§3.7 Joint Densities

Example 1. (Toss 2 fair 6-sided dice) Let X be the **difference** of the two numbers. Let Y be the **larger number** of the two numbers.

The sample space S has 36 sample points given by

$$S = \left\{ \begin{array}{cccccc} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{array} \right\}$$

From §3.5, the range of X is $X(S) = \{0, 1, 2, 3, 4, 5\}$ and the **pdf** of X is

$X = x$	0	1	2	3	4	5
$p_X(x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

Similarly, range of Y is $Y(S) = \{1, 2, 3, 4, 5, 6\}$ and the **pdf** of Y is

$Y = y$	1	2	3	4	5	6
$p_Y(y)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Definition. Discrete Joint Density

Let S is a **discrete** sample space. Let X and Y be two random variables on S. The **joint probability density function** (**joint pdf**) of X and Y is denoted by $p_{X,Y}(x, y)$ defined as

$$p_{X,Y}(x, y) := P(X = x, Y = y).$$

Here, $P(X = x, Y = y)$ is the probability when $X = x$ **and** $Y = y$.

Theorem.

The **joint pdf** $p_{X,Y}(x, y)$ satisfies

(1.) $p_{X,Y}(x, y) \geq 0$.

(2.) $\sum_{\text{All } x} \sum_{\text{All } y} p_{X,Y}(x, y) = 1$.

Question: Find the **joint pdf** of X and Y, $p_{X,Y}(x, y)$.

(1,1) (2,1) and (1,2)

$P_{X,Y}(x,y) = P(X=x, Y=y)$

$Y=y$ \ $X=x$	0	1	2	3	4	5
1	1/36	0	0	0	0	0
2	1/36	2/36	0	0	0	0
3	1/36	2/36	2/36	0	0	0
4	1/36	2/36	2/36	2/36	0	0
5	1/36	2/36	2/36	2/36	2/36	0
6	1/36	2/36	2/36	2/36	2/36	2/36

$P(Y=1)$
$P(Y=2)$
⋮

Sum col: $P(X=\ \ )$

Sum row: $P(Y=\ \ )$

formally written as

Theorem.

Let $p_{X,Y}(x,y)$ be the **joint pdf** of X and Y. Then

general marg. pdfs

$$p_X(x) = \sum_{\text{All } y} p_{X,Y}(x,y), \qquad \text{and} \qquad p_Y(y) = \sum_{\text{All } x} p_{X,Y}(x,y)$$

They are called the **marginal pdf**s of random variables X and Y respectively.

In particular,

$$p_X(a) = P(X=a) = \sum_{\text{All } y} P(X=a, Y=y)$$

$$p_Y(b) = P(Y=b) = \sum_{\text{All } x} P(X=x, Y=b)$$

Remark: In general, one can NOT recover joint pdf of X and Y, $p_{X,Y}(x,y)$ from the marginal pdfs $p_X(x)$ and $p_Y(y)$.

Question: Find the **marginal pdf**s for X and Y in Example 1.

Definition.

Two random variables X and Y are called **independent** if and only if $p_{X,Y}(x,y) = p_X(x)p_Y(y)$. for all x, y

Definition.

If X and Y are **continuous** random variables. the **joint pdf** $f_{X,Y}(x,y)$ of X and Y is a piecewise continuous multi-variable function satisfying

(1.) $f_{X,Y}(x,y) \geq 0$.

(2.) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = 1$. , $\iint_S f_{X,Y}(x,y)\, dA = 1$

The **probability** that X and Y are in a region R in the xy-plane $\mathbb{R}^2$ is given by

$$P\big((X,Y) \in R\big) = \iint_R f_{X,Y}(x,y)\ dxdy$$

$f_X(x) = \int_Y f(x,y)\,dy$

This involves the calculation the double integral $\iint_R f_{X,Y}(x,y)\ dxdy$ from Calculus 3. If you have not learned Calculus 3, we can learn and do some easy examples here.

Definition.

Let $f_{X,Y}(x,y)$ be the joint pdf of random variables X and Y. Then, the **marginal pdf** of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy$$

Sum $P(X=x)$ for all y values

really "all y"

and the **marginal pdf** of Y is

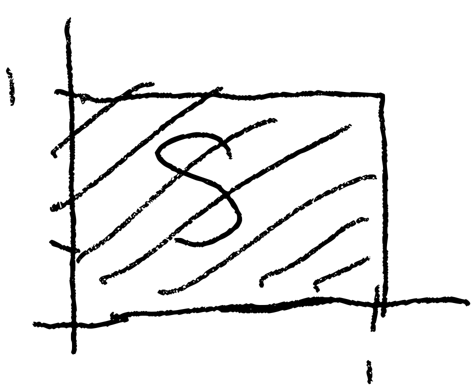
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx$$

really "all x"

$P(Y=y)$

Example 2. Suppose the **pdf** function is $f_{X,Y}(x,y) = c(x+y)$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

(1) Find c.

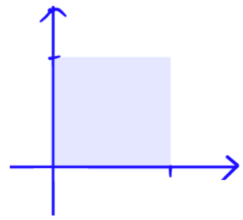


$\int_Y \int_X f_{X,Y}(x,y)\,dy\,dx$

$1 = c\int_0^1\int_0^1 x+y\ dy\,dx$

$= c\int_0^1 \left[xy + \frac{y^2}{2}\Big|_0^1\right] dx \Rightarrow \boxed{c=}$

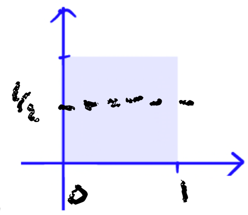
$\int_{-\infty}^{\infty} = \int_X$ since all $X \notin X$ $P(X=x) = 0$



(2) Find $P(Y \le \frac{1}{2})$

$$\int_0^{1/2}\int_0^{1} (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=1/2} dx = \int_0^1 \frac{1}{2}x + \frac{1}{8}\,dx$$

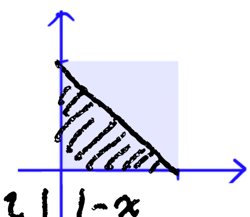


$$= \frac{x^2}{4} + \frac{1}{8}x \Big|_0^1 = \frac{1}{4} + \frac{1}{8}$$

(3) Find $P(X + Y \le 1)$

$0 \le x \le 1$

$0 \le y \le 1-x$



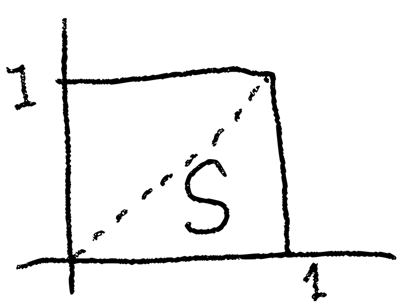
$$\int_0^1\int_0^{1-x} x+y\,dy\,dx = xy + \frac{y^2}{2}\Big|_0^{1-x}$$

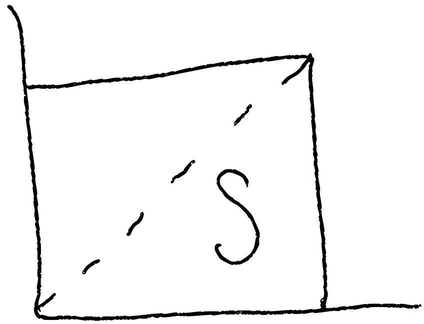
$=$

Example 3. Suppose the **pdf** function is $f_{X,Y}(x, y) = cxy$ for $0 \le x \le 1$, $0 \le y \le 1$ and $y < x$.

(1) Find c.

$$\iint_S cxy\,dA \overset{set}{=} 1$$





$0 \le y < x$

$0 \le x \le 1$

$$\iint_S cxy \, dy \, dx$$

$$c\int_0^1\int_0^x xy \, dy \, dx$$

$$c\int_0^1 x \cdot \frac{y^2}{2}\Big|_0^x$$

$$c\int_0^1 \frac{1}{2}x^3 \, dx = c \cdot \frac{x^4}{8}\Big|_0^1$$

$$= \frac{1}{8}c = 1$$

$$\boxed{c = 8}$$

$$\int xy + \frac{y^2}{2} \Big|_0^{1-x}$$

$$\int_0^1 x(1-x) + \frac{(1-x)^2}{2}\, dx$$

$$= \int_0^1 x - x^2 + \frac{1-2x+x^2}{2}\, dx$$

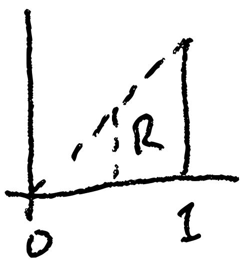
$$= \int_0^1 \cancel{x} - x^2 + \frac{1}{2} \cancel{-x} + \frac{1}{2}x^2\, dx$$

$$= \int_0^1 \frac{-1}{2}x^2 + \frac{1}{2}\, dx = \int_0^1 \frac{1}{2} - \frac{1}{2}x^2\, dx$$

$$= \frac{x}{2} - \frac{x^3}{6} \Big|_0^1 = \frac{1}{2} - \frac{1}{6}$$

$$= \frac{6}{12} - \frac{2}{12} = \frac{4}{12} = \boxed{\frac{1}{3}}$$

(2) Find $P(X > 1/2)$



$$P\left(X > \tfrac{1}{2}\right) = \iint_R f_{XY}(x,y)\,dA$$

$$8\int_{\frac{1}{2}}^{1}\int_0^{x} xy\,dy\,dx \Longrightarrow \frac{15}{16}$$

(3) Find marginal pdf of X and Y.

$$f_Y(y) = \int_X f_{XY}(x,y)\,dx = \int_y^1 8xy\,dx = \ldots \text{ integrate}$$

$$f_X(x) = \int_Y f_{XY}(x,y)\,dy = \int_0^x 8xy\,dy$$

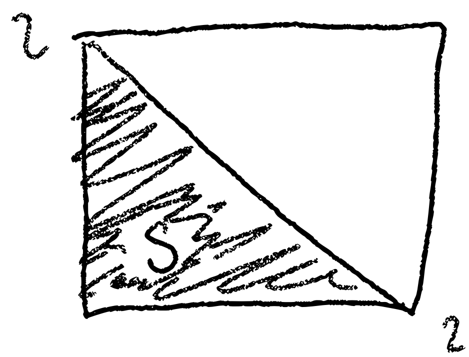
$$= 8x \cdot \frac{y^2}{2}\Big|_0^x = 4x^3,\ 0 \le x \le 1$$

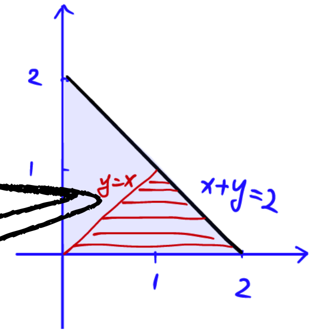
Example 4. (Practice at home.) Suppose the **pdf** function is $f_{X,Y}(x,y) = c(x+y)$ for $0 \le x \le 2$, $0 \le y \le 2$ and $x + y \le 2$.

(1) Find c.

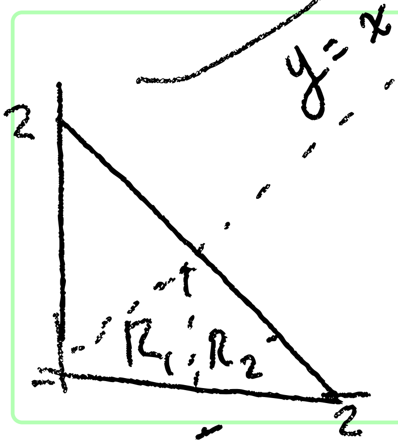
$$\iint_S c(x+y)\,dA = 1$$

$$\int_0^2\int_0^{2-x} c(x+y)\,dy\,dx \Rightarrow c = \frac{3}{8}$$





(2) Find $P(Y < X)$



Method 1: $dA = dy\,dx$

$$\iint_R \frac{3}{8}(x+y)\,dy\,dx$$

Split into 2 regions

$$\iint_{R_1} f_{XY}(x,y)\,dy\,dx +$$

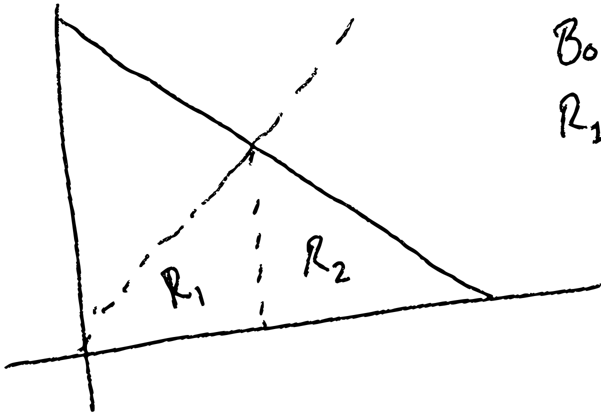
Method 2: $dA = dx\,dy$

See next page

(3) Find marginal pdf of X and Y.

Two random variables X and Y are called **independent** if and only if $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

If the pdf function $f_{X,Y}(x,y) = c$ in region R is constant, it is called bivariate **uniform** density. (HW3.7.10) This means all points are equally likely.



Bounds for $(dy\,dx)$

R_1: $0 \le y \le x$

$0 \le x \le 1$

Bounds for $(dy\,dx)$

R_2: $0 \le y \le 2-x$

$1 \le x \le 2$

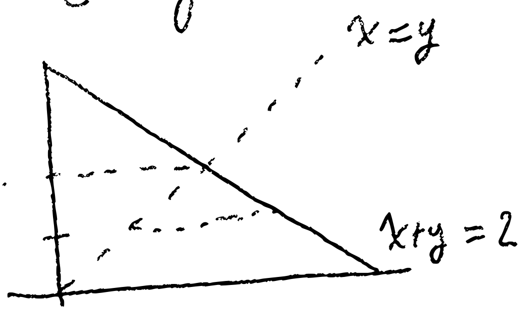
$$\iint_{R1} \frac{3}{8}(x+y)\,dy\,dx + \iint_{R2} \frac{3}{8}(x+y)\,dy\,dx$$

$$\int_0^1\int_0^x \frac{3}{8}(x+y)\,dy\,dx + \int_1^2\int_0^{2-x} \frac{3}{8}(x+y)\,dy\,dx$$

$dx\,dy$ method

$0 \le y \le 1$

$y \le x \le 2-y$



$$\int_0^1\int_{2-y}^{y} \frac{3}{8}(x+y)\,dx\,dy$$

Example 5. Two **independent** random variables X and Y both have **uniform** distributions: X is uniform on $[0, 20]$, Y is uniform on $[5, 10]$.

(1) Find the joint pdf $f_{X,Y}(x, y)$ for X and Y.

$X \sim \text{uni}(0, 20)$, $Y \sim \text{uni}(5, 10)$

$f_{XY}(x,y) = f_X(x) f_Y(y)$, $f_{XY}(x,y) = k$

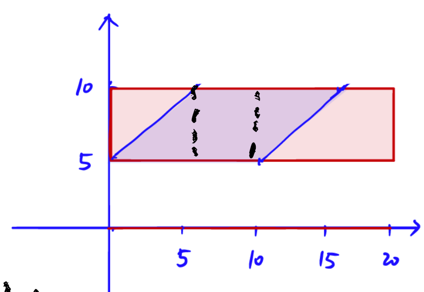
$f_X(x) = \frac{1}{20}$, $x \in [0, 20]$

$f_Y(y) = \frac{1}{5}$, $y \in [5, 10]$

$f_{XY}(x,y) = \frac{1}{100}$
$x \in [0, 20]$
$y \in [5, 10]$

(2) Find the probability that $|X - Y| \leq 5$.

$-5 \le X - Y \le 5$



geometry problem

Method 1: $P(|X-Y| \le 5) = \frac{\text{Area of } R}{\text{Area of } S} = \frac{50}{100}$

Method 2: $P(|X - Y| \leq 5) = \int_5^{10} \int_{y-5}^{y+5} \frac{1}{100} dx dy = \int_5^{10} \frac{1}{100}(10) dy = 0.5$, could use dydx, but would have to split

Theorem.

Two continuous random variables X and Y are **independent** if and only if there are functions $g(x)$ and $h(x)$ such that
$f_{X,Y}(x, y) = g(x)h(y)$, $\quad f_X(x) = g(x)$ and $f_Y(y) = h(y)$.

It is easy to see that the pdfs in Example 2 and 4 are not independent.

Example 6. Suppose two random variables X and Y are independent and $f_X(x) = 3x^2$ for $0 \leq x \leq 1$ and $f_Y(y) = \frac{1}{2}y$ for $0 \leq y \leq 2$. Find $P(Y > X)$.

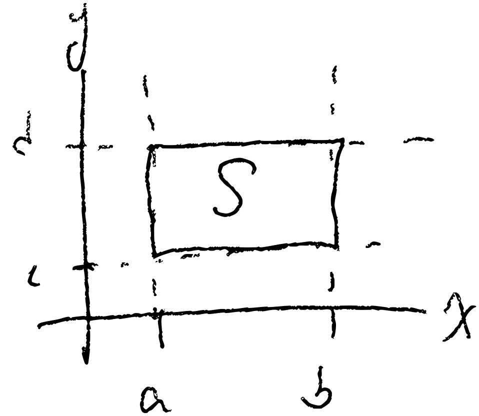
Joint uniform density for X and Y

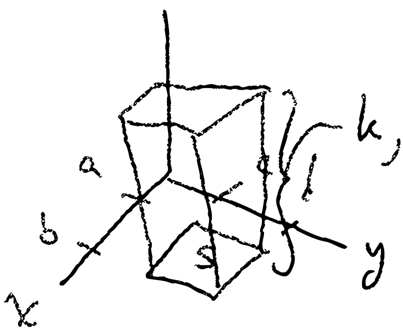
$$f_{XY}(x,y) = \begin{cases} k & \text{for } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

for instance:

$$S = [a,b] \times [c,d]$$

Surface is a flat plane





k, vol = 1 = base × height

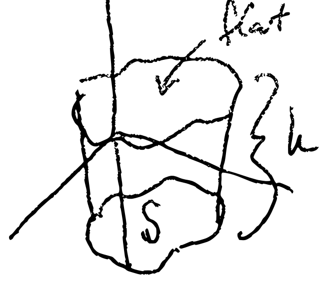
= area of S × k

$$k = \frac{1}{\text{area of } S}$$

$$= \frac{1}{(b-a)(d-c)}$$

for rectangular S

in general:

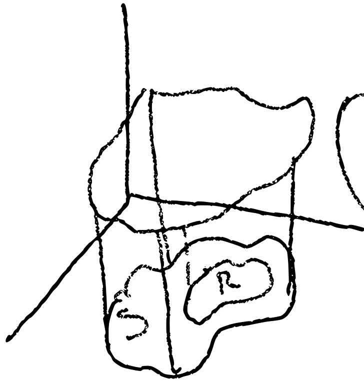


$P((x,y)\in R)$

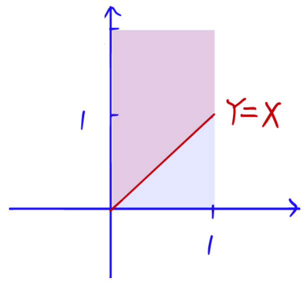
$$= \frac{\iint_R f_{x,y}(x,y)\,dA}{\iint_S f_{X,Y}(x,y)\,dA} =$$

$$= \frac{\iint_R k\,dA}{\iint_S k\,dA} = \frac{\iint_R dA}{\cancel{\iint_S dA}} \rightarrow 1$$

$$= \frac{\text{area of } R}{\text{area } S}$$



$$P((x,y)\in R) = \frac{\text{area}(R)}{\text{area}(S)}$$

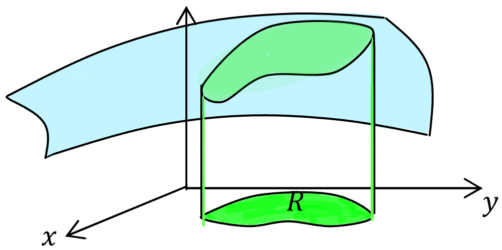


1
1
Y=X

Double Integrals

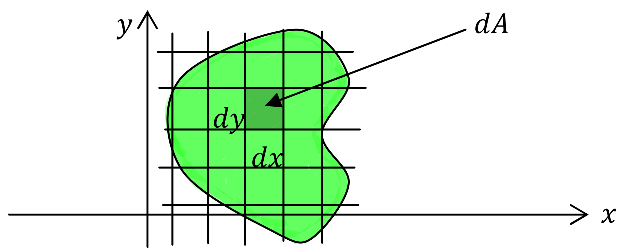
$f(x): \mathbb{R} \to \mathbb{R}$
$f(x,y): \mathbb{R}^2 \to \mathbb{R}$

Let's move now to a two-variable function $f(x,y)$, whose graph $z = f(x,y)$ is a surface in xyz-space. We may define a definite integral which gives the **signed volume** between the graph surface and xy-plane, for some region R on the xy-plane.



Recall that in the single-variable integral $\int_a^b f(x)\,dx$, the interval $[a,b]$ is divided into infinitesimal length elements dx. It is natural that the region R is divided into infinitesimal rectangular area elements

$$dA = dy\,dx$$



Since there are two independent variables x and y, it should not surprise us that the definite integral is a **double iterative integral**, with respect to x and y iteratively.

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dy\,dx$$

We would also expect FTC to work for this two-dimensional definite integral.

Example The signed volume above the rectangular region R,

$$-1 \le x \le 2, \qquad 0 \le y \le 1$$

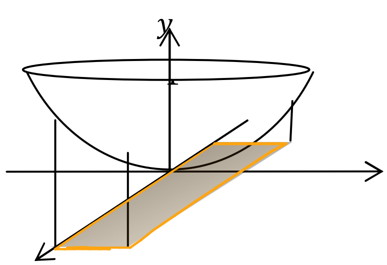
under the graph of

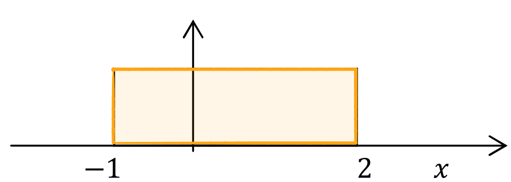
$$f(x,y) = 3x^2 + 6y^2$$

is given by the iterative double integral,

$$\iint_R f(x,y)\,dA = \int_0^1\int_{-1}^2 3x^2 + 6y^2\,dx\,dy$$

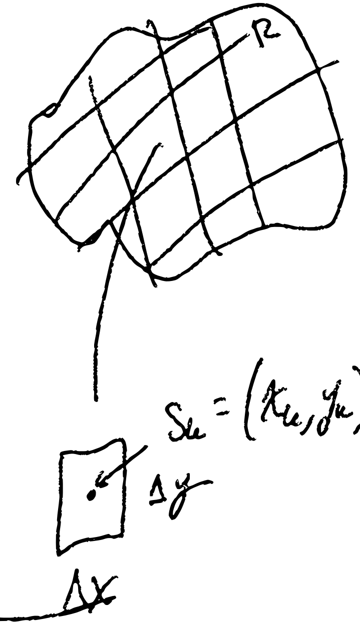
Note that the **inner integral** (in square brackets) is evaluated first, with the anti-derivative with respect to the inner variable y evaluated at the y-limits. We call y the **inner variable** and x the **outer variable** of the integral. The inner integral becomes a function of x, the outer variable, and the outer integral is the familiar single-variable integral with respect to x.

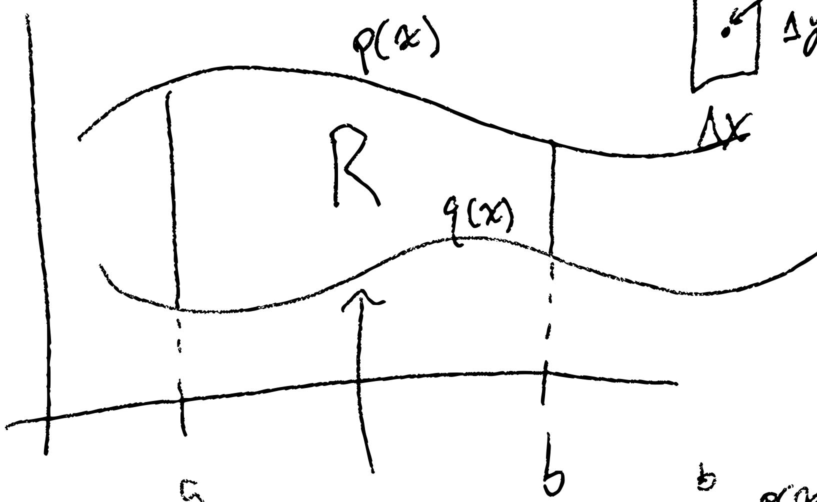




Reiman Sum

$$\lim_{n\to\infty}\sum_{k=1}^{n} \underbrace{f(S_k)}_{\text{height @ } x,y}\ \underbrace{\Delta A}_{\Delta x \Delta y}$$



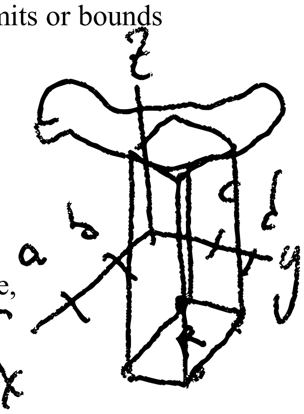


$$\iint_R f(x,y)\,dA = \int_a^b \int_{q(x)}^{p(x)} f(x,y)\,dy\,dx$$

a, b
constant

In the above example, the rectangular region of integration (shaded) corresponds to constant limits or bounds of integration. When we have a rectangular region of integration,

$$a \le x \le b, \qquad c \le y \le d$$



Fubini's Theorem says that

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Furthermore, there is a **useful trick** when the bounds are constant **and** the integrand is separable,

$$\int_a^b \int_c^d f(x)g(y)\,dy\,dx = \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)$$

Remark Be careful when applying the theorem or the trick only when the conditions are satisfied! In Example 7, we may apply Fubini's theorem but not the trick.

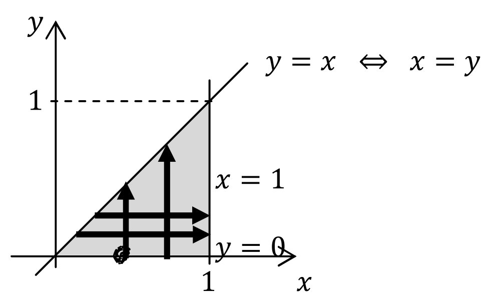
Example

$$\int_0^{\pi/2} \int_{-1}^{1} e^{-y} \cos x\,dy\,dx = \int_0^{\pi/2} \cos(x)\,dx \cdot \int^{1} e^{-y}\,dy$$

$= \cdots$ integrate like normal!...

We need to learn to specify bounds for a general region of integration in $\mathbb{R}^2$ (not just rectangles), in either order, $dy\,dx$ or $dx\,dy$.

Example Consider the following shaded triangular region, R.



To define the region, we need to have equations of all the boundaries. There are two ways to define the region.

Method 1: $dA = dy\,dx$

pick x @ random, hold constant

$0 \le y \le x$

$0 \le x \le 1$

$$\int_0^1 \int_0^x f(x,y)\,dy\,dx$$

Method 2: $dA = dx\,dy$

pick y @ random, hold const

$0 \le y \le 1$

$y \le x \le 1$

$$\int_0^1 \int_y^1 f(x,y)\,dx\,dy$$

$$\int_0^{\pi/2}\left[\int_{-1}^{1} e^{-y}\cos(x)\,dy\right]dx$$ ← x treated constant wrt y

$$\int_0^{\pi/2}\left[\cos(x)\cdot -e^{-y}\right]_{-1}^{1} dx$$

$$\int_0^{\pi/2}\cos(x)\cdot\left(e-\frac{1}{e}\right)$$

$$\left(e-\frac{1}{e}\right)\int_0^{\pi/2}\cos(x)\,dx$$

$$\left(e-\frac{1}{e}\right)\left(\sin(x)\Big|_0^{\pi/2}\right)$$

$$\left(e-\frac{1}{e}\right)(1-0)=\left(e-\frac{1}{e}\right)$$