if there are extreme values on both sides.

Definition.

Let X be a discrete random variable. The **median** of X is the number m such that $P(X \leq m) \geq 0.5$ and $P(X \geq m) \geq 0.5$.

Example 5. Find the median of the random variable Example 3.

k	0	1	2	3	4	5
$p_X(k)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

$k = 2$ is the median.

Definition.

Let Y be a continuous random variable. The **median** of Y is the number m such that

$$\int_{-\infty}^{m} f_Y(y)\, dy = 0.5$$

Example 6. Find the median of

exponential, $\lambda = 1$

$$f_X(x) = e^{-x} \text{ for } x \geq 0.$$

$$\int_0^m e^{-x}\,dx = 0.5$$

$$-e^{-x}\Big|_0^m = 0.5$$

$$(-e^{-m} + 1) = 0.5$$

$$-e^{-m} = -0.5$$

$$\ln(e^{-m}) = \ln(0.5)$$

$$\boxed{m = -\ln\left(\tfrac{1}{2}\right) = -\ln(2^{-1}) = \ln(2)}$$

§3.6 Variance ~ how far the data is spread out

Theorem.

$E(g(X)) = \sum_{x \in S} g(x) f_X(x)$ if X is a discrete random variable.

ex. $E[X^2] = \sum_{x \in S} x^2 f_X(x)$

$E(g(Y)) = \int_{-\infty}^{\infty} g(y) f_Y(y)$ if Y is a continuous random variable.

In particular, we care about the case when $g(X) = X^2$.

Warning: $E(X^2) \neq (E(X))^2$. *

Example 1. Calculate $E(X^2)$ for the random variable X in Example 2 in §3.5.

Solution:

$$E(X^2) = \frac{6}{36}(0) + \frac{10}{36}(1^2) + \frac{8}{36}(2^2) + \frac{6}{36}(3^2) + \frac{4}{36}(4^2) + \frac{2}{36}(5^2) = \frac{35}{6} \approx 5.83.$$

Example 2. Calculate $E(Y^2)$ for the random variable in Example 3 in §3.5.

$$\begin{aligned} E(Y^2) &= \int_{-1}^{1} y^2 \frac{3}{8}(y^2+1)dy \\ &= \frac{3}{8}\int_{-1}^{1} y^4 + y^2 dy \\ &= \frac{3}{8}\left[\frac{y^5}{5} + \frac{y^3}{3}\right]_{-1}^{1} \\ &= 2/5 \end{aligned}$$

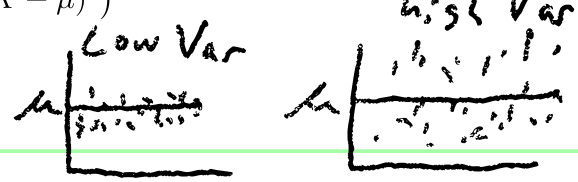
Definition. (Variance)

The **variance** of a random variable X is

avg. squared distance from the mean

$$\mathrm{Var}(X) := E\big((X-\mu)^2\big)$$

Here $\mu = E(X)$ is the mean of X.
The **standard deviation** is $\sigma := \sqrt{\mathrm{Var}(X)}$



Remark: This is the mean of the squared distance from the mean. It measures the spread of the data.

avg. dist. from the mean

Theorem.

Let X be a random variable.

$$\mathrm{Var}(X) = E(X^2) - \mu^2$$

μ ← $E(X)$

Proof on next page

Example 3. Calculate the **variance** for the random variable X in Example 1. discrete

$$\mathrm{Var}(X) = E(X^2) - \mu^2 = 5.83 - (1.94)^2 = 2.07$$

Example 4. Calculate the **variance** for the random variable in Example 2. continuous

$$\mathrm{Var}(Y) = E(Y^2) - \mu^2 = 0.4 - (0)^2 = 0.4$$

Example 5. Calculate the standard deviation of X with pdf

$$f_X(x) = \begin{cases} 2-x, & 1 \le x \le 2 \\ 1/2 & 3 \le x \le 4 \end{cases}$$

$$\mu = E[X] = \int_1^2 x(2-x)\,dx + \int_3^4 \frac{x}{2}\,dx$$

$$= 29/12$$

$$E[X^2] = \int_1^2 x^2(2-x)\,dx + \int_3^4 \frac{x^2}{2}\,dx = \frac{85}{12}$$

$$\mathrm{Var}(X) = E[X^2] - \mu^2 = \frac{85}{12} - \left(\frac{29}{12}\right)^2 \approx 1.22,$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{1.22} \; \boxed{= 1.1}$$

Example 6. (Homework 11) Let X be an exponential random variable with

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x \ge 0.$$

Find the variance and standard deviation of X.

Proof of thm:

$$Var(X) = E(X^2) - \mu^2$$

Proof:

$$Var(X) = E[(x-\mu)^2]$$

$$= E[x^2 - 2x\mu + \mu^2]$$

$$= E[X^2] - 2\mu E[X] + \mu^2$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

wwwww

QED

$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

jesus christ

From Example §3.5, we have $E(X) = \frac{1}{\lambda} = \mu$

$$Var(X) = E[X^2] - \mu^2 = E[X^2] - \frac{1}{\lambda^2}$$

$$E[X^2] = \int_0^\infty \underbrace{x^2}_{\text{poly.}} \cdot \underbrace{\lambda e^{-\lambda x}}_{\text{exp.}} \, dx \qquad u = x^2 \quad dv = \lambda e^{-\lambda x} dx \qquad du = 2x\,dx \quad v = -e^{-\lambda x}$$

$$= -x^2 e^{-\lambda x} + 2\int x e^{-\lambda x} \, dx \qquad u = x \quad dv = e^{-\lambda x} \qquad du = dx \quad v = -\frac{1}{\lambda} e^{-\lambda x}$$

$$= -x^2 e^{-\lambda x} + 2\left(\frac{-x}{\lambda} e^{-\lambda x} + \frac{1}{\lambda}\int e^{-\lambda x} \, dx\right)$$

$$\lim_{x\to\infty}\left(\cancel{\frac{-x^2}{e^{\lambda x}}}^{\to 0} - \cancel{\frac{2}{\lambda}\cdot\frac{x}{e^{\lambda x}}}^{\to 0} - \cancel{\frac{2}{\lambda^2} e^{-\lambda x}}^{0}\right) - \left(\frac{-2}{\lambda^2} e^0\right)$$

$$= \frac{2}{\lambda^2} \Rightarrow \boxed{Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}, \quad \sigma = \frac{1}{\lambda}}$$

Theorem.

$$\operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$$

Theorem.

Let X be the binomial random variable

$$\operatorname{Var}(X) = np(1 - p)$$

Proof:

$$Var(aX + b) = Var(aX) + Var(b)$$

$$= a^2 Var(X) + 0$$ ← variance of a constant is ZERO

$$= a^2 Var(X)$$

More about exponential random variable

Let X be an exponential random variable with **pdf** given by

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0$$

where λ is a fixed positive number.

In §3.4, we verified that it is a **pdf** with **cdf** $F_X(x) = 1 - e^{-\lambda x}$.
In §3.5, we computed the mean of X, which is $E(X) = \frac{1}{\lambda}$
In §3.6 we computed the variance of X, which is $\text{Var}(X) = \frac{1}{\lambda^2}$

This is a very useful random variable to model the **life time** of some objects, i.e., computer parts, electric equipment, etc.

Geometric Distribution.

There is a discrete random variable works similarly. For example, if we flip a unfair (biased) coin with $P(Head) = p$. Let Y denote the times until we get our first Head. The **pdf** of Y is

$$p_Y(k) = \underbrace{(1-p)^{k-1}}_{\text{fail } k-1 \text{ times}} p \leftarrow \text{succeed once}$$

for $k = 1, 2, 3, 4, ...$

The mean of Y is $\frac{1}{p}$.

Q_2: show $E(X \sim \text{geometric}) = \frac{1}{p}$

eg. Show: $\cancel{\int_1^\infty x(1-p)^{x-1} p \, dx = \frac{1}{p}}$ $\leftarrow$ Series, not integral,

$\cancel{p \int_1^\infty x(1-p)^{x-1} \, dx}$

Show:

$$\sum_{x=1}^{\infty} x(1-p)^{x-1} p = \frac{1}{p}$$

$$p \sum_{x=1}^{\infty} x(1-p)^{x-1} \leftarrow \text{geo. series}$$

§3.6 Variance

More example: Uniform Distribution

Let X be the **uniform distribution** on $[a, b]$. We already know the pdf function is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{for others} \end{cases}$$

Find the expected value and variance of X.

The expected value is

$$E(X) = \int_a^b \frac{1}{b-a} x dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \frac{a+b}{2}$$

$$E(X^2) = \int_a^b \frac{1}{b-a} x^2 dx = \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b = \frac{1}{3}(a^2 + ab + b^2)$$

The variance is

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{3}(a^2 + ab + b^2) - \frac{1}{4}(a^2 + 2ab + b^2) = \frac{1}{12}(b-a)^2.$$