Dylan Black

5.4 Properties of Estimators

Ex. $\hat{\mu} = \overline{X}$ is a point estimator for μ

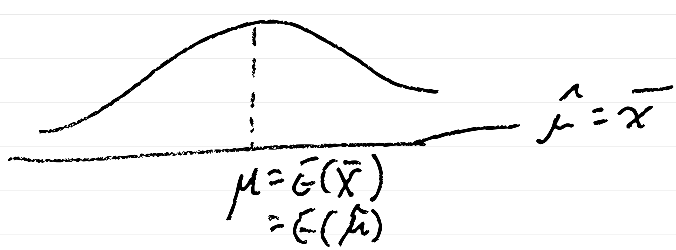
2) $\hat{\mu} = \frac{\sum_{i=1}^{n} x_i^2}{n}$

3) $\hat{\mu} = \overline{X} - 1$

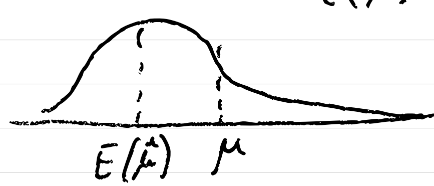
Ex $X \sim N(\mu, \sigma^2)$, μ is unknown

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

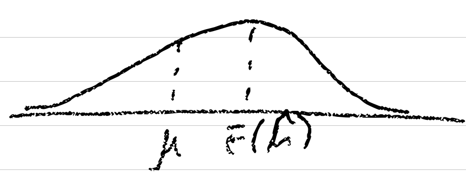
1:



2)



3



Def) $\hat{\theta}$ is unbiased if $E(\hat{\theta}) = \theta$

Ex. Show that for random sample $X_1, X_2 \ldots$ of size n, $\bar{X}$ is an unbiased estimator of the mean μ

<u>Proof</u> Need to show that $E(\bar{X}) = \mu$

$\Rightarrow \hat{\mu} = \bar{X}$

$$E(\bar{X}) = E\left(\frac{\sum_i X_i}{n}\right) = E\left(\frac{X_1 + X_2 \ldots X_n}{n}\right)$$

$$= \frac{1}{n}\left(E(X_1) + E(X_2) \ldots + E(X_n)\right)$$

$$= \frac{1}{n}(n\mu) = \mu \quad \text{QED}$$

recall, $E(X_i) = E(X) = \mu$

<u>Ex. 2)</u> Estimator includes Y_1 & Y_2

$$\hat{\mu} = cY_1 + (1-c)Y_2 \qquad 0 \leq c \leq 1$$

1) What c makes $\hat{\mu}$ unbiased?

$E(\hat{\mu}) = \mu$ by def. if $\hat{\mu}$ is unbiased

$$E(\hat{\mu}) = E(cY_1 + (1-c)Y_2) = cE(Y_1) + (1-c)E(Y_2)$$

$= c\mu + (1-c)\mu = \mu$

true for all $0 \le c \le 1$

2) $\hat{\mu} = 0.2Y_1 + 0.6Y_2$

$E(\hat{\mu}) = 0.2\mu + 0.6\mu = 0.8\mu$

$$0.8\mu \begin{cases} \neq \mu & \text{if } \mu \neq 0 \\ = \mu & \text{if } \mu = 0 \end{cases}$$

$\hat{\mu}$ is biased when $\mu \neq 0$, but is unbiased when $\mu = 0$.

$f_Y(y;\theta) = 2y\theta^2, \quad 0 < y < \frac{1}{\theta}$

$\widehat{\left(\frac{1}{\theta}\right)} = c(Y_1 + Y_2)$ ← note: my answer's right, but in notes, it's $2Y_2$, not Y_2.

$= cY_1 + cY_2$

$E\left(\widehat{\left(\frac{1}{\theta}\right)}\right) = E(cY_1 + cY_2) = cE(Y_1) + cE(Y_2) \overset{\text{set}}{=} \frac{1}{\theta}$

$E(Y_1) = E(Y_2)$

$$= \int_{-\infty}^{\infty} y f_Y(y ; \theta) \, dy$$

$$= \int_0^{\frac{1}{\theta}} y \cdot 2y\theta^2 \, dy$$

$$2\theta^2 \int_0^{1/\theta} y^2 \, dy = 2\theta^2 \cdot \left(\frac{y^3}{3}\right)\Big|_0^{\frac{1}{\theta}}$$

$$= 2\cancel{\theta^2} \cdot \frac{1}{3\theta^{\cancel{3}}} = \frac{2}{3\theta}$$

$$c \cdot \frac{2}{3\theta} + c \cdot \frac{2}{3\theta} = \frac{1}{\theta}$$

$$2 \cdot c \cdot \frac{2}{3\theta} = \frac{1}{\theta} \Rightarrow c \cdot \frac{4}{3\theta} = \frac{1}{\theta}$$

$$\boxed{c = \frac{3}{4} = 0.75}$$

Example 4:

given sample $Y_1, Y_2, \ldots Y_n$ from

$Y \sim N(\mu, \sigma^2)$ both unknown

MLE for σ^2:

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

is $\hat{\sigma^2}$ unbiased?

$$E(\hat{\sigma^2}) \overset{?}{=} \sigma^2$$

$$E(\hat{\sigma^2}) = E\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right)$$

$$= \frac{1}{n} E\left(\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right)$$

$$= \frac{1}{n} E\left((Y_1 - \bar{Y})^2 + (Y_2 - \bar{Y})^2 \ldots\right)$$

$$\cdots = \frac{n-1}{n}\sigma^2$$

$\hat{\sigma^2}$ is not unbiased!

Note: $E(\hat{\sigma^2}) = \frac{n-1}{n}\sigma^2 \quad \times \frac{n}{n-1}$

$$\sigma^2 = \frac{n}{n-1} E(\hat{\sigma^2}) = E\left(\frac{n}{n-1}\hat{\sigma^2}\right)$$

Let $S^2 = \frac{n}{n-1} \hat{\sigma}^2$

$$S^2 = \frac{\cancel{n}}{n-1} \cdot \frac{\sum_1^{!} (r_i - \bar{r})^2}{\cancel{n}} = \frac{\sum_1^{!} (r_i - \bar{r})^2}{n-1}$$

is an unbiased estimator for σ^2..

S^2 is the sample variance, S is sample std. deviation