

Homework 2

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Chapter 5

5.1

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \dots + n^3 = (n(n+1)/2)^2$ for the positive integer n .
- What is the statement $P(1)$?
 - Show that $P(1)$ is true, completing the basis step of the proof.
 - What is the inductive hypothesis?
 - What do you need to prove in the inductive step?
 - Complete the inductive step, identifying where you use the inductive hypothesis.
 - Explain why these steps show that this formula is true whenever n is a positive integer.

Answer:

- $P(1) : 1^3 = 1(1+1)/2^2$
- $(1(1+1)/2)^2 = (2/2)^2 = 1 = 1^3 = 1$
- Inductive Hypothesis: $1^3 + 2^3 + \dots + k^3 = (k(k+1)/2)^2$
- Need to prove that $P(k) \rightarrow P(k+1)$ for $k \geq 1$
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$$\begin{aligned}\sum_{n=1}^{k+1} i^3 &= \sum_{n=1}^k i^3 + (k+1)^3 \\ &= \left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3 && (InductiveHypothesis) \\ &= \frac{k^4 + 2k^3 + k^2}{4} + \frac{4(k+1)^3}{4} \\ &= \frac{k^4 + 2k^3 + k^2 + 4(k+1)^3}{4} \\ &= \frac{k^4 + 2k^3 + k^2 + 4(k^3 + 3k^2 + 3k + 1)}{4} \\ &= \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4} \\ &= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \\ &= \frac{(k^2 + 3k + 2)(k^2 + 3k + 2)}{4} \\ &= \frac{(k^2 + 3k + 2)^2}{2^2} \\ &= \frac{((k+1)(k+2))^2}{2^2} \\ &= \left(\frac{(k+1)(k+2)}{2} \right)^2 \\ &= \left(\frac{(k+1)((k+1)+1)}{2} \right)^2\end{aligned}$$

- Completed the base and inductive step, so by the principle of mathematical induction the statement is true for all positive integer n .
6. Prove that $1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Answer:

Proof. (Base Case) If $n = 1$ then the left side is $1 \cdot 1! = 1$ and the left side is $(2)! - 1 = 1$ so the formula holds for $n = 1$.
(Inductive Hypothesis) Assume that for $k \geq 1$ that the formula is true, that is $1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k! = (k+1)! - 1$.

(Inductive Step) Let $n = k + 1$ Then:

$$\begin{aligned}\sum_{n=1}^{k+1} i \cdot i! &= (k+1)(k+1)! + \sum_{n=1}^k i \cdot i! \\ &= (k+1)(k+1)! + (k+1)! - 1 && \text{(By inductive hypothesis)} \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1\end{aligned}$$

This is the right side that we want so it holds for $n = k + 1$. Thus by the principle of mathematical induction the theorem holds for all $n \in \mathbb{N}$. ■

5.2

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.

Answer:

Proof. Since we know that the first dominoes will fall we will assume that the first k dominoes will fall. If $k \leq 3$ then we already know that those will fall from what's given. We know that the $k - 2$ will fall by the inductive hypothesis. This means that the $(k - 2) + 3 = k + 1$ domino will fall. We have shown that if the k^{th} domino falls then the $k + 1$ domino will fall and Thus the statement is true by the principle of strong induction. ■

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.
- Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.
 - What is the inductive hypothesis of the proof?
 - What do you need to prove in the inductive step?
 - Complete the inductive step for $k \geq 21$.
 - Explain why these steps show that this statement is true whenever $n \geq 18$.

Answer:

- $P(18)$ is true because you can use two 7-cent stamps and a 4-cent stamp
 $P(19)$ is true with three 4-cent stamps and a 7-cent stamp.
 $P(20)$ is true with five 4-cent stamps.
 $P(21)$ is true with three 7-cent stamps.
- Inductive Hypothesis: Assume that $P(j)$ is true with $18 \leq j \leq k$ with $k \geq 21$
- Need to prove that if $P(k)$ is true then $P(k + 1)$ is also true with $k \geq 18$
- If $k \geq 21$ we know that $P(k - 3)$ is true since $k - 3 \geq 18$ by the inductive hypothesis This means that $P(k + 1)$ is true because we can add a 4-cent coin to the combination from $P(k - 3)$.
- Completed basis step and inductive step so it is true for all integers greater than 18.

5.3

2. Find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if $f(n)$ is defined recursively by $f(0) = 3$ and for $n = 0, 1, 2, \dots$
- $f(n + 1) = -2f(n)$.
 - $f(n + 1) = 3f(n) + 7$.
 - $f(n + 1) = f(n)^2 - 2f(n) - 2$.
 - $f(n + 1) = 3^{f(n)/3}$.

Answer:

- $f(1) = -6$, $f(2) = 12$, $f(3) = -24$, $f(4) = 48$, $f(5) = -96$
- $f(1) = 16$, $f(2) = 55$, $f(3) = 172$, $f(4) = 523$, $f(5) = 1576$
- $f(1) = 1$, $f(2) = -3$, $f(3) = 13$, $f(4) = 141$, $f(5) = 19597$

- d. $f(1) = 3, f(2) = 3, f(3) = 3, f(4) = 3, f(5) = 3$
4. Find $f(2), f(3), f(4),$ and $f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and for $n = 1, 2, \dots$
- $f(n+1) = f(n) - f(n-1).$
 - $f(n+1) = f(n)f(n-1).$
 - $f(n+1) = f(n)^2 + f(n-1)^3.$
 - $f(n+1) = f(n)/f(n-1).$

Answer:

- $f(2) = 0, f(3) = 1, f(4) = -1, f(5) = 0$
 - $f(2) = 1, f(3) = 1, f(4) = 1, f(5) = 1$
 - $f(2) = 2, f(3) = 5, f(4) = 33, f(5) = 1214$
 - $f(2) = 1, f(3) = 1, f(4) = 1, f(5) = 1$
8. Give a recursive definition of the sequence $\{a_n\}, n = 1, 2, 3, \dots$ if
- $a_n = 4n - 2.$
 - $a_n = 1 + (-1)^n.$
 - $a_n = n(n+1).$
 - $a_n = n^2.$

Answer:

- $f(1) = 2$
 $f(n+1) = f(n) + 4$
- $f(1) = 1$
 $f(n+1) = f(n) + (-1)^{f(n)}$
- $f(1) = 2, f(2) = 6$
 $f(n+1) = 2f(n) - f(n-1) + 2$
- $f(1) = 1, f(2) = 4$
 $f(n+1) = 2f(n) - f(n-1) + 2$

5.4

Chapter 6

Chapter 7

Chapter 8