§3.9 Further Properties of the Mean and the Variance

Theorem.

$$E(aX + bY) = aE(X) + bE(Y)$$

for **any** two random variables X and Y and numbers a and b.

Linearity of expectation

Proof of continuous random variables:

$$E(aX + bY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ax + by)f_{X,Y}(x, y)dxdy$$

$$= a\int_{-\infty}^{\infty} xf_X(x)dx + by\int_{-\infty}^{\infty} y\,f_Y(y)dy = aE(X) + bE(Y).$$

The above formula can be generalized to more random variables.

Theorem.

If X and Y are **independent**, then

$$E(XY) = E(X)E(Y).$$

Proof.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\,dy\,dx$$

$$= \underbrace{\int_{-\infty}^{\infty} x f_X(x)}_{E(X)} \underbrace{\int_{-\infty}^{\infty} y f_Y(y)\,dy}_{E(Y)}\,dx$$

$$= E(Y)E(X) = E(X)E(Y)$$ QED

$\text{Covariance}(X,Y) = E[X,Y] - E(X)E(Y)$

Theorem.

If X and Y are **independent**, then

$$\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$$

Proof.

$$\begin{aligned}
\text{Var}(aX + bY) &= E((aX + bY)^2) - (E(aX + bY))^2 \\
&= E((a^2X^2 + 2abXY + b^2Y^2) - (aE(X) + bE(Y))^2 \\
&= a^2E(X^2) + 2abE(XY) + b^2E(Y^2) - a^2E(X)^2 - 2abE(X)E(Y) - b^2E(Y)^2 \\
&= a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab(E(XY) - E(X)E(Y))
\end{aligned}$$

Since X and Y are independent, then $E(XY) - E(X)E(Y)$. Hence, $\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$.

In general, we have the following theorem.

Theorem.

$$\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2ab\,\text{Cov}(X, Y)$$

Here, $\text{Cov}(X, Y)$ is the **covariance** of X and Y defined as

$$\boxed{\text{Cov}(X, Y) := E(XY) - E(X)E(Y)}$$

If X and Y are independent, then $\text{Cov}(X, Y) = 0$.

The converse is not true: $\text{Cov}(X, Y) = 0$, but X and Y are NOT independent.

$n(S) = 5,$

Example 1. Consider the sample space $S = \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$, where each point is assumed to be equally likely. Define the random variable X to be the first component of a sample point and Y, the second. Then $X(-2, 4) = -2, Y(-2, 4) = 4$, and so on. Show that $\text{Cov}(X, Y) = 0$, but X and Y are dependent.

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$= 0$

$E(X) = -\frac{2}{5} - \frac{1}{5} - 0 + \frac{1}{5} + \frac{2}{5} = 0$

$E(Y) = 0 + \frac{2}{5} + \frac{8}{5} = 2$

$E(XY) = \frac{1}{5}(-2\cdot 4 + -1\cdot 1 + 0\cdot 0 + 1\cdot 1 + 2\cdot 4) = 0$

X	-2	-1	0	1	2
P	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Y	0	1	4
P	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

Example 2. Consider the experiment in homework 3.7.8 and 3.7.17.: Toss a fair coin 3 times. Let X denote the number of heads on the last flip. and let Y denote the total number of heads on the three flips. We already have the joint pdf $p_{X,Y}(x,y)$ given by

Outcome	X	Y
HHH	1	3
HHT	0	2
HTH	1	2
HTT	0	1
THH	1	2
THT	0	1
TTH	1	1
TTT	0	0

(x,y)	$p_{X,Y}$
(0,0)	1/8
(0,1)	2/8
(0,2)	1/8
(0,3)	0
(1,0)	0
(1,1)	1/8
(1,2)	2/8
(1,3)	1/8

y \ x	0	1
0	1/8	0
1	2/8	1/8
2	1/8	2/8
3	0	1/8

(0)(HW 3.7.17) Find the marginal pdfs of X and Y.

$Y \backslash X$	0	1	$P(Y)$
0	1/8	0	1/8
1	2/8	1/8	3/8
2	1/8	2/8	3/8
3	0	1/8	1/8
$P(X)$	1/2	1/2	$\Sigma = 1$

(1) Find the **mean** for X, Y, X^2, Y^2 and XY. Find the **variance** for X and Y.

$$E(X) = 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{2}\right) = \tfrac{1}{2}$$

$$E(Y) = 0\left(\tfrac{1}{8}\right) + 1\left(\tfrac{3}{8}\right) + 2\left(\tfrac{3}{8}\right) + 3\left(\tfrac{1}{8}\right) = \tfrac{3}{2}$$

$$E(XY) = 1\left(\tfrac{1}{8}\right) + 2\left(\tfrac{2}{8}\right) + 3\left(\tfrac{1}{8}\right) = 1$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 1 - \tfrac{1}{2}\cdot\tfrac{3}{2} = \tfrac{1}{4}$$

$$Var(X) = E(X^2) - E(X)^2 =$$

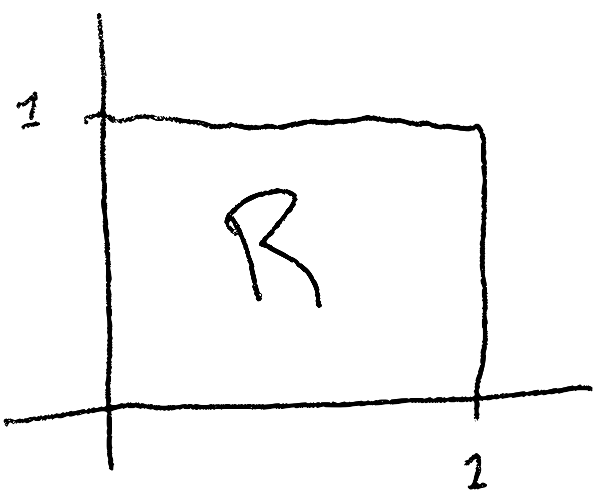
(2) ~~Find the covariance of X and Y.~~ $Var(Y) = E(Y^2) - E(Y)^2$

(3) Find the variance of $X + Y$.

$$Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Example 3. For the joint pdf $f_{X,Y}(x,y) = x+y$, for $0 \le x \le 1, 0 \le y \le 1$, find the variance of $X+Y$.

What is the variance of X?



$$Var(X+Y)$$
$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$f_X(x) = \int_0^1 x+y \, dy$$
$$= xy + \frac{y^2}{2}\Big|_0^1$$
$$= x + \frac{1}{2}, \quad 0 \le x \le 1$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{11}{144}$$
$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{11}{144}$$

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \frac{7}{12}$$

$$E(Y) = \int_0^1 y\left(y+\frac{1}{2}\right)dy = \frac{7}{12}$$

$$\boxed{\begin{aligned} &Var(X+Y) \\ &= \frac{11}{144} + \frac{11}{144} + 2\left(\frac{-1}{144}\right) \\ &= 5/36 \end{aligned}}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \frac{1}{144} = \frac{}{144}$$

$$E(XY) = \int_0^1\int_0^1 xy(x+y)\,dy\,dx = \int_0^1\int_0^1 x^2y + y^2x \,dy\,dx$$
$$= \int_0^1 \frac{x^2y^2}{2} + \frac{xy^3}{3}\Big|_0^1 dx = \int_0^1 \frac{x^2}{2} + \frac{x}{3}\,dx = \frac{x^3}{6} + \frac{x^2}{6}\Big|_0^1$$
$$= 1/3$$

Suppose random variables $X_1, X_2, \cdots, X_n$ are independent and each X_i has the same distribution. Consider the **sample sum**

$$X = X_1 + X_2 + \cdots + X_n$$

Example 4. In binomial distribution, we consider a series of n **independent** trials with **two** outcomes. The probability of "success" for each trial is **constant** p. Let X be the number of successes in n trials.

Let X_i be the number of success in the i-th trial. So, $X = X_1 + X_2 + \cdots + X_n$. The **pdf** of X_i is called **Bernoulli** distribution:

$X_i = k$	0	1
$P(X_i = k)$	$1-p$	p

Clearly, $E(X_i) = p$. Then

$E(X) = E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) = np.$

The variance of each X_i is $\mathrm{Var}(X_i) = p - p^2 = p(1-p)$. So,

$\mathrm{Var}(X) = \mathrm{Var}(X_1 + X_2 + \cdots + X_n) = np(1-p).$

Another important variable is the **sample mean**

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$$

As in our example, $E(\overline{X}) = p$ and $\mathrm{Var}(\overline{X}) = \frac{1}{n}p(1-p)$.

$$E(\overline{X}) = E\left(\frac{X_1 + X_2 \cdots X_n}{n}\right)$$
$$= \frac{1}{n}E(X_1 + X_2 \cdots X_n)$$
$$= \frac{1}{n}E(X) = \frac{1}{\cancel{n}}\cancel{n}p = p$$

$$\mathrm{Var}(\overline{X}) = \frac{1}{n^2}\mathrm{Var}(X) = \frac{1}{n^{\cancel{2}}}\cdot \cancel{n}p(1-p)$$
$$= \frac{1}{n}p(1-p)$$

The covariance $\mathrm{Cov}(X, Y)$ measures the association between X and Y. Another measure is the **correlation** of X and Y

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

The correlation is the normalized version of covariance, $-1 \le \mathrm{Corr}(X, Y) \le 1$. A positive correlation (or covariance) means that Y increases when X increases, while a negative correlation (or covariance) means that Y decreases when X increases.

In Example 2, the correlation of X and Y is

$$\mathrm{Corr}(X, Y) = \frac{1/4}{\sqrt{(\frac{1}{4})(\frac{3}{4})}} = \frac{1}{\sqrt{3}}$$

If $\mathrm{Corr}(X, Y) > 0$, X and Y are positively correlated and if $\mathrm{Corr}(X, Y) < 0$, X and Y are negatively correlated.

More question: $X_i \sim \text{Bernoulli}(p)$, $p = \frac{1}{200}$

Example 5. Suppose you attend the high school graduation party with 200 people each wearing the same hat. All people throw the hats into the center of the room and then each person randomly select a hat. (1) Find the probability you selects your own hat. (2) Find the expected number of people who select their own hats.

Let $X_i = 1$ if select own hat, $X_i = 0$ otherwise

$P(X_i = 1) = p = \frac{1}{200}$

$P(X_i = 0) = 1 - p = \frac{199}{200}$

X_i	1	0
$P(X_i)$	1/200	199/200

Let $X = X_1 + X_2 + X_3 \ldots + X_{200}$

$E(X) = E(X_1 + X_2 + X_3 \ldots + X_{200})$

$= E(X_1) + E(X_2) \cdots + E(X_{200}) = 200 \cdot \frac{1}{200}$

$= 1$

$X = 1, 2, 3, 4, \ldots 200$

$P(X = x) = \binom{200}{x} p^x q^{200-x}$