2.4 Conditional Probability

The conditional probability of an event A in relationship to an event B is the probability that event A occurs after event B has already occurred. The notation for conditional probability is P(A|B). This notation does not mean that A is divided by B; rather, it means the probability that event A occurs given that event B has already occurred.

Definition. Conditional probability

Probability that event A occurs **given** that event B already occurs, denoted by $P(A|B)$ is a **conditional probability**, defined by

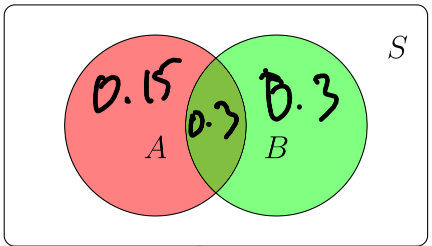
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

called **probability of** A **given** B.

If $\#S$ is finite, we can calculate conditional probability as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\#(A \cap B)/\#(S)}{\#(B)/\#(S)} = \frac{\#(A \cap B)}{\#(B)}.$$

Example Suppose $P(A) = 0.45$, $P(B) = 0.6$, and $P(A^c|B) = 0.5$. Find $P(A \cup B)$.



$$P(A^c|B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(A^c \cap B)}{0.6} = 0.5$$

$$P(A^c \cap B) = 0.3$$

$$P(A \cap B) = P(B) - P(A^c \cap B) = 0.6 - 0.3 = 0.3$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.45 + 0.6 - 0.3 = \boxed{0.75}$$

$$\boxed{0.75}$$

When we calculate probabilities involving one event AND another event occurring, we multiply their probabilities. In some cases, the first event happening impacts the probability of the second event. We call these dependent events. In other cases, the first event happening does not impact the probability of the seconds. We call these independent events.

Multiplication Rule: For any two events, we can say that

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$$

for independent A, B

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$

Note: Here are some examples of dependent events:

- Drawing a card from a deck, not replacing it, and then drawing a second card
- Selecting a ball from an urn, not replacing it, and then selecting a second ball

Example 2. Three cards are drawn from an ordinary deck and not replaced. Find the probability of these events.

1. Getting 3 jacks
2. Getting an ace, a king, and a queen in order
3. Getting a club, a spade, and a heart in order
4. Getting 3 clubs

1) $\frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = 0.0002$

2) $\frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = 0.0005$

3) $\frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} =$

4) $\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} =$

Example 3. A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip on the 1st draw *and* a white chip on the 2nd draw is $\frac{15}{56}$ and the probability of selecting a black chip on the first draw is $\frac{3}{8}$, find the probability of selecting the white chip on the second draw, *given* that the first chip selected was a black chip.

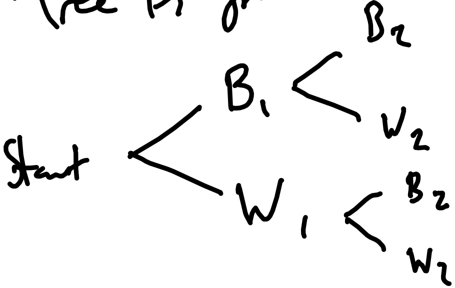
$P(B_1 \cap W_2) = \frac{15}{56}$

$P(B_1) = \frac{3}{8}$

$P(W_2 | B_1) = \frac{P(B_1 \cap W_2)}{P(B_1)} = \frac{\left(\frac{\cancel{15}^{\,5}}{\cancel{56}_{\,7}}\right)}{\left(\frac{\cancel{3}}{\cancel{8}}\right)}$

≈ 0.714

Tree Diagram



Example 4. A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

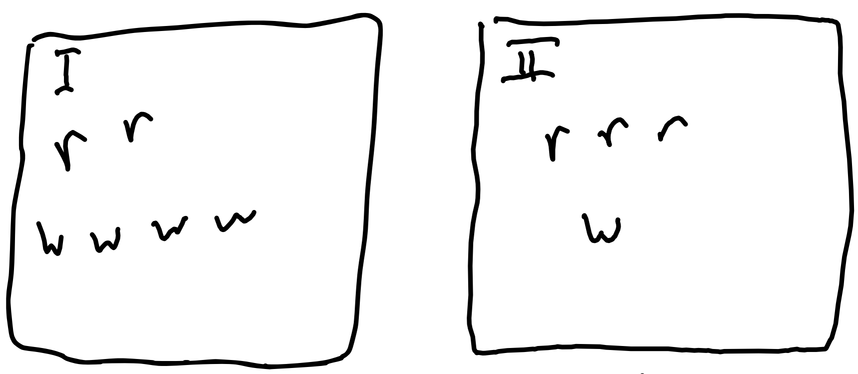
Find these probabilities.

a. The respondent answered yes, given that the respondent was a female. $\frac{8}{50}$

b. The respondent was a male, given that the respondent answered no. $\frac{18}{60}$

Conditions are constricting on the sample space

Example 5. Urn I contains 2 red chips and 4 white chips and urn II contains 3 red chips and 1 while chip. A chip is drawn at random from urn I and transferred to urn II. If a chip is drawn from urn II, find the probability that the chip drawn from urn II is red.

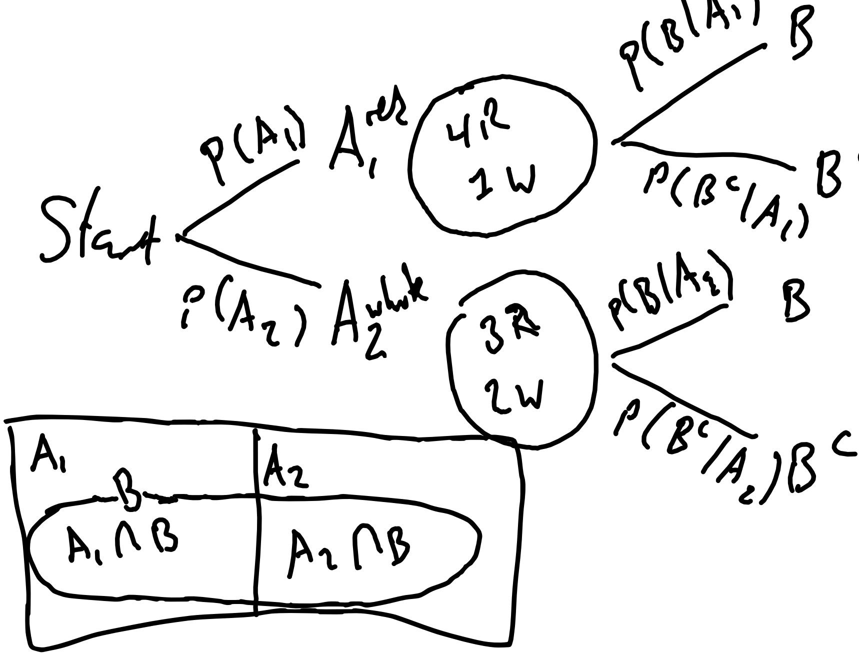


B = Urn 2 is red

A_1 = chip from 1 is red

A_2 = chip from 1 is white

find $P(B)$



$B = (A_1 \cap B) \dot{\cup} (A_2 \cap B)$

$P(B) = P(A_1 \cap B) + P(A_2 \cap B)$

$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$

$= \left(\frac{2}{6} \cdot \frac{4}{5}\right) + \left(\frac{4}{6} \cdot \frac{3}{5}\right)$

$= \frac{2}{3}$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' Theorem

Theorem. Bayes' Theorem

Let $A_1, A_2, \ldots, A_n$ be a sequence of events such that $S = \bigcup_{i=1}^{n} A_i$ and $A_i \cap A_j = \emptyset$. Then, for any event B,

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

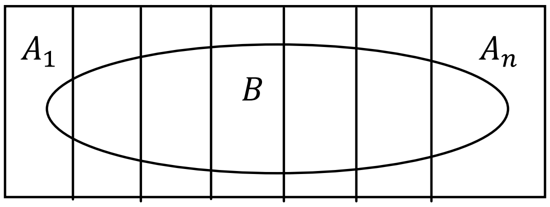
for any $j = 1, \cdots, n$.

$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) \ldots P(A_n)P(B|A_n)$

Proof is easy by the law of total probability:

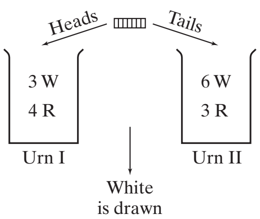
$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i) \Downarrow \sum_{i=1}^{n} P(B \cap A_i)$$



Example 6. A biased coin, twice as likely to come up heads as tails, is tossed once. If it shows heads, a chip is drawn from urn I, which contains three white chips and four red chips; if it shows tails, a chip is drawn from urn II, which contains six white chips and three red chips. Given that a white chip was drawn, what is the probability that the coin came up tails?

Sol.



$P(H) = 2P(T)$

$1 = P(H) + P(T) = 3P(T) \Rightarrow P(T) = \frac{1}{3}$, $P(H) = \frac{2}{3}$

Define the events.

B = W = white chip is drawn

A_1 = H = coin came up heads (i.e., chip came from urn I)

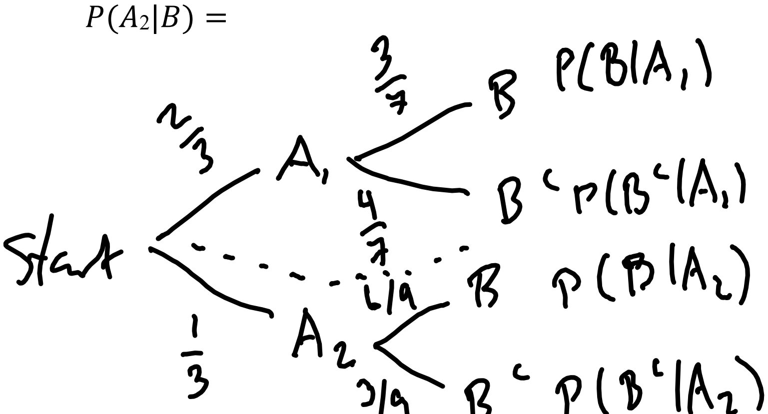
A_2 = T = coin came up tails (i.e., chip came from urn II)

$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$

Since $P(H) = 2P(T)$ and $P(H) + P(T) = 1$, it must be true that $P(H) =$ and $P(T) =$

Our objective is to find $P(A_2|B)$. From the figure, $P(B|A_1) =$ $P(B|A_2) =$

$P(A_2|B) =$

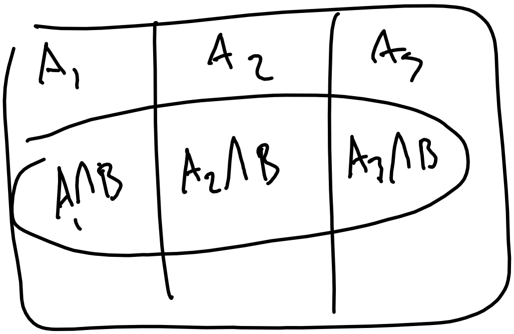


$$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{\frac{1}{3}\cdot\frac{2}{3}}{\underbrace{\frac{2}{3}\cdot\frac{3}{7}}_{A_1 \to B_1} + \underbrace{\frac{1}{3}\cdot\frac{2}{3}}_{A_2 \to B_1}}$$

$A_2 \to B_1$

Bayes' Thm

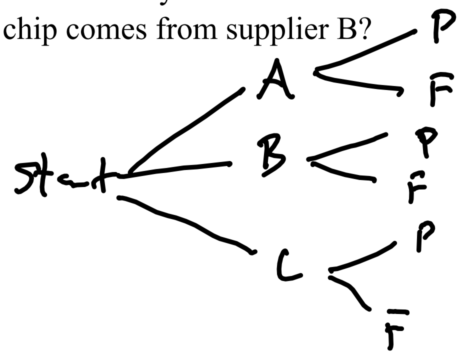
Lets say $n=3$



$$P(A_1 \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A_1)\,P(B \mid A_1)}{\sum_{i=1}^{3} P(A_i)\,P(B \mid A_i)}$$

Example 7. A computer manufacturer uses chips from three suppliers. Based on past performance, she knows that the chips from supplier A will fail with probability 0.01, the chips from supplier B will fail with probability 0.02, and the chips from supplier C will fail with probability 0.05. She buys 50% of her chips from supplier A, 40% from supplier B and 10% from supplier C. (1) What is the probability that a chip chosen randomly from her mixture will fail? (2) If she selects a chip that fails, what is the probability that the chip comes from supplier B?

1)



$$P(F) = P(A)P(F|A) + P(B)P(F|B) + P(C)P(F|C)$$

$$= 0.5(0.01) + 0.4(0.02) + 0.1(0.05)$$

$$\boxed{P(F) = 0.018}$$

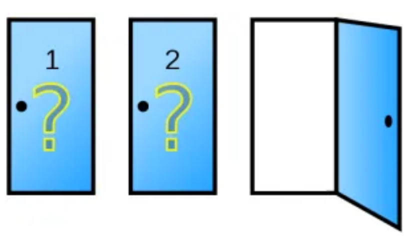
2) $P(B|F)$

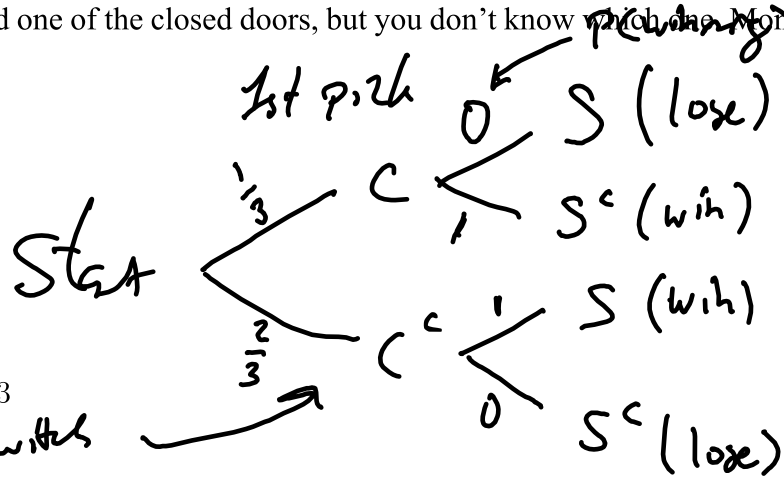
$$= \frac{P(B\cap F)}{P(F)}$$

$$= \frac{P(B)P(F|B)}{P(F)} = 0.444$$

Example 8. Prize behind door problem (Monty Hall problem)

Monty Hall asks you to choose one of three doors. One of the doors hides a prize and the other two doors have no prize. You state out loud which door you pick, but you don't open it right away. Monty opens one of the other two doors, and there is no prize behind it. At this moment, there are two closed doors, one of which you picked. The prize is behind one of the closed doors, but you don't know which one. Monty asks you, "Do you want to switch doors?"





C: 1st choice is correct. $P(C) = 1/3$
S: win after switch.
T: win without switch.

S = switch

$$P(S) = P(S|C)P(C) + P(S|C^c)P(C^c) = \left(0\cdot\tfrac{1}{3}\right) + \left(1\cdot\tfrac{2}{3}\right) = \tfrac{2}{3}$$

$$P(T) = P(T|C)P(C) + P(T|C^c)P(C^c) = \left(1\cdot\tfrac{1}{3}\right) + \left(0\cdot\tfrac{2}{3}\right) = \tfrac{1}{3}$$

You should switch.