10.7 Change of Basis (Optional)

In a given problem one selects what appears to be the most convenient basis, but at some stage of the analysis it may be desirable to switch to some other basis. For example, in studying the aerodynamics of a propeller it is generally most convenient to carry out the analysis (of the propeller-induced pressure field, for example) with respect to a propeller-fixed basis, one that rotates with the propeller, although eventually we may wish to relate quantities back to a stationary (nonrotating) basis. How do the coordinates (i.e., the components) of a given vector change as we change the basis? It is that question which we address in this section.

Let $B = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a given basis for the vector space V under consideration so that any given vector $\mathbf{x}$ in V can be expanded as

$$\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n. \tag{1}$$

If we switch to some other basis $B' = \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$, then we may, similarly, expand the same vector $\mathbf{x}$ as

$$\mathbf{x} = x'_1\mathbf{e}'_1 + \cdots + x'_n\mathbf{e}'_n. \tag{2}$$

How are the x'_j coordinates related to the x_j coordinates? Since B' is a basis, we may expand each of the $\mathbf{e}_j$'s in terms of B':

$$\begin{aligned} \mathbf{e}_1 &= q_{11}\mathbf{e}'_1 + \cdots + q_{n1}\mathbf{e}'_n, \\ &\vdots \\ \mathbf{e}_n &= q_{1n}\mathbf{e}'_1 + \cdots + q_{nn}\mathbf{e}'_n. \end{aligned} \tag{3}$$

Putting (3) into (1) gives

$$\begin{aligned} \mathbf{x} &= x_1\left(q_{11}\mathbf{e}'_1 + \cdots + q_{n1}\mathbf{e}'_n\right) + \cdots + x_n\left(q_{1n}\mathbf{e}'_1 + \cdots + q_{nn}\mathbf{e}'_n\right) \\ &= \left(x_1 q_{11} + \cdots + x_n q_{1n}\right)\mathbf{e}'_1 + \cdots + \left(x_1 q_{n1} + \cdots + x_n q_{nn}\right)\mathbf{e}'_n, \end{aligned} \tag{4}$$

and a comparison of (2) and (4) gives the desired relations

$$\begin{aligned} x'_1 &= q_{11}x_1 + \cdots + q_{1n}x_n, \\ &\vdots \\ x'_n &= q_{n1}x_1 + \cdots + q_{nn}x_n \end{aligned} \tag{5}$$

or, in matrix notation,

$$\boxed{[\mathbf{x}]_{B'} = \mathbf{Q}\,[\mathbf{x}]_B\,,} \tag{6}$$

where

$$\mathbf{Q} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \end{bmatrix} \tag{7}$$

and

$$[\mathbf{x}]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad [\mathbf{x}]_{B'} = \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}. \tag{8}$$

We call $[\mathbf{x}]_B$ the **coordinate vector** of the vector $\mathbf{x}$ with respect to the ordered basis B, and similarly for $[\mathbf{x}]_{B'}$, and we call $\mathbf{Q}$ the **coordinate transformation matrix** from B to B'.

Thus far, our results apply whether the bases are orthogonal or not. *In the remainder of this section we assume that both bases, B and B', are ON.* Thus, let us rewrite (3) as

$$\begin{aligned} \hat{\mathbf{e}}_1 &= q_{11}\hat{\mathbf{e}}'_1 + \cdots + q_{n1}\hat{\mathbf{e}}'_n, \\ &\vdots \\ \hat{\mathbf{e}}_n &= q_{1n}\hat{\mathbf{e}}'_1 + \cdots + q_{nn}\hat{\mathbf{e}}'_n, \end{aligned} \tag{9}$$

where the carets denote unit vectors, as usual. If we dot $\hat{\mathbf{e}}'_1$ into both sides of the first equation in (9), and remember that B' is ON, we obtain $q_{11} = \hat{\mathbf{e}}'_1 \cdot \hat{\mathbf{e}}_1$. Dotting $\hat{\mathbf{e}}'_2$ gives $q_{21} = \hat{\mathbf{e}}'_2 \cdot \hat{\mathbf{e}}_1, \dots,$ dotting $\hat{\mathbf{e}}'_n$ gives $q_{n1} = \hat{\mathbf{e}}'_n \cdot \hat{\mathbf{e}}_1$, and similarly for the second through nth equation in (9). The result is the formula

$$\boxed{q_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j,} \tag{10}$$

which tells us how to compute the transformation matrix $\mathbf{Q}$.

There are two properties of the $\mathbf{Q}$ matrix to address before turning to an example. To obtain the first of these, observe that

$$\begin{aligned} \mathbf{Q}^{\mathrm{T}}\mathbf{Q} &= \begin{bmatrix} q_{11} & \cdots & q_{n1} \\ \hline q_{12} & \cdots & q_{n2} \\ \hline & \vdots & \\ \hline q_{1n} & \cdots & q_{n2} \end{bmatrix} \left[\begin{array}{c|c|c|c} q_{11} & q_{12} & & q_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & & q_{nn} \end{array}\right] \\ &= \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 & \cdots & \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_n \\ \vdots & & \vdots \\ \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_1 & \cdots & \hat{\mathbf{e}}_n \cdot \hat{\mathbf{e}}_n \end{bmatrix} = \mathbf{I} \end{aligned} \tag{11}$$

so that

$$\boxed{\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}.} \tag{12}$$

It was useful to partition the $\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{Q}$ matrices in (11) because we can see from (9) that the columns of $\mathbf{Q}$ (and hence the rows of $\mathbf{Q}^{\mathrm{T}}$) are actually the $\hat{\mathbf{e}}_1, \dots, \hat{\mathbf{e}}_n$ vectors (in n-tuple form). Thus, from the way matrix multiplication is defined, we can see that the elements of the product matrix are dot products. Specifically, the i, j element of $\mathbf{Q}^{\mathrm{T}}\mathbf{Q}$ is $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j$, which is the Kronecker delta δ_{ij}. Hence, $\mathbf{Q}^{\mathrm{T}}\mathbf{Q}$ equals the identity matrix $\mathbf{I}$, so $\mathbf{Q}^{\mathrm{T}}$ must be the inverse of $\mathbf{Q}$, as stated in (12).

That result makes it easy for us to reverse equation (6) – that is, to solve for $[\mathbf{x}]_B$ in terms of $[\mathbf{x}]_{B'}$ for then $[\mathbf{x}]_B = \mathbf{Q}^{-1}[\mathbf{x}]_{B'} = \mathbf{Q}^{\mathrm{T}}[\mathbf{x}]_{B'}$. In other words, we do not need to face up to the evaluation of $\mathbf{Q}^{-1}$ since $\mathbf{Q}^{-1}$ is merely $\mathbf{Q}^{\mathrm{T}}$.

Any matrix with the useful property (12) is known as an **orthogonal matrix** because it follows from (12) that the column vectors in $\mathbf{Q}$ are orthonormal.

As the second property of $\mathbf{Q}$, observe that it also follows from (12) that

$$\boxed{\det\mathbf{Q} = \pm 1,} \tag{13}$$

that is, either $+1$ or -1 since $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ implies that $\det(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}) = \det\mathbf{I} = 1$. But $\det(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}) = (\det\mathbf{Q}^{\mathrm{T}})(\det\mathbf{Q}) = (\det\mathbf{Q})(\det\mathbf{Q}) = (\det\mathbf{Q})^2$. Hence, $\det\mathbf{Q}$ must be $+1$ or -1.

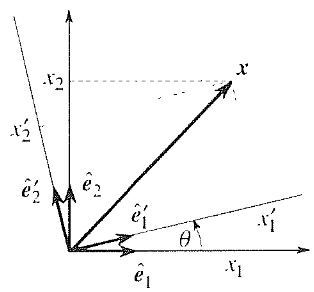

Figure 1. Rotation in the plane.

EXAMPLE 1. *Rotation in the Plane.* Consider the vector space $\mathbb{R}^2$, with the ON bases $B = \{\hat{\mathrm{e}}_1, \hat{\mathrm{e}}_2\}$ and $B' = \{\hat{\mathrm{e}}_1', \hat{\mathrm{e}}_2'\}$ shown in Fig. 1. B' is obtained from B by a counterclockwise rotation through an angle θ (or clockwise if θ is negative). From the figure,

$$\begin{aligned}
q_{11} &= \hat{\mathrm{e}}_1' \cdot \hat{\mathrm{e}}_1 = (1)(1)\cos\theta = \cos\theta, \\
q_{12} &= \hat{\mathrm{e}}_1' \cdot \hat{\mathrm{e}}_2 = (1)(1)\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \\
q_{21} &= \hat{\mathrm{e}}_2' \cdot \hat{\mathrm{e}}_1 = (1)(1)\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta, \\
q_{22} &= \hat{\mathrm{e}}_2' \cdot \hat{\mathrm{e}}_2 = (1)(1)\cos\theta = \cos\theta,
\end{aligned} \tag{14}$$

so that the coordinate transformation matrix is

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{15}$$

Hence,

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{16}$$

Or, the other way around,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}. \tag{17}$$

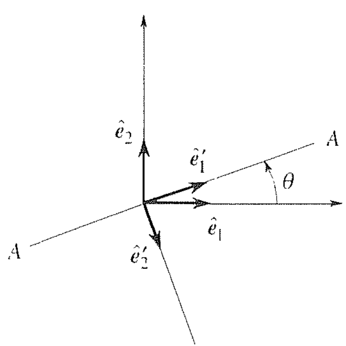

Figure 2. Rotation plus reflection.

COMMENT 1. It is easy enough to check (16) and (17) for one or two special cases. For example, if $\theta = 0$ then the two bases coincide so we should have $x_1' = x_1$ and $x_2' = x_2$, and that is what (16) and (17) give. Also, if $\theta = \pi/2$, say, we should have $x_1' = x_2$ and $x_2' = -x_1$ and, again, that is what (16) and (17) give.

COMMENT 2. Two ON bases in a plane are not necessarily related through a rotation. In this example, for instance, if we reverse the direction of $\hat{\mathrm{e}}_2'$, then $\{\hat{\mathrm{e}}_1', \hat{\mathrm{e}}_2'\}$ is still ON

(Fig. 2), but is not obtainable from $\{\hat{e}_1, \hat{e}_2\}$ by means of a rotation alone. Rather, we need a rotation *and a reflection*, a counterclockwise rotation through an angle θ, and then a reflection about AA (or, first a reflection about the $\hat{e}_1$ axis and then a counterclockwise rotation through an angle θ). In this case

$$\begin{aligned} q_{11} &= \hat{e}_1' \cdot \hat{e}_1 = \cos\theta, \\ q_{12} &= \hat{e}_1' \cdot \hat{e}_2 = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \\ q_{21} &= \hat{e}_2' \cdot \hat{e}_1 = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \\ q_{22} &= \hat{e}_2' \cdot \hat{e}_2 = \cos(\pi - \theta) = -\cos\theta \end{aligned}$$

so that $\mathbf{Q}$ is the orthogonal matrix

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}. \tag{18}$$

Recall from (13) that $\det\mathbf{Q}$ is either $+1$ or -1. For the case where B and B' are related through a pure rotation [$\mathbf{Q}$ given by (15)] $\det\mathbf{Q} = +1$, and for the case where they are related through a reflection and a rotation [$\mathbf{Q}$ given by (18)] $\det\mathbf{Q} = -1$. ∎

Closure. In this brief section we study the relationship between the components, or coordinates, of any given vector $\mathbf{x}$ expanded in terms of two different bases B and B'. We find the linear relationship (6), where the q_{ij} elements of the coordinate transformation matrix $\mathbf{Q}$ are the expansion coefficients of $\mathbf{e}_j$ in terms of $\mathbf{e}_1', \ldots, \mathbf{e}_n'$, as indicated by (3).

If B and B' are ON, then the q_{ij}'s are computed, simply, from (10), and $\mathbf{Q}$ admits the properties that $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}$ and that $\det\mathbf{Q}$ is $+1$ or -1. *Any* matrix $\mathbf{Q}$ having the property $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}$ has ON column vectors and is called an orthogonal matrix.

EXERCISES 10.7

1. Given that $\mathbf{e}_1 = \mathbf{e}_1' + 2\mathbf{e}_2'$ and $\mathbf{e}_2 = \mathbf{e}_1' - \mathbf{e}_2'$, find the coordinate transformation matrix $\mathbf{Q}$. Is $\mathbf{Q}$ orthogonal? If $[\mathbf{x}]_B = [5, -1]^{\mathrm{T}}$, find $[\mathbf{x}]_{B'}$. If $[\mathbf{x}]_{B'} = [2, 3]^{\mathrm{T}}$, find $[\mathbf{x}]_B$.

2. Given that $\mathbf{e}_1 = \mathbf{e}_1' + \mathbf{e}_2' - \mathbf{e}_3'$, $\mathbf{e}_2 = \mathbf{e}_1' - \mathbf{e}_2' + \mathbf{e}_3'$, and $\mathbf{e}_3 = -\mathbf{e}_1' + \mathbf{e}_2' + \mathbf{e}_3'$, find the coordinate transformation matrix $\mathbf{Q}$. Is $\mathbf{Q}$ orthogonal? If $[\mathbf{x}]_B = [4, 1, -2]^{\mathrm{T}}$, find $[\mathbf{x}]_{B'}$. If $[\mathbf{x}]_{B'} = [1, 0, 2]^{\mathrm{T}}$, find $[\mathbf{x}]_B$.

3. Let $\hat{e}_1 = [1, 0]^{\mathrm{T}}$, $\hat{e}_2 = [0, 1]^{\mathrm{T}}$, and $\hat{e}_1' = \frac{1}{\sqrt{5}}[2, 1]^{\mathrm{T}}$, $\hat{e}_2' = \frac{1}{\sqrt{5}}[1, -2]^{\mathrm{T}}$.

(a) Find the coordinate transformation matrix $\mathbf{Q}$. Is $\mathbf{Q}$ orthogonal?
(b) If $[\mathbf{x}]_B = [8, -6]^{\mathrm{T}}$, find $[\mathbf{x}]_{B'}$.
(c) If $[\mathbf{x}]_{B'} = [1, 3]^{\mathrm{T}}$, find $[\mathbf{x}]_B$.

4. Let $\hat{e}_1 = [1, 0, 0, 0]^{\mathrm{T}}$, $\hat{e}_2 = [0, 1, 0, 0]^{\mathrm{T}}$, $\hat{e}_3 = [0, 0, 1, 0]^{\mathrm{T}}$, $\hat{e}_4 = [0, 0, 0, 1]^{\mathrm{T}}$, and $\hat{e}_1' = \frac{1}{\sqrt{2}}[1, 1, 0, 0]^{\mathrm{T}}$, $\hat{e}_2' = [0, 0, 1, 0]^{\mathrm{T}}$, $\hat{e}_3' = \frac{1}{\sqrt{3}}[1, -1, 0, 1]^{\mathrm{T}}$, $\hat{e}_4' = \frac{1}{\sqrt{6}}[1, -1, 0, -2]^{\mathrm{T}}$.

(a) Find the coordinate transformation matrix $\mathbf{Q}$. Is $\mathbf{Q}$ orthogonal?
(b) If $[\mathbf{x}]_B = [1, 1, 2, 5]^{\mathrm{T}}$, find $[\mathbf{x}]_{B'}$.

(c) If $[\mathbf{x}]_{B'} = [1, 1, 2, 5]^{\mathrm{T}}$, find $[\mathbf{x}]_B$.

5. Show whether or not these matrices are orthogonal.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (f) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$

6. For the case of rotation in a plane, the transformation $\mathbf{Q}$ corresponded to a counterclockwise rotation of the basis through an angle θ. Does $\mathbf{Q}^{-1}$ correspond to the reverse of this, a *clockwise* rotation θ ? Prove or disprove.

7. (*Rotation and reflection*) (a) Show that *every* orthogonal coordinate transformation matrix of order 2 is of one of the following two types:

$$\mathbf{Q}_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{Q}_2 = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

i.e., as given by the pure rotation (15) or by the rotation plus reflection (18).

(b) Show that these two cases can be distinguished by the sign of the determinant, specifically, that $\det\mathbf{Q} = +1$ if $\mathbf{Q}$ corresponds to pure rotation, and that $\det\mathbf{Q} = -1$ if $\mathbf{Q}$ corresponds to rotation plus reflection.

8. (a) Prove that if $\mathbf{Q}$ is orthogonal, then so is $\mathbf{Q}^{\mathrm{T}}$.
(b) Prove that if $\mathbf{Q}$ is orthogonal, then so is $\mathbf{Q}^{-1}$.

9. Evaluate $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n$.

10.8 Vector Transformation (Optional)

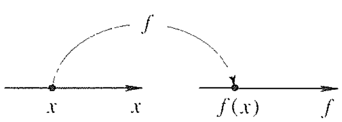

Figure 1. Function f as a transformation.

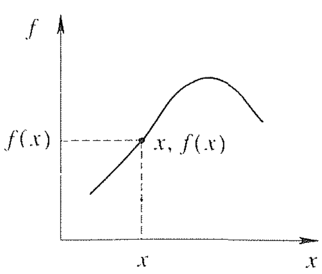

Figure 2. The graph of f.

Recall that a real-valued function f of a real variable x is a rule that assigns a uniquely determined value $f(x)$ to each specified value x as illustrated in Fig. 1. Thus, f is a transformation, or mapping, from points on an x axis to points on an f axis, and we view x as the "input" and $f(x)$ as the "output." [A more familiar graphical display of f, called the *graph* of f, can be obtained if, following *Descartes* (1596–1650), we arrange the x and f axes at right angles to each other and plot the set of points $x, f(x)$ as illustrated in Fig. 2.]

In this section we reconsider vectors and matrices from this transformation point of view. Specifically, we consider vector-valued functions $\mathbf{F}$ of a vector variable $\mathbf{x}$. That is, the "input" is now a vector $\mathbf{x}$ from some vector space V, and the function $\mathbf{F}$ assigns a uniquely determined "output" vector $\mathbf{F}(\mathbf{x})$ in some vector space W. We call $\mathbf{F}$ a **transformation**, or **mapping**, from V into W, and denote it as

$$\mathbf{F} : V \to W.$$

We call V the **domain of F** and W the **range of definition of F**. W may, but need not, be identical to V. If it *is* identical, then $\mathbf{F} : V \to V$ is called an **operator** on V.