## §3.2 Binomial Probabilities

Review: **Example.** Roll a biased (unfair) coin 8 times. Suppose the probability of getting Head is $P(H) = p$. Find: $P(\text{Exactly 3 Heads})$.

We know that the result is $\binom{8}{3} \cdot p^3(1-p)^5$. (Reason: There are $\binom{8}{3}$ possible outcomes that has exactly 3 heads. For each outcome, the probability is $p^3(1-p)^5$.)

This can be generalized to a series of $n$ independent trials with 2 outcomes: "success" or "failure".

**Theorem: (Binomial Distribution)**

Given a series of $n$ **independent** trials with **two** outcomes.
Suppose the probability of "success" for each trial is **constant** $p$. Then,

$$P(k \text{ successes}) = \binom{n}{k} \cdot p^k (1-p)^{n-k}$$

for $k = 0, 1, 2, \dots, n$.

n = # trials
p = prob success
k = # success

**Remarks:**

∘ Key for application: find correct $n$, $k$ and $p$.

∘ When $n = 1$, the binomial distribution is called **Bernoulli** variable (distribution).

∘ The assumption means that there are $n$ **independent** trials and each trial is identically the same distribution. (Here, each trial is Bernoulli distribution with probability $p$.) This assumption is called **IID** (Identical-Independent-Distributions).

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = P(S) = 1}$$

**Example 1.** A Roulette (a wheel gamble) has 18 red, 18 black, and 2 green.

(1) If you spin the wheel 20 times, what is the probability of getting 10 red.

Solution: For this question, we consider getting red as success and getting non-red as failure. So, it fits the binomial distribution with $n = 20$ and

$$p = P(red) = \frac{18}{38}$$

The probability of getting 10 red ($k = 10$ )is

$$P(10\ red) = \binom{20}{10}\left(\frac{18}{38}\right)^{10}\left(1-\frac{18}{38}\right)^{10} = 0.1713$$

(2) If you spin the wheel 10 times, what is the probability of getting 3 green.

Solution: For this question, we consider getting green as success and getting non-green as failure. So, it fits the binomial distribution with $n = 10$ and

$$p = P(\,green) = \frac{2}{38}$$

The probability of getting 3 green ($k = 3$ )is

$$P(3\ green) = \binom{10}{3}\left(\frac{2}{38}\right)^{3}\left(1-\frac{2}{38}\right)^{7} = 0.01148$$

**Example 2.** In NBA Final, the winner is the first team to get four victories. Suppose Boston Celticswill play with Team C for the final. For each game, Boston has 60% winning chance.

(1) What is the probability that Boston wins the championship within 5 games?

$X$: Number of games played until Boston wins the championship.
We want to calculate the probability that $X = 4$ or $X = 5$.
For $X = 4$, the game result should be (B,B,B,B)
$P(X = 4) = \binom{4}{4}(0.6)^4(0.4)^0 = 0.1296$
For $X = 5$, the game results should be $(\_,\_,\_,\_,B)$, where Boston need to win 3 in the first 4 games.
$P(X = 5) = \binom{4}{3}(0.6)^3(0.4)^1 \cdot 0.6 = 0.20736$
So, the probability that Boston wins the championship within 5 games is
$P(X \le 5) = 0.33696$

(2) What is the probability that Boston **wins** the championship using **7** games? (Win 3 of the first 6 games and win the 7th game)

$\underbrace{\_\ \ \_\ \ \_\ \ \_\ \ \_\ \ \_}_{\text{3 wins, 6 trials}}\ \ \underset{\_}{B}$

$$P(X = 7) = \binom{6}{3}(0.6)^3(0.4)^3 \cdot 0.6 = 0.1658$$

$X$: Number of games played until Boston win the championship.
For $X = 7$, the game results should be $(\_,\_,\_,\_,\_,\_,B)$, where Boston need to win 3 in the first 6 games.
So, the probability that Boston wins the championship in exactly 7 games is

$$P(X = 7) =$$ See last page

**Example 3.** There are 100 marbles in a box: 60 **red**, 40 other colors.

(1) Choose 5 with replacement, find $P$(two **red**)

X: the number of red.
X fits the binomial distribution with $n = 5$ and $p = P(red) = \frac{60}{100} = 0.6$. (Here, "with replacement" is necessary to make sure X is binomial. Compare with the next question.)

$$P(two\ red) = P(X = 2) = \binom{5}{2}(0.6)^2(1-0.6)^3$$

(2*) Choose 5 without replacement, find $P$(two **red**).

The probability that the first ball is red is $60/100 = 0.6$. Because there is no replacement, once we get the first red marble, the probability of getting the next red marble is changed. For example,

$$P(RROOO) = \left(\frac{60}{100}\cdot\frac{59}{99}\right)\cdot\left(\frac{40}{98}\cdot\frac{39}{97}\cdot\frac{38}{96}\right)$$

$$P(ROROO) = \left(\frac{60}{100}\cdot\frac{40}{99}\right)\cdot\left(\frac{59}{98}\cdot\frac{39}{97}\cdot\frac{38}{96}\right)$$

Same, just shifting around top

There are $\binom{5}{2}$ possible outcomes with exactly 2 red marbles, and they all have the same probability.
So,

$$P(2\ red) = \binom{5}{2}\frac{60\cdot 59\cdot 40\cdot 39\cdot 38}{100\cdot 99\cdot 98\cdot 97\cdot 96}$$

(The general distribution for this question is called hypergeometric distribution.)

**Example 4.** Toss **10** fair 6-sided dice. What is the probability that at least two $6's$ appeared?

$$P(X \geq 2) = 1 - P(X = 0 \text{ or } X = 1)$$
$$= 1 - [P(X=0) + P(X=1)]$$
$$= 1 - \left(\left[\left(\tfrac{5}{6}\right)^{10}\right] + \left[\binom{10}{1}\left(\tfrac{1}{6}\right)^1\left(\tfrac{5}{6}\right)^9\right]\right) = 0.515$$

Solution: success=6 appeared; failure=6 did not appear.
X: the number of $6's$ appeared.
X fits the binomial distribution with $n = 10$ and $p = P(success) = 1/6$.
The probability that at least two $6's$ appeared is

$$P(k \geq 2)$$

You can use calculator TI-83/ TI-84 (plus) to verify the calculation.

**2ED** $\rightarrow$ **Vars** $\rightarrow$ $A$ : **binompdf**

$P(X \leq k)$

$= P(x=0) + P(x=1) \ldots P(x=k)$

$= \sum_{i=0}^{k} P(x=i) \longrightarrow$ binomcdf

$P(x \leq k) = 1 - P(x > k)$

$$\binom{n}{r} = \binom{n}{n-r}$$

$N = \#$ of items

$n = \#$ selected w/o replacement

## Hypergeometric Probability

**Definition.** Hypergeometric probability $x$ successes, $n-x$ failures

Suppose a set (population) consists of $N$ items, $k$ of which are successes and $N-k$ of which are failures. If we randomly select $n$ items without replacement from a set of $N$ items of which:

- $k$ of the items are of one type (success)
- and $N-k$ of the items are of a second type (failure)

choose exactly $x$ k's

choose exactly $n-x$ failures out of a pool of $N-k$

Then the hypergeometric probability is $P(x \text{ successes}) = \dfrac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$.

ways to choose $n$ items overall, "restricted sample space"

| Succ | Fail |
|---|---|
| $k$ | $N-k$ |

$\}N$

Select $n$,

$$P(x \text{ success}) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

**Example 1.** Suppose we randomly select 5 cards without replacement from an ordinary deck of playing cards. What is the probability of getting exactly 2 red cards (i.e., hearts or diamonds)?

Sol. This is a hypergeometric experiment in which we know the following:
N = 52; since there are 52 cards in a deck.
k = 26; since there are 26 red cards in a deck.
n = 5; since we randomly select 5 cards from the deck.
x = 2; since 2 of the cards we select are red.

52

| 26 | 26 |
|---|---|

$5 = 2 + 3$

$$P(2 \text{ red cards}) = P(x = 2) = \frac{\binom{26}{2}\binom{26}{3}}{\binom{52}{5}} = 0.3251$$

**Example 2.** In the manufacture of car tires, a particular production process is known to yield 10 tires with defective walls in every batch of 100 tires produced. From a production batch of 100 tires, a sample of 4 is selected for testing to destruction. Find the probability that the sample contains at least 1 defective tire.

Sol. Sampling is clearly without replacement and we use the hypergeometric distribution with N = 100, k = 10, n = 4, x= number of defective tires. Hence:

$$P(\text{at least 1 defective tire}) = 1 - P(\text{no defectives})$$

$$= 1 - \frac{\binom{10}{0}\binom{90}{4}}{\binom{100}{4}} = 0.3484$$

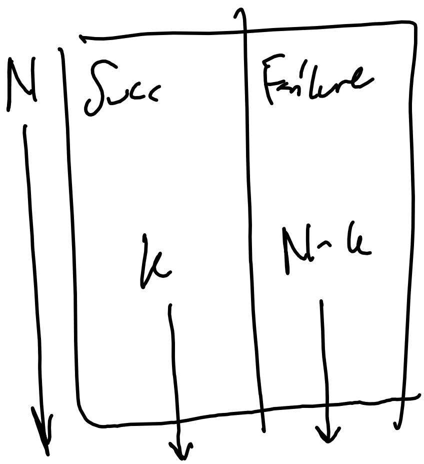
100

| Def | not def |
|---|---|
| 10 | 90 |

$4 = 0 + 4$

# Hypergeometric

#(choose $x$ succ $\wedge$ choose $n-x$ failures)



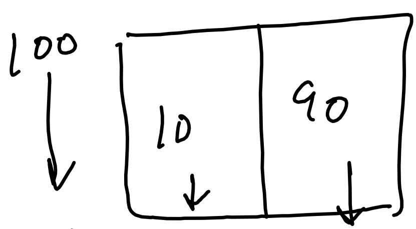

$$P(X=x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$n = x + (n-x)$

## Extra Question

P(at least 2 def)

$$P(X=2) + P(X=3) + P(X=4)$$




$$= \frac{\binom{10}{2}\binom{90}{2}}{\binom{100}{4}} + \frac{\binom{10}{3}\binom{90}{1}}{\binom{100}{4}} + \frac{\binom{10}{4}\binom{90}{0}}{\binom{100}{4}}$$

$4 = 2,3,4 + 2,1,0$

$$\boxed{= 0.0488}$$