§4.2 Poisson Distribution

With the Binomial, our random variable X was the number of successes out of n trials of an experiment. The Poisson distribution counts the number of occurrences per unit of measurement, for instance, a specific period of time or in a specific area or volume.

Characteristics of Poisson Distribution

(1) An experiment with a Poisson RV consists of counting the number of times a certain event occurs during a given unit of time, area or volume.

(2) The probability an event occurs in a given unit of measurement is the same for all similar units (i.e. if month is your unit, the probability is the same for every month).

(3) The number of times the event occurs in one unit of measurement is independent of the other like units.

Some Examples of Poisson Random Variables:

- X = The number of students who attend a seminar on Friday.
- X = The number of industrial accidents in a plant each month.
- X = The number of weeds growing in a one square foot section of yard.
- X = The number of worm larvae per acre on a farm.
- X = The number of cars that run a particular stop sign during one day.

Suppose that we can expect some independent event to occur λ times over a specified time interval. X: = the number of occurrences is the Poisson random variable.

Definition. (Poisson Distribution)

The **Poisson Distribution** Piosson(λ) is a discrete **pdf** function defined as

$$p_X(k) = P(X = k) := \frac{\lambda^k e^{-\lambda}}{k!}$$

for $k = 0, 1, 2, 3, ...$ Here, λ is a positive constant.

Theorem.

(1) It is a well defined **pdf** , i.e., $\sum_k p_X(k) = 1$
(2) The mean is $E(X) = \lambda$.
(3) The variance is $\mathrm{Var}(X) = \lambda$.

Proof:
(1)

$$\sum_{\text{all } k} p_X(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

(2)

$$E(X) = \sum_{\text{all } k} k p_X(k) = \sum_{k=0}^{\infty} \frac{k\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$
$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

(3) Similarly as (2) but more tricky.

Historically, Poisson distribution is used as an approximation for binomial distribution

$$p_Y(k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}, \text{ for } k = 0, 1, 2, \ldots, n.$$

Applications of Poisson distribution:

1. Poisson approximation for binomial distribution.

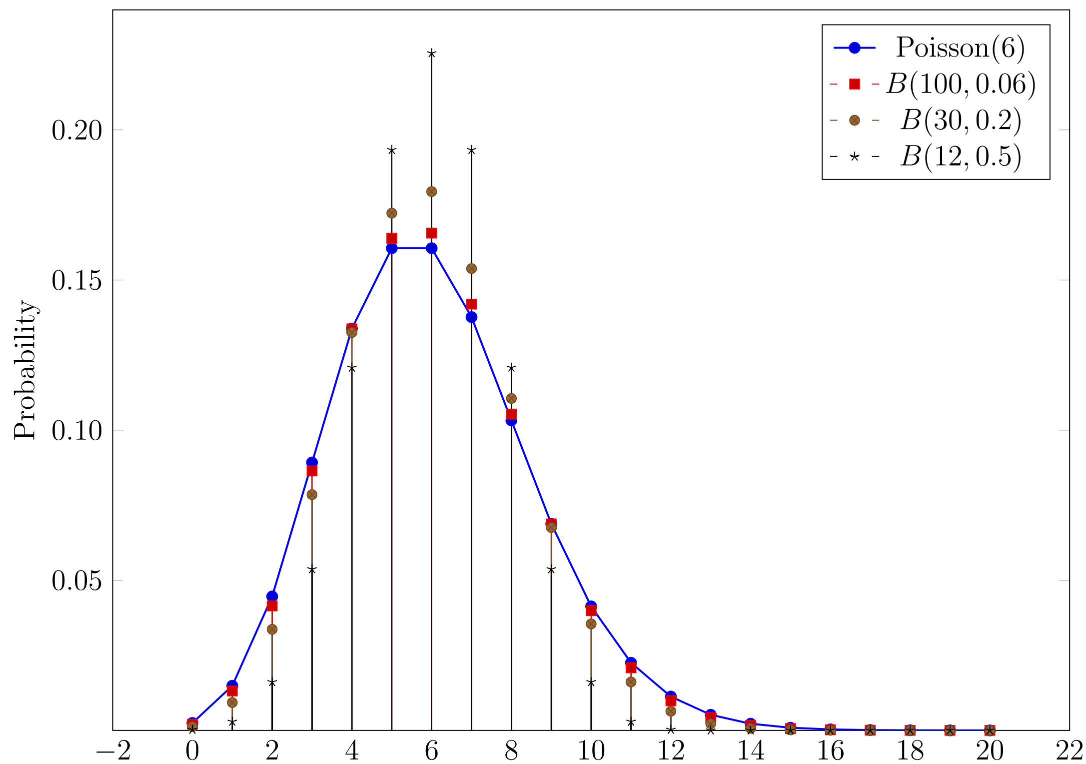
Theorem. Poisson limit

If n is large and p is small, then let $\lambda = np$, we have

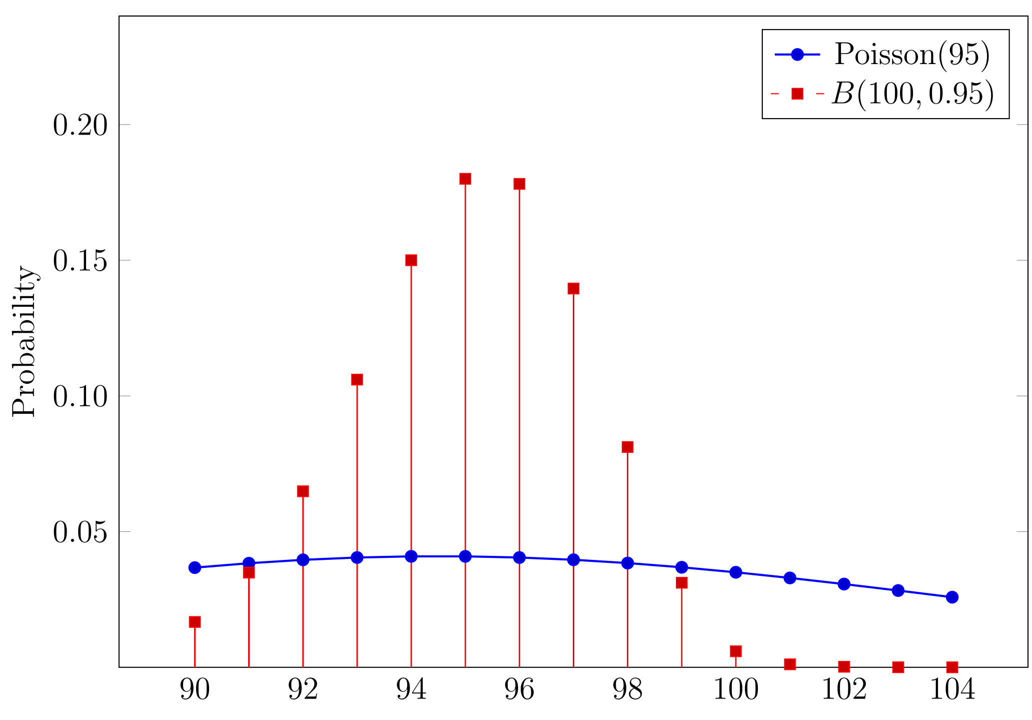
$$\frac{\lambda^k e^{-\lambda}}{k!} \approx \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

More precisely, if $np = \lambda$ is constant,

$$\lim_{\substack{n\to\infty \\ p\to 0}} \binom{n}{k} \cdot p^k (1-p)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}$$



Recall: When $n \to \infty$ and $p \to 0$ in such a way that $\lambda = np$ remains constant, the Poisson distribution appear as an approximation to Binomial distribution.



How to determine λ

λ = (the average number of occurrences per unit) * (the length of the observation period)

Example 1: Suppose that a fast-food restaurant can expect two customers every 3 minutes, on average. What is the probability that four or fewer patrons will enter the restaurant in a 9-minute period?

X = # of customers per 9 minute interval

λ = 2 customers × 3 per-minutes

$\lambda = 6$

$P(X \leq 4) = P(X=0) + P(X=1) + \ldots + P(X=4)$

$P(X=k) = \dfrac{6^k e^{-\lambda}}{k!} \Rightarrow \sum_{i=0}^{4} P(X=i) = \boxed{0.2851}$

$P(X \leq k) = \sum_{i=0}^{k} P(X=i) = \text{poissoncdf}(\lambda, k)$

Calculator TI-83/TI-84: 2ND → VARS → C:poissonpdf(

poissonpdf(λ, k)

Example 2: In a new fiber-optic communication system, transmission errors occur at the rate of 1.5 per ten seconds. What is the probability that more than two errors will occur during the next half-minute?

X = # errors per half-minute

$\lambda = 1.5 \cdot 3 = 4.5$

$P(X > 2) = 1 - P(X \leq 2) = 1 - \text{poissoncdf}(4.5, 2)$

$= 0.8264$

Example 3: Suppose that on-the-job injuries in a textile mill occur at the rate of 0.1 per day. What is the probability that two accidents will occur during the next five-day workweek?

X = # accidents in 5 days

$\lambda = 0.1 \cdot 5 = 0.5$

$P(X = 2) = \text{poissonpdf}(0.5, 2)$

$= 0.07582$

Example 4: A telephone is monitored for 1 hour, during which time the total number of phone calls received is 15. What is the probability that no phone calls will be received in the next 10 minutes?

$X = \#$ phone calls in 10 mins

$\lambda = (15 \text{ per hr})(\frac{1}{6} \text{ hr}) = \frac{15}{6}$

$P(X=0) = \text{poissonpdf}(\frac{15}{6}, 0)$

$= 0.0821$

Note: If $n \geq 100$, then the Poisson probability is a good approximation to binomial probability.

Example 5: According to an airline industry report, roughly 1 piece of luggage out of every 200 that are checked is lost. Suppose that a frequent flying business woman will be checking 120 bags over the course of the next year. Use the Poisson distribution to approximate the probability that she will lose 2 or more pieces of luggage.

1) Binomial prob

$X \sim B(n, p)$

$n = 120,\ p = \frac{1}{200}$

$P(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - \text{binomcdf}(120, \frac{1}{200}, 1)$

$= 1 - 0.8784 = 0.1216$

2) Poisson prob

$1 - \text{poissoncdf}(\frac{3}{5}, 1)$

$= 0.1219$