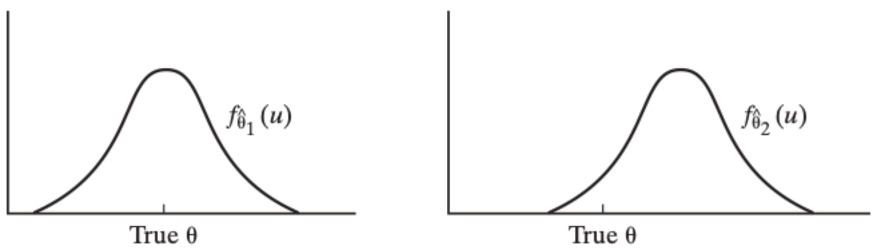
§5.4 Properties of Estimators

In §5.2, we use the maximum likelihood method to estimate parameter θ in the pdf $p_X(x;\theta)$ of a random variable X, based on a sample (observations) $X_1 = k_1$, $X_2 = k_2$, ... , $X_n = k_n$. Maximum likelihood gives one method for estimation θ_e. There are some other methods to estimate θ. We want to know which estimator is better.

The estimator $\widehat{\theta}$ itself is a random variable. The following figure shows the pdfs for two estimators, $\widehat{\theta}_1$ and $\widehat{\theta}_2$. Common sense tells us that $\widehat{\theta}_1$ is the better of the two because $f_{\widehat{\theta}_1}(u)$ is centered with respect to the true θ; $\widehat{\theta}_2$, on the other hand, will tend to give estimates that are too large because the bulk of $f_{\widehat{\theta}_2}(u)$ lies to the right of the true θ.



Definition.

An estimator $\widehat{\theta}$ is said to be an **unbiased** estimator for a parameter θ if $E(\widehat{\theta}) = \theta$.

Example 1. For a random sample, $X_1, X_2, X_3, \cdots, X_n$, of size n from some distribution with mean μ, the sample mean, $\overline{X}$, is an unbiased estimator of the mean μ.

Example 2. A sample of size two includes Y_1 and Y_2 from the same pdf. Suppose we have an estimator for the mean μ defined by

$$\widehat{\mu} = cY_1 + (1-c)Y_2, \quad 0 \le c \le 1.$$

(1) For which c, the above statistic is an unbiased estimator for μ?

(2) Is $\widehat{\mu} = 0.2Y_1 + 0.6Y_2$ an unbiased estimator for μ?

Example 3. A random sample of size 2, Y_1 and Y_2, is drawn from the pdf

$$f_Y(y;\theta) = 2y\theta^2, \quad 0 < y < \frac{1}{\theta}.$$

What must c equal if the statistic $c(Y_1 + 2Y_2)$ is to be an unbiased estimator for $\frac{1}{\theta}$.

Example 4. Given a random sample $Y_1, Y_2, \ldots, Y_n$ from a normal distribution whose parameters μ and σ^2 are both unknown, the maximum likelihood estimator for σ^2 is

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

Is $\widehat{\sigma^2}$ unbiased for σ^2? If not, what function of $\widehat{\sigma^2}$ does have an expected value equal to σ^2?

Note: S^2 is an unbiased estimator of σ^2.

$$S^2 = \text{sample variance} = \frac{n}{n-1} \cdot \frac{1}{n} \sum_{i=1}^{n} (Y_i - \overline{Y})^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2$$

$$S = \text{sample standard deviation} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \overline{Y})^2}$$