9.2 & 9.5 Two Sample Inferences on $\mu_1 - \mu_2$

§ 9.2 Test for the difference between population means using the *z* test or t test
§ 9.5 Confidence Intervals for $\mu_1 - \mu_2$

Comparing Two Means

Suppose you want to compare salaries between two companies. We will let $\boldsymbol{\mu_1}$ = mean salary from company 1 and $\boldsymbol{\mu_2}$ = mean salary from company 2. Since we want to compare these means, what "parameter" should we use? Intuitively, we could use the difference in means $\boldsymbol{\mu_1 - \mu_2}$. We can estimate this with the difference in sample means $\bar{X}_1 - \bar{X}_2$.

Two Sample Inferences on $\mu_1 - \mu_2$

■ For *known* sd's (σ_1, σ_2), use $\boldsymbol{z}$-distribution.

$$\bar{X}_1 - \bar{X}_2 \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$$

☆

H_0 is always

$\mu_1 = \mu_2$

Assumptions for the *z* Test or *z* CI to determine the difference between two means

1. Both samples are *random* samples.
2. The samples are *independent* of each other.
3. The standard deviation of both samples must be *known.*
4. If either of the sample sizes is less than 30, the populations must be *normal* or approximately *normally* distributed.

CI : $(\bar{X}_1 - \bar{X}_2) \pm z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

test statistic $z^* = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$

b/c of H_0

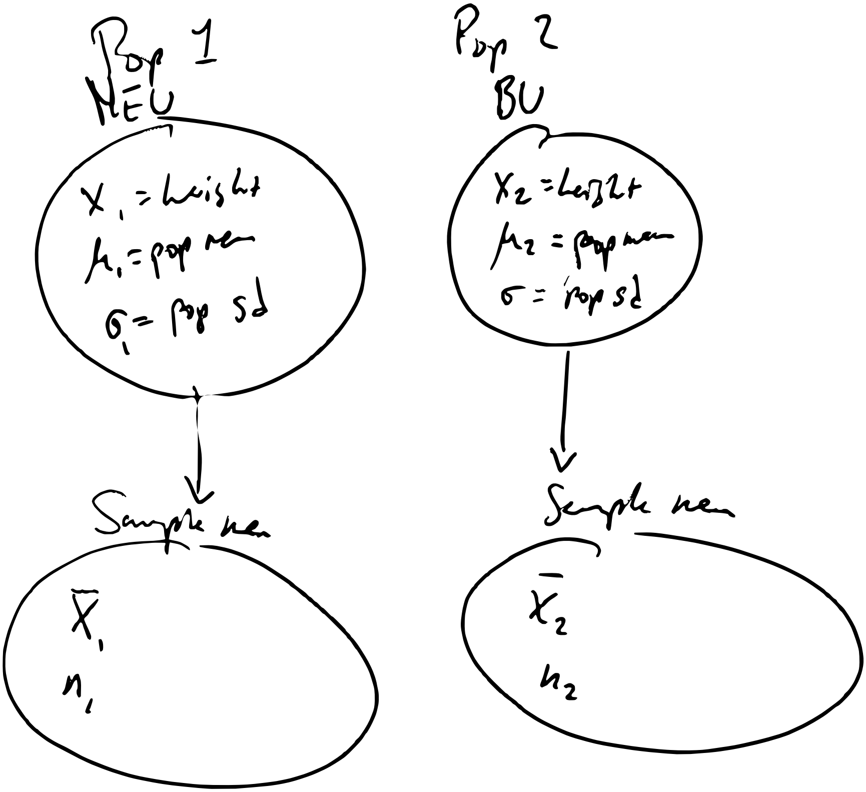
derived from $\frac{\bar{X} - \mu}{\sigma}$,

but $(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)$

↑ we assume $= 0$

Calculator: For hypothesis test, Stat >> Tests >> #3: 2-SampZTest
For CI, Stat >> Tests >> #9: 2-SampZInt

Comparing μ_1 and μ_2



$\mu_1 > \mu_2 \iff \mu_1 - \mu_2 > 0$

$\mu_1 < \mu_2 \iff \mu_1 - \mu_2 < 0$

$\mu_1 = \mu_2 \iff \mu_1 - \mu_2 = 0$

By CLT,

$\bar{X}_1 \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right)$

$\bar{X}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$

Best estimate for $\mu_i = \bar{X}_i$

$$E(\bar{X}_1 - \bar{X}_2)$$

$$= E(\bar{X}_1) - E(\bar{X}_2)$$

$$= \mu_1 - \mu_2$$

$\bar{X}_1$ and $\bar{X}_2$ are independent, so $\mathrm{Cov}(\bar{X}_1, \bar{X}_2) = 0$

$$Var(\bar{X}_1 - \bar{X}_2)$$

$$= Var(\bar{X}_1) + (-1)^2 (Var(\bar{X}_2))$$

$$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

$$\sigma(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\boxed{\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}$$

■ For *unknown* population standard deviations (σ_1, σ_2), use $\boldsymbol{t}$-distribution.

Assumptions for the *t* Test and t CI for determine the difference between two means

1. Both samples are *random* samples.
2. The samples are *independent* of each other.
3. The standard deviation of both samples must be *unknown*.
4. If either of the sample sizes is less than 30, the populations must be *normal* or approximately *normally* distributed.

1. If the population sd's are assumed to be equal (**pooled**):

CI: $(\bar{X}_1 - \bar{X}_2) \pm t_{\frac{\alpha}{2},df}\, s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$ $\sigma(\bar{x}_1 - \bar{x}_2)$ **test statistic** $t^* = \dfrac{\bar{X}_1 - \bar{X}_2}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$

Pooled sample sd. $s_p = \sqrt{\dfrac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}}$ and $df = n_1 + n_2 - 2$

Calculator: For hypothesis test, Stat >> Tests >> #4 2-SampTTest
For CI, Stat >> Tests >> #0 2-SampTInt then select Pooled: No Yes

2. If the population sd's are assumed to be **un**equal (**non-pooled**):

CI: $(\bar{X}_1 - \bar{X}_2) \pm t_{\frac{\alpha}{2},df}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},$ **test statistic** $t^* = \dfrac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}}$

$$df = \frac{\left(\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1-1}\left(\frac{s_1^2}{n_1}\right)^2+\frac{1}{n_2-1}\left(\frac{s_2^2}{n_2}\right)^2}$$

Calculator: For hypothesis test, Stat >> Tests >> #3 2-SampTTest
For CI, Stat >> Tests >> #0 2-SampTInt then select Pooled: No Yes

■ **Hypothesis Test**

Right-tailed test	**Left-tailed test**	**Two-tailed test**
H₀: $\mu_1 = \mu_2$ vs H₁: $\mu_1 > \mu_2$	H₀: $\mu_1 = \mu_2$ vs H₁: $\mu_1 < \mu_2$	H₀: $\mu_1 = \mu_2$ vs H₁: $\mu_1 \neq \mu_2$

$E(\bar{X}_1 - \bar{X}_2) = \mu_1 - \mu_2$ $\quad \bar{X}_1 \sim t\left(\mu_1, \frac{s_1^2}{n_1}\right)$

$\bar{X}_2 \sim t\left(\mu_2, \frac{s_2^2}{n_2}\right)$

1) Case 1: $\sigma_1 = \sigma_2$

$$Var(\bar{X}_1 - \bar{X}_2) = Var(\bar{X}_1) + Var(\bar{X}_2)$$

$$= \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}$$

$\sigma_1 = \sigma_2$

$\Rightarrow s_1 = s_2$

$= S_p$ pooled sample std deviation

$$= S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$S_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$$

$$\sigma(\bar{X}_1 - \bar{X}_2) = \sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

$$= S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

The basic format for hypothesis testing
Step 1: State the hypotheses and identify the claim.
Step 2: Compute the test value.
Step 3: Find the critical value(s)/p-value.
Step 4: Make the decision. CV and p-value methods are same as before.
Recall: *P*-value method

If *P*-value $\le \alpha$, reject H_0.
If *P*-value $> \alpha$, fail to reject H_0.

Example 1: A study using two random samples of 35 people each found that the average amount of time those in the age group of 26–35 years spent per week on leisure activities was 39.6 hours, and those in the age group of 46–55 years spent 35.4 hours. Assume that the population standard deviation for those in the first age group found by previous studies is 6.3 hours, and the population standard deviation of those in the second group found by previous studies was 5.8 hours. At $\alpha = 0.05$, can it be concluded that there is a significant difference in the average times each group spends on leisure activities?

$n_1 = 35$

$\bar{x}_1 = 39.6$

$\bar{x}_2 = 35.4$

$\sigma_1 = 6.3$

$\sigma_2 = 5.8$

$\alpha = 0.05$

$z_{\frac{\alpha}{2}} = \pm 1.96$

2-Samp Z Test:

$H_0: \mu_1 = \mu_2$, $H_a: \mu_1 \neq \mu_2$

$z^* = 2.9016$

P-value $= 0.0037 < 0.05$

$\Rightarrow$ reject H_0.

Example 2: Find the 95% confidence interval for the difference between the means in Example 1.

2-Sample Z Int:, using stats from

$\mu_1 - \mu_2 \in (1.363, 7.037)$

Relationship between P and CI for μ_1 / μ_2?

Since the confidence interval does not contain zero, the decision is to reject the null hypothesis, which agrees with the previous result.

Example 3: A researcher wishes to see if the average weights of newborn male infants are different from the average weights of newborn female infants. She selects a random sample of 10 male infants and finds the mean weight is 7 pounds 11 ounces and the standard deviation of the sample is 8 ounces. She selects a random sample of 8 female infants and finds that the mean weight is 7 pounds 4 ounces and the standard deviation of the sample is 5 ounces. Can it be concluded at α = 0.05 that the mean weight of the males is different from the mean weight of the females? Assume that the variables are normally distributed and the population standard deviations are unequal ($\sigma_1 \neq \sigma_2$) unpooled

$n_1 = 10$ $\quad n_2 = 8$ $\quad H_0: \mu_1 = \mu_2$, $H_a: \mu_1 \neq \mu_2$

$\bar{x}_1 = 123\text{ oz}$ $\quad \bar{x}_2 = 116\text{ oz}$

$s_1 = 8\text{ oz}$ $\quad s_2 = 5\text{ oz}$

2-SampTTest

$t^* = 2.268 > 2.13$ $\Downarrow$

$\alpha = 0.05$

$p = 0.038 < 0.05 \Rightarrow$ Reject H_0

$df = 15.25 \approx 15$

$\pm t_{\frac{\alpha}{2}} \overset{+}{=} \text{invT}(0.025, 15) = \pm 2.13$

Example 4: Find the 95% confidence interval for the data in Example 3.

Using 2-SampTInt:

$(0.43146, 13.569)$

Since 0 is not contained in the interval, there is enough evidence to support the claim that the mean weights are different.

Example 5: The mean tar content of a simple random sample of 25 unfiltered king size cigarettes is 21.1 mg, with a standard deviation of 3.2 mg. The mean tar content of a simple random sample of 25 filtered 100 mm cigarettes is 13.2 mg with a standard deviation of 3.7 mg. use a 0.05 significance level to test the claim that unfiltered king size cigarettes have a mean tar content greater than that of filtered 100 mm cigarettes. What does the result suggest about the effectiveness of cigarette filters? Assume that the population standard deviations are equal ($\sigma_1 = \sigma_2$).

$H_0: \mu_1 = \mu_2$ $\quad H_a: \mu_1 > \mu_2 \leftarrow$ right tailed test

2-SampTTest:

$\bar{x}_1 = 21.1$ $\quad \bar{x}_2 = 13.2$

$t^* = 8.074 > t_{0.05, 48}$

$n_1 = 25$ $\quad n_2 = 25$

$p = 8.49 \times 10^{-11} < 0.05$

$s_1 = 3.2$ $\quad s_2 = 3.7$

$df = 48$

$\alpha = 0.05$

Reject H_0

$-t_{0.05, 48} = \text{invT}(0.05, 48) = +1.677$

Example 6: A study using two samples of people found that the average amount of time those in the age group of 26-35 years spent per week on leisure activities was 35, 40, 42, 43, 38 hours, and those in the age group of 46-55 years spent 31, 39, 40, 34, 33, 35 hours. Assume that the population standard deviations are 3 hours and 3.5 hours respectively.

(1) Test at $\alpha = 0.05$ if there is a significant difference between the leisure times of the two groups.

From **STAT→ Edit**, input data L1: 35, 40, 42, 43, 38 and L2: 31, 39, 40, 34, 33, 35
STAT → TESTS → 2SampZTest (Data)

$\sigma_1 = 3 \quad \sigma_2 = 3.5$

$H_0: \mu_1 = \mu_2$

$H_a: \mu_1 \neq \mu_2$

2-Samp Z Test

$z^* = 2.176 > z_{\frac{\alpha}{2}}$

$p = 0.02949 < 0.05$

$\Rightarrow$ Reject

$z_{\frac{\alpha}{2}} = \text{invNorm}(0.025, 0, 1, \text{RIGHT}) = 1.96$

(2) Find the 95% confidence interval for the difference between the means.

2-Samp Z Int

(0.4251, 8.1082)

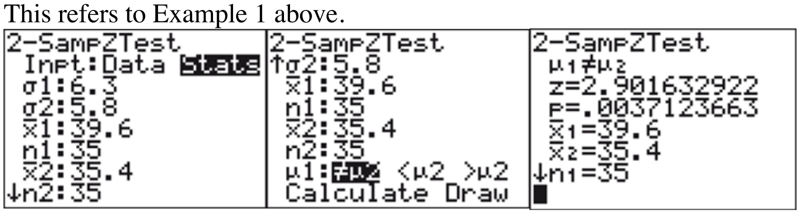
does not contain 0, so we are 95% sure $\mu_1 - \mu_2 \neq 0$

TI-84 Plus Step by Step

■ **Hypothesis Test** for the Difference Between Two Means and *z* Distribution (**Data**)
1. Enter the data values into L_1 and L_2.
2. Press **STAT** and move the cursor to TESTS.
3. Press **3** for **2-SampZTest**.
4. Move the cursor to Data and press **ENTER**.
5. Type in the appropriate values.
6. Move the cursor to the appropriate alternative hypothesis and press **ENTER**.
7. Move the cursor to Calculate and press **ENTER**.

■ **Hypothesis Test** for the Difference Between Two Means and *z* Distribution (**Statistics**)
1. Press **STAT** and move the cursor to TESTS.
2. Press **3** for **2-SampZTest**.
3. Move the cursor to Stats and press **ENTER**.
4. Type in the appropriate values.
5. Move the cursor to the appropriate alternative hypothesis and press **ENTER**.
6. Move the cursor to Calculate and press **ENTER**.

This refers to Example 1 above.



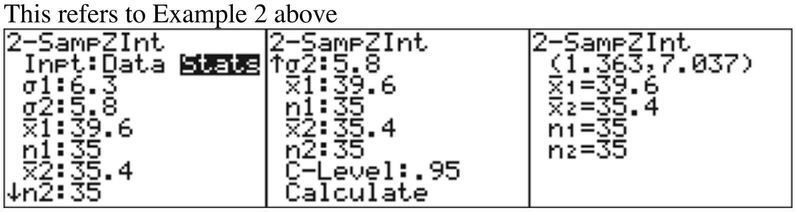
■ **Confidence Interval** for the Difference Between Two Means and *z* Distribution (**Data**)

1. Enter the data values into L_1 and L_2.
2. Press **STAT** and move the cursor to TESTS.
3. Press **9** for **2-SampZInt**.
4. Move the cursor to Data and press **ENTER**.
5. Type in the appropriate values.
6. Move the cursor to Calculate and press **ENTER**.

■ **Confidence Interval** for the Difference Between Two Means and *z* Distribution (**Statistics**)

1. Press **STAT** and move the cursor to TESTS.
2. Press **9** for **2-SampZInt**.
3. Move the cursor to Stats and press **ENTER**.
4. Type in the appropriate values.
5. Move the cursor to Calculate and press **ENTER**.

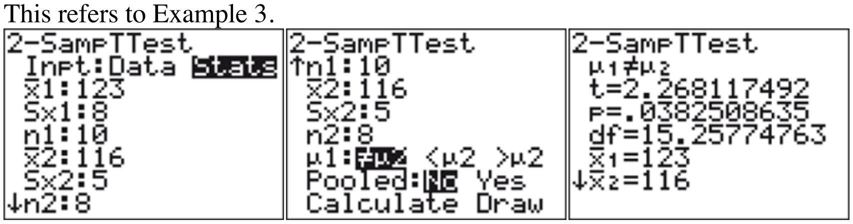
This refers to Example 2 above



■ Hypothesis Test for the Difference Between Two Means and ***t*** Distribution (Statistics)

1. Press **STAT** and move the cursor to TESTS.
2. Press **4** for 2-SampTTest.
3. Move the cursor to Stats and press **ENTER**.
4. Type in the appropriate values.
5. Move the cursor to the appropriate alternative hypothesis and press **ENTER**.
6. On the line for Pooled, move the cursor to No (standard deviations are assumed not equal) and press **ENTER**.
7. Move the cursor to Calculate and press **ENTER**.

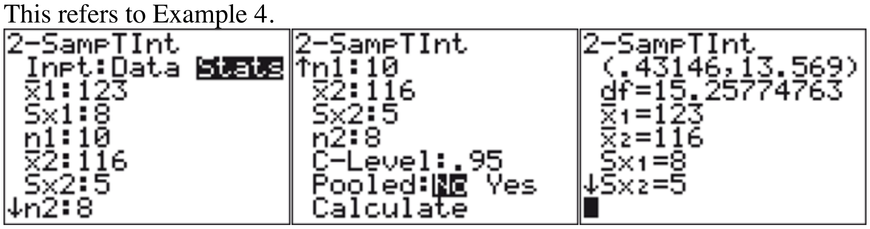
This refers to Example 3.



■ Confidence Interval for the Difference Between Two Means and ***t*** Distribution (Statistics)

1. Press **STAT** and move the cursor to TESTS.
2. Press **0** for 2-SampTInt.
3. Move the cursor to Stats and press **ENTER**.
4. Type in the appropriate values.
5. On the line for Pooled, move the cursor to No (standard deviations are assumed not equal) and press **ENTER**.
6. Move the cursor to Calculate and press **ENTER**.

This refers to Example 4.



■ **Extra Problems**

1. California and New York lead the list of average teachers' salaries. The California yearly average is \$64,421 while teachers in New York make an average annual salary of \$62,332. Random samples of 45 teachers from each state yielded the following.

	California	**New York**
Sample mean	64,510	62,900
Population standard deviation	8,200	7,800

At $\alpha = 0.10$, is there a difference in means of the salaries?

$\bar{X}_1 = 64510, \ \sigma_1 = 8200, \ n_1 = 45$

$\bar{X}_2 = 62900, \ \sigma_2 = 7800, \ n = 45$

$H_0: \mu_1 = \mu_2, \ H_a: \mu_1 \neq \mu_2$

$\alpha = 0.10 \Rightarrow Z_{\frac{\alpha}{2}} = \pm 1.645$

2-Sample Z Test:

$z^* = 0.954$

p-Value $= 0.3399$

Fail to reject

2. The average length of "short hospital stays" for men is slightly longer than that for women, 5.2 days versus 4.5 days. A random sample of recent hospital stays for both men and women revealed the following. At $\alpha = 0.01$, is there sufficient evidence to conclude that the average hospital stay for men is longer than the average hospital stay for women?

	Men	Women
Sample size	32	30
Sample mean	5.5 days	4.2 days
Population standard deviation	1.2 days	1.5 days

3. The average credit card debt for a recent year was \$9205. Five years earlier the average credit card debt was \$6618. Assume sample sizes of 35 were used and the population standard deviations of both samples were \$1928. Find the 95% confidence interval of the difference in means.

4. A tax collector wishes to see if the mean values of the tax-exempt properties are different for two cities. The values of the tax-exempt properties for the two random samples are shown. The data are given in millions of dollars. At $\alpha = 0.05$, is there enough evidence to support the tax collector's claim that the means are different? Use the non-pooled test.

City A				City B			
113	22	14	8	82	11	5	15
25	23	23	30	295	50	12	9
44	11	19	7	12	68	81	2
31	19	5	2	20	16	4	5

$H_0: \mu_1 = \mu_2$, $H_a: \mu_1 \neq \mu_2$

$\alpha = 0.05$

$\sigma_1 \neq \sigma_2$

2-SampTTest (Data)

$t^* = -0.94$

$p = 0.35 > 0.05 \Rightarrow$ fail to reject

df≈19

CI @ 95% $\Rightarrow$ Contains 0

CV: $t_{0.025,19} = \text{invT}(0.975, 19) = 2.093$

5. The mean age of a random sample of 25 people who were playing the slot machines is 48.7 years, and the standard deviation is 6.8 years. The mean age of a random sample of 35 people who were playing roulette is 55.3 with a standard deviation of 3.2 years. Can it be concluded at $\alpha = 0.05$ that the mean age of those playing the slot machines is less than those playing roulette? Use the non-pooled test.

6. A researcher claims that the mean of the salaries of elementary school teachers is greater than the mean of the salaries of secondary school teachers in a large school district. The mean of the salaries of a random sample of 26 elementary school teachers is \$48,256, and the sample standard deviation is \$3,912.40. The mean of the salaries of a random sample of 24 secondary school teachers is \$45,633. The sample standard deviation is \$5533. At $\alpha = 0.05$, can it be concluded that the mean of the salaries of the elementary school teachers is greater than the mean of the salaries of the secondary school teachers? Use the *P*-value method. Assume the population standard variances are not the same.

7. The following are scores on a standardized test for two independent random samples collected from two high schools. Assume that the samples were selected independently and $\boldsymbol{\sigma_1 = \sigma_2}$.

School A (sample of 10): 78, 84, 81, 78, 76, 83, 79, 75, 85, 81

School B (sample of 8): 85, 75, 83, 87, 80, 79, 88, 95

(a) At the .01 level, does it appear that school A has lower scores than school B?

$H_0: \mu_A = \mu_B$ $H_a: \mu_A < \mu_B$

2-SampTTest (Data) $t_{\alpha,16} = t_{0.01,16} = \text{invT}(0.01, 16)$

$= -2.58$

$t^* = -1.748$

$p = 0.0497$

$df = 16$

$\boxed{\text{fail to reject}}$

(b) Construct a 95% confidence interval for the difference in average score.