Homework 2

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Chapter 5

5.1

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \ldots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

   a. What is the statement P (1)?
   b. Show that P (1) is true, completing the basis step of the proof.
   c. What is the inductive hypothesis?
   d. What do you need to prove in the inductive step?
   e. Complete the inductive step, identifying where you use the inductive hypothesis.
   f. Explain why these steps show that this formula is true whenever n is a positive integer.

   Answer:

   a. $P(1) : 1^3 = 1(1(1+1)/2)^2$
   b. $(1(1+1)/2)^2 = (2/2)^2 = 1 = 1^3 = 1$
   c. Inductive Hypothesis: $1^3 + 2^3 + \ldots + k^3 = (k(k+1)/2)^2$
   d. Need to prove that $P(k) \to P(k+1)$ for $k \geq 1$
   e.

$$\begin{aligned}
\sum_{n=1}^{k+1} i^3 &= \sum_{n=1}^{k} i^3 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && (InductiveHypothesis) \\
&= \frac{k^4 + 2k^3 + k^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^4 + 2k^3 + k^2 + 4(k+1)^3}{4} \\
&= \frac{k^4 + 2k^3 + k^2 + 4(k^3 + 3k^2 + 3k + 1)}{4} \\
&= \frac{k^3 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \\
&= \frac{(k^2 + 3k + 2)(k^2 + 3k + 2)}{4} \\
&= \frac{(k^2 + 3k + 2)^2}{2^2} \\
&= \frac{((k+1)(k+2))^2}{2^2} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2
\end{aligned}$$

   f. Completed the base and inductive step, so by the principle of mathematical induction the statement is true for all positive integer n.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.
   Answer:

   Proof. (Base Case) If $n = 1$ then the left side is $1 \cdot 1! = 1$ and the left side is $(2)! - 1 = 1$ so the formula holds for n = 1. (Inductive Hypothesis) Assume that for $k \geq 1$ that the formula is true, that is $1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1$.

(Inductive Step) Let $n = k + 1$ Then:

$$\begin{aligned}\sum_{n=1}^{k+1} i \cdot i! &= (k+1)(k+1)! + \sum_{n=1}^{k} i \cdot i! \\ &= (k+1)(k+1)! + (k+1)! - 1 && \text{(By inductive hypothesis)} \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1\end{aligned}$$

This is the right side that we want so it holds for n = k +1. Thus by the principle of mathematical induction the theorem holds for all $n \in \mathbb{N}$. ■

5.2

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.
   Answer:

   Proof. Since we know that the first dominoes will fall we will assume that the first k dominoes will fall. If $k \leq 3$ then we already know that those will fall from what's given. We know that the $k - 2$ will fall by the inductive hypothesis. This means that the $(k-2)+3 = k+1$ domino will fall. We have shown that if the k^{th} domino falls then the $k+1$ domino will fall and Thus the statement is true by the principle of strong induction. ■

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

   a. Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.
   b. What is the inductive hypothesis of the proof?
   c. What do you need to prove in the inductive step?
   d. Complete the inductive step for $k \geq 21$.
   e. Explain why these steps show that this statement is true whenever $n \geq 18$.

   Answer:

   a. $P(18)$ is true because you can use two 7-cent stamps and a 4-cent stamp
      $P(19)$ is true with three 4-cent stamps and a 7-cent stamp.
      $P(20)$ is true with five 4-cent stamps.
      $P(21)$ is true with three 7-cent stamps.
   b. Inductive Hypothesis: Assume that $P(j)$ is true with $18 \leq j \leq k$ with $k \geq 21$
   c. Need to prove that if $P(k)$ is true then $P(k+1)$ is also true with $k \geq 18$
   d. If $k \geq 21$ we know that $P(k-3)$ is true since $k - 3 \geq 18$ by the inductive hypothesis This means that $P(k+1)$ is true because we can add a 4-cent coin to the combination from $P(k-3)$.
   e. Completed basis step and inductive step so it is true for all integers greater than 18.

5.3

2. Find $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f (n) is defined recursively by $f(0) = 3$ and for $n = 0, 1, 2, \ldots$

   a. $f(n+1) = -2f(n)$.
   b. $f(n+1) = 3f(n) + 7$.
   c. $f(n+1) = f(n)^2 - 2f(n) - 2$.
   d. $f(n+1) = 3^{f(n)/3}$.

   Answer:

   a. $f(1) = -6,\ f(2) = 12,\ f(3) - 24,\ f(4) = 48,\ f(5) = -96$
   b. $f(1) = 16,\ f(2) = 55,\ f(3) = 172,\ f(4) = 523,\ f(5) = 1576$
   c. $f(1) = 1,\ f(2) = -3,\ f(3) = 13,\ f(4) = 141,\ f(5) = 19597$

d. $f(1) = 3,\ f(2) = 3,\ f(3) = 3,\ f(4) = 3,\ f(5) = 3$

4. Find $f(2),\ f(3),\ f(4)$, and $f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and for $n = 1,\ 2,\ \ldots$

   a. $f(n+1) = f(n) - f(n-1)$.
   b. $f(n+1) = f(n)f(n-1)$.
   c. $f(n+1) = f(n)^2 + f(n-1)^3$.
   d. $f(n+1) = f(n)/f(n-1)$.

   Answer:

   1. $f(2) = 0,\ f(3) - 1,\ f(4) = -1,\ f(5) = 0$
   2. $f(2) = 1,\ f(3)1,\ f(4) = 1,\ f(5) = 1$
   3. $f(2) = 2,\ f(3)5,\ f(4) = 33,\ f(5) = 1214$
   4. $f(2) = 1,\ f(3)1,\ f(4) = 1,\ f(5) = 1$

8. Give a recursive definition of the sequence $\{a_n\},\ n = 1,\ 2,\ 3,\ \ldots$ if

   a. $a_n = 4n - 2$.
   b. $a_n = 1 + (-1)^n$.
   c. $a_n = n(n+1)$.
   d. $a_n = n^2$.

   Answer:

   a. $f(1) = 2$
      $f(n+1) = f(n) + 4$
   b. $f(1) = 1$
      $f(n+1) = f(n) + (-1)^{f(n)}$
   c. $f(1) = 2,\ f(2) = 6$
      $f(n+1) = 2f(n) - f(n-1) + 2$
   d. $f(1) = 1,\ f(2) = 4$
      $f(n+1) = 2f(n) - f(n-1) + 2$

5.4

2. Trace Algorithm 1 when it is given $n = 6$ as input. That is, show all steps used by Algorithm 1 to find 6!, as is done in Example 1 to find 4!.

   Answer:

   $6! = 6 \cdot 5!,\ 5! = 5 \cdot 4!,\ 4! = 4 \cdot 3!,\ 3! = 3 \cdot 2!,\ 2! = 2 \cdot 1!,\ 1! = 1 \cdot 0!$
   $0! = 1$ So $1! = 1 \cdot 1 = 1,\ 2! = 2 \cdot 1! = 2,\ 3! = 3 \cdot 2! = 6,\ 4! = 4 \cdot 3! = 24,\ 5! = 5 \cdot 4! = 120\ 6! = 6 \cdot 5! = 720$

8. Give a recursive algorithm for finding the sum of the first n positive integers.

   Answer:

   procedure: *sum_to_n*(n: nonnegative integer)
   if n = 0 **then return** 0
   else return $n+$ *sum_to_n*$(n-1)$

Chapter 6

6.1

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?
   Answer:
   $27 \cdot 37 = 999$

8. How many different three-letter initials with none of the letters repeated can people have?
   Answer:
   $26 \cdot 25 \cdot 24 = 15600$

30. How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?
Answer:
$26^3 \cdot 10^3 + 26^4 \cdot 10^2 = 63273600$

40. How many subsets of a set with 100 elements have more than one element?
Answer:
$2^{100} - 101$

44. How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?
Answer:
$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4} = 1260$

6.2

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.
Answer:
Since there are on 26 letters in the English alphabet, then if there are more than 26 students then at least 2 will have the same first letter in there last name.

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

   a. How many balls must she select to be sure of having at least three balls of the same color?

   b. How many balls must she select to be sure of having at least three blue balls?

   Answer:

   a. 13
   This is wrong, should be 5 balls, $\lceil 5/2 \rceil = 3$ balls that have the same color.

   b. 13

8. Show that if f is a function from S to T , where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.
Answer:
There are more elements in the domain than the codomain so by the pidgeon hole principle there must be 2 or more elements from the domain that map to the same element in the codomain thus the functin can't be one-to-one.

18. Suppose that there are nine students in a discrete mathe- matics class at a small college.

   a. Show that the class must have at least five male students or at least five female students.

   b. Show that the class must have at least three male students or at least seven female students.

   Answer:

   a. If there are 4 of one there must be 5 of the other.

   b. If there are only 2 male students then there must be $9 - 2 = 7 \geq 7$ female students and if there are only 6 female students then there must be $9 - 6 = 3 \geq 3$ male students.

6.3

4. Let $S = \{1, 2, 3, 4, 5\}$.

   a. List all the 3-permutations of S.

   b. List all the 3-combinations of S.

   Answer:

   a. 123, 124, 125, 132, 134, 135, 142, 143, 145, 152, 153, 154, 213, 214, 215, 231, 234, 235, 241, 243, 245, 251, 253, 254, 312, 314, 315, 321, 324, 325, 341, 342, 345, 351, 352, 354, 412, 413, 415, 421, 423, 425, 431, 432, 435, 451, 452, 453, 512, 513, 514, 521, 523, 524, 531, 532, 534, 541, 542, 543

   b. 123, 124, 125, 134, 135, 145, 234, 235, 245, 345

6. Find the value of each of these quantities.

1. C(5, 1)
2. C(5, 3)
3. C(8, 4)
4. C(8, 8)
5. C(8, 0)
6. C(12, 6)

Answer:

a. 5
b. 10
c. 70
d. 1
e. 1
f. 924

10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?
Answer:
$6! = 720$ different orders.

12. How many bit strings of length 12 contain

a. exactly three 1s?
b. at most three 1s?
c. at least three 1s?
d. an equal number of 0s and 1s?

Answer:

a. C(12, 3) = 220
b. C(12, 0) + C(12, 1) + C(12, 2) + C(12, 3) = 299
c. $\sum_{n=3}^{12} C(12, n) = 4017$
d. C(12, 6) = 924

6.4

2. Find the expansion of $(x + y)^5$

a. using combinatorial reasoning, as in Example 1.
b. using the binomial theorem.

Answer:

a. So $(x + y)^5 = (x + y)(x + y)(x + y)(x + y)(x + y)$ gives, x^5, x^4y, x^3y^2, x^2y^3, xy^4, and y^5 as the terms. To get x^5 and y^5 the only way is to chose x from all 5 or y from all 5 so they will both have 1 as their coefficient. To get x^4y x must be chosen from 4 of the 5 sums so it is 5 choose 4 which is 5, so x^4y must have a coefficient of 5 and then same for xy^4 by the same argument for the y. Then for x^3y^2 and x^2y^3 we choose 3 or 2 from the 5 sums, which is 10 so those will have coefficients of 10.

b. Binomial Theorem:
$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$
So it's $\binom{5}{0} = 1$ for x^5, $\binom{5}{1} = 5$ for x^4y, $\binom{5}{2} = 10$ for x^3y^2, $\binom{5}{3} = 10$ for x^2y^3, $\binom{5}{4} = 5$ for xy^4 and $\binom{5}{5} = 1$ for y^5.

6. What is the coefficient of x^7 in $(1 + x)^{11}$?
Answer:
$\binom{11}{7} = 330$

8. What is the coefficient of x^8y^9 in the expansion of $(3x + 2y)^{17}$?
Answer:
$\binom{17}{8} \cdot 3 \cdot 2 = 145860$
This is wrong should be $\binom{17}{8} \cdot 3^8 \cdot 2^9$

12. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}, 0 \leq k \leq 10$
1 10 45 120 210 252 210 120 45 10 1
Use Pascal's identity to produce the row immediately fol- lowing this row in Pascal's triangle.
Answer:
1 11 55 165 330 362 362 330 165 55 11 1
Don't add wrong, you tard, should be 462

Chapter 7

7.1

2. What is the probability that a fair die comes up six when it is rolled?
Answer:
1/6

8. What is the probability that a five-card poker hand contains the ace of hearts?
Answer:
In any group of 5 cards each card has a 1/52 chance of being the ace of hearts, so for a group of 5 there is $5 \cdot 1/52 = 5/52$ chance of there being an ace of hearts in that group.

30. What is the probability that a player of a lottery wins the prize offered for correctly choosing five (but not six) numbers out of six integers chosen at random from the integers between 1 and 40, inclusive?
Answer:
We know that there is $\binom{40}{6}$ choices for the 6 numbers. Then we have to choose 5 correct numbers out of 6. Then finally there is 34 numbers that we can choose that are not the correct one. So the answer is $\binom{6}{5} \cdot 34/\binom{40}{6} = 5.31 \cdot 10^{-5}$

40. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?
Answer:
Not changing doors : 0.25
Changing doors : $3/8 = 0.375$

7.2

2. Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.
Answer:
P(3) = 2/5 and the rest is 1/15 each.
This is wrong should be P(1) = P(2) = P(4) = P(5) = P(6) = x, then P(3) = 2x then 7x = 1, x = 1/7, P(3) = 2/7)

6. What is the probability of these events when we randomly select a permutation of 1, 2, 3?

   a. 1 precedes 3.
   b. 3 precedes 1.
   c. 3 precedes 1 and 3 precedes 2.

   Answer:
   Permutations: 123, 132, 213, 231, 312, 321

   a. 1/2
   b. 1/2
   c. 1/3

12. Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$.
Answer:
$p(E \cup F) = p(E) + p(F) - p(E \cap F) = 1.4 - p(E \cap F)$
But since the probability of an event can't be greater than 1 we know that $p(E \cap F) \geq 0.4$
Now to find the min of $p(E \cup F)$ we need to find the max of $p(E \cap F)$ it can't be higher than the probability of either of the events probability so it can't be greater than 0.6. So with the previous formula $p(E \cup F) = 1.4 - 0.6 = 0.8$ as the min. So $p(E \cup F) \geq 0.8$.

24. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?
Answer:
$p(E|F) = p(E \cap F)/p(F)$
Let F be first flip is tails then $p(F) = 1/2$
$p(E \cap F) = 1/2^5$ since there is only 1 sequence of flips that is 1 tails out of 2^5 combinations. So $p(E|F) = (1/32)/(1/2) = 1/16$

26. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?
Answer:
$p(E) = \left(\binom{3}{1} + \binom{3}{3}\right)/2^3 = 1/2$
$p(F) = 2^2/2^3 = 1/2$
$: p(E \cap F) = 2/8 = 1/4 = p(E) \cdot p(F)$
Thus E and F are independent.

Chapter 8

8.1

8. a. Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.
   b. What are the initial conditions?
   c. How many bit strings of length seven contain three consecutive 0s?

   Answer:

   a. Let a_n be the number of bit strings with 3 consecutive zeros.
   Case 1: Bit string with length n ends with a 1. Then there are a_{n-1} bit strings of length n with 3 consecutive zeros.
   Case 2: Bit string with length n that ends with 1 0. There are a_{n-2} bit strings of length n with 3 consecutive zeros.
   Case 3: Bit string with length n that ends with 1 0 0. There are a_{n-3} bit strings of length n with 3 consecutive zeros.
   Case 4: Bit strings with length n that ends with 0 0 0. There are 2^{n-3} bit strings of length n with 3 consecutive zeros.
   So $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$

   b. $a_0 = 0$
   $a_1 = 0$
   $a_2 = 0$

   c. $a_0 = 0$
   $a_1 = 0$
   $a_2 = 0$
   $a_3 = 1$
   $a_4 = 1 + 0 + 0 + 2^1 = 3$
   $a_5 = 3 + 1 + 0 + 2^2 = 8$
   $a_6 = 8 + 3 + 1 + 2^3 = 20$
   $a_7 = 20 + 8 + 3 + 2^4 = 47$

8.3

8. Suppose that $f(n) = 2f(n/2) + 3$ when n is an even positive integer, and $f(1) = 5$. Find

a. $f(2)$.
b. $f(8)$.
c. $f(64)$.
d. $f(1024)$.

Answer:

a. 13
b. 61
c. 509
d. 8189

10. Find $f(n)$ when $n = 2^k$, where f satisfies the recurrence relation $f(n) = f(n/2) + 1$ with $f(1) = 1$.
Answer:
$f(n) = \log_2(n) + 1 = k + 1$

14. Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are n/2 games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.
Answer:
$f(n) = f(n/2) + 1,\ f(1) = 0$

16. Solve the recurrence relation for the number of rounds in the tournament described in Exercise 14.
Answer:
$f(n) = \log_2(n) = k$