Metodos Numericos

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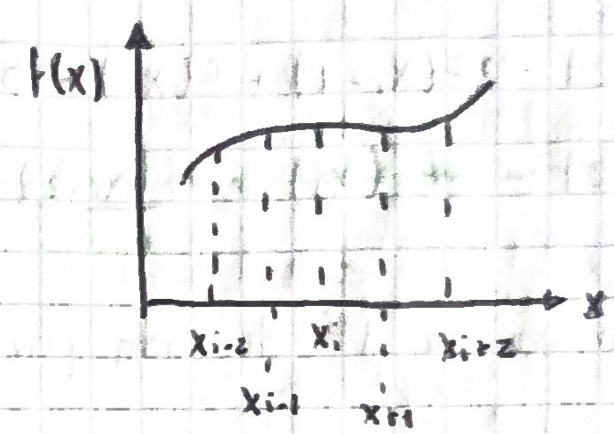
Demostrar que para la 1ª derivada por diferencias finitas hacia atrás es:

$$f'(x_i) = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2} + O(h)$$

Supongamos una función $f(x)$ que se comporta de la siguiente manera.



Por series de Taylor

① $f(x_{i-2}) = f(x_i) + f'(x_i)(x_{i-2} - x_i) + \frac{f''(x_i)(x_{i-2} - x_i)^2}{2!} + O(h)$

② $f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)(x_{i-1} - x_i)^2}{2!} + O(h)$

Donde: $x_{i-2} - x_i = -2h$ y $x_{i-1} - x_i = -h$

① $f(x_{i-2}) = f(x_i) + f'(x_i)(-2h) + \frac{f''(x_i)(-2h)^2}{2!} + O(h)$

② $f(x_{i-1}) = f(x_i) + f'(x_i)(-h) + \frac{f''(x_i)(-h)^2}{2!} + O(h)$

① $f(x_{i-2}) = f(x_i) - 2f'(x_i)h + \frac{\cancel{4}^{2} f''(x_i)h^2}{\cancel{2!}} + O(h)$

② $f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)h^2}{2!} + O(h)$

Multiplicamos la Ec 2 por -2

② $-2f(x_{i-1}) = -2f(x_i) + 2f'(x_i)h - \frac{\cancel{2}f''(x_i)h^2}{\cancel{2!}} + O(h)$

Sumamos Ec 2 + Ec 1.

$$② \quad -2f(x_{i-1}) = -2f(x_i) + \cancel{2f'(x_i)h}^{\,0} - f''(x_i)h^2 + O(h) \quad +$$
$$① \quad f(x_{i-2}) = f(x_i) - \cancel{2f'(x_i)h} + 2f''(x_i)h^2 + O(h)$$

$$f(x_{i-2}) - 2f(x_{i-1}) = -f(x_i) + f''(x_i)h^2 + O(h)$$

$$f(x_{i-2}) - 2f(x_{i-1}) + f(x_i) + O(h) = f''(x_i)h^2$$

$$\cancel{f(x_{i-2})} \quad f''(x_i) = \frac{f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)}{h^2} + O(h)$$

Demostrar que la 1era derivada para diferencias finitas centrada es igual a:

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)$$

Tomando x_{i+1} y x_{i-1}.

$$① \quad f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)(x_{i+1} - x_i)^2}{2!} + O(h^2)$$

$$② \quad f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)(x_{i-1} - x_i)^2}{2!} + O(h^2)$$

Donde $x_{i+1} - x_i = h$ y $x_{i-1} - x_i = -h$. $\therefore$

$$① \quad f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + O(h^2)$$

$$② \quad f(x_{i-1}) = f(x_i) + f'(x_i)(-h) + \frac{f''(x_i)(-h)^2}{2!} + O(h^2)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)(h^2)}{2!} + O(h^2)$$

Sumamos ① + ②

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + \cancel{f'(x_i)h - f'(x_i)h}^{\,0} + \frac{\cancel{2}f''(x_i)h^2}{\cancel{2!}} + O(h^2)$$

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)h^2 + O(h^2)$$

$$f''(x_i)h^2 = f(x_{i+1}) + f(x_{i-1}) - 2f(x_i) + O(h^2) \qquad \div h^2$$

$$f''(x_i) = \frac{f(x_{i+1}) + f(x_{i-1}) - 2f(x_i)}{h^2} + O(h^2)$$

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3ra derivada hacia adelante.

Se trunca la serie de Taylor hasta $f'''(x)$ para x_{i+1}, x_{i+2} y x_{i+3}

① $f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x)(x_{i+1} - x_i)^2}{2!} + \frac{f'''(x)(x_{i+1} - x_i)^3}{3!}$

$+ o(h)$.

② $f(x_{i+2}) = f(x_i) + f'(x_i)(x_{i+2} - x_i) + \frac{f''(x_i)(x_{i+2} - x_i)^2}{2!} + \frac{f'''(x_i)(x_{i+2} - x_i)^3}{3!}$

$+ o(h)$.

③ $f(x_{i+3}) = f(x_i) + f'(x_i)(x_{i+3} - x_i) + \frac{f''(x_i)(x_{i+3} - x_i)^2}{2!} + \frac{f'''(x_i)(x_{i+3} - x_i)^3}{3!}$

$+ o(h)$.

Donde $x_{i+1} - x_i = h$; $x_{i+2} - x_i = 2h$ y $x_{i+3} - x_i = 3h$

Reescribimos

① $f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)h^2}{2!} + \frac{f'''(x_i)h^3}{3!} + o(h)$.

② $f(x_{i+2}) = f(x_i) + 2f'(x_i)h + \frac{4f''(x_i)h^2}{2!} + \frac{8f'''(x_i)h^3}{3!} + o(h)$

③ $f(x_{i+3}) = f(x_i) + 3f'(x_i)h + \frac{9f''(x_i)h^2}{2!} + \frac{27f'''(x_i)h^3}{3!} + o(h)$

Sumamos Ec. 1. + Ec 2. y multiplicamos por -2 la Ec. 2.

④ $f(x_{i+3}) + f(x_{i+1}) = 2f(x_i) + 4f'(x_i)h + \frac{10f''(x_i)h^2}{2!} + o(h)$

$+ \frac{28f'''(x_i)h^3}{3!}$

⑤ $-2f(x_{i+2}) = -2f(x_i) - 4f'(x_i)h - \frac{8f''(x_i)h^2}{2!} - \frac{16f'''(x_i)h^3}{3!}$

$+ o(h)$

Sumamos Ec 4 + Ec 5

$$f(x_{i+3}) + f(x_{i+1}) - 2f(x_{i+2}) = \cancel{2f(x_i) - 2f(x_i)} + 4f'(x_i)h - 4f'(x_i)h + \frac{10f''(x_i)h^2}{2!} - \frac{8f''(x_i)h^2}{2!} + \frac{28f'''(x_i)h^3}{3!} - \frac{16f'''(x_i)h^3}{3!} + O(h)$$

$$f(x_{i+3}) + f(x_{i+1}) - 2f(x_{i+2}) = \frac{2f''(x_i)h^2}{2!} + \frac{12f'''(x_i)h^3}{3!} + O(h)$$

$$f(x_{i+3}) + f(x_{i+1}) - 2f(x_{i+2}) = f''(x_i)h^2 + 2f'''(x_i)h^3 + O(h)$$

Se conoce: $f''(x_i)h^2 = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i) + O(h)$

Sustituimos

$$f(x_{i+3}) + f(x_{i+1}) - 2f(x_{i+2}) = f(x_{i+2}) - 2f(x_{i+1}) + f(x_i) + 2f'''(x_i)h^3 + O(h)$$

$$2f'''(x_i)h^3 = f(x_{i+3}) + f(x_{i+1}) - 2f(x_{i+2}) - f(x_{i+2}) + 2f(x_{i+1}) - f(x_i) + O(h)$$

$$2f'''(x_i)h^3 = f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i) + O(h)$$

$$f'''(x_i) = \frac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{2h^3} + O(h)$$

$\therefore$ La 3^{ra} derivada para diferencias finitas hacia adelante es igual a:

$$\boxed{f^{(3)}(x_i) = \frac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{2h^3} + O(h)}$$

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3^{ra} derivada con diferencias finitas centrada.

Para x_{i+2} y x_{i-2}.

① $f(x_{i+2}) = f(x_i) + f'(x_i)\overbrace{(x_{i+2} - x_i)}^{2h} + \frac{f''(x_i)(x_{i+2} - x_i)^2}{2!} + \frac{f'''(x_i)(x_{i+2} - x_i)^3}{3!}$
$+ o(h^2)$

$f(x_{i+2}) = f(x_i) + 2f'(x_i)h + \frac{4f''(x_i)h^2}{2!} + \frac{8f'''(x_i)h^3}{3!} + o(h^2)$

② $f(x_{i-2}) = f(x_i) + f'(x_i)\overbrace{(x_{i-2} - x_i)}^{-2h} + \frac{f''(x_i)\overbrace{(x_{i-2} - x_i)}^{-2h}{}^2}{2!} + \frac{f'''(x_i)(x_{i-2} - x_i)^3}{3!}$
$+ o(h^2)$

$f(x_{i-2}) = f(x_i) - 2f'(x_i)h + \frac{4f''(x_i)h^2}{2!} - \frac{8f'''(x_i)h^3}{3!} + o(h^2)$

Multiplicar ② por -1

③ $-f(x_{i-2}) = -f(x_i) + 2f'(x_i)h - \frac{4f''(x_i)h^2}{2!} + \frac{8f'''(x_i)h^3}{3!} + o(h^2)$

① $f(x_{i+2}) = f(x_i) + 2f'(x_i)h + \frac{4f''(x_i)h^2}{2!} + \frac{8f'''(x_i)h^3}{3!} + o(h^2)$

Somando ① + ③

$f(x_{i+2}) - f(x_{i-2}) = 4f'(x_i)h + \frac{16f'''(x_i)h^3}{3 \cdot 2!} + o(h^2)$

Se conoce que $2f'(x_i)h = f(x_{i+1}) - f(x_{i-1}) + o(h^2)$ para diferencias finitas centrada. ∴

$f(x_{i+2}) - f(x_{i-2}) = 2f(x_{i+1}) - 2f(x_{i-1}) + \frac{8f'''(x_i)h^3}{3} + o(h^2)$

$\frac{8f'''(x_i)h^3}{3} = f(x_{i+2}) - f(x_{i-2}) - 2f(x_{i+1}) + 2f(x_{i-1}) + o(h^2)$

$$\boxed{f'''(x_i) = \frac{3f(x_{i+2}) - 3f(x_{i-2}) - 6f(x_{i+1}) + 6f(x_{i-1})}{8h^3} + o(h^2)}$$

3^{ra} derivada con diferencias finitas hacia atras

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)(x_{i-1} - x_i)^2}{2!}$$
$$\frac{f'''(x_i)(x_{i-1} - x_i)^3}{3!} + O(h)$$

$$f(x_{i-2}) = f(x_i) + f'(x_i)(x_{i-2} - x_i) + \frac{f''(x_i)(x_{i-2} - x_i)^2}{2!}$$
$$+ \frac{f'''(x_i)(x_{i-2} - x_i)^3}{3!} + O(h)$$

$$f(x_{i-3}) = f(x_i) + f'(x_i)(x_{i-3} - x_i) + \frac{f''(x_i)(x_{i-3} - x_i)^2}{2!}$$
$$+ \frac{f'''(x_i)(x_{i-3} - x_i)^3}{3!} + O(h)$$

Donde $x_{i-3} - x_i = -3h$
$x_{i-2} - x_i = -2h$
$x_{i-1} - x_i = -h$

① $f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)h^2}{2!} - \frac{f'''(x_i)h^3}{3!} + O(h)$

② $f(x_{i-2}) = f(x_i) - 2f'(x_i)h + \frac{4f''(x_i)h^2}{2!} - \frac{8f'''(x_i)h^3}{3!} + O(h)$

③ $f(x_{i-3}) = f(x_i) - 3f'(x_i)h + \frac{9f''(x_i)h^2}{2!} - \frac{27f'''(x_i)h^3}{3!} + O(h)$

Suma ① + ③

④ $f(x_{i-1}) + f(x_{i-3}) = 2f(x_i) - 4f'(x_i)h + \frac{10f''(x_i)h^2}{2!} - \frac{28f'''(x_i)h^3}{3!}$
$+ O(h)$

Multiplicamos ② por -2.

⑤ $-2f(x_{i-2}) = -2f(x_i) + 4f'(x_i)h - \frac{8f''(x_i)h^2}{2!} + \frac{16f'''(x_i)h^3}{3!} + O(h)$

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Sumar ① + ②

$$f(x_{i-3}) - 2f(x_{i-2}) + f(x_{i-1}) = 2f(\cancel{x_i}) - 2f(\cancel{x_i}) - 4f'(\cancel{x_i})h + 4f'(\cancel{x_i})h$$

$$+\frac{10f''(x_i)h^2}{2!} - \frac{8f''(x_i)h^2}{2!}$$

$$-\frac{16f'''(x_i)h^3}{3!} + \frac{28f'''(x_i)h^3}{3!} + O(h)$$

$$f(x_{i-3}) - 2f(x_{i-2}) + f(x_{i-1}) = \frac{2f''(x_i)h^2}{2!} + \frac{12f'''(x_i)h^3}{3!} + O(h)$$ ③

$$f(x_{i-3}) - 2f(x_{i-2}) + f(x_{i-1}) = f''(x_i)h^2 + 2f'''(x_i)h^3 + O(h)$$

Conocemos que $f''(x_i)h$ para diferencias finitas hacía atrás es igual a:

$$f''(x_i)h^2 = f(x_{i-2}) - 2f(x_{i-1}) + f(x_i)$$

$$f(x_{i-3}) - 2f(x_{i-2}) + f(x_{i-1}) = f(x_{i-2}) - 2f(x_{i-1}) + f(x_i) + 2f'''(x_i)h^3 + O(h)$$

$$f(x_{i-3}) - 2f(x_{i-2}) - f(x_{i-2}) + f(x_{i-1}) + 2f(x_{i-1}) - f(x_i) + O(h)$$
$$= 2f'''(x_i)h^3$$

$$2f'''(x_i)h^3 = f(x_{i-3}) - 3f(x_{i-2}) + 3f(x_{i-1}) - f(x_i) + O(h)$$

La 3^{ra} derivada con diferencias finitas hacía adelante es igual a:

$$\boxed{f''(x_i) = \frac{f(x_{i-3}) - 3f(x_{i-2}) + 3f(x_{i-1}) - f(x_i)}{2h^3} + O(h)}$$