Ch. 4.2, Poisson Distribution

of events in a certain $\underbrace{\text{time, area, volume}}_{\text{Some measurement}}$ etc.

Poisson - discrete dist.

$X \sim \text{Poisson}(\lambda)$ $X = 0, 1, 2 \ldots \infty$

- Counting # of things in a measured quantity
  - Ex: 50 emails / hour
- Prob of occurrence is the same for all items
- # of occurrences is independent across measured quantities

Ex. X = # of students who attend session on Friday

Y = # of accidents each month

Z = # of weeds growing in a 1 sq foot garden

W = # of worms per acre on a farm

Poisson Distribution

X = # of occurrences in specific unit of measurement

$$\text{pdf} = P(X=k) = P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$X \sim \text{Poisson}(\lambda)$

$$P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Thm:

$$\sum_{\text{all } k} P_X(k) = 1 \quad \left(\because \sum_{k=0}^{\infty} P_X(k) = 1 \right)$$

<u>Proof</u>

$$\sum_{k=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \cdots \right)$$

$$= e^{-\lambda} e^{\lambda} = e^0 = 1$$

2) $$E(X) = \sum_{k=0}^{\infty} k P_X(k) = \sum_{k=0}^{\infty} \cancel{k} \cdot \frac{\lambda^k e^{-\lambda}}{\cancel{k!}\,(k-1)!}$$

$$= e^{-\lambda} \lambda \cdot \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

$$= e^{-\lambda} \lambda \cdot e^{\lambda} = \lambda$$

Poisson to approximate Binomial RV.

$X \sim Bin(n,p)$ $\qquad$ $X \sim Poisson(\lambda)$

$\lambda = np$ — (n large, p small

$$\lim_{\substack{n\to\infty \\ p\to 0}} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}$$

How to determine λ

$\lambda = (\text{average \# of occurrences per unit})(\text{\# units})$

or rate

↑ p $\qquad$ ↑ n

Ex. 1) X = # of customers who will enter in a 9 minute period

$\lambda = (2 \text{ customers} / 3 \text{ min}) \cdot (3 \text{ 3min intervals})$

$= 6$

$$P_X(k) = \frac{6^k e^{-6}}{k!}$$

P(4 or fewer customers enter in 9 min)

$= P(4) + P(3) + P(2) + P(1) + P(0)$

≈ 0.2851