Stability and cycling near heteroclinic cycles and networks

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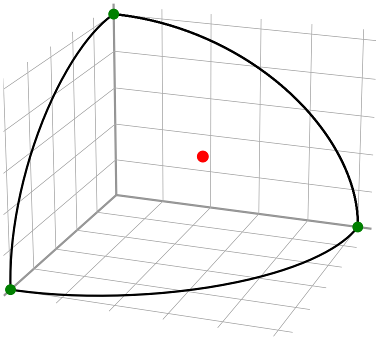
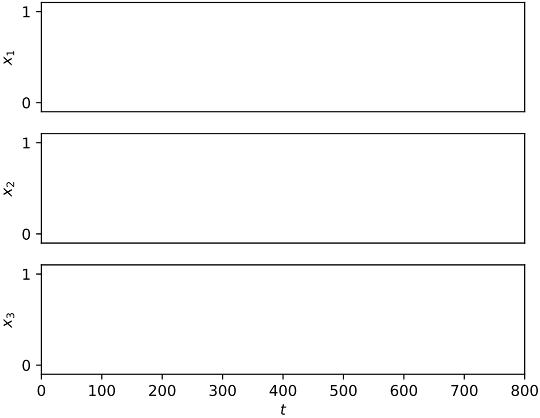
Supervised by Vivien Kirk and Claire Postlethwaite

Student Research Conference, June 7th, 2022

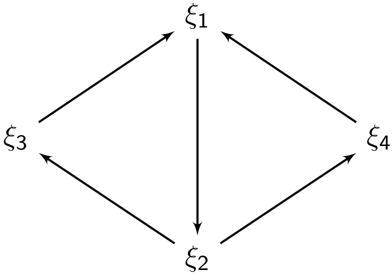
Competition between three species[1]





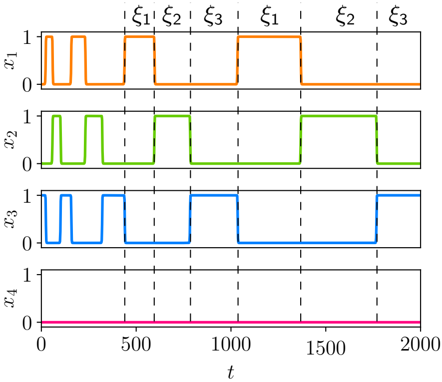
[1]May and Leonard, 1975

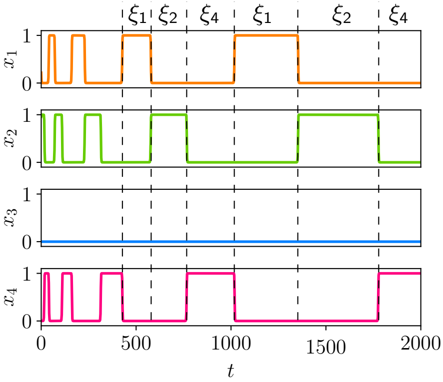
The Kirk-Silber network[2]

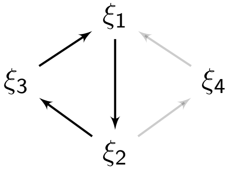


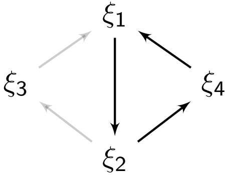
[2]Kirk and Silber, 1994

The Kirk-Silber network

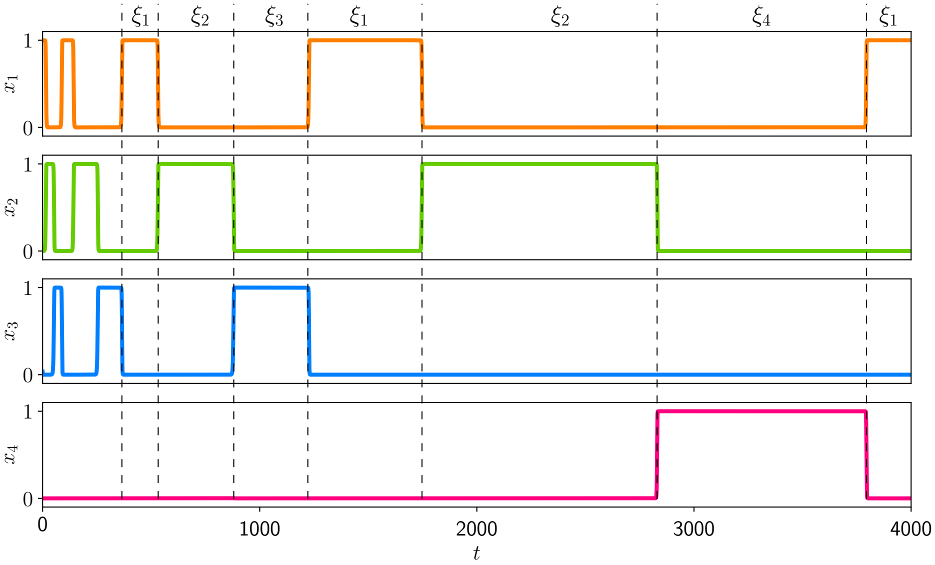




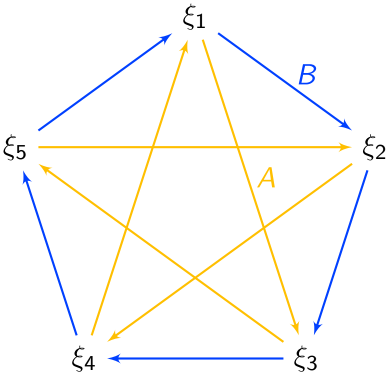




Switching in the Kirk-Silber network

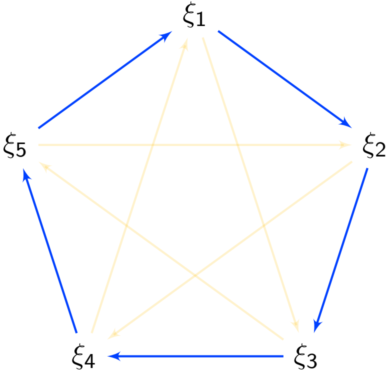


The Rock-Paper-Scissors-Lizard-Spock network[3]



[3] Postlethwaite and Rucklidge, 2022

The Rock-Paper-Scissors-Lizard-Spock network



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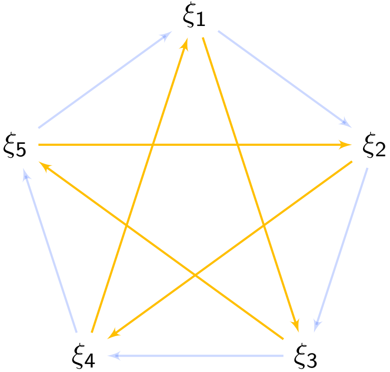

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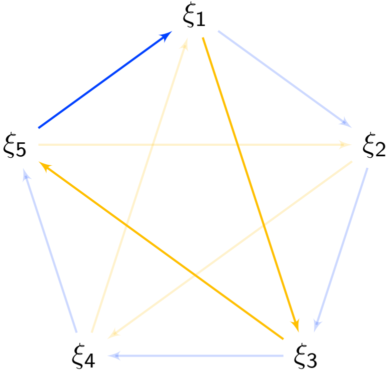
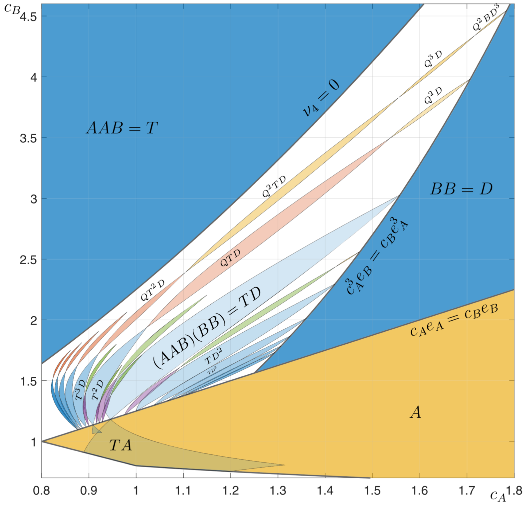
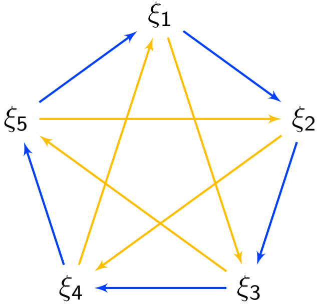


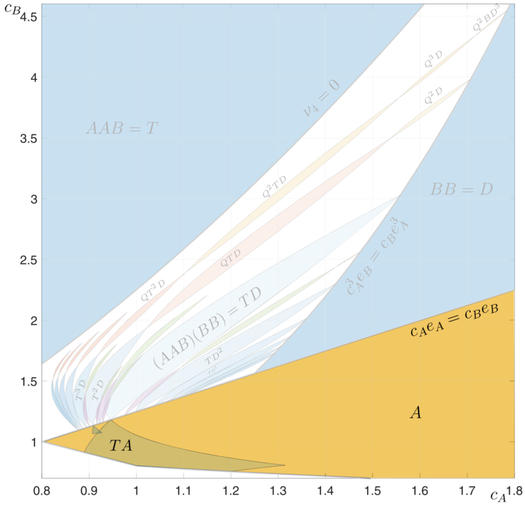
Cycling near the RPSLS network[3]

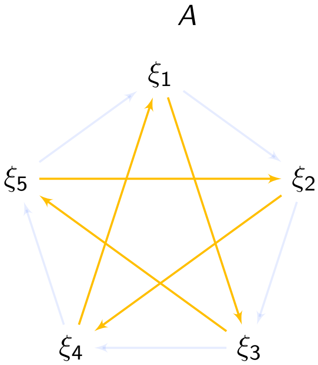




[3]Postlethwaite and Rucklidge, 2022

Cycling near the RPSLS network





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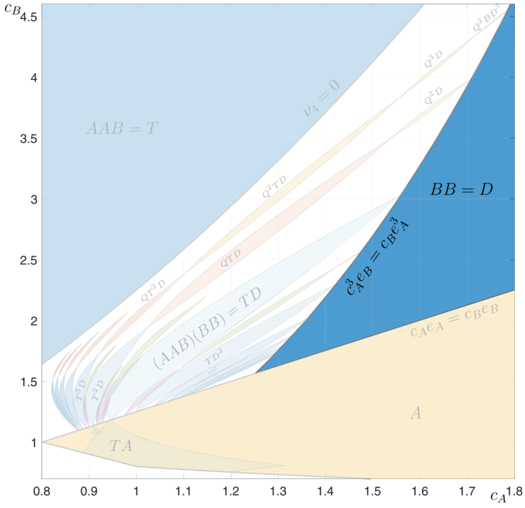

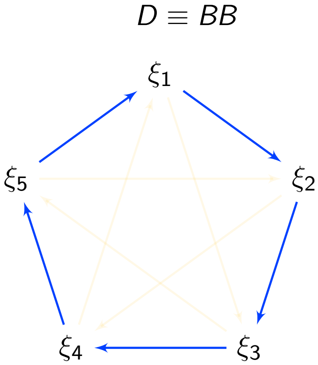

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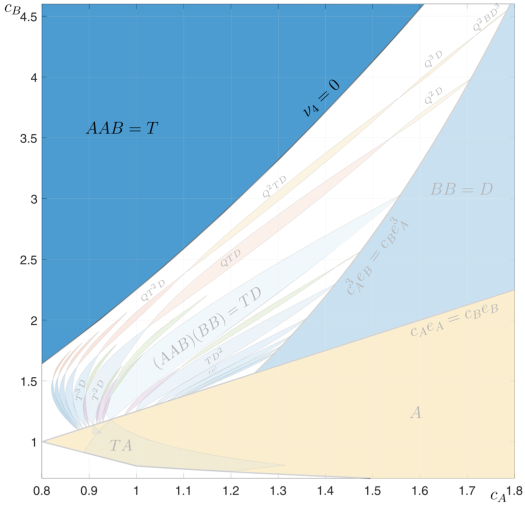
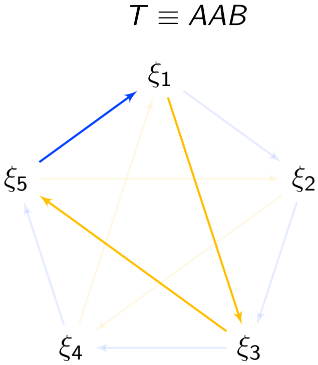
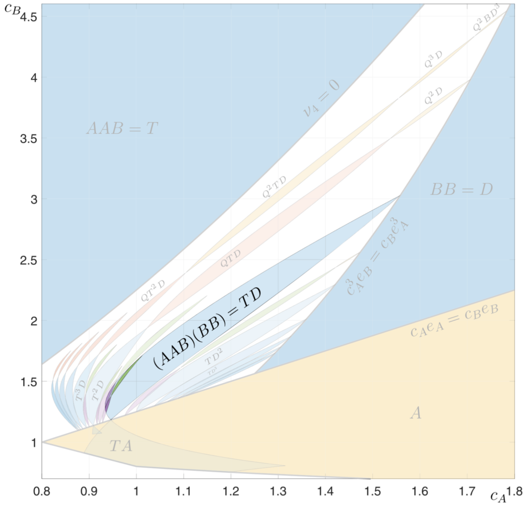
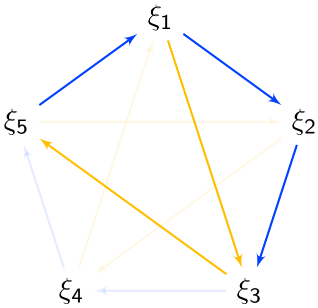




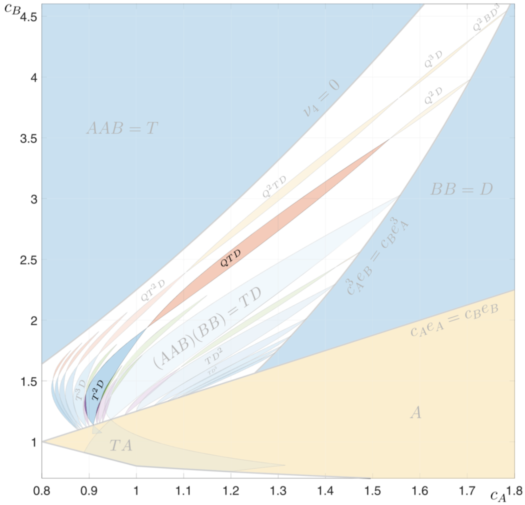
Cycling near the RPSLS network



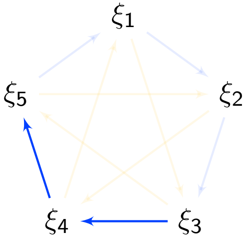
$TD \equiv AAB\ BB$



Shrinking points

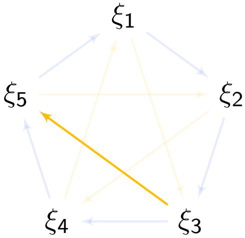


$QTD \equiv ABBB\ AAB\ BB$



$\Downarrow$

$T^2D \equiv AAB\ AAB\ BB$



Research questions

- Can we explain the chains of stability regions and other structure observed in this bifurcation diagram, and in the bifurcation diagrams of similar networks with more equilibria?
- Little is known about possible dynamics near large heteroclinic networks. Can we classify possible dynamics near classes of these networks based on the topology and symmetries of its representation as a graph?

Heteroclinic cycles and networks

Consider a system of ordinary differential equations defined by

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is an (at least) C^1 vector field. A *equilibrium* of (1) is a point $\xi \in \mathbb{R}^n$ such that $f(\xi) = 0$.

If ξ_j and ξ_{j+1} are equilibria of (1), then a solution $\phi_j(t)$ of (1) is a *heteroclinic orbit* if $\phi_j(t)$ is backward asymptotic to ξ_j and forward asymptotic to ξ_{j+1}.

A *heteroclinic cycle* is an invariant set $X \subset \mathbb{R}^n$ consisting of $m \geq 2$ equilibria, $\xi_1, \xi_2, \ldots, \xi_m$, and connecting heteroclinic orbits, $\phi_1(t), \phi_2(t), \ldots, \phi_m(t)$ (where we set $\xi_{m+1} \equiv \xi_1$).

A *heteroclinic network* is a connected union of heteroclinic cycles.

Asymptotic stability

Definition

An invariant set X is *asymptotically stable* if for every neighbourhood U of X there exists a neighbourhood V of X such that, for all $x(0) \in V$,

- $x(t) \in U$ for all $t > 0$; and
- $\lim_{t \to \infty} d(x(t), X) = 0$.

Definition

An invariant set X is *asymptotically stable relative to a set* B if for every neighbourhood U of X there exists a neighbourhood V of X such that, for all $x(0) \in V \cap B$,

- $x(t) \in U$ for all $t > 0$; and
- $\lim_{t \to \infty} d(x(t), X) = 0$.

Essential and fragmentary asymptotic stability

Definition

An invariant set X is *essentially asymptotically stable* if it is asymptotically stable relative to a set B, such that

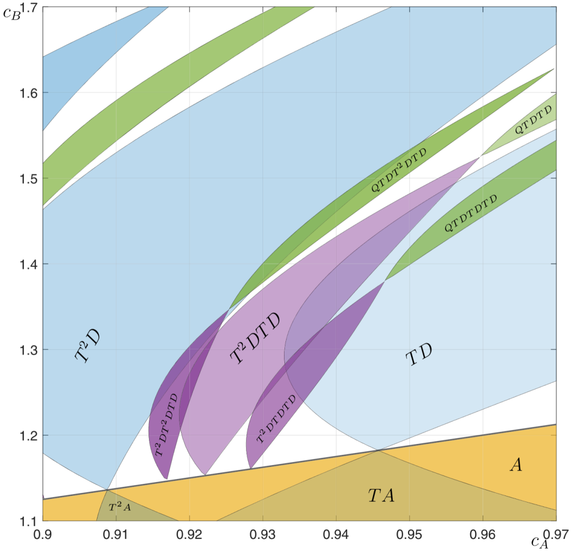
$$\lim_{r \to 0} \frac{\mu(N_r(X) \cap B)}{\mu(N_r(X))} = 1$$

where $N_r(X)$ is the generalised ball of radius r around X.

Definition

An invariant set X is *fragmentarily asymptotically stable* if it is asymptotically stable relative to a set B, such that $\mu(B \cap U) > 0$ for every neighbourhood U of X.

Farey-like patterns



Poincaré maps